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TESI DI DOTTORATO IN

Stochastic processes governed by the generalized telegraph process and by Brownian motion and their applications

S.S.D. MAT/06 Probabilità e Statistica Matematica

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Introduction

The study of stochastic processes derived from Brownian motion plays an increasingly important role in the scientific panorama given the considerable applications in physical and financial sciences.

Noteworthy, in recent times, the use of models for movements of the animals based on the Brownian bridge, first introduced by Horne et al. (2007), which quickly gained great popularity in the ecological community (see Lonergan et al. 2009, Willems and Hill 2009, Farmer et al. 2010, Takekawa et al. 2010, Yan et al. 2014), or the use of Brownian motions with alternating drift for the price of options in Finance (Di Crescenzo and Pellerey 2002; Kolesnik and Ratanov 2013; Di Crescenzo et al. 2014).

Brownian motion governed by a telegraph process is proposed as a natural and significant generalization of such stochastic processes.

The telegraph process, first studied by Cane (1959) in the context of ethology, is, in its most general form, a random motion $\{Y(t), t \geq 0\}$ characterized by two distinct velocities c and $-v$ (generally we consider $c, v > 0$) regulated by a generic alternating counting process with independent increments $\{N(t), t \geq 0\}$. It is a non-trivial probabilistic object and a large part of the results obtained so far are limited to the simplest case in which $N(t)$ is a process of Poisson (Perry et al. 1999, Stadje and Zacks 2004, Kolesnik and Ratanov 2013).

A first study of processes deriving from a combination of a standard Brownian motion and a generalized integrated telegraph process is given by Di Crescenzo and Zacks (2015), where three particular distributions for the inter-arrival times of the process were considered: exponential with constant intensity, exponential with linear intensities and Erlang distributions. Explicit expressions have been obtained for the transition density of the process and, for the first case, a system of differential equations characteristic for the transition density and the flow function (an explicit form of which is also given) which generalizes both Kolmogorov equations for the classical telegraph process (Kolesnik and Ratanov 2013) and the heat equation for the density of standard Brownian motion.

The aim of this research thesis is to broaden the treatment of this class of stochastic processes also showing an original application to the volcanic phenomenon of Phlegraean bradyseism. It consists in the periodic alternation of phases of raising and lowering the ground level in the territory of Campi Flegrei, in particular in Pozzuoli. The dynamics of this phenomenon, as well as the significant implications for the risk assessment in such a heavily man-made area, is very controversial and still the subject of hypotheses and surveys (De Vivo et al. 2009, De Natale et al. 2017).

This is the plan of the thesis: in Chapter 1 the classical telegraph process is presented. The Kolmogorov equations and the telegraph equation for the transition density are derived, as well as an explicit form for the latter. In Section 1.5 the results are extended to a generalization of the telegraph process that admits velocities with different absolute value, whose alternation is governed by a generic counting process with independent increments.

Chapter 2 presents a generalization of the telegraph process in which the turning rates depend on the current state of the motion. The process is defined in Section 2.1, in Section 2.2 the general expression of the transition density is obtained while in Section 2.3 the case in which the inter-arrival times have Gamma distribution is considered. A first passage time problem is addressed in Section 2.4. Part of these results is also reported in Di Crescenzo and Travaglino 2019 [37].

Chapter 3 deals with a stochastic process defined as the sum of a Brownian motion and a generalized integrated telegraph process, focusing in particular on the case in which the inter-arrival times have an exponential distribution. Explicit forms are derived for the transition density and flow function of the process, as well as a differential system that generalizes the Kolmogorov equations.

Chapter 4 presents an application to the phenomenon of bradyseism in Campi Flegrei, which carries on the one in Travaglino et al. 2018 [96], with more rigorous methods and updated data. After a brief description of the phenomenon in Section 4.1 the data sets available for analysis are presented. The stochastic model used, which is an appropriate generalization of the one seen in Chapter 3, is presented in Section 4.2. Section 4.3 is devoted to data analysis. In Section 4.3.1 a statistical procedure is presented to identify the points of changes in the trend of the motion and the corresponding inflation and deflation episodes. In Section 4.3.2 the velocities of motion, the turning rates and the infinitesimal variance of motion are estimated. The estimates obtained, combined with the knowledge of the probability laws that regulate the motion, allow us to make predictions on the position and velocity in future instants of time: this is done in Section 4.3.3. To verify the admissibility of the model, a statistical test on the Brownian component is performed in

Section 4.4. The chapter ends with some remarks on the obtained results and considerations on possible future developments.

A short appendix on Bessel functions closes the thesis. These functions appear in the explicit form of the transition density of the telegraph process, making the latter a very complex probabilistic object (for example, it is not even thinkable to use the likelihood function for parameter estimation). The process itself, however, appears as a conceptually simple object: throughout the thesis we have tried to exploit this point to obtain the results of interest in a simpler way, using methods designed *ad hoc*.

In this thesis the following software have been used for plots, analysis and statistical processing: R, Mathematica and MATLAB.

The following articles:

- Travaglino F, Di Crescenzo A, Martinucci B, Scarpa R (2018)
  A new model of Campi Flegrei inflation and deflation episodes based on Brownian motion driven by the telegraph process
  Mathematical Geosciences 50:961–975

- Di Crescenzo A, Travaglino F (2019)
  Probabilistic analysis of systems alternating for state-dependent dichotomous noise
  Mathematical Biosciences and Engineering 16:6386–6405

will be the main references of the entire thesis.

Chapter 1

Telegraph process

1.1 Definition and basic properties

Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t, t \geq 0\}, \mathbb{P})$ be a filtered probability space and $\{N(t), t \geq 0\}$ a Poisson process with intensity $\lambda > 0$ adapted to the filtration $\{\mathscr{F}_t, t \geq 0\}$. The probability distribution of $N(t)$ is given by

$$\begin{cases} \mathbb{P}\{N(0) = 0\} = 1, \\ \mathbb{P}\{N(t) = k\} = \dfrac{(\lambda t)^k}{k!} e^{-\lambda t}, \qquad t > 0,\, k = 0, 1, 2, \ldots \end{cases} \tag{1.1}$$

We consider the stochastic process $\{V(t), t \geq 0\}$ with random $V(0)$, independent from $N(t)$ and such that

$$\mathbb{P}\{V(0) = c\} = \mathbb{P}\{V(0) = -c\} = \frac{1}{2}, \tag{1.2}$$

where $c > 0$ is a fixed constant and

$$V(t) := V(0)(-1)^{N(t)}. \tag{1.3}$$

We give the following definition

Definition 1.1. The process $\{X(t), t \geq 0\}$ given by

$$X(t) := \int_0^t V(s) ds = V(0) \int_0^t (-1)^{N(s)} ds \tag{1.4}$$

is said Goldstein-Kac (*integrated*) *telegraph process.*

Such a process describes the position of a particle moving on the line $(-\infty, +\infty)$ with velocity $V(t)$ being constant in modulus (equal to a positive constant c). Each time a Poisson event occurs the sense of the motion changes. At time $t = 0$ the velocity is positive or negative with the same probability.

We can define the processes $X^+(t)$ and $X^-(t)$ such that

$$X^{\pm}(t) := \pm c \int_0^t (-1)^{N(s)} ds, \tag{1.5}$$

with fixed initial velocities $V(0) = \pm c$.

We observe that

$$\begin{aligned} &\mathbb{P}\{X^+(t) = +ct\} = \mathbb{P}\{X^-(t) = -ct\} = \mathbb{P}\bigg\{\int_0^t (-1)^{N(s)} ds = t\bigg\} \\ &= \mathbb{P}\{N(s) = 0 \quad \forall s \in [0, t]\} = \mathbb{P}\{N(t) = 0\} = e^{-\lambda t}, \end{aligned} \tag{1.6}$$

thus

$$\begin{aligned} &\mathbb{P}\{X(t) = \pm ct\} = \mathbb{P}\{X(t) = \pm ct, V(0) = \pm c\} \\ &= \mathbb{P}\{X(t) = \pm ct | V(0) = \pm c\}\mathbb{P}\{V(0) = \pm c\} \\ &= \frac{1}{2}\mathbb{P}\{X^{\pm}(t) = \pm ct\} = \frac{1}{2}e^{-\lambda t}. \end{aligned} \tag{1.7}$$

The distribution of $X(t)$ has two atoms at the points $\pm ct$, which correspond to the case in which no Poisson events occur until time t and thus the particle does not change its initial velocity, elsewhere

$$\mathbb{P}\{X(t) = x\} = 0 \qquad \forall x \neq \pm ct.$$

The distribution function $F(x,t) = \mathbb{P}\{X(t) < x\}$ is continuous for $(x,t) \in \mathbb{R}^2_+ \setminus \{|x| = ct\}$, where $\mathbb{R}^2_+ := \mathbb{R} \times (0, +\infty)$. Furthermore, being $c < +\infty$, $F(x,t) \equiv 0$ for $x < -ct, t > 0$ and $F(x,t) \equiv 1$ for $x > ct, t > 0$. For the same reason the density

$$p(x,t) = \frac{\mathbb{P}\{X(t) \in dx\}}{dx}, \qquad x \in \mathbb{R}, \quad t > 0, \tag{1.8}$$

has finite support $[-ct, ct]$.

The density (1.8) should be understood as a generalized function as it has a singular component. Indeed, one has

$$p(x,t) = \frac{1}{2}e^{-\lambda t}[\delta(x+ct) + \delta(x-ct)] + P(x,t)\mathbb{1}_{\{|x|<ct\}}(x,t) \tag{1.9}$$

for $(x,t) \in \mathbb{R}^2_+$, where δ is the Dirac delta function, P is the absolutely continuous component with support $[-ct, ct]$ and $\mathbb{1}_A(x)$ is the indicator function

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

For every continuous function $\varphi : \mathbb{R} \to \mathbb{R}$ we can compute the mean

$$\begin{aligned}\mathbb{E}[\varphi(x+X(t))] &= \int_{-\infty}^{+\infty} \varphi(x+y)p(y,t)dy \\ &= \frac{1}{2}e^{-\lambda t}\int_{-\infty}^{+\infty} \varphi(x+y)[\delta(y+ct)+\delta(y-ct)]dy \\ &+ \int_{-\infty}^{+\infty} \varphi(x+y)P(y,t)\mathbb{1}_{\{|y|<ct\}}(y,t)dy \\ &= \frac{1}{2}e^{-\lambda t}[\varphi(x-ct)+\varphi(x+ct)] + \int_{-ct}^{ct} \varphi(x+y)P(y,t)dy.\end{aligned} \tag{1.10}$$

Now let $p_+(x,t)$ and $p_-(x,t)$ be the densities conditioned by having fixed the initial velocity $V(0) = \pm c$,

$$p_\pm(x,t) = \frac{\mathbb{P}\{X^\pm(t) \in dx\}}{dx} = \frac{\mathbb{P}\{X(t) \in dx \,|\, V(0) = \pm c\}}{dx} \tag{1.11}$$

for $(x,t) \in \mathbb{R}^2_+$. One has

$$\begin{aligned}p(x,t) &= \frac{\mathbb{P}\{X(t) \in dx\}}{dx} \\ &= \frac{\mathbb{P}\{X(t) \in dx, V(0) = c\} + \mathbb{P}\{X(t) \in dx, V(0) = -c\}}{dx} \\ &= \frac{1}{2}\left[\frac{\mathbb{P}\{X(t) \in dx|V(0) = c\}}{dx} + \frac{\mathbb{P}\{X(t) \in dx|V(0) = -c\}}{dx}\right] \\ &= \frac{1}{2}\left[\, p_+(x,t) + p_-(x,t)\,\right]\end{aligned} \tag{1.12}$$

and, similarly to (1.9),

$$\begin{aligned}p_+(x,t) &= e^{-\lambda t}\delta(x-ct) + P_+(x,t)\mathbb{1}_{\{|x|<ct\}}(x,t), \\ p_-(x,t) &= e^{-\lambda t}\delta(x+ct) + P_-(x,t)\mathbb{1}_{\{|x|<ct\}}(x,t),\end{aligned} \tag{1.13}$$

where P_+ and P_- are the absolutely continuous components of these densities, with $P(x,t) = \frac{1}{2}[P_+(x,t) + P_-(x,t)]$.

1.2 Kolmogorov equations

For all $n = 1, 2, \dots$ we indicate with τ_n the *arrival time* of the n-th Poisson event

$$\tau_n := \inf\{t : N(t) = n\}.$$

Let τ be equal to τ_1. For all $t > 0$ the following equality in distribution conditional on having fixed the initial velocity $V(0) = v, v = \pm c$ holds:

$$X(t) \overset{d}{=} vt\mathbb{1}_{\{\tau>t\}} + \left[v\tau + \widetilde{X}(t-\tau)\right]\mathbb{1}_{\{\tau<t\}} \tag{1.14}$$

where $\{\widetilde{X}(t), t \geq 0\}$ is a telegraph process independent of X with opposite initial velocity $-v$.

From the (1.14) one has

$$\begin{aligned}
&\mathbb{P}\{X(t) \in dx|V(0) = v\} \\
&= \mathbb{P}\{X(t) \in dx, \tau > t|V(0) = v\} + \mathbb{P}\{X(t) \in dx, \tau < t|V(0) = v\} \\
&= \mathbb{P}\{vt \in dx, \tau > t\} + \mathbb{P}\{\widetilde{X}(t-\tau) \in d(x - v\tau), \tau < t\} \\
&= \mathbb{P}\{vt \in dx\}\mathbb{P}\{\tau > t\} + \int_0^t \mathbb{P}\{\widetilde{X}(t-\tau) \in d(x - v\tau)|\tau = s\}\mathbb{P}\{\tau \in ds\} \\
&= \mathbb{P}\{vt \in dx\}\mathbb{P}\{\tau > t\} + \int_0^t \mathbb{P}\{\widetilde{X}(t-s) \in d(x - vs)\}\mathbb{P}\{\tau \in ds\}.
\end{aligned}$$

Recalling that the first arrival time τ of the Poisson process has the exponential distribution of parameter λ we can conclude that the (1.14) is equivalent to the following two integral equations on conditional densities :

$$\begin{aligned}
p_+(x,t) &= e^{-\lambda t}\delta(x - ct) + \int_0^t p_-(x - cs, t - s)\lambda e^{-\lambda s} ds, \\
p_-(x,t) &= e^{-\lambda t}\delta(x + ct) + \int_0^t p_+(x + cs, t - s)\lambda e^{-\lambda s} ds.
\end{aligned} \tag{1.15}$$

Now let us define the last arrival time (in the case $N(t) > 0$) $\tau := \max\{\tau_n|\tau_n < t\}$. It is easy to see that

$$X(t) \overset{d}{=} V(t)t\,\mathbb{1}_{\{N(t)=0\}} + \left[V(t)(t-\tau) + X(\tau)\right]\mathbb{1}_{\{N(t)>0\}}. \tag{1.16}$$

Let us consider the joint probability densities of the position and the current direction of motion:

$$\begin{aligned}
f(x,t) &:= \frac{\mathbb{P}\{X(t) \in dx, V(t) = +c\}}{dx} \\
b(x,t) &:= \frac{\mathbb{P}\{X(t) \in dx, V(t) = -c\}}{dx}
\end{aligned} \tag{1.17}$$

defined for $(x,t) \in \mathbb{R}^2_+$.

Reasoning as before we obtain that the (1.16) is equivalent to the following integral equations for f and b:

$$\begin{aligned} f(x,t) &= \frac{1}{2}e^{-\lambda t}\delta(x-ct) + \int_0^t b(x-c(t-s),s)\lambda e^{-\lambda(t-s)}ds, \\ b(x,t) &= \frac{1}{2}e^{-\lambda t}\delta(x+ct) + \int_0^t f(x+c(t-s),s)\lambda e^{-\lambda(t-s)}ds. \end{aligned} \tag{1.18}$$

We immediately notice that, with the change of variables $s \to t-s$, the integrals in the equations (1.15) coincide with those in the (1.18).

Let us introduce the matrix operator:

$$\mathcal{L} := \begin{pmatrix} \frac{\partial}{\partial t} + c\frac{\partial}{\partial x} + \lambda & -\lambda \\ -\lambda & \frac{\partial}{\partial t} - c\frac{\partial}{\partial x} + \lambda \end{pmatrix}. \tag{1.19}$$

By differentiating the equations (1.15) and (1.18) we get the differential form of the Kolmogorov equations.

Theorem 1.1 (Kolmogorov equations)**.** *The functions* $\mathbf{p} = (p_+, p_-)^T$ *and* $\mathbf{p} = (f, b)^T$ *satisfy the equations*

$$\mathcal{L}\,\mathbf{p} = \mathbf{0}, \qquad |x| < ct. \tag{1.20}$$

For $|x| > ct$ *one has*

$$p_+(x,t) \equiv p_-(x,t) \equiv f(x,t) \equiv b(x,t) \equiv 0 \tag{1.21}$$

and the initial conditions are

$$\begin{aligned} p_+(x,0) &= p_-(x,0) = \delta(x) \\ f(x,0) &= b(x,0) = \frac{1}{2}\delta(x). \end{aligned} \tag{1.22}$$

Proof. The (1.21) is obvious while the initial conditions (1.22) immediately follow from the (1.15) and (1.18).

We will prove just the first of the (1.20)

$$\frac{\partial p_+}{\partial t}(x,t) + c\frac{\partial p_+}{\partial x}(x,t) = -\lambda p_+(x,t) + \lambda p_-(x,t) \tag{1.23}$$

only for the functions p_+ and p_-. The proof of the rest is similar.

Let us start by observing that

$$\begin{aligned} &\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left[e^{-\lambda t}\delta(x-ct)\right] \\ &= -\lambda e^{-\lambda t}\delta(x-ct) - ce^{-\lambda t}\delta'(x-ct) + ce^{-\lambda t}\delta'(x-ct) \\ &= -\lambda e^{-\lambda t}\delta(x-ct) \end{aligned}$$

and

$$\frac{d}{ds}\,p_-(x-cs,t-s) = -\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)[\,p_-(x-cs,t-s)\,].$$

Differentiating the first of (1.15) we get

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\big[p_+(x,t)\big] \\
&= -\lambda e^{-\lambda t}\delta(x-ct) + p_-(x-ct,0)\lambda e^{-\lambda t} - \int_0^t \frac{d}{ds}p_-(x-cs,t-s)\lambda e^{-\lambda s}ds.
\end{aligned}$$

Integrating by parts one has

$$\begin{aligned}
&\int_0^t \frac{d}{ds}p_-(x-cs,t-s)\lambda e^{-\lambda s}ds \\
&= p_-(x-cs,t-s)\lambda e^{-\lambda s}\Big|_{s=0}^{s=t} + \lambda\int_0^t p_-(x-cs,t-s)\lambda e^{-\lambda s}ds \\
&= p_-(x-ct,0)\lambda e^{-\lambda t} - p_-(x,t)\lambda + \lambda\int_0^t p_-(x-cs,t-s)\lambda e^{-\lambda s}ds
\end{aligned}$$

and thus

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\big[p_+(x,t)\big] \\
&= -\lambda e^{-\lambda t}\delta(x-ct) + \lambda p_-(x,t) - \lambda\int_0^t p_-(x-cs,t-s)\lambda e^{-\lambda s}ds \\
&= -\lambda\left[e^{-\lambda t}\delta(x-ct) + \int_0^t p_-(x-cs,t-s)\lambda e^{-\lambda s}ds\right] + \lambda p_-(x,t) \\
&= -\lambda p_+(x,t) + \lambda p_-(x,t).
\end{aligned}$$

□

System (1.20) is also known as *Cattaneo system* (see [58]).

From Theorem 1.1 we deduce that

$$f(x,t) = \frac{1}{2}\,p_+(x,t) \qquad \text{and} \qquad b(x,t) = \frac{1}{2}\,p_-(x,t). \tag{1.24}$$

Given a C^1 function $\varphi : \mathbb{R} \to \mathbb{R}$, we can introduce the Kolmogorov dual equations for the conditional means

$$u_\pm(x,t) := \mathbb{E}\big[\varphi(x+X(t))|V(0) = \pm c\big] = \int_{-\infty}^{+\infty} \varphi(x+y)p_\pm(y,t)dy \tag{1.25}$$

and the joint means

$$
\begin{aligned}
u^{+}(x,t) &:= \int_{-\infty}^{+\infty} \varphi(x+y) f(y,t)\,dy, \\
u^{-}(x,t) &:= \int_{-\infty}^{+\infty} \varphi(x+y) b(y,t)\,dy.
\end{aligned}
\tag{1.26}
$$

By differentiating the previous identity and taking into account the (1.20) we obtain that the functions $\mathbf{u} = (u_+, u_-)^T$ and $\mathbf{u} = (u^+, u^-)^T$ satisfies the equations

$$\mathcal{L}'\,\mathbf{u} = \mathbf{0}, \tag{1.27}$$

with the intial conditions

$$
\begin{aligned}
u_{\pm}(x,0) &= \varphi(x), \\
u^{\pm}(x,0) &= \frac{1}{2}\,\varphi(x).
\end{aligned}
\tag{1.28}
$$

$\mathcal{L}'$ is the dual operator of $\mathcal{L}$:

$$\mathcal{L}' := \begin{pmatrix} \frac{\partial}{\partial t} - c\frac{\partial}{\partial x} + \lambda & -\lambda \\ -\lambda & \frac{\partial}{\partial t} + c\frac{\partial}{\partial x} + \lambda \end{pmatrix}. \tag{1.29}$$

Let us consider now the conditional transition densities

$$p_{\pm}(y,t;x,s) := \frac{\mathbb{P}\{X(t) \in dy \,|\, X(s) = x, V(s) = \pm c\}}{dx} \tag{1.30}$$

for $s < t$, $x, y \in \mathbb{R}$. As for the Brownian motion, these functions only depend on the differences $y - x$ and $t - s$. To convince ourselves of this we first observe that, for $0 < s < t$,

$$V(t) = V(0)(-1)^{N(t)} = V(0)(-1)^{N(s)}(-1)^{N(t)-N(s)} = V(s)(-1)^{N(t)-N(s)}.$$

Let us now analyze the conditional increments

$$
\begin{aligned}
&\big[X(t) - X(s) | V(s) = v\big] \\
&= v\int_s^t (-1)^{N(\tau)-N(s)} d\tau \overset{d}{=} v\int_s^t (-1)^{N(\tau-s)} d\tau \\
&= v\int_0^{t-s} (-1)^{N(\tau)} d\tau = \big[X(t-s) | V(0) = v\big],
\end{aligned}
$$

with the distribution equality that follows from the stationarity of the increments of the Poisson process. It is also interesting to note that, for the same reason, conditional increments are independent. It is clear that

$$
\begin{aligned}
&p_{\pm}(y,t;x,s) = \mathbb{P}\{X(t) \in dy | X(s) = x, V(s) = \pm c\} \\
&= \mathbb{P}\{X(t) - X(s) \in d(y-x) | V(s) = \pm c\} \\
&= \mathbb{P}\{X(t-s) \in d(y-x) | V(0) = \pm c\} = p_{\pm}(y-x, t-s)
\end{aligned}
\tag{1.31}
$$

for every $s < t$ and $x, y \in \mathbb{R}$.

From the (1.15) we have then

$$\begin{aligned} & p_{\pm}(y,t;x,s) \\ & = e^{-\lambda(t-s)}\delta(y - x \mp c(t-s)) + \int_0^{t-s} p_{\mp}(y,t;x \pm c\tau, s+\tau)\lambda e^{-\lambda\tau} d\tau. \end{aligned} \tag{1.32}$$

By differentiating these equations in a similar way to the proof of Theorem (1.1) we obtain the backward Kolmogorov differential equations:

$$\begin{aligned} -\frac{\partial p_+}{\partial s}(y,t;x,s) - c\frac{\partial p_+}{\partial x}(y,t;x,s) &= -\lambda p_+(y,t;x,s) + \lambda p_-(y,t;x,s) \\ -\frac{\partial p_-}{\partial s}(y,t;x,s) + c\frac{\partial p_-}{\partial x}(y,t;x,s) &= -\lambda p_-(y,t;x,s) + \lambda p_+(y,t;x,s) \end{aligned} \tag{1.33}$$

for $s < t$ and $|y - x| < c(t-s)$.

1.3 Telegraph equation

In this section we will see that the first order hyperbolic systems (1.20), (1.27) and (1.33) are equivalent to just one second order hyperbolic differential equation.

Theorem 1.2. *The functions p, $p_{\pm}$, f and b are solutions of the telegraph differential equation:*

$$\frac{\partial^2 p}{\partial t^2}(x,t) + 2\lambda\frac{\partial p}{\partial t}(x,t) = c^2\frac{\partial^2 p}{\partial x^2}(x,t), \qquad (x,t) \in \mathbb{R}^2_+. \tag{1.34}$$

Proof. We will make use of the so called *Kac's trick*. We remember that

$$p(x,t) = \frac{p_+(x,t) + p_-(x,t)}{2}$$

and define

$$w(x,t) := \frac{p_+(x,t) - p_-(x,t)}{2}. \tag{1.35}$$

We see that

$$\begin{aligned} \frac{1}{2}(1,1) \cdot \mathcal{L}\mathbf{p} &= \frac{\partial p}{\partial t} + c\frac{\partial w}{\partial x} \\ \frac{1}{2}(1,-1) \cdot \mathcal{L}\mathbf{p} &= \frac{\partial w}{\partial t} + c\frac{\partial p}{\partial x} + 2\lambda w \end{aligned}$$

with $\mathbf{p} = (p_+, p_-)^T$ and $\mathcal{L}$ given by the (1.19).

System (1.20) is thus equivalent to

$$\begin{cases} \dfrac{\partial p}{\partial t} = -c\dfrac{\partial w}{\partial x}, \\ \dfrac{\partial w}{\partial t} = -c\dfrac{\partial p}{\partial x} - 2\lambda w. \end{cases} \tag{1.36}$$

Differentiating the first equation with respect to t and the second equation with respect to x we get

$$\frac{\partial^2 p}{\partial t^2} = -c\frac{\partial^2 w}{\partial x \partial t}, \qquad \frac{\partial^2 w}{\partial x \partial t} = -c\frac{\partial^2 p}{\partial x^2} - 2\lambda\frac{\partial w}{\partial x}$$

and therefore, eliminating the mixed derivative,

$$\frac{\partial^2 p}{\partial t^2} = c^2\frac{\partial^2 p}{\partial x^2} + 2\lambda c\frac{\partial w}{\partial x}.$$

From the first of the (1.36) it follows the (1.34) for p.

We can similarly verify that w satisfies the (1.34) by differentiating the first equations of system (1.36) with respect to x and the second with respect to t and again eliminating the mixed derivative.

The thesis then follows from the linearity of the telegraph equation (1.34), taking into account the relationships

$$p_+(x,t) = p(x,t) + w(x,t), \qquad p_-(x,t) = p(x,t) - w(x,t)$$

and the (1.24). $\square$

It is well known that the transition functions of the standard Brownian motion and of the derived processes can be obtained as solutions of suitable initial value problems for the heat equation. Similarly, the transition density of the telegraph process p, as well as the $p_\pm$, f and b, is the solution of an initial value problem for the telegraph equation.

To derive the initial conditions for the conditional densities $p_\pm$, we evaluate the equations (1.20) for $t = 0$, taking into account the (1.22):

$$\frac{\partial p_\pm}{\partial t}(x,0) \pm c\frac{\partial p_\pm}{\partial x}(x,0) = 0.$$

Furthermore, from the (1.15) it is easy to obtain

$$\frac{\partial p_\pm}{\partial x}(x,0) = \delta'(x).$$

So we have

$$p_{\pm}(x,0) = \delta(x), \qquad \frac{\partial p_{\pm}}{\partial t}(x,0) = \mp c\delta'(x). \tag{1.37}$$

Then it is easy to derive the initial conditions for $p = \frac{1}{2}(p_+ + p_-)$

$$p(x,0) = \delta(x), \qquad \frac{\partial p}{\partial t}(x,0) = 0 \tag{1.38}$$

and for f and b (taking into account the relationships (1.24))

$$\begin{aligned} f(x,0) &= \frac{1}{2}\delta(x), & \frac{\partial f}{\partial t}(x,0) &= -\frac{c}{2}\delta'(x), \\ b(x,0) &= \frac{1}{2}\delta(x), & \frac{\partial b}{\partial t}(x,0) &= +\frac{c}{2}\delta'(x). \end{aligned} \tag{1.39}$$

Now let $\varphi : \mathbb{R} \to \mathbb{R}$ be a C^2 function. We consider the conditional means (1.25); since

$$u_{\pm}(x,t) = \int_{-\infty}^{+\infty} \varphi(x+y)p_{\pm}(y,t)dy = \int_{-\infty}^{+\infty} \varphi(y)p_{\pm}(y-x,t)dy,$$

one has

$$\left(\frac{\partial^2}{\partial t^2} + 2\lambda\frac{\partial}{\partial t} - c^2\frac{\partial^2}{\partial x^2}\right)u_{\pm}(x,t) = \int_{-\infty}^{+\infty} \varphi(y)\left(\frac{\partial^2}{\partial t^2} + 2\lambda\frac{\partial}{\partial t} - c^2\frac{\partial^2}{\partial x^2}\right)p_{\pm}(y-x,t)dy$$

and thus the $u_{\pm}$ satisfy the (1.34), too.

We can therefore state the following theorem.

Theorem 1.3. *The functions $u_{\pm}$, as well as the function*

$$u(x,t) = \mathbb{E}[\varphi(x + X(t))] = \frac{1}{2}(u_+(x,t) + u_-(x,t)),$$

are solutions of the telegraph equation (1.34) with the initial conditions:

$$\begin{aligned} &u_+(x,0) = u_-(x,0) = u(x,0) = \varphi(x), \\ &\frac{\partial u_{\pm}}{\partial t}(x,0) = \pm c\varphi'(x), \qquad \frac{\partial u}{\partial t}(x,0) = 0. \end{aligned} \tag{1.40}$$

Finally, we observe that is possible to apply Kac's trick to the (1.33) too, obtaining the following version of the telegraph equation for the functions $p = p_{\pm}(y,t;x,s)$:

$$\frac{\partial^2 p}{\partial s^2} - 2\lambda\frac{\partial p}{\partial s} = c^2\frac{\partial^2 p}{\partial x^2}, \tag{1.41}$$

with the *final* conditions

$$p_{\pm}(y,t;x,t) = \delta(y-x), \qquad \frac{\partial p_{\pm}}{\partial s}(y,t;x,t) = \pm c\delta'(y-x). \tag{1.42}$$

Obviously, the previous identities can be obtained directly from the respective results for the functions $p_{\pm}(x,t)$, taking into account relationships (1.31).

We emphasize here that all the functions and derivatives treated must be understood as generalized functions.

In particular, in the space $\mathcal{D}'(\mathbb{R})$ of the distributions, the solutions of the (1.34) and of the equivalent hyperbolic systems having support included in $\overline{\mathbb{R}^2_+}$ are unique once the initial conditions are set.

1.4 Transition Density

In the following pages we will use the following lemma.

Lemma 1.1. *The solution $v(x,t)$ of the equation*

$$\frac{\partial^2 v}{\partial t^2}(x,t) - c^2\frac{\partial^2 v}{\partial x^2}(x,t) = \lambda^2 v(x,t), \qquad (x,t) \in \mathbb{R}^2_+, \tag{1.43}$$

with the initial conditions

$$v(x,0) = \varphi(x), \qquad \frac{\partial v}{\partial t}(x,0) = \psi(x), \tag{1.44}$$

can be written in the form

$$v(x,t) = Z(x,t;\psi) + \frac{\partial Z}{\partial t}(x,t;\varphi), \tag{1.45}$$

where, for every continuous function $\psi : \mathbb{R} \to \mathbb{R}$,

$$\begin{aligned} Z(x,t;\psi) &:= \frac{1}{2}\int_{-t}^{t} \psi(x+cs) I_0\Big(\lambda\sqrt{t^2-s^2}\Big)ds \\ &= \frac{1}{2}\int_0^t [\psi(x+cs) + \psi(x-cs)] I_0\Big(\lambda\sqrt{t^2-s^2}\Big)ds. \end{aligned} \tag{1.46}$$

Proof. Omitted. See [58, pp. 34-35]. □

The following theorem gives the explicit form of the transition density $p(x,t)$ of the telegraph process $\{X(t), t \geq 0\}$.

Theorem 1.4.

$$\begin{aligned} p(x,t) &= \frac{1}{2}e^{-\lambda t}[\delta(x+ct)+\delta(x-ct)] \\ &+\frac{1}{2c}e^{-\lambda t}\left[\lambda I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right)+\frac{\partial}{\partial t}I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right)\right]\mathbb{1}_{\{|x|<ct\}} \end{aligned} \tag{1.47}$$

for $(x,t)\in\mathbb{R}^2_+$, where $I_0(x)$ is the modified Bessel function.

Proof. As seen in the previous paragraph, it is sufficient to show that the density $p = p(x,t)$ defined by (1.47) is a solution of the (1.34) with the initial conditions (1.38). It is easy to see that this is equivalent to requiring that the function $v(x,t) = e^{\lambda t}p(x,t)$ satisfies the equation (1.43) with the initial conditions

$$v(x,0)=\delta(x), \qquad \frac{\partial v}{\partial t}(x,0)=\lambda\delta(x).$$

By applying Lemma 1.1 we therefore have

$$\begin{aligned} v(x,t) &= Z(x,t;\lambda\delta)+\frac{\partial Z}{\partial t}(x,t;\delta) \\ &= \frac{1}{2}\big[\delta(x-ct)+\delta(x+ct)\big] \\ &+\frac{1}{2}\int_{-t}^{t}\delta(x+cs)\Big[\lambda I_0\Big(\lambda\sqrt{t^2-s^2}\Big)+\frac{\partial}{\partial t}I_0\Big(\lambda\sqrt{t^2-s^2}\Big)\Big]ds \\ &= \frac{1}{2}\big[\delta(x-ct)+\delta(x+ct)\big] \\ &+\frac{1}{2c}\int_{-ct}^{ct}\delta(x+y)\left[\lambda I_0\left(\lambda\sqrt{t^2-\frac{y^2}{c^2}}\right)+\frac{\partial}{\partial t}I_0\left(\lambda\sqrt{t^2-\frac{y^2}{c^2}}\right)\right]dy \\ &= \frac{1}{2}\big[\delta(x-ct)+\delta(x+ct)\big] \\ &+\frac{1}{2c}\left[\lambda I_0\left(\lambda\sqrt{t^2-\frac{x^2}{c^2}}\right)+\frac{\partial}{\partial t}I_0\left(\lambda\sqrt{t^2-\frac{x^2}{c^2}}\right)\right]. \end{aligned}$$

Multiplying by $e^{-\lambda t}$ we get the thesis. □

Taking into account the well known identity for Bessel function $I_0'(z) = I_1(z)$, one has

$$\frac{\partial}{\partial t}I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right)=\frac{\lambda ct}{\sqrt{c^2t^2-x^2}}\,I_1\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right) \tag{1.48}$$

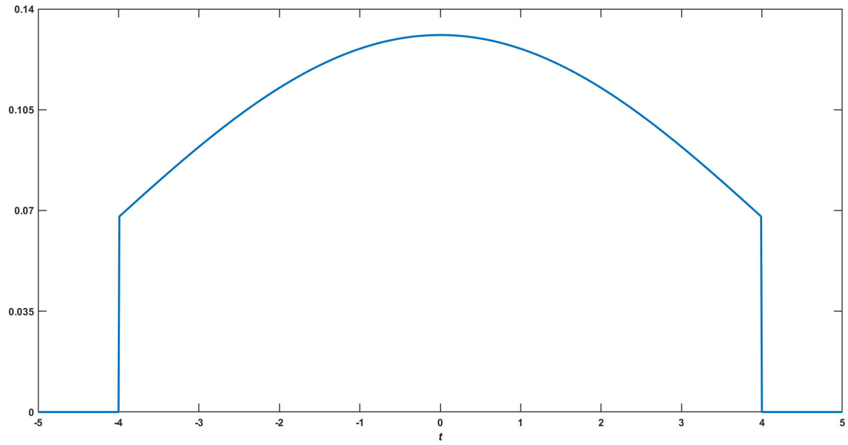

Figure 1.1: The absolutely continuous component (1.50) of the probability density (1.49), with $t = 2$, $\lambda = 1$, and $c = 2$.

and therefore the (1.47) can be written in the form:

$$\begin{aligned} p(x,t) &= \frac{1}{2}e^{-\lambda t}[\delta(x+ct)+\delta(x-ct)] \\ &+\frac{\lambda}{2c}e^{-\lambda t}\left[I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right)+\frac{ct}{\sqrt{c^2t^2-x^2}}\,I_1\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right)\right]\mathbb{1}_{\{|x|<ct\}}. \end{aligned} \tag{1.49}$$

Comparing the previous identity with the (1.9) we get the expression of the absolutely continuous component of the transition density of the telegraph process:

$$P(x,t) = \frac{\lambda}{2c}e^{-\lambda t}\left[I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right)+\frac{ct}{\sqrt{c^2t^2-x^2}}\,I_1\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right)\right] \tag{1.50}$$

for $(x,t)\in\mathbb{R}^2_+$, $|x|<ct$. Figure 1.1 shows a plot of the (1.50).

We can now reason as in Theorem 1.4 to derive the conditional densities $p_\pm(x,t)$. We notice then that, since the $p_\pm$ satisfy the (1.34) with the initial conditions (1.37), the functions $q_\pm(x,t) = e^{\lambda t}$ are solutions of the (1.43) with initial conditions

$$q_\pm(x,0) = \delta(x), \qquad \frac{\partial q_\pm}{\partial t}(x,0) = \lambda\delta(x) \mp c\delta'(x). \tag{1.51}$$

By applying again Lemma 1.1 we get the explicit form of the $p_\pm$:

$$\begin{aligned}
p_\pm(x,t) &= e^{-\lambda t}\left[Z(x,t;\lambda\delta \mp c\delta') + \frac{\partial Z}{\partial t}(x,t;\delta)\right] \\
&= e^{-\lambda t}\left[Z(x,t;\lambda\delta) + \frac{\partial Z}{\partial t}(x,t;\delta)\right] + e^{-\lambda t} Z(x,t;\mp c\delta') \\
&= p(x,t) \mp \frac{1}{2}e^{-\lambda t}\Big[\delta(x+ct) - \delta(x-ct)\Big] \\
&\pm \frac{1}{2c}e^{-\lambda t}\left[\frac{\lambda x}{\sqrt{c^2t^2-x^2}}\, I_1\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right)\mathbb{1}_{\{|x|<ct\}}\right] \\
&= e^{-\lambda t}\delta(x \mp ct) \\
&+ \frac{\lambda}{2c}e^{-\lambda t}\left[I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right) + \frac{ct \pm x}{\sqrt{c^2t^2-x^2}}\, I_1\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right)\right]\mathbb{1}_{\{|x|<ct\}}.
\end{aligned} \tag{1.52}$$

Comparing the previous identities with the (1.13) we obtain the absolutely continuous component:

$$P_\pm(x,t) = \frac{\lambda}{2c}e^{-\lambda t}\left[I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right) + \frac{ct \pm x}{\sqrt{c^2t^2-x^2}}\, I_1\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right)\right]. \tag{1.53}$$

We notice that

$$P_+(-x,t) = P_-(x,t) \qquad \text{and} \qquad p_+(-x,t) = p_-(x,t). \tag{1.54}$$

This is a consequence of the symmetry of the telegraph process; indeed

$$-X^+(t) = -c\int_0^t (-1)^{N(s)}ds = X^-(t). \tag{1.55}$$

We also get that $f(-x,t) = b(x,t)$, taking into account the (1.24).

1.5 Generalized telegraph process

So far we have considered a telegraph process with velocities c and $-c$, equal in the modulus, which alternate according to a Poisson process.

Let us now consider a process $\{X(t), \geq 0\}$ characterized by velocities c and $-v$, with $c, v > 0$ and $c \neq v$, governed by a generic alternating counting process $\{N(t), t \geq 0\}$ with independent increments. Let $V(t)$ denote the velocity of $X(t)$ at time instant t and assume that it is random at the initial instant:

$$\mathbb{P}\{V(0) = c\} = \mathbb{P}\{V(0) = -v\} = \frac{1}{2}. \tag{1.56}$$

Definition 1.2. The stochastic process

$$X(t) := \int_0^t V(s)ds, \qquad t \geq 0, \tag{1.57}$$

where

$$V(t) := V(0)(-1)^{N(t)} + \frac{c-v}{2}\Big[1-(-1)^{N(t)}\Big], \qquad t > 0, \tag{1.58}$$

is said *generalized telegraph process*.

The process $N(t)$ denotes the number of velocity changes in the interval $[0,t]$ and is governed by the sequences of random variables $\{U_1, U_2, \dots\}$ and $\{D_1, D_2, \dots\}$, independent of each other and of $V(0)$, where U_i (respectively D_i) describes the i-th period during which $X(t)$ has positive (respectively negative) velocity.

We assume that the variables U_i and D_i, $i = 1, 2, \dots,$ are absolutely continuous.

It is clear that $N(t)$ depends on $V(0)$, since the sequence of interarrival times is $U_1, D_1, U_2, D_2, \dots$ if $V(0) = c$ and $D_1, U_1, D_2, U_2, \dots$ if $V(0) = -v$.

For $i = 1, 2, \dots$ let F_{U_i} and F_{D_i} denote the cumulative distribution functions (CDF) of U_i and D_i, f_{U_i} and f_{D_i} the respective probability densitiy functions (PDF), $\overline{F}_{U_i} = 1 - F_{U_i}$ and $\overline{F}_{D_i} = 1 - F_{D_i}$ the survival functions.

Furthermore, for each $n = 1, 2, \dots,$ let us define the following sums

$$\begin{aligned} U^{(n)} &:= U_1 + U_2 + \cdots + U_n, \\ D^{(n)} &:= D_1 + D_2 + \cdots + D_n, \end{aligned} \tag{1.59}$$

and denote with $F_U^{(n)}$, $F_D^{(n)}$ and $f_U^{(n)}$, $f_D^{(n)}$ the relative CDFs and PDFs, respectively.

We notice that

$$\begin{aligned} &[V(t)|V(0) = c] \\ &= \begin{cases} c & \text{if } U^{(n)} + D^{(n)} \leq t < U^{(n+1)} + D^{(n)} \text{ for some } n = 0, 1, 2, \dots \\ -v & \text{if } U^{(n+1)} + D^{(n)} \leq t < U^{(n+1)} + D^{(n+1)} \text{ for some } n = 0, 1, 2, \dots \end{cases} \end{aligned} \tag{1.60}$$

$$\begin{aligned} &[V(t)|V(0) = -v] \\ &= \begin{cases} -v & \text{if } D^{(n)} + U^{(n)} \leq t < D^{(n+1)} + U^{(n)} \text{ for some } n = 0, 1, 2, \dots \\ c & \text{if } D^{(n+1)} + U^{(n)} \leq t < D^{(n+1)} + U^{(n+1)} \text{ for some } n = 0, 1, 2, \dots \end{cases} \end{aligned} \tag{1.61}$$

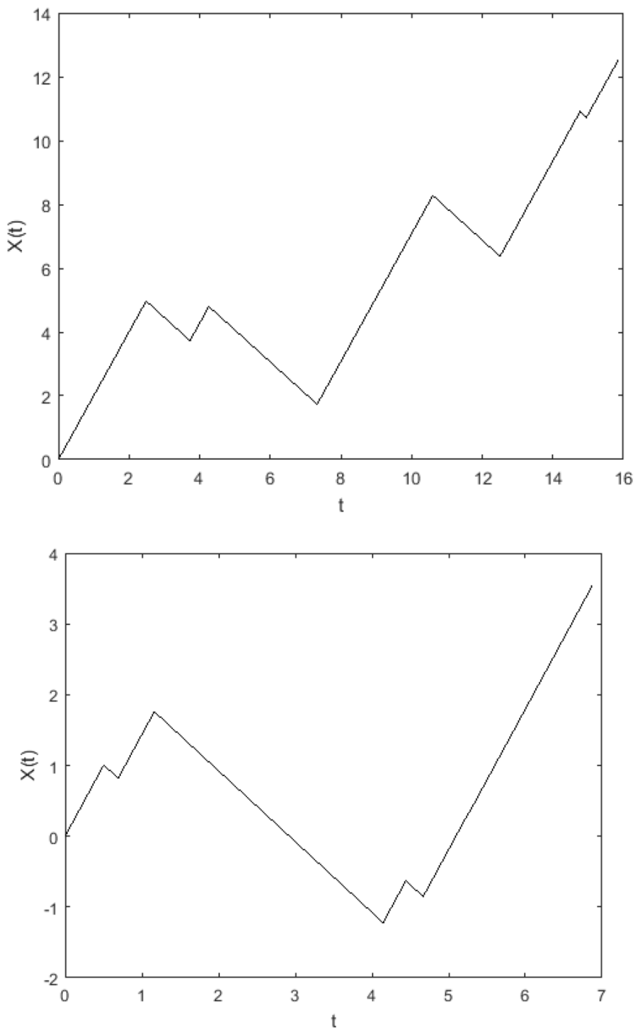

Figure 1.2: Simulated sample paths of a generalized telegraph process with $c = 2$ and $v = 1$.

having set in the previous identities $U^{(0)} = D^{(0)} = 0$.

To derive the probability law of $X(t)$, it is of fundamental importance to determine the distribution of the process

$$W(t) := \int_0^t \mathbb{1}_{\{V(s)=c\}}(s)\, ds, \qquad t \geq 0, \tag{1.62}$$

called *occupation time*, which represents the total amount of time in which $X(t)$ has positive velocity in the interval $[0,t]$.

It is clear that $\mathbb{P}\{0 \leq W(t) \leq t\} = 1$.

As we will see in the next theorem, the distribution of $W(t)$ has two atoms at the points 0 and t and an absolutely continuous component in the interval $(0,t)$. We therefore introduce, for $x \in (0,t)$ and $v_0, v_t \in \{c, -v\}$, the density

$$\psi(x,t) := \frac{\partial}{\partial x}\mathbb{P}\{W(t) \leq x\} = \frac{\mathbb{P}\{W(t) \in dx\}}{dx}, \tag{1.63}$$

the joint density

$$\psi(x,t;v_t) := \frac{\partial}{\partial x}\mathbb{P}\{W(t) \leq x, V(t) = v_t\}, \tag{1.64}$$

and the conditional joint density given the initial direction

$$\psi_{v_0}(x,t;v_t) := \frac{\partial}{\partial x}\mathbb{P}\{W(t) \leq x, V(t) = v_t \,|\, V(0) = v_0\}. \tag{1.65}$$

Let us first observe that

$$\begin{aligned}
&\mathbb{P}\{W(t) \leq x, V(t) = v_t\} \\
&= \mathbb{P}\{W(t) \leq x, V(t) = v_t, V(0) = c\} + \mathbb{P}\{W(t) \leq x, V(t) = v_t, V(0) = -v\} \\
&= \mathbb{P}\{W(t) \leq x, V(t) = v_t | V(0) = c\}\, \mathbb{P}\{V(0) = c\} \\
&\quad + \mathbb{P}\{W(t) \leq x, V(t) = v_t | V(0) = -v\}\, \mathbb{P}\{V(0) = -v\} \\
&= \frac{\mathbb{P}\{W(t) \leq x, V(t) = v_t | V(0) = c\}}{2} + \frac{\mathbb{P}\{W(t) \leq x, V(t) = v_t | V(0) = -v\}}{2}
\end{aligned}$$

hence

$$\psi(x,t;v_t) = \frac{1}{2}\Big[\psi_c(x,t;v_t) + \psi_{-v}(x,t;v_t)\Big]. \tag{1.66}$$

Of course one also have

$$\mathbb{P}\{W(t) \leq t\} = \mathbb{P}\{W(t) \leq t, V(t) = c\} + \mathbb{P}\{W(t) \leq t, V(t) = -v\}$$

and thus

$$\psi(x,t) = \psi(x,t;c) + \psi(x,t;-v). \tag{1.67}$$

Theorem 1.5. *For every $t > 0$*

$$\mathbb{P}\{W(t) = 0\} = \frac{1}{2}\overline{F}_{D_1}(t), \qquad \mathbb{P}\{W(t) = t\} = \frac{1}{2}\overline{F}_{U_1}(t). \tag{1.68}$$

Moreover, for $x \in (0, t)$,

$$\begin{aligned} \psi_c(x, t; c) &= \sum_{n=1}^{+\infty} \Big[F_U^{(n)}(x) - F_U^{(n+1)}(x)\Big] f_D^{(n)}(t - x), \\ \psi_c(x, t; -v) &= \sum_{n=0}^{+\infty} \Big[F_D^{(n)}(t - x) - F_D^{(n+1)}(t - x)\Big] f_U^{(n+1)}(x), \end{aligned} \tag{1.69}$$

and symmetrically

$$\begin{aligned} \psi_{-v}(x, t; c) &= \sum_{n=0}^{+\infty} \Big[F_U^{(n)}(x) - F_U^{(n+1)}(x)\Big] f_D^{(n+1)}(t - x), \\ \psi_{-v}(x, t; -v) &= \sum_{n=1}^{+\infty} \Big[F_D^{(n)}(t - x) - F_D^{(n+1)}(t - x)\Big] f_U^{(n)}(x). \end{aligned} \tag{1.70}$$

Proof. Since the random times U_i and D_i are positive, one has

$$\begin{aligned} W(t) = 0 &\iff V(0) = -v \text{ and } D_1 \geq t, \\ W(t) = 1 &\iff V(0) = c \text{ and } U_1 \geq t, \end{aligned}$$

therefore

$$\begin{aligned} \mathbb{P}\{W(t) = 0\} &= \mathbb{P}\{V(0) = -v, D_1 \geq t\} \\ &= \mathbb{P}\{V(0) = -v\}\mathbb{P}\{D_1 \geq t\} = \frac{1}{2}\mathbb{P}\{D_1 > t\} = \frac{1}{2}\overline{F}_{D_1}(t) \end{aligned}$$

where second equality follows from the independence of the variables D_i of $V(0)$ and third equality follows from being absolutely continuous D_1. Similarly

$$\mathbb{P}\{W(t) = t\} = \mathbb{P}\{V(0) = c\}\mathbb{P}\{U_1 > t\} = \frac{1}{2}\overline{F}_{U_1}(t).$$

Now let us prove the first of the (1.69).

$$\begin{aligned} \psi_c(x, t; c)dx :=& \mathbb{P}\{W(t) \in dx, V(t) = c | V(0) = c\} \\ =& \sum_{n=0}^{+\infty} \mathbb{P}\{W(t) \in dx, V(t) = c | V(0) = c, U^{(n)} < x \leq U^{(n+1)}\} \cdot \\ & \cdot \mathbb{P}\{U^{(n)} < x \leq U^{(n+1)}\} \\ =& \sum_{n=0}^{+\infty} \mathbb{P}\{W(t) \in dx, V(t) = c | V(0) = c, U^{(n)} < x \leq U^{(n+1)}\} \cdot \\ & \cdot \Big[F_U^{(n)}(x) - F_U^{(n+1)}(x)\Big]. \end{aligned}$$

Let us analyze the probabilities appearing as addends in the previous summation. Since they are conditional probabilities given $V(0) = c$, the interarrival times sequence is $U_1, D_1, \dots, U_n, D_n, U_{n+1}, \dots$.
From $W(t) \in dx$ and $U^{(n)} < x \le U^{(n+1)}$ we deduce that t is the sum of $U^{(n)}$, $D^{(n)}$ and a portion of $U^{(n+1)}$.
Then $t - W(t) = D^{(n)}$ and time instant t falls within the period U_{n+1}, so that condition $V(t) = c$ is superfluous. Hence

$$\begin{aligned}
&\mathbb{P}\{W(t) \in dx, V(t) = c | V(0) = c, U^{(n)} < x \le U^{(n+1)}\} \\
&= \mathbb{P}\{D^{(n)} \in d(t-x) | V(0) = c, U^{(n)} < x \le U^{(n+1)}\} \\
&= \mathbb{P}\{D^{(n)} \in d(t-x)\} = f_D^{(n)}(t-x)dx,
\end{aligned}$$

with third equality following from the independence of random times D_i of the variables $V(0)$ and U_i.
Then, taking into account that $0 < x < t$ and $D^{(0)} = 0$, one has $f_D^{(0)}(t-x) = 0$ and thus:

$$\begin{aligned}
\psi_c(x,t;c)dx &= \sum_{n=0}^{+\infty} \Big[F_U^{(n)}(x) - F_U^{(n+1)}(x)\Big] \cdot \\
&\cdot \mathbb{P}\{W(t) \in dx, V(t) = c | V(0) = c, U^{(n)} < x \le U^{(n+1)}\} \\
&= \sum_{n=1}^{+\infty} \Big[F_U^{(n)}(x) - F_U^{(n+1)}(x)\Big] f_D^{(n)}(t-x)dx.
\end{aligned}$$

The proof of the first of the (1.70) is similar, with the difference that, being $V(0) = -v$, the interarrival times sequence is $D_1, U_1, \dots, D_n, U_n, D_{n+1}, U_{n+1}, \dots$ and therefore $t - W(t) = D^{(n+1)}$:

$$\begin{aligned}
\psi_{-v}(x,t;c)dx &= \sum_{n=0}^{+\infty} \Big[F_U^{(n)}(x) - F_U^{(n+1)}(x)\Big] \cdot \\
&\cdot \mathbb{P}\{W(t) \in dx, V(t) = c | V(0) = -v, U^{(n)} < x \le U^{(n+1)}\} \\
&= \sum_{n=0}^{+\infty} \Big[F_U^{(n)}(x) - F_U^{(n+1)}(x)\Big] f_D^{(n+1)}(t-x)dx.
\end{aligned}$$

The remaining equations can be derived by symmetry by inverting the roles of processes $W(t)$ and $t - W(t)$. □

The previous theorem, along with the identities (1.66) and (1.67), gives us the explicit form of the absolutely continuous component $\psi(x,t)$ of the density of the process $W(t)$.

For every $(x,t) \in \mathbb{R}^2_+$, the generalized density of $W(t)$ is given by

$$\frac{\mathbb{P}\{W(t) \in dx\}}{dx} = \frac{1}{2}\overline{F}_{D_1}(t)\delta(x) + \frac{1}{2}\overline{F}_{U_1}(t)\delta(x-t) + \psi(x,t)\mathbb{1}_{\{0<x<t\}}. \quad (1.71)$$

From the proof of the previous theorem we notice that, for every $x \in [0,t]$, the random variable $W(t)$ is equal in distribution to $t - W(t)$ once the distributions of the processes $U^{(n)}$ and $D^{(n)}$ are inverted for each n. We can write:

$$\mathbb{P}\{W(t) \leq x\}|_{\{(U^{(n)},D^{(n)});n\in\mathbb{N}\}} = \mathbb{P}\{W(t) \geq t-x\}|_{\{(D^{(n)},U^{(n)});n\in\mathbb{N}\}}. \quad (1.72)$$

Corollary 1.1. *For every $t > 0$*

$$\mathbb{E}[W(t)] = \frac{1}{2}\int_0^t \Big[\overline{\Psi}_c(x,t) + \overline{\Psi}_{-v}(x,t)\Big]dx, \quad (1.73)$$

$$Var[W(t)] = \int_0^t x\Big[\overline{\Psi}_c(x,t) + \overline{\Psi}_{-v}(x,t)\Big]dx - \{\mathbb{E}[W(t)]\}^2, \quad (1.74)$$

where, for $0 < x < t$,

$$\begin{aligned}\overline{\Psi}_c(x,t) :&= \mathbb{P}\{W(t) > x | V(0) = c\} \\ &= \sum_{n=0}^{+\infty}\Big[F_U^{(n)}(x) - F_U^{(n+1)}(x)\Big]F_D^{(n)}(t-x),\end{aligned} \quad (1.75)$$

$$\begin{aligned}\overline{\Psi}_{-v}(x,t) :&= \mathbb{P}\{W(t) > x | V(0) = -v\} \\ &= \sum_{n=0}^{+\infty}\Big[F_D^{(n)}(t-x) - F_D^{(n+1)}(t-x)\Big]F_U^{(n)}(x).\end{aligned} \quad (1.76)$$

Proof. Let us start by proving the (1.73):

$$\begin{aligned}\mathbb{E}[W(t)] :&= \int_0^{+\infty}\mathbb{P}\{W(t) > x\}dx - \int_{-\infty}^0 \mathbb{P}\{W(t) \leq x\}dx \\ &= \int_0^t \mathbb{P}\{W(t) > x\}dx + \int_t^{+\infty}\mathbb{P}\{W(t) > x\}dx \\ &= \int_0^t \Big[\mathbb{P}\{W(t) > x, V(0) = c\} + \mathbb{P}\{W(t) > x, V(0) = -v\}\Big]dx \\ &= \frac{1}{2}\int_0^t \Big[\mathbb{P}\{W(t) > x | V(0) = c\} + \mathbb{P}\{W(t) > x | V(0) = -v\}\Big]dx \\ &= \frac{1}{2}\int_0^t \Big[\overline{\Psi}_c(x,t) + \overline{\Psi}_{-v}(x,t)\Big]dx.\end{aligned}$$

What remains to be proven are the (1.75), (1.76).

$$\begin{aligned}
&\mathbb{P}\{W(t) > x|V(0) = c\}\\
&= \mathbb{P}\{x < W(t) < t|V(0) = c\} + \mathbb{P}\{W(t) = t|V(0) = c\}\\
&= \mathbb{P}\{x < W(t) < t, V(t) = c|V(0) = c\}\\
&+ \mathbb{P}\{x < W(t) < t, V(t) = -v|V(0) = c\} + \mathbb{P}\{U_1 \geq t\}\\
&= \int_x^t \Big[\psi_c(y, t; c) + \psi_c(y, t; -v)\Big] dy + \overline{F}_{U_1}(t).
\end{aligned}$$

We compute the integral using (1.69):

$$\begin{aligned}
&\int_x^t \Big[\psi_c(y, t; c) + \psi_c(y, t; -v)\Big] dx\\
&= \int_x^t \sum_{n=0}^{+\infty} \Big\{ \Big[F_U^{(n+1)}(y) - F_U^{(n+2)}(y)\Big] f_D^{(n+1)}(t-y)\\
&\quad + \Big[F_D^{(n)}(t-y) - F_D^{(n+1)}(t-y)\Big] f_U^{(n+1)}(y)\Big\} dy\\
&= \sum_{n=0}^{+\infty} \int_x^t \Big\{ F_U^{(n+1)}(y) f_D^{(n+1)}(t-y) - F_D^{(n+1)}(t-y) f_U^{(n+1)}(y)\\
&\quad - F_U^{(n+2)}(y) f_D^{(n+1)}(t-y) + F_D^{(n+1)}(t-y) f_U^{(n+2)}(y)\Big\} dy\\
&\quad + \int_x^t F_D^{(0)}(t-y) f_U^{(1)}(y) dy\\
&= \sum_{n=0}^{+\infty} \int_x^t \frac{\partial}{\partial y}\Big[-F_U^{(n+1)}(y) F_D^{(n+1)}(t-y) + F_U^{(n+2)}(y) F_D^{(n+1)}(t-y)\Big] dy\\
&\quad + \int_x^t f_{U_1}(y) dy\\
&= \sum_{n=0}^{+\infty} \Big[F_U^{(n+1)}(x) F_D^{(n+1)}(t-x) - F_U^{(n+2)}(x) F_D^{(n+1)}(t-x)\Big] + F_{U_1}(t) - F_{U_1}(x)\\
&= \sum_{n=1}^{+\infty} \Big[F_U^{(n)}(x) - F_U^{(n+1)}(x)\Big] F_D^{(n)}(t-x) + F_{U_1}(t) - F_{U_1}(x).
\end{aligned}$$

Noticing that

$$F_{U_1}(t) - F_{U_1}(x) + \overline{F}_{U_1}(t) = 1 - F_{U_1}(x) = \Big[F_U^{(0)}(x) - F_U^{(1)}(x)\Big] F_D^{(0)}(t-x),$$

we get

$$
\begin{aligned}
\mathbb{P}\{W(t) > x | V(0) = c\} &= \int_x^t \Big[\psi_c(y,t;c) + \psi_c(y,t;-v)\Big] dy + \overline{F}_{U_1}(t) \\
&= \sum_{n=0}^{+\infty} \Big[F_U^{(n)}(x) - F_U^{(n+1)}(x)\Big] F_D^{(n)}(t-x).
\end{aligned}
$$

The (1.76) can be proven in a similar way.

To prove the (1.74) we use the identity (see [28, theorem 5.5]):

$$
\mathbb{E}[X^2] = \int_0^{+\infty} 2x\, \mathbb{P}\{X > x\} dx - \int_{-\infty}^0 2x\, \mathbb{P}\{X \le x\} dx. \tag{1.77}
$$

One has

$$
\begin{aligned}
\mathbb{E}\{[W(t)]^2\} &= \int_0^{+\infty} 2x\, \mathbb{P}\{W(t) > x\} dx - \int_{-\infty}^0 2x\, \mathbb{P}\{W(t) \le x\} dx \\
&= \int_0^t x \Big[\mathbb{P}\{W(t) > x | V(0) = c\} + \mathbb{P}\{W(t) > x | V(0) = -v\}\Big] dx \\
&= \int_0^t x \Big[\overline{\Psi}_c(x,t) + \overline{\Psi}_{-v}(x,t)\Big] dx
\end{aligned}
$$

and thus the thesis is proven, since $Var[X] = \mathbb{E}[X^2] - \{\mathbb{E}[X]\}^2$. □

We can now derive the distribution of the generalized telegraph process $X(t)$. To do so, we define the absolutely continuous component of the transition density:

$$
g(x,t) := \frac{\partial}{\partial x} \mathbb{P}\{X(t) \le x\}, \qquad -vt < x < ct, \quad t > 0. \tag{1.78}
$$

Theorem 1.6. *For every* $t > 0$

$$
\mathbb{P}\{X(t) = -vt\} = \frac{1}{2} \overline{F}_{D_1}(t), \qquad \mathbb{P}\{X(t) = ct\} = \frac{1}{2} \overline{F}_{U_1}(t), \tag{1.79}
$$

moreover, for $-vt < x < ct$,

$$
g(x,t) = \frac{1}{c+v}\, \psi\bigg(\frac{x+vt}{c+v}, t\bigg). \tag{1.80}
$$

Proof. From the definition of $W(t)$ the following relationship immediately follows:

$$
X(t) = c\, W(t) - v\big(t - W(t)\big) = (c+v) W(t) - vt. \tag{1.81}
$$

Hence

$$\begin{aligned}
&\mathbb{P}\{X(t) = -vt\} = \mathbb{P}\{W(t) = 0\},\\
&\mathbb{P}\{X(t) = ct\} = \mathbb{P}\{W(t) = t\},\\
&g(x,t) = \frac{\partial}{\partial x}\mathbb{P}\{X(t) \le x\} = \frac{\partial}{\partial x}\mathbb{P}\left\{W(t) \le \frac{x+vt}{c+v}\right\} = \frac{1}{c+v}\,\psi\left(\frac{x+vt}{c+v}, t\right).
\end{aligned}$$

The thesis follows from Theorem 1.5. □

From the previous theorem we can derive the generalized density of the process $\{X(t), t \ge 0\}$:

$$\begin{aligned}
p(x,t) &:= \frac{\mathbb{P}\{X(t) \in dx\}}{dx}\\
&= \mathbb{P}\{X(t) = -vt\}\delta(x+vt) + \mathbb{P}\{X(t) = ct\}\delta(x-ct) + g(x,t)\mathbb{1}_{\{-vt<x<ct\}}\\
&= \frac{1}{2}\overline{F}_{D_1}(t)\delta(x+vt) + \frac{1}{2}\overline{F}_{U_1}(t)\delta(x-ct) + \frac{1}{c+v}\,\psi\left(\frac{x+vt}{c+v}, t\right)\mathbb{1}_{\{-vt<x<ct\}}
\end{aligned} \tag{1.82}$$

for every $(x,t) \in \mathbb{R}^2_+$.

We conclude this section by showing a result for joint densities

$$\begin{aligned}
f(x,t) &:= \frac{\mathbb{P}\{X(t) \in dx, V(t) = c\}}{dx},\\
b(x,t) &:= \frac{\mathbb{P}\{X(t) \in dx, V(t) = -v\}}{dx},
\end{aligned} \tag{1.83}$$

defined for $(x,t) \in \mathbb{R}^2_+$, which generalizes Kolmogorov's equations (1.20) in the case in which the random times U_i and D_i are exponentially distributed.

Theorem 1.7. *For every $i = 1, 2, \dots,$ let the random variables U_i and D_i be exponentially distributed with parameters λ and μ, respectively. Then, for $t > 0$ and $-vt < x < ct$, the densities* (1.83) *satisfy the system of differential equations*

$$\begin{cases}
\dfrac{\partial f}{\partial t} + c\dfrac{\partial f}{\partial x} = -\lambda f + \mu b,\\
\dfrac{\partial b}{\partial t} - v\dfrac{\partial b}{\partial x} = -\mu b + \lambda f.
\end{cases} \tag{1.84}$$

Proof. Let us prove the first of the (1.84).

For fixed $t > 0$ and $x \in (-vt, ct)$ one has

$$\begin{aligned}
f(x, t+\Delta t) &:= \frac{\partial}{\partial x}\mathbb{P}\{X(t+\Delta t) \le x, V(t+\Delta t) = c\} \\
&= \frac{\partial}{\partial x}\mathbb{P}\{X(t) + c\Delta t \le x, V(t) = c, N(t+\Delta t) - N(t) = 0\} \\
&+ \frac{\partial}{\partial x}\mathbb{P}\{X(t) + \alpha\Delta t \le x, V(t) = -v, N(t+\Delta t) - N(t) = 1\} \\
&+ \frac{\partial}{\partial x}\mathbb{P}\{X(t+\Delta t) \le x, V(t+\Delta t) = c, N(t+\Delta t) - N(t) \ge 2\} \\
&= f(x - c\Delta t, t)(1 - \lambda\Delta t) + b(x - \alpha\Delta t, t)\mu\Delta t + o(\Delta t),
\end{aligned}$$

by having used well known properties of Poisson increments $N(t+\Delta t) - N(t)$.

We notice that the parameter $\alpha \in (-v, c)$ is random, as it is depends on the time instant $\theta \in (t, t+\Delta t]$ at which the velocity change occurs, but it is not necessary to make explicit the expression or the probability law since, as will be seen, it will be completely irrelevant in subsequent developments.

By developing in the Taylor series we get the previous identity

$$f + \Delta t\frac{\partial f}{\partial t} + o(\Delta t) = \left\{f - c\Delta t\frac{\partial f}{\partial x}\right\}(1 - \lambda\Delta t) + \left\{b - \alpha\Delta t\frac{\partial b}{\partial x}\right\}\mu\Delta t + o(\Delta t),$$

hence, dividing by Δt,

$$\frac{\partial f}{\partial t} = -c\frac{\partial f}{\partial x} - \lambda f - \lambda c\Delta t\frac{\partial f}{\partial x} + \mu b - \mu\alpha\Delta t\frac{\partial b}{\partial x} + \frac{o(\Delta t)}{\Delta t}.$$

The thesis follows by taking the limit for $\Delta t \to 0$.

The second equation can be proved similarly. □

Corollary 1.2. *Under the hypotheses of the previous theorem, the following system of differential equations holds*

$$\begin{cases}
\dfrac{\partial p}{\partial t} + \dfrac{c-v}{2}\dfrac{\partial p}{\partial x} + \dfrac{c+v}{2}\dfrac{\partial j}{\partial x} = 0, \\
\dfrac{\partial j}{\partial t} + \dfrac{c+v}{2}\dfrac{\partial p}{\partial x} + \dfrac{c-v}{2}\dfrac{\partial j}{\partial x} = -(\lambda - \mu)p - (\lambda + \mu)j,
\end{cases} \tag{1.85}$$

for $t > 0$ and $-vt < x < ct$, where $j(x,t) := f(x,t) - b(x,t)$.

Proof. It follows straightforwardly from the linearity of the system (1.84), taking into account that $p(x,t) = f(x,t) + b(x,t)$. □

Chapter 2

State-dependent telegraph process

In this chapter we consider a state-dependent telegraph process $\{X(t), t \geq 0\}$, which describes the alternating behavior of a suitable stochastic system, such as the motion of a particle running on the real line. The particle velocity, say $V(t)$, alternates randomly between the fixed values $c > 0$ and $-v < 0$. We assume that the initial position is $X(0) = 0$ and the initial velocity is $V(0) = c$.

Let $f(x,t)$ and $b(x,t)$ denote respectively the forward and backward transition densities of the motion, defined as

$$\begin{aligned} f(x,t) &:= \frac{\partial}{\partial x}\mathbb{P}\{X(t) \leq x, V(t) = c\}, \\ b(x,t) &:= \frac{\partial}{\partial x}\mathbb{P}\{X(t) \leq x, V(t) = -v\}. \end{aligned} \tag{2.1}$$

For $t > 0$ and $-vt < x < ct$, densities (2.1) satisfy the following partial differential equations:

$$\begin{cases} \dfrac{\partial}{\partial t}f(x,t) + c\dfrac{\partial}{\partial x}f(x,t) = -\lambda^+(x)f(x,t) + \lambda^-(x)b(x,t), \\ \dfrac{\partial}{\partial t}b(x,t) - v\dfrac{\partial}{\partial x}b(x,t) = -\lambda^-(x)b(x,t) + \lambda^+(x)f(x,t), \end{cases} \tag{2.2}$$

where $\lambda^+(x)$ and $\lambda^-(x)$ are nonnegative functions, for all $x \in \mathbb{R}$. The function $\lambda^+(x)$ represents the intensity of velocity changes when the particle occupies state x with current forward motion, and similarly $\lambda^-(x)$ represents the same intensity for the backward motion. Clearly, for the classical telegraph process the intensity functions $\lambda^+(x)$ and $\lambda^-(x)$ are constant, leading to exponentially distributed interarrival times between consecutive velocity changes. Instead,

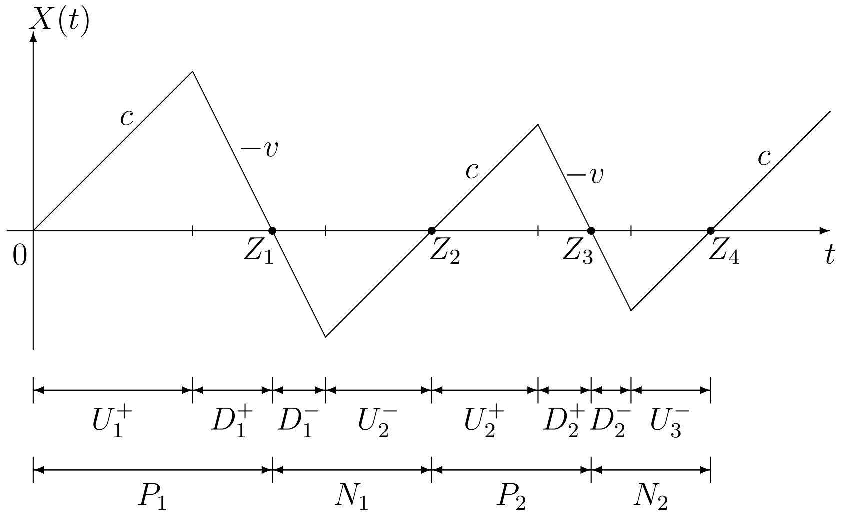

Figure 2.1: A sample path of $X(t)$ with indications of the relevant random variables and the velocities of the motion (after Di Crescenzo and Travaglino 2019).

herein we assume that they depend on the position x and satisfy the following assumptions:

$$\begin{cases} \lambda^+(x) = 0 & \text{if} \quad x \leq 0, \\ \lambda^-(x) = 0 & \text{if} \quad x \geq 0, \end{cases} \tag{2.3}$$

and

$$\int_0^{+\infty} \lambda^{\pm}(\pm x)\mathrm{d}x = +\infty. \tag{2.4}$$

The conditions (2.3) express that each instant a velocity change occurs, then the process is forced to return to the 0 state prior to the subsequent velocity change (see the sample path of $X(t)$ shown in Figure 2.1). Specifically, changes from positive to negative velocity occur only if the particle occupies a positive state x, whereas the opposite velocity changes occur only at negative states. The assumption (2.4) provide a bona fide condition, whose role will be clarified in the following. Clearly, the given assumptions imply that consecutive velocity changes of the motion are separated by passages through the zero state. The resulting state-dependent telegraph process is then useful to describe systems that alternate randomly around the 0 level.

We remark that other stochastic processes describing alternating motions governed by non-constant parameters have been treated recently by Garra and Orsingher [46]. Specifically, some cases of space-varying velocities and time-varying intensity are treated by means of suitable space-time transformations.

We point out that performing a transformation analogous to Eq. (2.3) of Beghin *et al.* [8], the equations (2.2) lead to a system of partial differential equations for the transition density $p(x,t)$ and the flow function $w(x,t)$ of the process $X(t)$, defined respectively as

$$\begin{aligned} p(x,t) &:= \frac{\partial}{\partial x}\mathbb{P}\{X(t) \le x\} = f(x,t) + b(x,t), \\ w(x,t) &:= f(x,t) - b(x,t). \end{aligned} \tag{2.5}$$

According to Orsingher [76], in a large ensemble of particles moving as specified, the function $w(x,t)$ can be viewed as a measure of the excess of particles moving forward with respect to those moving backward near point x at time t.

2.1 Probabilistic structure of the process

In this section we analyze the probabilistic structure of the random variables that describe the upward and downward particle motion.

For every $i \in \mathbb{N}$, we denote by U_i (respectively D_i) the random duration of the i-th time interval in which the motion has positive (negative) velocity. For any $i \in \mathbb{N}$, we express U_i as the sum of the random time length U_i^- during which $X(t) < 0$, and the random time length U_i^+ during which $X(t) > 0$. Clearly, since the initial velocity is positive, one has

$$U_1^- = 0 \quad \text{and} \quad U_1 = U_1^+.$$

Similarly we have (see Figure 2.1)

$$D_i = D_i^+ + D_i^-, \qquad i \in \mathbb{N}.$$

From the assumptions (2.3) and (2.4) it follows that the passages of $X(t)$ through the 0 state are regenerative alternating events, and the dynamics of the velocity changes do not depend on time. Hence, the sequence $\{U_i^+; i \in \mathbb{N}\}$ is formed by independent and identically distributed random variables. The same conclusion holds for the independent sequence $\{D_i^-; i \in \mathbb{N}\}$.

We denote by Z_i the i-th random instant in which the process is equal to 0, and by P_i (resp. N_i) the duration of the i-th time interval in which $X(t)$ is positive (negative), as shown in Figure 2.1. It is easy to verify that, for every $i \in \mathbb{N}$,

$$\begin{cases} P_i = Z_{2i-1} - Z_{2(i-1)}, \\ N_i = Z_{2i} - Z_{2i-1}, \end{cases} \tag{2.6}$$

where $Z_0 := 0$. From the assumptions on the motion, the following relationships are straightforward (as shown in Figure 2.1)

$$cU_i^+ - vD_i^+ = 0, \qquad -vD_i^- + cU_{i+1}^- = 0,$$

so that

$$\begin{cases} P_i = U_i^+ + D_i^+ = \dfrac{c+v}{v} U_i^+, \\ N_i = D_i^- + U_{i+1}^- = \dfrac{c+v}{c} D_i^-. \end{cases} \tag{2.7}$$

Thus, the sequences $\{P_i; i \in \mathbb{N}\}$ and $\{N_i; i \in \mathbb{N}\}$ are independent and, clearly, they are formed by independent and identically distributed random variables.

Since the motion proceeds with constant velocities, and the instants $\{Z_{2i}; i \in \mathbb{N}\}$ and $\{Z_{2i-1}; i \in \mathbb{N}\}$ are regenerative, a linear time-transformation allows the functions $\lambda^+(x)$ and $\lambda^-(x)$ in Eqs. (2.2) to represent the intensities of occurrence of velocity changes along the time axes. Hence, from classical arguments of renewal theory, it follows that $\lambda^+(x)$ and $\lambda^-(-x)$, for $x \geq 0$, are respectively the hazard rate functions of cU_i^+ and vD_i^- at $x \geq 0$, i.e.

$$\lambda^+(x) = \lim_{h \to 0^+} \frac{1}{h} \mathbb{P}\{cU_i^+ \leq x + h \,|\, cU_i^+ > x\},$$

$$\lambda^-(-x) = \lim_{h \to 0^+} \frac{1}{h} \mathbb{P}\{vD_i^- \leq x + h \,|\, vD_i^- > x\}.$$

We can thus introduce the following expressions, for $x \geq 0$,

$$\begin{aligned} \overline{F}_{cU_i^+}(x) &= \mathbb{P}\{cU_i^+ > x\} = \exp\left\{ -\int_0^x \lambda^+(y) dy \right\} = e^{-\Lambda^+(x)}, \\ \overline{F}_{vD_i^-}(x) &= \mathbb{P}\{vD_i^- > x\} = \exp\left\{ -\int_0^x \lambda^-(-y) dy \right\} = e^{-\Lambda^-(x)}, \end{aligned} \tag{2.8}$$

where

$$\Lambda^\pm(x) := \int_0^x \lambda^\pm(\pm y) dy, \qquad x \geq 0, \tag{2.9}$$

constitute the corresponding cumulative hazard rates. Hence, we immediately obtain the probability density functions of U_i^+ and D_i^-, namely

$$f_{U_i^+}(x) = c\lambda^+(cx)e^{-\Lambda^+(cx)}, \qquad f_{D_i^-}(x) = v\lambda^-(-vx)e^{-\Lambda^-(vx)}, \qquad x > 0.$$

Consequently, due to assumption (2.4), the random variables U_i^+ and D_i^- are nonnegative, absolutely continuous, honest random variables, for all $i \in \mathbb{N}$, with distribution functions

$$F_{U_i^+}(x) = 1 - e^{-\Lambda^+(cx)}, \qquad F_{D_i^-}(x) = 1 - e^{-\Lambda^-(vx)}, \qquad x > 0, \tag{2.10}$$

respectively. Finally, the relations (2.7) allow to express the complementary distribution functions of P_i and N_i, $i \in \mathbb{N}$, as follows, for $x \geq 0$,

$$\overline{F}_{P_i}(x) = \exp\left\{ -\Lambda^+\Big(\frac{cv}{c+v}x\Big)\right\}, \qquad \overline{F}_{N_i}(x) = \exp\left\{ -\Lambda^-\Big(\frac{cv}{c+v}x\Big)\right\}.$$

On the ground of the above results, we are now able to express the dependence of the process $X(t)$ on the regenerative random times Z_i. Indeed, if k velocity changes occurred in $[0,t]$, $t>0$, then

$$X(t) = V_k\,(t - Z_k), \tag{2.11}$$

where

$$V_k = \begin{cases} c, & k \text{ even} \\ -v, & k \text{ odd.} \end{cases}$$

Consequently, the process $X(t)$ can be expressed as

$$X(t) = \sum_{k=0}^{\infty} \mathbb{1}_{\{M(t)=k\}} V_k\,(t - Z_k), \qquad t > 0,$$

where $\mathbb{1}_A$ is the indicator function of A, i.e. $\mathbb{1}_A = 1$ if A is true, and $\mathbb{1}_A = 0$ otherwise, and where $M(t)$ is the alternating counting process that counts the number of velocity changes in $[0,t]$, whose interarrival times are $U_1, D_1, U_2, D_2, \ldots$

2.2 Transition densities

In this section we determine the formal expressions of the probability density functions that describe the motion during suitable time intervals. Specifically, with reference to the random variables introduced in Section 2.1, we deal with the following densities, for $n \in \mathbb{N}$,

$$\begin{aligned} f_n(x,t) &:= \frac{\partial}{\partial x}\mathbb{P}\{X(t) \leq x, Z_{2(n-1)} - U_n^- \leq t < Z_{2(n-1)} + U_n^+\}, \\ b_n(x,t) &:= \frac{\partial}{\partial x}\mathbb{P}\{X(t) \leq x, Z_{2n-1} - D_n^+ \leq t < Z_{2n-1} + D_n^-\}. \end{aligned} \tag{2.12}$$

Clearly, for each $n \in \mathbb{N}$, $f_n(x,t)$ (resp. $b_n(x,t)$) represents the forward (backward) density of the particle position at time t, during the n-th period in which the motion has positive (negative) velocity. Hence, the densities defined in (2.1) can be expressed by

$$f(x,t) = \sum_{n=1}^{\infty} f_n(x,t), \qquad b(x,t) = \sum_{n=1}^{\infty} b_n(x,t), \tag{2.13}$$

provided that the above series are convergent. Before determining the forward densities given in the first of (2.12) we recall that, since the initial velocity is positive, the state space of the particle position at time t is $(-vt, ct]$, and the probability law of $X(t)$ has a discrete component at point ct, so that the density $f_1(x,t)$ is degenerate.

Theorem 2.1. *The forward probability densities defined in the first of (2.12) can be expressed as follows:*

$$f_1(x,t) = \overline{F}_{U_1^+}(t)\delta(ct-x), \qquad x \in \mathbb{R}, \quad t \geq 0, \tag{2.14}$$

where $\delta(x)$ denotes the Dirac delta function, and, for $n = 2, 3, \dots,$

$$f_n(x,t) = \begin{cases} \dfrac{1}{c} f_{Z_{2(n-1)}}\left(t - \dfrac{x}{c}\right) \overline{F}_{U_n^+}\left(\dfrac{x}{c}\right), & 0 < x < ct, \\ \dfrac{1}{c+v} \displaystyle\int_0^{t+\frac{x}{v}} f_{D_{n-1}^-}\left(\dfrac{c(t-z)-x}{c+v}\right) f_{Z_{2n-3}}(z)\,\mathrm{d}z, & -vt < x < 0. \end{cases} \tag{2.15}$$

Proof. If $t < U_1^+$, then no changes of velocity have occurred up to time t, and so $X(t) = ct$. Thus, equation (2.14) follows immediately since the relevant distribution has an atom at the point $x = ct$. Note that the density $f_1(x,t)$ must be intended as a generalized function. It is clear that in the n-th period in which the velocity is positive $X(t) = c(t - Z_{2(n-1)})$, as stated in equation (2.11) for $k = 2n-1$, and confirmed by Figure 2.1. So that, for $n > 1$ and $0 < x < ct$ we have

$$\begin{aligned}
&\mathbb{P}\{X(t) \leq x, Z_{2(n-1)} < t < Z_{2(n-1)} + U_n^+\} \\
&= \mathbb{P}\{c(t - Z_{2(n-1)}) \leq x, U_n^+ > t - Z_{2(n-1)}, Z_{2(n-1)} < t\} \\
&= \int_0^t \mathbb{P}\{c(t - Z_{2(n-1)}) \leq x, U_n^+ > t - Z_{2(n-1)} | Z_{2(n-1)} = z\} \mathbb{P}\{Z_{2(n-1)} \in \mathrm{d}z\} \\
&= \int_0^t \mathbb{P}\{c(t-z) \leq x, U_n^+ > t - z\} \mathbb{P}\{Z_{2(n-1)} \in \mathrm{d}z\} \\
&= \int_0^t \mathbb{P}\{c(t-z) \leq x\} \mathbb{P}\{U_n^+ > t - z\} \mathbb{P}\{Z_{2(n-1)} \in \mathrm{d}z\} \\
&= \int_0^t \mathbb{P}\left\{z \geq t - \frac{x}{c}\right\} \mathbb{P}\{U_n^+ > t - z\} \mathbb{P}\{Z_{2(n-1)} \in \mathrm{d}z\} \\
&= \int_0^t \mathbb{1}_{\{z \geq t - \frac{x}{c}\}} \mathbb{P}\{U_n^+ > t - z\} \mathbb{P}\{Z_{2(n-1)} \in \mathrm{d}z\} \\
&= \int_{t-\frac{x}{c}}^t \overline{F}_{U_n^+}(t-z) f_{Z_{2(n-1)}}(z)\,\mathrm{d}z.
\end{aligned} \tag{2.16}$$

Differentiating with respect to x we thus obtain the density (2.15) for $0 < x < ct$. Furthermore, for $n = 2, 3, \dots$ and $-vt < x < 0$, in the n-th period in which the velocity is positive one has

$$\begin{aligned} X(t) &= -vD^-_{n-1} + c(t - Z_{2n-3} - D^-_{n-1}) \\ &= ct - cZ_{2n-3} - (c+v)D^-_{n-1}. \end{aligned}$$

So that, similarly to (2.16), we have

$$\begin{aligned} &\mathbb{P}\{X(t) \le x, Z_{2(n-1)} - U^-_n \le t < Z_{2(n-1)}\} \\ &= \mathbb{P}\{X(t) \le x, Z_{2n-3} + D^-_{n-1} \le t < Z_{2n-3} + D^-_{n-1} + U^-_n\} \\ &= \mathbb{P}\left\{X(t) \le x, Z_{2n-3} + D^-_{n-1} \le t < Z_{2n-3} + D^-_{n-1} + \frac{v}{c}D^-_{n-1}\right\} \\ &= \mathbb{P}\left\{X(t) \le x, Z_{2n-3} + D^-_{n-1} \le t < Z_{2n-3} + \frac{c+v}{c}D^-_{n-1}\right\} \\ &= \mathbb{P}\left\{ct - cZ_{2n-3} - (c+v)D^-_{n-1} \le x, D^-_{n-1} \le t - Z_{2n-3} < \frac{c+v}{c}D^-_{n-1}\right\} \\ &= \int_0^t \mathbb{P}\left\{c(t-z) - (c+v)D^-_{n-1} \le x, D^-_{n-1} \le t - z < \frac{c+v}{c}D^-_{n-1}\right\}\mathbb{P}\{Z_{2n-3} \in \mathrm{d}z\} \\ &= \int_0^t \mathbb{P}\left\{D^-_{n-1} \ge \frac{c(t-z)-x}{c+v}, D^-_{n-1} > \frac{c(t-z)}{c+v}, D^-_{n-1} \le t-z\right\}\mathbb{P}\{Z_{2n-3} \in \mathrm{d}z\} \\ &= \int_0^t \mathbb{P}\left\{D^-_{n-1} \ge \frac{c(t-z)-x}{c+v}, D^-_{n-1} \le t-z\right\}\mathbb{P}\{Z_{2n-3} \in \mathrm{d}z\} \\ &= \int_0^t \mathbb{1}_{\{\frac{c(t-z)-x}{c+v} \le t-z\}}\mathbb{P}\left\{\frac{c(t-z)-x}{c+v} \le D^-_{n-1} \le t-z\right\}\mathbb{P}\{Z_{2n-3} \in \mathrm{d}z\} \\ &= \int_0^{t+\frac{x}{v}} \left[F_{D^-_{n-1}}(t-z) - F_{D^-_{n-1}}\left(\frac{c(t-z)-x}{c+v}\right)\right] f_{Z_{2n-3}}(z)\,\mathrm{d}z. \end{aligned}$$

Differentiating with respect to x we finally get the density (2.15) for $-vt < x < 0$. □

A similar result can be obtained for the densities introduced in the second line of (2.12).

Theorem 2.2. *For $n \in \mathbb{N}$, the backward probability densities defined in the second of (2.12) can be expressed as*

$$b_n(x,t) = \begin{cases} \dfrac{1}{v} f_{Z_{2n-1}}\left(t + \dfrac{x}{v}\right)\overline{F}_{D^-_n}\left(-\dfrac{x}{v}\right), & -vt < x < 0, \\ \dfrac{1}{c+v}\displaystyle\int_0^{t-\frac{x}{c}} f_{U^+_n}\left(\dfrac{x + v(t-z)}{c+v}\right) f_{Z_{2(n-1)}}(z)\,\mathrm{d}z, & 0 < x < ct. \end{cases} \tag{2.17}$$

Proof. As stated in (2.11) for $k = 2n$, during the n-th period in which the velocity is negative The proof is omitted, being very similar to that of the previous theorem. $\square$

2.3 Gamma distributed interarrival times

In this section we aim to analyze in detail the case in which the upward and downward displacements performed by the particle after passages though the zero state have gamma distribution. Specifically, we assume that the random variables cU_i^+ and vD_i^-, $i \in \mathbb{N}$, are gamma distributed with shape parameters α and α^*, respectively, and equal rate parameters β, say

$$cU_i^+ \sim \text{Gamma}(\alpha, \beta), \qquad vD_i^- \sim \text{Gamma}(\alpha^*, \beta), \tag{2.18}$$

where $\alpha, \alpha^*, \beta > 0$. This assumption is similar to that in Di Crescenzo and Martinucci [26], in which a more classical telegraph process with gamma-distributed intertimes between velocity changes has been analyzed.

Clearly, the assumptions (2.18) correspond to the case in which the intensity functions $\lambda^+(x)$ and $\lambda^-(x)$ are given by

$$\lambda^+(x) = \frac{\beta^\alpha x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha, \beta x)}, \qquad \lambda^-(-x) = \frac{\beta^{\alpha^*} x^{\alpha^*-1} e^{-\beta x}}{\Gamma(\alpha^*, \beta x)}, \qquad x > 0, \tag{2.19}$$

where $\Gamma(\cdot, \cdot)$ denotes the upper incomplete gamma function. We remark that such intensity functions are strictly decreasing (increasing) in $|x|$ if $0 < \alpha < 1$ ($\alpha > 1$), and are constant if $\alpha = 1$, with

$$\lim_{x \to 0^+} \lambda^+(x) = \begin{cases} +\infty, & 0 < \alpha < 1, \\ \beta, & \alpha = 1, \\ 0, & \alpha > 1, \end{cases} \qquad \lim_{x \to +\infty} \lambda^+(x) = \beta,$$

with analogous limits holding for $\lambda^-(x)$. See Figure 2.2 for some plots of $\lambda^+(x)$. We remark that, from the assumptions given in (2.18), one also has

$$U_i^+ \sim \text{Gamma}(\alpha, c\beta), \qquad D_i^- \sim \text{Gamma}(\alpha^*, v\beta). \tag{2.20}$$

In the following theorems we obtain the forward and backward transition densities (2.1) in a special case. Such densities will be expressed in terms of the generalized (two-parameter) Mittag-Leffler function, defined as

$$E_{a,b}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(an + b)}. \tag{2.21}$$

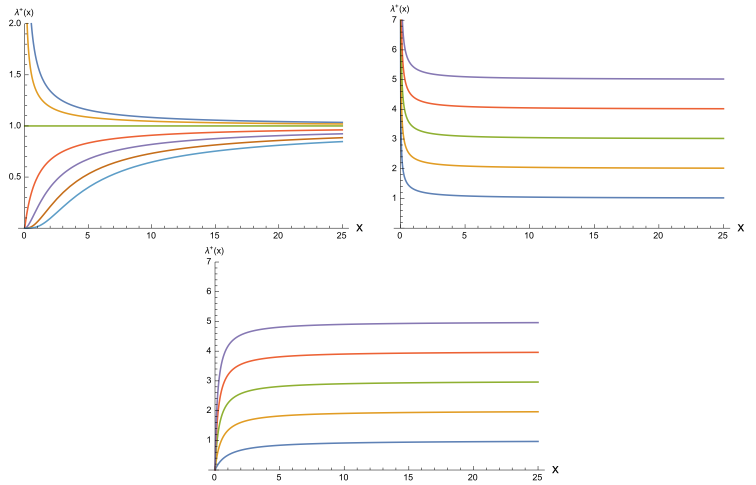

Figure 2.2: Intensity function $\lambda^+(x)$ given in (2.19), with $\beta = 1$ and $\alpha = 0.1, 0.5, 1, 2, 3, 4, 5$, from top to bottom (left plot), $\alpha = 0.5$ and $\beta = 1, 2, 3, 4, 5$, from bottom to top (central plot), $\alpha = 2$ and $\beta = 1, 2, 3, 4, 5$, from bottom to top (right plot), after Di Crescenzo and Travaglino (2019).

Properties and results on such Mittag-Leffler function can be found, for instance, in Gorenflo *et al.* [48], Haubold *et al.* [50] and references therein. We recall that function (2.21) has been used recently in the analysis of probability distributions of some birth-death type processes in Alipour *et al.* [2], Di Crescenzo *et al.* [34] and Orsingher and Polito [79].

Theorem 2.3. *Let $\{X(t), t \geq 0\}$ be the state-dependent telegraph process with intensity functions specified in (2.19). For $t > 0$, the forward transition density is given by*

$$f(x,t) = \frac{\Gamma(\alpha, \beta ct)}{\Gamma(\alpha)} \delta(ct - x) + \frac{\Gamma(\alpha, \beta x)}{\Gamma(\alpha)} \frac{1}{ct - x} \left(\frac{(ct - x)v\beta}{c + v} \right)^{\alpha + \alpha^*} \times \exp\left\{ - \frac{(ct - x)v\beta}{c + v} \right\} E_{\alpha + \alpha^*, \alpha + \alpha^*}\left(\left(\frac{(ct - x)v\beta}{c + v} \right)^{\alpha + \alpha^*} \right) \mathbb{1}_{\{x < ct\}}, \qquad 0 < x \leq ct, \tag{2.22}$$

$$f(x,t) = \frac{c^\alpha}{\Gamma(\alpha^*)} \left(\frac{v\beta}{c + v} \right)^{\alpha + \alpha^*} \exp\left\{ - \frac{v\beta(ct - x)}{c + v} \right\} \times \int_0^{t + \frac{x}{v}} (c(t - z) - x)^{\alpha^* - 1} z^{\alpha - 1} E_{\alpha + \alpha^*, \alpha}\left(\left(\frac{cv\beta z}{c + v} \right)^{\alpha + \alpha^*} \right) \mathrm{d}z, \qquad -vt < x < 0. \tag{2.23}$$

Proof. From (2.13), (2.14), the first case of (2.15) and (2.20) we easily obtain, for $0 < x \le ct$,

$$f(x,t) = \frac{\Gamma(\alpha, \beta ct)}{\Gamma(\alpha)}\delta(ct-x) + \frac{\Gamma(\alpha,\beta x)}{\Gamma(\alpha)}\frac{1}{c}\sum_{n=2}^{\infty} f_{Z_{2(n-1)}}\left(t-\frac{x}{c}\right). \tag{2.24}$$

In order to analyze the distribution of $Z_{2(n-1)}$ we note that, from the relationships (2.7), one has

$$P_i \sim \text{Gamma}\left(\alpha, \frac{cv}{c+v}\beta\right), \qquad N_i \sim \text{Gamma}\left(\alpha^*, \frac{cv}{c+v}\beta\right),$$

and thus

$$\sum_{i=1}^{n} P_i \sim \text{Gamma}\left(n\alpha, \frac{cv}{c+v}\beta\right), \qquad \sum_{i=1}^{n} N_i \sim \text{Gamma}\left(n\alpha^*, \frac{cv}{c+v}\beta\right). \tag{2.25}$$

We point out that, due to (2.6), for the regenerative random times Z_i one has

$$Z_{2n} = \sum_{i=1}^{n}(P_i + N_i), \qquad Z_{2n-1} = Z_{2(n-1)} + P_n, \qquad n \in \mathbb{N}.$$

Hence, since the gamma-distributed random variables in (2.25) have identical rates, we get

$$\begin{aligned} Z_{2n} &\sim \text{Gamma}\left(n(\alpha+\alpha^*), \frac{cv}{c+v}\beta\right), \\ Z_{2n-1} &\sim \text{Gamma}\left((n-1)(\alpha+\alpha^*)+\alpha, \frac{cv}{c+v}\beta\right). \end{aligned}$$

Such relations, thanks to (2.21), allow to compute the following sum:

$$\begin{aligned} &\frac{1}{c}\sum_{n=2}^{\infty} f_{Z_{2(n-1)}}\left(t-\frac{x}{c}\right) \\ &= \frac{1}{c}\exp\left\{-\frac{(ct-x)v\beta}{c+v}\right\}\sum_{n=2}^{\infty}\frac{\left(\frac{cv\beta}{c+v}\right)^{(n-2)(\alpha+\alpha^*)+(\alpha+\alpha^*)}}{\Gamma((n-2)(\alpha+\alpha^*)+(\alpha+\alpha^*))}\left(t-\frac{x}{c}\right)^{(n-2)(\alpha+\alpha^*)+(\alpha+\alpha^*)-1} \\ &= \frac{1}{ct-x}\exp\left\{-\frac{(ct-x)v\beta}{c+v}\right\}\left(\frac{(ct-x)v\beta}{c+v}\right)^{\alpha+\alpha^*}\sum_{n=0}^{\infty}\frac{\left(\left(\frac{(ct-x)v\beta}{c+v}\right)^{\alpha+\alpha^*}\right)^n}{\Gamma(n(\alpha+\alpha^*)+(\alpha+\alpha^*))} \\ &= \frac{1}{ct-x}\exp\left\{-\frac{(ct-x)v\beta}{c+v}\right\}\left(\frac{(ct-x)v\beta}{c+v}\right)^{\alpha+\alpha^*} E_{\alpha+\alpha^*,\alpha+\alpha^*}\left(\left(\frac{(ct-x)v\beta}{c+v}\right)^{\alpha+\alpha^*}\right). \end{aligned} \tag{2.26}$$

Substituting (2.26) in (2.24) we finally obtain (2.22). Similarly, from (2.13) and the second of (2.15) we obtain, for $-vt < x < 0$,

$$
\begin{aligned}
f(x,t) =& \frac{1}{c+v}\int_0^{t+\frac{x}{v}} \sum_{n=2}^{\infty} f_{D^-_{n-1}}\left(\frac{c(t-z)-x}{c+v}\right) f_{Z_{2n-3}}(z)\,\mathrm{d}z \\
=& \frac{1}{c+v}\int_0^{t+\frac{x}{v}} f_{D^-_1}\left(\frac{c(t-z)-x}{c+v}\right) \sum_{n=0}^{\infty} f_{Z_{2n+1}}(z)\,\mathrm{d}z \\
=& \frac{1}{c+v}\frac{(v\beta)^{\alpha^*}}{\Gamma(\alpha^*)}\int_0^{t+\frac{x}{v}} \left(\frac{c(t-z)-x}{c+v}\right)^{\alpha^*-1} \exp\left\{-\frac{v\beta(c(t-z)-x)}{c+v}\right\} \\
&\times \exp\left\{-\frac{cv\beta z}{c+v}\right\} \sum_{n=0}^{\infty} \frac{\left(\frac{cv\beta}{c+v}\right)^{n(\alpha+\alpha^*)+\alpha}}{\Gamma(n(\alpha+\alpha^*)+\alpha)} z^{n(\alpha+\alpha^*)+\alpha-1}\,\mathrm{d}z \\
=& \frac{1}{\Gamma(\alpha^*)}\left(\frac{v\beta}{c+v}\right)^{\alpha^*} \exp\left\{-\frac{v\beta(ct-x)}{c+v}\right\} \\
&\times \int_0^{t+\frac{x}{v}} (c(t-z)-x)^{\alpha^*-1}\left(\frac{cv\beta z}{c+v}\right)^{\alpha}\frac{1}{z}\sum_{n=0}^{\infty}\frac{\left(\left(\frac{cv\beta z}{c+v}\right)^{\alpha+\alpha^*}\right)^n}{\Gamma(n(\alpha+\alpha^*)+\alpha)}\,\mathrm{d}z \\
=& \frac{c^{\alpha}}{\Gamma(\alpha^*)}\left(\frac{v\beta}{c+v}\right)^{\alpha+\alpha^*} \exp\left\{-\frac{v\beta(ct-x)}{c+v}\right\} \\
&\times \int_0^{t+\frac{x}{v}} (c(t-z)-x)^{\alpha^*-1} z^{\alpha-1} E_{\alpha+\alpha^*,\alpha}\left(\left(\frac{cv\beta z}{c+v}\right)^{\alpha+\alpha^*}\right)\mathrm{d}z,
\end{aligned}
\tag{2.27}
$$

which finally gives Eq. (2.23). □

The following theorem is a companion of Theorem 2.3.

Theorem 2.4. *Let $\{X(t), t \geq 0\}$ be the state-dependent telegraph process with intensity functions specified in (2.19). For $t > 0$, the backward transition density is:*

$$
\begin{aligned}
b(x,t) =& \frac{\Gamma(\alpha^*, -\beta x)}{\Gamma(\alpha^*)}\frac{1}{vt+x}\left(\frac{c(vt+x)\beta}{c+v}\right)^{\alpha} \\
&\times \exp\left\{-\frac{c(vt+x)\beta}{c+v}\right\} E_{\alpha+\alpha^*,\alpha}\left(\left(\frac{c(vt+x)\beta}{c+v}\right)^{\alpha+\alpha^*}\right), \qquad -vt < x < 0,
\end{aligned}
\tag{2.28}
$$

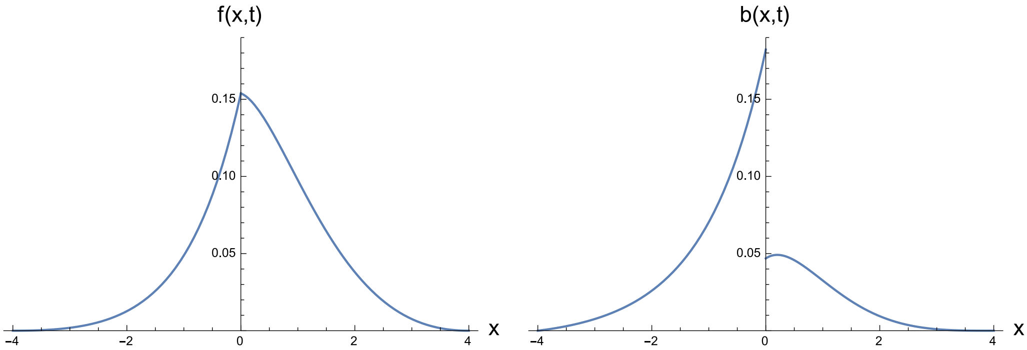

Figure 2.3: Absolutely continuous component of the densities obtained in Theorem 2.3 for $-vt < x < ct$, with $t = 2$, $\alpha = 2$, $\alpha^* = 1$, $\beta = 1$, $c = 2$, $v = 2$ (after Di Crescenzo and Travaglino 2019).

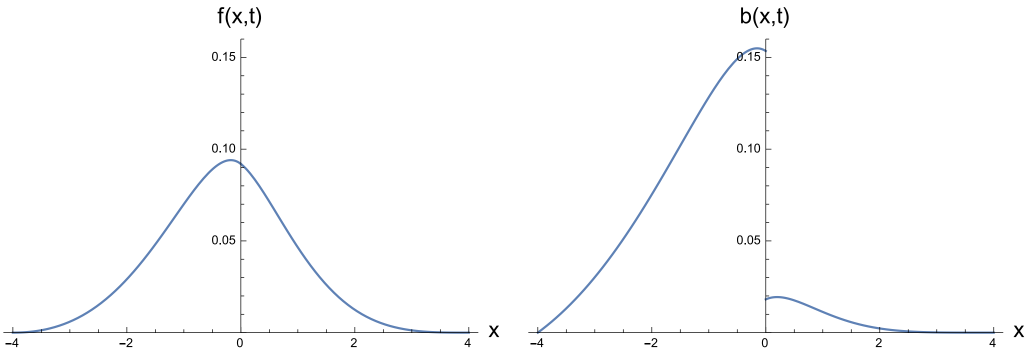

Figure 2.4: As for Figure 2.3, with $\alpha^* = 2$ (after Di Crescenzo and Travaglino 2019).

$$\begin{aligned} b(x,t) = {} & \frac{v^{\alpha+\alpha^*}}{\Gamma(\alpha)} \left(\frac{c\beta}{c+v} \right)^{2\alpha+\alpha^*} \exp\left\{ -\frac{c\beta(x+vt)}{c+v} \right\} \\ & \times \int_0^{t-\frac{x}{c}} (x+v(t-z))^{\alpha-1} z^{\alpha+\alpha^*-1} E_{\alpha+\alpha^*,\alpha+\alpha^*} \left(\left(\frac{cv\beta z}{c+v} \right)^{\alpha+\alpha^*} \right) \mathrm{d}z, \qquad 0 < x < ct. \end{aligned} \tag{2.29}$$

Proof. The proof is omitted, being similar to that of Theorem 2.3. □

Combining the results of the previous theorems with relationships (2.5) one can easily calculate the transition density and the flow function of the process $X(t)$.

As example, in Figure 2.3 and 2.4 we show some plots of the (absolutely continuous component of the) densities $f(x,t)$ and $b(x,t)$.

If $\alpha = \alpha^* = 1$, then the intensity rates (2.19) are constants and thus the corresponding cumulative hazard rates (2.9) are linear in x, i.e. the random times U_i^+ and D_i^- are exponentially distributed. This special case leads to simpler closed-form results, as shown hereafter.

Corollary 2.1. *Let $\{X(t), t \geq 0\}$ be the state-dependent telegraph process with intensity functions $\lambda^+(x) = \lambda^-(-x) = \beta$, $x > 0$. For $t > 0$, the forward and backward transition density are:*

$$f(x,t) = \begin{cases} e^{-\beta ct}\delta(ct-x) + \frac{1}{2}e^{-\beta x}\dfrac{v\beta}{c+v}\left(1 - \exp\left\{-\dfrac{2(ct-x)v\beta}{c+v}\right\}\right)\mathbb{1}_{\{x<ct\}}, & 0 < x \leq ct, \\ \dfrac{1}{2}e^{\beta x}\dfrac{v\beta}{c+v}\left(1 - \exp\left\{-\dfrac{2c(vt+x)\beta}{c+v}\right\}\right), & -vt < x < 0, \end{cases} \tag{2.30}$$

$$b(x,t) = \begin{cases} \dfrac{1}{2}e^{-\beta x}\dfrac{c\beta}{c+v}\left(1 - \exp\left\{-\dfrac{(ct-x)v\beta}{c+v}\right\}\right)^2, & 0 < x < ct, \\ \dfrac{1}{2}e^{\beta x}\dfrac{c\beta}{c+v}\left(1 + \exp\left\{-\dfrac{2c(vt+x)\beta}{c+v}\right\}\right), & -vt < x < 0. \end{cases} \tag{2.31}$$

Proof. The proof follows from Theorems 2.3 and 2.4 after some calculations, and noting that $E_{2,1}(z) = \cosh(\sqrt{z})$ and $E_{2,2}(z) = \sinh(\sqrt{z})/\sqrt{(z)}$. □

2.4 A first-passage-time problem

This section is devoted to a first-passage-time problem for the process $X(t)$, assuming the presence of two boundaries, say $\eta > 0$ and $-\xi < 0$. Hereafter we thoroughly assume that, in addition to assumptions (2.3) and (2.4), the intensity functions $\lambda^+(x)$ and $\lambda^-(x)$ satisfy the following condition:

$$\int_0^t \lambda^{\pm}(\pm x)\mathrm{d}x < +\infty \quad \text{for any } t > 0. \tag{2.32}$$

With reference with the notions introduced in Section 2.1, we denote by

$$M^+(t) = \sum_{n=0}^{\infty} \mathbb{1}_{\{Z_{2n} \leq t\}} \tag{2.33}$$

the right-continuous counting process whose increments occur at the random instants $0 = Z_0, Z_2, Z_4, \ldots$, so that

$$M^+(Z_{2n}) = n + 1, \qquad n \in \mathbb{N}_0.$$

We denote by

$$T_\zeta = \inf\{t > 0 : X(t) = \zeta\}, \qquad \zeta \neq 0$$

the first-passage time of $X(t)$ through the boundary $\zeta \neq 0$. Then we introduce the integer-valued random variable $M^+(T_\eta)$. Recalling that the i-th time interval in which the motion has positive velocity (upward period, say) has random duration U_i, $i \in \mathbb{N}$, let $M^+(T_\eta)$ denote the ordinal number of the first of such upward periods in which $X(t)$ crosses the boundary $\eta > 0$. Clearly, due to the first of (2.10) we have

$$\begin{aligned}\mathbb{P}\{M^+(T_\eta) = k\} &= \mathbb{P}\{cU_1^+ < \eta, \dots, cU_{k-1}^+ < \eta, cU_k^+ \geq \eta\} \\ &= \prod_{i=1}^{k-1} F_{U_i^+}\left(\frac{\eta}{c}\right) \overline{F}_{U_k^+}\left(\frac{\eta}{c}\right) = \left(1 - e^{-\Lambda^+(\eta)}\right)^{k-1} e^{-\Lambda^+(\eta)}, \qquad k \in \mathbb{N}.\end{aligned} \tag{2.34}$$

Hence, $M^+(T_\eta)$ has geometric distribution with parameter $e^{-\Lambda^+(\eta)}$, where $\Lambda^+(x)$ is defined in (2.9). It thus follows that

$$\mathbb{E}\big[M^+(T_\eta)\big] = e^{\Lambda^+(\eta)}. \tag{2.35}$$

Similarly,

$$M^-(t) = \sum_{n=0}^{\infty} \mathbb{1}_{\{Z_{2n+1} \leq t\}} \tag{2.36}$$

is the right-continuous counting process whose increments occur at the random instants $Z_1, Z_3, Z_5, \dots$, and thus

$$M^-(Z_{2n+1}) = n + 1, \qquad n \in \mathbb{N}_0.$$

For $\xi > 0$, we can thus introduce the integer-valued random variable $M^-(T_{-\xi})$. In analogy with $M^+(T_\eta)$, we assume that $M^-(T_{-\xi})$ gives the ordinal number of the first downward period in which $X(t)$ crosses the boundary $-\xi < 0$. Similarly as in (2.34), $M^-(T_{-\xi})$ has geometric distribution with parameter $e^{-\Lambda^-(\xi)}$, so that its expectation is

$$\mathbb{E}\big[M^-(T_{-\xi})\big] = e^{\Lambda^-(\xi)}. \tag{2.37}$$

Let us now consider the first-passage-time problem in the presence of two boundaries. We consider the random variable

$$M(-\xi, \eta) := \min\{M^+(T_\eta), M^-(T_{-\xi})\}. \tag{2.38}$$

Since, for $k \in \mathbb{N}$,

$$
\begin{aligned}
&\mathbb{P}\{M(-\xi,\eta)=k\} = \mathbb{P}\{cU_1^+ < \eta, vD_1^- < \xi, \ldots, cU_{k-1}^+ < \eta, vD_{k-1}^- < \xi, cU_k^+ \geq \eta\}\\
&+\mathbb{P}\{cU_1^+ < \eta, vD_1^- < \xi, \ldots, cU_{k-1}^+ < \eta, vD_{k-1}^- < \xi, cU_k^+ < \eta, vD_k^- \geq \xi\}\\
&=\left(1-e^{-\Lambda^+(\eta)}\right)^{k-1}\left(1-e^{-\Lambda^-(\xi)}\right)^{k-1}\left\{e^{-\Lambda^+(\eta)}+\left(1-e^{-\Lambda^+(\eta)}\right)e^{-\Lambda^-(\xi)}\right\}\\
&=\left(1-e^{-\Lambda^+(\eta)}-e^{-\Lambda^-(\xi)}+e^{-\Lambda^+(\eta)}e^{-\Lambda^-(\xi)}\right)^{k-1}\times\\
&\times\left\{e^{-\Lambda^+(\eta)}+e^{-\Lambda^-(\xi)}-e^{-\Lambda^+(\eta)}e^{-\Lambda^-(\xi)}\right\},
\end{aligned}
$$

it follows that $M(-\xi,\eta)$ has geometric distribution with parameter $e^{-\Lambda^+(\eta)}+e^{-\Lambda^-(\xi)}-e^{-\Lambda^+(\eta)}e^{-\Lambda^-(\xi)}$ and expectation

$$
\mathbb{E}\big[M(-\xi,\eta)\big]=\frac{1}{e^{-\Lambda^+(\eta)}+e^{-\Lambda^-(\xi)}-e^{-\Lambda^+(\eta)}e^{-\Lambda^-(\xi)}}. \tag{2.39}
$$

We remark that the assumption (2.32) ensures that the expectations given in (2.35), (2.37) and (2.39) are finite.

Let us now consider the special case of interarrival times having gamma distribution like in Section 2.3.

Proposition 2.1. *Let $\{X(t), t\geq 0\}$ be the state-dependent telegraph process with intensity functions specified in (2.19). Then, the expectations of $M^+(T_\eta)$ and $M^-(T_{-\xi})$ are expressed as*

$$
\mathbb{E}\big[M^+(T_\eta)\big]=\frac{\Gamma(\alpha)}{\Gamma(\alpha,\eta\beta)}, \qquad \mathbb{E}\big[M^-(T_{-\xi})\big]=\frac{\Gamma(\alpha^*)}{\Gamma(\alpha^*,\xi\beta)}. \tag{2.40}
$$

Hence, the expectation of (2.38) is

$$
\mathbb{E}\big[M(-\xi,\eta)\big]=\frac{\Gamma(\alpha)\Gamma(\alpha^*)}{\Gamma(\alpha,\eta\beta)\Gamma(\alpha^*)+\Gamma(\alpha^*,\xi\beta)\Gamma(\alpha)-\Gamma(\alpha,\eta\beta)\Gamma(\alpha^*,\xi\beta)}. \tag{2.41}
$$

Proof. The results immediately follow from Eqs. (2.35), (2.37), (2.39), and (2.8), by taking into account that for a Gamma(α,β)-distributed random variable the complementary distribution function at $x>0$ is $\Gamma(\alpha,\beta x)/\Gamma(\alpha)$. □

Recalling Eqs. (2.35) and (2.39), we can now introduce the following function:

$$
\mathcal{R}(-\xi,\eta):=\frac{\mathbb{E}\big[M(-\xi,\eta)\big]}{\mathbb{E}\big[M^+(T_\eta)\big]}=\frac{1}{1+e^{\Lambda^+(\eta)}e^{-\Lambda^-(\xi)}-e^{-\Lambda^-(\xi)}}. \tag{2.42}
$$

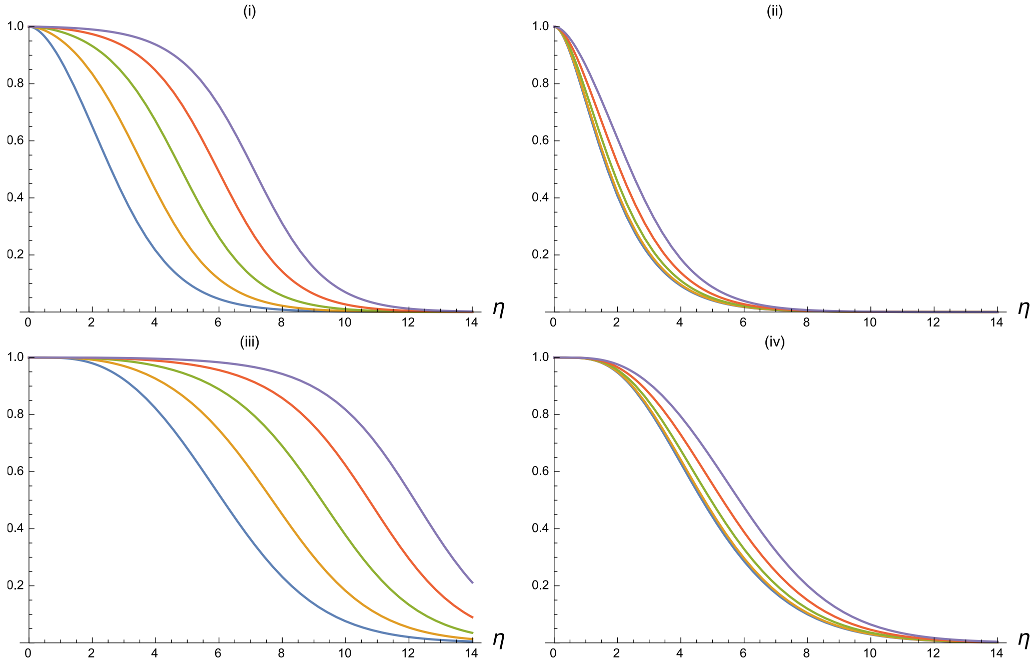

Figure 2.5: Plots of the ratio of mean values (2.43) for $0 \leq \eta \leq 14$, with $\beta = 1$, $\xi = 1, 2, 3, 4, 5$ (from bottom to top) and (i) $\alpha = 2$, $\alpha^* = 1$, (ii) $\alpha = 2$, $\alpha^* = 5$, (iii) $\alpha = 5$, $\alpha^* = 1$, (iv) $\alpha = 5$, $\alpha^* = 5$ (after Di Crescenzo and Travaglino 2019).

Clearly, since $M(-\xi, \eta)$ is stochastically smaller than $M^+(T_\eta)$, and both such random variables have finite means, from (2.42) we have

$$0 \leq \mathcal{R}(-\xi, \eta) \leq 1.$$

For the first-passage-time problem in the presence of the boundaries η and $-\xi$, the ratio of mean values introduced in (2.42) is a measure of the relevance of the boundary η with respect to the boundary $-\xi$, in the sense that $\mathcal{R}(-\xi, \eta)$ is close to 1 (0) if the first passage through the upper boundary η is expected much more (less) earlier than the first passage through $-\xi$.

Under the assumptions of Proposition 2.1, the ratio of mean values (2.42) becomes

$$\mathcal{R}(-\xi, \eta) = \frac{\Gamma(\alpha, \eta\beta)\Gamma(\alpha^*)}{\Gamma(\alpha, \eta\beta)\Gamma(\alpha^*) + \Gamma(\alpha^*, \xi\beta)\Gamma(\alpha) - \Gamma(\alpha, \eta\beta)\Gamma(\alpha^*, \xi\beta)}. \tag{2.43}$$

In Figures 2.5 and 2.6 some plots of $\mathcal{R}(-\xi, \eta)$ are shown for β equal to 1 and 2, respectively, for $\xi = 5$ and for different choices of the parameters α and α^*. As can be expected, the plots shows that $\mathcal{R}(-\xi, \eta)$ is decreasing in η and α^*,

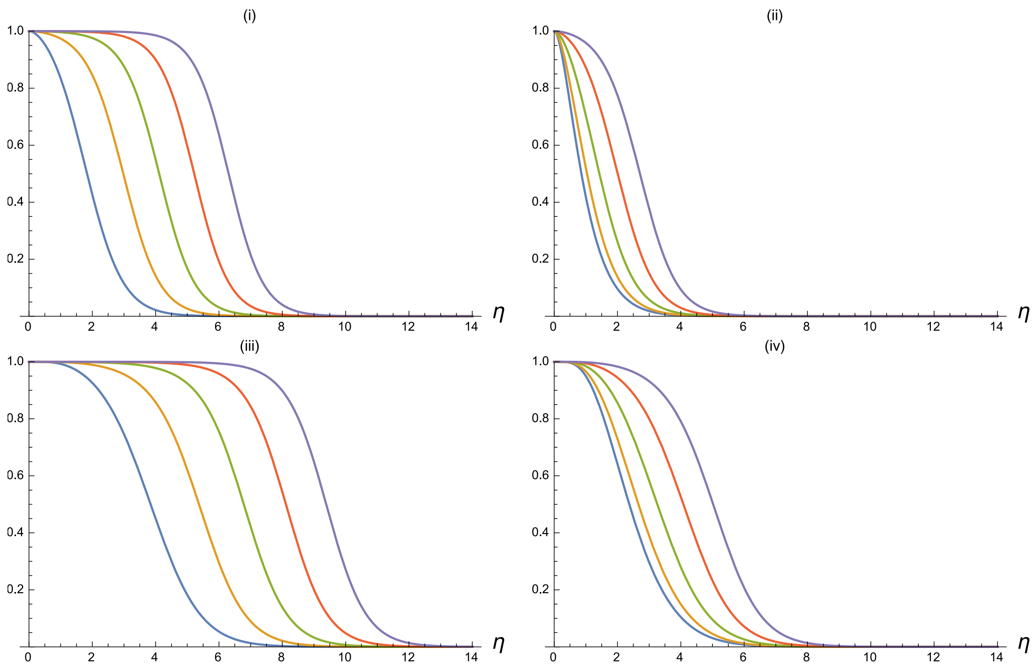

Figure 2.6: Plots of the ratio (2.43) for $0 \leq \eta \leq 14$, with $\beta = 2$ and the same values of the other parameters of Figure 2.5 (after Di Crescenzo and Travaglino 2019).

increasing in ξ and α. Recall that the mean of U_i^+ and D_i^- is proportional to α and α^*, respectively.

By taking advantage of the properties of the counting process (2.33) and (2.36), we can finally derive the explicit form for the probability law of the random first passage times. Let $\eta > 0$ be a threshold and T_η the random first passage time

$$T_\eta = \inf\{t > 0 : X(t) = \eta\}$$

Theorem 2.5.

$$\begin{aligned} \frac{\partial}{\partial t} P(T_\eta \leq t) = \frac{\Gamma(\alpha, \eta\beta)}{\Gamma(\alpha)} \frac{c}{ct - \eta} \exp\left\{ - \frac{(ct-\eta)v\beta}{c+v} \right\} \times \\ \times E_{\alpha+\alpha^*,0}\left(\frac{\gamma(\alpha, \eta\beta)}{\Gamma(\alpha)} \left(\frac{(ct-\eta)v\beta}{c+v} \right)^{(\alpha+\alpha^*)} \right) \mathbb{1}_{\left\{t > \frac{\eta}{c}\right\}} \end{aligned} \tag{2.44}$$

Proof. The case $t \leq \frac{\eta}{c}$ is trivial since it takes at least a time $t = \frac{\eta}{c}$ to cover a distance η at the velocity c. Now, taken a $k > 1$, we notice that

$$\begin{aligned} M^+(T_\eta) = k &\iff \eta \text{ crossed for the first time during the } k\text{-th upwards period} \\ &\iff T_\eta = Z_{2(k-1)} + \frac{\eta}{c}, \end{aligned}$$

so that, for $t > \frac{\eta}{c}$, we can compute the following probability:

$$\begin{aligned} P(T_\eta \leq t) &= \sum_{k=1}^{+\infty} P\{T_\eta \leq t, M^+(T_\eta) = k\} \\ &= \sum_{k=1}^{+\infty} P\{T_\eta \leq t | M^+(T_\eta) = k\} P\{M^+(T_\eta) = k\} \\ &= \sum_{k=1}^{+\infty} P\left\{T_\eta \leq t \middle| T_\eta = Z_{2(k-1)} + \frac{\eta}{c}\right\} P\{M^+(T_\eta) = k\} \\ &= \sum_{k=1}^{+\infty} P\left\{Z_{2(k-1)} \leq t - \frac{\eta}{c}\right\} P\{M^+(T_\eta) = k\} \\ &= \sum_{k=1}^{+\infty} F_{Z_{2(k-1)}}\left(t - \frac{\eta}{c}\right) \left[F_{U_1^+}\left(\frac{\eta}{c}\right)\right]^{k-1} \bar{F}_{U_1^+}\left(\frac{\eta}{c}\right). \end{aligned}$$

Remembering the gamma distribution of the random variables:

$$\begin{aligned} Z_{2(k-1)} &\sim \text{Gamma}\left((k-1)(\alpha + \alpha^*), \frac{cv}{c+v}\beta\right), \\ U_1^+ &\sim \text{Gamma}(\alpha, c\beta), \end{aligned}$$

we can compute the derivative

$$\begin{aligned}
\frac{\partial}{\partial t}P(T_\eta \le t) &= \sum_{k=1}^{+\infty} f_{Z_{2(k-1)}}\left(t-\frac{\eta}{c}\right)\left[F_{U_1^+}\left(\frac{\eta}{c}\right)\right]^{k-1}\bar{F}_{U_1^+}\left(\frac{\eta}{c}\right)\\
&= \sum_{k=1}^{+\infty} \frac{\left(\frac{cv\beta}{c+v}\right)^{(k-1)(\alpha+\alpha^*)}\left(t-\frac{\eta}{c}\right)^{(k-1)(\alpha+\alpha^*)-1}}{\Gamma\big((k-1)(\alpha+\alpha^*)\big)} \exp\left\{-\frac{cv\beta}{c+v}\left(t-\frac{\eta}{c}\right)\right\}\times\\
&\times \left[\frac{\gamma(\alpha,\eta\beta)}{\Gamma(\alpha)}\right]^{k-1}\frac{\Gamma(\alpha,\eta\beta)}{\Gamma(\alpha)}\\
&= \frac{\Gamma(\alpha,\eta\beta)}{\Gamma(\alpha)}\left(t-\frac{\eta}{c}\right)^{-1}\exp\left\{-\frac{(ct-\eta)v\beta}{c+v}\right\}\times\\
&\times \sum_{k=0}^{+\infty}\frac{\left[\left(\frac{cv\beta}{c+v}\right)\left(t-\frac{\eta}{c}\right)\right]^{k(\alpha+\alpha^*)}}{\Gamma\big(k(\alpha+\alpha^*)\big)}\left[\frac{\gamma(\alpha,\eta\beta)}{\Gamma(\alpha)}\right]^k\\
&= \frac{\Gamma(\alpha,\eta\beta)}{\Gamma(\alpha)}\left(\frac{c}{ct-\eta}\right)\exp\left\{-\frac{(ct-\eta)v\beta}{c+v}\right\}\times\\
&\times \sum_{k=0}^{+\infty}\frac{1}{\Gamma\big(k(\alpha+\alpha^*)\big)}\left[\left(\frac{(ct-\eta)v\beta}{c+v}\right)^{(\alpha+\alpha^*)}\frac{\gamma(\alpha,\eta\beta)}{\Gamma(\alpha)}\right]^k\\
&= \frac{\Gamma(\alpha,\eta\beta)}{\Gamma(\alpha)}\left(\frac{c}{ct-\eta}\right)\exp\left\{-\frac{(ct-\eta)v\beta}{c+v}\right\}\times\\
&\times E_{\alpha+\alpha^*,0}\left(\frac{\gamma(\alpha,\eta\beta)}{\Gamma(\alpha)}\left(\frac{(ct-\eta)v\beta}{c+v}\right)^{(\alpha+\alpha^*)}\right)
\end{aligned}$$

so that the theorem is proved. □

Similarly, the following theorem holds for the random first passage time:

$$T_{-\xi} := \inf\{t>0 : X(t) = -\xi\}, \quad \xi > 0.$$

Theorem 2.6.

$$\begin{aligned}
\frac{\partial}{\partial t}P(T_{-\xi}\le t) &= \frac{\Gamma(\alpha^*,\xi\beta)}{\Gamma(\alpha^*)}\left(\frac{v}{vt-\xi}\right)\left(\frac{c(vt-\xi)\beta}{c+v}\right)^{\alpha}\times\\
&\times \exp\left\{-\frac{c(vt-\xi)\beta}{c+v}\right\}E_{\alpha+\alpha^*,\alpha}\left(\frac{\gamma(\alpha^*,\xi\beta)}{\Gamma(\alpha^*)}\left[\frac{c(vt-\xi)\beta}{c+v}\right]^{(\alpha+\alpha^*)}\right)\mathbb{1}_{\left\{t>\frac{\xi}{v}\right\}}
\end{aligned} \tag{2.45}$$

Proof. Similarly to what we saw in the previous theorem, the following equivalence chain holds

$$\begin{aligned} M^-(T_{-\xi}) = k &\iff -\xi \text{ crossed for the first time during the } k\text{-th downwards period} \\ &\iff T_{-\xi} = Z_{2k-1} + \frac{\xi}{v} \end{aligned}$$

so that

$$\begin{aligned} P(T_{-\xi} \leq t) &= \sum_{k=1}^{+\infty} P\{T_{-\xi} \leq t, M^-(T_{-\xi}) = k\} \\ &= \sum_{k=1}^{+\infty} P\{T_{-\xi} \leq t | M^-(T_{-\xi}) = k\} P\{M^-(T_{-\xi}) = k\} \\ &= \sum_{k=1}^{+\infty} P\left\{T_{-\xi} \leq t \,\middle|\, T_{-\xi} = Z_{2k-1} + \frac{\xi}{v}\right\} P\{M^-(T_{-\xi}) = k\} \\ &= \sum_{k=1}^{+\infty} P\left\{Z_{2k-1} \leq t - \frac{\xi}{v}\right\} P\{M^-(T_{-\xi}) = k\} \\ &= \sum_{k=1}^{+\infty} F_{Z_{2k-1}}\left(t - \frac{\xi}{v}\right) \left[F_{D_1^-}\left(\frac{\xi}{v}\right)\right]^{k-1} \bar{F}_{D_1^-}\left(\frac{\xi}{v}\right). \end{aligned}$$

The random variables of interest have the following gamma distribution:

$$\begin{aligned} Z_{2k-1} &\sim \text{Gamma}\left((k-1)(\alpha + \alpha^*) + \alpha, \frac{cv}{c+v}\beta\right), \\ D_1^- &\sim \text{Gamma}(\alpha^*, v\beta). \end{aligned}$$

Thus, the theorem is proved by taking the derivative

$$
\begin{aligned}
&\frac{\partial}{\partial t} P(T_{-\xi} \leq t) = \sum_{k=1}^{+\infty} f_{Z_{2k-1}}\left(t - \frac{\xi}{v}\right)\left[F_{D_1^-}\left(\frac{\xi}{v}\right)\right]^{k-1} \bar{F}_{D_1^-}\left(\frac{\xi}{v}\right) \\
&= \sum_{k=1}^{+\infty} \frac{\left(\frac{cv\beta}{c+v}\right)^{(k-1)(\alpha+\alpha^*)+\alpha}\left(t - \frac{\xi}{v}\right)^{(k-1)(\alpha+\alpha^*)+\alpha-1}}{\Gamma\big((k-1)(\alpha+\alpha^*)+\alpha\big)} \times \\
&\times \exp\left\{-\frac{cv\beta}{c+v}\left(t - \frac{\xi}{v}\right)\right\}\left[\frac{\gamma(\alpha^*, \xi\beta)}{\Gamma(\alpha^*)}\right]^{k-1} \frac{\Gamma(\alpha^*, \xi\beta)}{\Gamma(\alpha^*)} \\
&= \frac{\Gamma(\alpha^*, \xi\beta)}{\Gamma(\alpha^*)}\left(t - \frac{\xi}{v}\right)^{\alpha-1}\left(\frac{cv\beta}{c+v}\right)^{\alpha} \exp\left\{-\frac{c(vt-\xi)\beta}{c+v}\right\} \times \\
&\times \sum_{k=0}^{+\infty} \frac{\left[\left(\frac{cv\beta}{c+v}\right)\left(t - \frac{\xi}{v}\right)\right]^{k(\alpha+\alpha^*)}}{\Gamma\big(k(\alpha+\alpha^*)+\alpha\big)}\left[\frac{\gamma(\alpha^*, \xi\beta)}{\Gamma(\alpha^*)}\right]^{k} \\
&= \frac{\Gamma(\alpha^*, \xi\beta)}{\Gamma(\alpha^*)}\left(\frac{v}{vt-\xi}\right)\left(\frac{c(vt-\xi)\beta}{c+v}\right)^{\alpha} \exp\left\{-\frac{c(vt-\xi)\beta}{c+v}\right\} \times \\
&\times \sum_{k=0}^{+\infty} \frac{1}{\Gamma\big(k(\alpha+\alpha^*)+\alpha\big)}\left[\left(\frac{c(vt-\xi)v\beta}{c+v}\right)^{(\alpha+\alpha^*)} \frac{\gamma(\alpha^*, \xi\beta)}{\Gamma(\alpha^*)}\right]^{k} \\
&= \frac{\Gamma(\alpha^*, \xi\beta)}{\Gamma(\alpha^*)}\left(\frac{v}{vt-\xi}\right)\left(\frac{c(vt-\xi)\beta}{c+v}\right)^{\alpha} \exp\left\{-\frac{c(vt-\xi)\beta}{c+v}\right\} \times \\
&\times E_{\alpha+\alpha^*, \alpha}\left(\frac{\gamma(\alpha^*, \xi\beta)}{\Gamma(\alpha^*)}\left(\frac{c(vt-\xi)\beta}{c+v}\right)^{(\alpha+\alpha^*)}\right).
\end{aligned}
$$

□

As example, in Figure 2.7 and 2.8 we show some plots of the densities (2.44) and (2.45).

2.5 Future developments

It would be interesting to study the limiting behaviour of the state-dependent telegraph process $\{X(t), t \geq 0\}$ as both the speed of the motion and the intensity of switchings tend to infinity. Specifically, one can ask if the process $X(t)$ converges in a weak sense to a certain stochastic process under suitable conditions on the parameters c, v, λ and μ.

For example, under the *Kac's scaling condition*, the classical telegraph process converges in distribution to a Brownian motion with zero drift (see

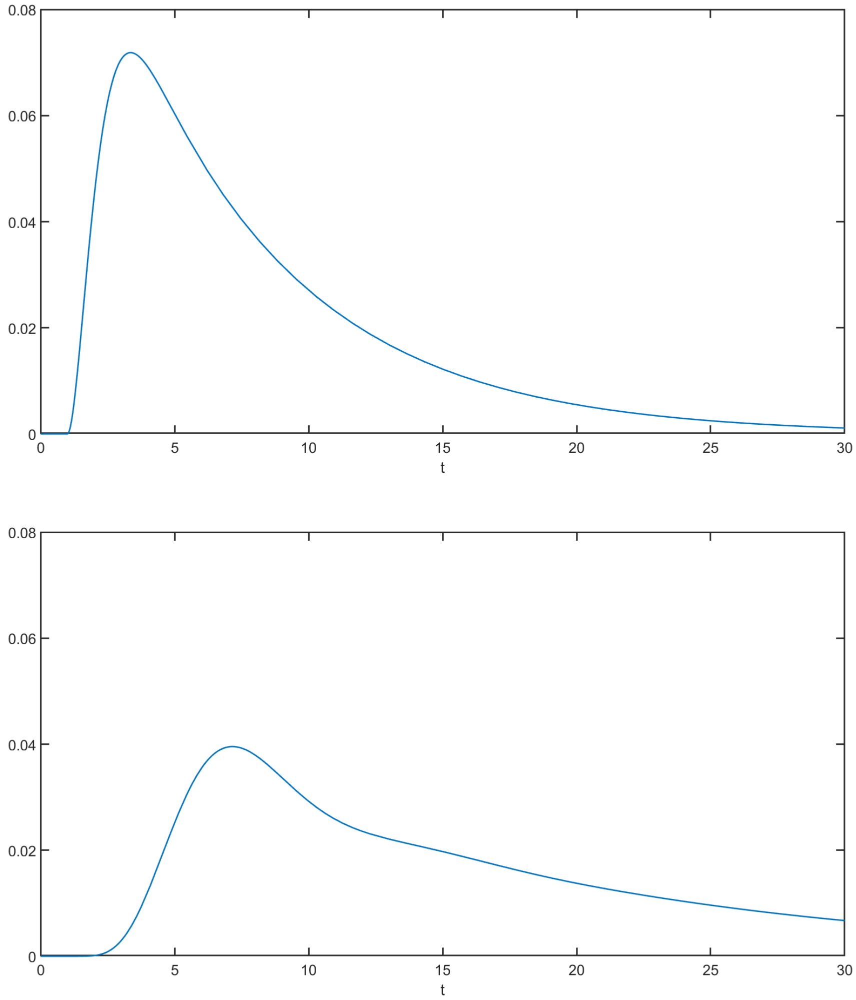

Figure 2.7: Probability density (2.44) of the first passage time, with $\eta = 2$, $\alpha = 2$, $\beta = 1$, $c = 2$, $v = 2$ and $\alpha^* = 1$ (above) or $\alpha^* = 5$ (below).

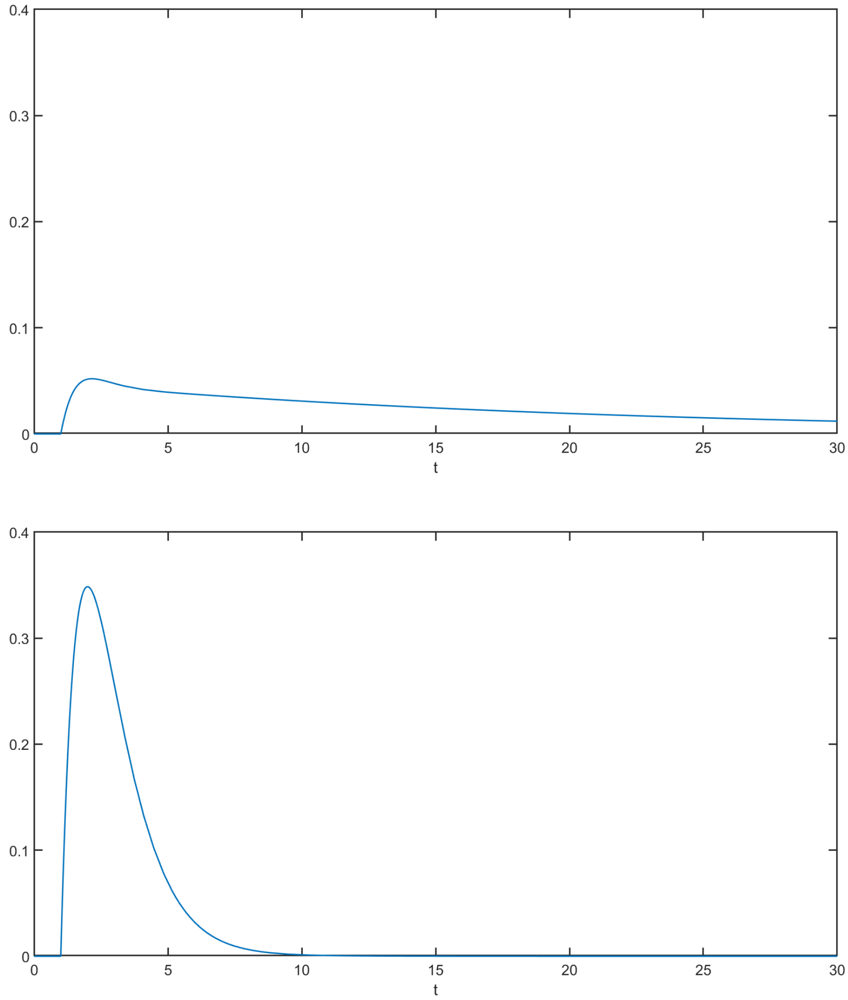

Figure 2.8: Probability density (2.45) of the first passage time, with $\xi = 2$, $\alpha = 2$, $\beta = 1$, $c = 2$, $v = 2$ and $\alpha^* = 1$ (above) or $\alpha^* = 5$ (below).

[58]). One can then verify if, under the following generalization of Kac's conditions

$$c \to \infty,\ v \to \infty,\ \lambda \to \infty,\ \mu \to \infty,\ \frac{c^2}{\lambda} \to \sigma_1^2,\ \frac{v^2}{\mu} \to \sigma_2^2,$$

the state-dependent telegraph process $X(t)$ converges in a weak sense to a Gaussian process with infinitesimal standard deviation alternating between σ_1 and σ_2, such as a Brownian motion modified in such a way to be attracted to the state zero, the origin of the motion.

This could be proved by verifying if the differential system (2.1) reduces to a generalization of the heat equation for the Wiener process. Despite the analysis carried out up to now, we are not yet able to give a rigorous proof for such a result.

Chapter 3

Brownian motion driven by a generalized telegraph process

In this chapter we will study a process deriving from the sum of a standard Brownian motion $\{B(t), t \geq 0\}$ and a generalized telegraph process $\{Y(t), t \geq 0\}$.

For the process $Y(t)$ we will use the same notations as in Section (1.5). Specifically, we will attribute the same meaning to the constants $c, v > 0$, to the processes $V(t)$ and $W(t)$, to the random times $\{U_1, U_2, \dots\}$, $\{D_1, D_2, \dots\}$ and to the densities $\psi(x,t)$, $\psi(x,t;v_t)$ and $\psi_{v_0}(x,t;v_t)$.

The main reference of this chapter will be Di Crescenzo and Zacks (2015) and Travaglino et al. (2018).

3.1 Definition and distribution of the process

Let us consider the process $\{X(t), t \geq 0\}$, defined as follows:

$$X(t) := Y(t) + B(t), \qquad t \geq 0, \tag{3.1}$$

where the standard Brownian motion $\{B(t), t \geq 0\}$ and the generalized telegraph process $\{Y(t), t \geq 0\}$ are independent of each other.

Such a process can be understood as a Brownian motion with drift $V(t)$ alternating the values c and $-v$.

Now let us define, for every $(x,t) \in \mathbb{R}^2_+ = \mathbb{R} \times (0, +\infty)$, the transition density

$$p(x,t) := \frac{\partial}{\partial x} \mathbb{P}\{X(t) \leq x\}. \tag{3.2}$$

The following theorem holds.

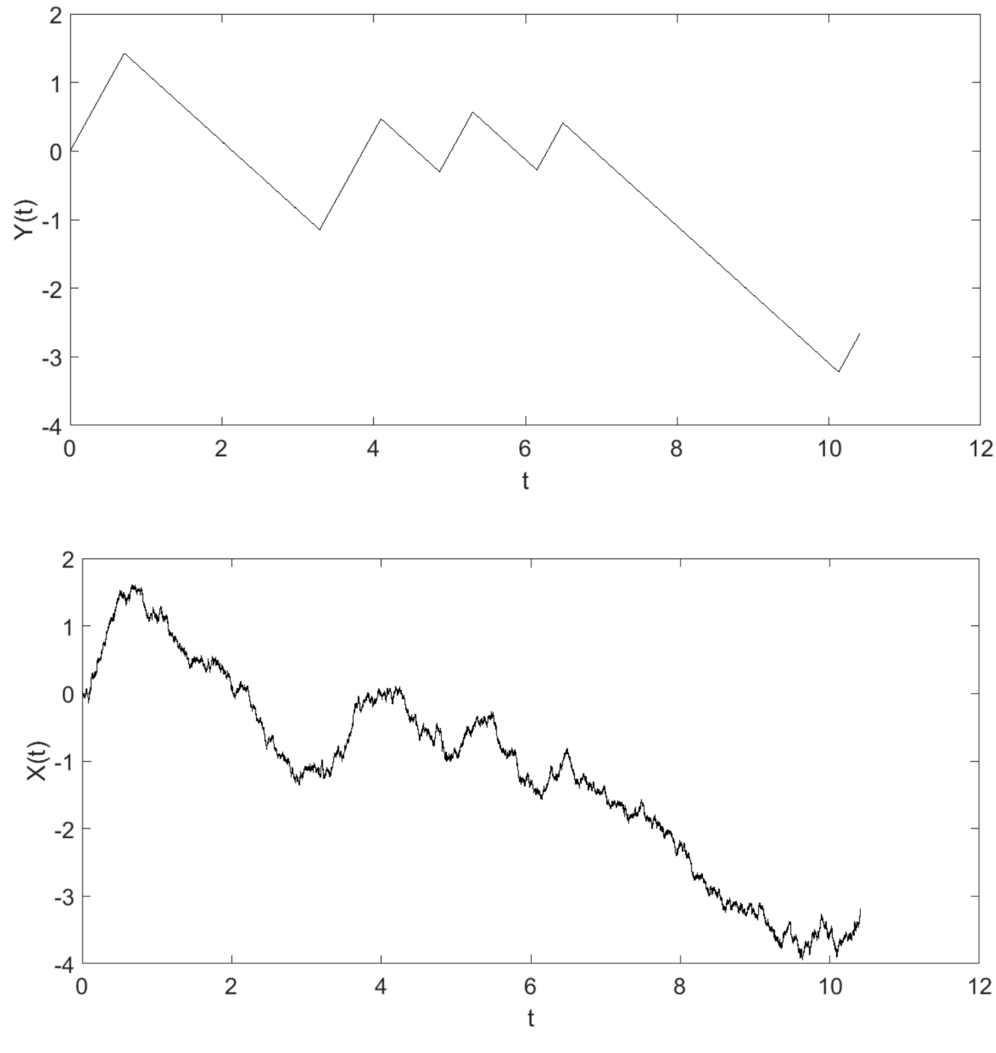

Figure 3.1: Simulated sample paths of $Y(t)$ and $X(t)$ for $c = 2$, $v = 1$, $\lambda = 1$, $\mu = 0.5$ and $\sigma = 0.5$

Theorem 3.1. *For every* $(x,t) \in \mathbb{R}^2_+$

$$\begin{aligned} p(x,t) =& \frac{1}{2\sqrt{t}}\left\{\overline{F}_{D_1}(t)\,\phi\left(\frac{x+vt}{\sqrt{t}}\right) + \overline{F}_{U_1}(t)\,\phi\left(\frac{x-ct}{\sqrt{t}}\right)\right\} \\ &+ \frac{1}{\sqrt{t}}\int_0^t \psi(w,t)\,\phi\left(\frac{x+vt-(c+v)w}{\sqrt{t}}\right)dw, \end{aligned} \tag{3.3}$$

where ϕ *is the standard normal density function.*

Proof. From the relationship (1.81) it easily follows that

$$X(t) = (c+v)W(t) - vt + B(t) \tag{3.4}$$

and thus

$$\big[X(t)\,|\,W(t) = w\big] = (c+v)w - vt + B(t) \overset{d}{=} (c+v)w - vt + \sqrt{t}\,B(1),$$

so that

$$\begin{aligned} \mathbb{P}\{X(t) \le x | W(t) = w\} &= \mathbb{P}\left\{B(1) \le \frac{x+vt-(c+v)w}{\sqrt{t}}\right\} \\ &= \Phi\left(\frac{x+vt-(c+v)w}{\sqrt{t}}\right). \end{aligned}$$

Therefore, by taking into account the (1.71), one has

$$\begin{aligned} \mathbb{P}\{X(t) \le x\} &= \int_{-\infty}^{+\infty} \mathbb{P}\{X(t) \le x | W(t) = w\}\mathbb{P}\{W(t) \in dw\} \\ &= \frac{1}{2}\overline{F}_{D_1}(t)\Phi\left(\frac{x+vt}{\sqrt{t}}\right) + \frac{1}{2}\overline{F}_{U_1}(t)\Phi\left(\frac{x-ct}{\sqrt{t}}\right) \\ &\quad + \int_0^t \psi(w,t)\Phi\left(\frac{x+vt-(c+v)w}{\sqrt{t}}\right)dw. \end{aligned}$$

The thesis follows by differentiating with respect to x. □

Corollary 3.1. *For every* $t > 0$

$$\mathbb{E}[X(t)] = (c+v)\mathbb{E}[W(t)] - vt, \tag{3.5}$$
$$Var[X(t)] = (c+v)^2 Var[W(t)] + t, \tag{3.6}$$

with $\mathbb{E}[W(t)]$ *and* $Var[W(t)]$ *given by the* (1.73) *and* (1.74).

Proof. It immediately follows from the (3.4). □

Let us introduce the joint probability densities:

$$
\begin{aligned}
f(x,t) &:= \frac{\partial}{\partial x}\mathbb{P}\{X(t) \le x, V(t) = c\}, \\
b(x,t) &:= \frac{\partial}{\partial x}\mathbb{P}\{X(t) \le x, V(t) = -v\},
\end{aligned}
\tag{3.7}
$$

for $(x,t) \in \mathbb{R}^2_+$.

Corollary 3.2. *For every* $(x,t) \in \mathbb{R}^2_+$

$$
\begin{aligned}
f(x,t) &= \frac{1}{\sqrt{t}}\left[\frac{\overline{F}_{U_1}(t)}{2}\phi\left(\frac{x-ct}{\sqrt{t}}\right) + \int_0^t \psi(w,t;c)\phi\left(\frac{x+vt-(c+v)w}{\sqrt{t}}\right)dw\right], \\
b(x,t) &= \frac{1}{\sqrt{t}}\left[\frac{\overline{F}_{D_1}(t)}{2}\phi\left(\frac{x+vt}{\sqrt{t}}\right) + \int_0^t \psi(w,t;-v)\phi\left(\frac{x+vt-(c+v)w}{\sqrt{t}}\right)dw\right].
\end{aligned}
\tag{3.8}
$$

Proof. Similar to the proof of Theorem 3.1. □

Finally, we define the *flow function* of $X(t)$:

$$
j(x,t) := f(x,t) - b(x,t), \qquad (x,t) \in \mathbb{R}^2_+. \tag{3.9}
$$

3.2 Exponentially distributed times

In this section we assume that the random times U_i and D_i are exponentially distributed with parameters λ and μ, where $\lambda, \mu > 0$.

The summations $U^{(n)}$ have thus Erlang distribution with parameters λ and n, briefly $U^{(n)} \sim \mathcal{E}(\lambda, n)$, and $D^{(n)} \sim \mathcal{E}(\mu, n)$. Therefore, one has, for $n = 1, 2, \dots$ and $x \ge 0$,

$$
\begin{aligned}
f_U^{(n)}(x) &= \frac{\lambda^n x^{n-1}}{(n-1)!}e^{-\lambda x}, \qquad F_U^{(n)}(x) = 1 - e^{-\lambda x}\sum_{j=0}^{n-1}\frac{(\lambda x)^j}{j!}, \\
f_D^{(n)}(x) &= \frac{\mu^n x^{n-1}}{(n-1)!}e^{-\mu x}, \qquad F_D^{(n)}(x) = 1 - e^{-\mu x}\sum_{j=0}^{n-1}\frac{(\mu x)^j}{j!}.
\end{aligned}
\tag{3.10}
$$

Theorem 3.2. *If the random variables U_i and D_i, $i = 1, 2, \dots$, are exponentially distributed with parameters λ and μ, respectively, then, for $t > 0$,*

$$
\mathbb{P}\{W(t) = 0\} = \frac{1}{2}e^{-\mu t}, \qquad \mathbb{P}\{W(t) = t\} = \frac{1}{2}e^{-\lambda t}, \tag{3.11}
$$

and, for $0 < x < t$,

$$\psi(x,t) = \frac{e^{-\lambda x - \mu(t-x)}}{2} \cdot \\ \cdot \left\{ (\lambda+\mu) I_0\Big(2\sqrt{\lambda\mu x(t-x)}\Big) + \frac{\sqrt{\lambda\mu}\, t}{\sqrt{x(t-x)}} I_1\Big(2\sqrt{\lambda\mu x(t-x)}\Big) \right\}. \tag{3.12}$$

Furthermore, for $(x,t) \in \mathbb{R}^2_+$,

$$\begin{aligned} p(x,t) = &\frac{1}{2\sqrt{t}} \left\{ e^{-\lambda t} \phi\Big(\frac{x-ct}{\sqrt{t}}\Big) + e^{-\mu t} \phi\Big(\frac{x+vt}{\sqrt{t}}\Big) \right\} \\ &+ \frac{1}{\sqrt{t}} \int_0^t \psi(w,t) \phi\Big(\frac{x+vt-(c+v)w}{\sqrt{t}}\Big) dw \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} j(x,t) = &\frac{1}{2\sqrt{t}} \left\{ e^{-\lambda t} \phi\Big(\frac{x-ct}{\sqrt{t}}\Big) - e^{-\mu t} \phi\Big(\frac{x+vt}{\sqrt{t}}\Big) \right\} \\ &+ \frac{1}{\sqrt{t}} \int_0^t \Big[\psi(w,t;c) - \psi(w,t;-v)\Big] \phi\Big(\frac{x+vt-(c+v)w}{\sqrt{t}}\Big) dw, \end{aligned} \tag{3.14}$$

where

$$\psi(x,t;c) - \psi(x,t;-v) = \frac{e^{-\lambda x - \mu(t-x)}}{2} \cdot \\ \cdot \left\{ (\mu-\lambda) I_0\Big(2\sqrt{\lambda\mu x(t-x)}\Big) + \frac{\sqrt{\lambda\mu}(2x-t)}{\sqrt{x(t-x)}} I_1\Big(2\sqrt{\lambda\mu x(t-x)}\Big) \right\}. \tag{3.15}$$

Proof. The (3.11) immediately follow from the (1.68), taking into account the exponential distribution of U_1 and D_1.

Similarly, the (3.13) follows from the (3.3) and the (3.14) immediately follows from the definition (3.9) and from Corollary 3.2.

From the (1.69) and (3.10) it follows that

$$
\begin{aligned}
&\psi_c(x,t;c)\\
&=\sum_{n=1}^{+\infty}\left[1-e^{-\lambda x}\sum_{j=0}^{n-1}\frac{(\lambda x)^j}{j!}-1+e^{-\lambda x}\sum_{j=0}^{n}\frac{(\lambda x)^j}{j!}\right]\frac{\mu^n(t-x)^{n-1}}{(n-1)!}e^{-\mu(t-x)}\\
&=\sum_{n=1}^{+\infty}\left[\frac{(\lambda x)^n}{n!}e^{-\lambda x}\right]\frac{\mu^n(t-x)^{n-1}}{(n-1)!}e^{-\mu(t-x)}\\
&=e^{-\lambda x-\mu(t-x)}\sum_{n=1}^{+\infty}\frac{(\lambda\mu x)^n(t-x)^{n-1}}{n!(n-1)!}\\
&=e^{-\lambda x-\mu(t-x)}(\lambda\mu x)\sum_{n=0}^{+\infty}\frac{\big(\lambda\mu x(t-x)\big)^n}{n!(n+1)!}\\
&=e^{-\lambda x-\mu(t-x)}\frac{\lambda\mu x}{\sqrt{\lambda\mu x(t-x)}}\sum_{n=0}^{+\infty}\frac{\big(\sqrt{\lambda\mu x(t-x)}\big)^{2n+1}}{n!(n+1)!}\\
&=e^{-\lambda x-\mu(t-x)}\frac{\sqrt{\lambda\mu}\,x}{\sqrt{x(t-x)}}\,I_1\Big(2\sqrt{\lambda\mu x(t-x)}\Big)
\end{aligned}
$$

as well as

$$
\begin{aligned}
\psi_c(x,t;-v)&=\sum_{n=0}^{+\infty}e^{-\mu(t-x)}\frac{\big(\mu(t-x)\big)^n}{n!}\frac{\lambda^{n+1}x^n}{n!}e^{-\lambda x}\\
&=\lambda e^{-\lambda x-\mu(t-x)}\sum_{n=0}^{+\infty}\frac{\big(\lambda\mu x(t-x)\big)^n}{n!\,n!}=\lambda e^{-\lambda x-\mu(t-x)}\,I_0\Big(2\sqrt{\lambda\mu x(t-x)}\Big).
\end{aligned}
$$

In the same way, by making use of the (1.70), one has

$$
\psi_{-v}(x,t;c)=\mu e^{-\lambda x-\mu(t-x)}\,I_0\Big(2\sqrt{\lambda\mu x(t-x)}\Big)
$$

as well as

$$
\psi_{-v}(x,t;-v)=e^{-\lambda x-\mu(t-x)}\frac{\sqrt{\lambda\mu}(t-x)}{\sqrt{x(t-x)}}\,I_1\Big(2\sqrt{\lambda\mu x(t-x)}\Big).
$$

The thesis follows from the relationships (1.66) and (1.67). □

Theorem 3.3. *Under the assumptions of the previous theorem one has, for $t\geq 0$,*

$$
\mathbb{E}[W(t)]=\frac{\mu}{\lambda+\mu}\,t+\frac{\lambda-\mu}{2(\lambda+\mu)^2}\Big(1-e^{-(\lambda+\mu)t}\Big). \tag{3.16}
$$

Proof. It follows from the second identity of Corollary 1 in [85]. □

Reasoning as in the proof of Theorem 1.7 we also get the following result.

Theorem 3.4. *For every $i = 1, 2, \dots,$ let the random variables U_i and D_i be exponentially distributed with parameters λ and μ, respectively. Then, for $(x,t) \in \mathbb{R}_+^2$, the densities $f(x,t)$ and $b(x,t)$ satisfy the system of differential equations*

$$\begin{cases} \dfrac{\partial f}{\partial t} + c\dfrac{\partial f}{\partial x} = \dfrac{1}{2}\dfrac{\partial^2 f}{\partial x^2} - \lambda f + \mu b, \\ \dfrac{\partial b}{\partial t} - v\dfrac{\partial b}{\partial x} = \dfrac{1}{2}\dfrac{\partial^2 b}{\partial x^2} - \mu b + \lambda f. \end{cases} \tag{3.17}$$

Corollary 3.3. *Under the hypothesis of the previous system, the following system of differential equations holds*

$$\begin{cases} \dfrac{\partial p}{\partial t} + \dfrac{c-v}{2}\dfrac{\partial p}{\partial x} + \dfrac{c+v}{2}\dfrac{\partial j}{\partial x} = \dfrac{1}{2}\dfrac{\partial^2 p}{\partial x^2}, \\ \dfrac{\partial j}{\partial t} + \dfrac{c+v}{2}\dfrac{\partial p}{\partial x} + \dfrac{c-v}{2}\dfrac{\partial j}{\partial x} = \dfrac{1}{2}\dfrac{\partial^2 j}{\partial x^2} - (\lambda-\mu)p - (\lambda+\mu)j, \end{cases} \tag{3.18}$$

for $(x,t) \in \mathbb{R}_+^2$.

Proof. It follows straightforwardly from the linearity of the system (3.17), taking into account that $p(x,t) = f(x,t) + b(x,t)$ and $j(x,t) = f(x,t) - b(x,t)$. □

We observe that by setting $c = v = 0$ in the system (3.18) we get the heat equation for the standard Brownian motion while if we disregard the diffusive terms $\frac{1}{2}\frac{\partial^2 p}{\partial x^2}$ and $\frac{1}{2}\frac{\partial^2 j}{\partial x^2}$ the system (3.18) is reduced to the system of differential equations (1.85) which governs the generalized telegraph process.

Chapter 4

A stochastic model for Campi Flegrei inflation and deflation episodes

This chapter is devoted to the construction of a suitable stochastic model, based on Brownian motion driven by the telegraph process, for the phenomenon of bradyseism in Campi Flegrei.

The idea first appeared in Travaglino et al. (2018) [96], from which the following introduction to the phenomenon is taken.

Campi Flegrei are an active caldera located near the city of Naples, Italy. This region is worldwide famous for its slow vertical motions recorded since roman times, providing one of the longest times series of ground deformation recorded near a volcanic region (cf. Guidoboni and Ciuccarelli (2011) and Orsi et al. (1999)). After some centuries of subsidence, following the last Monte Nuovo 1538 eruption, Campi Flegrei caldera has shown unrest episodes of activity since at least 1950 (Del Gaudio et al., 2010). The first recent uplift episode dates back in the period 1950-1953, amounting to 73 cm, without any report or record of seismic activity. In the period 1970-1972 and during 1982-1984 two strong inflation episodes occurred, the first accompanied by moderate low seismicity (Corrado et al., 1977), with only few events felt by residents, whereas the second has been accompanied by relatively intense swarms of Volcano Tectonic (VT) earthquakes, reaching up to magnitude 4 (Barberi et al., 1986). The seismic activity caused alarm in the population and a spontaneous nightly evacuation of part of the city of Pozzuoli (44.000 residents). Since this last episode, subsidence has been recorded for several years, interrupted by some small mini-uplift episodes, with a duration of several weeks, all accompanied by seismic swarms of low magnitude VT events. In recent years some high sensitivity instruments have been installed to detect slow earthquake transients and other mechanical/temperature low intensity precursory signals (Scarpa et al., 2007; Amoruso et al., 2015). Since late 2004 another moderate uplift is occurring at very small rate, amounting to about 1-2 cm/yr, showing the occurrence of clear Long Period (LP) events (Amoruso et al., 2007; Saccorotti et al., 2007; D'Auria et al., 2011). The aim of this chapter is to provide a quantitative model of this phenomenon based on the Brownian motion driven by a generalized telegraph process. This approach is very important for deriving a quantitative formulation of some basic parameters regulating the inflation/deflation processes, such as their velocities and relative time constants. This is of fundamental relevance for understanding the source process of this activity.

4.1 Data

As mentioned before there is a singular time series of ground deformation records in the Campi Flegrei region. Archeological ruins of Serapeo Roman temple demonstrate clearly that these three columns, builded above ground level around 200 b.C. have been below the sea level for centuries until the middle age. The dominant deflation has been interrupted around 88 a.C. and in early 1500, when a rapid uplift amounting to approximately 8 m preceded the last 1538 eruption several decades before. The subsidence is clearly documented since 1800 until 1950, when this trend has been interrupted by three main episodes of uplift and several minor inflation/deflation trends. Detailed reports of these processes have been made by Dvorak and Gasparini (1991), Dvorak and Berrino (1991) and more recently by Del Gaudio et al. (2010). Figure 4.1 illustrates the ground deformation reported in the last 2400 years (after Woo and Kilburn, 2010).

The quality of data on these vertical movements dramatically improved since 1970, due to the alarm in the population derived from the effects on the drainage of water in the local harbor and the occurrence of some felt earthquakes, particularly during the uplift occurred during 1982-1984. Leveling measurements improved in these years together with the installation of other geophysical monitoring networks, managed by Vesuvius Observatory. Starting from 2000, GPS continuous measurements were performed, integrated by InSAR techniques since 1992, thus providing a more complete and homogeneous picture of ground deformations. The data set under investigation is referred to weekly averaged GPS data recorded at the station RITE, located near the Pozzuoli harbor. These data are available in the period may 2000-july 2019. The values are referred to the average GPS data recorded at the Campi Flegrei network, composed by 10 stations. The precision of these data

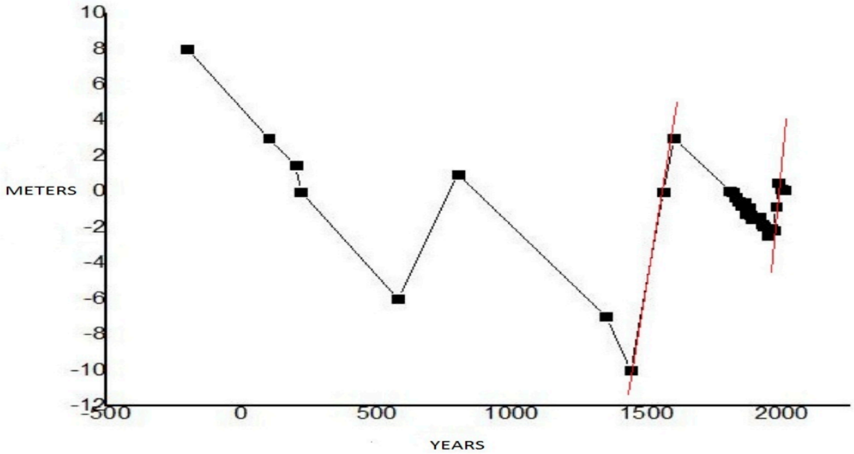

Figure 4.1: Secular behavior of uplift and subsidence observed in Campi Flegrei region observed at the Roman temple Serapeo, Pozzuoli in the period 200 bC-present (after Woo and Kilburn, 2010). It can be observed a comparable rate of uplift before the 1538 eruption and presently.

is close to 1-2 mm. Data have been retrieved from the technical reports of Vesuvius Observatory.

Other time series of ground deformation are available, but they are not homogeneous and they can basically divided in 3 parts: a first series, referred to vertical leveling, precision around 1 cm, in the period 1970-1995 derived from sea tide gaude data recorded at Pozzuoli harbor, compared with another tide gauge instrument located in the Napoli harbor. A second data set is referred to the measurements at a bench mark located near the Pozzuoli harbor deduced from high precision altimetric levelings in the period april 1950-july 2010. The precision of these data is around 1 cm and the data are referred to a bench marl located away from the center the caldera, near the centre of Napoli. A third data set is also deduced from leveling data, but for yearly measurements performed in the period march 1970-december 1994, corresponding to the most intense phase of unrest. However, our investigation is performed only on the data set concerning period may 2000-july 2019, whose measurements are homogenous and more reliable. Future investigations will be conducted also on the other time series, by taking into account their different precision and reliability.

In the rest of the chapter times are expressed in days and lengths are reported in cm.

4.2 The stochastic model

In order to describe the alternating random trend exhibited by the measurements performed in the Campi Flegrei area, we propose a suitable stochastic model defined as the superposition of a pure alternating trend process (telegraph process) and a diffusive noise component (Brownian motion). The available data do not exhibit drastic displacements, so that the considered model is characterized by continuous sample paths. As in the previous chapter, the model is based on a stochastic process $\{X(t), t \geq 0\}$ consisting in a Brownian motion whose drift alternates randomly between a positive and a negative value (c and $-v$), according to a generalized telegraph process. Formally, this assumes that the ground position at time t, with respect to the sea level, is described by

$$X(t) = x_0 + Y(t) + \sigma B(t), \qquad t \geq 0, \tag{4.1}$$

where
(i) $\{Y(t), t \geq 0\}$ is a generalized (integrated) telegraph process, with $Y(0) = 0$;
(ii) $\{B(t), t \geq 0\}$ is a standard Brownian motion;
(iii) $\{Y(t)\}$ and $\{B(t)\}$ are independent processes;
(iv) $x_0 \in \mathbb{R}$, $\sigma > 0$ and $X(0) = x_0$, since $Y(0) = B(0) = 0$.

In Eq. (4.1), $x_0 + Y(t)$ describes the randomly alternating trend of the observed displacements, whereas the term $\sigma B(t)$ is a diffusive noise component that represents the total sum of all other small independent perturbation factors that are involved in the considered phenomenon. Physically, these factors may include cumulative effects of small pressure perturbations in a shallow magma chamber as suggested by Amoruso *et al.* (2007). Clearly, the Gaussian distribution of $\sigma B(t)$ might be motivated by central-limit arguments. Similar arguments justify the role of the Brownian motion in other seismological contexts, as pointed out in Kagan and Knopoff (1987) and Matthews *et al.* (2002).

The telegraph process $Y(t)$ is characterized by velocities $c > 0$ and $-v < 0$, which alternate according to an independent alternating Poisson process $\{N(t), t \geq 0\}$. The latter process is governed by the sequences of positive independent random times $\{U_1, U_2, \ldots\}$ and $\{D_1, D_2, \ldots\}$, which in turn are assumed to be independent. The random variable U_i (resp. D_i) is exponentially distributed with parameter $\lambda > 0$ (resp. $\mu > 0$), and describes the duration of the i-th random period during which $Y(t)$ has positive (resp. negative) velocity, i.e. $X(t)$ has positive (resp. negative) drift. Let us denote by $V(t)$ the velocity of $Y(t)$ at time $t \geq 0$. As a general model, if the initial velocity of $Y(t)$ is randomly chosen by an independent Bernoulli trial, with

$$\mathbb{P}\{V(0)=c\}=\theta, \quad \mathbb{P}\{V(0)=-v\}=1-\theta, \qquad 0\le\theta\le 1, \tag{4.2}$$

then the following stochastic equations hold.

$$Y(t)=\int_0^t V(s)\,\mathrm{d}s, \qquad t>0, \tag{4.3}$$

$$V(t)=\frac{c-v}{2}+\mathrm{sgn}[V(0)]\,\frac{c+v}{2}\,(-1)^{N(t)}, \qquad t>0. \tag{4.4}$$

The alternating Poisson process $\{N(t)\}$, involved in the right-hand-side of (4.4), counts the number of velocity changes of $Y(t)$ in $[0,t]$. See, for instance, Figure 3.1 for simulated paths of $Y(t)$ and $X(t)$ when the initial velocity of $Y(t)$ is positive. Denoting by

$$p(x,t):=\frac{\partial}{\partial x}\mathbb{P}\{X(t)\le x\}$$

the transition density of $X(t)$, for all $(x,t)\in\mathbb{R}\times(0,+\infty)$ one has (it follows from the (3.13) after some simple generalizations)

$$\begin{aligned} p(x,t)=&\frac{1}{\sigma\sqrt{t}}\left\{\theta\,e^{-\lambda t}\phi\left(\frac{x-x_0-ct}{\sigma\sqrt{t}}\right)+(1-\theta)\,e^{-\mu t}\phi\left(\frac{x-x_0+vt}{\sigma\sqrt{t}}\right)\right\}\\ &+\frac{1}{\sigma\sqrt{t}}\int_0^t\psi(w,t)\phi\left(\frac{x-x_0+vt-(c+v)w}{\sigma\sqrt{t}}\right)dw, \end{aligned} \tag{4.5}$$

where $\phi(\cdot)$ is the standard normal density, and for $0<x<t$,

$$\begin{aligned} \psi(x,t)=&\,e^{-\lambda x-\mu(t-x)}\\ &\times\Bigg[\theta\left\{\lambda\,I_0\left(2\sqrt{\lambda\mu x(t-x)}\right)+\frac{\sqrt{\lambda\mu x}}{\sqrt{t-x}}\,I_1\left(2\sqrt{\lambda\mu x(t-x)}\right)\right\}\\ &+(1-\theta)\left\{\mu\,I_0\left(2\sqrt{\lambda\mu x(t-x)}\right)+\frac{\sqrt{\lambda\mu(t-x)}}{\sqrt{x}}\,I_1\left(2\sqrt{\lambda\mu x(t-x)}\right)\right\}\Bigg]. \end{aligned} \tag{4.6}$$

In Eq. (4.6), $I_0(z)$ and $I_1(z)$ denote the modified Bessel functions given by

$$I_0(z)=\sum_{k=0}^{+\infty}\frac{(z/2)^{2k}}{(k!)^2}, \qquad I_1(z)=\sum_{k=0}^{+\infty}\frac{(z/2)^{2k+1}}{k!\,(k+1)!}=I_0'(z). \tag{4.7}$$

The function (4.6) is the probability density of the sojourn time of $V(t)$ in the state c in the time interval $[0,t]$.

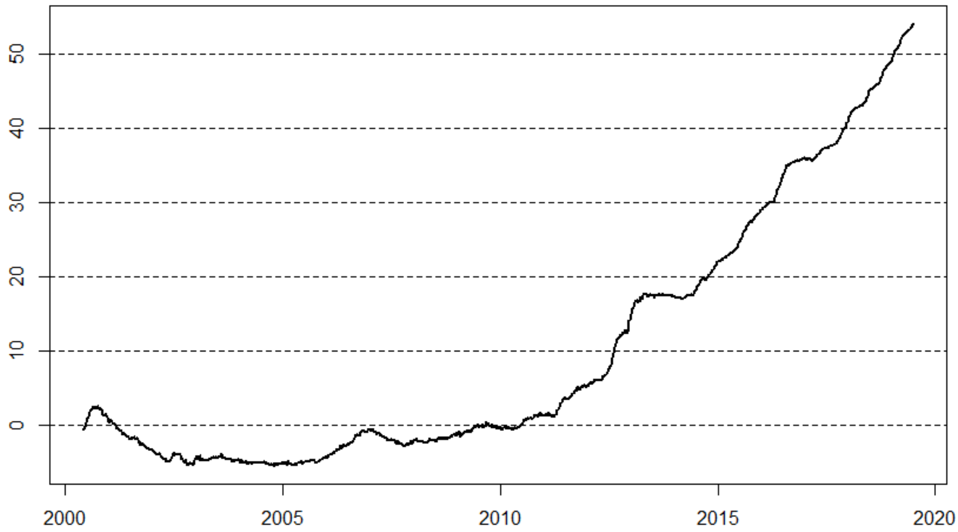

Figure 4.2: Observed ground position (in cm).

4.3 Data analysis

Various statistical analyses on the parameters of the (discretely observed) telegraph process have been performed in the past by means of a least-squares approach, pseudo maximum likelihood and moment (or approximate moment) estimations (see De Gregorio and Iacus (2008), (2011), Iacus and Yoshida (2009)). Since the model (4.1) involves also a Brownian motion component, a different approach is considered hereafter for the estimation of the relevant parameters.

4.3.1 Estimation of turning times

Recalling that the proposed position process (4.1) is the sum of a telegraph process $y_0 + Y(t)$ and an independent Brownian motion $\sigma B(t)$, let us now propose to interpret the available data (shown in Figure 4.2) as a trajectory of such alternating Brownian process $X(t)$.

The alternating behavior suggests to identify different time periods, corresponding to the inflation and deflation episodes. In order to estimate the turning times of the alternating periods we have made use of a suitable customized version of Muggeo's method in [73].

The main idea behind Muggeo's method is to linearize the following problem:

$$\begin{cases} x_i = \alpha_0^{(1)} + \alpha_1^{(1)} t_i + \epsilon_i & \text{if} \quad t_i \le r_1 \\ x_i = \alpha_0^{(2)} + \alpha_1^{(2)} t_i + \epsilon_i & \text{if} \quad r_1 < t_i \le r_2 \\ \vdots \\ x_i = \alpha_0^{(K+1)} + \alpha_1^{(K+1)} t_i + \epsilon_i & \text{if} \quad r_K < t_i \end{cases} \tag{4.8}$$

where the ϵ_i are the error terms and the $r_1, \ldots, r_K$ are the *breaking points* of the model. The linearization makes use of the following reparametrization:

$$x_i = \beta_0 + \beta_1 t_i + \sum_{j=1}^{K} \beta_{j+1} (t_i - r_j) \mathbb{1}_{\{t_i > r_j\}} + \epsilon_i. \tag{4.9}$$

Models (4.8) and (4.9) are equivalent when

$$\begin{cases} \alpha_0^{(1)} = \beta_0, \\ \alpha_0^{(k)} = \beta_0 - \sum_{j=1}^{k} \beta_{j+1} r_j \quad k = 1, 2, \ldots, K, \\ \alpha_1^{(k)} = \sum_{j=1}^{k} \beta_j \qquad k = 1, 2, \ldots, K+1. \end{cases} \tag{4.10}$$

In order to estimate the breaking points $(r_1, r_2, \ldots, r_K)$ one can adopt the following algorithm:

1. choose an initial estimate $(r_1^{(0)}, r_2^{(0)}, \ldots, r_K^{(0)})$,
2. use the first two terms of the Taylor expansion of the term $(t_i - r_j)\mathbb{1}_{\{t_i > r_j\}}$ in (4.9) around $r_j^{(0)}$:

$$(t_i - r_j)\mathbb{1}_{\{t_i > r_j\}} \approx (t_i - r_j^{(0)})\mathbb{1}_{\{t_i > r_j^{(0)}\}} + (r_j - r_j^{(0)})(-1)\mathbb{1}_{\{t_i > r_j^{(0)}\}}, \tag{4.11}$$

3. insert (4.11) in (4.9):

$$\begin{aligned} x_i = \beta_0 + \beta_1 t_i + \sum_{j=1}^{K} \beta_{j+1} (t_i - r_j^{(0)}) \mathbb{1}_{\{t_i > r_j^{(0)}\}} \\ + \sum_{j=1}^{K} \beta_{j+1} (r_j - r_j^{(0)})(-1) \mathbb{1}_{\{t_i > r_j^{(0)}\}}, \end{aligned} \tag{4.12}$$

4. define
$$\gamma_j = \beta_{j+1}(r_j - r_j^{(0)}) \tag{4.13}$$
and
$$U_{ji} = (t_i - r_j^{(0)})\mathbb{1}_{\{t_i > r_j^{(0)}\}}, \quad V_{ji} = -\mathbb{1}_{\{t_i > r_j^{(0)}\}}, \tag{4.14}$$
for $j = 1, 2, \ldots, K$,

5. solve the resulting multiple linear regression model:
$$x_i \approx \beta_0 + \beta_1 t_i + \sum_{j=1}^{K} \beta_{j+1} U_{ji} + \sum_{j=1}^{K} \gamma_j V_{ji} + \epsilon_i \tag{4.15}$$
to find the Least Squares estimates $\hat{\beta}_0, \hat{\beta}_1, \ldots, \hat{\beta}_{K+1}, \hat{\gamma}_1, \hat{\gamma}_2, \ldots, \hat{\gamma}_K$,

6. use Equation (4.13) to obtain the estimate of the breaking points
$$\hat{r}_j = r_j^{(0)} + \frac{\hat{\gamma}_j}{\hat{\beta}_{j+1}}, \quad j = 1, 2, \ldots, K, \tag{4.16}$$

7. if the $\hat{\gamma}_j$ are close to zero for each j stop, otherwise set $r_j^{(0)} = \hat{r}_j$ and repeat the steps until convergence occurs.

In our case, for every j, in
$$x_i = \alpha_0^{(j)} + \alpha_1^{(j)} t_i + \epsilon_i \tag{4.17}$$
the linear part $\alpha_0^{(j)} + \alpha_1^{(j)} t_i$ corresponds to the trajectory of the telegraph process $y_0 + Y(t)$. Since $Y(t)$ alternates just two velocities, system (4.9) can be written as:
$$x_i = \beta_0 + v_0 t_i + \sum_{j=1}^{K} \delta(-1)^{j+1}(t_i - r_j)\mathbb{1}_{\{t_i > r_j\}} + \epsilon_i, \tag{4.18}$$
where $v_0 := V(0)$ is the initial velocity of the process $Y(t)$ and $v_0 + \delta$ is the other possible value for $V(t)$. A customized version of the algorithm derives from this in which the following system must be solved at each step:
$$x_i \approx \beta_0 + v_0 t_i + \sum_{j=1}^{K} \delta(-1)^{j+1} U_{ji} + \sum_{j=1}^{K} \gamma_j V_{ji} + \epsilon_i \tag{4.19}$$
where the U_{ji} and V_{ji} are defined as in (4.14) and $\gamma_j := \delta(-1)^{j+1}(r_j - r_j^{(0)})$.

To find the right number of breaking points and the initial estimate for them an empirical approach has been carried out. First, as said in [73], to impose too many breaking points lacks of meaning; specifically, in our case we have observed that having too many breaking points causes these to "cross" with each other during the algorithm, thus resulting in the vector $(r_1, r_2, \ldots, r_K)$ no longer ordered. Given the above, in a preliminary approach, the turning times of the alternating periods have been estimated empirically by the local minima and maxima of the observed sample path; the corresponding estimates are:

$$\begin{gathered} r_1^{(0)} = 2005.03.29, \;\; r_2^{(0)} = 2007.01.23, \;\; r_3^{(0)} = 2007.10.16, \\ r_4^{(0)} = 2013.04.23, \;\; r_5^{(0)} = 2014.03.04. \end{gathered} \tag{4.20}$$

We point out that in our analysis it has been verified that a number of breaking points greater than five does not work for the above reasons. Then we use the (4.20) as initial estimate in the above algorithm, in which at each step an Ordinary Least Squares (OLS) regression is performed.

The final estimates of the breaking points are:

$$\begin{gathered} \hat{r}_1 = 2005.12.30, \;\; \hat{r}_2 = 2006.11.17, \;\; \hat{r}_3 = 2010.07.30, \\ \hat{r}_4 = 2013.08.02, \;\; \hat{r}_5 = 2014.04.12. \end{gathered} \tag{4.21}$$

The relevant estimated inflation and deflation periods are shown in Figure 4.3.

4.3.2 Estimation of the parameters of the motion

Once estimated the turning times, one can perform a further regression using General Least Squares (GLS). The error terms ϵ_i in the model (4.18) correspond to the Brownian component $\sigma B(t_i)$, so that they exhibit heteroskedasticity and autocorrelation. Specifically,

$$\mathbb{V}\mathrm{ar}(\epsilon_i) = \sigma^2 t_i, \tag{4.22}$$

$$\mathrm{Cov}(\epsilon_i, \epsilon_j) = \sigma^2 \min\{t_i, t_j\}, \tag{4.23}$$

or, in matrix form,

$$\mathbb{V}\mathrm{ar}(\epsilon) = \sigma^2 \Psi, \tag{4.24}$$

where $\Psi = (\min\{t_i, t_j\})$. If we denote with β, x and M the vector of the coefficients, the vector of the x_i and the matrix of the regressors, respectively, in model (4.18), then the GLS estimates are given by

$$\hat{\beta} = (M'\Psi^{-1}M)^{-1}M'\Psi^{-1}x. \tag{4.25}$$

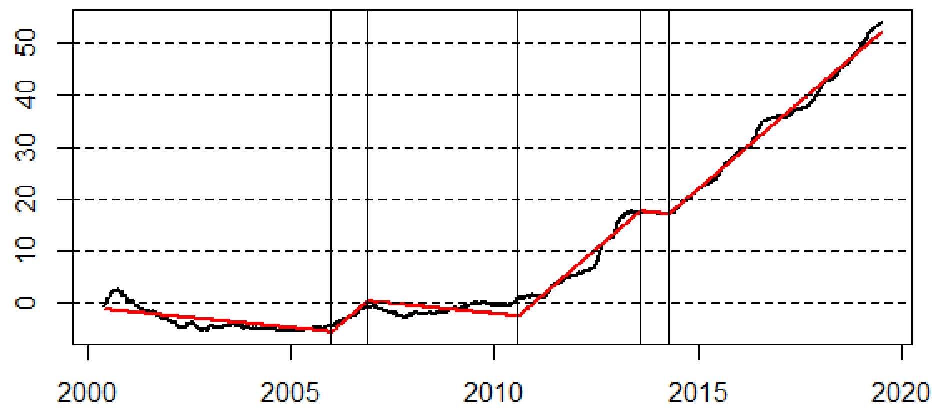

Figure 4.3: Estimated inflation and deflation periods. The red line corresponds to the regression line in the last step of the algorithm in Section 4.3.1.

Figure 4.4 shows the relevant trajectory of the telegraph process, compared with the observed data.

The infinitesimal variance σ^2 of the Brownian motion can be estimating by dividing the residual sum of squares by the number of observations minus the number of regressors, that is,

$$\hat{\sigma}^2 = \frac{1}{N-2}(x - M\hat{\beta})'\Psi^{-1}(x - M\hat{\beta}), \tag{4.26}$$

where $N = 942$ is the number of observation and being 2 the number of the regressors: those corresponding to v_0 and δ in (4.18). Moreover, the (4.26) can be used to estimate the covariance matrix of $\hat{\beta}$:

$$\mathbb{V}\mathrm{ar}(\hat{\beta}) = \sigma^2(M'\Psi^{-1}M)^{-1} \approx \hat{\sigma}^2(M'\Psi^{-1}M)^{-1}. \tag{4.27}$$

Thus, we can evaluate the standard deviation for $\hat{c}$ and $\hat{v}$, since

$$\begin{aligned} \hat{v} &= -\hat{\beta}_1, &\quad \mathbb{V}\mathrm{ar}(\hat{v}) &= \mathbb{V}\mathrm{ar}(\hat{\beta}_1) \\ \hat{c} &= \hat{\beta}_1 + \hat{\beta}_2, &\quad \mathbb{V}\mathrm{ar}(\hat{c}) &= \mathbb{V}\mathrm{ar}(\hat{\beta}_1) + \mathbb{V}\mathrm{ar}(\hat{\beta}_2) + 2\mathrm{Cov}(\hat{\beta}_1, \hat{\beta}_2). \end{aligned} \tag{4.28}$$

However, a t-ratio test for the significance of the regressors by means of the standard deviation of the coefficients would be meaningless, given the customized structure of the system (4.18).

Table 4.1: Estimates of the velocities of the telegraph process in the inflation and deflation episodes, of turning rates, and of infinitesimal standard deviation of the Brownian component, with the coefficient of determination for the telegraph process. For the velocities the estimated standard deviation has been reported too.

	estimates		std
c	$1.71 \cdot 10^{-2}$	cm/day	$1.3 \cdot 10^{-3}$
v	$6.17 \cdot 10^{-4}$	cm/day	$1.2 \cdot 10^{-3}$
λ	$6.01 \cdot 10^{-4}$	1/day	
μ	$5.49 \cdot 10^{-4}$	1/day	
σ	$7.47 \cdot 10^{-2}$	$\mathrm{cm}/\sqrt{\mathrm{day}}$	
R^2	0.95		

Furthermore, we can compute the Maximum Likelihood estimate of the turning rates λ and μ. The Likelihood function for λ is given by (see Figure 4.4 for simplicity)

$$L(\lambda) = \lambda e^{-\lambda(\hat{r}_2-\hat{r}_1)}\lambda e^{-\lambda(\hat{r}_4-\hat{r}_3)}e^{-\lambda(t_N-\hat{r}_5)} \tag{4.29}$$

where $t_N = 2019.07.02$ is the time instant corresponding to the last observation available. The log-likelihood is

$$l(\lambda) := \log(L(\lambda)) = 2\log(\lambda) - \lambda(\hat{r}_2 - \hat{r}_1 + \hat{r}_4 - \hat{r}_3 + t_N - \hat{r}_5). \tag{4.30}$$

Therefore, the resulting ML estimate is

$$\hat{\lambda} = \frac{2}{\hat{r}_2 - \hat{r}_1 + \hat{r}_4 - \hat{r}_2 + t_N - \hat{r}_5}. \tag{4.31}$$

Similarly, the log-likelihood for the rate μ is given by

$$l(\mu) := \log L(\mu) = 2\log(\mu) - \mu(\hat{r}_1 - t_0 + \hat{r}_3 - \hat{r}_2 + \hat{r}_5 - \hat{r}_4), \tag{4.32}$$

where $t_0 = 2000.05.30$ is the time instant corresponding to the first observation available, so that the resulting ML estimate is

$$\hat{\mu} = \frac{2}{\hat{r}_1 - t_0 + \hat{r}_3 - \hat{r}_2 + \hat{r}_5 - \hat{r}_4}. \tag{4.33}$$

A summary of the estimated parameters is reported in Table 4.1.

As a preliminary index of the goodness of the model, it is interesting to determine the coefficient of determination, given by $R^2 := 1 - S_{\mathrm{res}}/S_{\mathrm{tot}}$,

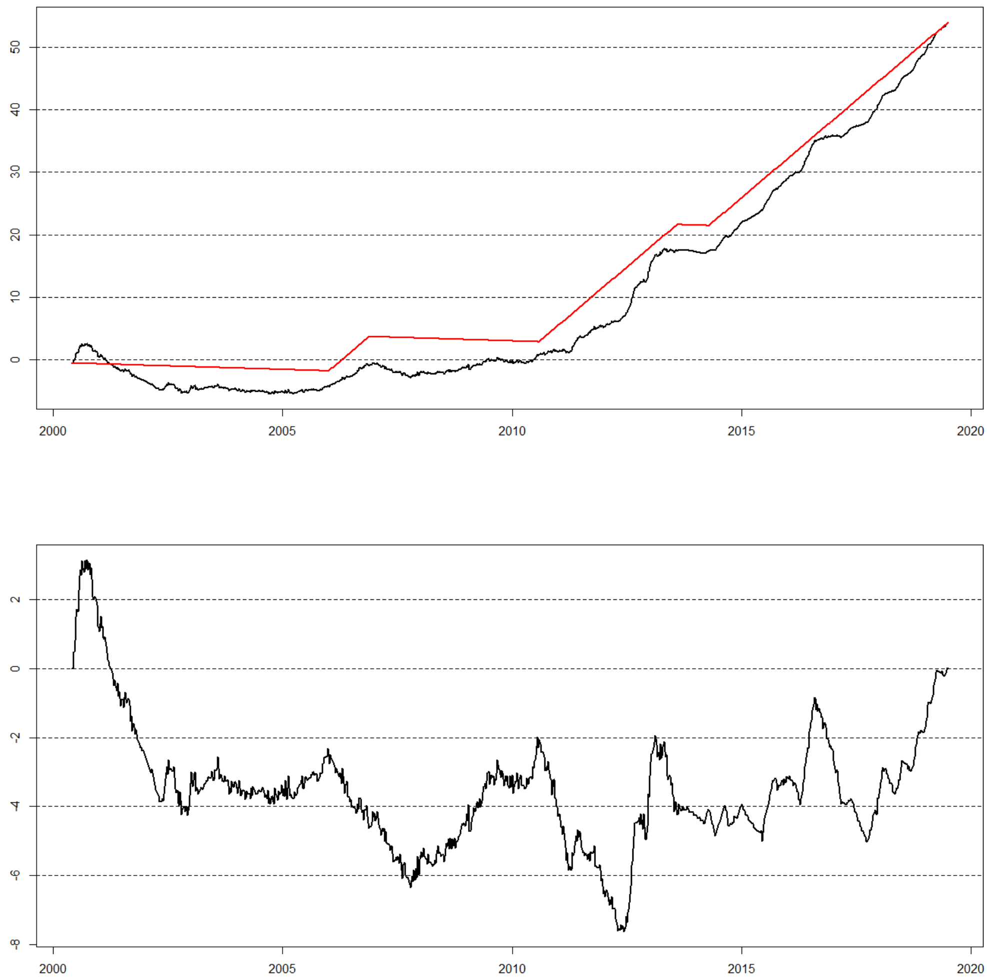

Figure 4.4: Estimated trajectory of the telegraph process $x_0 + Y(t)$ underlying the observed data (above) and the resulting Brownian motion $\sigma B(t)$ (below).

where S_{res} is the residual deviance and S_{tot} is the total deviance of data. Notice that the value of R^2 is referred to the approximation of the observed data by the telegraph process $x_0 + Y(t)$ only.

Due to Eq. (4.1), the trajectory of the Brownian component $\sigma B(t)$ is obtained simply as difference, and is reported in Figure 4.4.

Table 4.2: Probability that the current motion has no changes of tendency up to time t.

t	$P_0(t)$
2021.12.31	0.58
2022.12.31	0.46
2023.12.31	0.37
2024.12.31	0.30
2025.12.31	0.24

4.3.3 Some predictions

For the considered model it is now possible to perform some predictions on the ground of the results obtained so far.

The probability that the motion has no changes of tendency in time is evaluated first. To this purpose, let us denote by $P_0(t)$ the probability of having no changes of tendency of the motion in the time interval $(t_N, t]$, where $t_N = 2019.07.02$ is the last available observation time. According to the given assumptions, since at time t_N the trend is increasing (see Figure 4.3), $P_0(t)$ is concerning an exponential distribution with parameter λ. Hence, at time t, such probability is given by

$$P_0(t) = e^{-\lambda(t-t_N)}, \qquad t > t_N,$$

where the estimate $\lambda = 6.01 \cdot 10^{-4}$ is provided in Table 4.1. Some values of $P_0(t)$ are listed in Table 4.2. Due to the given results, during year 2022 such probability becomes smaller than 1/2, so that a change of tendency becomes more likely.

Let us now consider the problem of determining some predictive intervals, i.e. intervals where the estimated location of $X(t)$ is more likely, for suitable choices of t. Recalling (4.5), for $x_1 < x_2$ one has

$$\begin{aligned}
&\mathbb{P}[x_1 < X(t) < x_2] = \int_{x_1}^{x_2} p(x,t)dx \\
&= \theta\, e^{-\lambda t} \left[\Phi\left(\frac{x_2 - x_0 - ct}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{x_1 - x_0 - ct}{\sigma\sqrt{t}}\right)\right] \\
&+ (1-\theta)\, e^{-\mu t} \left[\Phi\left(\frac{x_2 - x_0 + vt}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{x_1 - x_0 + vt}{\sigma\sqrt{t}}\right)\right] \\
&+ \int_0^t \psi(w,t) \left[\Phi\left(\frac{x_2 - x_0 + vt - (c+v)w}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{x_1 - x_0 + vt - (c+v)w}{\sigma\sqrt{t}}\right)\right] dw,
\end{aligned} \tag{4.34}$$

where $\psi(w,t)$ is defined in (4.6), and $\Phi(\cdot)$ is the standard normal cumulative distribution function. Probability (4.34) conditional on the observed data is then evaluated for various choices of t and (x_1, x_2). This is performed by making use of the Markov property of the Brownian process, and taking into account that at time $t_N = 2019.07.02$ the position is $X(t_N) = x_N = 54$, and that the tendency is increasing (so that $\theta = 1$). Therefore

$$\begin{aligned}
&\mathbb{P}[x_1 < X(t) < x_2 \,|\, X(t_i) = x_i,\, i = 1, \dots, N] \\
&= \mathbb{P}[x_1 < X(t) < x_2 \,|\, X(t_N) = x_N, V(t_N) = c] \\
&= \mathbb{P}[x_1 < X(t - t_N) < x_2 \,|\, X(0) = x_N, V(0) = c] \\
&= e^{-\lambda(t-t_N)} \left[\Phi\left(\frac{x_2 - x_N - c(t-t_N)}{\sigma\sqrt{t-t_N}}\right) - \Phi\left(\frac{x_1 - x_N - c(t-t_N)}{\sigma\sqrt{t-t_N}}\right)\right] \\
&+ \int_0^{t-t_N} \psi(w, t-t_N) \left[\Phi\left(\frac{x_2 - x_N + v(t-t_N) - (c+v)w}{\sigma\sqrt{t-t_N}}\right) + \right. \\
&\qquad \left. - \Phi\left(\frac{x_1 - x_N + v(t-t_N) - (c+v)w}{\sigma\sqrt{t-t_N}}\right)\right] dw,
\end{aligned} \tag{4.35}$$

where the ψ is given by (see the (4.6) with $\theta = 1$)

$$\begin{aligned}
\psi(x,t) &= e^{-\lambda x - \mu(t-x)} \\
&\times \left\{\lambda\, I_0\left(2\sqrt{\lambda\mu x(t-x)}\right) + \frac{\sqrt{\lambda\mu x}}{\sqrt{t-x}}\, I_1\left(2\sqrt{\lambda\mu x(t-x)}\right)\right\}
\end{aligned} \tag{4.36}$$

and the involved parameters are chosen as in Table 4.1.

Finally, Table 4.3 shows the estimated intervals (x_1, x_2) for various values of the probability (4.35) conditional on the observed data. The extremes x_1 and x_2 are taken as $m \pm h$, where m is the mode of the conditional density

$$\begin{aligned}
p(x,t;x_N,t_N) &= \frac{e^{-\lambda(t-t_N)}}{\sigma\sqrt{t-t_N}}\,\phi\left(\frac{x - x_N - c(t-t_N)}{\sigma\sqrt{t-t_N}}\right) \\
&+ \frac{1}{\sigma\sqrt{t-t_N}} \int_0^{t-t_N} \psi(w, t-t_N)\phi\left(\frac{x - x_N + v(t-t_N) - (c+v)w}{\sigma\sqrt{t-t_N}}\right) dw.
\end{aligned} \tag{4.37}$$

By analysis of the density one has

$$m = \begin{cases} 69.6 & \text{for } t = 2021.12.31, \\ 75.9 & \text{for } t = 2022.12.31. \end{cases} \tag{4.38}$$

The density (4.37) is plotted in Figure 4.5 for the above specified choices of the parameters. The particular structure of the plot shows how the density

Table 4.3: Estimated intervals for $\mathbb{P}[x_1 < X(t) < x_2]$ conditional on the observed data.

probability $\mathbb{P}[x_1 < X(t) < x_2]$	(x_1, x_2) (cm) $t = 2021.12.31$	(x_1, x_2) (cm) $t = 2022.12.31$
0.70	(64.7 , 74.5)	(66.9 , 84.9)
0.80	(61.6 , 77.6)	(62.8 , 89.0)
0.90	(57.6 , 81.6)	(58.3 , 93.5)
0.95	(55.5 , 83.7)	(55.6 , 96.2)

$p(x,t)$ can be seen as a mixture of Gaussian densities. This is reflected in the structure of the (4.37): the highest peak corresponds to the first addendum in the right hand, which expresses the density of the process in the event that there are no changes in the trend of motion from time t_N to time t. The asymmetrical tail on the left corresponds to the combination of the infinite Gaussian densities that forms the second addend of the (4.37): each of them expresses the density of the process once the total time in which the motion has had positive velocity in the time interval (t_N, t) has been fixed (this time is given by the variable w). By conditioning on this time, indeed, the density of the process becomes normal (it is equivalent to considering the remaining Brownian motion with the resulting drift).

As mentioned above, the highest peak corresponds to the case of zero velocity changes: the mode (4.38) can therefore be found simply as $m = x_N + \hat{c}(t - t_N)$.

Finally, we point out that the adopted prediction procedure consists in a statistical method based on the hypothesis that no catastrophic event occurs in the reference time interval. Indeed, it is customary confirmed that catastrophic events modify the ground dynamics substantially.

4.4 Testing the Brownian component

For the stochastic model considered so far, denote by $\{D(t), t \geq 0\}$, the difference between the position process $X(t)$ and the telegraph (trend) process $x_0 + Y(t)$, plotted in Figure 4.4 (below). In this section, a statistical test is performed on the ground of observed data in order to verify if $D(t)$ is a Brownian motion. Specifically, the following test is considered

$$H_0 : \; D(t) = \sigma B(t) \qquad \text{vs} \qquad H_1 : \; D(t) \text{ is a } \left\{ \begin{array}{l} \text{confined} \\ \text{directed} \end{array} \right. \text{diffusion.}$$

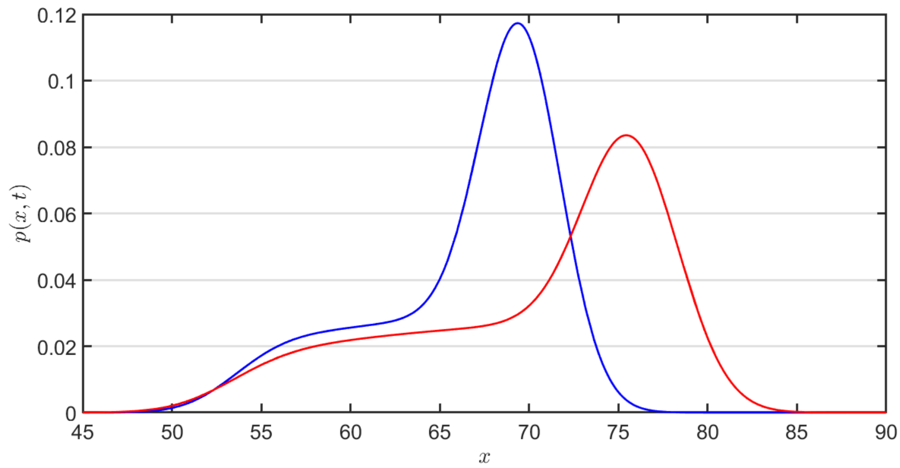

Figure 4.5: Estimated density $p(x,t)$ for $t = 2021.12.31$ (blue) and $t = 2022.12.31$ (red).

In a sense, this is aimed to test if the noise component is namely a Brownian motion or a confined or directed diffusion. The adopted method has been proposed recently by Briane *et al.* (2016). It is based on the data set concerning $D(t)$, denoted by $(D(\tau_0), D(\tau_1), \ldots, D(\tau_n))$ where τ_i's are the observation time instants, with $n + 1 = 942$. To this purpose, consider the following standardized statistic

$$T_D^n(\tau_n) = \frac{S_D^n(\tau_n)}{\hat{\sigma}_{MLE}\sqrt{\tau_n}}, \tag{4.39}$$

where

$$S_D^n(\tau_n) = \max_{i=1,2,\ldots,n} |D(\tau_i) - D(\tau_0)|$$

and

$$\hat{\sigma}_{MLE} = \left\{ \frac{1}{n} \sum_{i=1}^{n} \frac{[D(\tau_i) - D(\tau_{i-1})]^2}{\tau_i - \tau_{i-1}} \right\}^{1/2}. \tag{4.40}$$

Eq. (4.40) is the maximum likelihood estimator (MLE) of σ, which differs from the GLS estimator whose value is shown in Tables 4.1. However, their values are close, since, for the considered data,

$$\hat{\sigma}_{MLE} = 7.44 \cdot 10^{-2} \, \frac{\text{cm}}{\sqrt{\text{day}}}. \tag{4.41}$$

Being the sample size large, according to Briane *et al.* (2016), the asymptotic acceptance region of amplitude $1 - \alpha$ of the null hypotesis H_0 is defined as

$$\left\{ q\left(\frac{\alpha}{2}\right) \leq T_D^n(\tau_n) \leq q\left(1-\frac{\alpha}{2}\right) \right\}, \tag{4.42}$$

where $q(\alpha)$ is the lower quantile of level α of $\sup_{0\leq s\leq 1} |B(s) - B(0)|$. Since the estimate of (4.39) is

$$t_D^n(\tau_n) = 1.226$$

and since, for $\alpha = 0.05$, one has (cf. Table 1 of Briane *et al.* (2016))

$$q(0.025) = 0.834, \qquad q(0.975) = 2.940,$$

due to (4.42) the null hypotesis H_0 can be accepted with level 0.95. Therefore, this confirms that the observed trajectory of $D(t)$ can be viewed as a realization of a Brownian motion.

4.5 Some remarks

The data on the vertical ground motion of Campi Flegrei, provided by different methodologies, provide an unique example of unrest episodes in an active caldera due to the length of the available time series. It is very difficult to demonstrate if the current unrest episodes are a long-term precursor of a new eruption, and if, and when it will occur. The main results derived from this analysis provide a more precise quantification of uplift and subsidence rates, in good agreement with previous estimates made by different authors. The beginning of unrest dates back to 1950, and it has not been marked by significant seismicity, whereas later episodes (in particular the 1982-1984 event) are characterized by seismicity with progressively higher magnitudes and more significant seismic energy release. These values are still however quite low as compared to other calderas in the world experiencing eruption such as in Rabaul in 1994. The evolution of the deformation and seismicity seems to indicate a slow approach to more unstable conditions of the volcano. The time duration of the current unrest period, its size and its trend are similar to those that preceded the eruption of 1538 in the period 1400-1536 (i.e between 2,9 and 9.1 cm/y as estimated by Di Vito et al., 2016) and may have a similar conclusion, but it is quite difficult, in absence of additional data, have a more precise definition of the required time. It is only roughly expected, before the onset of the eruption, an accelerating rate of uplift and seismicity. Another relevant observation made by Amoruso et al (2015) is that seismicity occurs several minutes after inflation and deflation episodes. The quantitative model presented here offers a relevant possibility to precisely and quantitatively estimate the alternation of uplift and subsidence rates characterizing this volcanic region.

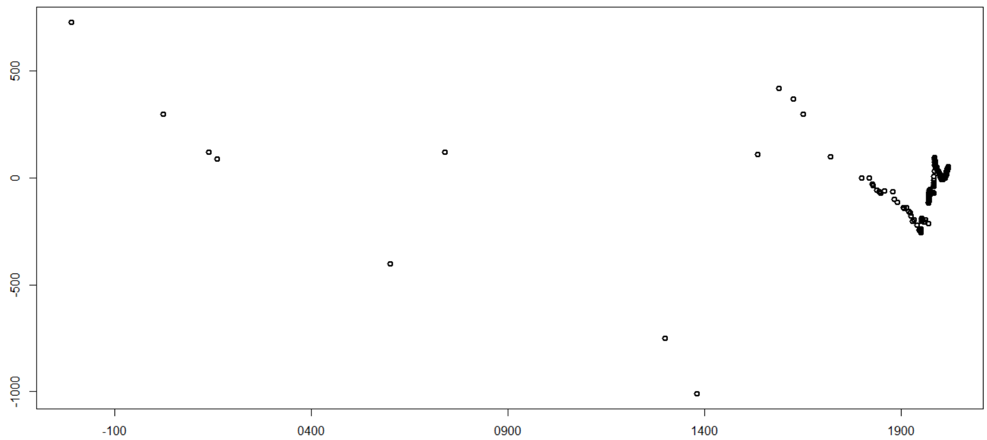

Figure 4.6: Historical series of ground deformation for Campi Flegrei, from 210 bC to the present day.

4.6 Future developments

The results of the analyzes in this chapter can be used to estimate the eruptive risk of Campi Flegrei as a function of the ground level. This can be done by studying the entire historical series of ground deformation for Campi Flegrei, reconstructed by putting together the different datasets described in Section 4.1 with the oldest observations carried out by means of carbon-14 dating performed on the columns of Serapeo Roman temple in Pozzuoli (see figures 4.1 and 4.6).

As mentioned, the last eruption occurred in 1538: the closest observations are $x = 110$ cm and $x = 420$ cm, corresponding to years 1536 and 1590, respectively. It is likely that the ground level had a very high peak of growth between 1536 and 1538 and then fell (very) slowly to the level recorded in 1590. Based on the current understanding of the phenomenon, according to which inflation episodes are due to the influx of magma in significant quantities at a very shallow depth level, it is possible to directly link the eruptive risk to the height of the ground. We can therefore consider the height recorded in 1590, which, based on what has been said above, should be slightly lower than that at the time of the eruption of Monte Nuovo, as a limit threshold: one could therefore study the eruptive risk as a first-passage-time problem for this barrier.

To do this, one needs to perform the statistical analysis seen in section 4.3 on the entire set of observations (plotted in figure 4.6).

However, the methods used herein, as well as the easier ones used in Travaglino et al. [96], do not perform well on such uneven data. For example, the algorithm used in Section 4.3.1 cannot work when there are only two or less observations between two breaking points. Furthermore, the precision of the measurements varies dramatically between the different observations: 1-2 mm for the most recent ones compared to a standard deviation of up to 1 meter for the oldest ones. The latter also present an uncertainty on the time axis (up to 10 years).

To overcome these issues, we are thinking of ways to separately process observations corresponding to different datasets and then recombine the results together, with an *ad hoc* designed procedure.

Appendix A

Bessel functions

The *Bessel functions* $J_\nu(z)$ with real argument are defined as the solutions of the *Bessel's differential equations*:

$$z^2 \frac{d^2u}{dz^2} + z\frac{du}{dz} + (z^2 - \nu^2)u = 0. \tag{A.1}$$

The $J_\nu(z)$ can be expressed as series:

$$J_\nu(z) = \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!\,\Gamma(k+\nu+1)} \left(\frac{z}{2}\right)^{2k+\nu}, \tag{A.2}$$

where $\Gamma(t) := \int_0^{+\infty} x^{t-1}e^{-x}dx$ is the *Euler's gamma function*.

Bessel functions can be evaluated even for complex z. Specifically, when z is a purely immaginary argument, one gets the *modified Bessel functions*, defined as follows:

$$I_\nu(z) := i^{-\nu} J_\nu(iz) = \sum_{k=0}^{+\infty} \frac{1}{k!\,\Gamma(k+\nu+1)} \left(\frac{z}{2}\right)^{2k+\nu}. \tag{A.3}$$

It is easy to verify that the $I_\nu(z)$s satisfy the *modified Bessel's equation*:

$$z^2 \frac{d^2u}{dz^2} + z\frac{du}{dz} - (z^2 + \nu^2)u = 0. \tag{A.4}$$

Form the (A.2) it follows that

$$I_\nu(0) = \begin{cases} 1 & \text{if } \nu = 0, \\ 0 & \text{if } \nu > 0. \end{cases} \tag{A.5}$$

For integer $v = n$, taking into account that $\Gamma(m+1) = m!$, one has

$$I_n(z) = \sum_{k=0}^{+\infty} \frac{(z/2)^{2k+n}}{k!\,(k+n)!} \tag{A.6}$$

and, specifically,

$$I_0(z) = \sum_{k=0}^{+\infty} \frac{(z/2)^{2k}}{(k!)^2}, \tag{A.7}$$

$$I_1(z) = \sum_{k=0}^{+\infty} \frac{(z/2)^{2k+1}}{k!\,(k+1)!} = I_0'(z), \tag{A.8}$$

$$I_2(z) = \sum_{k=0}^{+\infty} \frac{(z/2)^{2k+2}}{k!\,(k+2)!} = -\frac{2}{z} I_1(z) + I_0(z). \tag{A.9}$$

For $z \geq 0$ the following inequalities hold:

$$I_0(z) \leq e^z, \qquad \frac{I_1(z)}{z} \leq \frac{1}{2}\, e^z. \tag{A.10}$$

Indeed

$$I_0(z) = \sum_{k=0}^{+\infty} \left(\frac{(z/2)^k}{k!} \right)^2 \leq \left(\sum_{k=0}^{+\infty} \frac{(z/2)^k}{k!} \right)^2 = e^z,$$

$$\frac{I_1(z)}{z} = \frac{1}{2} \sum_{k=0}^{+\infty} \frac{1}{k!(k+1)!} \left(\frac{z}{2} \right)^{2k} \leq \frac{1}{2} \sum_{k=0}^{+\infty} \frac{1}{(k!)^2} \left(\frac{z}{2} \right)^{2k} = \frac{1}{2} I_0(z) \leq \frac{1}{2} e^z.$$

The asymptotic behaviour of Bessel functions is given by the following formula:

$$I_\nu(z) = \frac{e^z}{\sqrt{2\pi z}} \Big(1 + O(z^{-1}) \Big), \qquad \text{for } z \to +\infty. \tag{A.11}$$

As a matter of fact, the following asymptotic expansion holds for $z \to +\infty$

$$\begin{aligned} I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}} \bigg\{ 1 &- \frac{(4\nu^2 - 1)}{8z} + \frac{(4\nu^2-1)(4\nu^2-9)}{2!(8z)^2} + \\ &- \frac{(4\nu^2-1)(4\nu^2-9)(4\nu^2-25)}{3!(8z)^3} + \dots \bigg\}. \end{aligned} \tag{A.12}$$

Finally, we mention the following integral form

$$I_\nu(z) = \frac{z^\nu}{2^\nu \sqrt{\pi}\, \Gamma(\nu + \frac{1}{2})} \int_{-1}^{1} (1 - \zeta^2)^{\nu - \frac{1}{2}} \cosh(\zeta z) d\zeta, \tag{A.13}$$

for $\nu > -\frac{1}{2}$.

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