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ŁUKASIEWICZ LOGIC: ALGEBRAS AND SHEAVES

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Introduction

Classical logic arose from the need to study forms and laws of the human reasoning. But soon, it came out the difficulties of classical logic to formalize uncertain events and vague concepts, for which it is not possible to assert if a sentence is true or false.

In order to overcome these limits, at the beginning of the last century, non classical logics were introduced. In these logic it fails at least one among the basic principles of classical logic. For example, cutting out the principle of truth functionality (the true value of a sentence only depends on the truth values of its component more simpler sentences), we obtain modal logics for which the truth value of a sentence depends on the context where we are. In this case, the context is seen as a possible world of realization. Cutting out the principle of bivalence, we obtain many-valued logics instead.

The first among classical logician not to accept completely the principle of bivalence was Aristotele, who is, however, considered the father of classical logic. Indeed, Aristotele presented again the problem of *futuri contingenti*[1] introduced by Diodorus Cronus as exception to the principle of bivalence (see Chapter 9 in his De Intepretatione). The "futuri contingenti" are sentences talking about future events for which it is not possible to say if they are true or false. However, Aristotele didn't make up a system of many-valued logic

[1]Consider the sentence A:= "Tomorrow a sea battle will be fought". If the principle of bivalence holds, we can say that A is true and, in this case, it is already establishes the fact described by A (so it will happen necessarily), or A is false and so it is already established that the fact will not happen. But all this seems to lead to the fatalism and so to deny people's freedom of action.

able to overcome classical logic's limits.

In 1920, Jan Łukasiewicz, a Polish mathematician and philosopher, set up the present many-valued logic. Indeed, in an attempt to resolve the futuri contingenti problem, he introduced a third true value to indicate the possibility. In this way, he created his three-valued logic. This system involves not only a different definition of truth values, but an alteration to relationships among truth functional symbols too. Łukasiewicz introduced beside usual truth values one for the symbol $\diamond$ (i.e. "it is possible that") and, under suggestion of A. Tarski, he defined this symbol as $\diamond p = \neg p \to p$. These changes in the classical system together with the development of the truth tables method allowed him to made up his three-valued system. Afterward, together with many students, he generalized his system with n truth valued, with n finite or countable. Later, in 30's of the last century, J. Slupecki, D. A. Bochvar and J. B. Rosser studied many-valued Łukasiewicz logics, but only at the end of 50's Rosser and A. R. Turquette obtained a satisfying axiomatization of many-valued logics.

In the 30's, thank to algebraic construction introduced by Lindenbaum and Tarski, it was possible to associate with each logical theory T, an algebra made up by equivalence classes of formulas in T which are logically equivalent. This allowed research in Logic to acquire a distinguishing algebraic nature and it set up a bridge between the purely syntactical world and the algebraic semantics. In this way, algebraic properties in semantics can be translated automatically in properties of the associated logic and viceversa.

It is known that the Lindenbaum-Tarski algebra in classical logic is a Boolean algebra. However, Boolean algebras have not stayed glued only to logic, but they have found applications in other mathematical fields. In this way, Boolean algebras theory has become an independent theory.

Main fields of applications of Boolean algebras are:

- set theory (fields of sets);

- topology (compact Hausdorff zero-dimensional spaces);

- foundation set theory (Boolean valued models);

- measure theory (measure algebras: Boolean algebra with a measure);

- functional analysis (projections algebras);

- rings theory (Boolean rings).

This allows to translate a problem with Boolean algebras in terms of topology, analysis, logic etc or viceversa and to choose the simplest approach.

The Lindenbaum-Tarski algebra of infinite-valued Łukasiewicz logic is a free MV-algebra. MV-algebras were introduced by C. C. Chang (see [14]) in 1958 to give an algebraic proof of the completeness of Łukasiewicz logic reducing the problem to require the semisimplicity of the Lindenbaum-Tarski algebra. Once again purely logical concepts meet an algebraic counterpart. But, unlike Boolean algebras which are always semisimple, not every MV-algebra is semisimple. This leads to an enrichment of MV-algebras theory with respect to Boolean algebras theory. Indeed, as Łukasiewicz logic is a generalization of classical logic, MV-algebras are a generalization of Boolean algebras which are idempotent MV-algebras.

After their introduction by Chang, MV-algebras free themselves from the bonds of logic and become an autonomous mathematical discipline with deep connections to several other branches of mathematics. For example, in 1986 D. Mundici proved that the category of lattice ordered abelian groups with strong unit is categorical equivalent to the category of MV-algebras ([50]). This result is very important because lattice ordered abelian groups don't set up an equational variety unlike MV-algebras. In this way more complicated properties in groups language can became simpler in MV-algebras language. Moreover, the study of normal forms for Łukasiewicz logic brought to a deep relation between MV-algebras and toric varieties through the concept of Schauder bases, which are the affine versions of a complex of nonsingular cones.

For Boolean algebras, it holds Stone duality, according to that, maximal ideals spectrum with Zariski topology is a compact Hausdorff zero-dimensional space and each compact Hausdorff zero-dimensional space is the spectrum of a Boolean algebra. However, unlike Boolean algebras, for MV-algebras it doesn't exist any purely topological representation of the spectrum. Formerly, there were some attempts about it. MV-spaces were introduced, i.e. topological spaces which are homeomorphic to the spectrum of an MV-algebra. But they are still an unknown world.

In the representation theory of MV-algebras, one of the most important theorem is due to Chang (Theorem 1.3.3 of [18]): each MV-algebra can be embedded into the direct product of its quotients with respect to prime ideals and so, each MV-algebra is isomorphic to a subdirect product of MV-chains, that are totally ordered MV-algebras. For this, conditions over the spectrum divide MV-algebras in classes. It is worth to stress that Chang's theorem is the specialization of Birkhoff theorem in Universal Algebra at the case of MV-algebras: each not trivial algebra is isomorphic to a subdirect product of subdirectly irreducible algebras.

In 1991, Di Nola provided a representation of MV-algebras as algebra of functions ([24]). According to it, any MV-algebra A is, up to isomorphism, a subalgebra of algebra of $[0,1]^*$-valued functions over some suitable set X, where $[0,1]^*$ is an ultrapower of $[0,1]$, only depending on the cardinality of A. Recently, in [28] Di Nola, Lenzi and Spada have given a uniform version of this theorem which states the existence of an MV-algebra A of functions in $[0,1]^*$ such that any MV-algebra of a bounded cardinality embeds into A.

Noticing that rings theory and MV-algebras theory meet in Boolean algebras and for the spectrum of rings too it doesn't exist a Stone-like representation theorem, we convince ourselves of the impossibility to find such a representation for the spectrum of MV-algebras. Moreover, in both theories, the spectrum of prime ideals properly contains the spectrum of maximal ideals, whereas for Boolean algebras the two spectra coincide. So, it seems to be natural to be inspired by representation theory of rings. For rings, there exist representations which are not purely topological, but use a topological space together with a sheaf structure.

A sheaf is a triple (E, π, X), where X, E are topological spaces said *base space* and *total space* respectively and $\pi : E \to X$ is a local homeomorphism.

The notion of sheaf arose from the analytic continuity of functions. After World War II, J. Leray gave an explicit definition of a sheaf on a topological space in terms of the closed sets of the space ([41–44]). Later, H. Cartan gave the dual definition of a sheaf on a space in terms of its open sets ([13]).

Sheaves are a very useful tool in representation theory. Indeed, the basic idea of representation theorem is to decompose a given structure in more simple structures in such a way that properties of the given structure can be reduced to properties of more simple structures. In the case of the above-mentioned Birkhoff theorem, the more simple structures are subdirectly irreducible and the decomposition is subdirect. Moreover, this theorem is not easy-to-use. Indeed, it is usually very difficult to determine subdirectly irreducible factors of a given algebra and even when those are known, subdirect products are so “loose” that very little can be inferred from the properties of the factors. All this appears to have provided a major motivation to find an alternative representation to subdirect one and, so, to the development of sheaf representations. Actually, structures of global sections of sheaves are special subdirect products which are “tight” enough to allow significant conclusions to be drawn from the properties of the factors ([19]). This is possible because stalks can not be irreducible algebras.

There exist in literature representation theorems for several algebras as algebras of global sections of a sheaf, for example, for rings see [23], [49] and [52], for l-groups see [22] and [10]. These theorems evolved general constructions of sheaf representation for Universal Algebra: in 1973 Davey provided a method to obtain a sheaf representation from subdirect representation ([21]). In different classes of problems, sheaf representations of universal algebras are very useful since they reduced the study of algebras to the study of the stalks, which usually have a better known structure.

Finally, sheaf representations allow to establish a bridge between the represented structure and geometric objects as it happens in Algebraic Geometric for rings which are tied to the affine schemes introduced by Grothendieck.

In this context, they get in many attempts to set up a connection between MV-algebras and geometric objects. The used method are very similar to those used by Grothendieck in Algebraic Geometry, based on the categorical duality between rings and affine schemes and they involved the introduction of the so-called “MV-algebraic spaces”, which are the MV-algebraic version of ringed spaces. Indeed, an MV-algebraic space is a pair $(X,\ E)$ where X is a topological space and E is a sheaf of MV-algebras over X. An important result here is provided by Filipoiu and Georgescu ([32]) who presented a categorical duality between MV-algebras and a particular full subcategory of MV-algebraic spaces, the category of separating and local T_2 MV-algebraic spaces.

Recently, Dubuc and Poveda ([29]) provide a similar but weaker representation. Proceeding from Chang subdirect representation they obtain an adjoint functor between the category of MV-algebras and the category of MV-algebraic spaces with MV-chains as stalks.

These representations are different not only for stalks, but for the choice of base spaces too: in Filipoiu and Georgescu representation the base space is the spectrum of maximal MV-ideals topologized with Zariski topology, whereas in Dubuc and Poveda representation the base space is the spectrum of prime MV-ideals topologized with coZariski topology.

More recently, Di Nola, Esposito and Gerla ([25]) refined the results of [32] on MV-algebras presenting sheaf representations of classes of MV-algebras based on the choice of stalks from given classes of local MV-algebras.

In this thesis, inspired by methods of Bigard, Keimel and Wolfenstein [10], we develop an approach to sheaf representations of MV-algebras that is a mixture of the previous approach by Dubuc and Poveda and by Filipoiu and Georgescu. Following Davey’s approach, we use a subdirect representation of MV-algebras that is based on local MV-algebras that are a generalization of MV-chains. This led to a sheaf representation with local stalks and where the base space is $\mathrm{Spec}(A)$. As we can see, the base space is the same used by

Dubuc and Poveda except for the topology that is the dual, while stalks are local as in Filipoiu and Georgescu representation. This allowed us to obtain

- a representation of all MV-algebras as MV-algebras of all global sections of a sheaf of local MV-algebras on $\mathrm{Spec}(A)$;
- a representation of MV-algebras with $\mathrm{Min}(A)$ compact as MV-algebra of all global sections of a Hausdorff sheaf of MV-chains on $\mathrm{Min}(A)$, that is a Stone space;
- an adjunction between the category of all MV-algebras and the category of *local* T_1 *MV-algebraic spaces*.

It is worth to stress that, in general, MV-algebraic spaces are objects that are geometric in nature. Our approach provides a new class of objects, inherently geometric, which corresponds to MV-algebras. In this way we get an enrichment of the repertory of *geometric objects* which are deeply connected with MV-algebras.

Since more general stalks can represent even more tight classes, we have been drawn to study what happens when in Filipoiu and Georgescu representation the stalks are fixed in a particular class of MV-algebras. In this way, we have provided a partial classification of the variety of MV-algebras depending on stalks.

We have obtained the following results:

- each divisible MV-algebra is isomorphic to the MV-algebra of all global sections of a sheaf of local and divisible MV-algebras with a compact Hausdorff base space;
- each regular MV-algebra is isomorphic to the MV-algebra of all global sections of a sheaf of MV-chains with a Stone base space.

Using Filipoiu and Georgescu duality these results provide dualities with corresponding MV-algebraic spaces.

Trying to find other representations for MV-algebras, we have looked for their connections with other algebraic structures. Recently in [27], A. Di Nola and B. Gerla introduced the notion of MV-semirings, i.e. to each MV-algebra $(A, \oplus, \odot, *, 0, 1)$, we can associate a coupled semiring $\mathcal{A} = (\mathcal{R}_{\mathcal{A}}^{\vee}, \mathcal{R}_{\mathcal{A}}^{\wedge}, *)$, where $\mathcal{R}_{\mathcal{A}}^{\vee} = (A, \vee, 0, \odot, 1)$ and $\mathcal{R}_{\mathcal{A}}^{\wedge} = (A, \wedge, 1, \oplus, 0)$ and viceversa. In [35], B. Gerla shows how semirings can be used to describe, in terms of universal algebra, the connection between MV-algebras and l-groups given by Mundici's Γ functor (see Theorem 3.8, [35]). Moreover, the author develops the notion of BL-automaton, a $K - \Sigma$ automaton where K is the semiring reduct of a BL-algebra. Since MV-algebras can be defined as a subvariety of BL-algebras, B. Gerla introduces the notion of MV-automaton and proves that by MV-automata, it is possible to associate to each MV-algebra A the coupled semiring of the sets of all recognizable $\mathcal{R}_{\mathcal{A}}^{\vee}$-subsets and $\mathcal{R}_{\mathcal{A}}^{\wedge}$-subsets (see Theorem 4.4, [35]). In [4], the authors gave an alternative and equivalent definition of MV-semirings as commutative additively idempotent semirings such that for each element there exists the residuum with respect to 0.

In this thesis it is shown that MV-algebras and MV-semirings are isomorphic as categories. The next step have been to study the spectrum both of an MV-semiring and its semiring reduct, finding out connections with the spectrum of the associated MV-algebra. Indeed, although a semiring ideal is not an MV-ideal, we managed to establish a correspondence between an MV-semiring's prime spectrum and the associated MV-algebra's one: an MV-semiring's prime spectrum topologized with Zariski topology is homeomorphic to the associated MV-algebra's one topologized with coZariski topology.

The link between MV-algebras and semirings allows us to keep the inspiration and use new tools from semiring theory to analyze the class of MV-algebras. Indeed, we have used a sheaf representation for commutative semirings provided by Chermnykh ([17]) and specialized it in the case of MV-semirings. Actually, this representation is in analogy to the sheaf representation given by Grothendieck for rings. The base space is the prime spectrum of the semiring and the stalks are its localizations over prime ideals. In MV-semirings case, the stalks don't preserve the residua. Indeed, the localization of an MV-semiring over a prime ideal is a commutative additively idempotent and local semiring. However, the information about the residuum is preserved in some manner because the semiring of all global sections is an MV-semiring isomorphic to the represented MV-semiring. This seems to be surprising but we notice that the representation is in the category of commutative additively idempotent semirings and, gluing in a suitable manner, commutative additively idempotent and local semirings, MV-semirings are obtained.

From this sheaf representation of MV-semirings, using the categorical equivalence, we have obtained a representation of MV-algebras as MV-algebras of global sections of the Grothendieck sheaf of the associated semiring reduct.

Structure of the Work

The thesis is organized in five chapters.

Chapter 1 In this chapter we recall some definitions and properties which are involved in the theory developed.

Chapter 2 This chapter is dedicated to an overview of sheaves and sheaf representations. In particular, here we recall Davey's sheaf construction from subdirect products.

Chapter 3 This chapter contains results about the categorical isomorphism between MV-algebras and MV-semirings. Moreover we develop an ideals theory for MV-semirings.

Chapter 4 In this chapter, we present three different sheaf representations for MV-algebras. Moreover, using Chermnykh sheaf representation for commutative semirings, we present a sheaf representation for MV-semirings. Lastly, we provide an application of sheaf representation which allows us to obtain one of the possible embeddings in Di Nola's representation theorem for MV-algebras.

Chapter 5 In this chapter, we develop the MV-algebraic spaces theory. We recall the results obtained by Filipoiu and Georgescu firstly and then the results obtained by Dubuc and Poveda in this context. Moreover, we present an adjunction between the category of MV-algebras and the category of local T_1 MV-algebraic spaces. Lastly, using Filipoiu and Georgescu representation we provide representation for several subcategories of MV-algebras.

Chapter 1

Łukasiewicz logic and MV-algebras

In this chapter, we shall give some preliminary notions and results which will be useful in the sequel.
Many-valued logics arose from the need to overcome limits of classical logic about the formalization of uncertain events and vague concepts. They consist of logical systems for which the Bivalence Principle doesn't hold, that is, each sentence in these logics can assume a truth valued different from True or False.
The landscape of many-valued logics is variegated and finds applications in several fields such as Artificial Intelligence, Linguistic and Hardware Design.
In this thesis, we refer to ∞-valued logic developed by the Polish logician and philosopher Łukasiewicz in the early Twenties of the last century.

1.1 The syntax of $\mathcal{L}_\infty$

The *language* of the propositional calculus $\mathcal{L}_\infty$ consists of:

- countably infinite many propositional variables: $v_1, \ldots, v_n, \ldots$,

- logical connectives: $\to$ and $\neg$,

- parenthesis: (and).

In the sequel, V will denote the set of all the propositional variables.

The *formulas* are defined inductively as follows:

(f1) every propositional variable is a formula,

(f2) if φ is a formula then $\neg\varphi$ is a formula,

(f3) if φ and ψ are formulas then $\varphi \to \psi$ is a formula,

(f4) a string of symbols is a formula of $\mathcal{L}_\infty$ if and only if it can be shown to be a formula by a finite number of applications of **(f1)**, **(f2)**, and **(f3)**.

We will denote by $Form$ the set of all formulas of $\mathcal{L}_\infty$.

The particular four *axiom schemata* of this propositional calculus are:

(A1) $\varphi \to (\psi \to \varphi)$,

(A2) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$,

(A3) $(\varphi \to \psi) \to \psi) \to (\psi \to \varphi) \to \varphi)$,

(A4) $(\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$.

The *deduction rule* is *Modus Ponens*:

(MP) if φ and $\varphi \to \psi$ then ψ.

Definition 1.1.1. Let Θ be a set of formulas and φ a formula. A Θ-*proof* for φ is a finite sequence of formulas $\varphi_1, \ldots, \varphi_n = \varphi$ such that, for any $i \in \{1, \ldots, n\}$, one of the following conditions hold:

(c1) φ_i is an axiom,

(c2) $\varphi_i \in \Theta$,

(c3) there are $j,\ k < i$ such that φ_k is $\varphi_j \to \varphi_i$, that is the formula φ_i follows from φ_j and φ_k using Modus Ponens.

We will say that φ is a *syntactic consequence* of Θ or φ is *provable* from Θ if there exists a Θ-proof for φ. We will denote by $\Theta \vdash \varphi$. The set of all the syntactic consequences of Θ will be denoted by $Theor(\Theta)$.

A formula φ will be called a *theorem* or *provable formula* if it is provable from the empty set. This will be denoted by $\vdash \varphi$. In this case, a proof for φ will be a sequence of formulas $\varphi_1, \ldots, \varphi_n = \varphi$ such that for any $i \in \{1, \ldots, n\}$, one of the above conditions (c1) or (c3) is satisfied.

The set of all the theorems will be denoted by $Theor$.

Theorem 1.1.2. (Syntactic compactness)
If Θ is a set of formulas and φ is a formula such that $\Theta \vdash \varphi$, then $\Gamma \vdash \varphi$ for some finite subset $\Gamma \subseteq \Theta$.

We define other logical connectives as follows:

$$\varphi \oplus \psi := \neg\varphi \to \psi,$$

$$\varphi \odot \psi := \neg(\varphi \to \neg\psi),$$

$$\varphi \vee \psi := (\varphi \to \psi) \to \psi),$$

$$\varphi \wedge \psi := \varphi \odot (\varphi \to \psi),$$

$$\varphi^n := \underbrace{\varphi \odot \cdots \odot \varphi}_{n} \text{ for any } n \geq 1,$$

$$n\varphi := \underbrace{\varphi \oplus \cdots \oplus \varphi}_{n} \text{ for any } n \geq 1.$$

Theorem 1.1.3. (Deduction Theorem)
If $\Theta \subseteq Form$ and φ, $\psi \in Form$ then

$$\Theta \cup \{\varphi\} \vdash \psi \textit{ if and only if there is } n \geq 1 \textit{ such that } \Theta \vdash \varphi^n \to \psi.$$

1.2 MV-algebras

In literature, many-valued logics have been studied from two different and complementary points of view. Firstly, these logical systems are studied only from a propositional point of view charactering the behaviour of connectives by truth tables which generalize truth tables in classical logic. Lastly, opportune algebraic structures are introduced to evaluate formulas and semantic notions are defined to obtain correctness and completeness theorems and other metatheoretic results about formal theories [51].
This is the case of Łukasiewicz logic. Indeed, in order to prove the completeness theorem of Łukasiewicz infinite-valued logic, Chang introduced *MV-algebras* in [14]. Hence, defining the MV-algebras, Chang chose $\oplus$, $\odot$ and * as primary operations.
A main tool in Chang's proof of the completeness theorem was the bijective correspondence between the linearly ordered MV-algebras and the linearly ordered lattice ordered abelian groups with strong unit.

Below we give a simplified definition for MV-algebras, which is due to Mangani [47]. A standard reference is [18].

Definition 1.2.1. An *MV-algebra* is a structure $(A, \oplus, ^*, 0)$, where $\oplus$ is a binary operation, * is a unary operation and 0 is a constant such that the following axioms are satisfied for any a, $b \in A$:

MV1) $(A, \oplus, 0)$ is an abelian monoid,

MV2) $(a^*)^* = a$,

MV3) $0^* \oplus a = 0^*$

MV4) $(a^* \oplus b)^* \oplus b = (b^* \oplus a)^* \oplus a$.

In order to simplify the notation, an MV-algebra $(A, \oplus, ^*, 0)$ will be referred by its support set, A. An MV-algebra is *trivial* if its support is a singleton. On an MV-algebra A we define the constant 1 and the auxiliary operation $\odot$ as follows: On each MV-algebra A we define the constant 1 and the operation $\odot$ as follows:

i) $1 := 0^*$

ii) $a \odot b := (a^* \oplus b^*)^*$

for any $a, b \in A$. We shall also use the notation $a^{**} := (a^*)^*$. Unless otherwise specified by parentheses, the order of evaluation of these operations is first *, then $\odot$, and finally $\oplus$.

As proved by Chang, Boolean algebras are MV-algebras with the additional equation $a \oplus a = a$ (idempotency). In particular, the set $B(A)$ of all idempotent elements of an MV-algebra A is the largest Boolean algebra contained in A and is called the *boolean center* of A.

Notation. Let A be an MV-algebra, $a \in A$ and $n \in \omega$, where ω denotes the set of all the natural numbers. We introduce the following notations:

$$\begin{aligned} &0a = 0, \qquad na = a \oplus (n-1)a \text{ for any } n \geq 1, \\ &a^0 = 1, \qquad a^n = a \odot (a^{n-1}) \text{ for any } n \geq 1. \end{aligned}$$

We say that the element a *has order* n and we write $\mathrm{ord}(a) = n$, if n is the least natural number such that $na = 1$. We say that the element a *has a finite order*, and we write $\mathrm{ord}(a) < \infty$, if a has order n for some $n \in \omega$. If no such n exists, we say that a *has infinite order* and we write $\mathrm{ord}(a) = \infty$.

Definition 1.2.2. Let $(A, \oplus, ^*, 0)$ be an MV-algebra and $B \subseteq A$ such that the following conditions are satisfied:

(S1) $0 \in B$,

(S2) if $a, b \in B$ then $a \oplus b \in B$,

(S3) if $a \in B$ then $a^* \in B$.

Thus, if we consider the restriction of $\oplus$ and * to B, we get an MV-algebra $(B, \oplus, ^*, 0)$ which is an *MV-subalgebra* (or, simply, *subalgebra*) of the MV-algebra A.

If $S \subseteq A$ is a subset of A, then we shall denote by $\langle S \rangle$ the least subalgebra of A which includes S and it will be called *the subalgebra generated by S in A*. We will say that S is a *system of generators* for $\langle S \rangle$.

Example 1.2.3. In any MV-algebra A the subalgebra generated by the empty set is $\langle \emptyset \rangle = \{0, 1\}$.

Any MV-algebra A is equipped with the order relation

$$a \leq b \qquad \text{if and only if} \qquad a^* \oplus b = 1.$$

We introduce two auxiliary operations, $\vee$ and $\wedge$, by setting

$$a \vee b := a \oplus b \odot a^* = b \oplus a \odot b^* \text{ and } a \wedge b := a \odot (b \oplus a^*) = b \odot (a \oplus b^*).$$

Proposition 1.2.4. *The partially ordered set $(A, \leq)$ is a bounded lattice such that 0 is the first element, 1 is the last element and*

$$l.u.b\{a, b\} = a \vee b, \quad g.l.b\{a, b\} = a \wedge b,$$

for any $a, b \in A$.

We shall denote $L(A) = (A, \vee, \wedge, 0, 1)$, the lattice structure of A. We call $L(A)$ *the lattice reduct of* A.

Definition 1.2.5. An MV-algebra A is *complete* (σ-complete) if the lattice reduct of A is a complete (σ-complete) lattice.

Lemma 1.2.6 ([31])**.** *Let A be an MV-algebra and $x, y \in A$. Then*

$$x \wedge (y \oplus z) \leq (x \wedge y) \oplus (x \wedge z). \tag{1.1}$$

Proof. Since every MV-algebra is a subdirect product of MV-chains (Theorem 1.3.3 of [18]) we can assume that A is a chain. So

- if $x \leq y \leq z$, (1.1) becomes $x \leq 2x$;
- if $y \leq x \leq z$, (1.1) becomes $x \leq y \oplus x$;

- if $y \leq z \leq x \leq y \oplus z$, (1.1) becomes $x \leq y \oplus z$;
- if $y \oplus z \leq x$, (1.1) becomes $y \oplus z \leq y \oplus z$.

$\square$

Lemma 1.2.7 ([31]). *Let A be an MV-algebra. Then for every $a, b \in A$, $a \odot b^* \oplus (a \wedge b) = a$.*

Proof. $a \odot b^* \oplus (a \wedge b) = a \odot b^* \oplus a(a \odot b^*)^* = a \odot b^* \vee a = a.$ $\square$

In an MV-algebra $(A, \oplus, ^*, 0)$ we define *the distance function* $d : A \times A \to A$ by

$$d(a, b) := (a \odot b^*) \oplus (b \odot a^*).$$

1.3 Examples of MV-algebras

Example of MV-algebras are given by the following.

Example 1.3.1. (Boolean Algebras)
Any Boolean algebra is an MV-algebra in which the operations $\oplus$ and $\vee$ coincide.

Example 1.3.2. (The Lindenbaum-Tarski algebra Ł)
On $Form$, the set of all formulas of $\mathcal{L}_\infty$, we define the equivalence relation $\equiv$ by

$$\varphi \equiv \psi \text{ iff } \vdash \varphi \to \psi \text{ and } \vdash \psi \to \varphi.$$

Let us denote by $[\varphi]$ the equivalence class of the formula φ determined by $\equiv$ and by Ł the set of all the equivalence classes. On Ł we can define the following operations

$$[\varphi] \oplus [\psi] := [\neg\varphi \to \psi] \text{ and } [\varphi]^* := [\neg\varphi].$$

If we also define $0 := [\varphi]$ iff $\vdash \neg\varphi$, it's not difficult to prove that the structure $(\text{Ł}, \oplus, ^*, 0)$ is an MV-algebra.

Example 1.3.3. ($[0, 1]$, $Q \cap [0, 1]$, L_{n+1})
Let R denote the set of real numbers and let Q denote the set of rational numbers. For any $n \in \omega$, $n \geq 1$ we define $L_{n+1} = \{0, 1/n, \dots, (n-1)/n, 1\}$. If a and b are real numbers we define

$$a \oplus b := \min(a + b, 1), \text{ and } a^* := 1 - a.$$

One can easily see that the unit interval $[0, 1]$, the set $Q \cap [0, 1]$ and the set L_{n+1} with $n \geq 1$ are closed under the above defined operations. In particular, it results that $([0, 1], \oplus, ^*, 0)$, $(Q \cap [0, 1], \oplus, ^*, 0)$ and $(L_{n+1}, \oplus, ^*, 0)$ are MV-algebras, which will be simply denoted by $[0, 1]$, $Q \cap [0, 1]$ and L_{n+1} respectively.
If $n = 1$ then $L_{n+1} = L_2 = \{0, 1\}$, the Boolean Algebra with two elements. Moreover, the auxiliary operation $\odot$ is given by $a \odot b = \max(a + b - 1, 0)$ and the order is the natural order of the real numbers.

Example 1.3.4. (A^X, $[0, 1]^X$)
Let $(A, \oplus, ^*, 0)$ be an MV-algebra and X a nonempty set. The set A^X of all the functions $f : X \to A$ becomes an MV-algebra with the pointwise operations, i.e., if f, $g \in A^X$ then $(f \oplus g)(x) := f(x) \oplus g(x)$, $f^*(x) := f(x)^*$ for any $x \in X$ and $\mathbf{0}$ is the constant function associated with $0 \in A$.
Very important is the MV-algebra $[0, 1]^X$, where $[0, 1]$ is the MV-algebra defined in Example 1.3.3. Indeed, an element $f \in [0, 1]^X$ is called *fuzzy subset of* X and, for any $x \in X$, $f(x)$ represents the *degree of membership* of x to f. The subalgebras of $[0, 1]^X$ are called *bold algebras of fuzzy sets.*

Example 1.3.5. ($C(X)$)
Let X be a topological space and consider $[0, 1]$ the unit real interval equipped with the natural topology. We consider

$$C(X) = \{f : X \to [0, 1] : f \text{ is continuous}\}.$$

One can easily see that $C(X)$ is a subset of the MV-algebra $[0, 1]^X$ of the previous Example closed under the MV-algebra operations defined pointwise.

Thus, if $f,\ g \in C(X)$ then $f \oplus g$ and $f^* \in C(X)$ where $(f \oplus g)(x) = \min(f(x) + g(x), 1)$ and $f^*(x) = 1 - f(x)$ for any $x \in X$. We obtain the MV-algebra $(C(X), \oplus, ^*, \mathbf{0})$, where $\mathbf{0}$ is the constant function associated with $0 \in [0, 1]$.

Example 1.3.6. (Chang's MV-algebra **C**)
Let $\{c, 0, 1, +, -\}$ be a set of formal symbols. For any $n \in \omega$ we define the following abbreviations:

$$nc := \begin{cases} 0 & \text{if } n = 0, \\ c & \text{if } n = 1, \\ c + (n-1)c & \text{if } n > 1. \end{cases},$$

$$1 - nc := \begin{cases} 1 & \text{if } n = 0, \\ 1 - c & \text{if } n = 1, \\ 1 - (n-1)c - c & \text{if } n > 1. \end{cases}$$

We consider $\mathrm{C} = \{nc : n \in \omega\} \cup \{1 - nc : n \in \omega\}$ and we define the MV-algebra operations as follows:
($\oplus$1) if $x = nc$ and $y = mc$ then $x \oplus y := (m+n)c$,
($\oplus$2) if $x = 1 - nc$ and $y = 1 - mc$ then $x \oplus y := 1$,
($\oplus$3) if $x = nc$ and $y = 1 - mc$ and $m \leq n$ then $x \oplus y := 1$,
($\oplus$4) if $x = nc$ and $y = 1 - mc$ and $n < m$ then $x \oplus y := 1 - (m-n)c$,
($\oplus$5) if $x = 1 - mc$ and $y = nc$ and $m \leq n$ then $x \oplus y := 1$,
($\oplus$6) if $x = 1 - mc$ and $y = nc$ and $n < m$ then $x \oplus y := 1 - (m-n)c$,
(*1) if $x = nc$ then $x^* := 1 - nc$,
(*2) if $x = 1 - nc$ then $x^* := nc$.
Hence, the structure $(\mathrm{C}, \oplus, ^*, 0)$ is an MV-algebra, which is called *Chang's algebra* since it was defined by C.C. Chang [14]. The order relation is defined by:

$$x \leq y \text{ iff } \begin{cases} x = nc \text{ and} y = 1 - mc \text{ or} \\ x = nc \text{ and } y = mc \text{ and } n \leq m \text{ or} \\ x = 1 - nc \text{ and } y = 1 - mc \text{ and } m \leq n. \end{cases}$$

In conclusion, C is a linearly ordered MV-algebra:

$$0,\ c,\ \ldots,\ nc,\ \ldots,\ 1-nc,\ldots,\ 1-c,\ 1.$$

Example 1.3.7. ($^*[0,1]$)
Let R be the set of the real numbers, $\mathcal{P}(\omega)$ the Boolean algebra of all the subsets of ω and $\mathcal{F} \subseteq \mathcal{P}(\omega)$ the ultrafilter which contains all the cofinite subsets (i.e., the sets with finite complements). We denote $^*R := R^\omega/\mathcal{F}$, the ultrapower of R in the class of ℓ-groups. The elements of $^*R$ are called *nonstandard reals*. We shall briefly describe the structure of $^*R$. If $f, g : \omega \to R$ are two elements from R^ω we define $f \sim g$ iff $\{n \in \omega : f(n) = g(n)\} \in \mathcal{F}$. One can easily prove that $\sim$ is an equivalence, so we consider $^*R = R^\omega/\mathcal{F} = \{[f] : f \in R^\omega\}$ the set of all the equivalence classes with respect to $\sim$. If we define $[f]+[g] := [f+g]$ and $[f] \leq [g]$ iff $\{n \in \omega : f(n) \leq g(n)\} \in \mathcal{F}$ then $^*R$ becomes an ℓ-group. Moreover, since $\mathcal{F}$ is an ultrafilter, $^*R$ is linearly ordered. A *real* element of $^*R$ is an element of the form $[r]$ where r is a constant function R^ω. An *infinitesimal* is an element $\tau \in^* R$ such that $|\tau| \leq [1/n]$ for any $n \in \omega$, where $|\tau| = \max(\tau, -\tau)$ is the absolute value of τ. For example, if $t : \omega \to R$ by $t(0) = 0$ and $t(n) = 1/n$ for $n > 0$ then $\tau = [t]$ is an infinitesimal in $^*R$. Results from nonstandard analysis shows that any nonstandard real has one of the forms $[r]+\tau$ or $[r]-\tau$ where $[r]$ is a real and τ is an infinitesimal. Now, we consider the interval $^*[0,1] = \{[f] \in^* R : [0] \leq [f] \leq [1]\}$ and we define the operations

$$[f] \oplus [g] := \max([f]+[g],[1]) \text{ and } [f]^* := [1]-[f]$$

for any $[f]$, $[g] \in^* [0,1]$. As in Example 1.3.3, $(^*[0,1], \oplus, ^*, [0])$ is an MV-algebra.

1.4 Ideals and homomorphisms in MV-algebras

In this section, we introduce the notions of MV-ideals and of MV-homomorphisms and we provide some basic results that will be useful in the sequel.

Definition 1.4.1. Let A be an MV-algebra. A nonempty set $I \subseteq A$ is an *MV-ideal* if the following properties are satisfied

(I1) $a \leq b$ and $b \in I$ implies $a \in I$,

(I2) $a, b \in I$ implies $a \oplus b \in I$.

We shall denote by $\mathrm{Id}(A)$ the set of all the ideals of A.
An ideal is *proper* if it doesn't coincide with the entire algebra.

Remark 1.4.2. One can immediately prove the following:

i) $0 \in I$ for any ideal I of A,

ii) an ideal I is proper iff $1 \notin I$,

iii) if I is an ideal and $a, b \in I$ then $a \wedge b$, $a \odot b$, $a \vee b$ and $a \oplus b \in I$.

Example 1.4.3. If A is an MV-algebra, it trivially results that $\{0,\}$ and A are MV-ideals.

Example 1.4.4. (The ideals of [0,1])
Let $[0,1]$ be the MV-algebra from Example 1.3.3 and $I \subseteq [0,1]$ an MV-ideal. Suppose that there is $a \in I$ such that $a \neq 0$. It follows that there is $n \in \omega$ such that $\underbrace{a + \cdots + a}_{n \text{ times}} \geq 1$, where $+$ denotes the real numbers addition. We get $na = 1$, so $\mathrm{ord}(a) < \infty$. Since I is an MV-ideal, then $na = 1 \in I$ and $I = [0,1]$. We conclude that $\mathrm{Id}([0,1]) = \{\{0\}, [0,1]\}$.

Let A be an MV-algebra and I be an ideal of A. One can prove that the MV-subalgebra generated by I in A is $\langle I \rangle = I \cup I^*$, where $I^* = \{x^* : x \in I\}$.

Definition 1.4.5. Let S be a subset of A. We shall denote by $(S]$ the *MV-ideal generated by* S, i.e. the smallest ideal that includes S.
If $a \in A$ then the ideal generated by $\{a\}$ will be simply denoted $(a]$. An ideal I is called *principal* if there is $a \in A$ such that $I = (a]$.

It results that $(S] = \{a \in A : a \leq x_1 \oplus \cdots \oplus x_n$ with $n \in \omega$ and $x_1, \ldots, x_n \in S\}$. In particular, when $S = \{a\}$, it results that $(a] = \{x \in A : x \leq na$ for some $n \in \omega\}$.
In the sequel the MV-ideal generated by $I \cup J$ will be indicated by $I \oplus J$ and it results that $I \oplus J = \{x \leq a \oplus b \mid$ for some $x \in I,\ y \in J\}$.

Definition 1.4.6. If A and B are two MV-algebras, then an *MV-homomorphism* is a function $f : A \to B$ which satisfies the following conditions:

(M1) $f(0) = 0$,

(M2) $f(a \oplus b) = f(a) \oplus f(b)$ for any a, $b \in A$,

(M3) $f(a^*) = f(a)^*$ for any $a \in A$.

We shall denote by $\mathrm{Hom}(A, B)$ the set of all MV-homomorphisms $f : A \to B$.

Remark 1.4.7. Let $f : A \to B$ be an MV-homomorphism. One can immediately prove that:

(a) $f(1) = 1$,

(b) $f(a \odot b) = f(a) \odot f(b)$,

(c) $f(a \vee b) = f(a) \vee f(b)$,

(d) $f(a \wedge b) = f(a) \wedge f(b)$,

(e) $f(a \to b) = f(a) \to f(b)$,

for any a, $b \in A$. Thus, f is also a lattices homomorphism from $L(A)$ to $L(B)$. In particular, f is an increasing function.

Example 1.4.8. Let ${}^*[0,1]$ be the MV-algebra defined in Example 1.3.7 and $\tau \in {}^*[0,1]$ an infinitesimal. One can easily prove that the set $\{n\tau : n \in \omega\} \cup \{[1] - n\tau : n \in \omega\}$ is an MV-subalgebra of ${}^*[0,1]$ which is isomorphic to Chang's MV-algebra C from Example 1.3.6.

If $f : A \to B$ is an MV-homomorphism then the *kernel* of f is $\mathrm{Ker}(f) = f^{-1}(0) = \{a \in A : f(a) = 0\}$.

Proposition 1.4.9. *If $f : A \to B$ is an MV-homomorphism. The following assertions hold:*

(a) $\mathrm{Ker}(f)$ *is a proper ideal of A,*

(b) f is injective iff $\mathrm{Ker}(f) = \{0\}$,

(c) if $J \subseteq B$ is an ideal then $f^{-1}(J)$ is an ideal of A and $\mathrm{Ker}(f) \subseteq f^{-1}(J)$,

(d) if f is surjective and $I \subseteq A$ is an ideal such that $\mathrm{Ker}(f) \subseteq I$*, then $f(I)$ is an ideal of B.*

Now, we give the notion of MV-congruence which is strongly connected with the notion of quotient MV-algebras. It's worth to stress here that there is a bijective correspondence between the lattice of all the congruences defined on an MV-algebra A and the lattice of all the ideals of S.

Lemma 1.4.10. *If I is an MV-ideal then the relation $\equiv_I$ defined by*

$$a \equiv_I b \text{ iff } d(a,b) \in I$$

is a congruence on A.
Viceversa, if $\equiv$ is a congruence on A then the set

$$I_{\equiv} = \{a \in A : a \equiv 0\}$$

is an MV-ideal.

If I is an MV-ideal of A and $a \in A$ we shall denote by $[a]_I$ the congruence class of a with respect to $\equiv_I$, i.e. $[a]_I = \{b \in A : a \equiv_I b\}$.
One can easily see that $a \in I$ iff $[a]_I = [0]_I$. We shall denote by $A/I = \{[a]_I : a \in A\}$ the set of all the congruence classes determined by $\equiv_I$. Since $\equiv_I$ is a congruence relation, the MV-algebra operations on A/I given by

$$[a]_I \oplus [b]_I := [a \oplus b]_I,$$
$$([a]_I)^* := [a^*]_I,$$

are well defined. Hence, $(A/I, \oplus, ^*, [0]_I)$ is an MV-algebra which is called *the quotient of* A *by* I. The function $\pi_I : A \to A/I$ defined by $\pi_I(a) = [a]_I$ for any $a \in A$ is a surjective homomorphism, which is called *the canonical projection from* A *to* A/I. One can easily prove that $\mathrm{Ker}(\pi_I) = I$.

Theorem 1.4.11 (An extension of the Chinese remainder theorem)**.** *[[31]] Let* $I_1, I_2, \ldots, I_n$ *ideals of an MV-algebra* A *and* $a_1, a_2, \ldots, a_n$ *elements of* A *such that* $a_i \equiv a_j$ $(I_i \oplus I_j)$ *for* $i, j = 1, 2, \ldots n$. *Then there exists* $a \in A$ *such that* $a \equiv a_i$ (I_i) *for* $i = 1, 2, \ldots n$.

Proof. We will prove the theorem by induction. For $i = 1$ the theorem is true. Suppose the thesis true for $n-1$. Then there is $b \in A$ such that $b \equiv a_i$ (I_i), for $i = 1, 2, \ldots, n-1$. Since, by hypothesis $a_i \equiv a_n$ $(I_i \oplus I_n)$ for every $i = 1, 2, \ldots, n-1$, by transitivity,

$$b \equiv a_n \ \left(\bigcap_{i=1}^{n-1} (I_i \oplus I_n) \right). \tag{1.2}$$

Set $\bigcap_{i=1}^{n-1} I_i = J$, then

$$\bigcap_{i=1}^{n-1} (I_i \oplus I_n) = \bigcap_{i=1}^{n-1} I_i \oplus I_n = J \oplus I_n.$$

From (1.2), we have that $b \odot a_n^*,\ b^* \odot a_n \in J \oplus I_n$, that is $b \odot a_n^* \leq c \oplus c_n$ and $b^* \odot a_n \leq d \oplus d_n$, where $c, d \in J$ and $c_n, d_n \in I_n$. We can assume $c, c_n \leq b \odot a_n^*$ and $d, d_n \leq b^* \odot a_n$.

Set

$$a = b \odot a_n^* \odot c^* \oplus b^* \odot a_n \odot d_n^* \oplus (b \wedge a_n),$$

where

$$b^* \odot a_n \odot d_n^* \leq (d \oplus d_n) d_n^* = d \wedge d_n^* \in J \tag{1.3}$$

and

$$b \odot a_n^* \odot c^* \leq c^* \wedge c_n \in I_n. \tag{1.4}$$

We wish to prove that $a \equiv a_i(I_i)$ for $i = 1, 2, \ldots n$. For $i = n$, by (1.4) and Lemma 1.2.7, we get:

$$\frac{a}{I_n} = \frac{b \odot a_n^* \odot c^*}{I_n} \oplus \frac{b^* \odot a_n}{I_n} \odot \frac{d_n^*}{I_n} \oplus \frac{(b \wedge a_n)}{I_n}$$

$$= \frac{b^* \odot a_n}{I_n} \oplus \frac{(b \wedge a_n)}{I_n} = \frac{b^* \odot a_n \oplus (b \wedge a_n)}{I_n} = \frac{a_n}{I_n}.$$

So $a \equiv a_n(I_n)$.

To prove the remainder cases, we begin to show that $a \equiv b(J)$.

Indeed, by (1.3) and Lemma 1.2.7:

$$\frac{a}{J} = \frac{b \odot a_n^*}{J} \odot \frac{c^*}{J} \oplus \frac{b^* \odot a_n \odot d_n^*}{J} \oplus \frac{(b \wedge a_n)}{J}$$

$$= \frac{b \odot a_n^*}{J} \oplus \frac{(b \wedge a_n)}{J} = \frac{b \odot a_n^* \oplus (b \wedge a_n)}{J} = \frac{b}{J}.$$

Now, $a \equiv b(J)$ implies $a \equiv b(I_i)$, for $i = 1, 2, \ldots, n-1$. By induction hypothesis and transitivity $a \equiv a_i(I_i)$, for $i = 1, 2, \ldots, n-1$. So the theorem is completely proved. □

1.5 Prime ideals

In the structures belonging to the algebra of logic (i.e., algebraic structures that correspond to some logical system), the prime ideals are involved at least in three important matters, in algebra, topology and logic. They are extensively used for proving the algebraic representation theorems, as well as the topological duality results. The duals of the prime ideals (i.e. the prime filters) models the deduction in the corresponding logical system and they are frequently used in the algebraic proofs of the completeness theorems.

Proposition 1.5.1. *If P is an MV-ideal of A, then the following properties are equivalent:*

(a) for any a, $b \in A$, $a \odot b^* \in P$ or $a^* \odot b \in P$,

(b) for any a, $b \in A$, if $a \wedge b \in P$ then $a \in P$ or $b \in P$,

(c) for any I, $J \in \mathrm{Id}(A)$, if $I \cap J \subseteq P$ then $I \subseteq P$ or $J \subseteq P$.

Definition 1.5.2. An ideal of A is *prime* if it is proper and it satisfies one of the equivalent conditions from Proposition 1.5.1. We shall denote by $\mathrm{Spec}(A)$ the set of all the prime ideals of A.

Remark 1.5.3. If I and P are ideals of A such that $I \subseteq P$, then one can easily prove that

$$P \in \mathrm{Spec}(A) \text{ iff } \pi_I(P) \in \mathrm{Spec}(A/I).$$

Thus, there is a bijective correspondence between the prime ideals of A containing I and the prime ideals of A/I.

Definition 1.5.4. An MV-ideal M of A is *maximal* if it is a maximal element in the partially ordered set of all the proper ideals of A. This means that M is proper and, for any proper ideal I, if $M \subseteq I$ then $M = I$. We shall denote by $\mathrm{Max}(A)$ the set of all the maximal ideals of A.

Proposition 1.5.5. *If M is a proper MV-ideal of A then the following are equivalent:*

(a) M is maximal,

(b) for any $a \in A$, if $a \notin M$ then there is $n \in \omega$ such that $(a^*)^n \in M$.

Remark 1.5.6. If I and M are ideals of A such that $I \subseteq M$, then one can easily prove that

$$M \in \mathrm{Max}(A) \text{ iff } \pi_I(M) \in \mathrm{Max}(A/I).$$

Thus, there is a bijective correspondence between the maximal ideals of A containing I and the maximal ideals of A/I.

Lemma 1.5.7. *Any maximal ideal of an MV-algebra is a prime ideal.*

Proposition 1.5.8. *Any proper ideal of A can be extended to a maximal ideal. Moreover, for any prime ideal of A there is a unique maximal ideal containing it.*

Definition 1.5.9. An ideal P of an MV-algebra A is called *primary* if P is proper and there is a unique maximal ideal containing it.

Corollary 1.5.10. *In an MV-algebra any prime ideal is a primary ideal.*

Definition 1.5.11. The intersection of the maximal ideals of A is called the *radical* of A. It will be denoted by $\mathrm{Rad}(A)$. It is obvious that $\mathrm{Rad}(A)$ is an ideal, since a intersection of ideals is also an ideal.

Lemma 1.5.12. *For any a, $b \in \mathrm{Rad}(A)$, the following identities hold:*

(a) $a \odot b = 0$,

(b) $a \oplus b = a + b$,

(c) $a \leq b^*$.

Definition 1.5.13. An element a of A is called *infinitesimal* if $a \neq 0$ and $na \leq a^*$ for any $n \in \omega$.

Proposition 1.5.14. *For any $a \in A$, $a \neq 0$, the following are equivalent:*

(a) a is infinitesimal,

(b) $a \in \mathrm{Rad}(A)$,

(c) $(na)^2 = 0$ *for every* $n \in \omega$.

Definition 1.5.15. For a nonempty subset $X \subseteq A$, the set

$$X^{\perp} = \{a \in A : a \wedge x = 0 \text{ for any } x \in X\}$$

is called the *polar* or the *annihilator* of X. If $a \in A$ then the polar of $\{a\}$ will be simply denoted by $a^{\perp}$.

It results

Proposition 1.5.16 ([2])**.** *For each $a \in A$, $a^{\perp}$ is an ideal of A.*

Definition 1.5.17. An ideal P of A is a *minimal prime ideal* if it is a minimal element in $\mathrm{Spec}(A)$ ordered by inclusion. This means that P is a prime ideal and, whenever Q is a prime ideal such that $Q \subseteq P$, we get $P = Q$. We shall denote by $\mathrm{Min}(A)$ the set of all the minimal prime ideals of A.

According to [54] we prove the following similar results.

Lemma 1.5.18 ([31]). *Let A be an MV-algebra and $P \in \mathrm{Spec}(A)$. Then the following are equivalent:*

(a) P is a minimal prime ideal;

(b) for any $a \in A$, $a \in P$ iff there is $b \in A \setminus P$ such that $a \wedge b = 0$;

(c) $P = \bigcup\{b^{\perp} : b \notin P\}$.

Proof. $(a) \Rightarrow (b)$ Let P be a minimal prime ideal of A and $a \in A$. If $a \in P$ then, from Theorem 6.1.5 of [18], there exists an element $b \in A \setminus P$ such that $a \wedge b = 0$. The other implication follows by the fact that P is prime and $b \notin P$.

$(b) \Rightarrow (c)$ For any $b \notin P$, we have that $b^{\perp} \subseteq P$. So $\bigcup\{b^{\perp} : b \notin P\} \subseteq P$. Suppose now that $a \in P$ so, by (b), there exists an element $b \notin P$ such that $a \wedge b = 0$, i.e. $a \in a^{\perp}$. Hence $a \in \cup\{b^{\perp} : b \notin P\}$.

$(c) \Rightarrow (a)$ We shall prove that $P \in \mathrm{Min}(A)$.

Let Q be a prime ideal such that $Q \subseteq P$ and $a \in P$. Then, by hypothesis, there exists $b \notin P$ such that $a \wedge b = 0 \in Q$. Since Q is prime, $a \in Q$. Hence $P = Q$ and P is a minimal prime ideal of A. □

Proposition 1.5.19 ([31]). *Let A be an MV-algebra and U is an ultrafilter of the lattice $L(A) = (A, \vee, \wedge, 0, 1)$ then $A \setminus U \in \mathrm{Min}(A)$.*

Proof. First, we prove that $A \setminus U$ is a prime ideal of A.

Let $x, y \in A \setminus U$. Since U is an ultrafilter and $x, y \notin U$, $< U, x >=< U, y >= L(A)$. So $0 \in< U, x >=< U, y >$, then there are $a, b \in U$ and $n, m \in \omega$ such that $0 = a \wedge nx = b \wedge my$. Let $c = a \wedge b \in U$ and $r = \min\{n, m\}$, we have that $c \wedge rx = (a \wedge b) \wedge rx \leq (a \wedge b) \wedge nx = 0$. So $c \wedge rx = 0$. Analogously, we have that $c \wedge ry = 0$. By Lemma 1.2.6, $0 = (c \wedge rx) \oplus (c \wedge ry) \geq c \wedge (rx \oplus ry) = c \wedge r(x \oplus y) \geq c \wedge (x \oplus y)$. Hence $c \wedge (x \oplus y) = 0$. Since $c \in U$, $x \oplus y \notin U$ otherwise $0 \in U$, that is impossible. So $x \oplus y \in A \setminus U$.

The other properties follow easily.

To prove that $A \setminus U \in \mathrm{Min}(A)$, we use Lemma 1.5.18, proving that

$$a \in A \setminus U \text{ iff there exists } b \in U \text{ such that } a \wedge b = 0.$$

Let $a \notin U$, since U is an ultrafilter $< U, a >= L(A)$. So there exist $b \in U$ and $n \in \omega$ such that $0 = b \wedge na \geq b \wedge a$. So $a \wedge b = 0$. The other implication follows from the fact that P is prime and $b \in U$. □

Analogous results for other ordered structures are given by Theorem B of [39] and by Theorem 2.3 of [48].

Lemma 1.5.20 ([31]). *Let I be an ideal of an MV-algebra A. Then*

$$\bigcap \{P \in \mathrm{Spec}(A) : P \not\supseteq I\} = I^{\perp}.$$

Proof. Assume that $a \in I^{\perp}$ and $P \not\supseteq I$. Since $P \not\supseteq I$, there is some $y \in I \setminus P$. From $a \wedge y = 0$, it follows that $a \in P$. By generality of P $a \in \bigcap \{P \in \mathrm{Spec}(A) : P \not\supseteq I\}$.

Assume now $a \in \bigcap \{P \in \mathrm{Spec}(A) : P \not\supseteq I\}$ and $a \wedge x \neq 0$, for some $x \in I$. Then, by Lemma 2 of [15], there is a prime ideal Q such that $a \wedge x \notin Q$. Since $x \notin Q$, $Q \in U(I)$ that implies, by hypothesis, $a \in Q$, that is absurd. Thus $a \wedge x = 0$ for any $x \in I$. □

Proposition 1.5.21 ([31]). *Let I be an ideal of an MV-algebra A, then*

$$\bigcap \{P \in \mathrm{Spec}(A) : P \not\supseteq I\} = \bigcap \{m \in \mathrm{Min}(A) : m \not\supseteq I\}.$$

Proof. The inclusion

$$\bigcap \{P \in \mathrm{Spec}(A : P \not\supseteq I\} \subseteq \bigcap \{m \in \mathrm{Min}(A) : m \in U(I)\}$$

is trivial. Let $x \in \bigcap \{m \in \mathrm{Min}(A) : m \in U(I)\}$, $P \not\supseteq I$, $m \in \mathrm{Min}(A$ and $m \subseteq P$. Since $m \in U(I)$, $x \in m \subseteq P$. Then $x \in \bigcap \{P \in \mathrm{Spec}(A) : P \not\supseteq I\}$. □

Corollary 1.5.22 ([31]). *Let I be an ideal of an MV-algebra A, then*

$$I^{\perp} = \bigcap \{m \in \mathrm{Min}(A) : m \not\supseteq I\}.$$

Proof. It follows from Proposition 1.5.21 and Lemma 1.5.20. □

For every $P \in \mathrm{Spec}(A)$, we define $O(P) = \bigcap\{m \in \mathrm{Min}(A) : m \subseteq P\}$. It follows that $O(P)$ is an ideal of A such that $O(P) \subseteq P$.

Generalizing Lemma 5 of [9], we get:

Proposition 1.5.23 ([31]). *For each $P \in \mathrm{Spec}(A)$, $O(P) = \bigcup\{a^\perp : a \notin P\}$.*

Proof. Let $x \in \bigcup\{a^\perp : a \notin P\}$, so there exists $a \notin P$ such that $x \wedge a = 0$. Since $a \notin P$, then for each $m \in \mathrm{Min}(A)$, $m \subseteq P$ and $a \notin m$. By primality and generality of m, $x \in \bigcap\{m \in \mathrm{Min}(A) : m \subseteq P\}$.

Let $x \notin \bigcup\{a^\perp : a \notin P\}$, then $x \wedge a > 0$ for all $a \notin P$. So the filter F of the lattice $L(A)$ generated by $(A \setminus P) \cup \{x \wedge a : a \notin P\}$ is proper and, since in a distributive lattice every proper filter can be embedded in an ultrafilter ([1]), there exists an ultrafilter U such that $F \subseteq U$. By Proposition 1.5.19, $Q = A \setminus U \in \mathrm{Min}(A)$. Moreover $Q \subseteq P$ and $x \geq x \wedge a$, then $x \in F \subseteq U$ and $x \notin Q$. Hence $x \notin \bigcap\{m \in \mathrm{Min}(A) : m \subseteq P\}$. □

Theorem 1.5.24 ([31]). *For each $P \in \mathrm{Spec}(A)$, $O(P)$ is a primary ideal.*

Proof. We shall prove that there is a unique maximal ideal enclosing $O(P)$.

Let M_P denote the unique maximal ideal such that $O(P) \subseteq P \subseteq M_P$. Suppose there exists a maximal ideal $M \neq M_P$ enclosing $O(P)$. Then there exist $a \in M$ and $b \in M_P$ such that $a \oplus b = 1$, that is $a^* \odot b^* = 0$. Let $m \in \mathrm{Min}(A)$ with $m \subseteq P$, by Proposition 2.1 of [7], $(a^*)^2 \in m$ or $(b^*)^2 \in m$. If $(b^*)^2 \in m$ then $b \oplus (b^*)^2 = b \vee b^* = 1 \in M_P$, absurd. So $(a^*)^2 \in m$, for all $m \in \mathrm{Min}(A)$ with $m \subseteq P$, that is $(a^*)^2 \in O(P) \subseteq M$. Then $a \oplus (a^*)^2 = a \vee a^* = 1 \in M$, again it is absurd. Hence $O(P)$ is primary. □

The following proposition is due to Filipoiu and Georgescu, but here we give an alternative proof.

Proposition 1.5.25. *Let A be an MV-algebra. It results that $\bigcap\{O(M) \mid M \in \mathrm{Max}(A)\} = \{0\}$.*

Proof. Let $a \in \bigcap\{O(M) \mid M \in \mathrm{Max}(A)\}$, so for each $M \in \mathrm{Max}(A)$, $a \in O(M) = \cap\{m \in \mathrm{Min}(A) \mid m \subseteq M\}$. From this it follows that $a \in m$, for each $m \in \mathrm{Min}(A)$, i.e. $a \in \bigcap\{m \mid m \in \mathrm{Min}(A)\} = \{0\}$. Thus $a = 0$. □

For each $I \in \mathrm{Id}(A)$, we define $V(I) = \{x \in A \mid I \oplus x^{\perp} = A\}$. It results that $V(I)$ is an ideal of A for each $I \in \mathrm{Id}(A)$.

Proposition 1.5.26 ([32]). *Let A be an MV-algebra. It results that*

i) $V(I) \oplus V(J) = V(I \oplus J)$, *for each* $I,\ J \in \mathrm{Id}(A)$,

ii) $O(M) = V(M)$, *for each* $M \in \mathrm{Max}(A)$.

1.6 Spectral topology

Let $(A, \oplus, ^*, 0)$ be an MV-algebra. In this section, we define in classical manner the spectral topology on $\mathrm{Spec}(A)$, $\mathrm{Max}(A)$ and $\mathrm{Min}(A)$.
For any $I \in Id(A)$ we define

$$U(I) : = \{P \in \mathrm{Spec}(A) : I \nsubseteq P\}.$$

If $\tau : = \{U(I) : I \in Id(A)\}$ it results that $(\mathrm{Spec}(A), \tau)$ is a topological space. In the sequel τ will be referred as the *spectral topology* or the *Zariski topology*. For any $a \in A$ we define $U(a) : = \{P \in \mathrm{Spec}(A) : a \notin P\}$. It results that the family $\{U(a) : a \in A\}$ is a basis for the topology τ on $\mathrm{Spec}(A)$ and that the compact open subsets of $\mathrm{Spec}(A)$ are exactly the sets of the form $U(a)$ for some $a \in A$. From this, it follows that $(\mathrm{Spec}(A), \tau)$ is a compact T_1 space.
For each $a \in A$, set $H(a) = \{P \in \mathrm{Spec}(A) \mid a \in O(P)\}$. We .

Proposition 1.6.1 ([31]). *Let A be an MV-algebra, for each $a \in A$, $H(a)$ is an open set of* $\mathrm{Spec}(A)$.

Proof. We shall prove that every element of $H(a)$ has a neighbourhood enclosed in $H(a)$. Let $P \in H(a)$, so $a \in O(P)$, by Proposition 1.5.23 there is an element $b \in A \setminus P$ such that $a \wedge b = 0$. Thus $P \in U(b)$ and $U(b) \subseteq H(a)$. Indeed let $Q \in U(b)$, then $b \notin Q$ and, since $a \wedge b = 0$, $a \in m$ for each $m \in \mathrm{Min}(A)$ such that $m \subseteq Q$. So $a \in O(Q)$, $Q \in H(a)$. □

Since Max(A), Min$(A) \subseteq$ Spec(A), we can endow Max(A) and Min(A) with the topology induced by the spectral topology τ on Spec(A). This means that the open sets of Max(A) are

$$S(I) = U(I) \cap \mathrm{Max}(A) = \{M \in \mathrm{Max}(A) : I \not\subseteq M\}$$

and the open sets of Min(A) are

$$D(I) = U(I) \cap \mathrm{Min}(A) = \{m \in \mathrm{Min}(A) : I \not\subseteq m\}.$$

It results that Max(A) with the topology induced by the Zariski topology on Spec(A) is a compact Hausdorff topological space while Min(A) with the topology induced by the Zariski topology on Spec(A) is a Hausdorff zero-dimensional space, i.e Min(A) is a Hausdorff space with a basis of clopen sets.
It's worth to stress that in general Min(A) is not a compact space. Indeed, we can consider the following example.

Example 1.6.2. [[8]] Let C the MV-algebra described in the Example 1.3.6. Let $\omega = \{1, 2, \ldots\}$ and let

$$A_1 = \{(x_n)_{n\in\omega} : x_n \in C \text{ for every } n \in \omega\}.$$

A_1 is an MV-algebra under pointwise operations. Let

$$A_2 = \{(x_n)_{n\in\omega} : \text{ for every } n, \mathrm{ord}(x_n) = \infty, \text{ or for every } n, \mathrm{ord}(x_n) < \infty\}.$$

A_2 is clearly a subalgebra of A_1. Let $I = \{(x_n)_{n\in\omega} : \{n : x_n \neq 0\} \text{ is finite }\}$, i.e. I is the set of sequences which are almost everywhere null. It results I is an ideal of A_2. Let A the subalgebra of A_2 generated by I, thus $A = I \cup I^*$. Let $P \in$ Spec(A). Suppose for each $n \in \omega$ there is an $x \in P$ with $x_n \neq 0$. Let $y \in I$ and let $k = k(y)$ be the largest integer n such that $y_n \neq 0$, so $y_n = 0$ for each $n > k$. For each $n \leq k$ choose $a^{(n)} \in P$ such that $a_n^{(n)} \neq 0$. For $n \leq k$, $y_n = m_n c$. Let $m = \max m_1, \ldots, m_k$ and let $a = m(a^{(1)} + \ldots + a^{(k)})$. Then $a \in P$ and clearly $y \leq a$, so $y \in P$, i.e. $I \subseteq P$. If $x \in I^*$, then $x^* \in I$,

so $\{n : x_n^* \neq 0\}$ is finite. Since $I \subseteq A_2$, $x_n^* \neq 0$ implies $x_n^* = m_n c$ for some c. Thus for each $n \in \omega$, $x_n = 1 - m_n c$. It follows that $\mathrm{ord}(x) < \infty$, so I is the unique maximal ideal of A, hence $P = I$.
If $P \neq I$, then we see that there is an $n_0 \in \omega$ such that $x_{n_0} = 0$ for all $x \in P$. Moreover, since P is prime, n_0 is unique. We claim that P is minimal. For suppose $Q \subseteq P$, Q is prime. Then $x_{n_0} = 0$ for all $x \in Q$ and n_0 is unique for Q as well. So for $n \neq n_0$ there is an $x \in Q$ with $x_n \neq 0$. Arguing as in the preceding paragraph, we obtain $Q = P$.
Thus $\mathrm{Spec}(A) = \mathrm{Min}(A) \cup \{I\}$. Moreover it is clear that $\cup\,\mathrm{Min}(A) = I$. Now $\mathrm{Min}(A) \subseteq U(I)$. Were $\mathrm{Min}(A)$ compact there would be an $x \in I$ with $x \notin P$ for any $P \in \mathrm{Min}(A)$. Since $\cup\,\mathrm{Min}(A) = I$, this is impossible, hence $\mathrm{Min}(A)$ is not compact. One can also show that $\mathrm{Min}(A)$ is an infinite discrete space and therefore not compact.

Recently, Dubuc and Poveda have studied (see [29]) the properties of $\mathrm{Spec}(A)$ topologized with the CoZariski topology. As basis for this topology, one considers the family $\{W_a \mid a \in A\}$, where $W_a = \{P \in \mathrm{Spec}(A) \mid a \in P\}$. In particular it results that $W_0 = \mathrm{Spec}(A)$, $W_1 = \emptyset$ and $W_a \cap W_b = W_{a\oplus b}$.

Proposition 1.6.3 (([29])). *The spectrum of any MV-algebra with the CoZariski topology is sober, compact and has a basis of compact open sets.*

In the sequel, $\mathrm{Spec}(A)$ will be the spectrum of A with the Zariski topology whereas $\mathrm{coSpec}(A)$ will the spectrum of A with the CoZariski topology.

1.7 Chang representation theorem and subclasses of MV-algebras

In this section we recall some results about subclasses of MV-algebras which will be useful in the sequel. In particular we recall Chang representation theorem of MV-algebras ass subdirect product of MV-chains, that is linearly ordered MV-algebras.

1.7.1 Chang representation theorem and MV-chains

Definition 1.7.1. An MV-algebra A is an *MV-chain* if and only if A is linearly ordered.

It is possible to give the following characterizations for MV-chains.

Proposition 1.7.2. *For any MV-algebra A the following are equivalent:*

(a) A is an MV-chain,

(b) any proper ideal of A is prime,

(c) $\{0\}$ is a prime ideal,

(d) $\mathrm{Spec}(A)$ *is linearly ordered.*

Proposition 1.7.3. *If A is an MV-algebra and I is a proper ideal of A then the following are equivalent:*

(a) I is a prime ideal,

(b) A/I is an MV-chain.

Remembering that the intersection of all prime ideals is $\{0\}$, from Proposition 1.7.3 it follows the Chang representation theorem.

Theorem 1.7.4. *Every MV-algebra is a subdirect product of MV-chains.*

Hence, the algebraic calculus can be reduced to linearly ordered structures. An identity holds in any MV-algebra if and only if it holds considering all the possible orderings for the variables involved.

1.7.2 Local MV-algebras

Definition 1.7.5. An MV-algebra A is *local* if and only if A has only one maximal ideal.

In particular, it results that if A is local, then
$Rad(A)$ is the only maximal ideal of A.

For local MV-algebras, the following characterizations hold.

Proposition 1.7.6. *For an MV-algebra A and a proper ideal P of A, the following are equivalent:*

(a) P is a primary ideal,

(b) A/P is a local MV-algebra.

Proposition 1.7.7. *For an MV-algebra A, the following are equivalent:*

(a) A is local,

(b) any proper ideal of A is primary,

(c) $\{0\}$ is a primary ideal,

(d) $\mathrm{Rad}(A)$ contains a primary ideal.

1.7.3 Rank of an MV-algebra

Given a local MV-algebra A, it is known that $A/\mathrm{Rad}(A)$ is simple. Hence it results natural to classify a local MV-algebras by fixing properties of their quotients by the radical. In [25], the authors introduced the following definitions.

Definition 1.7.8. Let n be a positive integer. Then a local MV-algebra A is said to be of *rank n* iff $A/\mathrm{Rad}(A) \cong Ł_{n+1}$. A local MV-algebra A is said to be of *finite rank* iff A is of rank n for some integer n.

Definition 1.7.9. Let A be an MV-algebra, J a primary ideal of A and n a positive integer. We say that J has *rank n* ($rank(J) = n$) if and only if $rank(A/J) = n$.

Definition 1.7.10. Let A be an MV-algebra and k a positive integer. We say that A is *of k-bounded rank* if and only if $rank(O(M)) \leq k$, for each $M \in \mathrm{Max}(A)$.

Proposition 1.7.11 (Lemma 9.5, [25])**.** *Let A be a local MV-algebra and n a positive integer. Then the following conditions are equivalent:*

(1) $x \wedge (x^*)^n \in \mathrm{Rad}(A)$*, for each* $x \in A$*;*

(2) $rank(A) \leq n$*.*

Proposition 1.7.12 (Proposition 10.8, [25])**.** *Let A be an MV-algebra. Then the following statements are equivalent:*

(1) A is of k-bounded rank;

(2) $(x^* \vee (kx))^n \notin O(M)$*, for all* $n \in \omega$*, for all* $x \in A$ *and for each* $M \in \mathrm{Max}(A)$*.*

1.7.4 Divisible MV-algebras

Definition 1.7.13. An MV-algebra A is said to be *divisible* if and only if for any $a \in A$ and for any natural number $n \geq 1$ there is $x \in A$ such that $nx = a$ and $a^* \oplus (n-1)x = x^*$.

It's worth to stress that divisible MV-algebras hold an important role in the proof of Chang's Completeness Theorem for MV-algebras. Moreover, the class of divisible MV-algebras is closed under quotient. Indeed, the following proposition holds.

Proposition 1.7.14. *[26] Let A be a divisible MV-algebra and I an ideal of A. Then A/I is also divisible.*

Proof. We have to prove that for each $a/I \in A/I$ and $n \geq 1$ there exists $x/I \in A/I$ such that $n(x/I) = a/I$ and $(a/I)^* \oplus (n-1)x/I = (x/I)^*$.

Let $a \in A$ which is divisible. So for each $n \geq 1$ there exists $x \in A$ such that

$$nx = a \tag{1.5}$$

$$a^* \oplus (n-1)x = x^* \tag{1.6}$$

Since $\equiv_I$ is a congruence for $\oplus$, $\odot$, * and by (1.5) and (1.6), the following equalities hold

$$n(x/I) = \underbrace{x/I \odot \ldots \odot x/I}_{n\ times} = \underbrace{(x \odot \ldots \odot x)}_{n\ times}/I = (nx)/I = a/I$$

$$(a/I)^* \oplus (n-1)x/I = a^*/I \oplus ((n-1)x)/I = (a^* \oplus (n-1)x)/I = x^*/I = (x/I)^*$$

□

1.7.5 Regular MV-algebras

Definition 1.7.15. An MV-algebra A is said to be *regular* if and only if each minimal prime ideals is stonean of $L(A)$, i.e. it is generated by an idempotent element.

More useful in the sequel, it will the following characterization of regular MV-algebras as quasi-completely boolean dominated MV-algebras.

Definition 1.7.16. An MV-algebra A is said to be *quasi-completely boolean dominated* if and only if, for each $x, y \in A$ such that $x \wedge y = 0$, there exist $b_1, b_2 \in B(A)$ such that $b_1 \geq x$, $b_2 \geq y$ and $b_1 \wedge b_2 = 0$.

Proposition 1.7.17 ([40]). *Let A be an MV-algebra. A is quasi-completely boolean dominated if and only if A is regular.*

Proposition 1.7.18. *Let A, B MV-algebras and $f : A \to B$ an MV-isomorphism. If A is regular, then B is regular.*

Proof. To prove that B is regular, we shall prove that B is quasi-completely boolean dominated. Let $b,\ b' \in B$, such that $b \wedge b' = 0$. Since f is an isomorphism, there exist $a,\ a' \in A$ such that $b = f(a)$ and $b' = f(a')$. Now $0 = b \wedge b' = f(a) \wedge f(a') = f(a \wedge a')$, then $a \wedge a' = 0$. A is regular and so there exist $\alpha,\ \alpha' \in B(A)$ such that $\alpha \wedge \alpha' = 0$ and $a \leq \alpha$, $a' \leq \alpha'$. Being f an MV-homomorphism, we obtain that $f(\alpha),\ f(\alpha') \in B(B)$, $f(\alpha) \wedge f(\alpha') = 0$ and $b = f(a) \leq f(\alpha)$, $b' = f(a') \leq f(\alpha')$. Hence, B is regular. □

Proposition 1.7.19 ([8])**.** *Let A be a regular MV-algebra. For each $M \in \mathrm{Max}(A)$, the set $P_M = \{P \in \mathrm{Spec}(A) \mid P \subseteq M\}$ is linearly ordered with respect to set inclusion.*

Corollary 1.7.20. *Let A be a regular MV-algebra. For each $M \in \mathrm{Max}(A)$, $O(M)$ is a minimal prime ideal.*

Proof. Remember that $O(M) = \bigcap\{m \in \mathrm{Min}(A) \mid m \subseteq M\}$, for each $M \in \mathrm{Max}(A)$. From Proposition 1.7.19, there exists a unique minimal prime ideal $m \subseteq M$, and so $O(M) = m$. □

Proposition 1.7.21 ([8])**.** *Let A be a regular MV-algebra.* $\mathrm{Max}(A)$ *and* $\mathrm{Spec}(B(A))$ *are homeomorphic, therefore* $\mathrm{Max}(A)$ *is a Stone space.*

1.7.6 MV-algebras with $\mathrm{Min}(A)$ compact

Let A be an MV-algebra. In general, as we stressed in paragraph 1.6, the space of minimal prime ideals $\mathrm{Min}(A)$ is not compact with the spectral topology inherited by $\mathrm{Spec}(A)$. But when $\mathrm{Min}(A)$ is compact, it is always a Boolean space (or Stone space) even if it is not homeomorphic to $\mathrm{Spec}(B(A))$. In this section we give a characterization of MV-algebras with $\mathrm{Min}(A)$ compact.

Theorem 1.7.22 ([8], Theorem 29)**.** *In any MV-algebra A,* $\mathrm{Min}(A)$ *is compact iff for any $x \in A$, there is $y \in A$ such that $x^{\perp\perp} = y^{\perp}$.*

Chapter 2

Sheaves in Universal Algebra

In this chapter, we present some results contained in [21] about sheaves and sheaf representations in Universal Algebra which will be useful in the sequel.

The basic notions of categories theory can be found in [45].

2.1 Two different way to look at sheaves

In [21], Davey introduces two different definitions of sheaf: the first one is given in a topological manner (in this case the sheaf is denoted by $F(X, \mathbf{K})$), the second in the context of categories theory (in this case the sheaf is denoted by $\mathcal{F}(X, \mathbf{K})$).

Definition 2.1.1. A *sheaf space of sets over* X is a triple (F, π, X) where:

i) F and X are topological spaces;

ii) $\pi : F \to X$ is a local homeomorphism of F into X, i.e. π is a continuous map such that for each $a \in F$ there exists an open neighbourhood U of a such that $\pi_{|U} : U \to \pi(U)$ is a homeomorphism, being $\pi(U)$ an open subset of X.

Notation: For each Y subset of X, $\Gamma(Y, F)$ denotes the set of continuous maps $\sigma : Y \to F$ such that $\pi \cdot \sigma(y) = y$, for all $y \in Y$.

X is named *the base space*, F *the total space*, the elements of $\Gamma(Y, F)$ are called *sections over* Y and the elements of $\Gamma(X, F)$ are called *global sections*.

Proposition 2.1.2. *Let (F, π, X) be a sheaf space of sets.*

i) If $Y \subseteq X$ and $\sigma, \tau \in \Gamma(Y, F)$ then $\{y \in Y \mid \sigma(y) = \tau(y)\}$ is open in Y.

ii) For each $a \in F$ there is an open subset U of X and a section $\sigma(U, F)$ such that $\sigma(x) = a$ where $\pi(a) = x$.

iii) The set $\{\sigma(U) : U$ is open in X and $\sigma \in \Gamma(U, F)\}$ is a basis for the topology on F.

Definition 2.1.3. If (F, π, X) and (F', π', X) are two sheaf spaces of sets over X, then a map $g : F \to F'$ is *a sheaf space morphism* if g is continuous and such that the following diagram is commutative:

$$\begin{tikzcd} F \arrow[r, "\pi"] \arrow[d, "g"'] & X \\ F' \arrow[ur, "\pi'"'] & \end{tikzcd}$$

i.e. $\pi' \cdot g = \pi$.

Now we give other conditions to introduce sheaf spaces of algebras.

Definition 2.1.4. A sheaf space of sets (F, π, X) is *a sheaf space of algebras (of similarity type* τ if:

i) for each $x \in X$, $F_x := \pi^{-1}(x)$ is an algebra of similarity type τ;

ii) for each open subsets U of X, the set $\Gamma(U, F)$ is an algebra of similarity type τ under the pointwise operations: if f is a n-ary operation and $\sigma_1, \ldots, \sigma_n \in \Gamma(U, F)$, the map $f(\sigma_1, \ldots, \sigma_n) : U \to F$ defined by

$$f(\sigma_1, \ldots, \sigma_n)(x) = f(\sigma_1(x), \ldots, \sigma_n(x))$$

is a continuous section.

F_x is called *the stalk of* F *at* x.

If f^x is an n-ary operation on F_x for each $x \in X$, then a map $f : F^{(n)} \to F$, where $F^{(n)} = \{(a_1, \ldots, a_n) \in F^n \mid \pi(a_1) = \ldots = \pi(a_n)\}$, may be defined by

$$f(a_1, \ldots, a_n) = f^x(a_1, \ldots, a_n) \qquad \text{where } \pi(a_1) = \ldots = \pi(a_n) = x.$$

$F^{(n)}$ is endowed with the topology induced by the product topology on F^n.

The following Lemma summarizes important properties of sheaf spaces.

Lemma 2.1.5. *Let* (F, π, X) *be a sheaf space of sets such that each stalks* F_x *is an algebra of similiraty type* τ*, and assume that* $\Gamma(X, F)$ *is non-empty. Then the following are equivalent:*

(i) (F, π, X) *is a sheaf space of algebras;*

(ii) for each n*-ary operation* f *the map* $f : F^{(n)} \to F$ *is continuous;*

(iii) $\Gamma(Y, F)$ *is an algebra of similarity type* τ*, under the pointwise operations, for each subset* Y *of* X*.*

Definition 2.1.6. Let $\mathbf{K}$ be a class of algebras. (F, π, X) is *a sheaf space of* $\mathbf{K}$*-algebras* if:

i) for each $x \in X$, F_x is a $\mathbf{K}$-algebra;

ii) for each open subsets U of X, the set $\Gamma(U, F)$ is a $\mathbf{K}$-algebra.

If $\mathbf{K}$ is an equational class, the conditions i) and ii) are equivalent. In fact, $\Gamma(U, F)$ is clearly a subalgebra of

$$\prod_{x \in U} F_x.$$

Now we give the definition of sheaf in the context of categories theory

Definition 2.1.7. A *presheaf of* $\mathbf{K}$*-algebras over* X is a contravariant functor between the category $\mathcal{T}$ of the open subsets of X and a category $\mathbf{K}$ of algebras of a given kind, i.e. $\mathcal{F} : \mathcal{T} \to \mathbf{K}$.

NOTATION: If $U_j \subseteq U_i$ (so $f_{ij} : U_j \to U_i$), we denote the corresponding morphism $\mathcal{F}(f_{ij}) : \mathcal{F}(U_i) \to \mathcal{F}(U_j)$ by ϕ_{ij} and $\phi_{ij}(a)$ by $a_{|_{U_j}}$.
But the functor ΓF associated with a sheaf space (F, π, X) of $\mathbf{K}$-algebras have more properties than a presheaf. So, we have to introduce the following definition:

Definition 2.1.8. A presheaf $\mathcal{F}$ of $\mathbf{K}$-algebras is a *sheaf of* $\mathbf{K}$-*algebras* if, for any open subset U of X and any open cover $\{U_i : i \in I\}$ of U, we have:

i) given $a, b \in \mathcal{F}(U)$ if $a_{|_{U_i}} = b_{|_{U_i}}$ for all $i \in I$ then $a = b$, i.e. since $U_i \subseteq U$ for each $i \in I$, there exist $\varphi_i : U_i \to U$ morphisms in $\mathcal{T}$ for each $i \in I$. So we can consider $\mathcal{F}(f_i) : \mathcal{F}(U) \to \mathcal{F}(U_i)$. Now given $a\, b \in \mathcal{F}(U)$ we have $\mathcal{F}(f_i)(a) = a_{|_{U_i}}$ and $\mathcal{F}(f_i)(b) = b_{|_{U_i}}$. So, in a sheaf if a and b coincide on each open set in the cover of U, the they coincide over U;

ii) $\mathcal{F}$ satisfies the gluing axiom, i.e. for each $i \in I$ choose $a_i \in \mathcal{F}(U_i)$, we say that the sections in $\{a_i : i \in I\}$ are *compatible* if for each pair $j, k \in I$ we have $a_{j_{|_{U_{jk}}}} = a_{k_{|_{U_{jk}}}}$, where $U_{jk} = U_j \cap U_k$. The gluing axiom states:
for every set $\{a_i : i \in I\}$ of compatible sections on the cover $\{U_i : i \in I\}$, there exists a section $a \in \mathcal{F}(U)$ such that $a_{|_{U_i}} = a_i$ for each $i \in I$.

The section a is called the *gluing* of the sections $\{a_i\}$. For *i*) this section is unique.

NOTATION: Let us denote by $F(X, \mathbf{K})$ the category whose objects are sheaf spaces of K-algebras and whose morphisms are sheaf space morphisms, by $\mathcal{P}(X, \mathbf{K})$ the category whose objects are presheaves of $\mathbf{K}$-algebras and whose morphisms are natural transformations and by $\mathcal{F}(X, \mathbf{K})$ the full subcategory of sheaves.

Remark 2.1.9. In [21], Davey proved that the category $F(X, \mathbf{K})$ is equivalent to the category $\mathcal{F}(X, \mathbf{K})$. In this way, he gave a tool to pass from a definition to the other.
For this, in the following chapters we will call sheaf both sheaf spaces and contravariant functors.

2.2 Constructions of a sheaf space given a family of congruences

Let A be a fixed algebra. Let $\{\theta_x : x \in X\}$ be a family of congruences on A and $\mathcal{T}$ be a topology on X.
We're looking for a sheaf space of algebras (F_A, π, X) such that:

1. F_x is isomorphic to A/θ_x, for each $x \in X$;

2. the map $[a] : X \to F_A$ defined by $[a](x) = [a]_{\theta_x}$ is continuous for each $a \in A$ (so $[a]$ is a global section) and the map $\alpha : A \to \Gamma(X, F_A)$ defined by $\alpha(a) = [a]$ is a well defined homomorphism.

We may give two construction of the sheaf space (F_A, π, X). Firstly, we may define F_A as the disjoint union of the quotients A/θ_x, i.e.

$$F_A = \biguplus_{x \in X} A/\theta_x.$$

$\pi : F_A \to X$ is the natural projection defined in the following manner. Let $< a >\in F_A$, so there exists a unique $x \in X$ such that $< a >\in A/\theta_x$. Hence we define $\pi(a) = x$.

We consider the set $\{[a](U) : U \in \mathcal{T},\ a \in A\}$ as sub-basis for the topology on F_A.

Secondly, we may define a presheaf (remember that a presheaf is a contravariant functor) $\mathcal{F}_A : \mathcal{T} \to \mathbf{K}$. Let U be an object of $\mathcal{T}$, we define $\mathcal{F}_A(U)$ as $\mathcal{F}_A(U) = A/\theta_U$, where $\theta_U = \wedge\{\theta_x : x \in U\}$, i.e θ_U is the smallest congruence among the congruences indexed in U. Now we have to define $\mathcal{F}_A$ applied to the morphisms of $\mathcal{T}$. So let $U, V \in Ob(\mathcal{T})$ and $f \in Hom_{\mathcal{T}}(V, U)$ (so $V \subseteq U$), we denote with $\phi_{U,V}$ the image of f by $\mathcal{F}_A$. So $\phi_{U,V} : \mathcal{F}_A(U) \to \mathcal{F}_A(V)$, i.e $\phi_{U,V} : A/\theta_U \to A/\theta_V$ defined by $\phi_{U,V}([a]_{\theta_U}) = [a]_{\theta_V}$.

Remark 2.2.1. $\phi_{U,V}$ is well defined since $\theta_U \leqq \theta_V$, being $U \supseteq V$.

Once we have defined the presheaf $\mathcal{F}_A$ we can consider the sheaf space associated with the presheaf $\mathcal{F}_A$: $(S\mathcal{F}_a, \pi, X)$.

PROBLEMS: In the first construction F_x is isomorphic to A/θ_x, but (F_A, π, X) may not be a sheaf space: in fact π may not be a local homeomorphism. In the second construction $(S\mathcal{F}_A, \pi, X)$ is a sheaf space of algebras, but $g_x : \mathcal{F}_x \to A/\theta_x$ may not be an isomorphism.

These problems have a common solution: we need that the topology on X, i.e. the topology described by $\mathcal{T}$ satisfies the following property:

$(*)$ *if* $[a]_{\theta_x} = [b]_{\theta_x}$ *then there exists an open neighbourhood* U *of* x *such that* $[a]_{\theta_y} = [b]_{\theta_y}$ *and so* $[a]_{\theta_U} = [b]_{\theta_U}$.

If a topology on X satisfies the property $(*)$ then π is a local homeomorphism and g_x is an isomorphism.

Definition 2.2.2. If $\{\theta_x : x \in X\}$ is a family of congruences on an algebra A, then any topology on X which satisfies the property $(*)$ is called an S-topology.

So we have the following theorem.

Theorem 2.2.3. *If $\mathcal{T}$ is an S-topology with respect to the family $\{\theta_x : x \in X\}$ of congruences on A, then F_A and $S\mathcal{F}_A$ are isomorphic sheaf spaces of algebras for which the stalk at x is isomorphic to A/θ_x and $\alpha : A \to \Gamma(X, F_A) \cong \Gamma(X, S\mathcal{F}_A)$ is a homomorphism.*

2.3 Subdirect products

From the Universal algebra, we know that an algebra A is a subdirect product of algebras $\{A/\Theta_x : x \in X\}$ if and only if $\wedge\{\Theta_x : x \in X\} = \omega$, where ω is the least element of the lattice of all the congruences on A.

The following lemma assures that if an algebra A of given type τ is a subdirect product of algebras $\{A/\Theta_x : x \in X\}$, it is possible to construct a sheaf space of algebras of type τ such that A embeds in its algebra of global sections.

Lemma 2.3.1. *Let $\{\Theta_x : x \in X\}$ a family of congruences on a fixed algebra*

A. Then $\wedge\{\Theta_x : x \in X\} = \omega$ if and only if the map $\alpha : A \to \Gamma(X, S\mathcal{F}_A)$ defined above is a monomorphism.

Note that the previous lemma holds even if $\mathcal{T}$ is not a S-topology.

Theorem 2.3.2 (BASIC REPRESENTATION THEOREM). *Let $\{\Theta_x \mid x \in X\}$ be a family of congruences on an algebra A and assume A is a subdirect product of the algebras $\{A/\Theta_x : x \in X\}$. If $\mathcal{T}$ is an S-topology on X then there exists a presheaf of algebras $\mathcal{F}_A$ such that:*

i) the stalk $\mathcal{F}_x$ of the induced sheaf space $S\mathcal{F}_A$ is A/Θ_x (up to isomorphism);

ii) for each $a \in A$ the map $[a] : X \to S\mathcal{F}_A$, defined by $[a](x) = [a]_{\Theta_x}$, is a global section;

iii) $\{[a](U) : U \in \mathcal{T}, a \in A\}$ is a basis for the topology on $S\mathcal{F}_A$;

iv) the homomorphism $\alpha : A \to \Gamma(X, S\mathcal{F}_A)$, defined by $\alpha(a) = [a]$, embeds A into the algebra of global sections of $S\mathcal{F}_A$.

Hence, to embed an algebra A into the algebra Γ of global sections of some sheaf space of algebra, we need only:

i) obtain a representation of A as subdirect product of a family $\{A_x : x \in X\}$;

ii) define an S-topology on X.

Now, let $\{\Theta_x : x \in X\}$ an arbitrary family of congruences on an algebra A and let a, b be fixed elements of A, we denote with $U_{a,b}$ the set $\{x \in X :< a, b >\in \Theta_x\}$. So the family $\{U_{a,b} : a, b \in A\}$ is a sub-basis for the coarsest S-topology on X. In fact, if we have $[a]_{\Theta_x} = [b]_{\Theta_x}$, i.e. $< a, b >\in \Theta_x$ and $x \in U_{a,b}$, there exists $U_{a,b}$ which is an open neighbourhood of x such that $[a]_{\Theta_y} = [b]_{\Theta_y}$ for each $y \in U_{a,b}$ (for the definition of $U_{a,b}$).

Note that if $\{\Theta_x : x \in X\}$ is a family of congruences by prime ideals, this topology coincides with the dual spectral topology.

The following lemma proves that the coarsest S-topology is the best choice if we want to construct a sheaf space of algebras for which $\alpha : A \to \Gamma(X, S\mathcal{F}_A)$ is an isomorphism.

Lemma 2.3.3. *Let (F, π, X) be a sheaf space of sets and assume that the topologies on X and F are refined so that, with these new topologies, (F, π, X) is still a sheaf space. Then any section, continuous in the original topologies, is continuous in the new topologies.*

Definition 2.3.4. A family $\{c_x : x \in X\}$ of elements of A is said to be global with respect to a family $\{\Theta_x : x \in X\}$ of congruences on A if, for each $x \in X$, there exists $a_1^x, \ldots, a_n^x, b_1^x, \ldots, b_n^x \in A$ such that

i) $(a_j^x, b_j^x) \in \Theta_x$ for all $j = 1, \ldots, n$;

ii) if $(a_j^x, b_j^x) \in \Theta_y$ for all $j = 1, \ldots, n$ then $(c_y, c_x) \in \Theta_y$.

Theorem 2.3.5. *Let $\{\Theta_x \mid x \in X\}$ be a family of congruences on an algebra A and assume A is a subdirect product of the algebras $\{A/\Theta_x : x \in X\}$. Endow X with its coarsest S-topology. Then $\alpha : A \to \Gamma(X, S\mathcal{F}_A)$ is an isomorphism if and only if for each family $\{c_x : x \in X\}$ of elements of A, global with respect to $\{\Theta_x : x \in X\}$, there exists $c \in A$ with $(c_x, c) \in \Theta_x$ for all $x \in X$.*

Chapter 3

MV-algebras and other algebraic structures

In this chapter, we present the connection of MV-algebras with other algebraic structures.

3.1 MV-algebras and Wajsberg algebras

In this section we prove that the category of MV-algebras is isomorphic to the category of Wajsberg algebras. Wajsberg algebras are special algebraic structures that naturally arise from Łukasiewicz logic.
These results are contained in [33].

Definition 3.1.1. A *Wajsberg algebra* is a structure $(W, \to, *, 1)$, where $\to$ is a binary operation, * is a unary operation and 1 is a constant such that the following identities hold:

(W1) $1 \to a = a$,

(W2) $(a \to b) \to ((b \to c) \to (a \to c)) = 1$,

(W3) $(a \to b) \to b = (b \to a) \to a$,

(W4) $(a^* \to b^*) \to (b \to a) = 1$,

for each $a, b, c \in W$.

Let $(A, \oplus, ^*, 0)$ be an MV-algebra. It is possible to define the *implication* as $a \to b := a^* \oplus b$, for any $a, b \in A$. This leads us to obtain the following results.

Proposition 3.1.2. *If $(A, \oplus, ^*, 0)$ is an MV-algebra then $W_A = (A, \to, ^*, 1)$ is a Wajsberg algebra, where $\to$ is the MV-algebra implication and $1 = 0^*$.*

Proposition 3.1.3. *If $(W, \to, ^*, 1)$ is a Wajsberg algebra and we define $a \oplus b = a^* \to b$ and $0 = 1^*$ for any a, $b \in W$, then $A_W = (W, \oplus, ^*, 0)$ is an MV-algebra.*

From these results, it follows that the variety of MV-algebras and the variety of Wajsberg algebras are cryptoisomorphic (see [11]) and, so, the corresponding categories are isomorphic.

3.2 MV-algebras and semirings

Semirings are algebraic structures introduced by Vandiver ([55]) in 1934 and, later, deeply studied espacially in relation with their applications (see [36]).

Definition 3.2.1. A *semiring* S is a system $(S, +, \cdot, 0, 1)$ such that:

i) $(S, +, 0)$ is a commutative monoid;

ii) $(S, \cdot, 1)$ is a monoid;

iii) $\cdot$ is distributive over $+$;

iv) $0 \cdot s = 0 = s \cdot 0$, for each $s \in S$.

An *additively idempotent semiring* S is a semiring such that $s + s = s$, for each $s \in S$.
For an additively idempotent semiring S, there exists a natural order given by

$$s \leq t \qquad iff \qquad s + t = t,$$

with $s, t \in S$. A semiring S is *lattice-ordered* if and only if it also has the structure of a lattice such that for all $s, t \in S$

- $s + t = s \vee t$,
- $s \cdot t \leq s \wedge t$.

A semiring S is *dual lattice-ordered* if and only if it also has the structure of a lattice such that for all $s, t \in S$

- $s + t = s \wedge t$,
- $s \cdot t \geq s \vee t$.

Lattice-ordered semirings and dual-lattice ordered semirings are additively idempotent. In the following we shall use the name **lc-semiring** for lattice-ordered commutative semiring and **dual lc-semiring** for dual lattice-ordered commutative semirings.

We recall from the Universal Algebra that if A and B are algebras of some type τ, a homomorphism between A and B is a map which preserves all the operations in τ. So, when we introduce new structures, we should assume given the definition of homomorphism.

3.2.1 Coupled semirings

The first connection of MV-algebras with semirings is due to Di Nola and Gerla which in [27] introduced the notion of *coupled semiring* as follows.

Definition 3.2.2. A *coupled semiring* $\mathcal{A}$ is a triple (R_1, R_2, α) such that

CS1) $R_1 = (A, \vee, 0, \cdot, 1)$ and $R_2 = (A, \wedge, 0', \cdot', 1')$ are respectively an lc-semiring and a dual lc-semiring,

CS2) $0' = 1$ and $1' = 0$,

CS3) $\alpha : A \to A$ is a semiring isomorphism from R_1 into R_2, CS4) $\alpha(\alpha(x)) = x$, for every $x \in A$,

CS5) $x \vee y = x \cdot' (\alpha(x) \cdot y)$, for every $x, y \in A$.

It is possible to establish a correspondence between MV-algebras and coupled semirings as follows.

Proposition 3.2.3. *Let $\mathcal{A} = (R_1, R_2, \alpha)$ be a coupled semiring, where $R_1 = (A, \vee, 0, \cdot, 1)$ and $R_2 = (A, \wedge, 0', \cdot', 1')$. Then $(A, \cdot', \alpha, 0)$ is an MV-algebra.*

Proposition 3.2.4. *Let $(A, \oplus, ^*, 0)$ be an MV-algebra. Then the reducts $R_A^\vee = (A, \vee, 0, \odot, 1)$ and $R_A^\wedge = (A, \wedge, 1, \oplus, 0)$ are respectively an lc-semiring and a dual lc-semiring and $(R_A^\vee, R_A^\wedge, *)$ is a coupled semiring.*

3.2.2 Applications

In [35], B. Gerla provides some applications of coupled semirings to other fields. Here, we report the applications in Economics and in Mundici's categorical equivalence between MV-algebras and l-groups.

Belts and Blogs

We recall some of terminology used by Cuninghame-Green in [12]. Lc-semirings and dual lc-semirings are special cases of *belts* (i.e., lattice-ordered commutative semirings) and two semirings of the form $(R, \vee, \cdot)$ and $(R, \wedge, \cdot')$ are *dual* one with respect to the other. A *conjugation* between two belts $(R, +, \cdot)$ and $(S, +', \cdot')$ is an isomorphism of semirings $\alpha : R \to S$.
In this terminology a coupled semiring $\mathcal{A} = (R_1, R_2, \alpha)$ is a triple made up of two belts R_1 and R_2 and a conjugation α between them.
The principal interpretation of belts is given by *bounded lattice-ordered groups* or blog: given a lattice-ordered group G we consider $G^* = G \cup \{-\infty\} \cup \{\infty\}$ in such a way that $(G^*, \vee, -\infty, +, 0)$ and $(G^*, \wedge, \infty, +, 0)$ are both semirings, having as substructures respectively $(G \cup \{-\infty\}, \vee, -\infty, +, 0)$ and $(G \cup \{\infty\}, \wedge, \infty, +, 0)$.
Belts and blogs are the main algebraic structures to deal with *scheduling*

problems and they are used to model problems like the shortest path, activity networks and assignment problems.

Mundici's categorical equivalence

We show here how Mundici's categorical equivalence (see [18]) can be described in terms of universal algebra using semirings.
Let G be an abelian l-group and consider the set $G^* = G \cup \{-\infty\}$. So for every $g \in G$, $-\infty \leq g$ and $-\infty + g = g + (-\infty) = -\infty$. Then $(G^*, \vee, -\infty, +, 0)$ is a lc-semiring. In the same way if $G^{**} = G \cup \{\infty\}$, then $(G^{**}, \wedge, \infty, +, 0)$ is a dual lc-semiring.
For semplicity, we consider here only the case of linearly ordered groups. Let $G = (-\infty, \infty)$ be a totally ordered group and $[0, \infty)$ and $(-\infty, 0]$ its positive and negative cone respectively. We have that $[0, \infty]$ and $[-\infty, 0]$ are subsemirings of $G \cup \{-\infty\} \cup \{\infty\}$. Let u be a strong unit in G. On $[0, \infty]$ we consider the relation R such that xRy if and only if $x \wedge u = y \wedge u$. R is a congruence relation with respect to the semiring structure of $[0, \infty]$ and so, $[0, \infty]/R$ is still a semiring.
Consider the map $\varphi : [x] \in [0, \infty]/R \to x \wedge u \in [0, u]$. Setting $x \wedge y = \varphi([x] \wedge [y])$ and $x \oplus y = \varphi([x] + [y]) = (x + y) \wedge u$ for every $x, y \in [0, u]$, it results that $([0, u], \wedge, u, \oplus, 0)$ is a semiring.
The interval $[-\infty, u]$ has a structure of dual lc-semiring by setting $x \wedge y = x \wedge_{G^*} y$ and $x +' y = x +_{G^*} y - u$ for every $x, y \in [-\infty, u]$. Analogously, we consider a relation R' on $[-\infty, u]$ such that $xR'y$ if and only if $x \vee 0 = y \vee 0$. R' is a congruence relation with respect to the semiring structure of $[-\infty, u]$ and so, $[-\infty, u]/R'$ is still a semiring.
Consider the map $\varphi' : [x] \in [-\infty, u]/R' \to x \vee 0 \in [0, u]$. Setting $x \vee y = \varphi'([x] \vee [y])$ and $x \odot y = \varphi'([x] +' [y]) = (x + y - u) \vee 0$ for every $x, y \in [0, u]$, it results that $([0, u], \vee, 0, \odot, u)$ is a semiring.
In this way, on the interval $[0, u]$ we can give two different structures of semirings.

Theorem 3.2.5. *Let G a linearly ordered abelian group and $u \in G$ be a* *strong unit of G. Then $\mathcal{A}_1 = ([0,u], \vee, 0, \odot, u)$ and $\mathcal{A}_2 = ([0,u], \wedge, u, \oplus, 0)$ and the function $^* : x \in [0,u] \to u - x \in [0,u]$ makes $(\mathcal{A}_1, \mathcal{A}_2, ^*)$ a coupled semiring from which we can recover the MV-algebra $([0,u], \oplus, ^*, 0)$.*

3.3 MV-semirings

In [4], Belluce and Di Nola substitute the notion of coupled semiring associated with an MV-algebra for the notion of MV-semiring. But formerly, in [27], it was stressed the possibility to associate with an MV-algebra only a semiring which has the same properties of an MV-semiring. Here, we shall show that the category of MV-semirings is isomorphic to the category of MV-algebras.

The results collected here are contained in [5] and [6].

Definition 3.3.1. Let $S = (S, +, \cdot, 0, 1)$ be a commutative additively idempotent semiring. We call S a *MV-semiring* iff for each element $s \in S$, there exists the residuum with respect to 0, i.e. there exists a greatest element s^* such that $s \cdot s^* = 0$ and such that $s + t = (s^* \cdot (s^* \cdot t)^*)^*$, for each $s, t \in S$.
A semiring homomorphism is also a MV-semiring homomorphism if it preserves the residuum too.

We indicate by $\mathcal{MVS}$ the subcategory of the category of commutative additively idempotent semirings whose objects are MV-semirings and whose morphisms are MV-semiring homomorphisms.
Note that in the remainder of the paper we will often write xy for $x \cdot y$.

Proposition 3.3.2. *Let S be an MV-semiring, we have:*

1. $0^* = 1, \qquad 1^* = 0$;
2. $s^{**} = s$;
3. $s \leq 1$;
4. *if* $s \leq t$, *then* $t^* \leq s^*$,

5. $s \leq t$ *iff* $st^* = 0$,

for each $s, t \in S$.

Proof. *(1)* We have that $0 = 1 \cdot 1^* = 1^*$ and $1 = 1 + 1 = (1^* \cdot (1^* \cdot 1)^*)^* = (0 \cdot 0^*)^* = 0^*$.

(2) We have that $s = s + s = (s^* \cdot (s^* \cdot s)^*)^* = (s^* \cdot 0^*)^* = (s^* \cdot 1)^* = (s^*)^* = s^{**}$.

(3) To prove that $s \leq 1$ for each $s \in S$, we have to show that $s + 1 = 1$. We have $s + 1 = (s^*(s^* \cdot 1)^*)^* = (s^* s)^* = 0^* = 1$.

(4) Let $s \leq t$, for $s, t \in S$. To prove that $t^* \leq s^*$, we have to show that $s^* + t^* = s^*$. From $s \leq t$, it follows that $s + t = t$ or, equivalently $s^*(s^*t)^* = t^*$. So $s^* + t^* = s^* + s^*(s^*t)^* = s^*(1 + s^*t) = s^*(1) = s^*$.

(5) If $s \leq t$ then $s + t = t$. From this $t^* = (s + t)^* = s^*(s^*t)^*$. So $st^* = s(s^*(s^*t)^*) = 0(s^*t)^* = 0$.
Let $st^* = 0$, $s + t = (t^*(t^*s)^*)^* = (t^*1)^* = t$. □

Definition 3.3.3. [36] A semiring S is named *G-simple* iff $s + 1 = 1$, for each $s \in S$.

From Proposition 3.3.2 *(3)* we have that

Theorem 3.3.4. *Each MV-semiring is G-simple.* □

Definition 3.3.5. A semiring $(S, +, \cdot, 0, 1)$ is a *distributive lattice-ordered semiring* (DLO-semiring) iff it also has the structure of a lattice such that, for all $s, t \in S$:

(i) $s + t = s \vee t$;

(ii) $s \cdot t \leq s \wedge t$;

(iii) $\wedge$ distributes over $\vee$ from either side and viceversa.

where $\vee$ and $\wedge$ are respectively supremum and infimum.

For additively idempotent semiring the next proposition holds. For completeness, we report the proof.

Proposition 3.3.6. [36] *Let S be a G-simple and additively idempotent semiring satisfying the condition that*

$$if \qquad s \le t \qquad then \qquad s \in tS \cap St, \tag{3.1}$$

for each $s, t \in S$. Then we can define an operation $\wedge$ on S where $s \wedge t = s \cdot t'$, with $t = t' \cdot (s+t)$ such that S is a DLO-semiring. □

Proof. It is well known that if $+$ is a binary operation which is commutative, associative and idempotent and we define $s \le t$ if and only if $t = s + t$, then $s + t = s \vee t$, i. e., it is the least upper bound of $\{s, t\}$ with respect to $\le$. On the other hand, it follows from items (3) and (4) of Proposition 3.3.2 that $s \wedge t = (s^* + t^*)^* = s \cdot (s \cdot t^*)^*$. Hence S is a lattice. To prove that this lattice is distributive, it sufficies to show that $(s+t) \wedge u \le (s \wedge u) + (t \wedge u)$. Taking into account item (4) of Proposition 3.3.2: $(s + t) \wedge u = (s + t) \cdot ((s + t) \cdot u^*)^* = (s + t) \cdot ((s \cdot u^*) + (t \cdot u^*))^* = (s \cdot ((s \cdot u^*) + (t \cdot t^*))^*) + (t \cdot ((s \cdot u^*) + (t \cdot t^*))^*) \le (s \cdot (s \cdot u^*)^*) + (t \cdot t^*)^*) = (s \wedge u) + (t \wedge u)$. Finally, since $(s \cdot t) + s = (s \cdot t) + (s \cdot 1) = s \cdot (t + 1) = s$, one has that $s \cdot t \le s$, and this implies that $s \wedge t \le s \cdot t$. □

Theorem 3.3.7. *A MV-semiring is also a DLO-semiring.*

Proof. Let S be an MV-semiring. First we prove that S verifies the condition (3.1) of Proposition 3.3.6. Since $\cdot$ is commutative, we have to prove that

$$\mathrm{i}f \qquad s \le t \qquad \mathrm{t}hen \qquad s \in tS.$$

From $s \le t$ we have that $t^* \le s^*$, so $t^* + s^* = s^*$, i. e. $s^* = (t(t \cdot s^*)^*)^*$. Then $s = t(t \cdot s^*)^*$. Now $(t \cdot s^*)^* \in S$, so $s \in tS$. From Proposition 3.3.6, we have that $(S, +, \wedge)$ is a DLO-semiring.

Let see, how $\wedge$ is defined for MV-semirings. We have $t \le s+t$, so $(s+t)^* \le t^*$.

Then

$$\begin{aligned} t^* &= (s+t)^* + t^* \\ &= \Big((s+t)\big((s+t)t^*\big)^*\Big)^* \\ &= \big((s+t)(st^* + tt^*)^*\big)^* \\ &= \big((s+t)(st^* + 0)^*\big)^* \\ &= \big((s+t)(st^*)^*\big)^*. \end{aligned}$$

So $t = (s+t)(st^*)^*$. Then $t' = (st^*)^*$. By Proposition 3.3.6, we define $s \wedge t = s(st^*)^*$, i.e. $s \wedge t = (s^* + t^*)^*$. $\square$

By Proposition 3.3.2, it follows that if S is an MV-semiring then $(S, +, \wedge, 0, 1)$ is a bounded distributive lattice with first element 0 and last element 1.

An MV-algebra A is a system $(A, \oplus, \odot, *, 0, 1)$ where

- A is a set;
- $0, 1 \in A$;
- $(A, \oplus, 0)$ is a commutative monoid;
- $(A, \odot, 1)$ is a multiplicative monoid;
- $*$ is a unary operation on A that connects the two monoids by a de Morgan law, i.e $(x \oplus y)^* = x^* \odot y^*$ and $(x \odot y)^* = x^* \oplus y^*$, and such that $x^{**} = x$, $0^* = 1$.

Further, there are defined operations $\vee, \wedge$ assumed to be commutative by

1. $x \vee y = x \oplus (x^* \odot y)$;
2. $x \wedge y = x \odot (x^* \oplus y)$.

It can be shown that $(A, \vee, \wedge, 0, 1)$ is a distributive lattice with $0, 1$, where $x \leq y$ iff $y = x \vee y$ iff $x = x \wedge y$.

We indicate by $\mathcal{MV}$ the category whose objects are MV-algebras and whose morphisms are homomorphisms between MV-algebras.

Let $A = (A, \oplus, \odot, *, 0, 1)$ be an MV-algebra and consider the reduct $(A, \vee, \odot, 0, 1)$, we can prove

Lemma 3.3.8. $(A, \vee, \odot, 0, 1)$ *is an MV-semiring.*

Proof. It's easy to prove that $(A, \vee, \odot, 0, 1)$ is a semiring. The existence of the residuum with respect to 0 is a particular case of statement (iii) of Lemma 1.1.4 in [18]. □

Also, this semiring is additively idempotent with 0 as an additive identity and 1 as a multiplicative identity.

Lemma 3.3.9. *Let A and B be MV-algebras, a homomorphism $h : A \to B$ is also an MV-semiring homomorphism between the MV-semirings extracted from A and B.*

Proof. Trivial □

Let us consider the functor $\Delta : \mathcal{MV} \to \mathcal{MVS}$ defined as it follows: let A be an object of $\mathcal{MV}$, $\Delta(A) = (A, \vee, \odot, 0, 1)$ the MV-semiring extracted from A; let A, B be objects of $\mathcal{MV}$ and $f \in Hom_{\mathcal{MV}}(A, B)$, $\Delta(f) = f$. Lemmas 3.3.8 and 3.3.9 assure that Δ is a functor between $\mathcal{MV}$ and $\mathcal{MVS}$.
We shall prove that Δ is a natural equivalence between $\mathcal{MV}$ and $\mathcal{MVS}$ and give an explicit construction of an adjoint functor of Δ.

Remark 3.3.10. Let $S = (S, +, \cdot, 0, 1)$ be an MV-semiring, we define on S the operations $x \oplus y = (x^* \cdot y^*)^*$ and $x \odot y = x \cdot y$. It's straightforward to see that $(S, \oplus, \odot, *, 0, 1)$ is an MV-algebra. From this it follows that MV-semirings and MV-algebras are cryptoisomorphic (see [11]). This implies that the categories $\mathcal{MV}$ and $\mathcal{MVS}$ are isomorphic.

3.4 Ideals and the Prime spectrum of an MV-semiring

Let $S = (S,\ +,\ \cdot,\ 0,\ 1)$ be a commutative semiring with $0,\ 1$. An *ideal* of S is a non-empty subset $I \subseteq S$ such that I is closed under $+$ and such that if

$x \in I,\ y \in S$, then $x \cdot y \in I$. I is *proper* if $1 \notin I$.

It should be clear that if A is an MV-algebra and S the corresponding MV-semiring, then every ideal of A is an ideal of S. The converse is false. Moreover, in spite of the isomorphism between the categories of MV-algebras and MV-semirings, an ideal of A may sit differently in S. Specifically a prime ideal P of A is in general not prime in S; similarly, a maximal ideal M of A need not be maximal as an ideal of S. In another vein, the congruences determined in the standard semiring manner by ideals of S are different from the usual congruences determined by MV-ideals.

In this section we will prove some facts about the ideals of MV-semirings and in particular the nature of their prime ideal spaces.

It is well known that an *MV-space*, that is, the prime ideal space of an MV-algebra endowed with the Zariski topology, is a spectral space ([2]). However, every spectral space has a Hochster dual which is also a spectral space ([37])). In the context of MV-algebras the Hochster dual was exploited in the papers of [57] and [29]. In the latter the Hochster dual was used as the base space for a sheaf representation of a given MV-algebra.

Neither [57] or [29] explore the properties of the Hochster dual with respect to MV-semirings. We will give some results showing that the Hochster dual of an MV-space is in general not an MV-space, but is, however, the prime spectrum, under the Zariski topology, of an MV-semiring. This is similar to the situation in commutative rings where the Hochster dual of a prime ideal space under the co-Zariski topology is also the prime ideal space under the Zariski topology of some commutative ring.

Let $\mathrm{Id}(S),\ \mathrm{Id}(A)$ denote the set of ideals of $S,\ A$ respectively where S is a semiring and A an MV-algebra. Then if S is the associated MV-semiring of A we have, as mentioned above,

Proposition 3.4.1. $\mathrm{Id}(A) \subseteq \mathrm{Id}(S)$.

Proof. Let J be an MV-ideal of A and $x, y \in J$, since $\mathcal{MV}$ and $\mathcal{MVS}$ are isomorphic, the order of A and the order of S coincide, from this it follows that $x +_S y = x \vee_A y \leq x \oplus y \in J$. Hence $x + y \in J$. Now let $x \in S$ and $y \in J$, $xy \leq y \in J$ so $xy \in J$. $\square$

Recall that given a semiring $S = (S,\ +,\ \cdot,\ 0,\ 1)$, the congruence class determined by an ideal I is $E_S(I) = \{(x,\ y) \mid (\exists z \in I)(x + z = y + z)\}$. One can then form the quotient semiring $S/I = \{x/I \mid x \in S\}$ where x/I is the congruence class determined by the relation $E_S(I)$.

If A is an MV-algebra and $I \subseteq A$ an ideal, then I, determines a congruence relation as follows: $E_A(I) = \{(x,\ y) \mid xy^* \oplus x^*y \in I\}$.

Clearly,

Lemma 3.4.2. *If $I \in \mathrm{Id}(S),\ y \in I,\ x \leq y$, then $x \in I$.*

The following is also clear,

Proposition 3.4.3. *If $I \in \mathrm{Id}(S)$, then $x \in I$ iff $(x,\ 0) \in E_S(I)$.*

Next we describe $I \in \mathrm{Id}(S)$ when S/I is an MV-semiring.

Let S be an MV-semiring and I an ideal of S. Suppose I determines a congruence relation $E_S(I)$ with respect to $+,\ \cdot,\ ^*$ and that the associated set of equivalence classes, S/I is an MV-semiring homomorphic image of S. So the natural map $\pi : S \to S/I$ satisfies $\pi(x + y) = \pi(x) + \pi(y),\ \pi(xy) = \pi(x)\pi(y)$ and finally, $\pi(x^*) = (\pi(x))^*$.

Now we have for $x \in I$ that $\pi(x) = 0$ and so $\pi(x^*) = (\pi(x))^* = 1$. Thus if $x,\ y \in I$, then $\pi(x^*)\pi(y^*)) = 1$. Thus $(\pi(x)^*\pi(y)^*)^* = 0$. That is $\pi((x^*y^*)^*) = 0$ and so $(x^*y^*)^* \in I$. But this means that I is an MV-ideal.

That is,

Proposition 3.4.4. *For S/I to be an MV-semiring, naturally, we must have I an MV-ideal.*

So to resolve the question of when is S/I an MV-semiring naturally, we may confine our attention to those ideals of S that are also ideals of the associated MV-algebra A.

Remark 3.4.5. If $I \in \mathrm{Id}(A)$, then $E_S(I) \subseteq E_A(I)$. This follows from the fact that $x/I = y/I$ implies $xy^*/I = x^*y/I = 0$ iff $xy^*,\ x^*y \in I$ iff $xy^* \oplus x^*y \in I$ iff $(x,\ y) \in E_A(I)$.

We now obtain,

Proposition 3.4.6. *Suppose $I \in \mathrm{Id}(S)$ and $E_S(I)$ is a congruence relation with respect to $+,\ \cdot,\ ^*$. Then $E_S(I) = E_A(I)$.*

Proof. We know that I is an MV-ideal by Proposition 3.4.4. By the Remark 3.4.5 we have $E_S(I) \subseteq E_A(I)$. Conversely, suppose $(x,\ y) \in E_A(I)$ so that $xy^* \oplus x^*y \in I$. Then $xy^*,\ x^*y \in I$. So in S/I we have $xy^*/I = x^*y/I = 0$. Since S/I is an MV-semiring by assumption, we have that $x/I + y/I = y/I$. Thus $(x + y)/I = y/I$. By symmetry $(x + y)/I = x/I$, thus we have $x/I = y/I$. Therefore $(x,\ y) \in E_S(I)$ and we have $E_S(I) = E_A(I)$. □

In summary we have

Corollary 3.4.7. *Let S be an MV-semiring and I an ideal of S. Suppose that S/I is an MV-semiring homomorphic image of S under the natural map $S \to S/I$. Then I is an ideal of A. Also, the congruence on A determined by I is the same as the congruence on S determined by I. Moreover $S(A/I) \cong S/I$ where $S(A/I)$ is the MV-semiring associated to A/I.*

Proof. We need only show the last part. This is basically trivial; the congruence class $x/I \in A/I$ is identical with the class $x/I \in S/I$ since, as shown above, $E_S(I) = E_A(I)$. □

The above result makes the assumption that S/I is an MV-semiring, from which we inferred that I was an MV-ideal, that is an ideal of A. It certainly doesn't imply that if I is an MV-ideal, then S/I will be an MV-semiring. Indeed, consider the following example.

Example 3.4.8. Let C be the Chang algebra. That is, C is the MV-algebra defined by Chang in [14] with formal symbols $\{nc \mid n \geq 0\} \cup \{(nc)^* \mid n \geq 0\}$

with operations $nc \oplus mc = (m+n)c$, $nc \oplus (mc)^* = ((m-n)c)^*c$ if $n < m$ and equal to 1 if $m \leq n$, $0 = 0c$, $1 = 0^*$ together with the DeMorgan laws.

C has one maximal ideal, $M = \{nc \mid nc,\ n \geq 0\}$. Let $A = C \times C$ and let $I = M \times M$. Let $a = ((4c)^*,\ 3c)$, $b = ((5c)^*, c)$ where c is the atom of C. Then $b^* = (5c,\ c^*)$. Then $ab^* = ((4c)^*(5c),\ (3c)c^*) = (c,\ 2c) \in I$. So $(a/I)(b^*/I) = 0$. Thus, were S/I an MV-semiring we would have $a/I + b/I = b/I$. Therefore $a+b$ is congruent to b. Let $z = (z_1,\ z_2) \in I$. Then $a+b+z = ((4c)^* + (5c)^* + z_1,\ 3c + c + z_2) = ((4c)^*,\ 3c + z_2)$ since $z_1 \leq (5c)^* \leq (4c)^*$ for any $z_1 \in M$. On the other hand, $b + z = ((5c)^* + z_1, c + z_2) = ((5c)^*,\ c + z_2)$. Hence, as $(4c)^* \neq (5c)^*$ we see there is no $z \in I$ for which $a+b+z = b+z$ and so $a+b$ cannot be congruent to b. Consequently A/I is not an MV-semiring image of A.

This example raises the question of whether there are any non-zero ideals of S for which S/I is an MV-semiring (under the relation $E_S(I)$).

For a linearly ordered $S \neq \{0,\ 1\}$, we answer the above in the negative!

Proposition 3.4.9. *Let A, S be given, and I a non-zero ideal of S. If $S \neq \{0, 1\}$ is linearly ordered, then $E_S(I)$ cannot be a congruence on S with respect to $+$, $\cdot$, *.*

Proof. Suppose that the relation $E_S(I) = \{(x,\ y) \mid (\exists z \in I)(x + z = y + z)\}$ is a congruence on S with respect to $+$, $\cdot$, *. Then if $(x,\ y) \in E_S(I)$ we also have $(x^*,\ y^*) \in E_S(I)$. Let $x \in I,\ x \neq 0$. Then $(x,\ 0) \in E_S(I)$ since $x + x = 0 + x$. Thus $(x^*,\ 0^*) \in E_S(I)$; that is $(x^*,\ 1) \in E_S(I)$. So there is a $z \in I$ such that $x^* + z = 1 + z$. If A, hence S, is linearly ordered, this implies that $x^* = 1$ and so $x = 0$. □

Similarly we have the following

Proposition 3.4.10. *Suppose S an MV-semiring, and I a non-zero nil ideal. Then S/I is not an MV-semiring.*

Proof. Let $a \in I,\ a \neq 0$. So we have $a + a = 0 + a$. that is $(a,\ 0) \in E_S(I)$. Thus assuming S/I an MV-semiring we also have $(a^*,\ 1) \in E_S(I)$. So for some $z \in I$, $a^* + z = 1 + z = 1$. By Theorem 3.4 of [14] we have $(a^*)^n + z^n = 1$ for all $n > 0$. Since z is nilpotent we may infer there is an $n > 0$ such that $(a^*)^n = 1$. But this implies that $a^* = 1$ and so $a = 0$ contrary to assumption. □

Corollary 3.4.11. *If A is the MV-algebra associated with S and* $\mathrm{Rad}(A) \neq 0$*, then $S/\mathrm{Rad}(A)$ is not an MV-semiring.*

The following proposition shows that in some cases S/I is an MV- semiring.

Proposition 3.4.12. *Let S be an MV-semiring and let I be an ideal of S generated by multiplicative idempotents. Then S/I is an MV-semiring.*

Proof. Consider the relation $E_S(I) = \{(a, b) \mid (\exists z \in I)(a + z = b + z)\}$. We know this is a congruence with respect to $+$, $\cdot$. Assume $(a, b) \in E_S(I)$ so that $a + z = b + z$ for some $z \in I$. There are idempotent $e_1, \dots, e_n \in I$ with $z \leq e$, $e = e_1 + \cdots + e_n$. Moreover $a + z + e = b + z + e$ which gives $a + e = b + e$. Hence $a^*e^* = b^*e^*$. Therefore $e + a^*e^* = e + b^*e^*$. By the corollary above this gives $a^* + e = b^* + e$ and we have $(a^*, b^*) \in E_S(I)$. So $E_S(I)$ is also a congruence with respect to * and so $S/I = S/E_S(I)$ is an MV-semiring. □

Observe that if I is an ideal of S and is generated by multiplicative idempotents, then I is also an ideal of the associated MV-algebra A.

Suppose $S/I = S/E_S(I)$ is an MV-semiring. Then we have a morphism $S/I \to S(A/I)$; for notational reasons, write $S/I = S/E_S(I)$. Thus we have a morphism, $a/E_S(I) \to a/E$ where $E = \{(a, b) \in S \mid d(a, b) \in I\}$. Since we know that $E_S(I) \subseteq E$ the mapping is well defined; it is clearly surjective. Suppose then that $a/E = b/E$ so that a^*b, $ab^* \in I$. Then $(a^*b, 0) \in E_S(I)$ so that $a^*b/E_S(I) = a^*/E_S(I)b/E_S(I) = 0$. Since $S/E_S(I)$ is an MV-semiring naturally,this implies that $b/E_S(I) \leq a/E_S(I)$. By symmetry, $a/E_S(I) \leq b/E_S(I)$ and we may infer that the mapping $a/E_S(I) \to a/E$ is injective, hence $S/I \cong S(A/I)$.

The prime spectra

In what follows, A will be an MV-algebra and S the associated MV-semiring.

Definition 3.4.13. An ideal Q of a semiring S is *prime* if whenever $xy \in Q$, then $x \in Q$ or $y \in Q$.

Let $\mathrm{Spec}(S)$, $\mathrm{Spec}(A)$ denote respectively, the sets of prime ideals of S and of A.

We know every MV-ideal of A is a lattice ideal, thus an ideal of S. However a prime ideal of A need not be a prime ideal of S. For example, $\{0\}$ is a prime ideal of $[0,\ 1]$ the latter considered as an MV-algebra, but is not a prime ideal of $[0,\ 1]$ considered as an MV-semiring since $1/2 \cdot 1/2 = 0$. In fact we have,

Proposition 3.4.14. *If S is an MV-semiring, $S \neq \{0,\ 1\}$, then the zero ideal of S is not prime.*

Proof. Suppose $\{0\}$ is prime. Then for every $x \in S$ we have $xx^* = 0$ and so $x = 0$ or $x^* = 0$. That is $x \in \{0,\ 1\}$. Then $S = \{0,\ 1\}$ contrary to assumption. □

Proposition 3.4.15. *If M is a maximal ideal in an MV-semiring S, then M is prime.*

Proof. Suppose $xy \in M,\ x \notin M,\ y \notin M$. Then $M + id(x) = M + id(y) = S$. Thus for some $m \in M$ we have $1 = m + x = m + y$. Hence $1 = (m + x)(m + y) = m^2 = mx + my + xy \in M$ which is absurd. □

Proposition 3.4.16. *Let S be an MV-semiring and A the associated MV-algebra. A necessary condition for an ideal Q of S to be a prime ideal of S is that* $\mathrm{Rad}(A) \subseteq Q$.

Proof. Let Q be a prime ideal of S and let $x \in \mathrm{Rad}(A)$. Then as $x^2 = 0 \in P$ we must have $x \in P$. □

Recall that a maximal ideal M of an MV-algebra A is *supermaximal* if and only $A/M = \{0,\ 1\}$. Moreover, it results that M is supermaximal in A if and only if $A = M \cup M^*$.

Proposition 3.4.17. *Suppose that* $P \in \mathrm{Id}(A) \cap \mathrm{Spec}(S)$*. Then in A, P must be supermaximal.*

Proof. Let $x \in A$. Then $x \in S$ and $xx^* = 0$. Thus $x \in P$ or $x^* \in P$. □

Moreover,

Proposition 3.4.18. *If* $I \in Spec(S) \cap \mathrm{Id}(A)$*, then* S/I *is not an MV-semiring or* $S/I = \{0,\ 1\}$.

Proof. Again let's assume that S/I is an MV-semiring. Let $a/I \neq 0$. Since S/I is assumed to be an MV-semiring, $(a/I)^* \in S/I$. So for some $b \in S$ we have $b/I = (a/I)^*$. Hence $(a/I)(b/I) = 0 = (ab)/I$. So $ab \in I$ and as I is prime in S, either $a \in I$ or $b \in I$. If $b \in I$ implies $0 = b/I = (a/I)^*$ and so $a/I = 1$. If $a \in I$, then $a/I = 0$ contrary to assumption. Thus, $S/I = \{0,\ 1\}$. □

Remark 3.4.19. Observe that in general, if $S/I = \{0,\ 1\}$ this doesn't mean that the quotient S/I is an MV-semiring even though $\{0,\ 1\}$ can be made an MV-semiring. For let $S = [0,\ 1]$ and $I = [0,\ 1)$. If $a \in I,\ a \neq 0.$ then $a/I = 0$. But also, $a^* \in M$ and so $a^*/I = 0$. Therefore we can not have $(a/I)^* = a^*/I$. That is, S/I is not an MV-semiring naturally.

In fact we can say more. Suppose $S/I = \{0,\ 1\}$. Then for $a \in S$ either $a/I = 0$ or $a/I = 1$. The former implies $a \in I$. If S/I is an MV- semiring, then $a/I = 1$ implies $a^* \in I$. Hence if $S/I = \{0,\ 1\}$ and is an MV-semiring, then for $a \in S,\ a \in I$ or $a^* \in I$. But this means that I is a supermaximal ideal of A, hence by Proposition 3.4.18, S/I cannot be an MV-semiring.

Note that if I is an MV-ideal and prime as an S ideal, then I is also prime as an MV-ideal, thus is supermaximal. Of course I can be prime as an MV-ideal but not prime as an ideal of S.

Both $\mathrm{Spec}(A)$ and $\mathrm{Spec}(S)$ have as a topology a basis of compact open sets determined by the Zariski topology.

For A we have as basic open sets the collection of $U(x) = \{P \in \mathrm{Spec}(A) \mid x \notin P\}$ and for S we have the collection have $U_S(x) = \{Q \in \mathrm{Spec}(S) \mid x \notin Q\}$. We shall describe the relation between these topological spaces.

Let, then, $P \in \mathrm{Spec}(A)$; let $F_P = A - P$ and let $Q_P = (F_P)^* = \{x^* \mid x \in F_P\}$.

Proposition 3.4.20. *For $P \in \mathrm{Spec}(A)$, $Q_P \in \mathrm{Spec}(S)$.*

Proof. First let $x,\ y \in Q_P$. Then $x^*,\ y^* \in A - P$. Thus $x^*,\ y^* \notin P$ hence $x^* \wedge y^* \notin P$. So $x^* \wedge y^* \in F_P$ and therefore $(x^* \wedge y^*)^* \in Q_P$. But $(x^* \wedge y^*)^* = x + y$ and so $x + y \in Q_P$. Next, suppose $x \in Q_P,\ y \in S$. Then $x^* \in A - P$ and we have $x^* \notin P$. Therefore $x^* \oplus y^* \notin P$ and it follows that $(x^* \oplus y^*) \in F_P$. Hence $(x^* \oplus y^*)^* = x \cdot y \in Q_P$. So we see that Q_P is an ideal of S. Finally, suppose that $xy \in Q_P$. Then $(xy)^* \in A - P$ so that $(xy)^* = x^* \oplus y^* \notin P$. Therefore, say, $x^* \notin P$. Then $x^* \in A - P$ and we have $x \in Q_P$. So $Q_P \in \mathrm{Spec}(S)$. □

Proposition 3.4.21. *Let $Q \in \mathrm{Spec}(S)$. Then there is a unique $P \in \mathrm{Spec}(A)$ such that $Q = Q_P$.*

Proof. First let $F = Q^*$. If $x,\ y \in F$, then $x^*,\ y^* \in Q$ and so $x^* + y^* \in Q$. Hence $x \wedge y = (x^* + y^*)^* \in F$. If $x \in F,\ x \leq y$, then we have $y^* \leq x^* \in Q$ and so $y^* \in Q$ which yields $y \in F$. Suppose now that $x \oplus y \in F$. Then $x^* \cdot y^* \in Q$. But Q is prime so that, say, $x^* \in Q$ from which we have $x \in F$. Now let $P = A - F$. Since $0 \notin F$ we have $0 \in P$. Suppose $x,\ y \in P$. Then $x,\ y \notin F$. Therefore $x \oplus y \notin F$ and we have $x \oplus y \in P$. Now if $y \leq x \in P$ and $y \notin P$ we get $y \in F$ and so $x \in F$ which is false. Thus P is an ideal of A. Assume $x \wedge y \in P$. So $x \wedge y \notin F$. As F is closed under $\wedge$ it must be the case that $x \notin F$ or $y \notin F$; that is $x \in P$ or $y \in P$ and so $P \in \mathrm{Spec}(A)$. Since $P = A - F$ we have $F = A - P$. Thus $Q = (A - P)^*$. Suppose now that $P_0 \in \mathrm{Spec}(A)$ and $Q = (A - P_0)^*$. So $(A - P)^* = (A - P_0)^*$ from which we get $A - P = A - P_0$ and therefore $P = P_0$. □

Corollary 3.4.22. *There is a bijection* $\mathrm{Spec}(A) \leftrightarrow \mathrm{Spec}(S)$ as sets.

In general the above bijection will not be a homeomorphism. To obtain a homeomorphism between the relevant spaces we re-topologize $\mathrm{Spec}(A)$.

Consider the space $\mathrm{coSpec}(A)$ which has as elements prime ideals of A topologized as follows: for a basis of open sets take the sets $W(a) = \{P \in \mathrm{Spec}(A) \mid a \in P\},\ a \in A$. Now $W(a) \cap W(b) = W(a \oplus b)$, thus the basis is closed under finite intersection. Note also that $W(a) \cup W(b) = W(a \wedge b)$. This topology on the prime spectrum is known as the *coZariski* topology and the resulting space is called the *Hochster dual* of $\mathrm{Spec}(A)$. We shall denote the Hochster dual of $\mathrm{Spec}(A)$ by $\mathrm{coSpec}(A)$. Of course, every spectral space has a Hochster dual.

The next proposition establishes the relation between $\mathrm{coSpec}(A)$ and $\mathrm{Spec}(S)$.

Proposition 3.4.23. *If A is an MV-algebra and S the associated semiring, then the map $\phi : P \to Q_P, \mathrm{coSpec}(A) \to \mathrm{Spec}(S)$ is a homeomorphism.*

Proof. Corollary 3.4.22 establishes that ϕ is a bijection. Now $\phi^{-1}(U_S(a)) = \{P \in \mathrm{coSpec}(A) \mid a \notin (A - P)^*\}$. But $a \notin (A - P)^*$ iff $a^* \notin (A - P)$ iff $a^* \in P$. Thus $\phi^{-1}(U_S(a)) = W(a^*)$ which is open in $\mathrm{coSpec}(A)$. So ϕ is continuous. Similarly, $\phi(W(a)) = \{\phi(P) \mid a \in P\}$. But $a \in P$ iff $a \notin (A-P)$ iff $a^* \notin (A - P)^*$ iff $\phi(P) \in U_S(a^*)$. So $\phi(W(a)) = U_S(a^*)$ which is open in $\mathrm{Spec}(S)$. Therefore ϕ^{-1} is continuous and thus ϕ is homeomorphism. □

Corollary 3.4.24. $\mathrm{Spec}(S)$ *is a spectral space.*

Proof. It is well known ([37]) that the Hochster dual of a spectral space is a spectral space, thus $\mathrm{coSpec}(A)$ is a spectral space. □

To better understand the properties of $\mathrm{Spec}(S)$ and its relation to $\mathrm{Spec}(A)$ we need only to examine properties of $\mathrm{coSpec}(A)$.

Observe that the mapping $P \to Q_P$ of $\mathrm{Spec}(A)$ to $\mathrm{Spec}(S)$ reverses the order, $P \subseteq P'$ iff $Q_{P'} \subseteq Q_P$.

Two questions arise immediately:

1) when, if ever, are $\mathrm{Spec}(A)$ and $\mathrm{Spec}(S)$ homeomorphic?

2) when, if ever, is $\mathrm{Spec}(S)$ an MV-space?

From the above observation we have

Proposition 3.4.25. *Let A, A' be MV-algebras such that $\phi : \mathrm{coSpec}(A) \to \mathrm{Spec}(A')$ is a homeomorphism. Then ϕ is order reversing isomorphism from $\mathrm{Spec}(A)$ to $\mathrm{Spec}(A')$ as posets under $\subseteq$.*

Corollary 3.4.26. *In order for $\mathrm{Spec}(A)$ and $\mathrm{coSpec}(A)$ to be homeomorphic, it is necessary there be an order reversing isomorphism from $\mathrm{Spec}(A)$ to itself as a poset under $\subseteq$.*

As a consequence of this it is straightforward to see that for certain MV-algebras A that $\mathrm{Spec}(A)$ and $\mathrm{Spec}(S)$ cannot be homeomorphic, in particularly for a non-linearly ordered local MV-algebra.

In other words, if A is a non-linearly ordered local MV-algebra, then $\mathrm{coSpec}(A)$, and so $\mathrm{Spec}(S)$, is not an MV-space.

However for linearly ordered algebras we have,

Proposition 3.4.27. *If A is a linearly ordered MV-algebra, then $\mathrm{Spec}(S)$ is an MV-space.*

Proof. We know that $\mathrm{coSpec}(A)$ is a spectral space with order reversed from $\mathrm{Spec}(A)$. Hence it's a linearly ordered spectral space. It's known that a linearly ordered spectral space is an MV-space. Since $\mathrm{coSpec}(A)$ and $\mathrm{Spec}(S)$ are homeomorphic, the result follows. $\square$

This means that if A is linearly ordered there is a linearly ordered MV-algebra A' and an order reversing poset isomorphism of $\mathrm{Spec}(A)$ to $\mathrm{Spec}(A')$.

In general it can be shown that a necessary condition for $\mathrm{Spec}(S)$ to be an MV-space is that $\mathrm{Spec}(A)$ be a disjoint union of chains, that is, that A be hypernormal.

Theorem 3.4.28. *Let A be an MV-algebra. If $\mathrm{Spec}(\Delta(A))$ is an MV-space, then A is hypernormal, i.e. $\mathrm{Spec}(A)$ is a disjoint union of chains.*

Proof. Let A be an MV-algebra and $S = \Delta(A)$ the associated MV-semiring such that $\mathrm{Spec}(S)$ is an MV-space. By this and Proposition 3.4.23 there are an MV-algebra A' and a homeomorphism $\phi : \mathrm{coSpec}(A) \to \mathrm{Spec}(A')$. Moreover, from Proposition 3.4.25 ϕ is an order reversing isomorphism of $\mathrm{Spec}(A)$ onto $\mathrm{Spec}(A')$ as posets under $\subseteq$. Hence if $M \in \mathrm{Max}(A)$, then $\phi(M)$ must be minimal as a member of $\mathrm{Spec}(A')$. Let $P,\ Q \in \mathrm{Spec}(A)$ such that $P,\ Q \subseteq M$. Then $\phi(M) \subseteq \phi(P),\ \phi(Q)$. But $\phi(M)$ is a prime ideal in $\mathrm{Spec}(A')$ and since $\mathrm{Spec}(A')$ is a root system the prime ideals over $\phi(M)$ form a chain. Thus we have $\phi(P) \subseteq \phi(Q)$ or $\phi(Q) \subseteq \phi(P)$. Since ϕ is an order reversing isomorphism we get $P \subseteq Q$ or $Q \subseteq P$. Therefore the ideals below any maximal ideal of A form a chain and that means that A is hypernormal. □

Remark 3.4.29. Is the converse true, that is, if A is hypernormal is $\mathrm{coSpec}(A)$ an MV-space? That is true in the hyperarchimedean case. For here the order is the identity relation (remember that $\mathrm{Spec}(A) = \mathrm{Max}(A)$) and any bijection from the maximal ideal space to itself is order reversing trivially. However in this case $\mathrm{Spec}(A)$ is a boolean space and so $\mathrm{Spec}(A) = \mathrm{coSpec}(A)$.

Example 3.4.30. Another example is the MV-algebra $A = C \times C$ where C is the Chang algebra. Here we have $\mathrm{Spec}(A) = \{0 \times C,\ M \times C,\ C \times M.\ C \times 0\}$, where M is the radical of C. The bijection $0 \times C \leftrightarrow M \times C,\ C \times 0 \leftrightarrow C \times M$ is order reversing and a homeomorphism. This algebra is, of course, hypernormal.

For hypernormal algebras we have the following theorem, too.

Theorem 3.4.31. *Let A be an hypernormal MV-algebra and $S = \Delta(A)$ the associated MV-semiring. If* $\mathrm{Spec}(S_r)$ *is an MV-space, i.e.* $\mathrm{Spec}(S_r) \cong \mathrm{Spec}(A')$ *for some MV-algebra A', then A' is hypernormal, too.*

Proof. Suppose A is hypernormal and $\mathrm{Spec}(S) \cong \mathrm{Spec}(A')$. By Proposition 3.4.25, there is an order reversing poset isomorphism from $\mathrm{Spec}(A)$ onto $\mathrm{Spec}(A')$. Suppose $M' \in \mathrm{Max}(A')$. Then for some minimal ideal m of A we have $\phi(m) = M'$. Suppose $P', Q' \subseteq M'$. Then we must have $m \subseteq P, Q$ where $\phi(P) = P'$, $\phi(Q) = Q'$. As A is hypernormal we have, say, $P \subseteq Q$. Thus we arrive at $Q' \subseteq P'$. The result follows. $\square$

Chapter 4

MV-algebras and sheaves

Sheaves are a very useful tool in representation theory. In literature there exist representation theorems for many structures as algebras of global sections of a sheaf, for example, for rings (see [23], [49] and [52]), for l-groups (see [22] and [10]). In attempting to generalize the sheaf representation theorems for these different structures, many authors provided to give sheaf representation theorems in Universal Algebra (for example see [34], [38], [53], [56]).

In this chapter, we present some results present in literature and some new results about sheaf representation of MV-algebras.

4.1 Sheaf representations of MV-algebras by sheaves

In this paragraph, we provide three different representations of MV-algebras by sheaves. The first one is due to Filipoiu and Georgescu (see [32]) and represent MV-algebras as MV-algebras of global sections of a sheaf whose stalks are local MV-algebras and the base space is the space of maximal ideals of the represented MV-algebras with the Zariski topology. The second one is due to Dubuc and Poveda (see [29]) and represent MV-algebras as MV-algebras of global sections of a sheaf whose stalks are MV-chains and the base space is the spectrum of prime ideals with the CoZariski topology.

The last one (see [31]) is obtained from Filipoiu and Georgescu representation expanding the base space to the spectrum of prime ideals with the Zariski topology.

Being the class of MV-algebras an equational class, in according to Definition 2.1.6, we have the following

Definition 4.1.1. A sheaf of sets (F, π, X) is a sheaf of MV-algebras if and only if for each $x \in X$, the stalks F_x are MV-algebras.

4.1.1 Filipoiu and Georgescu sheaf representation for MV-algebras

In this section we give a revised version of the Filipoiu and Georgescu sheaf representation for MV-algebras.

Let A be an MV-algebra. In Chapter 1, we have proved that $\bigcap\{O(M) \mid M \in \mathrm{Max}(A)\} = \{0\}$, where $O(M) = \cap\{m \mid m \in \mathrm{Min}(A)\}$. This provides a representation of A as subdirect product of the family $\{A/O(M) \mid M \in \mathrm{Max}(A)\}$. Since for each $M \in Max(A)$, $O(M)$ is a primary ideal (see Theorem 1.5.24), it results that the corresponding quotient $A/O(M)$ is a local MV-algebra.

Using methods of Chapter 2, one can construct a sheaf of MV-algebras $(E_A, \pi, \mathrm{Max}(A))$ where E_A is the disjoint union of the quotients $A/O(M)$, i.e.

$$E_A = \left\{ \left(\frac{a}{O(M)}, M \right) \mid a \in A, M \in \mathrm{Max}(A) \right\},$$

$\pi : E_A \to \mathrm{Max}(A)$ is the natural projection.
In the sequel let a_M denote the pair $(\frac{a}{O(M)}, M)$, then

$$a_M = b_N \text{ if and only if } M = N \text{ and } d(a, b) \in O(M).$$

Proposition 4.1.2. *The Zariski topology on* $\mathrm{Max}(A)$ *is an* S*-topology with respect to the family* $\{O(M) \mid M \in \mathrm{Max}(A)\}$*.*

Proof. Let $M \in \mathrm{Max}(A)$, we have to prove that if $a_M = b_M$ then there exists an open neighbourhood U of M such that $a_N = b_N$ for each $N \in U$.

From $a_M = b_M$, we have that $d(a,b) \in O(M)$ and so $M \in H(d(a,b))$, which is an open set of $\mathrm{Spec}(A)$ (see Proposition 1.6.1). Denote by $H_M(d(a,b))$ the open set $H(d(a,b)) \cap \mathrm{Max}(A)$ of $\mathrm{Max}(A)$. Clearly $M \in H_M(d(a,b))$. Now let $N \in H_M(d(a,b))$, so $d(a,b) \in O(N)$, i.e. $a_n = b_n$. This proves that the Zariski topology on $\mathrm{Max}(A)$ is an S-topology. □

Theorem 4.1.3. *$(E_A, \pi, \mathrm{Max}(A))$ is a sheaf of MV-algebras such that for each $M \in \mathrm{Max}(A)$ the stalk E_M is isomorphic to the quotient $A/O(M)$.*

Proof. The thesis follows from Proposition 4.1.2 and Theorem 2.2.3. □

Remark 4.1.4. Being A a subdirect product of $\{A/O(M) \mid M \in \mathrm{Max}(A)\}$, from Theorem 2.3.2 we obtain that for each $a \in A$, the map $\hat{a} : \mathrm{Max}(A) \to E_A$ defined by $\hat{a}(M) = a_M$ is a global section and the map $\alpha : A \to \Gamma(\mathrm{Max}(A), E_A)$ is a monomorphism of MV-algebras. Moreover the family $\{\hat{a}(U) : a \in A, U \text{ open in } \mathrm{Max}(A)\}$ provides a basis for the topology on E_A.

Proposition 4.1.5. *For each $\sigma \in \Gamma(\mathrm{Max}(A), E_A)$ there is an element $a \in A$ such that $\sigma = \hat{a}$.*

Proof. Let $\sigma : \mathrm{Max}(A) \to E_A$ a global section. Then for each $M \in \mathrm{Max}(A)$ there exists $a^M \in A$ such that $\sigma(M) = (\frac{a^M}{O(M)}, M)$, i.e. $\sigma(M) = \widehat{a^M}(M)$. Hence, for Proposition 2.1.2 (i) there is a basic open neighbourhood $S_M = \{N \in \mathrm{Max}(A) \mid \sigma(N) = \widehat{a^M}(N)\}$.
The family $\{S_M\}_{M \in \mathrm{Max}(A)}$ is an open covering of the compact topological space $\mathrm{Max}(A)$. So it is possible to extract from $\{S_M\}_{M\in\mathrm{Max}(A)}$ a finite covering $S_1, \dots, S_n$ of $\mathrm{Max}(A)$. Let a_i the element of A which corresponds to S_i for each $i = 1, \dots, n$. Since S_i is a basic open of $\mathrm{Max}(A)$, there exists $b_i \in A$ such that $S_i = S(b_i) = \{N \in Max(A) \mid b_i \notin N\}$ for each $i = 1, \dots, n$ and $b_1 \vee \ldots \vee b_n = 1$.
Consider $S_i \cap S_i = S(b_i) \cap S(b_j) = S(b_i \wedge b_j)$. If $S(b_i \wedge b_j) = \emptyset$, set $H_{ij} = A$. Suppose, now, that $S(b_i \wedge b_j) \neq \emptyset$, so for each $N \in S(b_i \wedge b_j)$, $\widehat{a_i}(N) = \widehat{a_j}(N)$, i.e. $d(a_i, a_j) \in O(N)$ for each $N \in S(b_i \wedge b_j)$. Set $H_{ij} = \cap\{O(N) \mid N \in S(b_i \wedge b_j)\}$. So for each $i, j = 1, \dots, n$ it results that $d(a_i, a_j) \in H_{ij}$. Moreover for each $i, j = 1, \dots, n$, $d(a_i, a_j) \leq d(a_i, a_h) \oplus d(a_h, a_j) \in H_{ih} \oplus H_{hj}$, for

each $h = 1, \ldots n$. Remark that $\bigoplus_{k=1}^{n}(b_k] = (b_1 \vee \ldots \vee b_n] = A$. For each $k = 1, \ldots, n$, set $I_k = (b_k]$. Hence

$$\begin{aligned} d(a_i, a_j) &\in \bigcap_{h=1}^{n} (H_{ih} \oplus H_{hj}) \\ &= \left(\bigcap_{h=1}^{n} (H_{ih} \oplus H_{hj})\right) \bigcap \bigoplus_{k=1}^{n} I_k \\ &= \bigoplus_{k=1}^{n} \left(\bigcap_{h=1}^{n} (H_{ih} \oplus H_{hj}) \cap I_k\right) \\ &\subseteq \bigoplus_{k=1}^{n} ((H_{ik} \oplus H_{kj}) \cap I_k) \\ &= \left(\bigoplus_{k=1}^{n} (H_{ik} \cap I_k)\right) \oplus \left(\bigoplus_{k=1}^{n} (H_{kj} \cap I_k)\right). \end{aligned}$$

Set $\oplus_{k=1}^{n}(H_{ik} \cap I_k) = J_i$ and $\oplus_{k=1}^{n}(H_{kj} \cap I_k) = J_j$, then $a_i \equiv a_j\ (J_i \oplus J_j)$, for $i, j = 1, 2, \ldots, n$. By Theorem 1.4.11, there is $a \in A$, such that $a_i \equiv a\ (J_i)$, for $i = 1, 2, \ldots, n$.

It results that for each $i = 1, \ldots n$, $J_i \subseteq I_i^{\perp}$. To prove this, it is sufficient to prove that $H_{ik} \cap I_k \subseteq I_i^{\perp}$, for each $k = 1, \ldots, n$, with fixed i. Now, let $h = 1, \ldots, n$. If $S(b_i \wedge b_k) = \emptyset$, $H_{ik} = A$ and $b_i \wedge b_k = 0$, i.e. $b_k \in b_i^{\perp}$. So $H_{ik} \cap I_k = I_k = (b_k] \subseteq b_i^{\perp}$. If $S(b_i \wedge b_k) \neq \emptyset$, $H_{ik} = \cap\{O(N) \mid N \in S(b_i \wedge b_k)\}$ and $b_i \wedge b_k > 0$. Now suppose that there is $x \in b_i^{\perp}$ and $x \notin H_{ik} \cap I_k$. So $x \notin H_{ik}$ or $x \notin I_k$. If $x \notin H_{ik}$, $x \notin O(N)$ for each $N \in S(b_i \wedge b_k)$. Remember that $O(N) = \cup\{y^{\perp} \mid y \notin N\}$. So for each $N \in S(b_i \wedge b_k)$, $x \notin y^{\perp}$, for each $y \notin N$. Being $N \in S(b_i \wedge b_k)$, $b_i \wedge b_k \notin N$ and $b_i \notin N$. Hence $x \notin b_i^{\perp}$, that is absurd. So $x \notin I_k = (b_k]$, i.e. $b_k < x$. Therefore $b_k \wedge b_i < x \wedge b_i = 0$. This implies $b_k \wedge b_i = 0$ which conflicts with the assumption that $b_k \wedge b_i > 0$.

Hence for each $i = 1, \ldots, n$, $d(a_i, a) \in J_i \subseteq b_i^{\perp} \subseteq \cap\{O(N) \mid N \in S(b_i)\}$. Indeed, for each $N \in S(b_i)$, $b_i \notin N$ and so $b_i^{\perp} \subseteq O(N)$. Therefore $d(a_i, a) \in O(N)$, for each $N \in S_i$ and $i = 1, \ldots, n$. It follows that $\hat{a}_{|S_i} = \hat{a}_{i_{|S_i}} = \sigma_{|U_i}$ for every $i = 1, 2, \ldots, n$, that is $\sigma = \hat{a}$. □

From Remark 4.1.4 and Proposition 4.1.5 we obtain the following theorem.

Theorem 4.1.6. *Every MV-algebra A is isomorphic to the MV-algebra of all global sections of a sheaf with local MV-algebras as stalks and* $\mathrm{Max}(A)$ *as base space.*

4.1.2 Dubuc and Poveda sheaf representation for MV-algebras

In [29], Dubuc and Poveda provide a sheaf representation of MV-algebras proceeding from the Chang subdirect representation (Theorem 1.7.4).

Let A be an MV-algebra, it is possible to associate with A the sheaf $(F_A, \pi, \mathrm{coSpec}(A))$, where

- $\mathrm{coSpec}(A)$ is the spectrum of A with the coZariski topology (see Paragraph 1.6),
- F_A is the disjoint union of the MV-chains A/P, $P \in \mathrm{coSpec}(A)$, that is, $F_A = \{(a/P, P) \mid a \in A, P \in \mathrm{coSpec}(A)\}$,
- the map $\pi : F_A \to \mathrm{coSpec}(A)$ defined by $\pi(a/P, P) = P$ is the natural projection.

As in Filipoiu and Georgescu representation, each element $a \in A$ defines a global section (as a function of sets) $\hat{a} : \mathrm{coSpec}(A) \to F_A$ as $\hat{a}(P) = (a/P, P)$. As a basis for the topology in F_A, one consider the family $\{\hat{a}(W_b) \mid a, b \in A\}$. With this topology π becomes a local homeomorphism and every global section $\hat{a}$ a continuous and open function.

Proposition 4.1.7. *Let A be an MV-algebra.* $(F_A, \pi, \mathrm{coSpec}(A))$ *is a sheaf of MV-algebras with stalks that are linearly ordered.*

In particular, Dubuc and Poveda obtained the following representation theorem.

Theorem 4.1.8. *Every MV-algebra A is isomorphic to the MV-algebra of all global section of a sheaf with MV-chains as stalks and* $\mathrm{coSpec}(A)$ *as base space.*

4.1.3 A new sheaf representation of MV-algebras

In this section we present a new sheaf representation of MV-algebras inspired by methods of Bigard, Keimel and Wolfenstein [10]. This representation turns out to be a mixture of the previous approach by Filipoiu and Georgescu and by Dubuc and Poveda. Indeed, we extend the base space of the Filipoiu and Georgescu representation to the prime spectrum of an MV-algebra, as in Dubuc and Poveda representation up to the topology.
In this thesis we give a more brief presentation of this representation with respect to the paper [31].

Let an MV-algebra A, consider the family $\{O(P) \mid P \in \mathrm{Spec}(A)\}$. Even in this case, these ideals provide a representation of A as subdirect product of the family $\{A/O(P) \mid P \in \mathrm{Spec}(A)\}$. Each quotient is a local MV-algebra (see Section 4.1.1). So using methods of Chapter 2, we can construct a sheaf of MV-algebras $(E^s_A, \pi, \mathrm{Spec}(A))$ which is an extension of $(E_A, \pi, \mathrm{Max}(A))$ in Section 4.1.1. Hence, E^s_A is the disjoint union of the quotients $A/O(P)$, i.e. $E^s_A = \{a_P \mid a \in A, P \in \mathrm{Spec}(A)\}$ whereas $\pi : E^s_A \to \mathrm{Spec}(A)$ defined as $\pi(a_P) = P$ is the natural projection.
Moreover, we have that

$$a_P = b_Q \text{ if and only if } P = Q \text{ and } d(a, b) \in O(P).$$

In analogy to section 4.1.1 we obtain the following results.

Proposition 4.1.9. *The Zariski topology on* $\mathrm{Spec}(A)$ *is an S-topology with respect to the family* $\{O(P) \mid P \in \mathrm{Spec}(A)\}$.

Proof. Let $P \in \mathrm{Spec}(A)$, we have to prove that if $a_P = b_P$ then there exists an open neighbourhood U of P such that $a_Q = b_Q$ for each $Q \in U$.
From $a_P = b_P$, we have that $d(a, b) \in O(P)$ and so $P \in H(d(a, b))$, which

is an open set of $\mathrm{Spec}(A)$ (see Proposition 1.6.1). Now let $Q \in H(d(a,b))$, so $d(a,b) \in O(Q)$, i.e. $a_Q = b_Q$. This proves that the Zariski topology on $\mathrm{Spec}(A)$ is an S-topology. □

Theorem 4.1.10. $(E_A^s, \pi, \mathrm{Spec}(A))$ *is a sheaf of MV-algebras such that for each* $P \in \mathrm{Spec}(A)$ *the stalk* E_P^s *is isomorphic to the quotient* $A/O(P)$.

Proof. The thesis follows from Proposition 4.1.9 and Theorem 2.2.3. □

Remark 4.1.11. Being A a subdirect product of $\{A/O(P) \mid P \in \mathrm{Spec}(A)\}$, from Theorem 2.3.2 we obtain again that for each $a \in A$, the map $\hat{a}$: $\mathrm{Spec}(A) \to E_A^s$ defined by $\hat{a}(P) = a_P$ is a global section, the map $\alpha_s : A \to \Gamma(\mathrm{Spec}(A), E_A^s)$ is a monomorphism of MV-algebras and the set $\{\hat{a}(U) : a \in A, U \text{ open of } \mathrm{Spec}(A)\}$ is a basis for the topology on E_A^s.

Theorem 4.1.12. *For every global section* σ *of the sheaf* $(E_A^s, \pi, \mathrm{Spec}(A))$, *there exists* $a \in A$, *such that* $\sigma = \hat{a}$.

Proof. Let $\sigma : P \in \mathrm{Spec}(A) \to \sigma(P) \in E_A^s$ a global section of F. Fix $P_0 \in \mathrm{Spec}(A)$. $\sigma(P_0) \in E_{P_0}^s = \frac{A}{P_0} \times \{P_0\}$, then $\sigma(P_0) = a_{P_0}$, for some $a \in A$. Hence $\sigma(P_0) = \hat{a}(P_0)$.

Let $U(I)$ be a neighbourhood of P_0 and $U(I,a) = \{a_P : P \in U(I)\}$. Then $\sigma(P_0) \in U(I,a)$ and $\sigma^{-1}(\sigma(P_0)) = P_0 \in \sigma^{-1}(U(I,a))$. Since σ is continuous, $\sigma^{-1}(U(I,a))$ is open in $\mathrm{Spec}(A)$, so it is a neighbourhood of P_0.

If $P \in \sigma^{-1}(U(I,a))$, then it results $\sigma(P) \in \sigma(\sigma^{-1}(U(I,a))) \subseteq U(I,a)$. Moreover $\sigma(P) \in E_P^s$, so $\sigma(P) \in E_P^s \cap U(I,a)$, then $\sigma(P) = a_P$, that is $\sigma(P) = \hat{a}(P)$.

We have just proved that for every $P \in \mathrm{Spec}(A)$, there is a neighbourhood U_P of P and an element $a^P \in A$, such that $\sigma_{|U_P} = \widehat{a^P}_{|U_P}$.

The family $\{U_P\}_{P \in \mathrm{Spec}(A)}$ is an open covering of the compact topological space $\mathrm{Spec}(A)$. Let $U_{P_1}, U_{P_2}, \ldots, U_{P_n}$ be a finite covering of $\mathrm{Spec}(A)$ contained in $\{U_P\}_{P \in \mathrm{Spec}(A)}$. Set $a^{P_i} = a_i$ and $U_{P_i} = U_i = U(I_i)$.

$$\mathrm{Spec}(A) = \bigcup_{i=1}^{n} (U(I_i)) = U\left(\oplus_{i=1}^{n} I_i\right),$$

and $I_1 \oplus I_2 \oplus \ldots \oplus I_n = A$.

Suppose that $U_i \cap U_j \neq \emptyset$ and $P \in U_i \cap U_j = U(I_i \cap I_j)$, then $\sigma(P) = (a_i)_P = (a_j)_P$. Hence $a_i \equiv a_j\ (H_{ij})$, where $H_{ij} = \bigcap \{P : P \in U(I_i \cap I_j)\} = (I_i \cap I_j)^{\perp}$, by Lemma 1.5.20. For $U_i \cap U_j = \emptyset$, H_{ij} denotes the whole algebra A. Then for every $(i,j) \in \{1,2,\ldots,n\}^2$, $d(a_i,a_j) \leq d(a_i,a_h) \oplus d(a_h,a_j) \in H_{ih} \oplus H_{hj}$, for every $h \in \{1,2,\ldots,n\}$. Hence

$$
\begin{aligned}
d(a_i,a_j) &\in \bigcap_{h=1}^{n} (H_{ih} \oplus H_{hj}) \\
&= \left(\bigcap_{h=1}^{n} (H_{ih} \oplus H_{hj})\right) \bigcap \bigoplus_{k=1}^{n} I_k \\
&= \bigoplus_{k=1}^{n} \left(\bigcap_{h=1}^{n} (H_{ih} \oplus H_{hj}) \cap I_k\right) \\
&\subseteq \bigoplus_{k=1}^{n} ((H_{ik} \oplus H_{kj}) \cap I_k) \\
&= \left(\bigoplus_{k=1}^{n} (H_{ik} \cap I_k)\right) \oplus \left(\bigoplus_{k=1}^{n} (H_{kj} \cap I_k)\right).
\end{aligned}
$$

Set $\oplus_{k=1}^{n} (H_{ik} \cap I_k) = J_i$ and $\oplus_{k=1}^{n} (H_{kj} \cap I_k) = J_j$, then $a_i \equiv a_j\ (J_i \oplus J_j)$, for $i,j = 1,2,\ldots,n$. By Theorem 1.4.11, there is $a \in A$, such that $a_i \equiv a\ (J_i)$, for $i = 1,2,\ldots,n$.

Note that $J_i \cap I_i = (\oplus_{k=1}^{n} (H_{ik} \cap I_k)) \cap I_i = \oplus_{k=1}^{n} (H_{ik} \cap I_k \cap I_i)$. If $U_i \cap U_k \neq \emptyset$, then $H_{ik} = (I_i \cap I_k)^{\perp}$, hence $H_{ik} \cap I_k \cap I_i = \{0\}$. If $U_i \cap U_k = \emptyset$, then $U(I_i \cap I_k) = \emptyset$, hence $I_k \cap I_i = \{0\}$. In every case $H_{ik} \cap I_k \cap I_i = \{0\}$, hence $J_i \cap I_i = \{0\}$. Then $J_i \subseteq I_i^{\perp}$ and $d(a_i,a) \in I_i^{\perp} = \bigcap \{P \in \mathrm{Spec}(A) : P \in U(I_i)\} = \bigcap \{O(P) \in \mathrm{Spec}(A) : P \in U(I_i)\}$. Hence $a_i \equiv a\ (O(P))$ for every $P \in U(I_i)$ and for every $i = 1,2,\ldots,n$. It follows that $\hat{a}_{|U_i} = \hat{a}_{i|U_i} = \sigma_{|U_i}$ for every $i = 1,2,\ldots,n$, that is $\sigma = \hat{a}$. □

From Remark 4.1.11 and Proposition 4.1.12 we obtain the following theorem.

Theorem 4.1.13. *Every MV-algebra A is isomorphic to the MV-algebra of* *all global sections of a sheaf with local MV-algebras as stalks and* $\mathrm{Spec}(A)$ *as base space.*

4.2 Sheaf representation of MV-algebras with the minimal spectrum compact

At this point, we have a sheaf representation of MV-algebras with base space the maximal spectrum and the prime spectrum. It is natural to wonder what happens if we use the space of minimal prime ideals as base space in the Filipoiu and Georgescu representation. Nevertheless, in this way we are not able to represent all MV-algebras, but only MV-algebras with $\mathrm{Min}(A)$ compact.

As seen in Paragraph 1.6, for any MV-algebra A, $\mathrm{Min}(A)$ with the topology inherited from the Zariski topology on $\mathrm{Spec}(A)$ is a Hausdorff zero-dimensional space, i.e. $\mathrm{Min}(A)$ is a Hausdorff space with a basis of clopen sets of the form $D(a) = U(a) \cap \mathrm{Min}(A) = \{P \in \mathrm{Min}(A) : a \notin P\}$.

Remark 4.2.1. For each $P \in \mathrm{Min}(A)$ $O(P) = P$, so the quotient $\frac{A}{O(P)}$ is a MV-chain.

For each MV-algebra A, denote by $F_A^m = (E_A^m, \pi, \mathrm{Min}(A))$ the sheaf obtained from $F_A = (E_A, \pi, \mathrm{Max}(A))$ (see Theorem 4.1.3) by restricting the base space to $\mathrm{Min}(A)$.

Theorem 4.2.2. *For each MV-algebra A, the sheaf $(E_A^m, \pi, \mathrm{Min}(A))$ is a Hausdorff sheaf of MV-chains.*

Proof. We shall prove that the total space E_A^m is also Hausdorff. Let $a_P \neq b_Q$ in E_A^m.

If $P \neq Q$, since $\mathrm{Min}(A)$ is Hausdorff, there exist $U \ni P$ and $V \ni Q$ disjoint open sets in $\mathrm{Min}(A)$. Thus $\hat{a}(U) \ni a_P$ and $\hat{b}(V) \ni b_Q$ are disjoint open sets in E_A^m.

If $P = Q$ then $\frac{a}{P} \neq \frac{b}{P}$, so $d(a,b) \notin P$ and $P \in D(d(a,b))$. Then $\hat{a}(D(d(a,b))) \ni a_P$ and $\hat{b}(D(d(a,b))) \ni b_P$ are disjoint open sets in E_A^m.

So E_A^m is Hausdorff. □

It is easy to prove the following

Theorem 4.2.3. *For each MV-algebra A, the map $\alpha_m : A \to \Gamma(\mathrm{Min}(A), E_A^m)$ defined by $\alpha_m(a) = \hat{a}$ is a monomorphism.*

To prove that α_m is an isomorphism, we have to introduce a further condition on the MV-algebra A.

Consider MV-algebras with the minimal space $\mathrm{Min}(A)$ compact with respect to the Zariski topology inherited by $\mathrm{Spec}(A)$. When $\mathrm{Min}(A)$ is compact, $\mathrm{Min}(A)$ is a Stone space (see, Section 1.7.6). From this and Theorem 4.2.2 we have:

Theorem 4.2.4. *For each MV-algebra A with* $\mathrm{Min}(A)$ *compact, F_A^m is a Hausdorff sheaf of MV-chains over a Stone space.*

Theorem 4.2.5. *Let A be an MV-algebra with* $\mathrm{Min}(A)$ *compact. For every global section σ of the sheaf F_A^m, there exists $a \in A$ such that $\sigma = \hat{a}$.*

Proof. The proof is analogous to that of Theorem 4.1.12, using the compactness of $\mathrm{Min}(A)$ and Corollary 1.5.22. □

Theorem 4.2.6. *Every MV-algebra A, with* $\mathrm{Min}(A)$ *compact is isomorphic to the MV-algebra of all global sections of the sheaf F_A^m.*

Proof. It follows from Theorem 4.2.5 and Proposition 4.2.3. □

Theorem 4.2.7. *Every MV-algebra A with* $\mathrm{Min}(A)$ *compact is isomorphic to the MV-algebra of all global sections of a Hausdorff sheaf of MV-chains.*

Proof. It follows from Remark 4.2.1, Theorem 4.2.2 and Theorem 4.2.6. □

Let $\hat{A}_m$ denote the MV-algebra of all global sections of the sheaf F_A^m. We define an order relation on E_A^m inherited from A. Let $x_m, y_n \in E_A^m$, we have

$$x_m \le y_n \text{ iff } m = n \text{ and there exists } \frac{z}{m} \in \frac{A}{m} \text{ such that } \frac{x \oplus z}{m} = \frac{y}{m}.$$

It's easy to check that $\leq$ is an order relation, with

$$\left(\frac{x}{m}, m\right) \wedge \left(\frac{y}{m}, m\right) = \left(\frac{x \wedge y}{m}, m\right)$$

and

$$\left(\frac{x}{m}, m\right) \vee \left(\frac{y}{m}, m\right) = \left(\frac{x \vee y}{m}, m\right).$$

Lemma 4.2.8. *Let A be an MV-algebra and $x, y \in A$. Then*

$$\hat{x} \in \hat{y}^{\perp} \quad \textit{iff} \quad x \in y^{\perp}.$$

Proof. Let $\hat{x} \in \hat{y}^{\perp}$. For each $m \in \mathrm{Min}(A)$, we have $\hat{x}(m) \wedge \hat{y}(m) = 0$ iff $0 = x_m \wedge y_m = (x \wedge y)_m$ iff $x \wedge y \in m$. So $x \wedge y = 0$ iff $x \in y^{\perp}$. □

Set $\widehat{a^{\perp}} = \{\hat{x} : x \in a^{\perp}\}$. With the above notations, it is easy to prove that

$$\widehat{a^{\perp}} = \hat{a}^{\perp} \tag{4.1}$$

Theorem 4.2.9. *Let A be an MV-algebra with* $\mathrm{Min}(A)$ *compact, then the space* $\mathrm{Min}(\hat{A})_m$ *is compact.*

Proof. The thesis follows from Lemma 4.2.8, Theorem 1.7.22 and (4.1). □

Remark 4.2.10. We would like to stress here that limiting the base space to the minimal prime ideals space the representations described in Paragraph 4.1 coincide.

4.3 A sheaf representation for MV-semirings

In [17], Chermnykh gives a sheaf representation for commutative semirings in analogy to the sheaf representation given by Grothendieck for rings. We specialize this representation for MV-semirings. The main tool for such a representation is given by the localization of MV-semirings over prime ideals. Although these localizations are not MV-semirings, we still have a representation theorem for MV-semirings in terms of sections of sheaves that can be easily translated in an MV-algebraic fashion.

In this section we are going to give a representation of MV-algebras as MV-algebras of particular continuous functions. Firstly, we present a representation of MV-semirings by MV-semirings of continuous sections in a sheaf of commutative semirings. Using the categorical equivalence presented in section 2, we obtain a representation of MV-algebras.

Let S be a commutative idempotent semiring with unit and $D \subseteq S \setminus \{0\}$ a multiplicative monoid, i.e. $1 \in D$ and D is closed under $\cdot$. From S we can construct a semiring in the following way.

Let (a, b), $(c, d) \in S \times D$ and define $(a, b) \sim (c, d)$ if and only if there exists an element $k \in D$ such that $adk = bck$. It is easy to verify that $\sim$ is an equivalence relation. In the follow we denote by S_D the quotient of $S \times D$ by $\sim$ and by a/b the equivalence class of the pair (a, b). It results that S_D is a semiring with the following operations:

$$a/b+c/d = (ad + bc)/bd,$$

$$a/b \cdot c/d = ac/bd.$$

The $\mathbf{0}$ is the class $0/1$ and the $\mathbf{1}$ is the class $1/1$.

Remark 4.3.1. Let $s/t \in S_D$. It results that $s/t = 0/1$ if and only if there exists $k \in D$ such that $s \cdot 1 \cdot k = t \cdot 0 \cdot k$, i.e. $s \cdot k = 0$.

Proposition 4.3.2. *Let $(S, +, \cdot, 0, 1)$ be a commutative additively idempotent semiring and $D \subseteq S \setminus \{0\}$ a multiplicative monoid. Then $(S_D, +, \cdot, \mathbf{0}, \mathbf{1})$ is also a commutative additively idempotent semiring.*

Proof. It follows from the operations of S. □

Now let $P \in \mathrm{Spec}(S)$ and set $D = S \setminus P$. D is a multiplicative monoid and S_D is a local semiring (see [17]). We write S_P for S_D in this case and S_P is named *the localization of S at P*.

Proposition 4.3.3. *Let $(S, +, \cdot, 0, 1)$ be a commutative additively idempotent semiring and $P \in \mathrm{Spec}(S)$. The set $\mathbf{P} = \{a/x : \ a \in P, \ x \notin P\}$ is the maximal ideal of S_P.*

Proof. $\mathbf{P}$ is clearly an ideal of S_P. Now let $a/x,\ b/y \in S_P$ such that $ab/xy \in \mathbf{P}$. So there exists $c \in P$ and $z \notin P$ such that $ab/xy = c/z$, i.e. there exists $w \notin P$ such that $abzw = cxyw \in P$. For the primality of P it follows that $ab \in P$ and so $a \in P$ or $b \in P$. From this we may infer that $\mathbf{P} \in \mathrm{Spec}(S_P)$.

Now let $\mathbf{Q}$ be an ideal of S_P such that $\mathbf{P} \subset \mathbf{Q}$. Let $a/x \in \mathbf{Q} \setminus \mathbf{P}$, i.e. $a \notin P$ so $x/a \in S_P$. Hence $a/x{\cdot}x/a = ax/ax = \mathbf{1} \in \mathbf{Q}$ and $\mathbf{Q} = S_P$. It follows that $\mathbf{P}$ is maximal. □

Now let S be an MV-semiring and $P \in \mathrm{Spec}(S_r)$. It results

Proposition 4.3.4. *For any $s \in S,\ t \notin P$, $s/t = \mathbf{0}$ if and only if $s \in P$ and $s^* \notin P$.*

Proof. Let $s \in P$ such that $s^* \notin P$. Since $ss^* = 0$, $(s,t) \sim (0,1)$ for each $t \notin P$.

Now consider $s \notin P$. $S \setminus P$ is closed under $\cdot$, so for each $k \notin P$, $sk > 0$ and by the remark 4.3.1 $s/t \neq 0/1$ for each $t \notin P$.

Consider now $s,\ s^* \in P$. For each $k \in S$ such that $sk = 0$ it results that $k \leq s^* \in P$. By Lemma 3.4.2, $k \in P$ and so $s/t \neq 0/1$ for each $t \notin P$. □

If S is an MV-semiring and $P \in \mathrm{Spec}(S_r)$ in general S_P is not an MV-semiring. Consider the following example.

Example 4.3.5. *Let $S = (C,\ +,\ \cdot,\ 0,\ 1)$ be the Chang MV-semiring, i.e., C is the Chang algebra. Let M be the radical of C. Then M is prime in S, so $D = S \setminus M = \{(nc)^* \mid n \geq -\}$, c the atom of C. If S_M were an MV-semiring we would have for $n > 0$, $1/(nc)^* + 1/1 = 1/1$, so $(1 + (nc)^*)/(nc)^* = 1/1$. But $1 + (nc)^* = 1$ so we would have $1/(nc)^* = 1/1$. Thus for some $w \in D$ we have $w = (nc)^* w$. Now $w = (mc)^*$ for some $m \geq 0$. So we obtain $(mc)^* = (mc)^*(nc)^* = ((m+n)c)^*$. Hence $mc = (m+n)c$ and this implies $m + n = m$ so $n = 0$ contrary to assumption.*

Proposition 4.3.6. *Let S be an MV-semiring and $P \in \mathrm{Spec}(S_r)$. S_P is a commutative additively idempotent semiring.*

Proof. S_P is commutative since S is commutative. Now let $[(a,b)] \in S_P$

$$\begin{aligned} [(a,b)] + [(a,b)] &= [(ab+ba, bb)] \\ &= [(ab, bb)] \\ &= [(a,b)] \cdot [(b,b)] \\ &= [(a,b)] \end{aligned}$$

since $[(b,b)] = [(1,1)]$. □

In [17], Chermnykh proved the following theorem.

Theorem 4.3.7. *A commutative semiring is isomorphic to the semiring of all global sections of the Grothendieck sheaf.*

Let S be a commutative semiring, the Grothendieck sheaf of S is the triple $G(S) = (\mathrm{Spec}(S), E_S, \pi_S)$ where $E_S = \bigcup\{S_P \times \{P\} : P \in \mathrm{Spec}(S)\}$ and $\pi : E_S \to \mathrm{Spec}(S)$, defined as $\pi(a/b, P) = P$, is a local homeomorphism. In the follow, we denote by $[s/t]_P$ the element $(s/t, P) \in E_S$ and by $\widehat{S}$ the semiring of all global sections i.e. of the continuous maps of type $\widehat{s} \mid \mathrm{Spec}(S) \to E_S$ such that $\widehat{s}(P) = [s/1]_P \in S_P$.

$$\begin{aligned} (\hat{s}\widehat{+}\hat{t})(P) &= \hat{s}(P) + \hat{t}(P) = \widehat{s+t}(P) \\ (\hat{s}\hat{\cdot}\hat{t})(P) &= \hat{s}(P) \cdot \hat{t}(P) = \widehat{s \cdot t}(P) \end{aligned}$$

and

$$\begin{aligned} \hat{0} : P \in \mathrm{Spec}(S) &\to 0_P \in E_S \\ \hat{1} : P \in \mathrm{Spec}(S) &\to 1_P \in E_S \end{aligned}$$

The isomorphism between S and $\widehat{S}$ is given by $\varphi : s \in S \to \widehat{s} \in \widehat{S}$

Theorem 4.3.8. *Let $(S, +, \cdot, 0, 1)$ be an MV-semiring. $(\widehat{S}, \widehat{+}, \widehat{\cdot}, \widehat{0}, \widehat{1})$ is a commutative additively idempotent semiring.*

Proof. $\widehat{S}$ is trivially a commutative semiring with unit. We shall prove that $\widehat{S}$ is additively idempotent. Let $\hat{s} \in \widehat{S}$, for each $P \in \mathrm{Spec}(S_r)$ we have $(\hat{s} + \hat{s})(P) = \hat{s}(P) + \hat{s}(P) = [s/1]_P + [s/1]_P = (s/1 + s/1,\ P) = ((s+s)/1,\ P) = (s/1,\ P) = [s/1]_P = \hat{s}(P)$. □

Remark 4.3.9. Since $\widehat{S}$ is additively idempotent we can define an order in such a way that $\widehat{s}\widehat{\leq}\widehat{t}$ if and only if $\widehat{s}\widehat{+}\widehat{t} = \widehat{s+t} = \widehat{t}$ with $\widehat{s}, \widehat{t} \in \widehat{S}$.

Lemma 4.3.10. *Let S be an MV-semiring and $s, t \in S$ such that $s \leq t$. Then $\widehat{s}\widehat{\leq}\widehat{t}$.*

Proof. Trivial □

Theorem 4.3.11. *Let S be an MV-semiring. Then $(\widehat{S}, \widehat{+}, \widehat{\cdot}, \widehat{0}, \widehat{1})$ is an MV-semiring.*

Proof. We shall prove that for each $\widehat{s} \in \widehat{S}$ there is the greatest element $\widehat{s}^*$ such that $\widehat{s}\widehat{\cdot}\widehat{s}^* = \widehat{0}$. It results that $\hat{s}^* = \widehat{s^*}$ for each $\hat{s} \in \widehat{S}$. Indeed, for each $P \in \mathrm{Spec}(S)$, $\hat{s}(P) \cdot \widehat{s^*}(P) = \widehat{ss^*}(P) = \widehat{0}(P)$. So $\widehat{s} \cdot \widehat{s^*} = \widehat{0}$. Let $\widehat{t} \in \widehat{S}$ such that $\widehat{s}\widehat{\cdot}\widehat{t} = \widehat{0}$, i.e. $\widehat{s \cdot t} = \widehat{0}$. So $\varphi(s \cdot t) = \varphi(0)$ and since φ is a bijection between S and $\widehat{S}$ it follows that $s \cdot t = 0$ and $t \leq s^*$. By Lemma 4.3.10 we have $\widehat{t}\widehat{\leq}\widehat{s^*}$.

It is easy to verify that $\widehat{s}\widehat{+}\widehat{t} = (\widehat{s^*}\widehat{\cdot}(\widehat{s^*} \cdot \widehat{t})^*)^*$. Indeed

$$
\begin{aligned}
\widehat{s}\widehat{+}\widehat{t} &= \widehat{s+t} \\
&= s^* \cdot \widehat{(s^* \cdot t)^*})^* \\
&= (\widehat{s^*}\widehat{\cdot}(\widehat{s^*} \cdot \widehat{t})^*)^*.
\end{aligned}
$$

This prove that $\widehat{S}$ is an MV-semiring. □

Theorem 4.3.12. *An MV-semiring is isomorphic to the MV-semiring of all global sections of the Grothendieck sheaf.*

Proof. Let S be an MV-semiring. Since $\varphi(s)^* = \hat{s}^* = \widehat{s^*} = \varphi(s^*)$ for each $s \in S$, φ is an MV-semiring isomorphism. □

Let A be an MV-algebra and $\Delta(A)$ the MV-semiring associated. By Theorem 4.3.12, $\Delta(A)$ is isomorphic to the MV-semiring $\widehat{\Delta(A)} = \{\hat{a} : \mathrm{Spec}(\Delta(A)) \to E_{\Delta(A)} \mid \hat{a} \text{ is continuous and } \hat{a}(P) \in \Delta(A)_P, \text{ for each } P \in \mathrm{Spec}(\Delta(A)_r)\}$.

Theorem 4.3.13. *Each MV-algebra A is isomorphic to the MV-algebra of all global sections of the Grothendieck sheaf of the reduct semiring associated with A.*

Proof. It follows from the categorical equivalence between MV-algebras and MV-semirings. □

It is worth stressing again, that in our representation the stalks are not MV-semirings but only commutative additively idempotent semirings. Despite that, the algebra of all global sections in the sheaf representation is still an MV-semiring.

4.4 An Application of sheaf representation

In this paragraph, we obtain one of the possible embeddings in Di Nola's representation theorem for MV-algebras using Dubuc and Poveda sheaf representation.

Proposition 4.4.1. *Any MV-chain can be embedded into a divisible MV-chain.*

Proposition 4.4.2. *Any non-trivial divisible MV-chain is elementarily equivalent with $[0,1]_Q$.*

Theorem 4.4.3 (The joint embedding, [19])**.** *If $\mathcal{F}$ is a nonempty set of elementarily equivalent models, then there exists a model $\mathcal{A}$ such that every model $\mathcal{B}$ from $\mathcal{F}$ is elementarily embedded in $\mathcal{A}$.*

Theorem 4.4.4 (Frayne's Theorem, [16])**.** *Let $\mathcal{A}$, $\mathcal{B}$ be models for the language $\mathcal{L}$. It results that $\mathcal{A}$ is elementarily equivalent with $\mathcal{B}$ if and only if $\mathcal{A}$ is elementarily embedded in some ultrapower ${}^*\mathcal{B}$ of $\mathcal{B}$.*

Proposition 4.4.5. *Let A be an MV-algebra. For each $P \in \mathrm{Spec}(A)$, A/P embeds in an ultrapower ${}^*[0,1]$ of the MV-algebra $[0,1]$.*

Proof. For each $P \in \mathrm{Spec}(A)$, A/P is an MV-chain. So by Theorem 4.4.1 A/P can be embedded into a divisible MV-chain D_P. Consider now the set $\mathcal{F} = \{D_P \mid P \in \mathrm{Spec}(A)\}$. By Theorem 4.4.2, for each $P,\ Q \in \mathrm{Spec}(A)$, D_P is elementarily equivalent with D_Q. Hence by Theorem 1.4 there is a divisible MV-algebra D such that D_P can be elementarily embedded in D, for each $P \in \mathrm{Spec}(A)$. It also results that D is elementarily equivalent with the MV-algebras in $\mathcal{F}$. Since the MV-algebra $[0,1]$ is a divisible MV-chain, $[0,1]$ is also elementarily equivalent with the MV-algebras in $\mathcal{F}$. From Theorem 4.4.4 it follows that D elementarily embeds in an ultrapower $^*[0,1]$ of $[0,1]$.

Summarizing the embeddings, we obtain that for each $P \in \mathrm{Spec}(A)$, A/P embeds in D which embeds in $^*[0,1]$. This completes the proof. □

In the sequel, for each $P \in \mathrm{Spec}(A)$ we indicate by λ_P the embedding of A/P into $^*[0,1]$.

For each MV-algebra A, we can consider the sheaf representation by Dubuc and Poveda in Section 4.1.2. Let A be an MV-algebra. In the sequel, Spec_A will denote Dubuc and Poveda sheaf associated with A. By Theorem 4.1.8, there exists an MV-isomorphism $\varphi : A \to \widehat{A}$, where $\widehat{A}$ is the MV-algebra of all global sections in the sheaf Spec_A.

Remember that for each $a \in A$ the global section $\varphi(a)$ is the continuous map $\hat{a} : \mathrm{Spec}(A) \to E_A$ such that $\hat{a}(P) \in A/P$.

Now for each $a \in A$ consider the map $f_a : \mathrm{Spec}(A) \to^* [0,1]$ defined by $f_a(P) = \lambda_P(\hat{a}(P))$ and denote by $F(A) = \{f_a \mid a \in A\}$. For each $f_a,\ f_b \in F(A)$ we define

$$\begin{aligned} f_a \oplus_{\mathrm{F}} f_b &= f_{a\oplus b}, \\ (f_a)^{*_{\mathrm{F}}} &= f_{a^*}. \end{aligned}$$

Proposition 4.4.6. *$(F(A), \oplus_F, ^{*_F}, f_0)$ is an MV-algebra.*

Proof. The proof trivially follows from the properties of the operations of A. □

Theorem 4.4.7. *Each MV-algebra A is isomorphic to $F(A)$.*

Proof. Consider the map $\psi : A \to F(A)$ defined by $\psi(a) = f_a$. ψ is obviously surjective. Now let $a,\ b \in A$ such that $f_a = f_b$. So $f_a(P) = f_b(P)$ for each $P \in \mathrm{Spec}(A)$, i.e. $\lambda_P(\hat{a}(P)) = \lambda_P(\hat{b}(P))$ for each $P \in \mathrm{Spec}(A)$. Since λ_P is an embedding $\hat{a}(P) = \hat{b}(P)$ for each $P \in \mathrm{Spec}(A)$. By this $\hat{a} = \hat{b}$, i.e. $\varphi(a) = \varphi(b)$. Since φ is an isomorphism we obtain $a = b$.

Further it results that

$$\psi(a \oplus b) = f_{a\oplus b} = f_a \oplus_{\mathrm{F}} f_b = \psi(a) \oplus_{\mathrm{F}} \psi(b),$$

$$\psi(a^*) = f_{a^*} = (f_a)^{*_{\mathrm{F}}} = \psi(a)^{*_{\mathrm{F}}}.$$

Hence ψ is an MV-isomorphism between A and $\widehat{A}$. □

Theorem 4.4.8 (Di Nola's representation theorem for MV-algebras, [24])**.** *For any MV-algebra A there is an ultrapower $^*[0,1]$ of the MV-algebra $[0,1]$ such that A can be embedded into the product $(^*[0,1])^{\mathrm{Spec}(A)}$.*

Remark 4.4.9. For each MV-algebra A, $F(A)$ is obviously an MV-subalgebra of $(^*[0,1])^{\mathrm{Spec}(A)}$. So $F(A)$ can be seen as one of the possible embeddings in Di Nola's representation theorem for MV-algebras.

Chapter 5

MV-algebraic spaces

As already stressed before, it doesn't exist a merely topological representation of the spectrum of an MV-algebra and the spectrum of a ring. But thank to sheaf representations it is possible to establish a bridge between MV-algebras and geometric objects as it happens in Algebraic Geometry for rings which are tied to the affine schemes introduced by Grothendieck. This justifies the several results present in literature aimed to provide geometric objects in correspondence with MV-algebras. The methods used are just similar to the ones used by Grothendieck in Algebraic Geometry. This leads to the introduction of the so-called "MV-algebraic spaces"[1], which are the MV-algebraic version of ringed spaces. Indeed, an MV-algebraic space is a couple (X, F) where X is a topological space and F is a sheaf of MV-algebras on X.

In this chapter we provide three of such representations who arise from the sheaf representations in Paragraph 4.1. Indeed, from their sheaf representations Filipoiu and Georgescu obtained a categorical equivalence between MV-algebras and a particular full subcategory of MV-algebraic spaces ([32]). Dubuc and Poveda provide a similar but weaker representation obtaining an

[1]It must be remembered that up to now these spaces are called MV-spaces, but we prefer to call them MV-algebraic spaces for two reasons. Firstly, MV-spaces are already present in literature to indicate topological spaces which are homeomorphic to the spectrum of some MV-algebra. Lastly, in this way the connection with rings is more evident.

adjoint functor between the category of MV-algebras and the category of MV-algebraic spaces with MV-chains as stalks ([29]). From the last sheaf representation too, one obtains only an adjoint functor between the category of MV-algebras and the category of particular MV-algebraic spaces.
Moreover, we provide some categorical equivalences between subcategories of MV-algebras and MV-algebraic spaces.

Definition 5.0.10. An *MV-algebraic space* is a couple (X, F), where X is a compact topological space and $F = (F, \pi, X)$ is a sheaf of MV-algebras on X.

A morphism $\lambda : (X, F) \to (Y, G)$ between MV-algebraic spaces consists of

(i) a continuous map $g : Y \to X$;

(ii) a collection of MV-morphisms $\lambda_{U,V} : F(V) \to G(U)$, for all open sets $V \subseteq X$ and $U \subseteq Y$ with $U \subseteq g^{-1}(V)$, such that the following diagram is commutative:

$$\begin{tikzcd} F(V) \arrow[r, "\lambda_{U,V}"] \arrow[dr, "\lambda_{g^{-1}(V),V}"'] & G(U) \arrow[d] \\ & G(g^{-1}(V)) \end{tikzcd}$$

5.1 Filipoiu and Georgescu MV-algebraic spaces

In this section we present the categorical equivalence between MV-algebras and a full subcategory of MV-algebraic spaces that we are going to define.

Definition 5.1.1. A T_2 *MV-algebraic space* is an MV-algebraic space (X, F) such that X is a Hausdorff topological space.

An MV-algebraic space (X, F) will be called *separating* if for any $x \in X$ and any open neighbourhood U, $F(X) = K_x + \mathrm{Ker}(U)$, where $K_x = \{\sigma \in F(X) \mid \sigma(x) = 0\}$ and $\mathrm{Ker}(U) = \cap\{K_x \mid x \notin U\}$.

An MV-algebraic space $(X,\ F)$ will be called *local* if each stalk F_x is a local MV-algebra.

In particular, it results that

Lemma 5.1.2 ([32]). *(X, F) is a separating MV-algebraic space if and only if for each $x \in X$ and each closed set $D \subseteq X$, $x \notin D$, there exists $\sigma \in F(X)$ such that $\sigma(x) = 1$ and $\sigma_{|D} = 0$.*

Proposition 5.1.3. *For each MV-algebra A,$(\mathrm{Max}(A), E_A)$, where E_A is the sheaf constructed in section 4.1.1, is a separating and local T_2 MV-algebraic space.*

Proof. For each MV-algebra A, it is clear that $(\mathrm{Max}(A), E_A)$ is a local T_2 MV-algebraic space. Indeed, $\mathrm{Max}(A)$ is a Hausdorff topological space (see Paragraph 1.6) and for each $M \in \mathrm{Max}(A)$, the stalk E_M is isomorphic to $A/O(M)$ which is a local MV-algebra.
Now let $M \in \mathrm{Max}(A)$ and D a closed subset of $\mathrm{Max}(A)$ such that $M \notin D$. We have that, for each $N \in D$, $M \neq N$ and so, from Proposition 1.5.26 it follows that $O(M) \oplus O(N) = V(M) \oplus V(N) = V(M \oplus N) = A$. Hence, for each $N \in D$, there exist $x_N \in O(M)$ and $y_N \in O(N)$ such that $1 = x_N \oplus y_N$. Since for each $N \in D$, $x_N \in O(M)$, we have that $x = \oplus_{N \in D} x_N \in O(M)$ and $1 = x \oplus y_N$, that is $x^* \leq y_N$ for each $N \in D$. From $y_N \in O(N)$, we obtain that $x^* \in O(N)$, for each $N \in D$. Consider, now, the global section associated to x^*, $\widehat{x^*}$. It results that $d(x^*, 1) = x \in O(M)$, that is $\widehat{x^*}(M) = 1$ and $\widehat{x^*}(N) = x^*/O(N) = 0$, for each $N \in D$. $\square$

Let $\mathcal{MV}$ denote the category whose objects are MV-algebras and whose morphisms are the usual homomorphisms and $\mathcal{SLMVS}$ the full subcategory of separating and local T_2 MV-algebraic spaces.

We define the functor $Sc : \mathcal{SLMVS} \to \mathcal{MV}$ given by $Sc(X, F) = F(X)$ and if $\lambda : (X, F) \to (Y, G)$ is a morphism in $\mathcal{SLMVS}$ then $Sc(\lambda) = \lambda_{Y,X} : F(X) \to G(Y)$.

Now we define the functor $Q : \mathcal{MV} \to \mathcal{SLMVS}$ given by $Q(A) = (\mathrm{Max}(A), E_A)$ and for $f : A \to B$, $Q(f) : (\mathrm{Max}(A), E_A) \to (\mathrm{Max}(B), E_B)$ consists of

1. the continuous map $g : \mathrm{Max}(B) \to \mathrm{Max}(A)$ defined by $g(M) = f^{-1}(M)$ (Lemma 2.18 (i), [32])
2. for all $V \subseteq \mathrm{Max}(A)$ and $U \subseteq \mathrm{Max}(B)$ open such that $U \subseteq g^{-1}(V)$, the morphism of MV-algebras $Q(f)_{U,V} : E_A(V) \to E_B(U)$ is given by $Q(f)_{U,V}(s) = f^* \circ \tilde{s}$ where

    i) f^* is the continuous map $g^*(E_A) \xrightarrow{f^*} E_B$ defined by $f^*(x, M) = f(a)/O(M)$ being $x = a/O(g(M))$ [(2)],

    ii) $\tilde{s}$ is given in the following commutative diagram

$$\begin{tikzcd}
 & g^{-1}(V) \arrow[r, "g_{|V}"] \arrow[dd, "s^*"] & V \arrow[dd, "s"] \\
U \arrow[ur, hook] \arrow[dr, "\tilde{s}"'] & & \\
 & g^*(E_A) \arrow[r, "p_1"] \arrow[dl, "f^*"'] \arrow[dd, "p_2"] & E_A \arrow[dd, "\pi_A"] \\
E_B \arrow[dr, "\pi_B"'] & & \\
 & \mathrm{Max}(A) \arrow[r, "g"] & \mathrm{Max}(B)
\end{tikzcd}$$

where $s^* = (s(g(M)), M)$ for $M \in g^{-1}(V)$.

Theorem 5.1.4 (Theorem 2.22, [32])**.** *The functor $Q : \mathcal{MV} \to \mathcal{SLMVS}$ is an equivalence of categories.*

---

[2]Remember that $g^*(E_A)$ is the preimage of E_A along g defined by $g^*(E_A) = \{(x, M) \in E_A \times \mathrm{Max}(B) \mid \pi_A(x) = g(M)\}$

5.2 Dubuc and Poveda MV-algebraic spaces

In this section, we present the construction of an adjunction between the category of MV-algebras and a full subcategory of MV-algebraic spaces.

Definition 5.2.1. An MV-algebraic space $(X,\ F)$ is said *linearly ordered* if each stalk F_x is an MV-chain.

Proposition 5.2.2. *Let A be an MV-algebra. The couple* $(\mathrm{coSpec}(A), F_A)$ *is a linearly ordered MV-algebraic space, being F_A the sheaf constructed in section 4.1.2.*

Let $\mathcal{OMVS}$ denote the full subcategory of linearly ordered MV-algebraic spaces. In [29], the authors construct the following functors:

i) $Spec : \mathcal{MV} \to \mathcal{OMVS}$ defined by $Spec(A) = (\mathrm{coSpec}(A), F_A)$,

ii) $\Gamma : \mathcal{OMVS} \to \mathcal{MV}$ defined by $\Gamma(X, F) = F(X)$.

Theorem 5.2.3 ([29])**.** *The functors Spec and Γ are adjoint on the right.*

5.3 Local T_1 MV-algebraic spaces

In this section we present an adjunction between the category of MV-algebras and a full subcategory of MV-algebraic spaces which extends the functors defined in section 5.1.

Definition 5.3.1. An MV-algebraic space (X, F) is said to be T_1 if and only if X is a T_1 topological space.

Proposition 5.3.2. *Let A be an MV-algebra. The couple* $(\mathrm{Spec}(A), E^s_A)$ *is a local T_1 MV-algebraic space, being E^s_A the sheaf constructed in section 4.1.3, .*

Let $\mathcal{LMVS}$ denote the full subcategory of local T_1 MV-algebraic spaces. We consider the functor $Sc : \mathcal{LMVS} \to \mathcal{MV}$ given by $Sc((X, F)) = F(X)$

and if $\lambda : (X, F) \to (Y, G)$ is a morphism in $\mathcal{LMVS}$ then $Sc(\lambda) = \lambda_{Y,X} : F(X) \to G(Y)$.

Now we define the functor $Q : \mathcal{MV} \to \mathcal{LMVS}$ as $Q(A) = (\mathrm{Spec}(A), E^s_A)$ and, for $f : A \to B$, the definition of $Q(f) : (\mathrm{Spec}(A), E^s_A) \to (\mathrm{Spec}(B), E^s_B)$ is given in what follows.[3]

In the follow, for all $R \in \mathrm{Spec}(A)$, let A_R denote the set $\frac{A}{O(R)} \times \{R\}$.

Lemma 5.3.3. *Let A, B be two MV-algebras and $f : A \to B$ be a MV-morphism. If $P \in \mathrm{Spec}(B)$, there exists a unique morphism $f_P : A_{f^{-1}(P)} \longmapsto B_P$ such that the following diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ A_{f^{-1}(P)} \arrow[r, "f_P"] & B_P \end{tikzcd}$$

is commutative.

Proof. If $P \in \mathrm{Spec}(B)$ then $O(f^{-1}(P)) \subseteq f^{-1}(O(P))$. Indeed, if $x \in O(f^{-1}(P))$ then, by Proposition 1.5.23, there exists $y \notin f^{-1}(P)$ such that $x \wedge y = 0$. Since $f(x) \wedge f(y) = 0$ and $f(y) \notin P$ it follows $f(x) \in O(P)$, i.e. $x \in f^{-1}(O(P))$.

The map

$$f_P : a_{f^{-1}(P)} \in A_{f^{-1}(P)} \longmapsto f(a)_P \in B_P$$

where $a_{f^{-1}(P)} = \left(\frac{a}{O(f^{-1}(P))}, f^{-1}(P)\right)$ and $f(a)_P = \left(\frac{f(a)}{O(P)}, P\right)$ is well defined. Indeed, if $a_{f^{-1}(P)} = b_{f^{-1}(P)}$, then $d(a, b) \in O(f^{-1}(P))$, hence $d(f(a), f(b)) = f(d(a, b)) \in f(O(f^{-1}(P))) \subseteq f(f^{-1}(O(P))) \subseteq O(P)$. It's easy to prove that this morphism is unique making commutative the previous diagram. $\square$

Let $E^s_A = (E^s_A, \pi_A, \mathrm{Spec}(A))$ and $E^s_B = (E^s_B, \pi_B, \mathrm{Spec}(B))$ be the sheaves of MV-algebras associated with the MV-algebras A and B, $f : A \longmapsto B$ a given morphism and g the induced continuous map $g : P \in \mathrm{Spec}(B) \longmapsto f^{-1}(P) \in \mathrm{Spec}(A)$ (see [3]). We call *preimage of E^s_A along the map g* the set $g^*(E_A) =$

[3]Note that the functor defined here are an extension of the functor defined in section 5.1.

$\{(x,P) \in E_A^s \times \mathrm{Spec}(B) : \pi_A(x) = g(P)\}$. Recall that if $x = a_Q \in E_A^s$, then $\pi_A(x) = Q$ and $g(P) = f^{-1}(Q)$, so

$$g^*(E_A) = \left\{ \left(a_{f^{-1}(P)}, P\right) : a \in A, P \in \mathrm{Spec}(B) \right\}$$

, i.e.

$$g^*(E_A) = \bigcup_{P \in \mathrm{Spec}(B)} \left(A_{f^{-1}(P)} \times \{P\}\right). \tag{5.1}$$

$g^*(E_A)$ is a topological space with the topology induced from the product topology on $E_A^s \times \mathrm{Spec}(B)$. A basis for this topology is given by the sets $(U_A(I,a) \times U_B(J)) \cap g^*(E_A)$, where $U_A(I,a)$ is an open subset of E_A^s and $U_B(J)$ is an open subset of $\mathrm{Spec}(B)$. It is easy to verify that

$$(U_A(I,a) \times U_B(J)) \cap g^*(E_A) = \left\{ \left(a_{f^{-1}(P)}, P\right) : P \in U_B(J) \right\}.$$

In the next let $U_{J,a}$ denote the open subsets of $g^*(E_A)$, for each $J \in IdB$ and $a \in A$.

From previous remarks we have a commutative diagram of continuous maps:

$$\begin{tikzcd} g^*(E_A) \arrow[r, "p_1"] \arrow[d, "p_2"'] & E_A^s \arrow[d, "\pi_A"] \\ \mathrm{Spec}(B) \arrow[r, "g"] & \mathrm{Spec}(A) \end{tikzcd}$$

where p_1, p_2 are the canonical projections. From this and from (5.1), it follows that there exists an isomorphism ϑ_P between $(g^*(E_A))_P = p_2^{-1}(P)$ and the stalk at $f^{-1}(P)$, $A_{f^{-1}(P)}$. Using Lemma 5.3.3 we obtain for any $P \in \mathrm{Spec}(B)$ a map f_P^* such that the diagram

$$\begin{tikzcd} (g^*(E_A))_P \arrow[r, "\vartheta_P"] \arrow[dr, "f_P^*"'] & A_{f^{-1}(P)} \arrow[d, "f_P"] \\ & B_P \end{tikzcd}$$

is commutative, i.e. $f_P^* = f_P \circ \vartheta_P$. Since $E_B^s = \bigcup_{P \in \mathrm{Spec}(B)} B_P$, the family $(f_P^*)_{P \in \mathrm{Spec}(B)}$ induces a map $f^* : g^*(E_A) \longmapsto E_B^s$ such that the diagram

$$\begin{tikzcd} g^*(E_A) \arrow[r, "p_2"] \arrow[d, "f^*"'] & \mathrm{Spec}(B) \\ E_B^s \arrow[ur, "\pi_B"'] & \end{tikzcd}$$

is commutative. f^* can be defined pointwise as

$$\begin{aligned}
f^*\left(\left(a_{f^{-1}(P)}, P\right)\right) &= f_P^*\left(\left(a_{f^{-1}(P)}, P\right)\right) \\
&= f_P\left(\vartheta_P\left(a_{f^{-1}(P)}, P\right)\right) \\
&= f_P\left(a_{f^{-1}(P)}\right) \\
&= f(a)_P.
\end{aligned}$$

Let us prove that f^* is a continuous map. Let $U_B(J,b)$ be an open subset of E_B^s, we have to prove that $(f^*)^{-1}(U_B(J,b))$ is an open subset of $g^*(E_A)$. Let $(x,P) \in (f^*)^{-1}(U_B(J,b))$, then $f^*((x,P)) \in U_B(J,b)$. Since $(x,P) \in g^*(E_A)$ there exists $a \in A$ such that $x = a_{f^{-1}(P)}$. So $f^*((x,P)) = f(a)_P \in U_B(J,b)$. From this, $f(a) = b$ and $P \in U(J)$, i.e. $f^{-1}(P) \in U(f^{-1}(J))$ and so $\left(a_{f^{-1}(P)}, P\right) \in U_{J,a}$. Let us prove that $U_{J,a} \subseteq (f^*)^{-1}(U_B(J,b))$. Let $\left(a_{f^{-1}(S)}, S\right) \in U_{J,a}$, $f^*\left(\left(a_{f^{-1}(S)}, S\right)\right) = f(a)_S$. Since $f(a) = b$ and $S \in U(J)$, we have that $f(a)_S \in U(J,b)$. From this, it follows that f^* is continuous.

In this way we obtain a morphism from $(g^*(E_A), p_2, \mathrm{Spec}(B))$ to $(E_B^s, \pi_B, \mathrm{Spec}(B))$. Now we can construct $Q(f) : (\mathrm{Spec}(A), E_A^s) \longmapsto (\mathrm{Spec}(B), E_B^s)$. Let $V \subseteq \mathrm{Spec}(A)$, $U \subseteq \mathrm{Spec}(B)$ be open set such that $U \subseteq g^{-1}(V)$, the morphism of MV-algebras $Q(f)_{U,V} : E_A^s(V) \longmapsto E_B^s(U)$ is given by $Q(f)_{U,V}(s) = f^* \cdot s^*_{|U}$, where s^* is given in the following commutative diagram

$$\begin{tikzcd}
 & g^{-1}(V) \arrow[r, "g_{|V}"] \arrow[dd, "s^*"] & V \arrow[dd, "s"] \\
U \arrow[ur, hook] \arrow[dr, dotted, "s^*_{|U}"] \arrow[dd, "Q(f)_{U,V}(s)"'] & & \\
 & g^*(E_A) \arrow[r, "p_1"] \arrow[dl, "f^*"'] \arrow[dd, "p_2"] & E_A^s \arrow[dd, "\pi_A"] \\
E_B^s \arrow[dr, "\pi_B"'] & & \\
 & \mathrm{Spec}(B) \arrow[r, "g"] & \mathrm{Spec}(A)
\end{tikzcd}$$

where $s^*(P) = (s(g(P)), P)$, for $P \in g^{-1}(V)$.

In what follows we shall prove that the functors $Q : \mathcal{MV} \longmapsto \mathcal{LMVS}$ and $Sc : \mathcal{LMVS} \longmapsto \mathcal{MV}$ realise an adjunction of categories.

Definition 5.3.4. [45] Let A and X be categories. An *adjunction* from X to A is a triple $\langle F, G, \varphi \rangle : X \longmapsto A$, where F and G are functors

$$X \underset{G}{\overset{F}{\rightleftarrows}} A,$$

and φ is a function

$$\varphi : Ob(X) \times Ob(A) \longmapsto Hom_X(x, G(a))^{Hom_A(F(x),a)}$$

which assigns to each pair of objects (x, a) a bijection $\varphi(x, a) = \varphi_{x,a}$ such that to each arrow $f : F(x) \longmapsto a$ is associated an arrow $\varphi_{x,a}(f) : x \longmapsto G(a)$, the *right adjunct of* f, in such a way that the naturality conditions

$$\varphi_{x',a}(f \circ F(h)) = \varphi_{x,a}(f) \circ h, \qquad \varphi_{x,a'}(k \circ f) = G(k) \circ \varphi_{x,a}(f),$$

hold for all f and all arrows $h : x' \longmapsto x$ and $k : a \longmapsto a'$.

Lemma 5.3.5. *[45] Each adjunction $\langle F, G, \varphi \rangle : X \longmapsto A$ is completely determined by the functors F, G and a natural transformation $\eta : id_X \longmapsto G \circ F$ such that each $\eta_x : x \longmapsto GFx$ is universal from x to G. Then for each $f : Fx \longmapsto a$, φ is defined by $\varphi f = G(f) \circ \eta_x : x \longmapsto G(a)$.*

We obtain the MV-morphism $Sc(Q(f)) : \hat{A} \longmapsto \hat{B}$, where $\hat{A}$ and $\hat{B}$ are the MV-algebras of global sections respectively of $(\mathrm{Spec}(A), E^s_A)$ and of $(\mathrm{Spec}(B), E^s_B)$. By Lemma 3.3.10, we have the MV-isomorphism $\varphi_A : A \longmapsto \hat{A}$ given by $\varphi(a) = \hat{a}$, where $\hat{a}(P) = a_P$ for any $P \in \mathrm{Spec}(A)$.

Lemma 5.3.6. *For any MV-morphism $f : A \longmapsto B$ the diagram*

$$\begin{tikzcd} A \arrow[r, "\varphi_A"] \arrow[d, "f"'] & \hat{A} \arrow[d, "Sc(Q(f))"] \\ B \arrow[r, "\varphi_B"] & \hat{B} \end{tikzcd}$$

is commutative.

Proof. We shall prove that $Sc(Q(f)) \circ \varphi_A = \varphi_B \circ f$. Indeed it results that $Sc(Q(f))\varphi_A(a) = Sc(Q(f))(\hat{a}) = Q(f)_{\mathrm{Spec}(A),\mathrm{Spec}(B)}(\hat{a}) = f^* \circ \hat{a} \in \widehat{B}$ and is given by $f^*\hat{a}(P) = f^*(\hat{a}(P)) = f^*(a_{f^{-1}(P)}, P) = f(a)_P$.

Moreover $\varphi_B(f(a)) = \widehat{f(a)}$ that is given by $\widehat{f(a)}(P) = f(a)_P$. So the diagram is commutative in the category of MV-algebras. □

Remark 5.3.7. In this way, we have a natural isomorphism $\varphi : id_{\mathcal{MV}} \longmapsto Sc \circ Q$, that to each MV-algebras A assigns the isomorphism φ_A of the Lemma 3.3.10 and to each MV-morphism $f : A \longmapsto B$ the following diagram

$$\begin{tikzcd} A \arrow[r, "\varphi_A"] \arrow[d, "f"'] & \hat{A} \arrow[d, "Sc(Q(f))"] \\ B \arrow[r, "\varphi_B"] & \hat{B} \end{tikzcd}$$

that is commutative by Lemma 5.3.6.

Let (X, F) be a local and compact MV-algebraic space, for each $x \in X$, consider the set $K_x = \{\sigma \in F(X) : \sigma(x) = 0\}$.

Proposition 5.3.8. *For each $x \in X$, K_x is a primary ideal of $F(X)$.*

Proof. First we prove that K_x is an ideal of $F(X)$.

Let $\sigma, \tau \in K_x$, so $\sigma(x) = \tau(x) = 0$. Then $(\sigma \oplus \tau)(x) = \sigma(x) \oplus \tau(x) = 0$, i.e. $\sigma \widehat{\oplus} \tau \in K_x$.

Let $\sigma \in F(X)$ and $\tau \in K_x$ such that $\sigma \leq \tau$. Then $\sigma(x) \leq \tau(x) = 0$, so $\sigma(x) = 0$, i.e. $\sigma \in K_x$.

Now we are going to show that K_x is primary. Let $\sigma \widehat{\odot} \tau \in K_x$, so $0 = (\sigma \odot \tau)(x) = \sigma(x) \odot \tau(x) \in E_x$, hence $\sigma(x) \leq \tau(x)^*$. Now, if $ord(\sigma(x)) < \infty$, then there is $n \in \omega$ such that $n\tau(x)^* = 1$, i.e. $\tau(x)^n = 0$. So $\tau^n \in K_x$. If $ord(\sigma(x)) = \infty$, since E_x is a local MV-algebra, then $ord(\sigma(x)^*) < \infty$, hence there is $n \in \omega$ such that $n\sigma(x)^* = 1$, i.e. $\sigma(x)^n = 0$. So $\sigma^n \in K_x$. Hence K_x is primary. □

Remark 5.3.9. By Proposition 5.3.8, for each $x \in X$ there exists a unique maximal ideal M_x of $F(X)$ such that $K_x \subseteq M_x$. So we can define the map

$$g' : x \in X \longmapsto M_x \in \mathrm{Max}(F(X)). \tag{5.2}$$

We can prove that

Proposition 5.3.10. *The map g' defined in (5.2) is surjective and continuous.*

Proof. Let $M \in \mathrm{Max}(F(X)) \setminus g'(X)$. Then for each $x \in X$, there exists an element $\sigma^{(x)} \in K_x \setminus M$. Let $V_x = \{y \in X | \sigma^{(x)}(y) = 0\}$. Since $x \in V_x$ for each $x \in X$, by Proposition 2.1.2 (i), $\{V_x : x \in X\}$ is an open covering of X. By the compactness of X, $X = V_{x_1} \cup \cdots \cup V_{x_n}$ for some $x_1, \cdots, x_n \in X$. We claim that $\sigma^{x_1} \widehat{\wedge} \cdots \widehat{\wedge} \sigma^{x_n} = 0$. Indeed, if $\sigma^{x_1}(x) \wedge \cdots \wedge \sigma^{x_n}(x) > 0$ for some $x \in X$ then $\sigma^{x_i}(x) > 0$ for each $i = 1, \cdots, n$, i.e. $x \notin V_{x_i}$, that is impossible. So $\sigma^{x_1} \widehat{\wedge} \cdots \widehat{\wedge} \sigma^{x_n} \in M$ but $\sigma^{x_i} \notin M$ for each $i = 1, \cdots, n$, that is a contradiction. Hence g' is a surjective map.

Let us prove that g' is continuous. For any $a \in F(X)$, set $D(a) = U(a) \cap \mathrm{Max}(F(X))$, in this way $g'^{-1}(D(a)) = \{y \in Y : a \notin M_y\}$. By Theorem 4.7 of [14], there exists $m_y \in \omega$ such that $(a^*)^{m_y} \in M_y$. Consider the local MV-algebra $\frac{F(X)}{K_y}$ and the radical $\mathrm{Rad}\left(\frac{F(X)}{K_y}\right)$. The element $\frac{(a^*)^{m_y}}{K_y} \in \mathrm{Rad}\left(\frac{A}{K_y}\right)$. So $\frac{(a^*)^{2m_y}}{K_y} = 0$ then $(a^*)^{2m_y} \in K_y$, i.e. there exists $n_y \in \omega$ such that $(a^*)^{n_y} \in K_y$. Hence we have proved that $a \notin M_y$ if and only if there exists $n \in \omega$ such that $a^*)^n = 0$. So $g'^{-1}(D(a)) = \bigcup_{n=1}^{\infty} \{y \in X : (a^*)^n(y) = 0\}$ and the last is open in X. Therefore g' is continuous. $\square$

Theorem 5.3.11. *The triple $\langle Q, Sc, \varphi \rangle$ is an adjunction from $\mathcal{MV}$ and $\mathcal{LMVS}$.*

Proof. Let A be an MV-algebra. By Lemma 5.3.5, we have to prove that $\varphi_A : A \longmapsto \hat{A}$ is a universal arrow from A to Sc, i.e for each $f : A \longmapsto Sc(X, F)$, with (X, F) MV-algebraic space, there is a unique arrow $\lambda : (\mathrm{Spec}(A), E_A^s) \longmapsto (X, F)$ such that $f = Sc(\lambda) \circ \varphi_A$.

Now, we are going to construct λ. First we need a continuous map $g : X \longmapsto \mathrm{Spec}(A)$. We can consider the map $g' : x \in X \longmapsto M_x \in \mathrm{Spec}(F(X))$ defined in (5.2) and the continuous map $g_f : \mathrm{Spec}(F(X)) \longmapsto \mathrm{Spec}(A)$ associated with $Q(f)$. So $g = g_f \circ g'$.

Let U be an open subset of Spec(A), note that for each $\sigma \in E^s_A(U)$ there exists an element $a \in A$ such that $\sigma = \hat{a}_{|U}$. So if $V \subseteq X$ is an open subset of X such that $V \subseteq g^{-1}(U)$, we can define $\lambda_{U_V} : \hat{A}_{|U} \longmapsto F(U)$ by $\lambda_{U,V}(\hat{a}_{|U}) = f(a)_{|V}$.

It is easy to prove that λ is unique, so the triple $\langle Q, Sc, \varphi\rangle$ is an adjunction.

$\square$

5.4 Changes in the stalks

As seen just before, the best sheaf representation of MV-algebras is that by Filipoiu and Georgescu, since it is the unique which provides a categorical equivalences between MV-algebras and MV-algebraic spaces. So, it seems natural to ask what we can represent when we fix the stalks in some subcategories of MV-algebras.

In what follows we call Filipoiu and Georgescu representation a *Max-Sheaf representation of A* and say that A is *Max-Sheaf representable by local stalks.*

In this paragraph, we will prove the following results:

- the category of MV-algebras of k-bounded rank is equivalent to the category of k-bounded, separating and local T_2 MV-algebraic spaces;

- the category of divisible MV-algebras is equivalent to the category of divisible, separating and local T_2 MV-algebraic spaces;

- the category of regular MV-algebras is equivalent to the category of linearly ordered and separating Stone MV-algebraic spaces.

The results collected in this paragraph are contained in [26].

5.4.1 MV-algebraic spaces and MV-algebras of k-bounded rank

In this section we prove that the category of MV-algebras of k-bounded rank is equivalent to the category of k-bounded, separating and local T_2 MV-algebraic spaces.

In [25], the authors have proved the following result.

Theorem 5.4.1. *Let A be an MV-algebra. Then the following statements are equivalent:*

(1) A is of k-bounded rank;

(2) A is Max-Sheaf representable by local stalks of rank less than k.

Theorem 5.4.2. *Let (E, π, X) be a sheaf of MV-algebras where X is a Hausdorff topological space such that for each $x \in X$ the stalk $E_x = \pi^{-1}(x)$ is a local MV-algebra of rank less than k. The MV-algebra $E(X)$ of all global sections is of k-bounded rank.*

Proof. Using Proposition 1.7.12, to prove that $E(X)$ is of k-bounded rank, we have to show that for each $\sigma \in E(X)$, $(\sigma^* \vee (k\sigma))^n \notin O(M)$ (1), for all $n \in \omega$ and for each $M \in \mathrm{Max}(E(X))$. From Lemma 2.8 and Proposition 2.9 of [32], it follows that for each $M \in \mathrm{Max}(E(X))$ there exists $x \in X$ such that $O(M) = \{\tau \in E(X) \mid \tau(x) = 0\}$. Thus, (1) is equivalent to $(\sigma^* \vee (k\sigma))^n(x) > 0$ for each $x \in X$.
Now $\sigma(x) \in E_x$ which has rank less than k. So by Proposition 1.7.11, $\sigma(x) \wedge (\sigma(x)^*)^k \in \mathrm{Rad}(E_x)$. But E_x is also local, then $\mathrm{ord}(\sigma(x) \wedge (\sigma(x)^*)^k) = \infty$, that is for each $n \in \omega$, $n(\sigma(x) \wedge (\sigma(x)^*)^k) < 1$ which is equivalent to $(\sigma(x)^* \vee (k\sigma(x)))^n > 0$. Hence $(\sigma^* \vee (k\sigma))^n(x) > 0$ for each $x \in X$. $\square$

Definition 5.4.3. An MV-algebraic space (X, F) is said k-bounded if and only if for $x \in X$, the stalk F_x has rank less than k.

Remark 5.4.4. Let A be an MV-algebra of k-bounded rank. We can consider the separating and local T_2 MV-algebraic space $(\mathrm{Max}(A), E_A)$ associated with A in Filipoiu and Georgescu representation. From Proposition 5.4.1, it follows that the stalks of E_A have rank less than k, and so $\mathrm{Max}(A), E_A)$ is a k-bounded, separating and local T_2 MV-algebraic space.

In the sequel, we'll indicate by $\mathcal{BRMV}$ the full subcategory of all MV-algebras of k-bounded rank and by $\mathcal{BSLMVS}$ the full subcategory of all k-bounded, separating and local T_2 MV-algebraic spaces.
Consider $Sc_B : \mathcal{BSLMVS} \to \mathcal{BRMV}$ to be the restriction of the equivalence Sc defined in section 5.1. Sc is an equivalence of categories, Sc is full and faithful. Being $\mathcal{BSLMVS}$ and $\mathcal{BRMV}$ full subcategories, Sc_B is full and faithful too. Moreover, Sc_B is essentially surjective. Indeed, let A be an MV-algebra of k-bounded rank and consider the k-bounded, separating and local T_2 MV-algebraic space $(\mathrm{Max}(A), E_A)$. From Theorem 5.4.2, it follows that the MV-algebra $Sc_B(\mathrm{Max}(A), E_A) = E_A(\mathrm{Max}(A))$ of all global sections is of k-bounded rank and from Theorem 4.1.6 $E_A(\mathrm{Max}(A))$ is isomorphic to A. In this way, we obtain

Theorem 5.4.5. *The functor $Sc_B : \mathcal{BSLMVS} \to \mathcal{BRMV}$ is an equivalence of categories.*

5.4.2 MV-algebraic spaces and divisible MV-algebras

In this section we prove that the category of divisible MV-algebras is equivalent to the category of divisible, separating and local T_2 MV-algebraic spaces.

Theorem 5.4.6. *Let (E, π, X) be a sheaf of MV-algebras such that for each $x \in X$, the stalk $E_x = \pi^{-1}(x)$ is a divisible MV-algebra, the MV-algebra $E(X)$ of all global sections is divisible.*

Proof. To prove that $E(X)$ is divisible we have to show that for each $\sigma \in E(X)$ and $n \geq 1$, there is $\tau \in E(X)$ such that $\sigma = n\tau$ and $\sigma^* \oplus (n-1)\tau = \tau^*$. For each $x \in X$, $\sigma(x) \in E_x$ which is a divisible MV-algebra. Hence for each $n \geq 1$ there exists $k_x \in E_x$ such that $\sigma(x) = nk_x$ and $\sigma(x)^* \oplus (n-1)k_x = k_x^*$. From (ii) of Proposition 2.1.2, there exist an open set W_x such that $x \in W_x$

and a local section $\tau_x : W_x \to E$ such that $\tau_x(x) = k_x$. In this way we obtain an open cover $\{W_x \mid x \in X\}$ of X. Without loss of generality, we can assume these sets disjoint. Being (E, π, X) a sheaf, there exists a global section $\tau : X \to E$ such that $\tau_{|W_x} = \tau_x$. So for each $x \in X$ and $n \geq 1$, $\sigma(x) = n\tau(x)$ and $\sigma^*(x) \oplus (n-1)\tau(x) = \tau^*(x)$, i.e. $\sigma = n\tau$ and $\sigma^* \oplus (n-1)\tau = \tau^*$. This prove that $E(X)$ is a divisible MV-algebra. □

Theorem 5.4.7. *Let A be an MV-algebra. Then the following are equivalent:*

(1) A is divisible;

(2) A is Max-Sheaf representable by local and divisible stalks.

Proof. (1) ⇒ (2) From Proposition 1.7.14 we obtain that the class of divisible MV-algebras is closed under quotients and so the stalks in Filipoiu and Georgescu sheaf representation of A are also divisible.

(2) ⇒ (1) Let $a \in A$ and consider $\varphi(a) = \hat{a} \in E_A(\mathrm{Max}(A))$ that is a divisible MV-algebra by Theorem 5.4.6. So, for each $n \geq 1$ there exists a section $\sigma \in E_A(\mathrm{Max}(A))$ such that $\hat{a} = n\sigma$ and $\hat{a}^* \oplus (n-1)\sigma = \sigma^*$. Since φ is a surjective map, there exists $b \in A$ such that $\varphi(b) = \sigma$ and so $\varphi(a) = n\varphi(b)$ and $\varphi(a)^* \oplus (n-1)\varphi(b) = \varphi(b)^*$. Furthermore, φ is an MV-homomorphism, hence $\varphi(a) = \varphi(nb)$ and $\varphi(a^* \oplus (n-1)b) = \varphi(b^*)$. By the injective of φ, we obtain $a = nb$ and $(a^* \oplus (n-1)b = b^*$, i.e. A is divisible. □

Definition 5.4.8. An MV-algebraic space $(X,\ F)$ will be called *divisible* if every stalk F_x is a divisible MV-algebra.

Remark 5.4.9. Let A be a divisible MV-algebra and consider the separating and local T_2 MV-algebraic space MV-algebraic space $(\mathrm{Max}(A), E_A)$ associated with A in the Filipoiu and Georgescu representation. From Proposition 1.7.14 it follows that the stalks of F_A are divisible too and so $(\mathrm{Max}(A), E_A)$ is a divisible, separating and local T_2 MV-algebraic space.

In the sequel, we'll indicate by $\mathcal{DMV}$ the full subcategory of all divisible MV-algebras and by $\mathcal{DCLMV}$ the full subcategory of divisible, separating and local T_2 MV-algebraic spaces.

Consider $Sc_D : \mathcal{DCLMV} \to \mathcal{DMV}$ to be the restriction of the equivalence Sc defined in section 5.1. Since Sc is an equivalence of categories, Sc is full and faithful. Being $\mathcal{DCLMV}$ and $\mathcal{DMV}$ full subcategories, Sc_D is full and faithful too. Moreover, Sc_D is essentially surjective. Indeed, let A be a divisible MV-algebra and consider the divisible, separating and local T_2 MV-algebraic space $(\mathrm{Max}(A), E_A)$, from Theorem 5.4.6 it follows that the MV-algebra $Sc((\mathrm{Max}(A), E_A)) = E_A(\mathrm{Max}(A))$ of all global sections is divisible too and from Theorem 4.1.6, $E_A(\mathrm{Max}(A)) \cong A$. In this way, we obtain

Theorem 5.4.10. *The functor $Sc_D : \mathcal{DCLMV} \to \mathcal{DMV}$ is an equivalence of categories.*

5.4.3 MV-algebraic spaces and regular MV-algebras

In this section we prove that the category of regular MV-algebras is equivalent to the category of linearly ordered and separating Stone MV-algebraic spaces.

Theorem 5.4.11. *Let (E, π, X) be a sheaf of MV-algebras such that for each $x \in X$ the stalk $E_x = \pi^{-1}$ is an MV-chain and X is a Stone space. The MV-algebra $E(X)$ of all global sections is regular.*

Proof. Using Proposition 1.7.17, we shall prove that for each $\sigma,\ \tau \in E(X)$ such that $\sigma \wedge \tau = 0$, there exist $\alpha,\ \beta \in B(E(X))$ such that $\sigma \leq \alpha$, $\tau \leq \beta$ and $a \wedge \beta = 0$.
Since $\sigma \wedge \tau = 0$, for each $x \in X$, $\sigma(x) \wedge \tau(x) = 0$, with $\sigma(x),\ \tau(x) \in E_x$ which an MV-chain. So $\sigma(x) = 0$ or $\tau(x) = 0$. Moreover for Proposition 2.1.2 (ii), there exist a clopen neighbourhood V_1 of x and a section $\sigma_x \in E(V_1)$ such that $\sigma(x) = \sigma_x(x)$. Analogously for τ, there exist a clopen neighbourhood V_2 of x and a section $\tau_x \in E(V_1)$ such that $\tau(x) = \tau_x(x)$. Now if $\sigma(x) = 0_x \in E_x$ consider an open set V_3 of X, with $x \in V_3$ and a section $\alpha_x = \sigma_x$. If $\sigma(x) > 0_x$, consider the section $\alpha_x \in E(V_4)$, with V_4 open set of X, such that $\alpha_x(x) = 1_x$. In summary, we have

$$\alpha_x(x) := \begin{cases} 0_x, & \text{if } \sigma_x(x) = 0, \\ 1_x, & \text{if } \sigma_x(x) \neq 0. \end{cases}$$

Analogously, we define

$$\beta_x(x) := \begin{cases} 0_x, & \text{if } \tau_x(x) = 0, \\ 1_x, & \text{if } \tau_x(x) \neq 0. \end{cases}$$

where $\beta_x \in E(V_4)$, being V_4 an open neighbourhood of x.
Set $W_x = V_1 \cap V_2 \cap V_3 \cap V_4$. We have obtained that $\sigma_x(y) \wedge \tau_x(y) = 0$, $\sigma_x(y) \leq \alpha_x(y)$, $\tau_x(y) \leq \beta_x(y)$, $\alpha_x(y) \wedge \beta_x(y) = 0$, $\alpha_x(x),\ \beta_x(x) \in \{0_x, 1_x\}$.
Now $\{W_x \mid x \in X\}$ is an open covering of X. Without loss of generality, we can assume these sets disjoint. Being (E, π, X) is a sheaf, there exist $\alpha,\ \beta \in B(E(X))$ such that $\alpha_{|W_x} = \alpha_x$, $\beta_{|W_x} = \beta_x$, for each $x \in X$ and $\sigma \leq \alpha$, $\tau \leq \beta$ and $\alpha \wedge \beta = 0$. Hence $E(X)$ is regular. □

Theorem 5.4.12. *Let A be an MV-algebra. Then the following statements are equivalent:*

(1) A is regular;

(2) A is Max-Sheaf representable by linearly ordered stalks.

Proof. (1) ⇒ (2) From Corollary 1.7.20 we obtain that the stalks in Filipoiu and Georgescu sheaf representation of A are linearly ordered.

(2) ⇒ (1) From hypothesis it follows that $E_A(\mathrm{Max}(A)) \cong A$. Moreover $\mathrm{Max}(A)$ is a Stone space (see Proposition 1.7.19). By Theorem 5.4.11, $E_A(\mathrm{Max}(A))$ is regular and so A is regular for Proposition 1.7.18. □

Definition 5.4.13. An MV-algebraic space $(X,\ F)$ will be called *Stone* if and only if X is a Stone space.

Remark 5.4.14. Let A be a regular MV-algebra and consider the separating and local T_2 MV-algebraic space $(\mathrm{Max}(A), E_A)$ associated with A in the Filipoiu and Georgescu representation. From Theorem 5.4.12 it follows that $(\mathrm{Max}(A), E_A)$ is a linearly ordered and separating Stone MV-algebraic space.

In the sequel, we'll indicate by $\mathcal{RMV}$ the full subcategory of all divisible MV-algebras and by $\mathcal{ROMVS}$ the full subcategory of linearly ordered and separating Stone MV-algebraic spaces.

Consider $Sc_R : \mathcal{ROMVS} \to \mathcal{RMV}$ to be the restriction of the equivalence Sc defined in section 5.1. Since Sc is an equivalence of categories, Sc is full and faithful. Being $\mathcal{ROMVS}$ and $\mathcal{RMV}$ full subcategories, Sc_R is full and faithful too. Moreover, Sc_R is essentially surjective. Indeed, let A be a regular MV-algebra and consider the linearly ordered and separating Stone MV-algebraic space $(\mathrm{Max}(A), E_A)$, from Theorem 5.4.11 it follows that the MV-algebra $Sc((\mathrm{Max}(A), E_A)) = E_A(\mathrm{Max}(A))$ of all global sections is regular too and from Theorem 4.1.6, $E_A(\mathrm{Max}(A)) \cong A$. In this way, we obtain

Theorem 5.4.15. *The functor $Sc_R : \mathcal{ROMVS} \to \mathcal{RMV}$ is an equivalence of categories.*

5.5 Conclusions

Let $\mathcal{L}$ be a given logic. In [30] Feferman and Vaught provided an effective procedure to reduce the decidability of the theory of a generalized product of $\mathcal{L}$-structures to the decidability of the theories of the factors.

Later, in [20], Comer generalized these results for structures which can be represented by sections of sheaves such as rings. However, as Macintyre stressed in [46], the property of being model-complete is not preserved under products in general, but thereby he gave certain conditions under which model-completeness is preserved. Indeed Macintyre introduced, for a sheaf (X, π, E) of $\mathcal{L}$-structures, the notion of *stalk theory* as $\mathrm{Th}\left(\{\pi^{-1}(x) : x \in X\}\right)$ and the notion of *section theory* as $\mathrm{Th}\left(\Gamma(X, E)\right)$. Moreover he proved that, under particular conditions on the base space, the stalk theory and the logic, the model completeness of the stalk theory implies the model completeness of the section theory, i.e. the model completeness of the $\mathcal{L}$-structure represented by the sheaf considered.

Therefore our results give further tools which can be suitable for Łukasiewicz logic. The next step of our research is to verify if the above conditions hold for Łukasiewicz logic and MV-algebras using the representation by sections of Filipoiu and Georgescu.

Moreover, Lacava in [40] proved that an MV-algebra A is algebraically closed if and only if A is regular and divisible. So, gathering up the results in Sections 5.4.2 and 5.4.3, we obtain a representation of algebraically closed MV-algebras as linearly ordered, divisible and Stone MV-algebraic spaces. This can be the starting point to develop a sort of Algebraic Geometry based on MV-algebras.

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