On Vector-Valued Schrödinger Operators in L^p–spaces

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Abstract

We consider the following vector–valued Schrödinger operator

$$\mathcal{A}u = \mathrm{div}(Q\nabla u) - Vu = \left(\mathrm{div}(Q\nabla u_j) - \sum_{k=1}^{m} v_{jk}u_k\right)_{1\le j\le m}$$

acting on vector valued functions $u : \mathbb{R}^d \to \mathbb{C}^m$, where Q is a symmetric real matrix-valued function which is supposed to be bounded and satisfy the ellipticity condition, and V is a measurable *unbounded* matrix-valued function.

We construct a realization A_p of $\mathcal{A}$ in the spaces $L^p(\mathbb{R}^d, \mathbb{C}^m)$, $1 \le p < \infty$, that generates a contractive strongly continuous semigroup. First, by using form methods, we obtain generation of holomorphic semigroups when the potential V is symmetric. In the general case, we use some other techniques of functional analysis and operator theory to get a m-dissipative realization. But in this case the semigroup is not, in general, analytic.

We characterize the domain of the operator A_p in $L^p(\mathbb{R}^d, \mathbb{C}^m)$ by using firstly a non commutative version of the Dore-Venni theorem and then a perturbation theorem due to Okazawa.

We discuss some properties of the semigroup such as analyticity, compactness and positivity. We establish ultracontractivity and deduce that the semigroup is given by an integral kernel. Here, the kernel is actually a matrix whose entries satisfy Gaussian upper estimates.

Further estimates of the kernel entries are given for potentials with a diagonal of polynomial growth. Suitable estimates lead to the asymptotic behavior of the eigenvalues of the matrix Schrödinger operator when the potential is symmetric.