Contents

Introduction

In the study of several elliptic problems with solutions in a Sobolev space $S(\Omega)$ (with or without weight) on an open set Ω of $\mathbb{R}^n$, not necessarily bounded or regular, it is sometimes necessary to estabilish regularity results and a priori estimates for the solutions. These results often rely on the boundedness and possibly on the compactness of the multiplication operator

$$u \longrightarrow g\,u \qquad (i)$$

which is defined in $S(\Omega)$ and which takes values in a suitable Lebesgue space $L^p(\Omega)$, where g is a given function in a normed space V. Hence, it's necessary to obtain an estimate of the following type :

$$\|g\,u\|_{L^p(\Omega)} \leq c \cdot \|g\|_V \cdot \|u\|_{S(\Omega)}\,, \qquad (ii)$$

where $c \in \mathbb{R}_+$ depends on the regularity properties of Ω and on the summability exponents, and g satisfies suitable conditions.
If L is the differential operator associated to the corresponding elliptic problem, the estimate (ii) allows us, for instance, to prove the boundedness of the operator L, when

g is a coefficient of L. In some particular cases, it's not possible to obtain certain regularity results for the operator L itself, because of its non regular coefficients. Hence, there is the need to introduce a suitable class of operators L_h, whose coefficients, more regular, approximate the ones of L. This " deviation " of the coefficients of L_h from the ones of L needs to be done controlling the norms of the approximating coefficients with the norms of the given ones. Hence, it is necessary to obtain estimates where the dependence on the coefficients is expressed just in terms of their norms (in this case, for instance, there are no problems when passing to the limit). In other words, if g is a coefficient of operator L and g_h is a coefficient of L_h more regular, it's necessary to have a " good control" of the difference $g - g_h$. The introduction of decompositions for functions in suitable function spaces (where the coefficients of differential operator L belong) plays an important role in this approximation process.

Having this in mind, our purpose is to construct suitable decompositions for functions belonging to some specific functional spaces whose introduction is related to the solvability of certain elliptic problems of above mentioned type. As application, we want to study the boundeness and the compactness of an operator in Sobolev spaces with or without weight.

The idea of decomposition is to split a function in two summands, which are estimate in different (but fixed) norms. These norms are those of certain Banach spaces X and Y and all functions are defined on suitable domains in $\mathbb{R}^n$. Then a function f is split in the sum $f = g + h$, where $g \in X$ and $h \in Y$.

In literature there are several papers in which the authors have constructed decompositions with an appropriate couple (X, Y). In [10] Calderón and Zygmund have proved the classical decomposition (L^1, L^∞) for L^1, where a given function f in L^1 is decomposed, for any $t > 0$, in the sum of a " good " part $f_t \in L^\infty$ (whose norm can be controlled by $\|f_t\|_{L^\infty(\Omega)} < c(n) \cdot t$) and a remaining " bad " one $f - f_t \in L^1$. Analogous decompositions can be found also for different functional spaces (see for instance N. Kruglyak, E. A. Kuznetsov [37] and N. Kruglyak [36] for decompositions $(L^1, L^{1,\lambda})$, $(L^p, \text{Sobolev})$, (L^p, BMO)).

Our decompositions are done in the spirit of Calderón - Zygmund ones. Let F be a Banach space and F_0 be a subset of F, then we can consider the closure C of F_0 in F. The idea of our decomposition is to split a function $g \in C$ in the sum of a "good" part g_h, which is more regular, and of a " bad " part $g - g_h$ whose norm can be controlled by means of a continuity modulus of the function g itself.

In the previous considerations, we have put in evidence the need to prove boundedness and compactness of the operator (i) in the study of certain elliptic problems. Therefore the problem is to find the functional space V where the multiplication factor g has to belong. In literature there are several papers in which the authors have introduced suitable functional spaces in order to prove boundeness and compactness results for the operator (i) defined on the Sobolev space $S(\Omega)$. In [11], for instance, A. Canale, L. Caso, P. Di Gironimo introduced some weighted functional spaces where the weight is a function related to the distance from a fixed subset of the boundary of an open set of $\mathbb{R}^n$. As application, they obtained boundeness and compactness results for the operator (i) defined on weighted Sobolev spaces when the function g belongs to a suitable weighted space of above mentioned type. Moreover, in [58] M. Transirico, M. Troisi, A. Vitolo introduced some spaces of Morrey type and as application, they studied the operator (i) defined on a classical Sobolev space for a function g belonging to a suitable subspace of these spaces of Morrey type.

In the present work we want to analyze some of the issues above.

In the first part we want to deepen the study of some weighted functional spaces introduced in [11]. As application, using some decomposition results for functions belonging to such weighted spaces, we want to give a remarkable improvement of some results contained in [18], concerning some weighted norm inequalities on certain irregular domains of $\mathbb{R}^n$ and the boundedness and the compactness of the operator (i).

The structure of Chapter 1 and of Chapter 2 reflects the above purposes.

In Chapter 1 we describe some properties and applications of certain weighted Sobolev spaces which represent the setting of our main results.

If k is non - negative integer, p is a real number, $1 \leq p < +\infty$, Ω is a domain in $\mathbb{R}^n$ with boundary $\partial\Omega$, σ is a vector of non - negative (positive almost everywhere) measurable functions on Ω, which will be called weight, the weighted Sobolev space, usually denoted by $W^{k,p}(\Omega;\sigma)$, is defined as the set of all functions $u = u(x)$ which are defined *a.e.* on Ω and whose generalized (in the sense of distributions) derivatives $\partial^\alpha u$ of orders $|\alpha| \leq k$ satisfy

$$\int_\Omega |\partial^\alpha u(x)|^p \sigma_\alpha(x)\, dx < +\infty\,.$$

Sobolev spaces with weights have been intensively studied for more then forty years and their field of application has been constantly expanding. We have theoretical results concerning the structure of these spaces as well as applications to the theory of partial differential equations for the solution of boundary-value problems. The growing significance of these spaces is reflected by the number of papers devoted to them (see, for instance, D. E. Edmunds, W. D. Evans [27], A. Avantaggiati [3, 4], V. Benci,

D. Fortunato [6], V. G. Maz'ja [45], A. Kufner [39], A. Kufner, M. Sändig [43]).

In Chapter 2, we consider a class of weight functions denoted by $\mathcal{A}(\Omega)$ (introduced by M. Troisi in [60]) and the corresponding weighted Sobolev spaces defined on open subsets Ω of $\mathbb{R}^n$. More precisely, a measurable weight function $\rho : \Omega \to \mathbb{R}_+$ belongs to the class $\mathcal{A}(\Omega)$ if and only if there exists a costant $\gamma \in \mathbb{R}_+$, independent on x and y, such that

$$\gamma^{-1}\,\rho(y) \leq \rho(x) \leq \gamma\,\rho(y)\,, \qquad \forall\, y \in \Omega, \quad \forall\, x \in \Omega \cap B(y, \rho(y)),$$

Let S_ρ be a non empty subset of $\partial\Omega$ such that

$$\lim_{x \to z} \rho(x) = 0\,, \qquad \forall \rho \in \mathcal{A}(\Omega),\ \ \forall z \in S_\rho\,.$$

It's well known that a weight function $\rho \in \mathcal{A}(\Omega)$ is related to the distance function from S_ρ (see, e.g., M. Troisi [61]).

For more details on weight functions as distance functions from a nonempty subset of the boundary of a bounded open set of $\mathbb{R}^n$ or weight functions related to these distance functions, and for related problems see, i.e., A. Kufner [39], A. Kufner, O. John, S. Fucík [40], I. E. Egorov [28], Yu. D. Salmanov [49], M. Troisi [60].

For examples and properties of functions $\rho \in \mathcal{A}(\Omega)$ we refer to M. Troisi [60] and also to A. Canale, L. Caso, P. Di Gironimo [11], L. Caso, M. Transirico [18].

In some papers (see, e.g. , D. Fortunato [30], R. Schianchi [51], S. Matarasso, M. Troisi [44], M. Troisi [59], A. Canale, L. Caso, P. Di Gironimo [11]) some classes of weighted Sobolev spaces have been studied, where the weight function is a power of a function

$\rho \in \mathcal{A}(\Omega)$.

In particular in [11] the authors introduced a functional space, denoted by $K^r_t(\Omega)$ ($r \in [1, +\infty[$, $t \in \mathbb{R}$), as the class of all functions g, locally belonging to $L^r(\Omega)$, such that

$$\sup_{\Omega} \left(\rho^{t-\frac{n}{r}}(x) \, \|g\|_{L^r(\Omega \cap B(x,\rho(x)))} \right) < +\infty \, ,$$

where the weight function ρ belongs to the class $\mathcal{A}(\Omega)$. Moreover, they studied two subspaces of $K^r_t(\Omega)$, denoted by $\tilde{K}^r_t(\Omega)$ and $\mathring{K}^r_t(\Omega)$, defined respectively as the closure of $L^\infty_t(\Omega)$ and $C^\infty_o(\Omega)$ in $K^r_t(\Omega)$ (the space $L^\infty_t(\Omega)$ is the space of all functions g such that $\rho^t \, g \in L^\infty(\Omega)$).

In the first part of Chapter 2 we deepen the study of the spaces $K^r_t(\Omega)$ and of their subspaces. In particular, we construct suitable decompositions of functions $g \in \tilde{K}^r_t(\Omega)$ and of functions $g \in \mathring{K}^r_t(\Omega)$ (see L. Caso, R. D'Ambrosio [16]).

In the framework of spaces $K^r_t(\Omega)$, in [18] (see also [11]) the authors studied the operator (i) when $S(\Omega)$ is the weighted Sobolev space $W^{k,p}_s(\Omega)$, $k \in \mathbb{N}_0$, $s \in \mathbb{R}$, $1 \leq p \leq +\infty$, of the distributions u on Ω such that $\rho^{s+|\alpha|-k} \, \partial^\alpha \, u \in L^p(\Omega)$ for $|\alpha| \leq k$ and equipped with the norm

$$\|u\|_{W^{k,p}_s(\Omega)} = \sum_{|\alpha| \leq k} \|\rho^{s+|\alpha|-k} \, \partial^\alpha \, u\|_{L^p(\Omega)} \, ,$$

where ρ is a weight function belonging to the class $\mathcal{A}(\Omega)$. They gave different conditions on the function $g \in K^r_t(\Omega)$ necessary to obtain the estimate (ii) and the boundedness and compactness of the above mentioned operator.

These functional spaces were then used in the study of Dirichlet problems for linear second order elliptic equations in non regular domains and in weighted Sobolev spaces (see L. Caso, M. Transirico [19, 21, 20], L. Caso [12], L. Caso [13]). The main results of these papers are based on the properties of the operator (i) defined on $W_s^{k,p}(\Omega)$, in the two cases $g \in \widetilde{K}_t^r(\Omega)$ and $g \in \overset{\circ}{K}{}_t^r(\Omega)$ with appropriate conditions on p, r, s and t.

We study the operator (i) defined on weighted Sobolev space $W_s^{k,p}(\Omega)$ and taking values in $L^q(\Omega)$ with appropriate $q \in [p, r[$ obtaining a remarkable improvement of some results of [18]. We give suitable conditions on p, q, s, r, ρ, Ω and on the function $g \in K_t^r(\Omega)$ so that the following estimate holds

$$\|g\, u\|_{L^q(\Omega)} \leq c \cdot \|g\|_{K_t^r(\Omega)} \cdot \|u\|_{W_s^{k,p}(\Omega)}\,, \qquad (iii)$$

If $g \in \widetilde{K}_t^r(\Omega)$ or $g \in \overset{\circ}{K}{}_t^r(\Omega)$, from (iii) we deduce boundedness and compactness results for the considered operator. The use of our decompositions in these results allows us to put in evidence how the bad part $(g - g_h)$ of the function g in $\widetilde{K}_t^r(\Omega)$ or in $\overset{\circ}{K}{}_t^r(\Omega)$, affects the estimate.

The details of these proofs are contained in L. Caso, R. D'Ambrosio [16].

In the study of the above mentioned Dirichlet problems on irregular or unbounded domains, there is the need to put some conditions at the infinity on the lower order coefficients of the elliptic differential operator. Such conditions are ensured, for instance, by the assumption that the coefficients belong to space $\overset{\circ}{K}{}_t^r(\Omega)$. This also gives the compactness of the operator (i).

In view of these last considerations, we put in evidence a new characterization of the spaces $\overset{\circ}{K}{}_t^r(\Omega)$ by means of the introduction a new subspace of $K_t^r(\Omega)$, denoted by $\overset{*}{K}{}^r_t(\Omega)$ (see L. Caso, R. D'Ambrosio [16]). We state that under suitable conditions on the weight function $\rho \in \mathcal{A}(\Omega)$ the space $\overset{*}{K}{}^r_t(\Omega)$ is settled between $\overset{\circ}{K}{}^r_t(\Omega)$ and $\widetilde{K}{}^r_t(\Omega)$. In particular we give a condition on the weight function in order to obtain that $\overset{*}{K}{}^r_t(\Omega) = \overset{\circ}{K}{}^r_t(\Omega)$.

In the last part of this work we want to deepen the study of spaces of Morrey type introduced by M. Transirico, M. Troisi and A. Vitolo in [58]. Also in this case, using some decomposition results for functions belonging to a suitable subspace of a space of Morrey type we want to deduce a further compactness result for the operator (i) defined on Sobolev spaces without weight.

In Chapter 3 we analyze this aspect. Let Ω be an unbounded open subset of $\mathbb{R}^n$, $n \geq 2$. For $p \in [1, +\infty[$ and $\lambda \in [0, n[$, we consider the space $M^{p,\lambda}(\Omega)$ of the functions g in $L^p_{loc}(\overline{\Omega})$ such that

$$\|g\|^p_{M^{p,\lambda}(\Omega)} = \sup_{\substack{\tau \in]0,1] \\ x \in \Omega}} \tau^{-\lambda} \int_{\Omega \cap B(x,\tau)} |g(y)|^p \, dy < +\infty,$$

where $B(x, \tau)$ is the open ball with center x and radius τ.

This space of Morrey type is a generalization of the classical Morrey space $L^{p,\lambda}$ (see A. Kufner, O. John, S. Fucík [40]). It strictly contains $L^{p,\lambda}(\mathbb{R}^n)$ when $\Omega = \mathbb{R}^n$ and it is smaller than the class of the spaces $M^p(\Omega)$ of M. Transirico, M. Troisi [57, 55]. We remark that if the weight function $\rho \in \mathcal{A}(\Omega)$ is a positive constant, then the spaces $K^r_t(\Omega)$ are equal to the spaces $M^p(\Omega)$. The introduction of the spaces of Morrey type is related to the solvability of certain elliptic problems with discontinuous coefficients in the case of unbounded domains and in Sobolev spaces (see for instance M. Transirico, M. Troisi, A. Vitolo [58], P. Cavaliere, M. Longobardi, A. Vitolo [23],

L. Caso, P. Cavaliere, M. Transirico [15], L. Caso, P. Cavaliere, M. Transirico [14]).

In the first part of Chapter 3, we turn our attention to the density property of Morrey type spaces. The example $|x|^{(\lambda-n+1)/p}$ shows, the space $L^\infty(\Omega)$ is not dense in the space $M^{p,\lambda}(\Omega)$. So, it's important and useful to give a new characterization of functions in the closure of $L^\infty(\Omega)$ and $C_o^\infty(\Omega)$ in $M^{p,\lambda}(\Omega)$ (which are respectively denoted with $\widetilde{M}^{p,\lambda}(\Omega)$ and $M_0^{p,\lambda}(\Omega)$). By means of such chacterization lemmas we are allowed to construct suitable decompositions of functions in $\widetilde{M}^{p,\lambda}(\Omega)$ and $M_0^{p,\lambda}(\Omega)$ (see L. Caso, R. D'Ambrosio, S. Monsurrò [17]) .

In the framework of Morrey type spaces, in [58] the authors considered the operator defined in (i) when $S(\Omega)$ is the Sobolev space $W^{k,p}(\Omega)$ ($p \in [1, +\infty[$, $k \in \mathbb{N}$). In particular, they studied such operator for $k = 1$, generalizing a well known result proved by C. Fefferman in [30] (see also F. Chiarenza, M. Frasca [26]). They established conditions for the boundedness and compactness of this operator. In P. Cavaliere, M. Longobardi, A. Vitolo [23], the boundedness result and the straightforward estimates have been extended to more general results for any $k \in \mathbb{N}$.

The second part of Chapter 3 is devoted to a further analysis of the following multiplication operator

$$u \in W^{k,p}(\Omega) \to g\,u \in L^q(\Omega)$$

with a suitable q greater than p and g belonging to a space of Morrey type $M^{p,\lambda}(\Omega)$. By means of our decomposition results we are allowed to deduce a compactness result for the above mentioned operator. The details of these proofs are contained in L. Caso, R. D'Ambrosio, S. Monsurrò [17].

The deeper examination of the structure of $M^{p,\lambda}(\Omega)$ and of its subspaces lead us to the definition of a new functional space, that is a weighted Morrey type space, denoted by $M^{p,\lambda}_{\rho}(\Omega)$. In literature several authors have considered different kinds of weighted spaces of Morrey type and their applications to the study of elliptic equations, both in the degenerate case and in the non-degenerate one (see, for instance, C. Vitanza, P. Zamboni [62], C. Yemin [24] and Y. Komori, S. Shirai [34]).

In Chapter 3 we consider another class of weight functions, denoted by $\mathcal{G}(\Omega)$ (introduced by M. Troisi in [61]), and we define the corresponding weighted space $M^{p,\lambda}_{\rho}(\Omega)$ (see L. Caso, R. D'Ambrosio, S. Monsurró [17]). More precisely, let $d \in \mathbb{R}_+$, a measurable weight function $\rho : \Omega \to \mathbb{R}_+$ belongs to the class $\mathcal{G}(\Omega,\, d)$ if and only if there exists $\gamma \in \mathbb{R}_+$, independent on x and y, such that

$$\gamma^{-1}\,\rho(y) \le \rho(x) \le \gamma\,\rho(y)\,, \qquad \forall\, y \in \Omega, \quad \forall\, x \in \Omega(y,d).$$

We put

$$\mathcal{G}(\Omega) = \bigcup_{d>0} \mathcal{G}(\Omega, d).$$

For examples and properties of functions $\rho \in \mathcal{G}(\Omega)$ we refer to M. Troisi [61] and also to S. Boccia, M. Salvato, M. Transirico [8].

Let $\rho \in \mathcal{G}(\Omega) \cap L^\infty(\Omega)$ and let d be the positive real number such that $\rho \in \mathcal{G}(\Omega, d)$. Fix a Lebesgue measurable subset E of Ω, for $p \in [1, +\infty[$, $\lambda \in [0, n[$ we denote by $M^{p,\lambda}_{\rho}(\Omega)$ the space of all functions $g \in M^{p,\lambda}(\Omega)$ such that

$$\lim_{h \to +\infty} \Big(\sup_{\substack{E \in \Sigma(\Omega) \\ \sup\limits_{\substack{x \in \Omega \\ \tau \in]0,d]}} \tau^{-\lambda} \rho(x) |E(x,\tau)| \le \frac{1}{h}}} \| g\, \chi_E \|_{M^{p,\lambda}(\Omega)} \Big) = 0,$$

We prove that the space $M_\rho^{p,\lambda}(\Omega)$ is settled between $M_0^{p,\lambda}(\Omega)$ and $\widetilde{M}^{p,\lambda}(\Omega)$. In particular, we provide some conditions on ρ that entail $M_0^{p,\lambda}(\Omega) = M_\rho^{p,\lambda}(\Omega)$.

We remark that the results of this work can be used in the study of elliptic problems. More precisely, the estimates obtained in Chapter 2 can be used, for instance, in the study of some elliptic problems on irregular domains (i.e. domains with singular boundary) and in weighted Sobolev spaces $W_s^{k,p}$ to prove that the considered operators (whose lower order coefficients belong to weighted functional spaces K_t^r) have closed range or are semi-Fredholm. The estimates obtained in Chapter 3 can be useful, for instance, in the study of Dirichlet problems concerning elliptic equations in unbounded domains (whose boundary is sufficiently smooth) and in classical Sobolev spaces to estabilish a priori estimates for differential operator whose lower order coefficients belong to spaces of Morrey type.

Moreover we put in evidence that the introduction of spaces $\overset{*}{K}{}_t^r(\Omega)$ and $M_\rho^{p,\lambda}(\Omega)$ offers new points of views in the approach to the study of some classes of elliptic problems with discontinuos coefficients.

Finally, I warmly thank Maria Transirico, Loredana Caso and Sara Monsurrò for the useful suggestions and comments.

Chapter 1

Preliminaries

In this chapter we introduce some notations used throughout this work and we recall the definitions of some function spaces needed in the sequel.

1.1 Notations

Let G be a Lebesgue measurable subset of $\mathbb{R}^n$ and $\Sigma(G)$ be the σ-algebra of all Lebesgue measurable subsets of G. Given $F \in \Sigma(G)$ we denote by $|F|$ its Lebesgue measure and by χ_F its characteristic function. For every $x \in F$ and every $t \in \mathbb{R}_+$ we set $F(x,t) = F \cap B(x,t)$, where $B(x,t)$ is the open ball with center x and radius t and in particular we put $F(x) = F(x,1)$.

The class of restrictions to F of functions $\zeta \in C_\circ^\infty(\mathbb{R}^n)$ with $\overline{F} \cap \operatorname{supp} \zeta \subseteq F$ is denoted by $\mathfrak{D}(F)$ and, for $p \in [1, +\infty[$, $L^p_{\text{loc}}(F)$ is the class of all functions $g : F \to \mathbb{R}$ such that $\zeta\, g \in L^p(F)$ for any $\zeta \in \mathfrak{D}(F)$.

1.2 Weighted Sobolev Spaces

Weighted Sobolev spaces[1] are usually denoted by

$$W^{k,p}(\Omega;\sigma)$$

where

k is a non-negative integer, i.e. $k \in \mathbb{N}_0$,

p is a real number, $1 \le p < +\infty$,

Ω is a domain in $\mathbb{R}^n$ with a boundary $\partial\Omega$,

σ is a vector of non-negative (positive almost everywhere) measurable functions on Ω, which will be called a *weight*, i. e.

$$\sigma \;=\; \{\sigma_\alpha = \sigma_\alpha(x)\ ,\ x\in\Omega\ ,\ |\alpha|\le k\}\ , \tag{1.2.1}$$

α is a multiindex, i. e., $\alpha \in \mathbb{N}_0^n$ or equivalently

$$\alpha \;=\; (\alpha_1,\alpha_2,............,\alpha_n)\quad,\quad \alpha_i\in\mathbb{N}_0\,, \tag{1.2.2}$$

$$|\alpha| \;=\; \alpha_1+\alpha_2+.........+\alpha_n\,. \tag{1.2.3}$$

The space $W^{k,p}(\Omega;\sigma)$ is defined as the set of all functions $u = u(x)$ which are defined *a.e.* on Ω and whose generalized (in the sense of distributions) derivatives $\partial^\alpha u$ of orders $|\alpha| \le k$ satisfy

$$\int_\Omega |\partial^\alpha u(x)|^p \sigma_\alpha(x)\,dx < +\infty\,. \tag{1.2.4}$$

[1] See, *e.g.*, A. Kufner [39] and A. Kufner, A. M. Sändig [43].

It is a normed linear space if equipped with the norm

$$||u||_{W^{k,p}(\Omega;\sigma)} = \left(\sum_{|\alpha| \le k} \int_\Omega |\partial^\alpha u(x)|^p \sigma_\alpha(x) dx \right)^{1/p} , \tag{1.2.5}$$

or, equivalently, with the norm

$$||u||_{W^{k,p}(\Omega;\sigma)} = \sum_{|\alpha| \le k} \left(\int_\Omega |\partial^\alpha u(x)|^p \sigma_\alpha(x) dx \right)^{1/p} . \tag{1.2.6}$$

If

$$\sigma_\alpha^{-\frac{1}{p}} \in L^q_{loc}(\Omega) \quad \text{for} \quad |\alpha| \le k \ , \tag{1.2.7}$$

where q is the conjugate index of p, then the space $W^{k,p}(\Omega;\sigma)$ is a Banach space[2]. We observe that condition (1.2.7) is necessary to have the completeness; for instance, in A. Kufner, B. Opic [41] is proved that if $n = 1$, $\Omega = (-1, 1)$, $p = 2$, $\lambda, \mu \in \mathbb{R}$ and $\sigma = \left\{ \sigma_0 = |x|^\lambda, \sigma_1 = |x|^\mu \right\}$, the space $W^{1,2}(\Omega;\sigma)$ is non-complete if the parameters λ, μ are suitably chosen (it is easy to check that, for $\lambda \ge 1$ and $\mu \ge 1$, condition (1.2.7) is not true).

For $k = 0$ we introduce the following notation: we write

$$W^{0,p}(\Omega;\sigma) = L^p(\Omega;\sigma)$$

and denote

$$||u||_{L^p(\Omega;\sigma)} = \left(\int_\Omega |u(x)|^p \sigma(x) dx \right)^{1/p} , \tag{1.2.8}$$

[2] See A. Kufner, B. Opic [41, 42].

so that

$$||u||_{W^{k,p}(\Omega;\sigma)} = \left(\sum_{|\alpha| \le k} ||\partial^\alpha u||^p_{L^p(\Omega;\sigma_\alpha)} \right)^{1/p} .$$

Clearly the classical Sobolev spaces $W^{k,p}(\Omega)$ represent a special case of the weighted spaces $W^{k,p}(\Omega;\sigma)$: they can be obtained by the choice

$$\sigma_\alpha(x) = 1 \quad \text{for} \quad |\alpha| \le k.$$

The norm of a function $u \in W^{k,p}(\Omega)$ will be denoted by $||u||_{W^{k,p}(\Omega)}$[3].

Let us suppose that

$$\sigma_\alpha \in \text{Ł}^1_{loc}(\Omega) \quad \text{for} \quad |\alpha| \le k.$$

Then all functions in $C_0^\infty(\Omega)$ belong to $W^{k,p}(\Omega;\sigma)$ and it is meaningful to introduce the space $\overset{0}{W}{}^{k,p}(\Omega;\sigma)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm $||\cdot||_{W^{k,p}(\Omega;\sigma)}$. This space is again a Banach space if additionally (1.2.7) is satisfied.

There are several possibilities of application of Sobolev spaces with weights.

The first one concerns elliptic boundary value problems on domains whose boundary has various singularities as for example corners or edges. In the vicinity of a corner or an edge the solution u of the boundary-value problem may have a singularity which can be often very suitably characterized by an appropriate *weight.* This weight is most usually a power of the distance from the " singular set" on the boundary of domain. Hence, in this case, the weight functions make it possible to describe in more detail the qualitative properties of the solution, first of all as concerns its regularity.

[3] More information about the spaces $W^{k,p}(\Omega)$ can be found, for example, in S. L. Sobolev [52], J. Necăs [46], R. A. Adams [2], A. Kufner, O. John, S. Fucík [40].

On behalf of a number of papers devoted to these problems let us mention the paper by V. A. Kondrat'ev [35], and the paper by B. Kawohl [32].

A second field of application of weighted spaces concerns the study of functions defined on unbounded domains, which solve certain boundary value problems. Let us consider an unbounded domain Ω; for instance, let Ω be the exterior of the unit ball in $\mathbb{R}^n$. It is well known that - when solving boundary value problems - it is in this case necessary to give not only conditions on $\partial\Omega$, but also conditions at infinity, which prescribe the behaviour of the solution $u(x)$ for $|x| \to +\infty$. These conditions can again be described in a very convenient form in terms of weight functions, for example by means of functions of the form

$$(1+|x|)^\epsilon\,, \quad \epsilon \in \mathbb{R}_+\,.$$

It is evident that the condition

$$\int_{|x|>1} |u(x)|^2(1+|x|)^\epsilon\, dx < +\infty$$

characterizes the behaviour of the function $u(x)$ for large x. A typical representant of this direction is L. D. Kudrjavcev whose monograph [38] represents the first systematic exposition of properties of certain weighted spaces and of their applications. There is a number of groups and individuals working in this field - apart from Kudrjavcev and his successors let us mention for example B. Hanouzet [31], A. Avantaggiati, M. Troisi [5], as well as R. A. Adams [1].

Another domain of employment of weighted Sobolev spaces concerns more theoretical applications, namely existence theorems for elliptic differential equations and further for problems of the type of degenerate equations and equations with singular coefficients. Even in this field, the weighted spaces can provide an useful tool enlarging the scope of boundary value problems solvable by functional-analytical methods. A typical representant of this direction of applications of weighted Sobolev spaces is I. A. Kiprijanov [33]. The same topics are studied by S. M. Nikol'skii [47] and by a numerous French group (P. Bolley, J. Camus [9]).

We have introduced the weighted space $W^{k,p}(\Omega;\sigma)$ without making too many assumptions about the domain Ω and the weight σ whose components σ_α - see (1.2.1) - can be different for a different α. In what follows we will consider some weighted Sobolev spaces defined on an open set Ω in $\mathbb{R}^n$ and we will consider both weight vectors with different components and weight vectors with equal components. In the first case we will have a weighted norm in which the weight function varies if the order of derivation of function u varies, in the second case we will have a weighted norm in which the weight function is independent from the order of derivation of the function u.

Chapter 2

Weighted spaces and weighted norm inequalities on irregular domains

In this chapter we deepen the study of certain weighted spaces, denoted by $K^p_s(\Omega)$, defined on open subsets Ω of $\mathbb{R}^n$ when the weight is a function related to the distance from a subset of $\partial\Omega$. We also introduce a new weighted subspace of K^p_s. Moreover, we construct decompositions for functions belonging to some particular subspaces of K^p_s and as application, we prove boundedness and compactness results for an operator in weighted Sobolev spaces.

2.1 Weight functions and weighted spaces

Let Ω be an open subset of $\mathbb{R}^n$. We denote by $\mathcal{A}(\Omega)$[4] the class of measurable weight functions $\rho : \Omega \to \mathbb{R}_+$ such that

$$\sup_{\substack{x,y\in\Omega \\ |x-y|<\rho(y)}} \left| \log \frac{\rho(x)}{\rho(y)} \right| < +\infty \, .$$

It is easy to show[5] that $\rho \in \mathcal{A}(\Omega)$ if and only if there exists $\gamma \in \mathbb{R}_+$, independent on x and y, such that

$$\gamma^{-1}\rho(y) \leq \rho(x) \leq \gamma\, \rho(y)\,, \qquad \forall\, y \in \Omega\,, \quad \forall\, x \in \Omega \cap B(y, \rho(y))\,. \tag{2.1.1}$$

We remark that $\mathcal{A}(\Omega)$ contains the class of all functions $\rho : \Omega \to \mathbb{R}_+$ which are Lipschitz continuous in Ω with Lipschitz constant less than 1. Moreover, if $\rho \in \mathcal{A}(\Omega)$ and $a \in]0,1[$, then the function $\omega(x) = a\, \rho(x)$ $(x \in \Omega)$ belongs to $\mathcal{A}(\Omega)$.

Typical examples of functions $\rho \in \mathcal{A}(\Omega)$ are the function

$$x \in \mathbb{R}^n \to 1 + a\,|x|\,, \quad a \in\,]0,1[\,, \tag{2.1.2}$$

and, if $\Omega \neq \mathbb{R}^n$ and S is a nonempty subset of $\partial\Omega$, the function

$$x \in \Omega \to a\, \mathrm{dist}(x, S), \quad a \in]0,1[\,. \tag{2.1.3}$$

[4] The class $\mathcal{A}(\Omega)$ has been introduced by M. Troisi in [60].
[5] See, *e.g.*, M.Troisi [60].

Moreover if $\rho \in \mathcal{A}(\Omega)$ then for any $b \in \mathbb{R}_+$ and for any $s \in \mathbb{R}$ the function

$$x \in \Omega \;\to\; \frac{\rho(x)}{1 + b\,\rho^s(x)} \tag{2.1.4}$$

is in $\mathcal{A}(\Omega)$ (see, also M. Troisi [60]).

For any weight function $\rho \in \mathcal{A}(\Omega)$ we put

$$S_\rho = \{z \in \partial\Omega \,\big|\, \rho(x) \le |x - z| \quad \forall\, x \in \Omega\}\,. \tag{2.1.5}$$

We recall some properties of the set S_ρ[6].

Lemma 2.1.1 *For any $\rho \in \mathcal{A}(\Omega)$, the set S_ρ is a closed subset of $\partial\Omega$. Moreover we have*

$$z \in S_\rho \quad \Longleftrightarrow \quad \lim_{x \to z} \rho(x) = 0$$

$$z \in \partial\Omega \setminus S_\rho \quad \Longleftrightarrow \quad \exists\, r \in \mathbb{R}_+ \;:\; \inf_{\substack{x \in \Omega \\ |x-z|<r}} \rho(x) > 0\,. \qquad \blacksquare$$

From Lemma 2.1.1 it follows that, if $S_\rho \neq \emptyset$, then

$$\rho(x) \le \mathrm{dist}(x, S_\rho) \qquad \forall\, x \in \Omega\,, \tag{2.1.6}$$

$$\overline{B(x, \rho(x)) \cap \Omega} \,\cap\, S_\rho = \emptyset \qquad \forall\, x \in \Omega\,, \tag{2.1.7}$$

$$\rho \in L^\infty_{\mathrm{loc}}(\overline{\Omega})\,, \qquad \rho^{-1} \in L^\infty_{\mathrm{loc}}(\overline{\Omega} \setminus S_\rho)\,. \tag{2.1.8}$$

[6] See Lemma 1.1 and Theorem 1.1 in L. Caso, M. Transirico [18].

We recall now a regularization result for a function $\rho \in \mathcal{A}(\Omega)$, needed in the sequel. Let $\rho \in \mathcal{A}(\Omega)$, for any $x \in \Omega$ and for any $\lambda \in \mathbb{R}_+$, we set

$$E_\lambda(x) = \{y \in \Omega : |y - x| < \lambda\, \rho(y)\} \ , \ E(x) = E_1(x) \, .$$

$$I_\lambda(x) = \Omega \cap B(x, \lambda\, \rho(x)) \ , \ I(x) = I_1(x) \, .$$

It's easy to prove that

$$x \in E_\lambda(y) \Leftrightarrow y \in I_\lambda(x) \, .$$

For any $x \in \Omega$ and for any $\lambda \in \mathbb{R}_+$ we put

$$\chi_\lambda(x) = \rho^{-n}(x) |E_\lambda(x)| \, , \qquad \chi(x) = \chi_1(x) \, .$$

In M. Troisi [60] is proved that

$$\sup_{x \in \Omega} \chi(x) < +\infty \, . \tag{2.1.9}$$

Let us suppose

$$\inf_{x \in \Omega} \chi_\lambda(x) > 0 \qquad \text{for} \quad \lambda \in]0, 1[. \tag{2.1.10}$$

We remark that the condition (2.1.10) holds for any $\rho \in \mathcal{A}(\Omega)$ in the following cases: $\Omega = \mathbb{R}^n$, $\Omega = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$; Ω bounded domain with cone property[7]. Moreover, (2.1.10) is satisfied if Ω is an unbounded domain with the cone property and ρ is a bounded function in Ω. We remark that, if suitable conditions hold for

[7] A domain Ω with cone property means that there exists a finite cone C such that each point $x \in \overline{\Omega}$ is the vertex of a finite cone C_x contained in Ω and congruent to C .

the function ρ, (2.1.10) can hold for a domain Ω which has not the cone property. For instance, if one has

$$\rho(x) \cong \mathrm{dist}(x, \partial\Omega) \qquad \forall x \in \Omega \ \ ,$$

then the (2.1.10) holds for any domain Ω.
Moreover, it's easy to show that (2.1.10) holds if exist $\theta \in]0, \pi/2[$ and $c \in \mathbb{R}_+$ such that any $x \in \Omega$ is the vertex of a cone with opening θ and height $c\,\rho(x)$ which is contained in Ω.

Remark 2.1.2 *We observe that if the condition (2.1.10) holds, we have that for any $\rho \in \mathcal{A}(\Omega)$ there exists a function $\sigma \in \mathcal{A}(\Omega) \cap C^\infty(\Omega) \cap C^{0,1}(\overline{\Omega})$ such that*

$$c'\rho(x) \leq \sigma(x) \leq c''\rho(x) \qquad \forall x \in \Omega \,, \tag{2.1.11}$$

$$|\partial^\alpha \sigma(x)| \leq c_\alpha \sigma^{1-|\alpha|}(x) \qquad \forall x \in \Omega \quad \textit{and} \ \ \forall \ \alpha \in \mathbb{N}_0^n \,, \tag{2.1.12}$$

where $c', c'', c_\alpha \in \mathbb{R}_+$ are independent on x[8] *.*

Further properties of the class $\mathcal{A}(\Omega)$ can be found in M.Troisi [60] and L. Caso, M. Transirico [18].
If $k \in \mathbb{N}_0$, $1 \leq p \leq +\infty$, $s \in \mathbb{R}$ and $\rho \in \mathcal{A}(\Omega)$, we denote by $W^{k,p}_s(\Omega)$ the space of distributions u on Ω such that $\rho^{s+|\alpha|-k}\,\partial^\alpha u \in L^p(\Omega)$ for $|\alpha| \leq k$. Equipped with the norm

$$\|u\|_{W^{k,p}_s(\Omega)} = \sum_{|\alpha|\leq k} \|\rho^{s+|\alpha|-k}\,\partial^\alpha u\|_{L^p(\Omega)} \,, \tag{2.1.13}$$

$W^{k,p}_s(\Omega)$ is a Banach space. Moreover, it is separable if $1 \leq p < +\infty$, and, in particu-

[8] See Theorem 3.2 in M. Troisi [60].

lar, $W^{k,2}_s(\Omega)$ is a separable Hilbert space. We also denote by $\overset{\circ}{W}{}^{k,p}_s(\Omega)$ the closure of $C^\infty_o(\Omega)$ in $W^{k,p}_s(\Omega)$. The spaces just introduced are an example of weighted Sobolev spaces. A detailed account of properties of the above defined weighted Sobolev spaces can be found in D. E. Edmunds, W. D. Evans [27], V. Benci, D. Fortunato [6] and M. Troisi [61].

For $k = 0$ we put

$$W^{0,p}_s(\Omega) = L^p_s(\Omega)$$

From well known results[9] we deduce that, for $1 \le p < +\infty$ and $s \in \mathbb{R}$, the space $C^\infty_o(\Omega)$ is dense in $L^p_s(\Omega)$.

Clearly the following imbeddings hold:

$$\overset{\circ}{W}{}^{k,p}_s(\Omega) \hookrightarrow W^{k,p}_s(\Omega) \hookrightarrow L^p_{s-k}(\Omega)\,.$$

2.2 The spaces $K^p_s(\Omega)$

The purpose of this section is to deepen the study of the weighted spaces K^p_s and their properties[10]. Let us introduce some definitions which are essential to study such spaces.

Let Ω be an open subset of $\mathbb{R}^n$ and let $\rho \in \mathcal{A}(\Omega)$. We fix f in $\mathfrak{D}(\overline{\mathbb{R}}_+)$ satisfying the conditions

$$0 \le f \le 1\,, \qquad f(t) = 1 \quad \text{if} \quad t \le \frac{1}{2}\,, \qquad f(t) = 0 \quad \text{if} \quad t \ge 1\,,$$

[9] See, *e.g.*, D. E. Edmunds, W. D. Evans [27] and M. Troisi [61].

[10] The spaces K^p_s were studied, *e.g.*, by A. Canale, L. Caso, P. Di Gironimo in [11] and by L. Caso, M. Transirico in [18].

and $\alpha \in C^\infty(\Omega) \cap C^{0,1}(\overline{\Omega})$ equivalent to dist $(\cdot, \partial\Omega)$[11]. Hence, for $h \in \mathbb{N}$ we put

$$\psi_h : x \in \overline{\Omega} \to \Big(1 - f\big(h\,\alpha(x)\big)\Big)\, f\big(|x|/2h\big)\,. \tag{2.2.1}$$

It is easy to prove that ψ_h belongs to $\mathfrak{D}(\bar{\Omega} \setminus S_\rho)$ for any $h \in \mathbb{N}$ and

$$0 \leq \psi_h \leq 1\,, \qquad {\psi_h}_{|\overline{\Omega}_h} = 1\,, \qquad \operatorname{supp} \psi_h \subset \overline{\Omega}_{2h}\,, \tag{2.2.2}$$

where

$$\Omega_h = \big\{x \in \Omega \;\big|\; |x| < h,\ \alpha(x) > 1/h\big\}. \tag{2.2.3}$$

For any $x \in \Omega$, let $G(x)$ be an open subset of $\mathbb{R}^n$ such that

$$x \in G(x) \subseteq \Omega \cap B\big(x, \rho(x)\big)\,. \tag{2.2.4}$$

For $1 \leq p < +\infty$ and $s \in \mathbb{R}$, we denote by $K^p_s(\Omega)$ the class of functions $g \in L^p_{\mathrm{loc}}(\bar{\Omega} \setminus S_\rho)$ such that

$$\|g\|_{K^p_s(\Omega)} = \sup_{\Omega} \Big(\rho^{s-\frac{n}{p}}(x)\, \|g\|_{L^p(G(x))}\Big) < +\infty\,. \tag{2.2.5}$$

Obviously $K^p_s(\Omega)$ is a Banach space with the norm defined by (2.2.5). It is easy to prove that the spaces $L^\infty_s(\Omega)$ and $C^\infty_o(\Omega)$ are subsets of $K^p_s(\Omega)$[12]. Therefore, we can define two new spaces of functions $\widetilde{K}^p_s(\Omega)$ and $\overset{\circ}{K}{}^p_s(\Omega)$ as the closures of $L^\infty_s(\Omega)$ and $C^\infty_o(\Omega)$, respectively, in $K^p_s(\Omega)$.

[11] For more details on the existence of such an α see, *e.g.*, Theorem 2, Cap. VI in E. M. Stein [53] and Lemma 3.6.1 in W. P. Ziemer [63].

[12] In fact from (2.1.8) we have that $C^\infty_0(\Omega)$ is a subset of $L^\infty_s(\Omega)$ for every $s \in R$. Moreover the space $L^\infty_s(\Omega)$ is imbedded in $K^p_s(\Omega)$ for any $p \in [1, +\infty[$ and for any $s \in R$, see, *e.g.*,(14) in A. Canale, L. Caso, P. Di Gironimo [11].

For all $s \in \mathbb{R}$ the following inclusions hold[13]

$$L^r_{s-\frac{n}{r}}(\Omega) \hookrightarrow K^q_s(\Omega) \hookrightarrow K^p_s(\Omega) \qquad 1 \le p \le q \le r \le +\infty\,, \tag{2.2.6}$$

$$L^q_{s-\frac{n}{q}}(\Omega) \subset \overset{\circ}{K}{}^p_s(\Omega) \subset \widetilde{K}^p_s(\Omega) \qquad 1 \le p \le q < +\infty\,, \tag{2.2.7}$$

$$K^q_s(\Omega) \subset \widetilde{K}^p_s(\Omega) \qquad 1 \le p < q < +\infty\,. \tag{2.2.8}$$

We put

$$K^p(\Omega) = K^p_{n/p}(\Omega)\,, \tag{2.2.9}$$

and, in the same way, we define the spaces $\overset{\circ}{K}{}^p(\Omega)$ and $\widetilde{K}^p(\Omega)$. From (2.2.6), (2.2.7) and (2.2.8) we have

$$L^p(\Omega) \subset \overset{\circ}{K}{}^p(\Omega) \subset \widetilde{K}^p(\Omega) \subset K^p(\Omega)\,. \tag{2.2.10}$$

It' s possible to show, with some counterexamples, that the three inclusions in (2.2.10) can be strict (see [11]).

We recall now some characterizations of the spaces $\widetilde{K}^p_s(\Omega)$ and $\overset{\circ}{K}{}^p_s(\Omega)$[14] needed in sequel.

Lemma 2.2.1 *A function $g \in K^p_s(\Omega)$ belongs to $\widetilde{K}^p_s(\Omega)$ if and only if*

$$\lim_{h\to+\infty}\Bigg(\sup_{\substack{E\in\Sigma(\Omega)\\ \sup_{x\in\Omega}\frac{|G(x)\cap E|}{\rho^n(x)}\le 1/h}} \|g\,\chi_E\|_{K^p_s(\Omega)}\Bigg) = 0,$$

where χ_E denotes the characteristic function of the set E. ∎

[13] See, *e.g.*, (18), (19), (20) in A. Canale, L. Caso, P. Di Gironimo [11].
[14] See Lemma 3 and Lemma 2 of [11].

Lemma 2.2.2 *A function* $g \in K^p_s(\Omega)$ *belongs to* $\overset{\circ}{K}{}^p_s(\Omega)$ *if and only if*

$$\lim_{h \to +\infty} \|\big(1 - \psi_h\big)\, g\|_{K^p_s(\Omega)} = 0. \qquad \blacksquare$$

In L. Caso, M. Transirico [18] is proved the following condition for a function in $K^p_s(\Omega)$ to belong to $\overset{\circ}{K}{}^p_s(\Omega)$.

Lemma 2.2.3 *Let* $g \in K^p_s(\Omega)$*. If*

$$\lim_{|x| \to +\infty} \rho^s(x)\, g(x) = \lim_{x \to z} \rho^s(x)\, g(x) = 0 \qquad \forall\, z \in S_\rho\,,$$

then $g \in \overset{\circ}{K}{}^p_s(\Omega)$. $\blacksquare$

In the following Lemma we give a necessary condition for a function $g \in K^p_s(\Omega)$ to belong to $\overset{\circ}{K}{}^p_s(\Omega)$.

Lemma 2.2.4 *If* $g \in \overset{\circ}{K}{}^p_s(\Omega)$ *then*

$$\lim_{|x| \to +\infty} ||g||_{L^p_{s-\frac{n}{p}}(G(x))} = 0 \tag{2.2.11}$$

Proof $-$ From Lemma 2.2.2 and (2.1.1) we have

$$||(1 - \psi_h)\, g||_{L^p_{s-\frac{n}{p}}(G(x))} < \epsilon\,, \qquad \forall h \geq h_\epsilon\,.$$

Using (2.2.2) we have

$$||g||_{L^p_{s-\frac{n}{p}}(G(x))} \leq ||(1 - \psi_{h_\epsilon})\, g||_{L^p_{s-\frac{n}{p}}(G(x))} < \epsilon\,, \quad \forall\, |x| \geq 2\, h_\epsilon\,.$$

From the previous inequalities we obtain (2.2.11). ■

In view of Lemma 2.2.3, we can give[15] a condition on a weight function ρ and on $s\,,\tau \in \mathbb{R}$ so that $L^\infty_\tau(\Omega)$ is a subspace of $\overset{\circ}{K}{}^p_s(\Omega)$.

Lemma 2.2.5 *Let* $1 \le p < +\infty$ *and* $s,\ \tau \in \mathbb{R}$ *with* $\tau < s$. *Suppose that* $\lim\limits_{|x|\to+\infty} \rho(x) = 0$, *then* $L^\infty_\tau(\Omega)$ *is a subspace of* $\overset{\circ}{K}{}^p_s(\Omega)$.

Proof $-$ First we observe that the hypothesis on ρ and (2.1.8) give that $\rho \in L^\infty(\Omega)$. By (2.2.6) we have that $L^\infty_\tau(\Omega) \subset K^p_\tau(\Omega)$. We prove that $L^\infty_\tau(\Omega) \subset K^p_s(\Omega)$. In fact, fixed $g \in L^\infty_\tau(\Omega) \subset K^p_\tau(\Omega)$, we have

$$\|g\|_{K^p_s(\Omega)} = \sup_\Omega \left(\rho^{s-\frac{n}{p}}(x)\, \|g\|_{L^p(G(x))}\right) \le$$

$$\|\rho\|^{s-\tau}_{L^\infty(\Omega)} \cdot \sup_\Omega \left(\rho^{\tau-\frac{n}{p}}(x)\, \|g\|_{L^p(G(x))}\right) = \|\rho\|^{s-\tau}_{L^\infty(\Omega)} \cdot \|g\|_{K^p_\tau(\Omega)}\,.$$

Using again the hypothesis on ρ, we have

$$\lim_{|x|\to+\infty} |\rho^s(x)\, g(x)| \le \lim_{|x|\to+\infty} \rho^{s-\tau}(x)\, \|g\|_{L^\infty_\tau(\Omega)} = 0\,,$$

$$\lim_{x\to z} |\rho^s(x)\, g(x)| \le \lim_{x\to z} \rho^{s-\tau}(x)\, \|g\|_{L^\infty_\tau(\Omega)} = 0 \qquad \forall\, z \in S_\rho\,.$$

Then the result easily follows from Lemma 2.2.3. ■

[15] See Lemma 3.4 in L. Caso, R. D'Ambrosio [16].

2.2.1 The spaces $\overset{*}{K}{}^p_s(\Omega)$

Let us introduce a new subspace of $K^p_s(\Omega)$. Let Ω be an open subset of $\mathbb{R}^n$, for any $p \in [1, +\infty[$ and $s \in \mathbb{R}$ we denote by $\overset{*}{K}{}^p_s(\Omega)$[16] the space of all functions $g \in K^p_s(\Omega)$ such that

$$\lim_{h \to +\infty} \left(\sup_{\substack{E \in \Sigma(\Omega) \\ \sup_{x \in \Omega} |G(x) \cap E| \leq 1/h}} \| g \, \chi_E \|_{K^p_s(\Omega)} \right) = 0. \tag{2.2.12}$$

Now we prove some properties of the space $\overset{*}{K}{}^p_s(\Omega)$ and then we examine the relations between all the subspaces of $K^p_s(\Omega)$. The shown results are in L. Caso, R. D'Ambrosio [16].

At first we rewrite the space $\overset{*}{K}{}^p_s(\Omega)$ as a closure of an appropriate subspace of $K^p_s(\Omega)$.

Lemma 2.2.6 *Let $1 \leq p < +\infty$ and $s \in \mathbb{R}$. Then $\overset{*}{K}{}^p_s(\Omega)$ is the closure of the space $L^\infty_{s-\frac{n}{p}}(\Omega) \cap K^p_s(\Omega)$ in $K^p_s(\Omega)$.*

Proof − Fix $g \in \overset{*}{K}{}^p_s(\Omega)$. By (2.2.12), for any $\epsilon > 0$ there exists $t_\epsilon \in]0, 1[$ such that, if $E \in \Sigma(\Omega)$ with $\sup_\Omega |G(x) \cap E| \leq t_\epsilon$, then $\| g \, \chi_E \|_{K^p_s(\Omega)} < \epsilon$. For each $k \in \mathbb{R}_+$ put

$$E_k = \left\{ x \in \Omega \;\middle|\; \rho^{s-\frac{n}{p}}(x) \, |g(x)| \geq k \right\}.$$

Thus, by (2.2.4) and (2.1.1), we have

$$\begin{gathered} \|g\|_{K^p_s(\Omega)} \geq c_1 \sup_\Omega \|\rho^{s-\frac{n}{p}} \, g\|_{L^p(G(x))} \geq \\ c_1 \sup_\Omega \|\rho^{s-\frac{n}{p}} \, g\|_{L^p(G(x) \cap E_k)} \geq c_1 \, k \sup_\Omega |G(x) \cap E_k|^{\frac{1}{p}}, \end{gathered} \tag{2.2.13}$$

[16] See L. Caso, R. D'Ambrosio [16].

where $c_1 \in \mathbb{R}_+$ depends on ρ, s, n and p. If we set

$$k_\epsilon = \frac{\|g\|_{K^p_s(\Omega)}}{c_1\,(t_\epsilon)^{\frac{1}{p}}}\,,$$

from (2.2.13) it follows that

$$\sup_\Omega |G(x) \cap E_{k_\epsilon}| \le \left(\frac{\|g\|_{K^p_s(\Omega)}}{c_1\,k_\epsilon}\right)^p,$$

and then

$$\|g\,\chi_{E_{k_\epsilon}}\|_{K^p_s(\Omega)} < \epsilon\,. \tag{2.2.14}$$

Now define $g_\epsilon = g - g\,\chi_{E_{k_\epsilon}}$ and observe that $g_\epsilon \in L^\infty_{s-\frac{n}{p}}(\Omega) \cap K^p_s(\Omega)$. Therefore from (2.2.14) we deduce that $\|g - g_\epsilon\|_{K^p_s(\Omega)} < \epsilon$.
Suppose conversely that g belongs to the closure of $L^\infty_{s-\frac{n}{p}}(\Omega)\cap K^p_s(\Omega)$ in $K^p_s(\Omega)$. Therefore for any fixed $\epsilon > 0$ there exists $g_\epsilon \in L^\infty_{s-\frac{n}{p}}(\Omega) \cap K^p_s(\Omega)$ for which

$$\|g - g_\epsilon\|_{K^p_s(\Omega)} < \frac{\epsilon}{2}\,. \tag{2.2.15}$$

Fixed $E \in \Sigma(\Omega)$, we observe that by (2.2.15) we get

$$\|g\,\chi_E\|_{K^p_s(\Omega)} \le \|(g - g_\epsilon)\,\chi_E\|_{K^p_s(\Omega)} + \|g_\epsilon\,\chi_E\|_{K^p_s(\Omega)} < \frac{\epsilon}{2} + \|g_\epsilon\,\chi_E\|_{K^p_s(\Omega)}\,. \tag{2.2.16}$$

On the other hand, (2.2.4) and (2.1.1) imply

$$\|g_\epsilon\, \chi_E\|_{K^p_s(\Omega)} \le c_2 \sup_\Omega \|\rho^{s-\frac{n}{p}}\, g_\epsilon\, \chi_E\|_{L^p(G(x))} \le$$

$$c_2\, \|g_\epsilon\|_{L^\infty_{s-\frac{n}{p}}(\Omega)} \sup_\Omega |G(x)\cap E|^{\frac{1}{p}}\,, \tag{2.2.17}$$

where $c_2 \in \mathbb{R}_+$ depends on ρ, s, n and p. If we set

$$t_\epsilon = \left(\frac{\epsilon}{2\, c_2\, \|g_\epsilon\|_{L^\infty_{s-\frac{n}{p}}(\Omega)}}\right)^p,$$

from (2.2.17) we deduce that, if $\sup_\Omega |G(x)\cap E| \le t_\epsilon$ then

$$\|g_\epsilon\, \chi_E\|_{K^p_s(\Omega)} \le \frac{\epsilon}{2}\,. \tag{2.2.18}$$

In view of (2.2.16) and (2.2.18), it follows that $g \in \overset{*}{K}{}^p_s(\Omega)$. ■

We can prove now the following result which is similar to Lemma 2.2.5, but with no additional hypothesis on the weight function ρ.

Lemma 2.2.7 *Let $1 \le p < +\infty$ and $s, \tau \in \mathbb{R}$ with $\tau < s$. If $g \in L^\infty_\tau(\Omega) \cap K^p_s(\Omega)$, then $g \in \overset{*}{K}{}^p_s(\Omega)$.*

Proof − Let $g \in L^\infty_\tau(\Omega) \cap K^p_s(\Omega)$. Fix $E \in \Sigma(\Omega)$ and $t \in]0,1[$ such that

$$\sup_\Omega |G(x)\cap E| \le t\,. \tag{2.2.19}$$

From (2.2.4), (2.1.1) and (2.2.19), we obtain

$$\|g\,\chi_E\|_{K^p_s(\Omega)} \le c_1 \sup_\Omega \rho^{s-\tau-\frac{n}{p}}(x)\|\rho^\tau\, g\,\chi_E\|_{L^p(G(x))} \le$$
$$c_1\,\|g\|_{L^\infty_\tau(\Omega)} \sup_\Omega \rho^{s-\tau-\frac{n}{p}}(x)|G(x)\cap E|^{\frac{1}{p}} \le c_2\,\|g\|_{L^\infty_\tau(\Omega)}\, t^{\frac{s-\tau}{n}}\,, \tag{2.2.20}$$

where $c_1 \in \mathbb{R}_+$ depends on ρ, τ, p and $c_2 \in \mathbb{R}_+$ depends on the same parameters as c_1 and on n.

Since $\tau < s$, from (2.2.20) it follows that

$$\lim_{t\to 0}\left(\sup_{\substack{E\in\Sigma(\Omega)\\ \sup\limits_{x\in\Omega}|G(x)\cap E|\le t}} \|g\,\chi_E\|_{K^p_s(\Omega)}\right) = 0,$$

and so $g \in \overset{*}{K}{}^p_s(\Omega)$. ■

Combining Lemmas 2.2.6 and 2.2.7, we can obtain a new characterization of the space $\overset{*}{K}{}^p_s(\Omega)$.

Lemma 2.2.8 *Let* $1 \le p < +\infty$ *and* $s \in \mathbb{R}$. *Then* $\overset{*}{K}{}^p_s(\Omega)$ *is the closure of the space* $\bigcup\limits_{\tau<s} L^\infty_\tau(\Omega)\cap K^p_s(\Omega)$ *in* $K^p_s(\Omega)$.

Proof $-$ Fix $g \in \overset{*}{K}{}^p_s(\Omega)$. By Lemma 2.2.6, g belongs to the closure of $L^\infty_{s-\frac{n}{p}}(\Omega)\cap K^p_s(\Omega)$ in $K^p_s(\Omega)$. Since $L^\infty_{s-\frac{n}{p}}(\Omega) \subset \bigcup\limits_{\tau<s} L^\infty_\tau(\Omega)$, we easily deduce one of our assertions. In order to prove the converse statement, fix g in the closure of $\bigcup\limits_{\tau<s} L^\infty_\tau(\Omega)\cap K^p_s(\Omega)$ in $K^p_s(\Omega)$. Therefore for each $\epsilon > 0$ there exist $\tau < s$ and a function $g_\epsilon \in L^\infty_\tau(\Omega)\cap K^p_s(\Omega)$ such that

$$\|g - g_\epsilon\|_{K^p_s(\Omega)} \le \frac{\epsilon}{2}\,. \tag{2.2.21}$$

Obviously, from (2.2.21), for any $E \in \Sigma(\Omega)$ we have

$$\|g\,\chi_E\|_{K^p_s(\Omega)} \leq \|(g-g_\epsilon)\,\chi_E\|_{K^p_s(\Omega)} + \|g_\epsilon\,\chi_E\|_{K^p_s(\Omega)} \leq \frac{\epsilon}{2} + \|g_\epsilon\,\chi_E\|_{K^p_s(\Omega)}\,. \tag{2.2.22}$$

On the other hand, from Lemma 2.2.7, we deduce that there exists $t_o \in]0,1[$ such that for $t \in]0, t_o[$ we obtain

$$\sup_{\substack{E\in\Sigma(\Omega)\\ \sup_{x\in\Omega}|G(x)\cap E|\leq t}} \|g_\epsilon\,\chi_E\|_{K^p_s(\Omega)} < \frac{\epsilon}{2}\,. \tag{2.2.23}$$

From (2.2.22), it follows that

$$\sup_{\substack{E\in\Sigma(\Omega)\\ \sup_{x\in\Omega}|G(x)\cap E|\leq t}} \|g\,\chi_E\|_{K^p_s(\Omega)} < \epsilon\,,$$

for any $t \in]0, t_o[$ and then $g \in \overset{*}{K}{}^p_s(\Omega)$. ■

We look now at the connections between the spaces $\overset{*}{K}{}^p_s(\Omega)$ and $\overset{\circ}{K}{}^p_s(\Omega)$ or $\widetilde{K}{}^p_s(\Omega)$.

Lemma 2.2.9 *Let* $1 \leq p < +\infty$ *and* $s \in \mathbb{R}$. *Then* $\overset{\circ}{K}{}^p_s(\Omega) \subset \overset{*}{K}{}^p_s(\Omega)$. *If moreover* $\lim_{|x|\to+\infty} \rho(x) = 0$, *then* $\overset{\circ}{K}{}^p_s(\Omega) = \overset{*}{K}{}^p_s(\Omega)$.

Proof – We observe that $C^\infty_o(\Omega)$ is a subspace of $L^\infty_{s-\frac{n}{p}}(\Omega) \cap K^p_s(\Omega)$. So, from Lemma 2.2.6, we easily deduce that $\overset{\circ}{K}{}^p_s(\Omega) \subset \overset{*}{K}{}^p_s(\Omega)$.
Suppose now that $\lim_{|x|\to+\infty} \rho(x) = 0$. From Lemma 2.2.5 we know that the closure of $\bigcup_{\tau<s} L^\infty_\tau(\Omega)$ in $K^p_s(\Omega)$ is a subspace of $\overset{\circ}{K}{}^p_s(\Omega)$. This assertion and Lemma 2.2.8 complete the proof. ■

Now we give a condition on weight function ρ so that a function g in $\overset{*}{K}{}^p_s(\Omega)$ is in $\widetilde{K}^p_s(\Omega)$.

Lemma 2.2.10 *Let $1 \le p < +\infty$ and $s \in \mathbb{R}$. If $\rho \in L^\infty(\Omega)$, then $\overset{*}{K}{}^p_s(\Omega) \subset \widetilde{K}^p_s(\Omega)$.*

PROOF – Since $\rho \in L^\infty(\Omega)$, then $L^\infty_{s-\frac{n}{p}}(\Omega) \subset L^\infty_s(\Omega)$. The statement easily follows from Lemma 2.2.6. ∎

We define now $\overset{*}{K}{}^p(\Omega)$ in the same way of the space $K^p(\Omega)$ (see 2.2.9).

From Lemma 2.2.9 we have

$$\overset{\circ}{K}{}^p(\Omega) \subset \overset{*}{K}{}^p(\Omega) \tag{2.2.24}$$

and if $\rho \in L^\infty(\Omega)$ then from Lemma 2.2.10 one has

$$\overset{*}{K}{}^p(\Omega) \subset \widetilde{K}^p(\Omega) \tag{2.2.25}$$

We want to show, with some counterexamples, that the two inclusions in (2.2.24) and (2.2.25) can be strict.

For semplicity we will assume in the following that $G(x) = \Omega(x)\,,\ \forall x \in \Omega$.

Example 1

Let

$$\Omega = \left\{x \in \mathbb{R}^2 \,:\, x_1 > 1\,,\ \ 0 < x_2 < x_1\right\}\,, \qquad \rho : x \in \Omega \to \frac{2+|x|}{4+|x|}\,. \tag{2.2.26}$$

We remark that

$$\rho(x) = \frac{\sigma(x)}{1+\sigma(x)}$$

where the function $\sigma(x) = 1 + \frac{1}{2}|x| \in \mathcal{A}(\Omega)$ for any $x \in \Omega$ (see 2.1.2). Hence from (2.1.4) we have that the function $\rho \in \mathcal{A}(\Omega)$.

Let us consider the function

$$g \, : \, x \in \Omega \to \frac{|x|}{1+|x|} \tag{2.2.27}$$

We have

$$g \in \overset{*}{K}{}^1(\Omega) \setminus \overset{\circ}{K}{}^1(\Omega)\,.$$

In fact, since the functions g and ρ belong, obviously, to the space $L^\infty(\Omega)$ and the imbedding (2.2.6) holds, one has

$$L^\infty(\Omega) \hookrightarrow L^\infty_2(\Omega) \hookrightarrow K^1(\Omega)$$

Hence

$$g \in L^\infty(\Omega) \cap K^1(\Omega)$$

and from Lemma 2.2.7 we obtain that $g \in \overset{*}{K}{}^1(\Omega)$.

We want to show now that $g \notin \overset{\circ}{K}{}^1(\Omega)$.

We remark that

$$|\Omega(x)| \approx \rho^2(x) = \left(\frac{2+|x|}{4+|x|}\right)^2 , \quad x_1 < |x| < \sqrt{2}\, x_1 \quad \forall x \in \Omega$$

and

$$\frac{1}{1+\sqrt{2}} < g(x) < 1\,, \quad \forall x \in \Omega \tag{2.2.28}$$

We have

$$\int_{\Omega(x)} g(y)\,dy \approx |\Omega(x)| \approx \left(\frac{2+|x|}{4+|x|}\right)^2 . \tag{2.2.29}$$

Hence

$$\lim_{|x|\to+\infty} ||g||_{L^1(\Omega(x))} = 1 .$$

Using Lemma 2.2.4 with $s = 2$, $n = 2$ and $p = 1$, we deduce that $g \notin \overset{\circ}{K}{}^1(\Omega)$.

Example 2

Let $t \in]1, +\infty[$, $\Omega = \left\{x \in \mathbb{R}^2 \,:\, 0 < x_1 < 1 ,\ 0 < x_2 < x_1^t\right\}$ and $\rho(x) = \frac{1}{2}\, x_1^t$.

Evidently, $\rho \in \mathcal{A}(\Omega) \cap L^\infty(\Omega)$ (see, also Example 1 of A. Canale, L. Caso, P. Di Gironimo [11]). Moreover for a fixed $\theta \in]0, \frac{\pi}{2}[$ and for any $x \in \Omega$, there exists $C_\theta(x)$ such that $\overline{C_\theta(x, \rho(x))} \subset \Omega$. Let $\Omega(x)$, $x \in \Omega$, be the open subset of $\mathbb{R}^2$ union of the open cones $C \subset\subset \Omega$ with height $\rho(x)$, opening θ and such that $x \in \Omega$.

If we consider the function

$$g_\alpha \,:\, x \in \Omega \to |x|^{-\alpha} , \quad \alpha \in \mathbb{R}_+ ,$$

we have

$$g_\alpha \in \overset{*}{K}{}^1(\Omega) , \quad \text{if } 0 < \alpha < 2t ,$$

$$g_{2t} \in \widetilde{K}{}^1(\Omega) \setminus \overset{*}{K}{}^1(\Omega) .$$

Since $\rho \in L^\infty(\Omega)$ and Ω is a bounded open subset of $\mathbb{R}^2$, from Lemma 2.2.9 one has

$$\overset{\circ}{K}{}^p(\Omega) = \overset{*}{K}{}^p(\Omega) . \tag{2.2.30}$$

We remark that, obviously, it holds

$$g \in \overset{\circ}{K}{}^p(\Omega) \Leftrightarrow g \in K^p(\Omega) \text{ and } \lim_{x \to 0} ||g||_{L^p(\Omega(x))} = 0\,, \tag{2.2.31}$$

and

$$|\Omega(x)| \approx \rho^2(x) = \frac{1}{4} x_1^{2t} \approx |x|^{2t}\,, \quad \forall x \in \Omega\,. \tag{2.2.32}$$

We have

$$\int_{\Omega(x)} g_\alpha(y)\, dy \approx x_1^{-\alpha} |\Omega(x)| \approx |x|^{-\alpha+2t} \qquad \forall x \in \Omega\,. \tag{2.2.33}$$

From (2.2.31) and (2.2.33) we deduce that

$$g_\alpha \in \overset{\circ}{K}{}^1(\Omega) \Leftrightarrow 0 < \alpha < 2t\,, \tag{2.2.34}$$

so from (2.2.30) we obtain that $g_\alpha \in \overset{*}{K}{}^1(\Omega)$ for any $0 < \alpha < 2t$.

On the other hand for all $E \in \Sigma(\Omega)$ we have

$$\int_{\Omega(x) \cap E} g_{2t}(y)\, dy \approx x_1^{-2t} |\Omega(x) \cap E| \approx \rho^{-2}(x) |\Omega(x) \cap E|\,, \quad \forall x \in \Omega \tag{2.2.35}$$

from (2.2.35) and Lemma 2.2.1 we deduce that $g_{2t} \in \widetilde{K}{}^1(\Omega)$.

2.3 Decompositions of functions in $\widetilde{K}{}^p_s(\Omega)$, $\overset{*}{K}{}^p_s(\Omega)$, $\overset{\circ}{K}{}^p_s(\Omega)$

We now introduce some continuous functions related to the characterizations of the subspaces of $K^p_s(\Omega)$. We will assume in the following that $p \in [1, +\infty[$ and $s \in \mathbb{R}$.

Let $g \in \widetilde{K}{}^p_s(\Omega)$. We define *modulus of continuity* of g in $\widetilde{K}{}^p_s(\Omega)$ a map $\widetilde{\omega}{}^p_s[g] : \mathbb{R}_+ \to \mathbb{R}_+$

such that

$$\begin{gathered}\sup_{\substack{E\in\Sigma(\Omega)\\ \sup_{x\in\Omega}\frac{|G(x)\cap E|}{\rho^n(x)}\le 1/h}} \|g\,\chi_E\|_{K^p_s(\Omega)} \le \widetilde{\omega}^p_s[g](h)\\ \lim_{h\to+\infty}\widetilde{\omega}^p_s[g](h)=0\,.\end{gathered} \tag{2.3.1}$$

Let $g\in \overset{*}{K}{}^p_s(\Omega)$. We define *modulus of continuity* of g in $\overset{*}{K}{}^p_s(\Omega)$ a map $\overset{*}{\omega}{}^p_s[g]:\mathbb{R}_+\to\mathbb{R}_+$ such that

$$\begin{gathered}\sup_{\substack{E\in\Sigma(\Omega)\\ \sup_{x\in\Omega}|G(x)\cap E|\le 1/h}} \|g\,\chi_E\|_{K^p_s(\Omega)} \le \overset{*}{\omega}{}^p_s[g](h)\\ \lim_{h\to+\infty}\overset{*}{\omega}{}^p_s[g](h)=0\,.\end{gathered} \tag{2.3.2}$$

Finally, let $g\in \overset{\circ}{K}{}^p_s(\Omega)$. We call *modulus of continuity* of g in $\overset{\circ}{K}{}^p_s(\Omega)$ a map $\overset{\circ}{\omega}{}^p_s[g]:\mathbb{N}\to\mathbb{N}$ such that

$$\begin{gathered}\|\big(1-\psi_h\big)\,g\|_{K^p_s(\Omega)}+\sup_{\substack{E\in\Sigma(\Omega)\\ \sup_{x\in\Omega}|G(x)\cap E|\le 1/h}} \|\psi_h\,g\,\chi_E\|_{K^p_s(\Omega)} \le \overset{\circ}{\omega}{}^p_s[g](h)\\ \lim_{h\to+\infty}\overset{\circ}{\omega}{}^p_s[g](h)=0\,,\end{gathered} \tag{2.3.3}$$

where ψ_h $(h\in\mathbb{N})$ are defined in (2.2.1).

Let us now show[17] that any function g which belongs to one of the previous subspaces can be expressed as a sum of two particular functions. As said in the introduction, in these decompositions the first function is less regular than the second function and it can be controlled by means of a continuous modulus of the function g itself.

[17] See L. Caso, R. D'Ambrosio [16].

These decompositions involve the modulus of continuity and the characterizations of such subspaces.

Lemma 2.3.1 *Let* $g \in \widetilde{K}^p_s(\Omega)$*; then, for any* $h \in \mathbb{R}_+$*, we have*

$$g = g'_h + g''_h \qquad in\ \ \Omega\,, \tag{2.3.4}$$

with $g''_h \in L^\infty_s(\Omega)$ *and*

$$\|g'_h\|_{K^p_s(\Omega)} \le \widetilde{\omega}^p_s[g](h)\,, \qquad\qquad \|g''_h\|_{L^\infty_s(\Omega)} \le \gamma^{|s|}\,\|g\|_{K^p_s(\Omega)}\,h^{\frac{1}{p}}\,, \tag{2.3.5}$$

where γ *is given in* (2.1.1).

Proof − Fix $h \in \mathbb{R}_+$ and set

$$E_h = \{x \in \Omega \mid |\rho^s(x)\,g(x)| \ge \gamma^{|s|}\,\|g\|_{K^p_s(\Omega)}\,h^{\frac{1}{p}}\}\,. \tag{2.3.6}$$

We observe that

$$\begin{aligned} &\frac{|G(x)\cap E_h|}{\rho^n(x)} \le \frac{1}{\rho^n(x)}\int_{G(x)\cap E_h}\Big|\frac{\rho^s(y)\,g(y)}{\gamma^{|s|}\,\|g\|_{K^p_s(\Omega)}\,h^{\frac{1}{p}}}\Big|^p dy \le \\ &\frac{\rho^{s\,p-n}(x)}{\|g\|^p_{K^p_s(\Omega)}\,h}\int_{G(x)}|g(y)|^p\,dy \le \frac{1}{h}\,. \end{aligned} \tag{2.3.7}$$

If we define

$$g'_h = g\,\chi_{E_h} = \begin{cases} g & \text{if } x \in E_h \\ 0 & \text{if } x \in \Omega \setminus E_h\,, \end{cases} \qquad g''_h = g - g\,\chi_{E_h} = \begin{cases} 0 & \text{if } x \in E_h \\ g & \text{if } x \in \Omega \setminus E_h\,, \end{cases}$$

in view of (2.3.6) and (2.3.7), we obtain the result. ■

Lemma 2.3.2 *Let* $g \in \overset{*}{K}{}^p_s(\Omega)$*; then, for any* $h \in \mathbb{R}_+$*, we have*

$$g = \varphi'_h + \varphi''_h \qquad in \ \ \Omega\,, \tag{2.3.8}$$

with $\varphi''_h \in L^\infty_{s-\frac{n}{p}}(\Omega)$ *and*

$$\|\varphi'_h\|_{K^p_s(\Omega)} \leq \overset{*}{\omega}{}^p_s[g](h)\,, \qquad\qquad \|\varphi''_h\|_{L^\infty_{s-\frac{n}{p}}(\Omega)} \leq \gamma^{|s-\frac{n}{p}|}\, \|g\|_{K^p_s(\Omega)}\, h^{\frac{1}{p}}\,. \tag{2.3.9}$$

Proof $-$ For any $h \in \mathbb{R}_+$ we set

$$F_h = \{x \in \Omega \mid |\rho^{s-\frac{n}{p}}(x)\, g(x)| \geq \gamma^{|s-\frac{n}{p}|}\, \|g\|_{K^p_s(\Omega)}\, h^{\frac{1}{p}}\}\,, \tag{2.3.10}$$

and observe that

$$\begin{aligned} |G(x) \cap F_h| &\leq \int_{G(x)\cap F_h} \Big| \frac{\rho^{s-\frac{n}{p}}(y)\, g(y)}{\gamma^{|s-\frac{n}{p}|}\, \|g\|_{K^p_s(\Omega)}\, h^{\frac{1}{p}}} \Big|^p dy \leq \\ &\frac{\rho^{s\,p-n}(x)}{\|g\|^p_{K^p_s(\Omega)}\, h} \int_{G(x)} |g(y)|^p\, dy \leq \frac{1}{h}\,. \end{aligned} \tag{2.3.11}$$

Now, if we define

$$\varphi'_h = g\,\chi_{F_h} = \begin{cases} g & \text{if } x \in F_h \\ 0 & \text{if } x \in \Omega \setminus F_h\,, \end{cases} \qquad \varphi''_h = g - g\,\chi_{F_h} = \begin{cases} 0 & \text{if } x \in F_h \\ g & \text{if } x \in \Omega \setminus F_h\,, \end{cases}$$

by (2.3.10) and (2.3.11) we deduce the result. ■

Lemma 2.3.3 *Let $g \in \overset{\circ}{K}{}^p_s(\Omega)$; then, for any $h \in \mathbb{N}$, we have*

$$g = \phi'_h + \phi''_h \qquad in\ \ \Omega\,, \tag{2.3.12}$$

with

$$\|\phi'_h\|_{K^p_s(\Omega)} \le \overset{\circ}{\omega}{}^p_s[g](h)\,, \qquad\qquad |\phi''_h(x)| \le \psi_h(x)\, \rho^{-s+\frac{n}{p}}(x)\, \gamma^{|s-\frac{n}{p}|}\, \|g\|_{K^p_s(\Omega)}\, h^{\frac{1}{p}}\,, \tag{2.3.13}$$

and where ψ_h is given in (2.2.1).

Proof − Let us write for any $h \in \mathbb{N}$

$$\phi'_h = g\,(1-\psi_h) + \psi_h\, g\, \chi_{F_h} = \begin{cases} g & \text{if } x \in F_h \\ g\,(1-\psi_h) & \text{if } x \in \Omega \setminus F_h\,, \end{cases}$$

$$\phi''_h = \psi_h\,(g - g\,\chi_{F_h}) = \begin{cases} 0 & \text{if } x \in F_h \\ g\,\psi_h & \text{if } x \in \Omega \setminus F_h\,, \end{cases}$$

where F_h is defined by (2.3.10). Using (2.3.11) and (2.3.10), we deduce the result also in this case. ■

2.4 Imbedding and compactness results

In this section, as application, we study the operator of multiplication

$$u \longrightarrow g\,u, \tag{2.4.1}$$

as an operator defined on a weighted Sobolev space $W^{k,p}_s(\Omega)$ and which takes values in $L^q(\Omega)$ with suitable $r \in [1, +\infty[$ and $q \in [p, r[$. We give conditions on Ω, k, p, s, q, r and g in order that the operator is bounded and compact. The obtained estimates are in L. Caso, R. D'Ambrosio [16].

We consider the following condition on Ω:

h_1) There exists $\theta \in]0, \frac{\pi}{2}[$ such that

$$\forall\, x \in \Omega \qquad \exists\, C_\theta(x) \quad : \quad \overline{C_\theta(x, \rho(x))} \subset \Omega\,,$$

where $C_\theta(x)$ is an indefinite cone with vertex at x and opening θ, and $C_\theta(x, \rho(x)) = C_\theta(x) \cap B(x, \rho(x))$.

Remark 2.4.1 *We observe*[18] *that if, for example, $\rho \in L^\infty(\Omega)$, and there exists an open subset Ω_o of $\mathbb{R}^n$ with the cone property such that*

$$\Omega \subset \Omega_o\,, \qquad\qquad \partial\Omega \setminus S_\rho \subset \partial\Omega_o\,,$$

then the condition h_1) holds. ■

Remark 2.4.2 *We note that if the condition h_1) holds, fixed a weight function $\rho \in \mathcal{A}(\Omega)$ it's possible to find a continuous weight function in $\mathcal{A}(\Omega)$ which is equivalent to ρ (see Remark 2.1.2).*

For any fixed $x \in \Omega$ we denote by $\Omega(x)$ the union of all open cones C with opening θ and height $\rho(x)$ such that $C \subset\subset \Omega$ and $x \in C$. For simplicity, we will assume in the following that $G(x) = \Omega(x)$, $\forall\, x \in \Omega$.

[18] See Remark 3.1 in L. Caso, M. Transirico [18].

For reader's convenience, we recall the following Lemma needed in the sequel. It is a particular case of a more general result proved in Lemma 3.4 of S. Boccia, L. Caso [7] for continuous weight functions belonging to $\mathcal{A}(\Omega)$.

Lemma 2.4.3 *Suppose that condition $h_1)$ holds and fix a function φ which verifies the following conditions:*

$$\varphi(x) > 0 \qquad \forall x \in \Omega$$

$$\exists\, \mu \in \mathbb{R}_+ \ : \mu^{-1}\varphi(y) \le \varphi(x) \le \mu\, \varphi(y) \qquad \forall x \in \Omega,\ \forall y \in \Omega(x)\,.$$

then for any $p, q \in [1, +\infty[$, with $q \ge p$, there exist $c_1\,, c_2 \in \mathbb{R}_+$ such that

$$\int_\Omega \varphi(x)\, \rho^{-n}(x)\, ||u||^p_{L^p(\Omega(x))}\, dx \ge c_1 \int_\Omega \varphi(x)\, |u(x)|^p\, dx\,, \tag{2.4.2}$$

$$\int_\Omega \varphi^{\frac{q}{p}}(x)\, \rho^{-n}(x)\, ||u||^q_{L^p(\Omega(x))}\, dx \le c_2 \Big(\int_\Omega \varphi(x)\, |u(x)|^p\, dx \Big)^{\frac{q}{p}}\,, \tag{2.4.3}$$

for any $u \in L^p_{loc}(\bar\Omega \setminus S_\rho)$, where c_1 depends on $\theta, n, \mu\,, c'$ and c_2 depends on $n, \mu,\ c'', p$ and q.(We specify that c' and c'' are the constants of Remark 2.1.2.)

For each fixed $x \in \Omega$, we consider the map

$$\Psi^x : y \in \Omega \to \Psi^x(y) = x + \frac{y - x}{\rho(x)}\,.$$

By construction the set $\Omega^*(x) = \Psi^x(\Omega(x))$ is an open set with the cone property with opening and height independent of x. In the following for any function f defined on Ω we write

$$f^* = \big(f^x\big)^* : z \in \Omega^*(x) \to f^*(z) = f\big(x + \rho(x)(z - x)\big)\,.$$

Fix $s \in \mathbb{R}$ and let k, r, p, q be real numbers such that

$$h_2) \qquad k \in \mathbb{N}\,, \quad 1 \le p \le q \le r\,, \quad r > q \quad \text{if } p = \frac{n}{k}\,, \quad \frac{1}{q} \ge \frac{1}{r} + \frac{1}{p} - \frac{k}{n}\,.$$

Let $u \in W^{k,p}_s(\Omega)$; we observe that for any multiindex of order $|\alpha| = k$, we have

$$\partial^\alpha u^*(z) = \rho^{|\alpha|}(x)\, \partial^\alpha u(y)\,, \qquad z = \Psi^x(y)\,.$$

By (2.1.7) and (2.1.8), we deduce that $u^* \in W^{k,p}(\Omega^*(x))$. Consequently, from Sobolev imbedding theorem we also obtain that $u^* \in L^{\frac{q\,r}{r-q}}(\Omega^*(x))$ and

$$\|u^*\|_{L^{\frac{q\,r}{r-q}}(\Omega^*(x))} \le c_o\, \|u^*\|_{W^{k,p}(\Omega^*(x))}\,, \tag{2.4.4}$$

where $c_o \in \mathbb{R}_+$ depends only on n, k, p, q, r and on the cone determining the cone property of $\Omega^*(x)$.

We now establish our main result .

Theorem 2.4.4 *Suppose that conditions $h_1)$ and $h_2)$ hold. For all $u \in W^{k,p}_s(\Omega)$ and for all $g \in K^r_{-s+k+n(\frac{1}{q}-\frac{1}{p})}(\Omega)$ we have $g\,u \in L^q(\Omega)$. Moreover there exists $c \in \mathbb{R}_+$ such that*

$$\|g\,u\|_{L^q(\Omega)} \le c\, \|g\|_{K^r_{-s+k+n(\frac{1}{q}-\frac{1}{p})}(\Omega)}\, \|u\|_{W^{k,p}_s(\Omega)}\,, \tag{2.4.5}$$

where c depends on n, k, p, q, r, ρ and θ.

Proof − Let $u \in W^{k,p}_s(\Omega)$ and $g \in K^r_{-s+k+n(\frac{1}{q}-\frac{1}{p})}(\Omega)$. Using the Hölder's inequality, we obtain

$$\int_{\Omega^*(x)} |g^* u^*|^q \, dz \le \Big(\int_{\Omega^*(x)} |g^*|^r \, dz \Big)^{\frac{q}{r}} \cdot \Big(\int_{\Omega^*(x)} |u^*|^{\frac{q\,r}{r-q}} \, dz \Big)^{\frac{r-q}{r\,q}\,q} .$$

Thus in view of (2.4.4) there exists a constant $c_1 \in \mathbb{R}_+$, depending on n, k, p, q, r and on the cone determining the cone property of $\Omega^*(x)$, such that

$$\int_{\Omega^*(x)} |g^* u^*|^q \, dz \le c_1 \Big(\int_{\Omega^*(x)} |g^*|^r \, dz \Big)^{\frac{q}{r}} \cdot \|u^*\|^q_{W^{k,p}(\Omega^*(x))} \, . \tag{2.4.6}$$

Then, converting back to the y-variables ($z = \Psi^x(y)$) and using (2.1.1), we obtain

$$\rho^{-n}(x) \int_{\Omega(x)} |g\, u|^q \, dy \le$$

$$c_2\, \rho^{-\frac{n\,q}{r}}(x) \Big(\int_{\Omega(x)} |g|^r \, dy \Big)^{\frac{q}{r}} \cdot \Big[\rho^{-\frac{n\,q}{p}}(x) \cdot \sum_{|\alpha| \le k} \rho^{|\alpha|\,q}(x) \, \|\partial^\alpha u\|^q_{L^p(\Omega(x))} \Big] ,$$

where $c_2 \in \mathbb{R}_+$ depends only on n, k, p, q, r, ρ and θ.

Integrating the above inequality over Ω, we obtain

$$\int_\Omega \rho^{-n}(x) \, \|g\, u\|^q_{L^q(\Omega(x))} \, dx \le \tag{2.4.7}$$

$$c_2 \int_\Omega \Big(\rho^{-s+k+n(\frac{1}{q}-\frac{1}{p})-\frac{n}{r}}(x) \, \|g\|_{L^r(\Omega(x))} \Big)^q \cdot \Big[\sum_{|\alpha| \le k} \rho^{q(s+|\alpha|-k)-n}(x) \, \|\partial^\alpha u\|^q_{L^p(\Omega(x))} \Big] \, dx \, .$$

Using (2.4.2) of Lemma 2.4.3 to the left hand side we have

$$\|gu\|^q_{L^q(\Omega)} = \int_\Omega |g\, u|^q \, dx \le c_3 \int_\Omega \rho^{-n}(x) \, \|g\, u\|^q_{L^q(\Omega(x))} \, dx \le \tag{2.4.8}$$

$$c_4 \sup_{\Omega} \left(\rho^{-s+k+n(\frac{1}{q}-\frac{1}{p})-\frac{n}{r}}(x)\, \|g\|_{L^r(\Omega(x))} \right)^q \cdot \sum_{|\alpha|\le k} \int_\Omega \rho^{q(s+|\alpha|-k)-n}(x)\, \|\partial^\alpha u\|^q_{L^p(\Omega(x))}\, dx \le$$

$$c_4\, \|g\|^q_{K^r_{-s+k+n(\frac{1}{q}-\frac{1}{p})}(\Omega)} \cdot \sum_{|\alpha|\le k} \int_\Omega \rho^{q(s+|\alpha|-k)-n}(x)\, \|\partial^\alpha u\|^q_{L^p(\Omega(x))}\, dx\,,$$

where c_3, $c_4 \in \mathbb{R}_+$ depend on the same parameters as c_2.

Now applying the (2.4.3) of Lemma 2.4.3 to the last side of previous inequality, we finally obtain

$$\|gu\|^q_{L^q(\Omega)} \le c_5\, \|g\|^q_{K^r_{-s+k+n(\frac{1}{q}-\frac{1}{p})}(\Omega)} \cdot \sum_{|\alpha|\le k} \left(\int_\Omega \rho^{p(s+|\alpha|-k)}(x)\, |\partial^\alpha u|^p\, dx \right)^{\frac{q}{p}}, \qquad (2.4.9)$$

where $c_5 \in \mathbb{R}_+$ depends on the same parameters as c_2. The result easily follows from (2.4.9). ∎

We observe that, under conditions h_1) and h_2), if $u \in W^{k,p}_s(\Omega)$ then, from Theorem 4.4 of S. Boccia , L. Caso [7], $u \in L^q_{s-k-n(\frac{1}{q}-\frac{1}{p})}(\Omega)$.

In the following result we prove an upper bound on the operator of multiplication in the case $g \in \widetilde{K}^r_{-s+k+n(\frac{1}{q}-\frac{1}{p})}(\Omega)$.

Corollary 2.4.5 *If conditions h_1) and h_2) hold, then for any $g \in \widetilde{K}^r_{-s+k+n(\frac{1}{q}-\frac{1}{p})}(\Omega)$ and $h \in \mathbb{R}_+$ there exists a constant $c_1 \in \mathbb{R}_+$, depending on n, ρ, s, k, p, q, $\|g\|_{K^r_{-s+k+n(\frac{1}{q}-\frac{1}{p})}(\Omega)}$, such that*

$$\|g\, u\|_{L^q(\Omega)} \le c \cdot \widetilde{\omega}^r_{-s+k+n(\frac{1}{q}-\frac{1}{p})}[g](h) \cdot \|u\|_{W^{k,p}_s(\Omega)} + c_1 \cdot h^{\frac{1}{r}} \cdot \|u\|_{L^q_{s-k-n(\frac{1}{q}-\frac{1}{p})}(\Omega)}\,, \quad (2.4.10)$$

for each function $u \in W^{k,p}_s(\Omega)$, where $c \in \mathbb{R}_+$ is the constant in (2.4.5).

Proof – Fix $g \in \widetilde{K}^r_{-s+k+n(\frac{1}{q}-\frac{1}{p})}(\Omega)$. By Lemma 2.3.1 and Theorem 2.4.4, for any

$u \in W_s^{k,p}(\Omega)$ it follows that

$$
\begin{aligned}
&\|g\,u\|_{L^q(\Omega)} \le \|g'_h\,u\|_{L^q(\Omega)} + \|g''_h\,u\|_{L^q(\Omega)} \le \\
&c\,\|g'_h\|_{K^r_{-s+k+n(\frac{1}{q}-\frac{1}{p})}(\Omega)} \cdot \|u\|_{W_s^{k,p}(\Omega)} + \|g''_h\,u\|_{L^q(\Omega)} \le \\
&c \cdot \widetilde{\omega}^r_{-s+k+n(\frac{1}{q}-\frac{1}{p})}[g](h) \cdot \|u\|_{W_s^{k,p}(\Omega)} + \|g''_h\,u\|_{L^q(\Omega)} \le \\
&c \cdot \widetilde{\omega}^r_{-s+k+n(\frac{1}{q}-\frac{1}{p})}[g](h) \cdot \|u\|_{W_s^{k,p}(\Omega)} + c_1 \cdot \; h^{\frac{1}{r}} \cdot \|u\|_{L^q_{s-k-n(\frac{1}{q}-\frac{1}{p})}(\Omega)} \,.
\end{aligned}
\tag{2.4.11}
$$

■

In the next Corollary we prove a different bound on the same operator of multiplication in the case $g \in \overset{\circ}{K}{}^r_{-s+k+n(\frac{1}{q}-\frac{1}{p})}(\Omega)$.

Corollary 2.4.6 *If conditions $h_1)$ and $h_2)$ hold, then for any $g \in \overset{\circ}{K}{}^r_{-s+k+n(\frac{1}{q}-\frac{1}{p})}(\Omega)$ and $h \in \mathbb{N}$ there exist a constant $c_1 \in \mathbb{R}_+$, depending on h, n, ρ, s, r, k, p, q, $\|g\|_{K^r_{-s+k+n(\frac{1}{q}-\frac{1}{p})}(\Omega)}$, and an open set $A_h \subset\subset \Omega$ with the cone property, such that*

$$
\|g\,u\|_{L^q(\Omega)} \le c \cdot \overset{\circ}{\omega}{}^r_{-s+k+n(\frac{1}{q}-\frac{1}{p})}[g](h) \cdot \|u\|_{W_s^{k,p}(\Omega)} + c_1 \cdot h^{\frac{1}{r}} \cdot \|u\|_{L^q(A_h)}\,, \tag{2.4.12}
$$

for each function $u \in W_s^{k,p}(\Omega)$, where $c \in \mathbb{R}_+$ is the constant in (2.4.5).

Proof − Fix $g \in \overset{\circ}{K}{}^r_{-s+k+n(\frac{1}{q}-\frac{1}{p})}(\Omega)$ and $h \in \mathbb{N}$. By Lemma 2.3.3 and Theorem 2.4.4, for any $u \in W^{k,p}_s(\Omega)$ it follows that

$$\|g\,u\|_{L^q(\Omega)} \le \|\phi'_h\,u\|_{L^q(\Omega)} + \|\phi''_h\,u\|_{L^q(\Omega)} \le$$

$$c\,\|\phi'_h\|_{K^r_{-s+k+n(\frac{1}{q}-\frac{1}{p})}(\Omega)} \cdot \|u\|_{W^{k,p}_s(\Omega)} + \|\phi''_h\,u\|_{L^q(\Omega)} \le \tag{2.4.13}$$

$$c \cdot \overset{\circ}{\omega}{}^r_{-s+k+n(\frac{1}{q}-\frac{1}{p})}[g](h) \cdot \|u\|_{W^{k,p}_s(\Omega)} + \|\phi''_h\,u\|_{L^q(\Omega)}\,.$$

According to Lemma 2.3.3 we also have

$$\|\phi''_h\,u\|_{L^q(\Omega)} \le \gamma^{|-s+k+n(\frac{1}{q}-\frac{1}{p})-\frac{n}{r}|}\,\|g\|_{K^r_{-s+k+n(\frac{1}{q}-\frac{1}{p})}(\Omega)}\,h^{\frac{1}{r}}\left(\int_\Omega \left|\psi_h\,\rho^{s-k-n(\frac{1}{q}-\frac{1}{p})+\frac{n}{r}}\,u\right|^q dx\right)^{\frac{1}{q}} \le$$

$$c_2\,h^{\frac{1}{r}}\left(\int_{\mathrm{supp}\,\psi_h} \left|\rho^{s-k-n(\frac{1}{q}-\frac{1}{p}-\frac{1}{r})}\,u\right|^q dx\right)^{\frac{1}{q}},$$

where $c_2 \in \mathbb{R}_+$ depends on n, ρ, s, k, p, q and $\|g\|_{K^r_{-s+k+n(\frac{1}{q}-\frac{1}{p})}(\Omega)}$. So, from (2.1.8), we deduce that there exists $c_3 \in \mathbb{R}_+$, depending on the same parameters as c_2 and on h, such that

$$\|\phi''_h\,u\|_{L^q(\Omega)} \le c_3 \cdot h^{\frac{1}{r}} \cdot \|u\|_{L^q(\mathrm{supp}\,\psi_h)}. \tag{2.4.14}$$

Now fix $d_h \in \left]0, \frac{\mathrm{dist}\,(\mathrm{supp}\,\psi_h, \partial\Omega)}{2}\right[$ and $\theta \in]0, \frac{\pi}{2}[$. Let A_h be the set of $\mathbb{R}^n$ union of the open cones $C \subset\subset \Omega$ with opening θ, height d_h and such that $C \cap \mathrm{supp}\,\psi_h \ne \emptyset$. Therefore the result follows from (2.4.13) and (2.4.14). ■

In the following result we prove that if $g \in \overset{\circ}{K}{}^r_{-s+k+n(\frac{1}{q}-\frac{1}{p})}(\Omega)$ then the operator of multiplication is compact.

Corollary 2.4.7 *Suppose that conditions $h_1)$ and $h_2)$ hold and fix $g \in \overset{\circ}{K}{}^r_{-s+k+n(\frac{1}{q}-\frac{1}{p})}(\Omega)$. Then the operator*

$$u \in W^{k,p}_s(\Omega) \longrightarrow g\,u \in L^q(\Omega) \tag{2.4.15}$$

is compact.

Proof – First we observe that, in view of (2.1.8), if $\Omega' \subset\subset \Omega$ is an open bounded set, then the operator

$$u \in W^{k,p}_s(\Omega) \longrightarrow u_{|_{\Omega'}} \in W^{k,p}(\Omega')$$

is linear and bounded. Moreover, if Ω' has the cone property, by Rellich - Kondrachov Theorem, we know that the operator

$$w \in W^{k,p}(\Omega') \longrightarrow w \in L^q(\Omega')$$

is compact. So we deduce that, if $\Omega' \subset\subset \Omega$ is a bounded open set with the cone property, the operator

$$u \in W^{k,p}_s(\Omega) \longrightarrow u \in L^q(\Omega')$$

is compact.

Let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence in $W^{k,p}_s(\Omega)$ and let $M \in \mathbb{R}_+$ be such that $\|u_n\|_{W^{k,p}_s(\Omega)} \le M \quad \forall n \in \mathbb{N}$. Fixed $g \in \overset{\circ}{K}{}^r_{-s+k+n(\frac{1}{q}-\frac{1}{p})}(\Omega)$ and $h \in \mathbb{N}$, from Corollary 2.4.6 we deduce that there exist $c(h) \in \mathbb{R}_+$, independent of n, and an open set $A_h \subset\subset \Omega$ with the cone property, such that

$$\|g\,u_n\|_{L^q(\Omega)} \le c \cdot \overset{\circ}{\omega}{}^r_{-s+k+n(\frac{1}{q}-\frac{1}{p})}[g](h) \cdot \|u_n\|_{W^{k,p}_s(\Omega)} + c(h) \cdot h^{\frac{1}{r}} \cdot \|u_n\|_{L^q(A_h)}\,. \tag{2.4.16}$$

On the other hand, according to the above considerations, there exist a subsequence $(u_{m_n})_{n\in\mathbb{N}}$ and $\nu \in \mathbb{N}$ such that

$$\|u_{m_n} - u_{m_l}\|_{L^q(A_h)} \leq \frac{c \cdot \overset{\circ}{\omega}{}^r_{-s+k+n(\frac{1}{q}-\frac{1}{p})}[g](h)}{c(h) \cdot h^{\frac{1}{r}}} \qquad \forall\, n, l > \nu\,. \tag{2.4.17}$$

From (2.4.16) and (2.4.17) we obtain, for $n, l > \nu$,

$$\|g\, u_{m_n} - g\, u_{m_l}\|_{L^q(\Omega)} \leq c \cdot \overset{\circ}{\omega}{}^r_{-s+k+n(\frac{1}{q}-\frac{1}{p})}[g](h) \cdot \big(2M + 1\big)\,. \tag{2.4.18}$$

From (2.4.18) and (2.3.3) we deduce that $(g\, u_{m_n})_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^q(\Omega)$ and so the operator defined by (2.4.15) is compact. ■

Chapter 3

Some remarks on spaces of Morrey type

In this chapter we deepen the study of some Morrey type spaces, denoted by $M^{p,\lambda}(\Omega)$, defined on an unbounded open subset Ω of $\mathbb{R}^n$. In particular, we construct decompositions for functions belonging to two different subspaces of $M^{p,\lambda}(\Omega)$, which allow us to prove a compactness result for an operator in Sobolev spaces. We also introduce a weighted Morrey type space, settled between the above mentioned subspaces.

3.1 Some preliminary results

Let us recall the definition of the classical Morrey space $L^{p,\lambda}(\mathbb{R}^n)$[19].

For $n \geq 2$, $\lambda \in [0, n[$ and $p \in [1, +\infty[$, $L^{p,\lambda}(\mathbb{R}^n)$ is the set of the functions $g \in L^p_{loc}(\mathbb{R}^n)$

[19] See, for instance, A. Kufner, O. John, S. Fucík [40] and F. Chiarenza, M. Frasca [26] .

such that

$$\|g\|_{L^{p,\lambda}(\mathbb{R}^n)} = \sup_{\substack{\tau>0\\ x\in\mathbb{R}^n}} \tau^{-\frac{\lambda}{p}} \|g\|_{L^p(B(x,\tau))} < +\infty, \tag{3.1.1}$$

equipped with the norm defined by (3.1.1).

If Ω is an unbounded open subset of $\mathbb{R}^n$ and t is fixed in $\mathbb{R}_+$, we can consider the space $M^{p,\lambda}(\Omega,t)$, which is larger than $L^{p,\lambda}(\mathbb{R}^n)$ when $\Omega=\mathbb{R}^n$. More precisely, $M^{p,\lambda}(\Omega,t)$ is the set of all functions g in $L^p_{loc}(\overline{\Omega})$ such that

$$\|g\|_{M^{p,\lambda}(\Omega,t)} = \sup_{\substack{\tau\in]0,t]\\ x\in\Omega}} \tau^{-\lambda/p} \|g\|_{L^p(\Omega(x,\tau))} < +\infty, \tag{3.1.2}$$

endowed with the norm defined in (3.1.2).

We explicitly observe that a diadic decomposition[20] gives for every $t_1, t_2 \in \mathbb{R}_+$ the existence of $c_1, c_2 \in \mathbb{R}_+$, depending only on t_1, t_2 and n, such that

$$c_1 \|g\|_{M^{p,\lambda}(\Omega,t_1)} \leq \|g\|_{M^{p,\lambda}(\Omega,t_2)} \leq c_2 \|g\|_{M^{p,\lambda}(\Omega,t_1)}, \quad \forall\, g \in M^{p,\lambda}(\Omega,t_1). \tag{3.1.3}$$

All the norms being equivalent, from now on we consider the space

$$M^{p,\lambda}(\Omega) = M^{p,\lambda}(\Omega,1).$$

Moreover, we put

$$M^{p,0}(\Omega) = M^p(\Omega)^{21}. \tag{3.1.4}$$

[20] See Proposition 1.1.4 in P. Cavaliere [22].

[21] For more informations about spaces $M^p(\Omega)$ and its applications to elliptic Pde's, see M. Transirico, M. Troisi [55],[56].

It is easily seen that:

$$M^{p_0,\lambda_0}(\Omega) \hookrightarrow M^{p,\lambda}(\Omega) \qquad \text{if } p \le p_0 \text{ and } \frac{\lambda - n}{p} \le \frac{\lambda_0 - n}{p_0} \tag{3.1.5}$$

with λ, $\lambda_0 \in [0, n[$. (See also L. C. Piccinini [48], A. Kufner, O. John, J. Fucík [40]).

For reader's convenience, we briefly recall some properties of functions in $L^{p,\lambda}(\mathbb{R}^n)$ and $M^{p,\lambda}(\Omega)$ needed in the sequel.
The first lemma is a particular case of a more general result proved in Proposition 3 of C. T. Zorko [64].

Lemma 3.1.1 *Let $(J_h)_{h\in\mathbb{N}}$ be a sequence of mollifiers in $\mathbb{R}^n$. If $g \in L^{p,\lambda}(\mathbb{R}^n)$ and*

$$\lim_{y\to 0} \|g(x-y) - g(x)\|_{L^{p,\lambda}(\mathbb{R}^n)} = 0,$$

then

$$\lim_{h\to+\infty} \|g - J_h * g\|_{L^{p,\lambda}(\mathbb{R}^n)} = 0.$$

The second results concerns the zero extensions of functions in $M^{p,\lambda}(\Omega)$[22].

Remark 3.1.2 *Let $g \in M^{p,\lambda}(\Omega)$. If we denote by g_0 the zero extension of g outside Ω, then $g_0 \in M^{p,\lambda}(\mathbb{R}^n)$ and for every τ in $]0,1]$*

$$\|g_0\|_{M^{p,\lambda}(\mathbb{R}^n,\tau)} \le c_1 \|g\|_{M^{p,\lambda}(\Omega,\tau)}, \tag{3.1.6}$$

where $c_1 \in \mathbb{R}_+$ is a constant independent of g, Ω and τ.

[22] See also Remark 2.4 of M. Transirico, M. Troisi, A. Vitolo [58].

Furthermore if $diam(\Omega) < +\infty$, then $g_0 \in L^{p,\lambda}(\mathbb{R}^n)$ and

$$\|g_0\|_{L^{p,\lambda}(\mathbb{R}^n)} \leq c_2 \|g\|_{M^{p,\lambda}(\Omega)}, \tag{3.1.7}$$

where $c_2 \in \mathbb{R}_+$ is a constant independent of g and Ω.

For a general survey on Morrey and Morrey type spaces we refer to A. Kufner, O. John, S. Fucík [40], L. C. Piccinini [48], M. Transirico, M. Troisi, A. Vitolo [58] and P. Cavaliere, M. Longobardi, A. Vitolo [23].

3.2 The spaces $\widetilde{M}^{p,\lambda}(\Omega)$ and $M_o^{p,\lambda}(\Omega)$

This section is devoted to the study of two subspaces of $M^{p,\lambda}(\Omega)$, denoted by $\widetilde{M}^{p,\lambda}(\Omega)$ and $M_o^{p,\lambda}(\Omega)$. Here, we point out the peculiar characteristics of functions belonging to these sets by means of two characterization lemmas[23].

Let us put, for $h \in \mathbb{R}_+$ and $g \in M^{p,\lambda}(\Omega)$,

$$F[g](h) = \sup_{\substack{E \in \Sigma(\Omega) \\ \sup_{x\in\Omega} |E(x)| \leq \frac{1}{h}}} \|g\,\chi_E\|_{M^{p,\lambda}(\Omega)}.$$

Lemma 3.2.1 *Let $\lambda \in [0, n[$, $p \in [1, +\infty[$ and $g \in M^{p,\lambda}(\Omega)$. The following properties are equivalent:*

$$g \text{ is in the closure of } L^\infty(\Omega) \text{ in } M^{p,\lambda}(\Omega), \tag{3.2.1}$$

$$\lim_{h \to +\infty} F[g](h) = 0\,, \tag{3.2.2}$$

[23] See L. Caso, R. D'Ambrosio, S. Monsurrò [17].

$$\lim_{h\to+\infty}\left(\sup_{\substack{E\in\Sigma(\Omega)\\ \sup\limits_{\substack{x\in\Omega\\ \tau\in]0,1]}}\tau^{-\lambda}|E(x,\tau)|\le\frac{1}{h}}}\|g\,\chi_E\|_{M^{p,\lambda}(\Omega)}\right)=0\,. \tag{3.2.3}$$

We denote by $\widetilde{M}^{p,\lambda}(\Omega)$ the subspace of $M^{p,\lambda}(\Omega)$ made up of functions verifying one of the above properties.

Proof − The equivalence between (3.2.1) and (3.2.2) is proved in Lemma 1.3 of M. Transirico, M. Troisi, A. Vitolo [58]. Let us show that (3.2.1) entails (3.2.3) and vice versa.

Fix g in the closure of $L^\infty(\Omega)$ in $M^{p,\lambda}(\Omega)$, then for each $\varepsilon>0$ there exists a function $g_\varepsilon\in L^\infty(\Omega)$ such that

$$\|g-g_\varepsilon\|_{M^{p,\lambda}(\Omega)}<\frac{\varepsilon}{2}\,. \tag{3.2.4}$$

Fixed $E\in\Sigma(\Omega)$, from (3.2.4) it easily follows that

$$\|g\,\chi_E\|_{M^{p,\lambda}(\Omega)}\le\|(g-g_\varepsilon)\,\chi_E\|_{M^{p,\lambda}(\Omega)}+\|g_\varepsilon\,\chi_E\|_{M^{p,\lambda}(\Omega)}<\frac{\varepsilon}{2}+\|g_\varepsilon\,\chi_E\|_{M^{p,\lambda}(\Omega)}\,. \tag{3.2.5}$$

On the other hand

$$\begin{gathered}\|g_\varepsilon\,\chi_E\|_{M^{p,\lambda}(\Omega)}=\sup_{\substack{\tau\in]0,1]\\ x\in\Omega}}\tau^{-\frac{\lambda}{p}}\|g_\varepsilon\,\chi_E\|_{L^p(\Omega(x,\tau))}\le\\ \|g_\varepsilon\|_{L^\infty(\Omega)}\sup_{\substack{\tau\in]0,1]\\ x\in\Omega}}(\tau^{-\lambda}|E(x,\tau)|)^{\frac{1}{p}}\,.\end{gathered} \tag{3.2.6}$$

Therefore, if we set

$$\frac{1}{h_\varepsilon}=\left(\frac{\varepsilon}{2\,\|g_\varepsilon\|_{L^\infty(\Omega)}}\right)^p,$$

from (3.2.6) we deduce that, if $\sup\limits_{\substack{\tau\in]0,1]\\ x\in\Omega}} \tau^{-\lambda}|E(x,\tau)| \le \dfrac{1}{h_\varepsilon}$, then

$$\|g_\varepsilon\, \chi_E\|_{M^{p,\lambda}(\Omega)} \le \frac{\varepsilon}{2}\,. \tag{3.2.7}$$

Putting together (3.2.5) and (3.2.7) we get (3.2.3).

Conversely, if we take a function $g \in M^{p,\lambda}(\Omega)$ satisfying (3.2.3), for any $\varepsilon > 0$ there exists $h_\varepsilon \in \mathbb{R}_+$ such that, if $E \in \Sigma(\Omega)$ with $\sup\limits_{\substack{\tau\in]0,1]\\ x\in\Omega}} \tau^{-\lambda}|E(x,\tau)| \le \dfrac{1}{h_\varepsilon}$, then $\|g\, \chi_E\|_{M^{p,\lambda}(\Omega)} < \varepsilon$.

For each $k \in \mathbb{R}_+$ we set

$$E_k = \left\{x \in \Omega \;\middle|\; |g(x)| \ge k\right\}.$$

Observe that

$$\begin{aligned}\|g\|_{M^{p,\lambda}(\Omega)} \ge \sup_{\substack{\tau\in]0,1]\\ x\in\Omega}} \tau^{-\frac{\lambda}{p}}\|g\|_{L^p(E_k(x,\tau))} \ge \\ k \sup_{\substack{\tau\in]0,1]\\ x\in\Omega}} \left(\tau^{-\lambda}|E_k(x,\tau)|\right)^{\frac{1}{p}}.\end{aligned} \tag{3.2.8}$$

Therefore, if we put

$$k_\varepsilon = \|g\|_{M^{p,\lambda}(\Omega)} h_\varepsilon^{\frac{1}{p}}\,,$$

from (3.2.8) it follows that

$$\sup_{\substack{\tau\in]0,1]\\ x\in\Omega}} \tau^{-\lambda}|E_{k_\varepsilon}(x,\tau)| \le \frac{1}{h_\varepsilon}$$

and then

$$\|g\, \chi_{E_{k_\varepsilon}}\|_{M^{p,\lambda}(\Omega)} < \varepsilon\,. \tag{3.2.9}$$

To end the proof we define the function $g_\varepsilon = g - g\,\chi_{E_{k_\varepsilon}}$. Indeed, by construction, $g_\varepsilon \in L^\infty(\Omega)$ and by (3.2.9) one gets that $\|g - g_\varepsilon\|_{M^{p,\lambda}(\Omega)} < \varepsilon$. ■

Remark 3.2.2 *It is easily seen that*[24] *if* $g \in \widetilde{M}^{p,\lambda}(\Omega)$, *then*

$$\lim_{t\to 0} \|g\|_{M^{p,\lambda}(\Omega,t)} = 0.$$

Now we introduce two classes of applications needed in the sequel.

For $h \in \mathbb{R}_+$ we denote by ζ_h a function of class $C_o^\infty(\mathbb{R}^n)$ such that

$$0 \le \zeta_h \le 1\,, \qquad \zeta_h|_{\overline{B(0,h)}} = 1\,, \qquad \operatorname{supp}\zeta_h \subset B(0,2h). \tag{3.2.10}$$

The second class is made up of the applications ψ_h defined in (2.2.1), for $h \in \mathbb{R}_+$. It is easy to prove that ψ_h belongs to $C_o^\infty(\Omega)$, for any $h \in \mathbb{R}_+$. Moreover (2.2.2) holds for any $h \in \mathbb{R}_+$.

Lemma 3.2.3 *Let* $\lambda \in [0,n[$, $p \in [1,+\infty[$ *and* $g \in M^{p,\lambda}(\Omega)$. *The following properties are equivalent:*

$$g \text{ is in the closure of } C_o^\infty(\Omega) \text{ in } M^{p,\lambda}(\Omega), \tag{3.2.11}$$

$$\lim_{h\to+\infty} (\|(1-\zeta_h)\, g\|_{M^{p,\lambda}(\Omega)} + F[g](h)) = 0, \tag{3.2.12}$$

$$\lim_{h\to+\infty} (\|(1-\psi_h)\, g\|_{M^{p,\lambda}(\Omega)} + F[g](h)) = 0, \tag{3.2.13}$$

$$\lim_{t\to 0} \|g\|_{M^{p,\lambda}(\Omega,\,t)} + \lim_{|x|\to+\infty} \Big(\sup_{\tau\in]0,1]} \tau^{-\frac{\lambda}{p}} \|g\|_{L^p(\Omega(x,\tau))}\Big) = 0, \tag{3.2.14}$$

$$g \in \widetilde{M}^{p,\lambda}(\Omega) \text{ and } \lim_{|x|\to+\infty} \Big(\sup_{\tau\in]0,1]} \tau^{-\frac{\lambda}{p}} \|g\|_{L^p(\Omega(x,\tau))}\Big) = 0. \tag{3.2.15}$$

[24] See also M. Transirico, M. Troisi, A. Vitolo [58].

The subspace of $M^{p,\lambda}(\Omega)$ of the functions satisfying one of the above properties will be denoted by $M_o^{p,\lambda}(\Omega)$.

Proof − The equivalence between (3.2.11) and (3.2.12) is a consequence of (3.2.2) and of Lemmas 2.1 and 2.5 of M. Transirico, M. Troisi, A. Vitolo [58]. The one between (3.2.11) and (3.2.14) follows from Remark 2.2 of [58]. Always in [58], see Lemma 2.1 and Remark 2.2, it is proved (3.2.11) entails (3.2.15) and vice versa. Let us show that (3.2.11) and (3.2.13) are equivalent too.

Let us firstly assume that g belongs to the closure of $C_o^\infty(\Omega)$ in $M^{p,\lambda}(\Omega)$.

Clearly, this entails that g is in the closure of $L^\infty(\Omega)$ in $M^{p,\lambda}(\Omega)$, thus by Lemma 3.2.1 one has that

$$\lim_{h\to+\infty} F[g](h) = 0.$$

It remains to show that

$$\lim_{h\to+\infty} \|(1-\psi_h)\, g\|_{M^{p,\lambda}(\Omega)} = 0. \tag{3.2.16}$$

To this aim observe that fixed $\varepsilon > 0$ there exists $g_\varepsilon \in C_o^\infty(\Omega)$ such that

$$\|g - g_\varepsilon\|_{M^{p,\lambda}(\Omega)} < \varepsilon\,. \tag{3.2.17}$$

On the other hand, if we consider the sets Ω_h defined in (2.2.3) for $h \in \mathbb{R}_+$, one has

$$\Omega\setminus\Omega_h = \big\{x\in\Omega \;\big|\; |x| \geq h\big\} \cup \big\{x\in\Omega \;\big|\; \alpha(x) \leq 1/h\big\}.$$

Therefore, since g_ε has a compact support, there exists $h_\varepsilon \in \mathbb{R}_+$

$$(\Omega \setminus \Omega_h) \cap \text{ supp } g_\varepsilon = \emptyset \qquad \forall\, h \geq h_\varepsilon\,.$$

Then, since $\psi_{h|\Omega_h} = 1$ one has that $\text{supp}(1-\psi_h) \subset \Omega \setminus \Omega_h$, hence $(1-\psi_h)g_\varepsilon = 0$ $\forall\, h \geq h_\varepsilon$.

The above considerations together with (3.2.17) give, for any $h \geq h_\varepsilon$,

$$\|(1-\psi_h)\; g\|_{M^{p,\lambda}(\Omega)} = \|(1-\psi_h)\; (g - g_\varepsilon)\|_{M^{p,\lambda}(\Omega)} \leq \|g - g_\varepsilon\|_{M^{p,\lambda}(\Omega)} < \varepsilon,$$

that is (3.2.16). Conversely, assume that $g \in M^{p,\lambda}(\Omega)$ and that (3.2.13) holds.

First of all we observe that, denoted by g_o the zero extension of g to $\mathbb{R}^n$, by (3.1.6) of Remark 3.1.2 there exists a positive constant c_1, independent of g, ψ_h and of Ω, such that

$$\|(1-\psi_h)\; g_o\|_{M^{p,\lambda}(\mathbb{R}^n)} \leq c_1 \|(1-\psi_h)\; g\|_{M^{p,\lambda}(\Omega)}.$$

Furthermore, by (3.2.13) we get that fixed $\varepsilon > 0$ there exists h_ε such that

$$\|(1-\psi_{h_\varepsilon})\; g\|_{M^{p,\lambda}(\Omega)} < \frac{\varepsilon}{2\, c_1}.$$

Therefore,

$$\|(1-\psi_{h_\varepsilon})\; g_o\|_{M^{p,\lambda}(\mathbb{R}^n)} < \frac{\varepsilon}{2}. \tag{3.2.18}$$

Set

$$\Phi_\varepsilon = \psi_{h_\varepsilon}\, g_o$$

by construction

$$\text{supp } \Phi_\varepsilon \subset \text{supp } \psi_{h_\varepsilon} \subset \overline{\Omega}_{2h_\varepsilon}. \tag{3.2.19}$$

Hence, taking into account (3.1.7) of Remark 3.1.2, one has that

$$\Phi_\varepsilon \in L^{p,\lambda}(\mathbb{R}^n). \tag{3.2.20}$$

On the other hand, assumption (3.2.13) together with Lemma 3.2.1 give that $g \in \widetilde{M}^{p,\lambda}(\Omega)$, then from Remark 3.2.2 we get

$$\lim_{t\to 0} \|g\|_{M^{p,\lambda}(\Omega,\, t)} = 0.$$

So, using (3.1.6) of Remark 3.1.2 we have that $\Phi_\varepsilon \in M^{p,\lambda}(\mathbb{R}^n)$ and

$$\lim_{t\to 0} \|\Phi_\varepsilon\|_{M^{p,\lambda}(\mathbb{R}^n,\, t)} = 0. \tag{3.2.21}$$

Arguing as in Lemma 1.2 of F. Chiarenza, M. Franciosi [25], from (3.2.19) - (3.2.21) we conclude that

$$\lim_{y\to 0} \|\Phi_\varepsilon(x-y) - \Phi_\varepsilon(x)\|_{L^{p,\lambda}(\mathbb{R}^n)} = 0.$$

We are now in the hypotheses of Lemma 3.1.1. Hence, denoted by $(J_k)_{k\in\mathbb{N}}$ a sequence of mollifiers in $\mathbb{R}^n$, we can find a positive integer $k_\varepsilon > h_\varepsilon$ such that

$$\|\Phi_\varepsilon - J_{k_\varepsilon} * \Phi_\varepsilon\|_{L^{p,\lambda}(\mathbb{R}^n)} < \frac{\varepsilon}{2}. \tag{3.2.22}$$

Set $g_\varepsilon = J_{k_\varepsilon} * \Phi_\varepsilon$ one has $g_\varepsilon \in C_o^\infty(\Omega)$. Furthermore, using (3.2.18) and (3.2.22) we get

$$\|g - g_\varepsilon\|_{M^{p,\lambda}(\Omega)} \leq \|g_o - J_{k_\varepsilon} * \Phi_\varepsilon\|_{M^{p,\lambda}(\mathbb{R}^n)} \leq$$

$$\|g_o - \Phi_\varepsilon\|_{M^{p,\lambda}(\mathbb{R}^n)} + \|\Phi_\varepsilon - J_{k_\varepsilon} * \Phi_\varepsilon\|_{M^{p,\lambda}(\mathbb{R}^n)} \leq$$

$$\|g_o - \psi_{h_\varepsilon} g_o\|_{M^{p,\lambda}(\mathbb{R}^n)} + \|\Phi_\varepsilon - J_{k_\varepsilon} * \Phi_\varepsilon\|_{L^{p,\lambda}(\mathbb{R}^n)} \leq$$

$$\|(1 - \psi_{h_\varepsilon}) g_o\|_{M^{p,\lambda}(\mathbb{R}^n)} + \frac{\varepsilon}{2} < \varepsilon \,,$$

this concludes the proof . ■

3.3 Decompositions of functions in $\widetilde{M}^{p,\lambda}(\Omega)$ and $M_o^{p,\lambda}(\Omega)$

The characterizations of the spaces $\widetilde{M}^{p,\lambda}(\Omega)$ and $M_o^{p,\lambda}(\Omega)$ naturally lead us to the introduction of the following *moduli of continuity.*

Let g be a function in $\widetilde{M}^{p,\lambda}(\Omega)$. A modulus of continuity of g in $\widetilde{M}^{p,\lambda}(\Omega)$ is a map $\widetilde{\sigma}^{p,\lambda}[g] : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\begin{gathered} F[g](h) \leq \widetilde{\sigma}^{p,\lambda}[g](h) \\ \lim_{h \to +\infty} \widetilde{\sigma}^{p,\lambda}[g](h) = 0 \,. \end{gathered} \tag{3.3.1}$$

If g belongs to $M_o^{p,\lambda}(\Omega)$, a modulus of continuity of g in $M_o^{p,\lambda}(\Omega)$ is an application $\sigma_o^{p,\lambda}[g] : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|(1-\zeta_h)\, g\|_{M^{p,\lambda}(\Omega)} + F[g](h) \leq \sigma_o^{p,\lambda}[g](h)$$

$$\lim_{h\to+\infty} \sigma_o^{p,\lambda}[g](h) = 0\,. \tag{3.3.2}$$

Let us show now the decomposition results[25]. Also in this case, any function g in $\widetilde{M}^{p,\lambda}(\Omega)$ or in $M_0^{p,\lambda}(\Omega)$ can be written as the sum of two functions : the first function is less regular than the second and it is controlled by means of a continuity modulus of the function g itself.

Lemma 3.3.1 *Let $\lambda \in [0, n[$, $p \in [1, +\infty[$ and $g \in \widetilde{M}^{p,\lambda}(\Omega)$. For any $h \in \mathbb{R}_+$, we have*

$$g = g'_h + g''_h, \tag{3.3.3}$$

with $g''_h \in L^\infty(\Omega)$ and

$$\|g'_h\|_{M^{p,\lambda}(\Omega)} \leq \widetilde{\sigma}^{p,\lambda}[g](h)\,, \qquad \|g''_h\|_{L^\infty(\Omega)} \leq\ h^{\frac{1}{p}}\|g\|_{M^{p,\lambda}(\Omega)}\,. \tag{3.3.4}$$

Proof $-$ Given $g \in \widetilde{M}^{p,\lambda}(\Omega)$ and $h \in \mathbb{R}_+$, we introduce the set

$$E_h = \{x \in \Omega \mid |g(x)| \geq\ h^{\frac{1}{p}}\, \|g\|_{M^{p,\lambda}(\Omega)}\}\,. \tag{3.3.5}$$

[25] See L. Caso, R. D'Ambrosio, S. Monsurrò [17].

Observe that

$$|E_h(x)| \leq \int_{\Omega(x)\cap E_h} \frac{|\,g(y)|^p}{\|g\|^p_{M^{p,\lambda}(\Omega)} h}\, dy \leq$$

$$\frac{1}{\|g\|^p_{M^{p,\lambda}(\Omega)} h} \int_{\Omega(x)} |\,g(y)|^p\, dy \leq \frac{1}{\|g\|^p_{M^{p,\lambda}(\Omega)} h} \sup_{\substack{\tau\in\,]0,1] \\ x\in\Omega}} \tau^{-\lambda} \|g\|^p_{L^p(\Omega(x,\tau))} = \frac{1}{h}. \tag{3.3.6}$$

Set

$$g'_h = g\,\chi_{E_h} = \begin{cases} g & \text{if } x \in E_h \\ 0 & \text{if } x \in \Omega \setminus E_h\,, \end{cases} \qquad g''_h = g - g\,\chi_{E_h} = \begin{cases} 0 & \text{if } x \in E_h \\ g & \text{if } x \in \Omega \setminus E_h\,. \end{cases}$$

In view of (3.3.6)

$$\|g'_h\|_{M^{p,\lambda}(\Omega)} = \|g\,\chi_{E_h}\|_{M^{p,\lambda}(\Omega)} \leq F[g](h) \leq \widetilde{\sigma}^{p,\lambda}[g](h)\,,$$

this gives the first inequality in (3.3.4), the second one easily follows from (3.3.5). ■

Lemma 3.3.2 *Let* $\lambda \in [0, n[$, $p \in [1, +\infty[$ *and* $g \in M_0^{p,\lambda}(\Omega)$. *For any* $h \in \mathbb{R}_+$, *we have*

$$g = \phi'_h + \phi''_h, \tag{3.3.7}$$

with

$$\|\phi'_h\|_{M^{p,\lambda}(\Omega)} \leq \sigma_o^{p,\lambda}[g](h)\,, \qquad |\phi''_h| \leq \zeta_h\, h^{\frac{1}{p}} \|g\|_{M^{p,\lambda}(\Omega)}\,. \tag{3.3.8}$$

Proof − To prove this second decomposition result we exploit again the definition of the set E_h introduced in (3.3.5) and inequality (3.3.6). In this case, for any $h \in \mathbb{R}_+$, we define the functions

$$\phi_h' = g\,(1-\zeta_h) + \zeta_h\, g\,\chi_{E_h} = \begin{cases} g & \text{if } x \in E_h \\ g\,(1-\zeta_h) & \text{if } x \in \Omega \setminus E_h\,, \end{cases}$$

$$\phi_h'' = \zeta_h\,(g - g\,\chi_{E_h}) = \begin{cases} 0 & \text{if } x \in E_h \\ g\,\zeta_h & \text{if } x \in \Omega \setminus E_h. \end{cases}$$

To obtain the first inequality in (3.3.8) we observe that (3.3.6) gives

$$\|\phi_h'\|_{M^{p,\lambda}(\Omega)} \le \|g\,(1-\zeta_h)\|_{M^{p,\lambda}(\Omega)} + \|\zeta_h\, g\,\chi_{E_h}\|_{M^{p,\lambda}(\Omega)} \le \|g\,(1-\zeta_h)\|_{M^{p,\lambda}(\Omega)} +$$

$$\|\, g\,\chi_{E_h}\|_{M^{p,\lambda}(\Omega)} \le \|g\,(1-\zeta_h)\|_{M^{p,\lambda}(\Omega)} + F[g](h) \le \sigma_o^{p,\lambda}[g](h)\,.$$

The second one is a consequence of (3.3.5). ∎

3.4 A compactness result

In this section, as application, we use the previous results to prove a compactness result for a multiplication operator defined on Sobolev spaces $W^{k,p}(\Omega)$ and which takes value in a suitable Lebesgue space.

To this aim let us recall an imbedding theorem proved in Lemma 2.2 of P. Cavaliere, M. Longobardi, A. Vitolo [23], which gives a boundeness result for such multiplication operator when $\Omega = \mathbb{R}^n$.

Let us specify the assumptions:

$h_1)$ Ω is an open subset of $\mathbb{R}^n$ having the cone property with cone C,

the parameters k, r, p, q, λ satisfy one of the following conditions:

$h_2)$ $k \in \mathbb{N}\,, \quad 1 \leq p \leq q \leq r < +\infty, \quad 0 \leq \lambda < n\,, \quad \gamma = \dfrac{1}{q} - \dfrac{1}{p} + \dfrac{k}{n} > 0\,,$

with $r > q$ when $p = \dfrac{n}{k} > 1$ and $\lambda = 0$, and with $\lambda > n(1 - r\gamma)$ when $r\gamma < 1$;

$h_3)$ $k = 1\,, \quad 1 < p = q < r \leq n, \quad \lambda = n{-}r.$

Lemma 3.4.1 *Under hypothesis $h_1)$ and if $h_2)$ or $h_3)$ holds, for any $u \in W^{k,p}(\mathbb{R}^n)$ and for any $g \in M^{r,\lambda}(\mathbb{R}^n)$ we have $g\,u \in L^q(\mathbb{R}^n)$. Moreover there exists a constant $c_1 \in \mathbb{R}_+$, depending on n, k, p, q, r and λ such that*

$$\|g\,u\|_{L^q(\mathbb{R}^n)} \leq c_1 \, \|g\|_{M^{r,\lambda}(\mathbb{R}^n)} \, \|u\|_{W^{k,p}(\mathbb{R}^n)}\,. \tag{3.4.1}$$

Proof − If (h_3) holds, we deduce the result from a theorem of C. Fefferman [29][26].
Otherwise, we have to consider the two different cases: $\lambda = 0$ and $\lambda > 0$.
In the first case, by assumptions, we have that $r\,\gamma \geq 1$, from which

$$\frac{r-q}{rq} \geq \frac{n-kp}{np}\,.$$

[26] The result can be also deduced from a theorem of F. Chiarenza, M. Frasca [26] which is a simplified proof of an imbedding theorem by C. Fefferman [29], for more details see P. Cavaliere [22].

Let $u \in W^{k,p}(\mathbb{R}^n)$, from Sobolev imbedding theorem we also have that $u \in L^{\frac{qr}{r-q}}(\mathbb{R}^n)$ and

$$||u||_{L^{\frac{qr}{r-q}}(\mathbb{R}^n)} \leq c_0 \, ||u||_{W^{k,p}(\mathbb{R}^n)} \,, \tag{3.4.2}$$

where $c_0 \in \mathbb{R}_+$ depends only on n, k, p, q, r and on the cone determining the cone property of $\mathbb{R}^n$.

Then, using Holder's inequality and the (3.4.2) we have

$$\int_{\mathbb{R}^n} |g\, u|^q \, dx \leq \Big(\int_{\mathbb{R}^n} |g|^r \, dx \Big)^{\frac{q}{r}} \cdot \Big(\int_{\mathbb{R}^n} |u|^{\frac{q\,r}{r-q}} \, dx \Big)^{\frac{r-q}{r\,q}\, q} \; \leq \; c_0 ||g||^q_{M^{r,0}(\mathbb{R}^n)} \cdot ||u||^q_{W^{k,p}(\mathbb{R}^n)} \,.$$

Using (3.1.4) and the imbedding of the space $M^{r,\lambda}(\mathbb{R}^n)$ in $M^r(\mathbb{R}^n)$, we deduce the result.

In the second case (i.e. $\lambda > 0$), we observe that there exists ϵ_0 such that

$$n - \lambda < \epsilon_0 < n\, r\, \gamma \,, \qquad \epsilon_0 < n \,. \tag{3.4.3}$$

In fact, if $r\,\gamma \; \geq 1$, we can take $\epsilon_0 \in \,]n - \lambda, \, n[$, if $r\,\gamma < 1$ then, by assumption, we have

$$n - \lambda < n\, r\, \gamma < n$$

and again there exists ϵ_0 satisfying (3.4.3). Let us consider the following application

$$H : g \in M^{r,\lambda}(\mathbb{R}^n) \rightarrow \sup_{x \in \mathbb{R}^n} \Big(\int_{B_1(x)} |g(y)|^r \, |x-y|^{\epsilon_0 - n} \, dy \Big)^{\frac{1}{r}} \,.$$

It is bounded application, in fact

$$\int_{B_1(x)} |g(y)|^r \, |x-y|^{\epsilon_0 - n} \, dy \; = \sum_{k \in \mathbb{N}} \int_{B(x,2^{-k+1}) \setminus B(x,2^{-k})} |g(y)|^r \; |x-y|^{\epsilon_0} \, \cdot \, |x-y|^{-n} dy$$

$$\leq \sum_{k \in \mathbb{N}} \left(\frac{1}{2^{k-1}} \right)^{\epsilon_0} \left(\frac{1}{2^k} \right)^{-n} \int_{B(x,2^{-k+1})} |g(y)|^r \, dy$$

$$\leq 2^{\epsilon_0 + \lambda} \left(\sum_{k \in \mathbb{N}} 2^{(n-\lambda-\epsilon_0)\,k} \right) ||g||^r_{M^{r,\lambda}(\mathbb{R}^n)} \;\; , \qquad \forall x \in \mathbb{R}^n \, .$$

From previous inequalities it turns out that

$$H(g) \leq c_3 \, ||g||_{M^{r,\lambda}(\mathbb{R}^n)} \, , \tag{3.4.4}$$

where c_3 depends only on n, r, λ and ϵ_0.

Moreover, from Theorem 1 of M. Schechter [50] we obtain

$$||gu||_{L^q(\mathbb{R}^n)} \leq c_4 \, H(g) \, ||u||_{W^{k,p}(\mathbb{R}^n)} \, ,$$

where c_4 depends only on k, p, q, r and ϵ_0. Using (3.4.4) in the previous inequality we deduce (3.4.1). ∎

In M. Transirico, M. Troisi, A. Vitolo [58] (see also R. A. Adams [2], E. M. Stein [54]) the following result has been shown

Lemma 3.4.2 *For every open subset Ω of $\mathbb{R}^n$ having the cone property with cone C*

and every $d_0 \in \mathbb{R}_+$ *there exists a sequence* $(\Omega_i)_{i\in\mathbb{N}}$ *of open subsets of* $\mathbb{R}^n$ *such that*

$i_1)$ $\displaystyle\bigcup_{i\in\mathbb{N}} \Omega_i = \Omega$;

$i_2)$ $diam\ \Omega_i \le d_0 \qquad \forall i \in \mathbb{N}$;

$i_3)$ *there exists* $m \in \mathbb{N}$ *such that every collection of* $m+1$ *elements of the sequence* $(\Omega_i)_{i\in\mathbb{N}}$ *has empty intersection ;*

$i_4)$ Ω_i, $i \in \mathbb{N}$, *has locally Lipschitz boundary with Lipschitz coefficient depending only on* C *;*

$i_5)$ *for each* $i \in \mathbb{N}$, *there exists a linear extension operator*

$$E_i : W^{k,p}(\Omega_i) \to W^{k,p}(\mathbb{R}^n), \;\; k \in \mathbb{N}, \;\; p \in [1, +\infty]\,,$$

such that

$$||E_i(u)||_{W^{k,p}(\mathbb{R}^n)} \le c_2 ||u||_{W^{k,p}(\Omega_i)}\,,$$

where c_2 *depends only on* n, p, m, k, C, d_0.

Remark 3.4.3 *We remark that*[27] *if* Ω *is an unbounded open subset of* $\mathbb{R}^n$ *having the cone property and* $F_0 = \{\Omega_i,\, i \in \mathbb{N}\}$ *is a sequence of open subsets of* $\mathbb{R}^n$ *satisfying the properties of Lemma 3.4.2 for any fixed* $d_0 \in \mathbb{R}_+$, *then*

$$\sum_{i\in\mathbb{N}} \int_{\Omega_i} |f| \le (m+2) \int_\Omega |f| \;\;, \quad \forall f \in L^1(\Omega) \tag{3.4.5}$$

[27] See Proposition 2.0.3 in P. Cavaliere [22].

where $m \in \mathbb{N}$ satisfies i_3).

We recall now an imbedding theorem proved in Theorem 3.2 of P. Cavaliere, M. Longobardi, A. Vitolo [23], which gives a boundeness result for multiplication operator when $\Omega \neq \mathbb{R}^n$.

Theorem 3.4.4 *Under hypothesis h_1) and if h_2) or h_3) holds, for any $u \in W^{k,p}(\Omega)$ and for any $g \in M^{r,\lambda}(\Omega)$ we have $g\,u \in L^q(\Omega)$. Moreover there exists a constant $c \in \mathbb{R}_+$, depending on n, k, p, q, r, λ and C, such that*

$$||g\,u||_{L^q(\Omega)} \leq c\,||g||_{M^{r,\lambda}(\Omega)}\,||u||_{W^{k,p}(\Omega)}\,. \tag{3.4.6}$$

Proof − Since Ω has cone property, fixed $d_0 \in]0,1]$, from Lemma 3.4.2 there exist a sequence $(\Omega_i)_{i\in N}$ of open subsets of $\mathbb{R}^n$ and a sequence $(E_i)_{i\in N}$ of linear extension operators which satisfy the properties of Lemma 3.4.2. Moreover we recall (see Remark 3.1.2) that, if $g \in M^{r,\lambda}(\Omega)$, then the zero extension g_0 of g outside Ω belongs to $M^{r,\lambda}(\mathbb{R}^n)$ and the following estimate holds:

$$||g_0||_{M^{r,\lambda}(\mathbb{R}^n)} \leq\, c_0\,||g||_{M^{r,\lambda}(\Omega)}\,,$$

where c_0 depends only on n, r and λ. Since

$$||gu||^q_{L^q(\Omega)} \leq \sum_{i\in\mathbb{N}} \int_{\Omega_i} |gu|^q = \sum_{i\in\mathbb{N}} \int_{\mathbb{R}^n} |\chi_{\Omega_i}\, g_0|^q\, |E_i(u)|^q\,,$$

using Lemma 3.4.1, (i_5) and Remark 3.4.3, it turns out that

$$||gu||^q_{L^q(\Omega)} \leq (c_0\, c_1)^q\, ||g||^q_{M^{r,\lambda}(\Omega)} \Big(\sum_{i\in\mathbb{N}} ||E_i(u)||^q_{W^{k,p}(\mathbb{R}^n)}\Big) \leq$$

$$\leq (c_0\, c_1\, c_2)^q\, ||g||^q_{M^{r,\lambda}(\Omega)} \Big(\sum_{i\in\mathbb{N}} ||u||^q_{W^{k,p}(\Omega_i)}\Big) \leq$$

$$\leq (c_0\, c_1\, c_2\, (m+2))^q\, ||g||^q_{M^{r,\lambda}(\Omega)}\, ||u||^q_{W^{k,p}(\Omega)} \,.$$

this concludes the proof . ■

Putting together Lemma 3.3.1 and Theorem 3.4.4, we easily have the following result[28]

Corollary 3.4.5 *Under hypothesis $h_1)$ and if $h_2)$ or $h_3)$ holds, for any $g \in \widetilde{M}^{r,\lambda}(\Omega)$ and for any $h \in \mathbb{R}_+$ we have*

$$\|g\, u\|_{L^q(\Omega)} \leq c \cdot \widetilde{\sigma}^{r,\lambda}[g](h) \cdot \|u\|_{W^{k,p}(\Omega)} + h^{\frac{1}{r}} \cdot \|g\|_{M^{r,\lambda}(\Omega)} \cdot \|u\|_{L^q(\Omega)}\,, \tag{3.4.7}$$

for each $u \in W^{k,p}(\Omega)$, where $c \in \mathbb{R}_+$ is the constant of (3.4.6).

Proof − Fix $g \in \widetilde{M}^{r,\lambda}(\Omega)$ and $h \in \mathbb{R}_+$. In view of Lemma 3.3.1 and Theorem 3.4.4 for any $u \in W^{k,p}(\Omega)$ we have

$$\|g\, u\|_{L^q(\Omega)} \leq \|g'_h\, u\|_{L^q(\Omega)} + \|g''_h\, u\|_{L^q(\Omega)} \leq$$

$$c\, \|g'_h\|_{M^{r,\lambda}(\Omega)} \cdot \|u\|_{W^{k,p}(\Omega)} + \|g''_h\|_{L^\infty(\Omega)} \cdot \|u\|_{L^q(\Omega)} \leq \tag{3.4.8}$$

$$c \cdot \widetilde{\sigma}^{r,\lambda}[g](h) \cdot \|u\|_{W^{k,p}(\Omega)} + h^{\frac{1}{r}} \cdot \|g\|_{M^{r,\lambda}(\Omega)} \cdot \|u\|_{L^q(\Omega)}\,.$$ ■

[28] See L. Caso, R. D'Ambrosio, S. Monsurrò [17].

If g is in $M_0^{r,\lambda}(\Omega)$ the previous estimate can be improved as showed in the corollary below.

Corollary 3.4.6 *Under hypothesis h_1) and if h_2) or h_3) holds, for any $g \in M_0^{r,\lambda}(\Omega)$ and for any $h \in \mathbb{R}_+$ there exists an open set $A_h \subset\subset \Omega$ with the cone property, such that*

$$\|g\,u\|_{L^q(\Omega)} \le c \cdot \sigma_o^{r,\lambda}[g](h) \cdot \|u\|_{W^{k,p}(\Omega)} + h^{\frac{1}{r}} \cdot \|g\|_{M^{r,\lambda}(\Omega)} \cdot \|u\|_{L^q(A_h)}\,, \tag{3.4.9}$$

for each $u \in W^{k,p}(\Omega)$, where $c \in \mathbb{R}_+$ is the constant of (3.4.6).

Proof − Fix $g \in M_0^{r,\lambda}(\Omega)$ and $h \in \mathbb{R}_+$. In view of Lemma 3.3.2 and Theorem 3.4.4 for any $u \in W^{k,p}(\Omega)$ we have

$$\begin{gathered}\|g\,u\|_{L^q(\Omega)} \le \|\phi_h'\,u\|_{L^q(\Omega)} + \|\phi_h''\,u\|_{L^q(\Omega)} \le \\ c\,\|\phi_h'\|_{M^{r,\lambda}(\Omega)} \cdot \|u\|_{W^{k,p}(\Omega)} + \|\phi_h''\,u\|_{L^q(\Omega)} \le \\ c \cdot \sigma_o^{r,\lambda}[g](h) \cdot \|u\|_{W^{k,p}(\Omega)} + \|\phi_h''\,u\|_{L^q(\Omega)}\,.\end{gathered} \tag{3.4.10}$$

Using again Lemma 3.3.2 we obtain

$$\begin{gathered}\|\phi_h''\,u\|_{L^q(\Omega)} \le \|g\|_{M^{r,\lambda}(\Omega)}\,h^{\frac{1}{r}} \left(\int_\Omega \left|\zeta_h\,u\right|^q dx\right)^{\frac{1}{q}} \le \\ \|g\|_{M^{r,\lambda}(\Omega)}\,h^{\frac{1}{r}} \left(\int_{\operatorname{supp}\zeta_h} \left|\,u\right|^q dx\right)^{\frac{1}{q}}.\end{gathered} \tag{3.4.11}$$

Putting together (3.4.10) and (3.4.11) we get estimate (3.4.9), with A_h obtained as follows: fixed $d_h \in \left]0, \frac{\operatorname{dist}(\operatorname{supp}\zeta_h, \partial\Omega)}{2}\right[$ and $\theta \in]0, \frac{\pi}{2}[$, the set A_h is union of the open cones $\mathcal{C} \subset\subset \Omega$ with opening θ, height d_h and such that $\mathcal{C} \cap \operatorname{supp}\zeta_h \neq \emptyset$. ■

We are now in position to prove the compactness result[29].

Corollary 3.4.7 *Suppose that condition h_1) is satisfied, that h_2) or h_3) holds and fix $g \in M_0^{r,\lambda}(\Omega)$. Then the operator*

$$u \in W^{k,p}(\Omega) \longrightarrow g\,u \in L^q(\Omega) \tag{3.4.12}$$

is compact.

Proof – Observe that, if $\Omega' \subset\subset \Omega$ is a bounded open set with the cone property, the operator

$$u \in W^{k,p}(\Omega) \longrightarrow u \in L^q(\Omega')$$

is compact.

Indeed, if $\Omega' \subset\subset \Omega$ is a bounded open set the operator

$$u \in W^{k,p}(\Omega) \longrightarrow u_{|\Omega'} \in W^{k,p}(\Omega')$$

is linear and bounded. Moreover, since Ω' has the cone property, Rellich - Kondrachov Theorem (see, for instance R. A. Adams [2]) applies and gives that the operator

$$w \in W^{k,p}(\Omega') \longrightarrow w \in L^q(\Omega')$$

[29] See L. Caso, R. D'Ambrosio, S. Monsurrò [17].

is compact.

Let us consider now a sequence $(u_n)_{n\in\mathbb{N}}$ bounded in $W^{k,p}(\Omega)$ and let $M \in \mathbb{R}_+$ be such that $\|u_n\|_{W^{k,p}(\Omega)} \le M \quad \forall\, n \in \mathbb{N}$. According to the above considerations, fixed $\varepsilon > 0$ there exist a subsequence $(u_{n_m})_{m\in\mathbb{N}}$ and $\nu \in \mathbb{N}$ such that

$$\|u_{n_m} - u_{n_l}\|_{L^q(\Omega')} \le \varepsilon \qquad \forall\, m, l > \nu\,. \tag{3.4.13}$$

On the other hand, given $g \in M_0^{r,\lambda}(\Omega)$ and $h \in \mathbb{R}_+$, in view of Corollary 3.4.6 there exist a constant $c \in \mathbb{R}_+$ and an open set $A_h \subset\subset \Omega$ with the cone property, independent of u_n, such that

$$\|g\, u_n\|_{L^q(\Omega)} \le c \cdot \sigma_o^{r,\lambda}[g](h) \cdot \|u_n\|_{W^{k,p}(\Omega)} + h^{\frac{1}{r}} \cdot \|g\|_{M^{r,\lambda}(\Omega)} \cdot \|u_n\|_{L^q(A_h)}\,. \tag{3.4.14}$$

From (3.4.14) and (3.4.13) written for $\varepsilon = \dfrac{c \cdot \sigma_o^{r,\lambda}[g](h)}{h^{\frac{1}{r}} \cdot \|g\|_{M^{r,\lambda}(\Omega)}}$ and $\Omega' = A_h$, for $m, l > \nu$ one has

$$\|g\, u_{n_m} - g\, u_{n_l}\|_{L^q(\Omega)} \le c \cdot \sigma_o^{r,\lambda}[g](h) \cdot \big(2M + 1\big)\,. \tag{3.4.15}$$

By (3.4.15) and (3.3.2) we conclude that $(g\, u_{n_m})_{m\in\mathbb{N}}$ is a Cauchy sequence in $L^q(\Omega)$, which gives the compactness of the operator defined in (3.4.12). ■

3.5 The space $M_\rho^{p,\lambda}(\Omega)$

In this section we introduce some weighted spaces of Morrey type settled between $M_0^{p,\lambda}(\Omega)$ and $\widetilde{M}^{p,\lambda}(\Omega)$. To this aim, given $d \in \mathbb{R}_+$, we consider the set $G(\Omega,\, d)$ defined in M. Troisi [61] as the class of measurable weight functions $\rho : \Omega \to \mathbb{R}_+$ such that

$$\sup_{\substack{x,y\in\Omega \\ |x-y|<d}} \frac{\rho(x)}{\rho(y)} < +\infty \,. \tag{3.5.1}$$

We note that the definition (3.5.1) is equivalent to the following

$$\sup_{\substack{x,y\in\Omega \\ |x-y|<d}} \left| \log \frac{\rho(x)}{\rho(y)} \right| < +\infty \,.$$

It is easy to show that $\rho \in \mathcal{G}(\Omega,\, d)$ if and only if there exists $\gamma \in \mathbb{R}_+$, independent on x and y, such that

$$\gamma^{-1}\,\rho(y) \le \rho(x) \le \gamma\,\rho(y)\,, \qquad \forall\, y \in \Omega, \quad \forall\, x \in \Omega(y,d). \tag{3.5.2}$$

Furthermore, in S. Boccia, M. Salvato, M. Transirico [8] is proved that

$$\rho, \rho^{-1} \in L^\infty_{\mathrm{loc}}(\overline{\Omega}). \tag{3.5.3}$$

We put

$$\mathcal{G}(\Omega) = \bigcup_{d>0} \mathcal{G}(\Omega, d).$$

It is easy to verify that the functions

$$\rho(x) = e^{t|x|}\,, \qquad \rho(x) = (1+|x|^2)^t\,, \qquad (\forall\, x \in \Omega)\,(\forall\, t \in \mathbb{R}) \tag{3.5.4}$$

belong to the class $\mathcal{G}(\Omega)$.

Moreover, we observe that if $\rho \in \mathcal{A}(\Omega)$ and $\inf\limits_{\Omega} \rho > 0$ then $\rho \in \mathcal{G}(\Omega)$.

Remark 3.5.1 *We remark that if* $\sigma \in \mathcal{G}(\Omega)$ *then for any* $b \in \mathbb{R}_+$ *and for any* $s \in \mathbb{R}$ *the function*

$$\rho(x) = \frac{\sigma(x)}{1 + b\,\sigma^s(x)}$$

is in $\mathcal{G}(\Omega)$.

In fact if $\sigma \in \mathcal{G}(\Omega)$ *for any fixed* $d \in \mathbb{R}_+$ *there exists a constant* $\nu > 1$ *such that*

$$\nu^{-1}\sigma(y) \le \sigma(x) \le \nu\,\sigma(y) \qquad \forall y \in \Omega, \forall x \in \Omega(y,d)\,.$$

Let

$$t = \begin{cases} -s & \text{if } s > 0 \\ s & \text{if } s < 0 \end{cases}$$

For any $y \in \Omega$ *and for any* $x \in \Omega(y,d)$, *we have*

$$\rho(x) = \frac{\sigma(x)}{1 + b\,\sigma^s(x)} \le \frac{\nu\,\sigma(y)}{1 + \nu^t\,b\,\sigma^s(y)} \le \frac{\nu\,\sigma(y)}{\nu^t + \nu^t\,b\,\sigma^s(y)} = \frac{\nu}{\nu^t}\,\rho(y) = \gamma\,\rho(y)\,, \quad (3.5.5)$$

and

$$\rho(x) = \frac{\sigma(x)}{1 + b\,\sigma^s(x)} \ge \frac{\nu^{-1}\,\sigma(y)}{1 + \nu^{-t}\,b\,\sigma^s(y)} \ge \frac{\nu^{-1}\,\sigma(y)}{\nu^{-t} + \nu^{-t}\,b\,\sigma^s(y)} = \frac{\nu^{-1}}{\nu^{-t}}\,\rho(y) = \gamma^{-1}\,\rho(y)\,. \quad (3.5.6)$$

From (3.5.5), (3.5.6) and (3.5.2) we obtain that $\rho \in \mathcal{G}(\Omega)$.

Moreover, it's easy to prove that if $s \in [1, +\infty[$ *and* $b \in \mathbb{R}_+$ *then the function* $\rho \in L^\infty(\Omega)$. ■

Further properties of the class $\mathcal{G}(\Omega)$ can be found, for instance, in M. Troisi [61] and in S. Boccia, M. Salvato, M. Transirico [8].

If $\rho \in \mathcal{G}(\Omega)$, $k \in \mathbb{N}_0$, $1 \leq p \leq +\infty$ and $s \in \mathbb{R}$ we consider the space $U^{k,p}_s(\Omega)$ of distributions u on Ω such that $\rho^s\,\partial^\alpha u \in L^p(\Omega)$ for $|\alpha| \leq k$, equipped with the norm

$$\|u\|_{U^{k,p}_s(\Omega)} = \sum_{|\alpha|\leq k} \|\rho^s\,\partial^\alpha u\|_{L^p(\Omega)}\,. \tag{3.5.7}$$

We put

$$U^{0,p}_s(\Omega) = L^p_s(\Omega)\,.$$

It can be proved that, for $1 \leq p < +\infty$ and $s \in \mathbb{R}$, the space $C^\infty_0(\Omega)$ is dense in $L^p_s(\Omega)$[30]. A more detailed account of properties of the spaces $U^{k,p}_s(\Omega)$ can be found, for instance in M. Troisi [61]. The space just introduced is another example of weighted Sobolev space.

From now on we consider $\rho \in G(\Omega) \cap L^\infty(\Omega)$ and we denote by d the positive real number such that $\rho \in G(\Omega, d)$.

The shown results are in L. Caso, R. D'Ambrosio, S. Monsurrò [17].

Lemma 3.5.2 *Let $\lambda \in [0, n[$, $p \in [1, +\infty[$ and $g \in M^{p,\lambda}(\Omega)$. The following properties are equivalent:*

$$g \text{ is in the closure of } L^\infty_{-\frac{1}{p}}(\Omega) \text{ in } M^{p,\lambda}(\Omega), \tag{3.5.8}$$

$$\lim_{h\to+\infty}\left(\sup_{\substack{x\in\Omega\\ \tau\in]0,d]}}\ \sup_{\substack{E\in\Sigma(\Omega)\\ \tau^{-\lambda}\rho(x)|E(x,\tau)|\leq\frac{1}{h}}} \|g\,\chi_E\|_{M^{p,\lambda}(\Omega)}\right) = 0, \tag{3.5.9}$$

$$\lim_{h\to+\infty}\left(\sup_{x\in\Omega}\ \sup_{\substack{E\in\Sigma(\Omega)\\ \rho(x)|E(x,d)|\leq\frac{1}{h}}} \|g\,\chi_E\|_{M^{p,\lambda}(\Omega)}\right) = 0. \tag{3.5.10}$$

[30] See, for instance, D. E. Edmunds, W. D. Evans [27], M. Troisi [61].

We denote by $M^{p,\lambda}_\rho(\Omega)$ the set of functions satisfying one of the above properties.

Proof − We start proving the equivalence between (3.5.8) and (3.5.9). This proof is in the spirit of the one of Lemma 3.2.1. For reader's convenience, we write down just few lines pointing out the main differences.

If (3.5.8) holds, fixed $\varepsilon > 0$ there exists a function $g_\varepsilon \in L^\infty_{-\frac{1}{p}}(\Omega)$ such that

$$\|g - g_\varepsilon\|_{M^{p,\lambda}(\Omega)} < \frac{\varepsilon}{2}\,. \tag{3.5.11}$$

From (3.5.11) we get that, for any $E \in \Sigma(\Omega)$,

$$\|g\, \chi_E\|_{M^{p,\lambda}(\Omega)} < \frac{\varepsilon}{2} + \|g_\varepsilon\, \chi_E\|_{M^{p,\lambda}(\Omega)}\,. \tag{3.5.12}$$

Furthermore, in view of the equivalence of the spaces $M^{p,\lambda}(\Omega,d)$ and $M^{p,\lambda}(\Omega)$ given by (3.1.3) and taking into account (3.5.2),

$$\begin{gathered}\|g_\varepsilon\, \chi_E\|_{M^{p,\lambda}(\Omega)} \le c_1\, \|g_\varepsilon\, \chi_E\|_{M^{p,\lambda}(\Omega,d)} = c_1 \sup_{\substack{\tau\in]0,d]\\ x\in\Omega}} \tau^{-\frac{\lambda}{p}} \|g_\varepsilon\, \chi_E\|_{L^p(\Omega(x,\tau))} \le \\ c_1\, \gamma^{\frac{1}{p}}\, \|g_\varepsilon\|_{L^\infty_{-\frac{1}{p}}(\Omega)} \sup_{\substack{\tau\in]0,d]\\ x\in\Omega}} (\tau^{-\lambda} \rho(x)|E(x,\tau)|)^{\frac{1}{p}}\,,\end{gathered} \tag{3.5.13}$$

where $c_1 \in \mathbb{R}_+$ depends only on n and d. Hence, set

$$\frac{1}{h_\varepsilon} = \left(\frac{\varepsilon}{2\, c_1\, \gamma^{\frac{1}{p}}\, \|g_\varepsilon\|_{L^\infty_{-\frac{1}{p}}(\Omega)}}\right)^p,$$

from (3.5.13) we deduce that, if $\sup\limits_{\substack{\tau\in]0,d]\\ x\in\Omega}} \tau^{-\lambda}\rho(x)|E(x,\tau)| \le \dfrac{1}{h_\varepsilon}$, then

$$\|g_\varepsilon\, \chi_E\|_{M^{p,\lambda}(\Omega)} \le \frac{\varepsilon}{2}\,. \tag{3.5.14}$$

Putting together (3.5.12) and (3.5.14) we obtain (3.5.9).

Now assume that g is a function in $M^{p,\lambda}(\Omega)$ and that (3.5.9) holds. Then, for any $\varepsilon > 0$ there exists $h_\varepsilon \in \mathbb{R}_+$ such that, if $E \in \Sigma(\Omega)$ with $\sup\limits_{\substack{\tau\in]0,d]\\ x\in\Omega}} \tau^{-\lambda}\rho(x)|E(x,\tau)| \le \dfrac{1}{h_\varepsilon}$, then $\|g\, \chi_E\|_{M^{p,\lambda}(\Omega)} < \varepsilon$.

For each $k \in \mathbb{R}_+$ we define the set

$$G_k = \left\{x \in \Omega \;\middle|\; \rho^{-\frac{1}{p}}(x)\,|g(x)| \ge k\right\}. \tag{3.5.15}$$

Using again (3.1.3), there exists $c_2 \in \mathbb{R}_+$ depending on the same parameters as c_1 such that

$$\begin{aligned} \|g\|_{M^{p,\lambda}(\Omega)} &\ge c_2\, \|g\|_{M^{p,\lambda}(\Omega,d)} \ge c_2 \sup_{\substack{\tau\in]0,d]\\ x\in\Omega}} \tau^{-\frac{\lambda}{p}}\|g\|_{L^p(G_k(x,\tau))} \ge \\ & c_2\,\gamma^{-\frac{1}{p}}\, k \sup_{\substack{\tau\in]0,d]\\ x\in\Omega}} \left(\tau^{-\lambda}\,\rho(x)\,|G_k(x,\tau)|\right)^{\frac{1}{p}}. \end{aligned} \tag{3.5.16}$$

Therefore, if we put

$$k_\varepsilon = \frac{\gamma^{\frac{1}{p}} h_\varepsilon^{\frac{1}{p}} \|g\|_{M^{p,\lambda}(\Omega)}}{c_2}\,,$$

from (3.5.16) we obtain

$$\sup_{\substack{\tau\in]0,d]\\ x\in\Omega}} \tau^{-\lambda}\,\rho(x)\,|G_{k_\varepsilon}(x,\tau)| \le \frac{1}{h_\varepsilon}$$

and then

$$\|g\,\chi_{G_{k_\varepsilon}}\|_{M^{p,\lambda}(\Omega)} < \varepsilon\,. \tag{3.5.17}$$

We conclude setting $g_\varepsilon = g - g\,\chi_{G_{k_\varepsilon}}$. Indeed by (3.5.15) $g_\varepsilon \in L^\infty_{-\frac{1}{p}}(\Omega)$ and (3.5.17) gives that $\|g - g_\varepsilon\|_{M^{p,\lambda}(\Omega)} < \varepsilon$.

Arguing similarly we prove also that (3.5.8) entails (3.5.10) and vice versa. Indeed, if $g \in M^{p,\lambda}(\Omega)$ and (3.5.8) holds, we can obtain as before (3.5.12) and (3.5.13).

On the other hand, there exists a constant $c_3 = c_3(n)$ such that

$$\sup_{\substack{\tau\in]0,d]\\ x\in\Omega}} (\tau^{-\lambda}\cdot\rho(x)\cdot|E(x,\tau)|)^{\frac{1}{p}} \le$$

$$\|\rho\|^{\frac{\lambda}{np}}_{L^\infty(\Omega)} \sup_{\substack{\tau\in]0,d]\\ x\in\Omega}} \tau^{-\frac{\lambda}{p}}\cdot\rho^{\frac{n-\lambda}{np}}(x)\;\cdot|E(x,\tau)|^{\frac{\lambda}{np}}\;\cdot|E(x,\tau)|^{\frac{n-\lambda}{np}} \le \tag{3.5.18}$$

$$c_3\cdot\|\rho\|^{\frac{\lambda}{np}}_{L^\infty(\Omega)} \sup_{\substack{\tau\in]0,d]\\ x\in\Omega}} (\rho(x)\;\cdot|E(x,\tau)|)^{\frac{n-\lambda}{np}}.$$

Putting together (3.5.13) and (3.5.18) we obtain

$$\|g_\varepsilon\,\chi_E\|_{M^{p,\lambda}(\Omega)} \le$$

$$c_4\,\gamma^{\frac{1}{p}}\,\|g_\varepsilon\|_{L^\infty_{-\frac{1}{p}}(\Omega)}\,\|\rho\|^{\frac{\lambda}{np}}_{L^\infty(\Omega)} \sup_{\substack{\tau\in]0,d]\\ x\in\Omega}} (\rho(x)|E(x,\tau)|)^{\frac{n-\lambda}{np}}\,, \tag{3.5.19}$$

where $c_4 = c_1 \cdot c_3$. Now, set

$$\frac{1}{h_\varepsilon} = \left(\frac{\varepsilon}{2\, c_4\, \gamma^{\frac{1}{p}}\, \|g_\varepsilon\|_{L^\infty_{-\frac{1}{p}}(\Omega)}\, \|\rho\|^{\frac{\lambda}{np}}_{L^\infty(\Omega)}} \right)^{\frac{np}{n-\lambda}},$$

from (3.5.19) we deduce that, if $\sup\limits_{\substack{\tau\in]0,d] \\ x\in\Omega}} \rho(x)|E(x,\tau)| \le \dfrac{1}{h_\varepsilon}$ then

$$\|g_\varepsilon\, \chi_E\|_{M^{p,\lambda}(\Omega)} \le \frac{\varepsilon}{2}\,. \tag{3.5.20}$$

From (3.5.12) and (3.5.20) we obtain (3.5.10).

Conversely, assume that (3.5.10) holds. We consider again the sets G_k introduced in (3.5.15). From (3.5.16) we get

$$\|g\|_{M^{p,\lambda}(\Omega)} \ge c_2\, \|g\|_{M^{p,\lambda}(\Omega,d)} \ge c_2\, d^{-\frac{\lambda}{p}} \gamma^{-\frac{1}{p}}\, k \sup_{x\in\Omega} (\rho(x)\, |G_k(x,d)|)^{\frac{1}{p}}. \tag{3.5.21}$$

Therefore, if we put

$$k_\varepsilon = \frac{d^{\frac{\lambda}{p}}\, \gamma^{\frac{1}{p}} h_\varepsilon^{\frac{1}{p}} \|g\|_{M^{p,\lambda}(\Omega)}}{c_2}\,,$$

from (3.5.21) we obtain

$$\sup_{x\in\Omega} \rho(x)\, |G_{k_\varepsilon}(x,d)| \le \frac{1}{h_\varepsilon}$$

and then, (3.5.10) being verified,

$$\|g\, \chi_{G_{k_\varepsilon}}\|_{M^{p,\lambda}(\Omega)} < \varepsilon\,. \tag{3.5.22}$$

We conclude the proof setting $g_\varepsilon = g - g\,\chi_{G_{k_\varepsilon}}$. Indeed, clearly $g_\varepsilon \in L^\infty_{-\frac{1}{p}}(\Omega)$ and (3.5.22) gives $\|g - g_\varepsilon\|_{M^{p,\lambda}(\Omega)} < \varepsilon$. ■

Arguing in the spirit of §3.3, we want to obtain a decomposition result also for functions in $M^{p,\lambda}_\rho(\Omega)$. To this aim we put, for $h \in \mathbb{R}_+$ and $g \in M^{p,\lambda}(\Omega)$

$$D[g](h) = \sup_{\substack{E\in\Sigma(\Omega) \\ \sup\limits_{x\in\Omega} \rho(x)|E(x,d)| \le \frac{1}{h}}} \|g\,\chi_E\|_{M^{p,\lambda}(\Omega)}.$$

In view of the previous lemma, we can define a modulus of continuity of a function g in $M^{p,\lambda}_\rho(\Omega)$ as a map $\sigma^{p,\lambda}_\rho[g] : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$D[g](h) \le \sigma^{p,\lambda}_\rho[g](h)$$

$$\lim_{h\to+\infty} \sigma^{p,\lambda}_\rho[g](h) = 0\,.$$

Lemma 3.5.3 *Let $\lambda \in [0, n[$, $p \in [1, +\infty[$ and $g \in M^{p,\lambda}_\rho(\Omega)$. For any $h \in \mathbb{R}_+$, we have*

$$g = \varphi'_h + \varphi''_h, \tag{3.5.23}$$

with $\varphi''_h \in L^\infty_{-\frac{1}{p}}(\Omega)$ and

$$\|\varphi'_h\|_{M^{p,\lambda}(\Omega)} \le \sigma^{p,\lambda}_\rho[g](h)\,, \qquad \|\varphi''_h\|_{L^\infty_{-\frac{1}{p}}(\Omega)} \le c\,\gamma^{\frac{1}{p}}\,h^{\frac{1}{p}}\|g\|_{M^{p,\lambda}(\Omega)}\,, \tag{3.5.24}$$

where c is a positive constant only depending on n, d, p and λ and where γ is that of (3.5.2).

Proof – Fix $g \in M^{p,\lambda}_\rho(\Omega)$, for any $h \in \mathbb{R}_+$ we set

$$\varphi'_h = g\,\chi_{G_h} = \begin{cases} g & \text{if } x \in G_h \\ 0 & \text{if } x \in \Omega \setminus G_h\,, \end{cases} \qquad \varphi''_h = g - g\,\chi_{G_h} = \begin{cases} 0 & \text{if } x \in G_h \\ g & \text{if } x \in \Omega \setminus G_h\,, \end{cases}$$

where

$$G_h = \left\{ x \in \Omega \;\middle|\; \rho^{-\frac{1}{p}}(x)\,|g(x)| \geq \gamma^{\frac{1}{p}}\, d^{\frac{\lambda}{p}}\, h^{\frac{1}{p}} \|g\|_{M^{p,\lambda}(\Omega,d)} \right\}.$$

The thesis follows the by (3.5.2) and (3.1.3) arguing as in the proof of Lemma 3.3.1 ■

Let us show the following inclusion:

Lemma 3.5.4 *Let $\lambda \in [0, n[$ and $p \in [1, +\infty[$. Then $L^\infty_{-\alpha}(\Omega) \cap M^{p,\lambda}(\Omega) \subset M^{p,\lambda}_\rho(\Omega)$, $\forall\, \alpha \in \mathbb{R}_+$.*

Proof – For $\alpha \geq 1/p$, clearly $L^\infty_{-\alpha}(\Omega) \subset L^\infty_{-\frac{1}{p}}(\Omega)$ and then (3.5.8) holds. On the other hand, for $\alpha < 1/p$ we can show that if $g \in L^\infty_{-\alpha}(\Omega) \cap M^{p,\lambda}(\Omega)$, then (3.5.9) holds. Indeed observe that, by (3.1.3) there exists a constant $c_1 = c_1(n, d)$ such that for any $E \in \Sigma(\Omega)$

$$\|g\,\chi_E\|_{M^{p,\lambda}(\Omega)} \leq c_1\,\|g\,\chi_E\|_{M^{p,\lambda}(\Omega,d)} = c_1 \sup_{\substack{\tau \in]0,d] \\ x \in \Omega}} \tau^{-\frac{\lambda}{p}} \|g\,\chi_E\|_{L^p(\Omega(x,\tau))} \leq$$

$$c_1\,\gamma^\alpha\,\|g\|_{L^\infty_{-\alpha}(\Omega)} \sup_{\substack{\tau \in]0,d] \\ x \in \Omega}} \tau^{-\frac{\lambda}{p}} \rho^\alpha(x) |E(x,\tau)|^{\frac{1}{p}}\,.$$

Moreover there exists a constant $c_2 = c_2(n)$ such that

$$\sup_{\substack{\tau\in]0,d]\\ x\in\Omega}} \tau^{-\frac{\lambda}{p}} \rho^{\alpha}(x)|E(x,\tau)|^{\frac{1}{p}} =$$

$$\sup_{\substack{\tau\in]0,d]\\ x\in\Omega}} \Big(\tau^{-\lambda}\,\rho(x)\,|E(x,\tau)|\Big)^{\alpha} \Big(\tau^{-\lambda}\,|E(x,\tau)|\Big)^{\frac{1}{p}-\alpha} \leq$$

$$c_2\, d^{(n-\lambda)\left(\frac{1}{p}-\alpha\right)} \sup_{\substack{\tau\in]0,d]\\ x\in\Omega}} \Big(\tau^{-\lambda}\,\rho(x)\,|E(x,\tau)|\Big)^{\alpha}.$$

Hence, fixed $\varepsilon > 0$ and set

$$\frac{1}{h_\varepsilon} = \left(\frac{\varepsilon}{c_1 \cdot c_2\, \gamma^{\alpha}\, \|g\|_{L^{\infty}_{-\alpha}(\Omega)}\, d^{(n-\lambda)\left(\frac{1}{p}-\alpha\right)}}\right)^{\frac{1}{\alpha}},$$

we deduce that, if $\sup_{\substack{\tau\in]0,d]\\ x\in\Omega}} \tau^{-\lambda}\rho(x)|E(x,\tau)| \leq \dfrac{1}{h_\varepsilon}$ then $\|g\,\chi_E\|_{M^{p,\lambda}(\Omega)} \leq \varepsilon\,.$ ■

Now we can prove a further characterization of $M^{p,\lambda}_{\rho}(\Omega)$.

Lemma 3.5.5 *Let $\lambda \in [0,n[$ and $p \in [1,+\infty[$. Then $M^{p,\lambda}_{\rho}(\Omega)$ is the closure of $\bigcup_{\alpha\in\mathbb{R}_+} L^{\infty}_{-\alpha}(\Omega) \cap M^{p,\lambda}(\Omega)$ in $M^{p,\lambda}(\Omega)$.*

Proof − Clearly if $g \in M^{p,\lambda}_{\rho}(\Omega)$ by (3.5.8) one has also that g is in the closure of $\bigcup_{\alpha\in\mathbb{R}_+} L^{\infty}_{-\alpha}(\Omega) \cap M^{p,\lambda}(\Omega)$ in $M^{p,\lambda}(\Omega)$.

Conversely, let us prove that if g belongs to the closure of $\bigcup_{\alpha\in\mathbb{R}_+} L^{\infty}_{-\alpha}(\Omega)\cap M^{p,\lambda}(\Omega)$ in $M^{p,\lambda}(\Omega)$ then (3.5.10) holds. Indeed, given $\varepsilon > 0$ there exists a function $g_\varepsilon \in L^{\infty}_{-\alpha}(\Omega)\cap M^{p,\lambda}(\Omega)$, for an $\alpha \in \mathbb{R}_+$, such that

$$\|g - g_\varepsilon\|_{M^{p,\lambda}(\Omega)} < \frac{\varepsilon}{2}\,.$$

Hence, given $E \in \Sigma(\Omega)$

$$\|g\,\chi_E\|_{M^{p,\lambda}(\Omega)} \leq \|(g-g_\varepsilon)\,\chi_E\|_{M^{p,\lambda}(\Omega)} + \|g_\varepsilon\,\chi_E\|_{M^{p,\lambda}(\Omega)} < \frac{\varepsilon}{2} + \|g_\varepsilon\,\chi_E\|_{M^{p,\lambda}(\Omega)}\,. \quad (3.5.25)$$

Now observe that since $g_\varepsilon \in L^\infty_{-\alpha}(\Omega) \cap M^{p,\lambda}(\Omega)$ by Lemma 3.5.4 we get $g_\varepsilon \in M^{p,\lambda}_\rho(\Omega)$ and therefore using (3.5.10) of Lemma 3.5.2 we obtain that if $\sup\limits_{x\in\Omega} \rho(x)\,|E(x,d)| \leq \dfrac{1}{h}$ then

$$\|g_\varepsilon\,\chi_E\|_{M^{p,\lambda}(\Omega)} \leq \frac{\varepsilon}{2}\,.$$

This, together with (3.5.25), ends the proof. ■

A straightforward consequence of the definitions (3.2.11) of Lemma 3.2.3, (3.5.8) of Lemma 3.5.2 and (3.2.1) of Lemma 3.2.1 is given by the following result:

Lemma 3.5.6 *Let $\lambda \in [0,n[$ and $p \in [1,+\infty[$. Then $M^{p,\lambda}_0(\Omega) \subset M^{p,\lambda}_\rho(\Omega) \subset \widetilde{M}^{p,\lambda}(\Omega)$.*

Let us show that if ρ vanishes at infinity the first inclusion stated in the lemma above becomes an identity.

Lemma 3.5.7 *Let $\lambda \in [0,n[$ and $p \in [1,+\infty[$. If ρ is such that*

$$\lim_{|x|\to+\infty} \rho(x) = 0, \quad (3.5.26)$$

then $M^{p,\lambda}_0(\Omega) = M^{p,\lambda}_\rho(\Omega)$.

Proof − We show the inclusion $M^{p,\lambda}_\rho(\Omega) \subset M^{p,\lambda}_0(\Omega)$, the converse being stated in Lemma 3.5.6. In view of Lemma 3.5.5, it is enough to verify that if (3.5.26) holds, then $L^\infty_{-\alpha}(\Omega) \cap M^{p,\lambda}(\Omega) \subset M^{p,\lambda}_0(\Omega)$, for any $\alpha \in \mathbb{R}_+$.

To this aim, given $\alpha \in \mathbb{R}_+$, we fix $g \in L^\infty_{-\alpha}(\Omega) \cap M^{p,\lambda}(\Omega)$ and we prove that (3.2.15) is satisfied. Observe that by Lemmas 3.5.4 and 3.5.6 $g \in \widetilde{M}^{p,\lambda}(\Omega)$. Moreover, for any $x \in \Omega$ and if $1 \leq d$ there exists a constant $c = c(n)$ such that

$$\sup_{\tau\in]0,1]} \tau^{-\frac{\lambda}{p}} \|g\|_{L^p(\Omega(x,\tau))} \leq \gamma^\alpha \|g\|_{L^\infty_{-\alpha}(\Omega)} \sup_{\tau\in]0,1]} \tau^{-\frac{\lambda}{p}} \rho^\alpha(x)\, |\Omega(x,\tau)|^{\frac{1}{p}} \leq \tag{3.5.27}$$

$$c\,\gamma^\alpha \|g\|_{L^\infty_{-\alpha}(\Omega)} \sup_{\tau\in]0,1]} \tau^{\frac{n-\lambda}{p}} \rho^\alpha(x) = c\,\gamma^\alpha \|g\|_{L^\infty_{-\alpha}(\Omega)} \rho^\alpha(x).$$

On the other hand, if $d < 1$, clearly one has

$$\sup_{\tau\in]0,1]} \tau^{-\frac{\lambda}{p}} \|g\|_{L^p(\Omega(x,\tau))} = \tag{3.5.28}$$

$$\max\left\{ \sup_{\tau\in]0,d]} \tau^{-\frac{\lambda}{p}} \|g\|_{L^p(\Omega(x,\tau))},\ \sup_{\tau\in]d,1]} \tau^{-\frac{\lambda}{p}} \|g\|_{L^p(\Omega(x,\tau))} \right\}.$$

We can treat the first term on the right-hand side of this last equality as done in (3.5.27) obtaining

$$\sup_{\tau\in]0,d]} \tau^{-\frac{\lambda}{p}} \|g\|_{L^p(\Omega(x,\tau))} \leq d^{\frac{n-\lambda}{p}} c\,\gamma^\alpha \|g\|_{L^\infty_{-\alpha}(\Omega)} \rho^\alpha(x), \tag{3.5.29}$$

the constant $c = c(n)$ being the one of (3.5.27).

Concerning the second one, observe that for any $x \in \Omega$ and $\tau \in]d,1]$ we have the inclusion $\Omega(x,\tau) \subset Q(x,\tau)$, where $Q(x,\tau)$ denotes an n-dimensional cube of center x and edge 2τ. Now, there exists a positive integer k such that we can decompose the cube $Q(x,1)$ in k cubes of edge less than $d/2$ and center x_i, with $x_i \in \Omega$ for

$i = 1, ..., k$. Therefore $Q(x,1) \subset \bigcup_{i=1}^{k} B(x_i, d/2)$. Hence for any $x \in \Omega$ and $\tau \in]d,1]$ we have, arguing as before with opportune modifications,

$$\tau^{-\frac{\lambda}{p}} \|g\|_{L^p(\Omega(x,\tau))} \le d^{-\frac{\lambda}{p}} \sum_{i=1}^{k} \|g\|_{L^p(\Omega(x_i,d/2))} \le k\, d^{\frac{n-\lambda}{p}}\, c\,\gamma^\alpha \|g\|_{L^\infty_{-\alpha}(\Omega)} \rho^\alpha(x), \quad (3.5.30)$$

the constant $c = c(n)$ being the same of (3.5.27).

The thesis follows then from (3.5.27), (3.5.28), (3.5.29) and (3.5.30) passing to the limit as $|x| \to +\infty$, as a consequence of hypothesis (3.5.26). ■

From the latter result we easily obtain the following lemma:

Lemma 3.5.8 *Let $\lambda \in [0, n[$ and $p \in [1, +\infty[$. If ρ, $\sigma \in G(\Omega) \cap L^\infty(\Omega)$ and*

$$\lim_{|x|\to+\infty} \rho(x) = \lim_{|x|\to+\infty} \sigma(x) = 0,$$

then $M^{p,\lambda}_\rho(\Omega) = M^{p,\lambda}_\sigma(\Omega)$.

Finally, we want to show with some counterexamples that the inclusions between the spaces $M^{p,\lambda}_0(\Omega)$ and $M^{p,\lambda}_\rho(\Omega)$ or $\widetilde{M}^{p,\lambda}(\Omega)$ of Lemma 3.5.6 can be strict.

Example 3

Let Ω be an unbounded open subset of $\mathbb{R}^2$ defined in (2.2.26) and

$$\rho \,:\, x \in \Omega \,\to\, \frac{e^{\,|x|}}{1 + e^{\,|x|}} \quad (3.5.31)$$

We remark that

$$\rho(x) = \frac{\sigma(x)}{1 + \sigma(x)}$$

where $\sigma(x) = e^{|x|} \in \mathcal{G}(\Omega)$ for any $x \in \Omega$. From Remark 3.5.1 we deduce that the function $\rho \in \mathcal{G}(\Omega) \cap L^\infty(\Omega)$.

Let us consider the function g defined in (2.2.27). We have

$$g \in M_\rho^{1,1}(\Omega) \setminus M_0^{1,1}(\Omega)\,.$$

In fact, since the functions g, ρ and ρ^{-1} belong, obviously, to the space $L^\infty(\Omega)$, one has

$$g \in L^\infty_{-1}(\Omega) \cap M^{1,1}(\Omega)$$

so from Lemma 3.5.4 we obtain that $g \in M_\rho^{1,1}(\Omega)$.

We want to show now that $g \notin M_0^{1,1}(\Omega)$. Using the estimate (2.2.28) we have

$$\sup_{\tau \in]0,1]} (\tau^{-1} ||g||_{L^1(\Omega(x,\tau))}) \approx \sup_{\tau \in]0,1]} \tau = 1\,. \tag{3.5.32}$$

Hence

$$\lim_{|x| \to +\infty} \left(\sup_{\tau \in]0,1]} \tau^{-1} \|g\|_{L^1(\Omega(x,\tau))} \right) = 1\,. \tag{3.5.33}$$

From (3.2.14) of Lemma 3.2.3 we can deduce that $g \notin M_0^{1,1}(\Omega)$.

Example 4

Let Ω be an unbounded open subset of $\mathbb{R}^2$ defined in (2.2.26) and

$$\rho\,:\, x \in \Omega \to (1 + |x|^2)^{-1}$$

Let us consider the function g defined in (2.2.27). We have

$$g \in \widetilde{M}^{1,1}(\Omega) \setminus M_\rho^{1,1}(\Omega)\,.$$

In fact, we remark that, obviously $\rho \in \mathcal{G}(\Omega) \cap L^\infty(\Omega)$ and

$$\lim_{|x|\to+\infty} \rho(x) = 0\,, \tag{3.5.34}$$

In particular, from Lemma 3.5.7 one has

$$M_0^{1,1}(\Omega) = M_\rho^{1,1}(\Omega)\,. \tag{3.5.35}$$

Since $g \in L^\infty(\Omega)$ we have that $g \in \widetilde{M}^{1,1}(\Omega)$. Using again (3.5.32), (3.5.33) and (3.2.14) of Lemma 3.2.3 we can deduce that $g \notin M_0^{1,1}(\Omega)$. From (3.5.35) we have that $g \notin M_\rho^{1,1}(\Omega)$.

Bibliography

[1] R. A. Adams, Compact imbeddings of weighted Sobolev spaces on unbounded domains, *J. Differential Equations* 9 (1971), 325–334.

[2] R. A. Adams, Sobolev Spaces, *Academic Press*, New York (1975).

[3] A. Avantaggiati, Spazi di Sobolev con peso ed alcune applicazioni, *Boll. Un. Mat. Ital. (5)* (1976), 1 – 52.

[4] A. Avantaggiati, On compact embedding theorems in weighted Sobolev spaces, *Czechoslovak Math. J.* 29 (1979), 635 – 648.

[5] A. Avantaggiati - M. Troisi, Spazi di Sobolev con peso e problemi ellittici in un angolo III, *Ann. Mat. Pura Appl. (4)* 99 (1974), 1 – 51.

[6] V. Benci - D. Fortunato, Weighted Sobolev spaces and the nonlinear Dirichlet problem in unbounded domains, *Ann. Mat. Pura Appl. (4)* 121 (1979), 319 – 336.

[7] S. Boccia - L. Caso, Interpolation inequalities in weighted Sobolev Spaces, *J. Math. Inequal. 2* (2008), 309 – 322.

[8] S. Boccia - M. Salvato - M.Transirico, A Priori Bounds for Elliptic Operators in Weighted Sobolev Spaces, submitted.

[9] P. Bolley - J. Camus, Sur une classe d'opérateurs elliptiques et dégénérés à une variable, (French) *J. Math. Pures Appl. (9)* 51 (1972), 429 – 463.

[10] A.P.Calderón - A. Zygmund, On the existence of certain singular integrals, *Acta Math.* 88 (1952), 85 – 139.

[11] A. Canale - L. Caso - P. Di Gironimo, Weighted norm inequalities on irregular domains, *Rend. Accad. Naz. Sci. XL Mem. Mat. (5)* 16 (1992), 193 – 209.

[12] L. Caso, Regularity results for elliptic problems with singular data, *J. Funct. Spaces Appl. (4)* (2006), 243 – 259.

[13] L. Caso, Properties of weak solutions of an elliptic problem in weighted spaces, *Int. J. Pure Appl. Math.* 62 (2010), 151 – 166.

[14] L.Caso - P.Cavaliere - M. Transirico, Uniqueness results for elliptic equations with *VMO* - coefficients, *Int. J. Pure. Appl. Math.* 13 (2004), 499 – 512.

[15] L.Caso - P.Cavaliere - M. Transirico, An existence result for elliptic equations with *VMO* - coefficients, *J. Math. Anal. Appl.* 325 (2007), 1095 – 1102.

[16] L. Caso - R. D'Ambrosio, Weighted spaces and weighted norm inequalities on irregular domains, submitted.

[17] L. Caso - R. D'Ambrosio - S. Monsurrò, Some remarks on spaces of Morrey type, *Abstract and Applied Analysis*, (2010), ID 242079, 22 pages.

[18] L. Caso - M. Transirico, Some remarks on a class of weight functions, *Comment. Math. Univ. Carolin.* 37 (1996), 469 – 477.

[19] L. Caso - M. Transirico, The Dirichlet problem for second order elliptic equations with singular data, *Acta Math. Hungar.* 76 (1997), 1 – 16.

[20] L. Caso - M. Transirico, The Dirichlet problem for elliptic equations in weighted Sobolev spaces, *J. Anal. Appl.* 5 (2007), 167 – 183.

[21] L. Caso - M. Transirico, A priori estimates for elliptic equations in weighted Sobolev spaces, *Math. Inequal. Appl.* 13 (2010), 655 – 666.

[22] P.Cavaliere, Spazi di tipo Morrey ed applicazioni alle equazioni ellittiche, Tesi di dottorato, XI Ciclo, Napoli, 1999.

[23] P.Cavaliere - M. Longobardi - A. Vitolo, Imbedding estimates and elliptic equations with discontinuous coefficients in unbounded domains, *Matematiche (Catania)* 51 (1996), 87–104 .

[24] Chen, Yemin, Regularity of solutions to the Dirichlet problem for degenerate elliptic equations, *Chinese Ann. Math. Ser. B*, 24 (2003), 529 – 540.

[25] F.Chiarenza - M. Franciosi, A generalization of a theorem by C. Miranda, *Ann. Mat. Pura Appl. (4)* 161 (1992), 285 – 297.

[26] F.Chiarenza - M. Frasca, A remark on a paper by C. Fefferman, *Proc. Amer. Math. Soc.* 108 (1990), 407 – 409.

[27] D. E. Edmunds - W.D. Evans, Elliptic and degenerate - elliptic operators in unbounded domains, *Ann. Scuola Norm. Sup. di Pisa* 27 (1973), 591 – 640 .

[28] I. E. Egorov, Weighted spaces of Sobolev type and degenerate elliptic equations, *Casopis Pest. Mat.* 109 (1984), 74 – 85.

[29] C. FEFFERMAN, The uncertainly principle, *Bull. Amer. Math. Soc.* 9 (1983), 129 – 206.

[30] D. FORTUNATO, Spazi di Sobolev con peso ed applicazioni ai problemi ellittici, *Rend. Accad. Sc. Fis. Mat. di Napoli (4)* 41 (1974), 245 – 289.

[31] B. HANOUZET, Espaces de Sobolev avec poids. Application an probléme de Dirichlet dans une demi espace, (French), *Rend. Sem. Mat. Univ. Padova* 46 (1971), 227 – 272.

[32] B. KAWOHL, On nonlinear mixed boundary value problems for second order elliptic differential equations on domains with corners, *Proc. Roy. Soc. Edinburgh Sect. A* 87 (1980/81), 35 – 51.

[33] I.A. KIPRIJANOV, A certain class of singular elliptic operators II, (Russian), *Sibirsk. Mat. Ž* 14 (1973), 560 – 568.

[34] Y. KOMORI - S. SHIRAI, Weighted Morrey spaces and a singular integral operator, *Math. Narchr.* 282 (2009), 219 – 231.

[35] V.A. KONDRAT'EV, Boundary value problems for elliptic equations in domains with conical or angular points, (Russian), *Trudy Moskov. Mat. Obšč.* 16 (1967), 209 – 292.

[36] N. KRUGLYAK, Calderón - Zygmund type decompositions and applications, *Proc. Estonian Acad. Sci. Phys. Math.* 55 (2006), 170 – 173.

[37] N. KRUGLYAK - E.A. KUZNETSOV, Smooth and non-smooth Calderón - Zygmund type decompositions for Morrey spaces, *J. Fourier Anal. Appl.* 11 (2005), 697 – 714.

[38] L.D. KUDRJAVCEV, Direct and inverse imbedding theorems. Applications to the solution of elliptic equations by variational methods, (Russian), *Trudy Mat. Inst. Steklov* 55 (1959), 1 – 182.

[39] A. Kufner, Weighted Sobolev spaces, *Teubner Texts in Mathematics* 31 (1980).

[40] A. Kufner - O. John - S. Fucík, Function Spaces, *Noordhoff International Publishing*, Leyden (1977).

[41] A. Kufner - B. Opic, How to define reasonably weighted Sobolev space, *Comment. Math. Univ. Carolin.* 25 (1984), 537 – 554.

[42] A. Kufner - B. Opic, Some remarks on the definition of weighted Sobolev spaces, (Russian), *Partial Differential Equations*, 119 – 126, Novosibirsk (1986).

[43] A. Kufner - A.M. Sändig, Some Applications of Weighted Sobolev Spaces, *Teubner Texte zur Mathematik*, Leipzig (1987).

[44] S. Matarasso - M. Troisi, Teoremi di compattezza in domini non limitati, *Boll. Un. Mat. Ital. (5)*, 18-B (1981), 517 – 537.

[45] V. G. Maz'ja, Sobolev spaces, *Springer - Verlag, Berlin Heidelberg New York Tokyo*, (1980).

[46] J. Necǎs , Les méthodes directes en théorie des équations elliptiques, (French), *Masson et Cie Éditeurs, Paris; Academia, Éditeurs, Prague* 1967, 351 pp.

[47] S.M. Nikol'skii, On a boundary value problem of the first kind with a strong degeneracy, *Dokl. Akad. Nauk SSSR*, Russian (1975).

[48] L. C. Piccinini, Proprietà di inclusione e interpolazione tra spazi di Morrey e loro generalizzazione, *Sc. Norm. Sup. Pisa Cl. Sci.* (1969).

[49] Yu. D. Salmanov, On certain imbedding and extension theorems for a weighted class of functions, *Anal. Math.* 19 (1993), 273 – 295.

[50] M. Schechter, Imbedding estimates involving new norms and applications, *Bull. Am. Math. Soc.* 11 (1984), 163 – 166.

[51] R. Schianchi, Spazi di Sobolev dissimetrici e con peso, *Rend. Accad. Sc. Fis. Mat. di Napoli (4)* 42 (1975), 349 – 388.

[52] S.L. Sobolev, Applications of functional analysis in mathematical physics, *Americ. Math. Soc.*, Providence (1963).

[53] E. M. Stein, Singular Integrals and Differentiability Properties of Functions, *Princenton Univ. Press*, Princenton, New Jersey (1970).

[54] E. M. Stein, Harmonic Analysis: Real variable methods, Orthogonality and Oscillatory Integrals, *Princenton Univ. Press*, Princenton, New Jersey (1993).

[55] M. Transirico - M. Troisi , Equazioni ellittiche del secondo ordine di tipo non variazionale in aperti non limitati, *Ann. Mat. Pura Appl. (4)* 152 (1988), 209 – 226.

[56] M. Transirico - M. Troisi, Sul problema di Dirichlet per le equazioni ellittiche a coefficienti discontinui, *Note Mat.* 7 (1987), 271 – 309 (1988).

[57] M. Transirico - M. Troisi , Equazioni ellittiche del secondo ordine a coefficienti discontinui e di tipo variazionale in aperti non limitati, *Boll. Un. Mat. Ital. B (7)* 2 (1988), 385 – 389.

[58] M. Transirico - M. Troisi - A. Vitolo, Spaces of Morrey type and elliptic equations in divergence form on unbounded domains, *Boll. Un. Mat. Ital.B (7)* 9 (1995), 153 – 174.

[59] M. Troisi, Teoremi di inclusione negli spazi di Sobolev con peso, *Ric. di Mat.* 18 (1969), 49 – 74.

[60] M. Troisi, Su una classe di funzioni peso, *Rend. Accad. Naz. Sci. XL Mem. Mat. (5)* 10 (1986), 141 – 152.

[61] M. Troisi, Su una classe di spazi di Sobolev con peso, *Rend. Accad. Naz. Sci. XL Mem. Mat. (5)* 10 (1986), 177 – 189.

[62] C. Vitanza - P. Zamboni, Regularity and existence results for degenerate elliptic operators, *Variational analysis and applications*, 1129 – 1140, *Nonconvex Optim. Appl.* 79 Springer (2005).

[63] W. P. Ziemer, Weakly Differentiable Functions, *Springer*, Berlin (1989).

[64] C. T. Zorko, Morrey space, *Proc. Amer. Math. Soc.*, 98 (1986), 586 – 592.