UNIVERSITÀ DI SALERNO

TESI DI DOTTORATO

Uniqueness and Partition of Energy for Thermomicrostretch Elastic Solids Backward in Time

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Contents

Introduction

The theory of micromorphic bodies, developed by Eringen in [1,2], considers a material point as endowed with three deformable directors. When the directors are constrained to have only breathing-type microdeformations, then the body is a microstretch continuum [3–5]. The points of these materials can stretch and contract independently of their translations and rotations. The microstretch continuum defines a model useful for the study of composite materials reinforced with chopped elastic fibers, porous media whose pores are filled with gas or liquid, asphalt, etc. Further, the theory of microstretch continua is an adequate tool to describe the behavior of porous materials. In fact, if the microrotation vector field is neglected, then the linear equations that describe the behavior of a microstretch elastic body become similar to the equations of an elastic material with voids, as established by Cowin and Nunziato [6].

In [7] Iesan and Quintanilla establish existence and uniqueness results for the basic boundary-value problems of elastostatics for the microstretch elastic solids. Moreover, they use some results of the semigroups theory to prove an existence theorem in the dynamic theory. In the context of the theory of thermo-microstretch elastic solids, Bofill and Quintanilla studied existence and uniqueness results [8] and for a semi-infinite cylinder with the boundary lateral surface at null temperature Quintanilla [9] established a spatial decay estimate controlled by an exponential of a polynomial of second degree.

The problem of asymptotic partition of energy has been studied by Goldstein [10, 11]; applying the semigroup theory the author proves an equipartition theorem. Using the Lagrange identity method, Levine [12] shows that the difference of Cesaro means of kinetic and potential energies vanishes as time goes to infinity. Day [13] establishes the asymptotic equipartition of kinetic and strain energies in linear elastodynamics. This last result is extended by Chirita [14] to the theory of linear thermoelasticity.

Following the methods developed by Chirita [14], and Chirita e Ciarletta [15], Ciarletta and Scalia [16] obtain a complete analysis of the spatial behavior of the solutions with time-dependent and time-independent decay and growth rates. In this context they study the spatial behavior for large and short values of time.

On the other hand, the backward in time problems have been initially considered by Serrin [17] who established uniqueness results for the Navier-Stokes equations. Explicit uniqueness and stability criteria for classical Navier-Stokes equations backward in time have been further established by Knops and Payne [18] and Galdi and Straughan [19] (see also Payne and Straughan [20] for a class of improperly posed problems for parabolic partial differential equations). Such back in time problems have been considered also by Ames and Payne [21] in order to obtain stabilizing criteria for solutions of the boundary-final value problem. It is well known that this type of problem is ill posed.

In [22], Ciarletta established uniqueness and continuous dependence results upon mild requirements concerning the thermoelastic coefficients; in particular the author considers hypotheses not realistic from the physical point of view, such as a positive semidefinite elasticity tensor or a non positive heat capacity. Moreover, introducing an appropriate time-weighted volume measure, Ciarletta and Chirita [23] established the spatial estimate describing the spatial exponential decay of the thermoelastic process backward in time. Recently Quintanilla [24, 25] improved the uniqueness result obtained by Ciarletta, giving a proof based on more concrete assumptions, in particular considering a strictly positive heat capacity. The author use such a proof to show in an elegant way the impossibility of localization in time of the solutions of the forward in time problem for the linear thermoelasticity of Green and Naghdi (see [26, 27]), and for the linear thermoelasticity with voids (see [6, 28, 29]).

In the context of the linear theory of thermoelasticity, in [22] and [30] Ciarletta and Chirita investigate the past history of a thermoelastic process by using the final set of data. In particular, in [30] the authors introduce the Cesaro means of various parts of the total energy and then establish the relations that describe the asymptotic behavior of the mean energies, provided suitable constraint restriction is supposed on the backward in time process.

The backward in time problem for porous elastic materials has been studied by Quintanilla [25] and by Iovane and Passarella [31]; such backward in time problem has never been studied for the general case of thermo-microstretch elastic solids.

In this work we consider the boundary-final value problem associated with the linear theory of thermo-microstretch elastic materials. In Chapter 1 we give a brief introduction to the basic concepts of machanics of continua. In Chapter 2 we extend these concepts to the theory of microstretch materials. In Chapter 3, we formulate the backward in time problem for such material, where the final data are given at $t = 0$ and then we are interested in extrapolating the solution to all previous times, and we present some auxiliary Lagrange identities [32, 33] which will be used in the following chapters. In Chapter 4, using some of these Lagrange identities and energy arguments, we prove the uniqueness of the solutions for the backward in time problem assuming that the heat capacity a is strictly positive. This result implies the impossibility of the localization in time of the solutions of the forward in time problem, and this is proved in Theorem 12. In Chapter 5, the Cesaro means of various parts of the total energy are introduced. Then, extending the method developed by Chirita [14] for the classical case, the relations describing the asymptotic behavior of the mean energies follow, provided some mild restrictions are imposed on the backward in time process.

We have to outline that the present paper considers nonstandard problems concerning the general theory of microstretch thermoelastic materials. Such important nonstandard problems are intensively studied in literature (see, for example, the paper by Quintanilla and Straughan [34] and the papers cited there).

Chapter 1

A Short Introduction to Continuum Mechanics

In this first chapter a short survey on the fundamental concepts of the Continuum Mechanics will be given.

1.1 Kinematics

1.1.1 Description of motion

A continuum body is a set of particles in one-to-one relation with points of a three-dimensional domain. We assume at time t_0 the body occupies a domain B_0 of the three-dimensional space, called the *reference configuration*. The motion of the body is given by the time evolution of the position of each of his points. Let be X_i, $i = 1, 2, 3$, the coordinates with respect to an orthogonal frame of the point P_0 where a given particle is located at time t_0, and x_i the coordinates of the point P where the same particle is located at time t. The description of the motion of the body is given by

$$x_i = x_i\left(X_1, X_2, X_3, t\right)$$

We assume such set of functions is sufficiently smooth and has inverse for each fixed value of t, so that

$$J = \det\left(\frac{\partial x_i}{\partial X_j}\right) \neq 0;$$

for $t = t_0$ we have $J = 1$, so for continuity it should be positive for each $t \geq t_0$.

The domain B occupied by the body at time t is called the *current configuration.* The transformation from the reference configuration to the current configuration is called *deformation* of the body. The coordinates X_i are called *material* or *lagrangian* coordinates, the coordinates x_i are called *spatial* or *eulerian* coordinates. Each physical quantity can be equivalently expressed in terms of X_i or in terms of x_i; the first description is called material or lagrangian, the second is called spatial or eulerian, and is particularly useful in describing the motion of fluids.

The displacement vector field is defined as

$$u_i = x_i - X_i.$$

1.1.2 Deformation tensors

The tensor defined by

$$F_{ij} = \frac{\partial x_i}{\partial X_j}$$

is called *material gradient of deformation*; its inverse

$$F_{ij}^{-1} = \frac{\partial X_i}{\partial x_j}$$

is called *spatial gradient of deformation.* We can express the square length of a deformed infinitesimal vector as

$$(d\mathbf{x})^2 = dx_i dx_i = \frac{\partial x_i}{\partial X_r}\frac{\partial x_i}{\partial X_s} dX_r dX_s = F_{ri}^T F_{is} dX_r dX_s$$

and the tensor

$$C_{rs} = F_{ri}^T F_{is}$$

is called *Cauchy-Green tensor of deformation* (or *Cauchy right tensor*).

The difference of square length of a deformed and undeformed infinitesiam vector is given by

$$(d\mathbf{x})^2 - (d\mathbf{X})^2 = (C_{ij} - \delta_{ij})\, dX_i dX_j = 2E_{ij} dX_i dX_j$$

where we have defined the *Green tensor of deformation* (or *Green strain tensor*).

$$E_{ij} = \frac{1}{2}\left(C_{ij} - \delta_{ij}\right).$$

1.1.3 Velocity of deformation

Another important tensor useful in many questions is the *gradient of velocity*

$$L_{ij} = \frac{\partial v_i}{\partial x_j}.$$

Its symmetric part

$$D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

is called *velocity of deformation tensor*; the antisymmetric part

$$W_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right)$$

is called *spint tensor*

1.1.4 Linear theory of deformation

If we assume that the motion of the body depends upon some parameter ε, supposing the value of this parameter is sufficiently small, we can expand in Taylor series each physical quantity and take only first order terms with respect to ε. In this view, the Green tensor of deformation is reduced to the *infinitesimal strain tensor*

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right)$$

that is the symmetric part of the material gradient of u_i.

1.2 Laws of balance

1.2.1 Conservation of the mass

We assume that on the current configuration is defined a positive real function ρ, called *mass density*, such that

$$m(P) = \int_P \rho dv$$

where P is a part of the domain B, and $m(P)$ is its mass. The conservation of mass in the deformation of the body imply

$$\rho J = \rho_0$$

where ρ_0 is the density function in the reference configuration. It can be seen that, in the spatial description, the law of conservation of mass is written

$$\dot{\rho} + \rho \operatorname{div} \mathbf{v} = 0$$

where $\mathbf{v}$ is the eulerian velocity vector field and the divergence is with respect to eulerian coordinates.

1.2.2 Balance of the inpulse

The law of balance of inpulse state that for each part P we have

$$\frac{d}{dt} \int_P \rho \mathbf{v} dv = \int_P \rho \mathbf{f} dv + \int_{\partial P} \mathbf{t} da$$

where $\mathbf{f}$ is the *body force* per unit mass and $\mathbf{t}$ the force per unit surface acting on the boundary of P and due to the contact with the rest of B, also called *stress vector*. It can be shown that $\mathbf{t}$ is linear with respect to the surface normal unit vector $\mathbf{n}$ (*Cauchy theorem*)

$$t_i(\mathbf{n}) = t_{ji} n_j$$

and the tensor t_{ij} is the *Chauchy stress tensor*. The local formulation of the conservation of impulse is

$$\frac{\partial t_{ji}}{\partial x_j} + \rho f_i = \rho \ddot{u}_i$$

An analogous expression can be given in the lagrangian formulation, yielding

$$\frac{\partial T_{ji}}{\partial X_j} + \rho_0 f_i = \rho_0 \ddot{u}_i$$

where T_{ij} is the *first Piola-Kirchhoff stress tensor*.

1.2.3 Balance of the angular momentum

The law of balance of angular momentum state that for each part P we have

$$\frac{d}{dt}\int_P \rho \mathbf{x} \times \mathbf{v} dv = \int_P \rho \mathbf{x} \times \mathbf{f} dv + \int_{\partial P} \mathbf{x} \times \mathbf{t} da.$$

The local form of such law state the symmetry of the stress tensor

$$t_{ij} = t_{ji}.$$

The lagrangian formulation of this conservation law is

$$S_{ij} = S_{ji}$$

where S_{ij} is the *second Piola-Kirchhoff stress tensor*, defined in terms of the first through the relation

$$T_{ki} = \frac{\partial x_i}{\partial X_j} S_{kj}.$$

1.2.4 Conservation of the energy

The conservation of energy expresses the first law of thermodynamics, and can be written as

$$\frac{d}{dt} \int_P \rho \left(\frac{1}{2} v^2 + \varepsilon \right) dv = \int_P \rho \left(\mathbf{f} \cdot \mathbf{v} + s \right) dv + \int_{\partial P} \left(\mathbf{t} \cdot \mathbf{v} + h \right) da,$$

where ε is the density of *internal energy* per unit mass, s the density of *heat sources* per unit mass and per unit time and h the *heat flux* through ∂P per unit surface and per unit time. The heat flux can be expressed as

$$h \left(\mathbf{n} \right) = q_i n_i$$

where the vector q_i is called *heat flux vector*.

The local form of the conservation of the energy is

$$\rho \dot{\epsilon} = t_{ji} \frac{\partial v_i}{\partial x_j} + \rho s + \frac{\partial q_i}{\partial x_i}$$

or also

$$\rho\dot{\epsilon} = t_{ji}D_{ij} + \rho s + \frac{\partial q_i}{\partial x_i}.$$

The local form in lagrangian formulation is

$$\rho_0\dot{\epsilon} = T_{ji}\frac{\partial v_i}{\partial X_j} + \rho_0 s + \frac{\partial Q_i}{\partial X_i}$$

or also

$$\rho_0\dot{\epsilon} = S_{ji}\dot{E}_{ij} + \rho_0 s + \frac{\partial Q_i}{\partial X_i},$$

where Q_i is the heat flux vector in the lagrangian representation.

1.2.5 Clausius-Duhem inequality

The second law of thermodynamics state that

$$\frac{d}{dt}\int_P \rho\eta dv \geq \int_P \rho\frac{s}{T}dv + \int_{\partial P}\frac{h}{T}da$$

where η is the entropy per unit mass and T the absolute temperature. The local form of this law is called Clausius-Duhem inequality, and its expression in eulerian and lagrangian form is, respectively,

$$\rho T\dot{\eta} \geq \rho s + \frac{\partial q_i}{\partial x_i} - \frac{q_i}{T}\frac{\partial T}{\partial x_i},$$

$$\rho_0 T\dot{\eta} \geq \rho_0 s + \frac{\partial Q_i}{\partial X_i} - \frac{Q_i}{T}\frac{\partial T}{\partial X_i}.$$

1.3 Constitutive equations

1.3.1 Introduction

The equation introduced in the preceding sections should be satisfied during the motion of a continuum body. But such equations are not sufficient to determine the motion of the body, as can be seen making a simple check of the balance among number of unknown functions and equations.

In the development of general equation until here, we have not taken into account the characteristics of the material of the body. So it is not a surprise the indetermination we have to deal to, and it is clear that to have a well posed problem we must to consider some other equations that give us the mathematical representation of the behaviour of the material.

Such equation are called *constitutive equation* and define classes of material that represent, up to some limitations and under suitable conditions, the behaviour of real materials.

1.3.2 Elastic solids

An *elastic solid* is defined by the following constitutive equations

$$\begin{cases} \varepsilon = \varepsilon\left(\dfrac{\partial x_i}{\partial X_j}, X_k\right), \\ \\ S_{pq} = S_{pq}\left(\dfrac{\partial x_i}{\partial X_j}, X_k\right). \end{cases}$$

If such equations do not depend on X_k the material is said *homogeneous*.

Because of the *objectivity principle*, one of the most important postulates constitutive equations bust obey, the first of these equations became

$$\varepsilon = \varepsilon \left(E_{ij}, X_k \right),$$

while for a purely mechanic theory the second equation became

$$S_{ij} = \frac{1}{2}\rho_0 \left(\frac{\partial \varepsilon}{\partial E_{ij}} + \frac{\partial \varepsilon}{\partial E_{ji}} \right).$$

1.3.3 Linear theory of elasticity

In the linear theory of deformation the internal energy has the form

$$\varepsilon = \frac{1}{2} C_{ijrs} \varepsilon_{ij} \varepsilon_{rs}.$$

This can be interpreted as a Taylor series expansion of ε with respect to the component of the infinitesimal strain tensor ε_{ij}; the zero order term is null with a suitable choice of the additive constant of the energy; the first order term is null if we assume the stress tensor is null in the reference configuration.

From this we deduce the contitutive equations

$$t_{ij} = C_{ijrs} \varepsilon_{rs}$$

known as *generalized Hooke law.* The *elasticity tensor* C_{ijrs} does not depend on X_i for a homogeneous material, and has the following symmetry properties

$$C_{ijrs} = C_{rsij} = C_{jirs}.$$

1.3.4 Elastic moduli

In the linear theory of elasticity, the internal energy of an isotropic body is a function of the invariants of the tensor ε_{ij}. The expression of the elasticity tensor is

$$C_{ijrs} = \lambda \delta_{ij} \delta_{rs} + \mu \left(\delta_{ir} \delta_{js} + \delta_{is} \delta_{jr} \right)$$

where λ, μ are the Lam elastic moduli, and for physical reason is commonly assumed that they obey the following conditions

$$\mu > 0, \quad 3\lambda + 2\mu > 0.$$

The constitutive equation for the stress tensor became

$$t_{ij} = \lambda \varepsilon_{rr} \delta_{ij} + 2\mu \varepsilon_{ij},$$

and can be inverted to obtain the deformation in function of stresses

$$\varepsilon_{ij} = \frac{1+\nu}{E} t_{ij} - \frac{\nu}{E} t_{rr} \delta_{ij},$$

where

$$E = \frac{\mu \left(3\lambda + 2\mu \right)}{\lambda + \mu}$$

is the *Young modulo*, and

$$\nu = \frac{\lambda}{2 \left(\lambda + \mu \right)}$$

is the *Poisson ratio.*

If the continuum is homogeneous λ, μ are constants, and the equations of motion can be written

$$\mu \Delta u_i + (\lambda + \mu)\, u_{r,ri} + \rho_0 f_i = \rho_0 \ddot{u}_i,$$

called *Navier-Chauchy equations.*

Chapter 2

Microstretch matrials

In the atomic scale, crystalline solids possess primitive cells in the form of geometrical figures (lattice structures) like cubes, hexagonsm, etc. In crystalline solids there are many different arrangements with ions occupying many different positions in their primitive lattices. This is also true for more complex structures consisting of molecules. There exist also fluids characterized by oriented molecules. For example liquid crystals possess dipolar elements in the form of short bars and platelets. As an example, we can talk about animal blood which carry deformable platelets. In general all these media, such as blood, clouds with smoke, bubbly fluids, granular solids, concrete, composite materials, can be all considered examples of media characterized by microstrucures. In alla these examples we notice that the primitive elements of the media are stable elements. These stable elements are considered deformable but not destructible. For brevity we shall call these stable elements *particles.*

Definition 1. *A microcontinuum is a continuous collection of deformable* *point particles.*

Physically, the particles are point particles, i.e., they are infinitesimal in size. They do not violate continuity of matter and yet, they are deformable. Clearly, the deformability of the material point places microcontinuum theories beyond the scope of the classical continuum theory. We want represent the intrinsic deformation of point particles.

A particle P is identified by its position vector (or its coordinates X_K) $K = 1, 2, 3$, in the reference state B and vectors attached to P, representing the inner structure of P by $\mathbf{\Xi}$. Both $\mathbf{X}$ and $\mathbf{\Xi}$ have their own motion.

2.1 Motions and deformations

A physical body B is considered to be a collection of a set of material particles $\{P\}$. The body is embedded in a three dimensional Euclidean space E^3, at all times. A material point $P\ (\mathbf{X}, \mathbf{\Xi}) \in B$ is characterized by its centroid C and vector $\mathbf{\Xi}$ attached to C. The point C is identified by its rectangular coordinates X_1, X_2, X_3 in a coordinate frame $X_K, K = 1, 2, 3$ and the vector $\mathbf{\Xi}$ by its components Ξ_1, Ξ_2, Ξ_3 (in short Ξ_K) in the coordinate frame X_K. Deformation carries $P\,(\mathbf{X}, \mathbf{\Xi})$ to $p\,(\mathbf{x}, \boldsymbol{\xi})$ in a spatial frame of reference b so that $X_K \longrightarrow x_k$, $\Xi_K \longrightarrow \xi_k$ $(K = 1, 2, 3;\ k = 1, 2, 3)$. These mappings are expressed by

$$\mathbf{X} \longrightarrow \mathbf{x} = \widehat{\mathbf{x}}\,(\mathbf{X}, t) \qquad or \qquad x_k = \widehat{x}_k\,(X_K, t)\,, \tag{2.1}$$

$$\mathbf{\Xi} \longrightarrow \boldsymbol{\xi} = \widehat{\xi}\,(\mathbf{X}, \mathbf{\Xi}, t) \qquad or \qquad \xi_k = \widehat{\xi}_k\,(X_K, \Xi_K, t)\,. \tag{2.2}$$

The mapping 2.1.1 is called the *macromotion* (or simply the motion) and 2.1.2 the micromotion. Material particles are considered to be of very small size as compared to macroscopic scales of the body. Consequently a linear approximation in $\boldsymbol{\Xi}$ is permissible for the micromotion 2.1.2 replacing it by

$$\xi_k = \chi_{kK}\left(\mathbf{X}, t\right)\Xi_K, \tag{2.3}$$

where, hanceforth, the summation convention on repeated indices is understood.

Definition 2. *A material body is called a micromorphic continuum if its motions are described by 2.1.1 and 2.1.3 which posses continuous partial derivatives with respect to X_K and t, and they are invertible uniquely.*

The tensor χ_{kK} is called microdeformation. The matemathical idealization 2.1.3 is valid from the continuum viewpoint, only when the particles are considered to be infinitesimally small, so that the continuity of matter is not violated. In order to retain the right-hand screw orientations of the frames-of-reference we assume

$$J = \det\left(\frac{\partial x_k}{\partial X_K}\right) > 0, \qquad j = \det \chi_{kK} > 0. \tag{2.4}$$

A material point in the body is now considered to possess three deformable directors, which represent the degrees of freedom arising from microdeformation of the physical particle. Thus, a micromorphic continuum is none other than a classical continum endowed with extra degrees of freedom represented by the deformable directors $\boldsymbol{\chi}_K$.

2.2 Rotation

According to a theorem of Cauchy, a non singular matrix $\mathbf{F}$ may be decomposed as a product of two matrices, one of which is ortogonal and the other is a symmetric matrix

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR}, \tag{2.5}$$

with

$$\mathbf{U}^2 = \mathbf{F}^T\mathbf{F}, \qquad \mathbf{V}^2 = \mathbf{F}\mathbf{F}^T, \tag{2.6}$$

where a superscript T denotes transpose. If we take $F_{kK} = x_{k,K}$, then R represent a classical macrorotation tensor. U and V are called right and left stretch tensors for macro and microdeformations. In the case of microdeformations the above equations read:

$$\boldsymbol{\chi} = \mathbf{ru} = \mathbf{vr}, \tag{2.7}$$

and

$$\mathbf{u}^2 = \boldsymbol{\chi}^T\boldsymbol{\chi}, \qquad \mathbf{v}^2 = \boldsymbol{\chi}\boldsymbol{\chi}^T. \tag{2.8}$$

Definition 3. *(Microstretch continuum) A micromorphic continuum is called microstretch if satisfy*

$$\chi_{kK}\chi_{lK} = j^2\delta_{kl}, \qquad \chi_{kK}\chi_{kL} = j^2\delta_{KL}. \tag{2.9}$$

A microstretch continuum is a micromorphic continuum that is constrained to undergo microrotation and microstretch (expansion and contraction) without microshearing (breathing microrotations).

Definition 4. *(Micropolar continuum). A micromorphic continuum is called micropolar if its directors are orthonormal, i.e.*

$$\chi_{kK}\chi_{lK} = \delta_{kl}. \tag{2.10}$$

Amongst the substancies that can be modeled by Microstrech Continua Model we can classify: animal lungs, bubbly fluids, polluted air, slurries, springy suspension, mixtures with breathing elements, porous media, lattices with base, biological fluids, smal animal etc.

As a consequence, a micromorphic continuum, that is constrained to undergo a uniform microstretch (a breathing motion) represented with ν and rigid microrotation, represented with ν_k, is a microstretch continuum.

2.3 The balance laws of microstretch continua

The balance laws of microstretch continua may be obtained by imposing the Galilean invariance requirement to the energy equation, see [5].

So we obtain the conservation of mass:

$$\dot{\rho} + \rho\nabla \cdot \mathbf{v} = 0, \tag{2.11}$$

the conservation of microstretch inertia:

$$\frac{Dj_0}{Dt} - 2j_0\nu = 0, \tag{2.12}$$

$$\frac{Dj_{kl}}{Dt} - 2\nu j_{kl} + \left(\epsilon_{kpr} j_{lp} + \epsilon_{lpr} j_{kp}\right) \nu_r = 0, \tag{2.13}$$

the balance of momentum:

$$t_{kl,k} + \rho \left(f_l - \dot{v}_l \right) = 0, \tag{2.14}$$

and finally the balance of momentum moments:

$$m_{kl,k} + \epsilon_{lmn} t_{mn} + \rho \left(l_l - \sigma_l \right) = 0, \tag{2.15}$$

$$m_{k,k} + t - s + \rho (l - \sigma) = 0. \tag{2.16}$$

Chapter 3

Backward in time Problem for Microstretch Materials

3.1 Formulation of the problem

Throughout this paper we shall denote by B a bounded regular region of the physical space E^3, whose boundary is the piecewise smooth surface ∂B. Identifying E^3 with the associated vector space, we introduce an orthonormal system of reference so that vectors and tensors will have components denoted by the usual Latin subscripts ranging over 1, 2, 3. Summation over repeated subscripts and other typical conventions for differential operations are implied, such as a superposed dot or a comma followed by a subscript to denote partial derivative with respect to time or the corresponding Cartesian coordinate. All involved functions are supposed sufficiently regular as necessary.

We suppose that B is filled by an anisotropic and inhomogeneous thermoelastic material with stretch. We consider the problem associated with the linear theory of thermo-microstretch elastic solids, as established by Eringen [4], on the time interval I. Thus, in the absence of supply terms, the fundamental system of field equations consists of the evolution equations, in $B \times I$

$$\begin{aligned} &\rho \ddot{u}_i = t_{ji,j}\,, \qquad && I_{ij}\ddot{\phi}_j = m_{ji,j} + \varepsilon_{irs}t_{rs}\,, \\ &J\ddot{\phi} = \lambda_{i,i} - \omega\,, \qquad && \rho T_0 \dot{\eta} = q_{i,i}\,, \end{aligned} \tag{3.1}$$

the constitutive equations, in $\bar{B} \times I$

$$\begin{aligned} t_{ij} &= A_{ijrs}e_{rs} + B_{ijrs}\kappa_{rs} + D_{ijr}\gamma_r + A_{ij}\phi - \beta_{ij}\theta\,, \\ m_{ij} &= B_{rsij}e_{rs} + C_{ijrs}\kappa_{rs} + E_{ijr}\gamma_r + B_{ij}\phi - C_{ij}\theta\,, \\ 3\lambda_i &= D_{rsi}e_{rs} + E_{rsi}\kappa_{rs} + D_{ij}\gamma_j + d_i\phi - \xi_i\theta\,, \\ 3\omega &= A_{rs}e_{rs} + B_{rs}\kappa_{rs} + d_i\gamma_i + m\phi - \zeta\theta\,, \\ \rho\eta &= \beta_{rs}e_{rs} + C_{rs}\kappa_{rs} + \xi_i\gamma_i + \zeta\phi + a\theta\,, \\ q_i &= k_{ij}\theta_{,j}\,, \end{aligned} \tag{3.2}$$

and the geometrical relations, on $\bar{B} \times I$

$$e_{ij} = u_{j,i} + \varepsilon_{jik}\phi_k\,, \qquad \kappa_{ij} = \phi_{j,i}\,, \qquad \gamma_i = \phi_{,i}\,. \tag{3.3}$$

In the above equations we have used the following notations: t_{ij} is the stress tensor, m_{ij} is the couple stress tensor, λ_i is the microstress vector, ω is the microstress function, u_i is the displacement, ϕ_i is the microrotation, ϕ is the microstretch function, η is the specific entropy, ρ is the mass density, T_0 is the absolute temperature in the reference configuration, q_i is the heat flux vector, θ is the temperature variation from the temperature T_0, I_{ij} is the symmetric microinertia tensor, J is equal to $I_{ii}/2$ and ε_{ijk} is the alternating symbol.

The constitutive coefficients and I_{ij} are prescribed functions of the spatial variables with the following symmetries

$$A_{ijrs} = A_{rsij}\,, \qquad C_{ijrs} = C_{rsij}\,, \qquad D_{ij} = D_{ji}\,, \qquad k_{ij} = k_{ji}\,. \tag{3.4}$$

The internal energy density W associated with the kinematic fields u_i, ϕ_i, ϕ is defined by

$$\begin{aligned} W = \frac{1}{2}\left(A_{ijrs}e_{ij}e_{rs} + C_{ijrs}\kappa_{ij}\kappa_{rs} + D_{ij}\gamma_i\gamma_j + m\phi^2\right) + \\ + B_{ijrs}e_{ij}\kappa_{rs} + D_{ijk}e_{ij}\gamma_k + A_{ij}e_{ij}\phi + E_{ijr}\kappa_{ij}\gamma_r + B_{ij}\kappa_{ij}\phi + d_i\gamma_i\phi\,. \end{aligned} \tag{3.5}$$

We denote by $I_m(\mathbf{x})$ the minimum eigenvalue of $I_{ij}(\mathbf{x})$.

We assume that ρ and I_{ij} are continuous functions and the constitutive coefficients are continuously differentiable functions on $\bar{B}$. Furthermore, we suppose that

[(i)]

1. $\rho(\mathbf{x}) \geqslant \rho_0$, $I_m(\mathbf{x}) \geqslant I_0$, $J(\mathbf{x}) \geqslant J_0$, $a(\mathbf{x}) \geqslant a_0$, where ρ_0, I_0, J_0, a_0 are positive constants;

2. k_{ij} is a positive definite tensor;

3. W is a positive definite quadratic form.

It is obvious to see that, as a consequence of (1), I_{ij} is a positive definite tensor.

The hypotheses (2) and (3) imply that there exist positive constants k_m, k_M, μ_m and μ_M such that

$$k_m\theta_{,i}\theta_{,i} \leqslant k_{ij}\theta_{,i}\theta_{,j} \leqslant k_M\theta_{,i}\theta_{,i}\,, \tag{3.6}$$

$$\mu_m\left(e_{ij}e_{ij}+\phi^2+\frac{I_0}{\rho_0}\kappa_{ij}\kappa_{ij}+\frac{3J_0}{\rho_0}\gamma_i\gamma_i\right) \leqslant 2W \leqslant \mu_M\left(e_{ij}e_{ij}+\phi^2+\frac{I_0}{\rho_0}\kappa_{ij}\kappa_{ij}+\frac{3J_0}{\rho_0}\gamma_i\gamma_i\right). \tag{3.7}$$

Now we consider $I=(-\infty,0]$ and so we study the boundary-final value problem $\mathcal{P}$ defined by the relations (3.1) to (3.3), the homogeneous boundary conditions

$$\begin{aligned}
u_i &= 0 \quad \text{on} \quad \Sigma_1^{(1)}\times(-\infty,0]\,, & t_i &= 0 \quad \text{on} \quad \Sigma_2^{(1)}\times(-\infty,0]\,,\\
\phi_i &= 0 \quad \text{on} \quad \Sigma_1^{(2)}\times(-\infty,0]\,, & m_i &= 0 \quad \text{on} \quad \Sigma_2^{(2)}\times(-\infty,0]\,,\\
\phi &= 0 \quad \text{on} \quad \Sigma_1^{(3)}\times(-\infty,0]\,, & h &= 0 \quad \text{on} \quad \Sigma_2^{(3)}\times(-\infty,0]\,,\\
\theta &= 0 \quad \text{on} \quad \Sigma_1^{(4)}\times(-\infty,0]\,, & q &= 0 \quad \text{on} \quad \Sigma_2^{(4)}\times(-\infty,0]\,,
\end{aligned} \tag{3.8}$$

and the final conditions in $\bar{B}$

$$\begin{aligned}
&u_i(\mathbf{x},0)=u_i^0(\mathbf{x})\,, \qquad \phi_i(\mathbf{x},0)=\phi_i^0(\mathbf{x})\,, \qquad \phi(\mathbf{x},0)=\phi^0(\mathbf{x})\,, \qquad \theta(\mathbf{x},0)=\theta^0(\mathbf{x})\,,\\
&\dot{u}_i(\mathbf{x},0)=\dot{u}_i^0(\mathbf{x})\,, \qquad \dot{\phi}_i(\mathbf{x},0)=\dot{\phi}_i^0(\mathbf{x})\,, \qquad \dot{\phi}(\mathbf{x},0)=\dot{\phi}^0(\mathbf{x})\,,
\end{aligned} \tag{3.9}$$

where

$$t_i=t_{ji}n_j\,, \qquad m_i=m_{ji}n_j\,, \qquad h=3\lambda_jn_j\,, \qquad q=q_jn_j\,.$$

In these relations, n_j is the outward unit normal vector to the boundary surface and, for each $i = 1, \dots, 4$, we have that $\Sigma_1^{(i)}$, $\Sigma_2^{(i)}$ are subsurfaces of ∂B such that

$$\Sigma_1^{(i)} \cap \Sigma_2^{(i)} = \varnothing\,, \qquad \bar{\Sigma}_1^{(i)} \cup \bar{\Sigma}_2^{(i)} = \partial B\,,$$

where the closure is relative to ∂B, and where u_i^0, $\dot{u}_i^0$, ϕ_i^0, $\dot{\phi}_i^0$, ϕ^0, $\dot{\phi}^0$, θ^0 are prescribed continuous functions compatible with (3.8) on the appropriate subsurfaces of ∂B.

For further convenience, we use an appropriate change of variables and notations suitably chosen in order to transform the boundary-final value problem $\mathcal{P}$ into the boundary-initial value problem $\mathcal{P}^*$. In particular, for every function depending on time $f(t)$ we set $f^*(t^*) = f(t)$, with $t^* = -t$. Removing the star signs from notations for sake of simplicity, we have the boundary-initial value problem $\mathcal{P}^*$ defined by the following equations

$$\begin{aligned} &\rho\ddot{u}_i = t_{ji,j}\,, \qquad && I_{ij}\ddot{\phi}_j = m_{ji,j} + \varepsilon_{irs}t_{rs}\,, \\ &J\ddot{\phi} = \lambda_{i,i} - \omega\,, && \rho T_0\dot{\eta} = -q_{i,i}\,, \end{aligned} \tag{3.10}$$

in $B \times [0, +\infty)$, equations (3.2) in $\bar{B} \times [0, +\infty)$, and equations (3.3) on $\bar{B} \times [0, +\infty)$, with the boundary conditions

$$\begin{aligned} &u_i = 0 \quad \text{on} \quad \Sigma_1^{(1)} \times [0, +\infty)\,, \qquad && t_i = 0 \quad \text{on} \quad \Sigma_2^{(1)} \times [0, +\infty)\,, \\ &\phi_i = 0 \quad \text{on} \quad \Sigma_1^{(2)} \times [0, +\infty)\,, && m_i = 0 \quad \text{on} \quad \Sigma_2^{(2)} \times [0, +\infty)\,, \\ &\phi = 0 \quad \text{on} \quad \Sigma_1^{(3)} \times [0, +\infty)\,, && h = 0 \quad \text{on} \quad \Sigma_2^{(3)} \times [0, +\infty)\,, \\ &\theta = 0 \quad \text{on} \quad \Sigma_1^{(4)} \times [0, +\infty)\,, && q = 0 \quad \text{on} \quad \Sigma_2^{(4)} \times [0, +\infty)\,, \end{aligned} \tag{3.11}$$

and the initial conditions (3.9).

An array field $\mathcal{U} = (\mathbf{u}, \boldsymbol{\phi}, \phi, \theta)$ meeting equations (3.10), (3.2), (3.3), (3.11) and (3.9) will be referred to as a solution of the boundary-initial value problem $\mathcal{P}^*$.

3.2 Auxiliary integral identities

In this section we establish some integral identities of Lagrange type [32,33], that we will use in the next sections. To this end, we introduce the following energetic terms

$$\mathcal{K}(t) = \frac{1}{2}\left[\rho\dot{u}_i(t)\dot{u}_i(t) + I_{ij}\dot{\phi}_i(t)\dot{\phi}_j(t) + 3J\dot{\phi}^2(t)\right], \tag{3.12}$$

$$\mathcal{E}(t) = \int_B \left[\mathcal{K}(t) + W(t) + \frac{1}{2}a\theta^2(t)\right] \mathrm{d}v\,, \tag{3.13}$$

$$\mathcal{E}^*(t) = \int_B \left[\mathcal{K}(t) + W(t) - \frac{1}{2}a\theta^2(t)\right] \mathrm{d}v\,. \tag{3.14}$$

We can prove the following lemmas

Lemma 5. *Let $\mathcal{U}$ be a solution of the boundary-initial value problem $\mathcal{P}^*$. Then, for all $t \geqslant 0$, we have*

$$\mathcal{E}(t) = \mathcal{E}(0) + \int_0^t\!\!\int_B \frac{1}{T_0}k_{ij}\theta_{,i}(s)\theta_{,j}(s)\,\mathrm{d}v\,\mathrm{d}s\,. \tag{3.15}$$

Proof. Starting from the expression of $\mathcal{K}$

$$\mathcal{K}(t) = \frac{1}{2}\left[\rho\dot{u}_i(t)\dot{u}_i(t) + I_{ij}\dot{\phi}_i(t)\dot{\phi}_j(t) + 3J\dot{\phi}^2(t)\right], \tag{3.16}$$

if we consider the time rate of the kinetic energy per unit volume $\mathcal{K}$

$$\frac{\partial \mathcal{K}}{\partial t} = \rho \dot{u}_i(t)\ddot{u}_i(t) + I_{ij}\dot{\phi}_i(t)\ddot{\phi}_j(t) + 3J\dot{\phi}\ddot{\phi}(t) \tag{3.17}$$

and take into account the equations of motion (3.10) and the geometric relations (3.3), we have

$$\frac{\partial \mathcal{K}}{\partial t} = \dot{u}_i(t_{ji,j}) + \dot{\phi}_i(m_{ji,j} + \varepsilon_{irs}t_{rs}) + 3\dot{\phi}(\lambda_{i,i} - \omega).$$

$$\frac{\partial \mathcal{K}}{\partial t} = (\dot{u}_i t_{ji})_{,j} - \dot{u}_{i,j}t_{ji} + (\dot{\phi}_i m_{ji})_{,j} - \dot{\phi}_{i,j}m_{ji} + \dot{\phi}_i\varepsilon_{irs}t_{rs} + 3(\dot{\phi}\lambda_j)_j - 3\dot{\phi}_{,i}\lambda_i - 3\dot{\phi}\omega.$$

$$\frac{\partial \mathcal{K}}{\partial t} = [\dot{u}_i t_{ji} + \dot{\phi}_i m_{ji} + 3\dot{\phi}\lambda_j]_{,j} - \dot{u}_{i,j}t_{ji} - \dot{\phi}_{i,j}m_{ji} + \dot{\phi}_i\varepsilon_{irs}t_{rs} - 3\dot{\phi}_{,i}\lambda_i - 3\dot{\phi}\omega.$$

$$\frac{\partial \mathcal{K}}{\partial t} = [\dot{u}_i t_{ji} + \dot{\phi}_i m_{ji} + 3\dot{\phi}\lambda_j]_{,j} - t_{ij}(\dot{u}_{j,i} - \dot{\phi}_k\varepsilon_{kij}) - \dot{\phi}_{j,i}m_{ji} - 3\dot{\phi}_i\lambda_i - 3\dot{\phi}\omega.$$

$$\frac{\partial \mathcal{K}}{\partial t} = [\dot{u}_i t_{ji} + \dot{\phi}_i m_{ji} + 3\dot{\phi}\lambda_j]_{,j} - t_{ij}\dot{e}_{ij} - m_{ji}\dot{k}_{ij} - 3\lambda_i\dot{\gamma}_i - 3\dot{\phi}\omega.$$

if we observe that

$$\begin{aligned} t_{ij}\dot{e}_{ij} + m_{ji}\dot{k}_{ij} + 3\lambda_i\dot{\gamma}_i + 3\dot{\phi}\omega = {} & (A_{ijrs}e_{rs} + B_{ijrs}\kappa_{rs} + D_{ijr}\gamma_r + A_{ij}\phi - \beta_{ij}\theta)\dot{e}_{ij} + \\ & + (B_{rsij}e_{rs} + C_{ijrs}\kappa_{rs} + E_{ijr}\gamma_r + B_{ij}\phi - C_{ij}\theta)\dot{k}_{ij} + \\ & + (D_{rsi}e_{rs} + E_{rsi}\kappa_{rs} + D_{ij}\gamma_j + d_i\phi - \xi_i\theta\,,)\dot{\gamma}_i + \\ & + (A_{rs}e_{rs} + B_{rs}\kappa_{rs} + d_r\gamma_r + m\phi - \zeta\theta)\dot{\phi}, \end{aligned}$$

and

$$\frac{\partial \mathcal{W}}{\partial t} = A_{ijrs}\dot{e}_{ij}e_{rs} + C_{ijrs}\dot{\kappa}_{ij}\kappa_{rs} + D_{ij}\dot{\gamma}_i\gamma_r + m\phi\dot{\phi} +$$

$$+ B_{ijrs}\dot{e}_{ij}\kappa_{rs} + B_{ijrs}e_{ij}\dot{\kappa}_{rs} + D_{ijr}\dot{e}_{ij}\gamma r + D_{ijk}e_{ij}\dot{\gamma}_r +$$

$$+ A_{ij}\dot{e}_{ij}\phi + A_{ij}e_{ij}\dot{\phi} + E_{ijr}\dot{\kappa}_{ij}\gamma_r + E_{ijr}\kappa_{ij}\dot{\gamma}_r +$$

$$+ B_{ij}\dot{\kappa}_{ij}\phi + B_{ij}\kappa_{ij}\dot{\phi} + d_r\dot{\gamma}_r\phi + d_r\gamma_r\dot{\phi},$$

then we have

$$t_{ij}\dot{e}_{ij} + m_{ji}\dot{k}_{ij} + 3\lambda_i\dot{\gamma}_i + 3\dot{\phi}\omega = \frac{\partial\mathcal{W}}{\partial t} - (\beta_{ij}\dot{e}_{ij} + C_{ij}\dot{k}_{ij} + \xi_i\dot{\gamma}_i + \varsigma\dot{\phi})\theta,$$

and also

$$\rho\dot{\eta} \quad = \beta_{rs}\dot{e}_{rs} + C_{rs}\dot{k}_{rs} + \xi_i\dot{\gamma}_i + \varsigma\dot{\phi} + a\dot{\theta},$$

$$\rho\dot{\eta} - a\dot{\theta} = -\frac{q_{i,i}}{T_0} - a\dot{\theta}.$$

So

$$t_{ij}\dot{e}_{ij} + m_{ji}\dot{k}_{ij} + 3\lambda_i\dot{\gamma}_i + 3\dot{\phi}\omega = \frac{\partial\mathcal{W}}{\partial t} - \left(\frac{q_{i,i}}{T_0} - a\dot{\theta}\right)\theta,$$

$$t_{ij}\dot{e}_{ij} + m_{ji}\dot{k}_{ij} + 3\lambda_i\dot{\gamma}_i + 3\dot{\phi}\omega = \frac{\partial\mathcal{W}}{\partial t} + a\dot{\theta}\theta + \frac{1}{T_0}q_{i,i}\theta,$$

$$t_{ij}\dot{e}_{ij} + m_{ji}\dot{k}_{ij} + 3\lambda_i\dot{\gamma}_i + 3\dot{\phi}\omega = \frac{\partial\mathcal{W}}{\partial t} + \frac{\partial}{\partial t}\frac{1}{2}a\theta^2 + \frac{1}{T_0}[(q_i\theta)_{,i} - q_i\theta_{,i}],$$

$$t_{ij}\dot{e}_{ij} + m_{ji}\dot{k}_{ij} + 3\lambda_i\dot{\gamma}_i + 3\dot{\phi}\omega = \frac{\partial}{\partial t}\left(W + \frac{1}{2}a\theta^2\right) + (\frac{1}{T_0}q_j\theta)_{,j} - \frac{1}{T_0}q_i\theta_{,i}$$

$$t_{ij}\dot{e}_{ij} + m_{ji}\dot{k}_{ij} + 3\lambda_i\dot{\gamma}_i + 3\dot{\phi}\omega = \frac{\partial}{\partial t}\left(W + \frac{1}{2}a\theta^2\right) + \left(\frac{1}{T_0}q_j\theta\right)_{,j} - \frac{1}{T_0}k_{ij}\theta_{,i}\theta_{,j}.$$

Then, finally we have

$$\frac{\partial\mathcal{K}}{\partial t} = [\dot{u}_i t_{ji} + \dot{\phi}_i m_{ji} + 3\dot{\phi}\lambda_j]_{,j} - \frac{\partial}{\partial t}\left(W + \frac{1}{2}a\theta^2\right) - \left(\frac{1}{T_0}q_j\theta\right)_{,j} + \frac{1}{T_0}k_{ij}\theta_{,i}\theta_{,j},$$

$$\frac{\partial}{\partial t}\left(K + W + \frac{1}{2}a\theta^2\right) = \left(\dot{u}_i t_{ji} + \dot{\phi}_i m_{ji} + 3\dot{\phi}\lambda_j - \frac{1}{T_0}q_j\theta\right)_{,j} + \frac{1}{T_0}k_{ij}\theta_{,i}\theta_{,j}.$$

Since

$$\mathcal{E}(t) = \int_B \left[\mathcal{K}(t) + W(t) + \frac{1}{2}a\theta^2(t)\right] \mathrm{d}v\,,$$

$$\frac{\partial}{\partial t}\mathcal{E}(t) = \int_B \left[\dot{u}_i t_{ji} + \dot{\phi}_i m_{ji} + 3\dot{\phi}\lambda_j - \frac{1}{T_0}q_j\theta\right]_{,j} \mathrm{d}v + \frac{1}{T_0}\int_B k_{ij}\theta_{,i}\theta_{,j}\,\mathrm{d}v$$

Then, by an integration of the result over $\bar{B} \times [0, t]$ and by using the divergence theorem and the boundary conditions (3.11), we deduce:

$$\frac{\partial}{\partial t}\mathcal{E}(t) = \int\limits_{\partial B} \left[\dot{u}_i t_{ji} + \dot{\phi}_i m_{ji} + 3\dot{\phi}\lambda_j - \frac{1}{T_0}q_j\theta\right] n_j\,\mathrm{d}\,a + \frac{1}{T_0}\int_B k_{ij}\theta_{,i}\theta_{,j}\,\mathrm{d}v$$

$$[\mathcal{E}(t)]_0^t = \frac{1}{T_0}\int\limits_0^t \int_B k_{ij}\theta_{,i}\theta_{,j}\,\mathrm{d}v\,\mathrm{d}s$$

and the proof is complete. □

By the same procedure we can obtain the identity expressed in the following result.

Corollary 6. *Let $\mathcal{U}$ be a solution of the boundary-initial value problem $\mathcal{P}^*$. Then, for all $t \geqslant 0$, we have*

$$\mathcal{E}^*(t) = \mathcal{E}^*(0) - \int_0^t\!\!\int_B \Big\{2\Big[\dot{u}_i(s)\,[\beta_{ji}\theta(s)]_{,j} + \dot{\phi}_k(s)\,[C_{jk}\theta(s)]_{,j} + \varepsilon_{jik}\beta_{ji}\dot{\phi}_k(s)\theta(s) + \\ + \dot{\phi}(s)\,[\xi_j\theta(s)]_{,j} - \zeta\dot{\phi}(s)\theta(s)\Big] + \frac{1}{T_0}k_{ij}\theta_{,i}(s)\theta_{,j}(s)\Big\}\,\mathrm{d}v\,\mathrm{d}s\,. \tag{3.18}$$

Proof. As we have already seen

$$\frac{\partial \mathcal{K}}{\partial t} = [\dot{u}_i t_{ji} + \dot{\phi}_i m_{ji} + 3\dot{\phi}\lambda_j]_{,j} - \left[\frac{\partial W}{\partial t}\left(\frac{1}{T_0} q_j \theta\right)_j + a\dot{\theta}\theta - \frac{1}{T_0} k_{ij}\theta_{,i}\theta_{,j}\right]$$

$$\frac{\partial \mathcal{K}}{\partial t} = [\dot{u}_i t_{ji} + \dot{\phi}_i m_{ji} + 3\dot{\phi}\lambda_j - \frac{1}{T_0} q_j\theta]_{,j} - \left[\frac{\partial W}{\partial t} + 2a\dot{\theta}\theta - a\dot{\theta}\theta\right]_j + \frac{1}{T_0} k_{ij}\theta_{,i}\theta_{,j}$$

$$\frac{\partial}{\partial t}\left(K + W - \frac{1}{2}a\theta^2\right) = [\dot{u}_i t_{ji} + \dot{\phi}_i m_{ji} + 3\dot{\phi}\lambda_j - \frac{1}{T_0} q_j\theta]_{,j} - 2a\dot{\theta}\theta + \frac{1}{T_0} k_{ij}\theta_{,i}\theta_{,j}$$

where

$$-2a\dot{\theta}\theta + \frac{1}{T_0} k_{ij}\theta_{,i}\theta_{,j} = -2\theta\left[\rho\dot{\eta} - \beta_{rs}\dot{e}_{rs} - C_{rs}\dot{k}_{rs} - \xi_i\dot{\gamma}_i - \zeta\dot{\phi}\right] + \frac{1}{T_0} k_{ij}\theta_{,i}\theta_{,j}$$

$$-2a\dot{\theta}\theta + \frac{1}{T_0} k_{ij}\theta_{,i}\theta_{,j} = -2\theta\rho\dot{\eta} + 2\theta\left[\beta_{rs}\left(\dot{u}_{s,r} + \varepsilon_{srk}\dot{\phi}_k\right) + C_{rs}\dot{\phi}_{s,r} + \xi_i\dot{\phi}_{,i} + \zeta\dot{\phi}\theta\right] + \frac{1}{T_0} k_{ij}\theta_{,i}\theta_{,j}$$

$$-2a\dot{\theta}\theta + \frac{1}{T_0} k_{ij}\theta_{,i}\theta_{,j} = 2\theta\frac{1}{T_0} q_{i,i} + 2\left[\beta_{rs}\theta\dot{u}_{s,r} + \varepsilon_{srk}\beta_{rs}\theta\dot{\phi}_k + C_{rs}\dot{\phi}_{s,r}\theta + \xi_i\dot{\phi}_{,i}\theta + \zeta\dot{\phi}\theta\right] + \frac{1}{T_0} k_{ij}\theta_{,i}\theta_{,j}$$

$$\begin{aligned} -2a\dot{\theta}\theta + \frac{1}{T_0} k_{ij}\theta_{,i}\theta_{,j} = {} & \frac{2}{T_0}(q_i\theta)_{,i} - \frac{2}{T_0} q_i\theta_{,i} + \\ & + 2\left[(\beta_{rs}\theta\dot{u}_s)_{,r} - (\beta_{rs}\theta)_{,r}\dot{u}_s + \left(C_{rs}\dot{\phi}_s\theta\right)_{,r} - (C_{rs}\theta)_{,r}\dot{\phi}_s + \left(\xi_i\dot{\phi}_s\theta\right)_{,r} - (\xi_i\theta)_{,r}\dot{\phi}_s \right. \\ & + \frac{1}{T_0} k_{ij}\theta_{,i}\theta_{,j} - 2a\dot{\theta}\theta + \frac{1}{T_0} k_{ij}\theta_{,i}\theta_{,j} = \frac{2}{T_0}(q_i\theta)_{,i} - \frac{2}{T_0} q_i\theta_{,i} + \end{aligned} \tag{3.19}$$

$$+ 2\left[(\beta_{rs}\theta\dot{u}_s)_{,r} - (\beta_{rs}\theta)_{,r}\dot{u}_s + \left(C_{rs}\dot{\phi}_s\theta\right)_{,r} - (C_{rs}\theta)_{,r}\dot{\phi}_s + \left(\xi_i\dot{\phi}_s\theta\right)_{,r} - (\xi_i\theta)_{,r}\dot{\phi}_s + \varepsilon_{srk}\beta_{rs}\dot{\phi}_k\theta + \zeta\dot{\phi}\theta\right.$$

$$+ \frac{1}{T_0} k_{ij}\theta_{,i}\theta_{,j}$$

$$\frac{\partial}{\partial t}\mathcal{E}^*(t) = \frac{\partial}{\partial t}\left(K + W - \frac{1}{2}a\theta^2\right) = [\dot{u}_i t_{ji} + \dot{\phi}_i m_{ji} + 3\dot{\phi}\lambda_j - \frac{1}{T_0}q_j\theta]_{,j} + 2\left[(\beta_{rs}\theta\dot{u}_s)_{,r} + \left(C_{rs}\dot{\phi}_s\theta\right)_{,r} \right.$$

$$\left. - 2\left[-\frac{1}{T_0}(q_i\theta)_{,i} + (\beta_{rs}\theta)_{,r}\dot{u}_s + (C_{rs}\theta)_{,r}\dot{\phi}_s + (\xi_i\theta)_{,r}\dot{\phi}_s - \varepsilon_{srk}\beta_{rs}\dot{\phi}_k\theta - \zeta\dot{\phi}\theta\right] - \frac{2}{T_0}q_i\theta_{,i} + \frac{1}{T_0}\right.$$

$$[\mathcal{E}^*(t)]_0^t = -\int_0^t\int_B 2\left\{(\beta_{rs}\theta\dot{u}_s)_{,r} + \left(C_{rs}\dot{\phi}_s\theta\right)_{,r} + \left(\xi_i\dot{\phi}_s\theta\right)_{,r} + \varepsilon_{srk}\beta_{rs}\dot{\phi}_k\theta - \zeta\dot{\phi}\theta + \frac{1}{T_0}k_{ij}\theta_{,i}\theta_{,j}\right\} dvds$$

and the proof is complete.

Another useful result is expressed in the following Lemma □

Lemma 7. *Let $\mathcal{U}$ be a solution of the problem determined by* (3.10), (3.2), (3.3), *boundary conditions* (3.11) *and null initial conditions. Then, for all $t \geqslant 0$, we have*

$$\int_B \left(\mathcal{K}(t) - \frac{1}{2}a\theta^2(t)\right)\mathrm{d}v = \int_B W(t)\,\mathrm{d}v\,. \tag{3.20}$$

Proof. Using the Lagrange identity method and the identity

$$\frac{\partial}{\partial s}\left\{\rho\dot{u}_i(s)\dot{u}_i(2t-s) + I_{ij}\dot{\phi}_i(s)\dot{\phi}_j(2t-s) + 3J\dot{\phi}(s)\dot{\phi}(2t-s) - a\theta(s)\theta(2t-s)\right\} =$$

$$= \rho\ddot{u}_i(s)\dot{u}_i(2t-s) + I_{ij}\ddot{\phi}_i(s)\dot{\phi}_j(2t-s) + 3J\ddot{\phi}(s)\dot{\phi}(2t-s) + a\theta(s)\dot{\theta}(2t-s) -$$

$$- \rho\dot{u}_i(s)\ddot{u}_i(2t-s) - I_{ij}\dot{\phi}_i(s)\ddot{\phi}_j(2t-s) - 3J\dot{\phi}(s)\ddot{\phi}(2t-s) - a\dot{\theta}(s)\theta(2t-s)$$

for a fixed $t \in (0,T)$, the equations (3.10), (3.2), (3.3) and (3.4) imply

$$\frac{\partial}{\partial s}\left\{\rho\dot{u}_i(s)\dot{u}_i(2t-s) + I_{ij}\dot{\phi}_i(s)\dot{\phi}_j(2t-s) + 3J\dot{\phi}(s)\dot{\phi}(2t-s) - a\theta(s)\theta(2t-s)\right\} =$$
$$= 2\frac{\partial}{\partial s}\mathcal{L}(s,2t-s) + \Big[t_{ji}(s)\dot{u}_i(2t-s) - t_{ji}(2t-s)\dot{u}_i(s) + m_{ji}(s)\dot{\phi}_i(2t-s) -$$
$$- m_{ji}(2t-s)\dot{\phi}_i(s) + 3\lambda_j(s)\dot{\phi}(2t-s) - 3\lambda_j(2t-s)\dot{\phi}(s) - \frac{1}{T_0}\theta(s)q_j(2t-s) +$$
$$+ \frac{1}{T_0}\theta(2t-s)q_j(s)\Big]_{,j} \tag{3.21}$$

where

$$2\mathcal{L}(r,s) = A_{ijrs}e_{ij}(s)e_{rs}(r) + C_{ijrs}\kappa_{ij}(s)\kappa_{rs}(r) + D_{ij}\gamma_i(s)\gamma_j(r) + m\phi(s)\phi(r) +$$
$$+ B_{ijrs}\left[e_{ij}(s)\kappa_{rs}(r) + e_{ij}(r)\kappa_{rs}(s)\right] + D_{ijk}\left[e_{ij}(s)\gamma_k(r) + e_{ij}(r)\gamma_k(s)\right] +$$
$$+ A_{ij}\left[e_{ij}(s)\phi(r) + e_{ij}(r)\phi(s)\right] + E_{ijr}\left[\kappa_{ij}(s)\gamma_r(r) + \kappa_{ij}(r)\gamma_r(s)\right] +$$
$$+ B_{ij}\left[\kappa_{ij}(s)\phi(r) + \kappa_{ij}(r)\phi(s)\right] + d_i\left[\gamma_i(s)\phi(r) + \gamma_i(r)\phi(s)\right]\,.$$

It is trivial to observe that

$$\mathcal{L}(s,r) = \mathcal{L}(r,s) \qquad \text{and} \qquad \mathcal{L}(t,t) = W(t)\,.$$

Since the initial and boundary conditions are null, integrating (3.21) over $\bar{B} \times [0,t]$ and using definitions (3.5) and (3.12), we arrive to (3.20). $\square$

Lemma 8. *Let $\mathcal{U}$ be a solution of the boundary-initial value problem $\mathcal{P}^*$.*

Then, for all $t \geqslant 0$, we have

$$\int_B \left[\left(\rho u_i(t)\dot{u}_i(t) + I_{ij}\phi_i(t)\dot{\phi}_j(t) + 3J\phi(t)\dot{\phi}(t) \right) - \frac{1}{2T_0} k_{ij}\tau_{,i}(t)\tau_{,j}(t) \right] \mathrm{d}v =$$
$$= \int_B \left(\rho u_i(0)\dot{u}_i(0) + I_{ij}\phi_i(0)\dot{\phi}_j(0) + 3J\phi(0)\dot{\phi}(0) \right) \mathrm{d}v + \int_0^t\!\!\int_B \rho\eta(0)\theta(s)\,\mathrm{d}v\,\mathrm{d}s +$$
$$+ 2\int_0^t\!\!\int_B \left[\frac{1}{2}\left(\rho\dot{u}_i(s)\dot{u}_i(s) + I_{ij}\dot{\phi}_i(s)\dot{\phi}_j(s) + 3J\dot{\phi}^2(s) \right) - W(s) - \frac{1}{2}a\theta^2(s) \right] \mathrm{d}v\,\mathrm{d}s\,, \tag{3.22}$$

where we have defined

$$\tau(t) = \int_0^t \theta(s)\,\mathrm{d}s\,.$$

Proof. If we take into account the equations of motion (3.10), the constitutive equations (3.2) and the geometric relations (3.3), we have

$$\frac{\partial}{\partial t}\left(\rho u_i\dot{u}_i + I_{ij}\phi_i\dot{\phi}_j + 3J\phi\dot{\phi} \right) = \left(\rho\dot{u}_i\dot{u}_i + I_{ij}\dot{\phi}_i\dot{\phi}_j + 3J\dot{\phi}^2 \right) +$$
$$+ \left(t_{ji}u_i + m_{ji}\phi_i + 3\lambda_j\phi \right)_{,j} - \left(t_{ji}e_{ij} + m_{ji}\kappa_{ij} + 3\lambda_j\gamma_j + 3\omega\phi \right).$$

The last term in parentheses can be expressed as

$$t_{ji}e_{ij} + m_{ji}\kappa_{ij} + 3\lambda_j\gamma_j + 3\omega\phi = 2\left(W + \frac{1}{2}a\theta^2 \right) + \left(\frac{1}{T_0}Q_j\theta \right)_{,j} - \frac{1}{T_0}k_{ij}\tau_{,i}\tau_{,j} - \rho\eta(0)\theta\,,$$

where we have defined

$$Q_i = \int_0^t q_i\,\mathrm{d}s\,,$$

and used the expression of η obtained by integrating with respect to time its evolution equation. Then, by an integration of the result over $\bar{B} \times [0, t]$ and by using the divergence theorem and the boundary conditions (3.11), we deduce the relation (3.22) and the proof is complete. □

Lemma 9. *Let $\mathcal{U}$ be a solution of the boundary-initial value problem $\mathcal{P}^*$. Then, for all $t \geqslant 0$, we have*

$$
\begin{aligned}
2\int_B \Big[\Big(\rho u_i(t)\dot{u}_i(t) + I_{ij}\phi_i(t)\dot{\phi}_j(t) + 3J\phi(t)\dot{\phi}(t)\Big) - \frac{1}{2T_0}k_{ij}\tau_{,i}(t)\tau_{,j}(t)\Big]\,\mathrm{d}v = \\
= \int_B \Big(\rho\dot{u}_i(0)u_i(2t) + I_{ij}\dot{\phi}_i(0)\phi_j(2t) + 3J\dot{\phi}(0)\phi(2t)\Big)\,\mathrm{d}v + \\
+ \int_B \Big(\rho u_i(0)\dot{u}_i(2t) + I_{ij}\phi_i(0)\dot{\phi}_j(2t) + 3J\phi(0)\dot{\phi}(2t)\Big)\,\mathrm{d}v - \\
- \int_0^t\!\!\int_B \rho\eta(0)\big(\theta(t+s) - \theta(t-s)\big)\,\mathrm{d}v\,\mathrm{d}s\,,
\end{aligned}
\tag{3.23}
$$

where we have defined τ as in Lemma 8.

Proof. Using the equations of motion (3.10), and the geometric relations (3.3), we have

$$
\begin{aligned}
&\frac{\partial}{\partial s}\Big\{\rho\big[\dot{u}_i(t+s)u_i(t-s) + u_i(t+s)\dot{u}_i(t-s)\big] + I_{ij}\big[\dot{\phi}_i(t+s)\phi_j(t-s) + \\
&\quad + \phi_i(t+s)\dot{\phi}_j(t-s)\big] + 3J\big[\dot{\phi}(t+s)\phi(t-s) + \phi(t+s)\dot{\phi}(t-s)\big]\Big\} = \\
&\quad \big[t_{ji}(t+s)u_i(t-s) - t_{ji}(t-s)u_i(t+s) + m_{ji}(t+s)\phi_i(t-s) - \\
&\quad - m_{ji}(t-s)\phi_i(t+s) + 3\lambda_j(t+s)\phi(t-s) - 3\lambda_j(t-s)\phi(t+s)\big]_{,j} - \\
&\quad - \big[t_{ji}(t+s)e_{ij}(t-s) - t_{ji}(t-s)e_{ij}(t+s) + m_{ji}(t+s)\kappa_{ij}(t-s) - \\
&\quad - m_{ji}(t-s)\kappa_{ij}(t+s) + 3\lambda_j(t+s)\gamma_j(t-s) - 3\lambda_j(t-s)\gamma_j(t+s) + \\
&\quad + 3\omega(t+s)\phi(t-s) - 3\omega(t-s)\phi(t+s)\big]\,.
\end{aligned}
$$

Using the constitutive equations (3.2) and (3.4), the last term in parentheses can be expressed as

$$\begin{aligned}
&\big[t_{ji}(t+s)e_{ij}(t-s) - t_{ji}(t-s)e_{ij}(t+s) + m_{ji}(t+s)\kappa_{ij}(t-s) - m_{ji}(t-s)\kappa_{ij}(t+s) + \\
&\quad + 3\lambda_j(t+s)\gamma_j(t-s) - 3\lambda_j(t-s)\gamma_j(t+s) + 3\omega(t+s)\phi(t-s) - \\
&\quad - 3\omega(t-s)\phi(t+s)\big] = \frac{1}{T_0}\big[\theta(t+s)Q_j(t-s) - \theta(t-s)Q_j(t+s)\big]_{,j} - \\
&\quad - \rho\eta(0)\big(\theta(t+s) - \theta(t-s)\big) - \frac{1}{T_0}k_{ij}\big[\dot{\tau}_{,i}(t+s)\tau_{,j}(t-s) - \tau_{,i}(t+s)\dot{\tau}_{,j}(t-s)\big]\,,
\end{aligned}$$

where we have defined Q_i as in the proof of Lemma 8 and used the expression of η obtained by integrating with respect to time its evolution equation. Then, by an integration of the result over $\bar{B}\times[0,t]$ and by using the divergence theorem and the boundary conditions (3.11), we deduce the relation (3.23) and the proof is complete. □

Corollary 10. *Let $\mathcal{U}$ be a solution of the boundary-initial value problem $\mathcal{P}^*$. Then, for all $t \geqslant 0$, we have*

$$\begin{aligned}
&2\int_0^t\!\!\int_B \Big[\Big(\rho\dot{u}_i(s)\dot{u}_i(s) + I_{ij}\dot{\phi}_i(s)\dot{\phi}_j(s) + 3J\dot{\phi}^2(s)\Big) - 2W(s) - a\theta^2(s)\Big]\,\mathrm{d}v\,\mathrm{d}s = \\
&\quad = -2\int_B \Big(\rho u_i(0)\dot{u}_i(0) + I_{ij}\phi_i(0)\dot{\phi}_j(0) + 3J\phi(0)\dot{\phi}(0)\Big)\,\mathrm{d}v + \\
&\quad + \int_B \Big(\rho\dot{u}_i(0)u_i(2t) + I_{ij}\dot{\phi}_i(0)\phi_j(2t) + 3J\dot{\phi}(0)\phi(2t)\Big)\,\mathrm{d}v + \\
&\quad + \int_B \Big(\rho u_i(0)\dot{u}_i(2t) + I_{ij}\phi_i(0)\dot{\phi}_j(2t) + 3J\phi(0)\dot{\phi}(2t)\Big)\,\mathrm{d}v - \\
&\quad - \int_0^t\!\!\int_B \rho\eta(0)\Big(2\theta(s) + \theta(t+s) - \theta(t-s)\Big)\,\mathrm{d}v\,\mathrm{d}s\,,
\end{aligned} \tag{3.24}$$

Proof. A combination of (3.22) and (3.23) implies the identity (3.24) and the proof is complete. □

Chapter 4

Uniqueness and Impossibility of time localization

The aim of this section is to establish the uniqueness of the backward in time problem, and, consequently, to prove the impossibly of localization of the solutions of the forward in time problem.

We begin by proving the following uniqueness theorem.

Theorem 11. *The boundary-initial value problem $\mathcal{P}^*$ has at most one solution.*

Proof. Thanks to the linearity of the problem in concern, we only need to show that null data imply null solution or, in other words, that the null solution is the solution corresponding to null data. Taking into account (3.14) and (3.20) we obtain

$$\mathcal{E}^*(t) = \int_B 2W(t)\,\mathrm{d}v\,. \tag{4.1}$$

If we consider the following function

$$F(t) = \mathcal{E}^*(t) + \varepsilon \mathcal{E}(t)\,,$$

we have from (3.13) and (4.1)

$$F(t) = \int_B \Big[\varepsilon \mathcal{K}(t) + (\varepsilon + 2) W(t) + \frac{\varepsilon}{2} a\theta^2(t)\Big]\, \mathrm{d}v\,, \tag{4.2}$$

where ε is a positive constant, and in what follows we take $0 < \varepsilon < 1$. The hypotheses (1)-(3) imply that F is a positive function and it defines a measure of the solution. On the other side, since the initial and boundary conditions are null, we can rewrite (3.15) and (3.18) as

$$\begin{aligned}
\mathcal{E}(t) &= \int_0^t\!\!\int_B \frac{1}{T_0} k_{ij}\theta_{,i}(s)\theta_{,j}(s)\,\mathrm{d}v\,\mathrm{d}s\,,\\
\mathcal{E}^*(t) &= -\int_0^t\!\!\int_B \Big\{2\Big[\dot{u}_i(s)[\beta_{ji}\theta(s)]_{,j} + \dot{\phi}_k(s)[C_{jk}\theta(s)]_{,j} + \varepsilon_{jik}\beta_{ji}\dot{\phi}_k(s)\theta(s) +\\
&\quad + \dot{\phi}(s)[\xi_j\theta(s)]_{,j} - \zeta\dot{\phi}(s)\theta(s)\Big] + \frac{1}{T_0}k_{ij}\theta_{,i}(s)\theta_{,j}(s)\Big\}\,\mathrm{d}v\,\mathrm{d}s\,;
\end{aligned}$$

consequently, it is

$$\begin{aligned}
F(t) = &-\int_0^t\!\!\int_B \Big\{2\Big[\dot{u}_i(s)[\beta_{ji}\theta(s)]_{,j} + \dot{\phi}_k(s)[C_{jk}\theta(s)]_{,j} + \varepsilon_{ijk}\beta_{ji}\dot{\phi}_k(s)\theta(s)\\
&+ \dot{\phi}(s)[\xi_j\theta(s)]_{,j} - \zeta\dot{\phi}(s)\theta(s)\Big] + \frac{1-\varepsilon}{T_0}k_{ij}\theta_{,i}(s)\theta_{,j}(s)\Big\}\,\mathrm{d}v\,\mathrm{d}s\,.
\end{aligned} \tag{4.3}$$

If we consider the time rate of (4.3), we obtain

$$\dot{F}(t) = -2\int_B \Big[\dot{u}_i(t)[\beta_{ji}\theta(t)]_{,j} + [C_{kj}\theta(t)]_{,j}\dot{\phi}_k(t) + \varepsilon_{jik}\beta_{ji}\dot{\phi}_k(t)\theta(t) + \\ + \dot{\phi}(t)[\xi_j\theta(t)]_{,j} - \zeta\dot{\phi}(t)\theta(t) + \frac{1-\varepsilon}{2T_0}k_{ij}\theta_{,i}(t)\theta_{,j}(t)\Big]\,\mathrm{d}v\,.$$

Using the arithmetic-geometric mean inequality we have

$$\begin{aligned} &-2\Big[\beta_{ji,j}\dot{u}_i + (C_{kj,j} + \varepsilon_{jik}\beta_{ji})\dot{\phi}_k + (\xi_{j,j} + \zeta)\dot{\phi}\Big]\theta \leqslant \delta_1(\rho_0\dot{u}_i\dot{u}_i + I_0\dot{\phi}_k\dot{\phi}_k + 3J_0\dot{\phi}^2) + \frac{A}{\delta_1}a_0\theta^2\,,\\ &-2\Big[\beta_{ji}\dot{u}_i + C_{kj}\dot{\phi}_k + \xi_j\dot{\phi}\Big]\theta_{,j} \leqslant \delta_2(\rho_0\dot{u}_i\dot{u}_i + I_0\dot{\phi}_k\dot{\phi}_k + 3J_0\dot{\phi}^2) + \frac{B}{\delta_2}\frac{k_m}{T_0}\theta_{,i}\theta_{,i}\,, \end{aligned} \tag{4.4}$$

where δ_1, δ_2 are positive constants, and

$$\begin{aligned} A &= \frac{1}{a_0}\max_{\bar{B}}\left\{\frac{\beta_{ji,j}\beta_{ki,k}}{\rho_0} + \frac{(C_{kj,j} + \varepsilon_{jik}\beta_{ji})(C_{kr,r} + \varepsilon_{rsk}\beta_{rs})}{I_0} + \frac{(\xi_{j,j} + \zeta)^2}{3J_0}\right\}\,,\\ B &= \frac{T_0}{k_m}\max_{\bar{B}}\left\{\frac{\sqrt{\beta_{ji}\beta_{ki}\beta_{jh}\beta_{kh}}}{\rho_0} + \frac{\sqrt{C_{kj}C_{ki}C_{hj}C_{hi}}}{I_0} + \frac{\xi_j\xi_j}{3J_0}\right\}\,. \end{aligned}$$

We set δ_1 and δ_2 as follow

$$\delta_1 = \frac{A}{\varkappa}\,, \qquad \delta_2 = \frac{B}{\varepsilon_1}\,,$$

where ε_1 is an arbitrary positive constant and $\varkappa$ is the positive solution of the equation

$$\varepsilon_1\varkappa^2 - B\varkappa - \varepsilon_1 a = 0\,.$$

We note that the constant $\varkappa$ is expressed in terms of the constitutive coefficients and in terms of ε_1. We can see that, with this choice of δ_1, δ_2, $\varkappa$ and considering (1), (2), and (4.4), we have

$$\dot{F}(t) \leqslant 2\varkappa \int_B \left[\mathcal{K}(t) + \frac{1}{2}a\theta^2(t)\right] \mathrm{d}v + \frac{\varepsilon_1 - (1-\varepsilon)}{T_0} \int_B k_{ij}\theta_{,i}(t)\theta_{,j}(t)\,\mathrm{d}v\,.$$

If we take into account (4.2), hypothesis (3) and choose $\varepsilon_1 \leqslant 1-\varepsilon$, we can obtain

$$\dot{F}(t) \leqslant \frac{2\varkappa}{\varepsilon} \int_B \varepsilon \left[\mathcal{K}(t) + \frac{1}{2}a\theta^2(t)\right] \mathrm{d}v \leqslant \frac{2\varkappa}{\varepsilon}F\,.$$

A solution of this differential inequality is such that

$$0 \leqslant F(t) \leqslant F(0)e^{(2\varkappa/\varepsilon)t}\,.$$

Since we are considering homogeneous boundary-initial data, we have $F(0) = 0$, and so

$$F(t) = 0 \qquad \forall t \geqslant 0\,.$$

In conclusion, the definition of F as a measure imply that the only solution of the considered problem is

$$u_i(t) = 0\,, \qquad \phi_i(t) = 0\,, \qquad \phi(t) = 0 \qquad \forall t > 0\,. \qquad \square$$

We now consider the boundary-initial value problem $\widehat{\mathcal{P}}$ defined by the relations (3.1) to (3.3), homogeneous boundary conditions and initial conditions (3.9). We can show the impossibility of localization in time of this problem. In particular we can prove that the only solution for this problem that vanishes after a finite time is the null solution.

Theorem 12. *Let $\mathcal{U}$ be a solution of the boundary-initial value problem $\widehat{\mathcal{P}}$*

that vanishes after a finite time $T \geqslant 0$

$$u_i(t) = 0\,, \qquad \phi_i(t) = 0\,, \qquad \phi(t) = 0\,, \qquad \forall t \geqslant T\,.$$

Then, this solution is the null solution.

Proof. We can consider the corresponding backward in time problem in the time interval $(-\infty, T]$, defined by the relations (3.1) to (3.3), homogeneous boundary conditions and null final conditions:

$$\begin{aligned}
&u_i(\mathbf{x}, T) = 0\,, \qquad \phi_i(\mathbf{x}, T) = 0\,, \qquad \phi(\mathbf{x}, T) = 0\,, \qquad \theta(\mathbf{x}, T) = 0\,,\\
&\dot{u}_i(\mathbf{x}, T) = 0\,, \qquad \dot{\phi}_i(\mathbf{x}, T) = 0\,, \qquad \dot{\phi}(\mathbf{x}, T) = 0\,.
\end{aligned}$$

According to Theorem 11 the only solution is the null solution. $\square$

Chapter 5

The asymptotic partition of energy

In this section we derive the relations which exhibit the asymptotic partition of the energy, provided only that the thermoelastodynamics process is constrained to lie in the set $\mathcal{M}$ of all thermoelastodymanics processes defined on $B \times [0, +\infty)$ which satisfy

$$\int_0^t \int_B \frac{1}{T_0} k_{ij} \theta_{,i}(s) \theta_{,j}(s) \,\mathrm{d}v \,\mathrm{d}s \leqslant M \,, \tag{5.1}$$

where M is a positive constant.

If $\mathcal{U}$ is a solution of the boundary-initial value problem $\mathcal{P}^*$, then, for the various associated energies, we introduce the following Cesaro means:

$$\begin{aligned}
\mathcal{K}_C(t) &= \frac{1}{t}\int_0^t\!\!\int_B \frac{1}{2}\Big(\rho\dot{u}_i(s)\dot{u}_i(s) + I_{ij}\dot{\phi}_i(s)\dot{\phi}_j(s) + 3J\dot{\phi}^2(s)\Big)\,\mathrm{d}v\,\mathrm{d}s\,,\\
\mathcal{S}_C(t) &= \frac{1}{t}\int_0^t\!\!\int_B W(s)\,\mathrm{d}v\,\mathrm{d}s\,,\\
\mathcal{T}_C(t) &= \frac{1}{t}\int_0^t\!\!\int_B \frac{1}{2}a\theta^2(s)\,\mathrm{d}v\,\mathrm{d}s\,,\\
\mathcal{D}_C(t) &= \frac{1}{t}\int_0^t\!\!\int_0^s\!\!\int_B \frac{1}{T_0}k_{ij}\theta_{,i}(\iota)\theta_{,j}(\iota)\,\mathrm{d}v\,\mathrm{d}\iota\,\mathrm{d}s\,.
\end{aligned} \tag{5.2}$$

We observe that if $\mathrm{meas}\big(\Sigma_1^{(1)}\big) = \mathrm{meas}\big(\Sigma_1^{(2)}\big) = 0$, then there exists a set of rigid motions, null temperatures and null microstretch functions such that the equations (3.2), (3.3), (3.10), (3.11) are satisfied, so that it is possible to write the initial data u_i^0, $\dot{u}_i^0$, ϕ_i^0, $\dot{\phi}_i^0$ as

$$\begin{aligned}
u_i^0 &= u_i^* + U_i^0\,, &\qquad \phi_i^0 &= \phi_i^* + \Phi_i^0\,,\\
\dot{u}_i^0 &= \dot{u}_i^* + \dot{U}_i^0\,, &\qquad \dot{\phi}_i^0 &= \dot{\phi}_i^* + \dot{\Phi}_i^0\,,
\end{aligned} \tag{5.3}$$

where u_i^*, $\dot{u}_i^*$, Φ_i^*, $\dot{\Phi}_i^*$ are rigid displacements determined in such a way that, defined

$$I_i^{(1)}(\mathbf{v}) = \int_B \rho v_i\,\mathrm{d}v\,, \qquad I_i^{(2)}(\mathbf{v},\boldsymbol{\psi}) = \int_B \rho\left(\varepsilon_{ijk}x_j v_k + \psi_i\right)\mathrm{d}v\,,$$

we have

$$\begin{aligned}
I_i^{(1)}\big(\mathbf{U}^0\big) &= 0\,, &\qquad I_i^{(2)}\big(\mathbf{U}^0,\boldsymbol{\Phi}^0\big) &= 0\,,\\
I_i^{(1)}\big(\dot{\mathbf{U}}^0\big) &= 0\,, &\qquad I_i^{(2)}\big(\dot{\mathbf{U}}^0,\dot{\boldsymbol{\Phi}}^0\big) &= 0\,,
\end{aligned} \tag{5.4}$$

Let us introduce the following notations

$$\hat{\mathbf{C}}^1(B) = \Big\{ (\mathbf{v}, \boldsymbol{\psi}) \in C^1\left(\bar{B}\right)^3 \times C^1\left(\bar{B}\right)^3 : v_i = 0 \text{ on } \Sigma_1^{(1)} \text{ and } \psi_i = 0 \text{ on } \Sigma_1^{(2)}$$
$$\text{and if } \operatorname{meas}\big(\Sigma_1^{(1)}\big) = \operatorname{meas}\big(\Sigma_1^{(2)}\big) = 0 \text{ then } I_i^{(1)}(\mathbf{v}) = I_i^{(2)}(\mathbf{v}, \boldsymbol{\psi}) = 0 \Big\},$$
$$\hat{C}^1(B) = \Big\{ \varphi \in C^1\left(\bar{B}\right) : \varphi = 0 \text{ on } \Sigma_1^{(3)} \Big\},$$
$$\tilde{C}^1(B) = \Big\{ \vartheta \in C^1\left(\bar{B}\right) : \vartheta = 0 \text{ on } \Sigma_1^{(4)} \Big\},$$

and

$\hat{\mathbf{W}}_1(B)$ the completion of $\hat{\mathbf{C}}^1(B)$ by means of the norm of $\mathbf{W}_1(B)^2$,

$\hat{W}_1(B)$ the completion of $\hat{C}^1(B)$ by means of the norm of $W_1(B)$,

$\tilde{W}_1(B)$ the completion of $\tilde{C}^1(B)$ by means of the norm of $W_1(B)$,

where $C^1(\bar{B})$ represents the set of continuously differentiable functions on $\bar{B}$; moreover $W_m(B)$ represents the familiar Sobolev space [35] and $\mathbf{W}_m(B) = W_m(B)^3$.

We note that the hypothesis (3.7) assures that, for every $\mathbf{V} = (\mathbf{v}, \boldsymbol{\psi}, \varphi) \in \hat{\mathbf{W}}_1(B) \times \hat{W}_1(B)$ the following inequality [36, 37] holds

$$\int_B 2W(\mathbf{V})\,\mathrm{d}v \geqslant m_1 \int_B \left(v_i v_i + I_{ij}\psi_i\psi_j + 3J\varphi^2 \right) \mathrm{d}v \tag{5.5}$$

where m_1 is a suitable positive constant and $W(\mathbf{V})$ is defined through the relation (3.5). Furthermore, from relation (3.6) we obtain, for every $\vartheta \in$

$\tilde{W}_1(B)$, the following Poincar inequality

$$\int_B k_{ij}\vartheta_{,i}\vartheta_{,j}\,\mathrm{d}v \geqslant m_2 \int_B \vartheta^2\,\mathrm{d}v \tag{5.6}$$

where m_2 is a suitable positive constant.

If $\mathrm{meas}\big(\Sigma_1^{(1)}\big) = \mathrm{meas}\big(\Sigma_1^{(2)}\big) = 0$, then we can decompose the solution of the problem $\mathcal{P}^*$ in the form

$$\begin{aligned} u_i(\mathbf{x},t) &= v_i(\mathbf{x},t) \,+ u_i^*(\mathbf{x}) + t\dot{u}_i^*(\mathbf{x})\,, \qquad & \phi(\mathbf{x},t) &= \varphi_(\mathbf{x},t)\,,\\ \phi_i(\mathbf{x},t) &= \psi_i(\mathbf{x},t) + \phi_i^*(\mathbf{x}) + t\dot{\phi}_i^*(\mathbf{x})\,, & \theta(\mathbf{x},t) &= \vartheta(\mathbf{x},t)\,, \end{aligned} \tag{5.7}$$

where $(\mathbf{v},\boldsymbol{\psi},\varphi,\vartheta) \in \hat{\mathbf{W}}_1\,(B)\times\hat{W}_1\,(B)\times\tilde{W}_1\,(B)$ is the solution of the problem $\mathcal{P}^*$ according to the following initial conditions

$$\begin{aligned} &v_i(\mathbf{x},0) = U_i^0(\mathbf{x})\,, \qquad \psi_i(\mathbf{x},0) = \Phi_i^0(\mathbf{x})\,, \qquad \varphi(\mathbf{x},0) = \phi^0(\mathbf{x})\,, \qquad \vartheta(\mathbf{x},0) = \theta^0(\mathbf{x})\,,\\ &\dot{v}_i(\mathbf{x},0) = \dot{U}_i^0(\mathbf{x})\,, \qquad \dot{\psi}_i(\mathbf{x},0) = \dot{\Phi}_i^0(\mathbf{x})\,, \qquad \dot{\varphi}(\mathbf{x},0) = \dot{\phi}^0(\mathbf{x})\,, \end{aligned}$$

From the above elements it is possible to derive the asymptotic partition in terms of the Cesaro means defined by the relations (5.2).

Theorem 13. *Let $\mathcal{U}$ be a solution of the boundary-initial value problem $\mathcal{P}^*$. Then, for all choices of initial data*

$$\begin{aligned} &\mathbf{u}^0 \in \mathbf{W}_1(B)\,, \qquad \boldsymbol{\phi}^0 \in \mathbf{W}_1(B)\,, \qquad \phi^0 \in W_1(B)\,, \qquad \theta^0 \in W_0(B)\,,\\ &\dot{\mathbf{u}}^0 \in \mathbf{W}_0(B)\,, \qquad \dot{\boldsymbol{\phi}}^0 \in \mathbf{W}_0(B)\,, \qquad \dot{\phi}^0 \in W_0(B)\,, \end{aligned}$$

we have

$$\lim_{t\to\infty} \mathcal{T}_C(t) = 0\,, \tag{5.8}$$

and it follows that

P_1) *if* $\text{meas}\big(\Sigma_1^{(1)}\big) \neq 0$ *or* $\text{meas}\big(\Sigma_1^{(2)}\big) \neq 0$*, then*

$$\begin{aligned}
&\lim_{t\to\infty} \mathcal{K}_C(t) = \lim_{t\to\infty} \mathcal{S}_C(t)\,, \\
&\lim_{t\to\infty} \mathcal{D}_C(t) = 2\lim_{t\to\infty} \mathcal{K}_C(t) - \mathcal{E}(0) = 2\lim_{t\to\infty} \mathcal{S}_C(t) - \mathcal{E}(0)\,;
\end{aligned} \tag{5.9}$$

P_2) *if* $\text{meas}\big(\Sigma_1^{(1)}\big) = \text{meas}\big(\Sigma_1^{(2)}\big) = 0$*, then*

$$\lim_{t\to\infty} \mathcal{K}_C(t) = \lim_{t\to\infty} \mathcal{S}_C(t) + \frac{1}{2}\int_B \rho \dot{u}_i^* \dot{u}_i^* \,\mathrm{d}v + \frac{1}{2}\int_B I_{ij}\dot{\phi}_i^*\dot{\phi}_j^* \,\mathrm{d}v\,, \tag{5.10}$$

$$\begin{aligned}
\lim_{t\to\infty} \mathcal{D}_C(t) &= 2\lim_{t\to\infty} \mathcal{K}_C(t) - \mathcal{E}(0) - \frac{1}{2}\int_B \rho \dot{u}_i^* \dot{u}_i^* \,\mathrm{d}v - \frac{1}{2}\int_B I_{ij}\dot{\phi}_i^*\dot{\phi}_i^* \,\mathrm{d}v = \\
&= 2\lim_{t\to\infty} \mathcal{S}_C(t) - \mathcal{E}(0) + \frac{1}{2}\int_B \rho \dot{u}_i^* \dot{u}_i^* \,\mathrm{d}v + \frac{1}{2}\int_B I_{ij}\dot{\phi}_i^*\dot{\phi}_i^* \,\mathrm{d}v\,.
\end{aligned} \tag{5.11}$$

Proof. Using the relations (5.2) and (3.15), we deduce

$$\mathcal{K}_C(t) + \mathcal{S}_C(t) + \mathcal{T}_C(t) = \mathcal{E}(0) + \mathcal{D}_C(t)\,, \qquad t > 0\,. \tag{5.12}$$

On the basis of the relations (5.1), (5.2), (5.6) and (3.13) it results

$$\mathcal{T}_C(t) \leqslant \frac{T_0 M}{2m_2 t}\max_{\bar{B}}\{a(\mathbf{x})\}\,, \qquad t > 0\,,$$

and hence by making t to tend to infinity, we get relation (5.8). On the other end, from relations (3.24) and (5.2) we get

$$
\begin{aligned}
\mathcal{K}_C(t) - \mathcal{S}_C(t) - \mathcal{T}_C(t) = &-\frac{1}{2t}\int_B \Big(\rho u_i(0)\dot{u}_i(0) + I_{ij}\phi_i(0)\dot{\phi}_j(0) + 3J\phi(0)\dot{\phi}(0)\Big)\,\mathrm{d}v + \\
&+ \frac{1}{4t}\int_B \Big(\rho\dot{u}_i(0)u_i(2t) + I_{ij}\dot{\phi}_i(0)\phi_j(2t) + 3J\dot{\phi}(0)\phi(2t)\Big)\,\mathrm{d}v + \\
&+ \frac{1}{4t}\int_B \Big(\rho u_i(0)\dot{u}_i(2t) + I_{ij}\phi_i(0)\dot{\phi}_j(2t) + 3J\phi(0)\dot{\phi}(2t)\Big)\,\mathrm{d}v - \\
&- \frac{1}{4t}\int_0^t\!\!\int_B \rho\eta(0)\Big(2\theta(s) + \theta(t+s) - \theta(t-s)\Big)\,\mathrm{d}v\,\mathrm{d}s\,,
\end{aligned}
\tag{5.13}
$$

Further, the relations (3.7), (5.1), (5.6) and (3.15) give

$$
\begin{aligned}
&\int_B \rho\dot{u}_i(t)\dot{u}_i(t)\,\mathrm{d}v \leqslant 2\big(\mathcal{E}(0)+M\big)\,, && \mu_m\int_B \phi^2(t)\,\mathrm{d}v \leqslant 2\big(\mathcal{E}(0)+M\big)\,,\\
&\int_B I_{ij}\dot{\phi}_i(t)\dot{\phi}_j(t)\,\mathrm{d}v \leqslant 2\big(\mathcal{E}(0)+M\big)\,, && a_0\int_B \theta^2(t)\,\mathrm{d}v \leqslant 2\big(\mathcal{E}(0)+M\big)\,,\\
&\int_B 3J\dot{\phi}^2(t)\,\mathrm{d}v \leqslant 2\big(\mathcal{E}(0)+M\big)\,,\\
&\int_B 2W(t)\,\mathrm{d}v \leqslant 2\big(\mathcal{E}(0)+M\big)\,.
\end{aligned}
\tag{5.14}
$$

Thus, by using the Schwarz's inequality and the relations (5.8) and (5.14) into relation (5.13), we obtain

$$
\lim_{t\to\infty}\mathcal{K}_C(t) - \lim_{t\to\infty}\mathcal{S}_C(t) = \lim_{t\to\infty}\frac{1}{4t}\int_B \Big(\rho\dot{u}_i(0)u_i(2t) + I_{ij}\dot{\phi}_i(0)\phi_j(2t)\Big)\,\mathrm{d}v\,,
\tag{5.15}
$$

Let us first consider the point P_1. Since $\mathrm{meas}\big(\Sigma_1^{(1)}\big) \neq 0$, $\mathrm{meas}\big(\Sigma_1^{(2)}\big) \neq 0$ and $(\mathbf{u},\boldsymbol{\phi}) \in \hat{\mathbf{W}}_1(B)$, from equations (5.5), (3.13) and (3.15) we deduce that

$$
m_1\int_B u_i(t)u_i(t)\,\mathrm{d}v \leqslant \int_B 2W(t)\,\mathrm{d}v \leqslant 2\big(\mathcal{E}(0)+M\big)\,,
$$

$$m_1 \int_B I_{ij}\phi_i(t)\phi_j(t)\,\mathrm{d}v \leqslant \int_B 2W(t)\,\mathrm{d}v \leqslant 2\big(\mathcal{E}(0) + M\big)\,.$$

so that, by means of the Schwarz's inequality, we obtain

$$\lim_{t\to\infty} \frac{1}{4t} \int_B \Big(\rho\dot{u}_i(0)u_i(2t) + I_{ij}\dot{\phi}_i(0)\phi_j(2t)\Big)\,\mathrm{d}v = 0\,. \tag{5.16}$$

Consequently, the relations (5.15), (5.16) imply the relation (5.9) and, by taking into account the relation (5.12), we obtain the desired result. Let us further consider the point P_2. Since $\operatorname{meas}\big(\Sigma_1^{(1)}\big) = \operatorname{meas}\big(\Sigma_1^{(2)}\big) = 0$, then the decompositions (5.3), (5.7) and the relation (5.4) give

$$\begin{aligned}
&\frac{1}{4t} \int_B \Big(\rho\dot{u}_i(0)u_i(2t) + I_{ij}\dot{\phi}_i(0)\phi_j(2t)\Big)\,\mathrm{d}v = \\
&\quad \frac{1}{4t}\int_B \rho\dot{u}_i^* u_i^*\,\mathrm{d}v + \frac{1}{4t}\int_B \rho(\dot{u}_i^* + \dot{U}_i^0)v_i(2t)\,\mathrm{d}v + \frac{1}{2}\int_B \rho\dot{u}_i^*\dot{u}_i^*\,\mathrm{d}v + \\
&\quad \frac{1}{4t}\int_B I_{ij}\dot{\phi}_i^*\phi_j^*\,\mathrm{d}v + \frac{1}{4t}\int_B I_{ij}(\dot{\phi}_i^* + \dot{\Phi}_i^0)\psi_j(2t)\,\mathrm{d}v + \frac{1}{2}\int_B I_{ij}\dot{\phi}_i^*\dot{\phi}_j^*\,\mathrm{d}v\,.
\end{aligned} \tag{5.17}$$

Now, from relations (5.5), (3.13) and (3.15) we deduce

$$\begin{aligned}
m_1 \int_B v_i(t)v_i(t)\,\mathrm{d}v &\leqslant 2\big(\mathcal{E}(0) + M\big)\,, \\
m_1 \int_B I_{ij}\psi_i(t)\psi_j(t)\,\mathrm{d}v &\leqslant 2\big(\mathcal{E}(0) + M\big)\,,
\end{aligned}$$

and from equation (5.17) we obtain

$$\lim_{t\to\infty} \frac{1}{4t} \int_B \Big(\rho\dot{u}_i(0)u_i(2t) + I_{ij}\dot{\phi}_i(0)\phi_j(2t)\Big)\,\mathrm{d}v = \frac{1}{2}\int_B \rho\dot{u}_i^*\dot{u}_i^*\,\mathrm{d}v + \frac{1}{2}\int_B I_{ij}\dot{\phi}_i^*\dot{\phi}_j^*\,\mathrm{d}v\,.$$

Therefore, if we substitute the last relation into equation (5.15) we deduce the relation (5.10). Moreover, taking into account the relations (5.8), (5.10) and (5.12), we obtain the relation (5.11), thus the proof of the theorem is complete. $\square$

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