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XII Ciclo

# Bornological Convergences on Local Proximity Spaces and $\omega_\mu$–metric Spaces

Tesi di Dottorato in Geometria

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Anno Accademico 2014/2015

*”Those who wonder*
*discover that this is*
*in itself a wonder.”*
*M.C. Escher*

# Contents

# 1 Introduction

The main topics of this thesis are *local proximity spaces* jointly with some *bornological convergences* naturally related to them, and $\omega_\mu$*–metric spaces*, in particular those which are *Atsuji spaces*, jointly with their hyperstructures. The interest in these was born looking at extensions of topological spaces preserving some structure. In fact, studying *topological groups of homeomorphisms* for the master thesis, we recognized the importance of constructing local $T_2$ compactifications and at the same time became familiar with non-standard analysis. This interest was then generalized. On one side, we started to look at *local proximity spaces* because of the one-to-one correspondence with local $T_2$ compactifications, [27]. The study of hypertopologies on them led us to consider a special generalization of metric spaces, $\omega_\mu$–metric spaces, [28], which are non-Archimedean when $\omega_\mu$ is different from $\omega_0$. On the other side our attention was captured by the metric spaces, called *Atsuji spaces*, for which any continuity is uniform, because of their relationship with metric complete and metric compact spaces, their nice structure and their relevant properties, [29].

Local proximities spaces carry with them two particular features: proximity [48] and boundedness [37], [40]. Proximities allow us to deal with a concept of nearness even though not providing a metric. Proximity spaces are located between topological and metric spaces. Boundedness is a natural generalization of the metric boundedness, that is a family closed under finite unions and by taking subsets, known in literature as a bornology when it is a cover. A *local proximity space* $(X, \delta, \mathscr{B})$ consists of a non-empty set $X$ together with a proximity $\delta$ on $X$ and a boundedness $\mathscr{B}$ in $X$ that are subject to suitable compatibility conditions.

In general, the global proximity $\delta$ is weak but locally is fine. Locally it is an Efremovič proximity . Essentially, a proximity is Efremovič when two sets are far if and only if can be separated by a real-valued uniformly continuous function. So, when trying to *refer macroscopic phenomena to local structures*, local proximity spaces appear as a very attractive option. For that, jointly with Prof. A. Di Concilio, in a first step we displayed a uniform procedure as an exhaustive method of generating all local proximity spaces. A related result is:

**Theorem 1.1.** *Let $X$ be a Tychonoff space, $\mathscr{U}$ a uniformity compatible with $X$, $\delta_{\mathscr{U}}$ its natural proximity and $\mathscr{B}$ a bornology of $X$. Then, the triple $(X, \delta_{\mathscr{U},\mathscr{B}}, \mathscr{B})$ is a local proximity space on the space $X$, for which the local proximity $\delta_{\mathscr{U},\mathscr{B}}$ agrees with $\delta_{\mathscr{U}}$ when restricted to any bounded set, if and only if $\mathscr{B}$ is stable under small enlargements.*

Recall that a family $\mathscr{B}$ is *stable under small enlargements with respect to a uniformity* $\mathscr{U}$ if for each $B \in \mathscr{B}$ there is $U \in \mathscr{U}$ so that $U[B]$, the $U-$enlargement of $B$, belongs again to $\mathscr{B}$.

After denoting as $\delta_{\mathscr{U}}$ the natural proximity associated with $\mathscr{U}$, we introduced over $\mathscr{P}(X)$ the following proximity $\delta_{\mathscr{U},\mathscr{B}}$ :

$A, B \subset X$, $A\ \delta_{\mathscr{U},\mathscr{B}}\ B$ if and only if there exists $C \in \mathscr{B}$ such that $A \cap C\ \delta_{\mathscr{U}}\ B \cap C$.

After that, we looked at suitable topologies for the hyperspace of a local proximity space. In contrast with the proximity case, in which there is no canonical way of equipping the hyperspaces with a uniformity, the same with a proximity, the local proximity case is simpler.

Apparently, at the beginning, we have two natural different ways to topologize the hyperspace $CL(X)$ of all closed non-empty subsets of $X$. A first option calls upon the dense embedding of $X$ in the natural $T_2$ local compactification $\ell(X)$, while a second one stems from joining together proximity and bornology in a hit and far-miss topology. We showed they match in just one case. In a first choice, we identified the hyperspace $CL(X)$ of $X$ with the subspace $\{ Cl_{\ell(X)}A : A \in CL(X) \}$ of the hyperspace $CL(\ell(X))$ of $\ell(X)$ when carrying the Fell topology.

We defined the *local Fell topology,* $\tau_{loc,F}$, by saying that:

If $\{A_\lambda\}_{\lambda\in\Lambda}$ stands for a net in $CL(X)$, $A\in CL(X)$, and $\{Cl_{\ell(X)}A_\lambda\}_{\lambda\in\Lambda}$, $Cl_{\ell(X)}A$ are their closures in $\ell(X)$, then:

$$\{A_\lambda\} \overset{\tau_{loc,F}}{\longrightarrow} A \quad \text{if and only if} \quad \{Cl_{\ell(X)}A_\lambda\} \overset{\tau_F}{\longrightarrow} Cl_{\ell(X)}A$$

where $\tau_F$ denotes the Fell topology on $CL(\ell(X))$.

Then, coming up as a natural mixture, we recasted the hit and far-miss topology associated with the proximity $\delta$ and the bornology $\mathscr{B}$ as the topology induced by the weak uniformity generated by infimal value functionals of the real functions on $X$ which preserve proximity and boundedness and, moreover, have a bounded support. In particular we obtained the following matching result.

**Theorem 1.2.** *Let $(X,\delta,\mathscr{B})$ be a local proximity space. The local Fell topology $\tau_{loc,F}$ on $CL(X)$ agrees with the hit and far miss topology $\tau_{\delta,\mathscr{B}}$ associated with the proximity $\delta$ and the bornology $\mathscr{B}$.*

Finally, bringing up the underlying uniform characters, we connected local proximity spaces to bornological convergences. Uniform bornological convergences are a mixture of uniformity and bornology allowing an *interplay between large and small*, [14, 19].

Let $(X,\mathscr{U})$ be a uniform space and $\mathscr{B}$ a family of subsets of $X$.

The *upper uniform bornological convergence associated with $\mathscr{U}$ and $\mathscr{B}$*, which we denote as $\mathscr{U}^+_{\mathscr{B}}$, is defined as follows:

$\{A_\lambda\} \overset{\mathscr{U}^+_{\mathscr{B}}}{\to} A$ iff for each $U\in\mathscr{U}$ and each $B\in\mathscr{B}$, then $A_\lambda\cap B \subset U[A]$, residually.

Next, the *lower uniform bornological convergence associated with $\mathscr{U}$ and $\mathscr{B}$*, which we denote as $\mathscr{U}^-_{\mathscr{B}}$, is defined as follows:

$\{A_\lambda\} \overset{\mathscr{U}^-_{\mathscr{B}}}{\to} A$ iff for each $U\in\mathscr{U}$ and each $B\in\mathscr{B}$, then $A\cap B \subset U[A_\lambda]$, residually.

The *two-sided uniform bornological convergence associated to $\mathscr{U}$ and $\mathscr{B}$*, in short, *uniform bornological convergence*, which we denote as $\mathscr{U}_{\mathscr{B}}$, is the join of the upper and lower uniform bornological convergences related to $\mathscr{U}$ and $\mathscr{B}$.

In the seminal paper [42] Lechicki, Levi and Spakowski studied a family of uniform bornological convergences in the hyperspace of a metric space, including the Attouch-Wets, Fell, and Hausdorff metric topologies. We proved the uniform counterpart of metric case by essentially miming the proofs performed in that by using essentially only uniform features of a metric [19].

Furthermore, we obtained a result that makes the local Fell topology on the hyperspace $CL(X)$ of a local proximity space $(X, \delta, \mathscr{B})$ as the most natural one we can associate with it.

**Theorem 1.3.** *Let $(X, \delta, \mathscr{B})$ a local proximity space and $\mathscr{U}_{\mathcal{A}}^*$ the relative Alexandroff uniformity. Then, the local Fell topology, $\tau_{loc,F}$ is the topology of the two-sided uniform bornological convergence associated with $\mathscr{U}_{\mathcal{A}}^*$ and $\mathscr{B}$.*

In the light of the previous local proximity results, we looked for necessary and sufficient conditions of uniform nature for two different uniform bornological convergences to match. Since the metric case is essentially based on two facts, the former: any metrizable uniformity is the finest one in its proximity class, or, in other words, is total; the latter: the bornology of metrically bounded sets is stable under small metric enlargements, we identified the key properties on one hand for uniformities to be total when localized on bounded sets and, on the other hand, for bornologies to be stable under small enlargements. We emphasize that any proximity function from a total uniform space $(X, \mathscr{U})$ towards any proximity space is uniformly continuous.

The described key properties led us to focus on a special class of uniformities: those with a linearly ordered base. A uniformity $\mathscr{U}$ has a linearly ordered base if it admits a base $\{U_\alpha : \alpha \in A\}$ of diagonal neighbourhoods, where $\alpha$ runs over an ordered set $(A, <)$ and $U_\alpha$ contains $U_\beta$, when $\alpha < \beta$. As proved in [3], every uniformity with a linearly ordered base is the finest one in its proximity class, and not only that, any uniform subspace admits in its turn a linearly ordered base. In the pioneering paper [61], Sikorski proved that any uniform space $(X, \mathscr{U})$ with a linearly ordered base carries a distance $\rho$ valued in an ordered abelian additive group $G$ satisfying the usual formal properties of a real metric having as a base the starting one. We recall that every uniform space $X$ with a linearly ordered base of power $\aleph_\mu$ is $\omega_\mu$–metrizable, where $\omega_\mu$ is the least ordinal of cardinality $\aleph_\mu$. That is: there is a linearly ordered abelian group $G$ which has a decreasing $\omega_\mu$–sequence convergent to $0$ in the order topology and a distance function $\rho : X \times X \rightarrow G$ sharing the usual properties with the real metrics.

Carrying a richer structure than usual uniform spaces, they share some properties with the usual metric spaces, but for some other aspects they reveal themselves really far from the others. As an example we can consider the generalized versions of compactness, completeness and total boundedness. In this frame it is not always true that an $\omega_\mu$–totally bounded and complete space is $\omega_\mu$–compact.

Anyway, it appears very natural to introduce the Attouch-Wets convergence on $CL(X)$, relative to an $\omega_\mu$–metric space $(X, \rho)$ and the collection of the $\rho$–bounded sets definable in the usual way [12, 17]. So, and among others, we achieved the following two issues in the $\omega_\mu$–metric setting. The former: the Attouch-Wets topologies associated with two $\omega_\mu$-metrics on a same space agree if and only if those ones have the same bounded sets and are proximally equivalent on any bounded set. The latter: the Attouch-Wets topologies associated with two $\omega_\mu$-metrics on a same space agree if and only if their hit and bounded far-miss topologies agree.

**Theorem 1.4.** *Let $X$ stand for an $\omega_\mu$–metrizable space, $\rho_1, \rho_2$ two compatible $\omega_\mu$–metrics on $X$ and $\mathscr{B}_1$, $\mathscr{B}_2$ the bornologies of their bounded sets, respectively. Then the following are equivalent:*

*$a)$ The Attouch-Wets topologies relative to $\rho_1$ and $\rho_2$ match.*

*$b)$ $\mathscr{B}_1$ and $\mathscr{B}_2$ agree and $\rho_1, \rho_2$ are proximally equivalent on any bounded set.*

*$c)$ The hit and far-miss topology associated with the natural proximity of $\rho_1$ and $\mathscr{B}_1$ agrees with the hit and far-miss topology associated with the natural proximity of $\rho_2$ and $\mathscr{B}_2$.*

Furthermore, in relation with $\omega_\mu$–metric spaces, we looked at generalizations of well known hyperspace convergences, as Hausdorff and Kuratowski convergences obtaining analogue results with respect to the standard case, [28].

Finally, we dealt with *Atsuji spaces*. The interest in this topic follows the same line of that in Local Proximity Spaces. In fact we were interested in the problem of constructing a dense extension $Y$ of a given topological space $X$, which is *Atsuji* and in which $X$ is topologically embedded. When such an extension there exists, we say that the space $X$ is *Atsuji extendable*. *Atsuji spaces* play an important role above all because they allow us to deal with a very nice structure when we concentrate on the most significant part of the space, that is the derived set. Moreover, we know that each continuous function between metric or uniform spaces is uniformly continuous on compact sets. It is possible to have an analogous property on a larger class of topological spaces, Atsuji spaces. They are situated between complete metric spaces and compact ones.

We proved a necessary and sufficient condition for a metrizable space $X$ to be *Atsuji extendable*.

**Theorem 1.5.** *Let $X$ be a metrizable space. $X$ is* Atsuji extendable *if and only if $X'$ is separable.*

Moreover we looked at conditions under which a continuous function $f : X \to \mathbb{R}$ can be continuously extended to the *Atsuji extension* $Y$ of $X$.
Uc metric spaces admit a very long list of equivalent formulations. We extended many of these to the class of $\omega_\mu-$metric spaces.
The results are contained in [29].

During the last three months there was the opportunity to work with Professor J.F. Peters as Research Assistant at the University of Manitoba (Canada). Our research involved the study of more general proximities leading to a kind of *strong farness*, [52]. Strong proximities are associated with Lodato proximities and the Efremovič property.We say that $A$ and $B$ are $\delta-$*strongly far*, where $\delta$ is a Lodato proximity, and we write $\underset{\Join}{\not\delta}$ if and only if $A \not\delta B$ and there exists a subset $C$ of $X$ such that $A \not\delta X \setminus C$ and $C \not\delta B$, that is the Efremovič property holds on $A$ and $B$. Related to this idea we defined also a new concept of *strong nearness*, [53]. Starting by these new kinds of proximities we introduced also new kinds of hit-and-miss hypertopologies, concepts of strongly proximal continuity and strong connectedness. Finally we looked at some applications that in our opinion might reveal interesting.

# 2 Proximity Spaces

The idea of nearness was first formulated mathematically by F. Riesz, who communicated a paper in 1908 on the nearness of two sets [57], initiating a field of study later known as proximity spaces [25, 48, 49, 64]. The concept of nearness is easy to understand, in fact it is part of every day life and so, as Joseph Loius Lagrange said, it can be explained to "the first person one meets in the street". Furthermore Mathematics can describe it in a rigorous way.

Proximity spaces are located between topological spaces and metric spaces. In fact they have an associated topological structure and they allow us to talk about nearness even if without assigning numbers, so without using distances.

We can deal with nearness on several levels. We can start from nearness between a point and a set, or we can start from nearness between pair of sets, or also from nearness of a number of families. We will analyse only the first two approaches and we will see that in the first case we can obtain *topological spaces*, while in the second one we obtain *proximity spaces*.

## 2.1 Topological spaces by nearness relations

We know that a topological space can be obtained by the *Kuratowski closure operator* that is defined by the following well-known axioms:

Let $X$ be a nonempty set and let $B, C$ be arbitrary subsets of $X$. A closure operator *cl* is a self-map on the power set of $X$ which satisfies the following:

1. $cl\varnothing = \varnothing$
2. $B \subset clB$

3. $cl(B \cup C) = clB \cup clC$

4. $cl(clB) = clB$

Now we want to rewrite the previous axioms by using a nearness relation between points and sets. So we define $x\delta B \Leftrightarrow x \in clB$ and we say that *the point $x$ is near the subset $B$*. Then we obtain:

T1) $x\delta B \Rightarrow B \neq \varnothing$

T2) $\{x\} \cap B \neq \varnothing \Rightarrow x\delta B$

T3) $x\delta(B \cup C) \Leftrightarrow x\delta B$ or $x\delta C$

T4) $x\delta B$ and $b\delta C$ for each $b \in B \Rightarrow x\delta C$

If we have such a relation, we have a topological space defined by closed sets. In fact a subset $B$ is closed if and only if it coincides with its closure $clB = \{x \in X : x\delta B\}$.

It is particularly fashinating to see how continuity is formulated by using this relation. If $X$ and $Y$ are two sets with closure relations $\delta, \delta'$ respectively, it is easily shown that a function $f : X \to Y$ is continuous if and only if

$$\forall x \in X, B \subset X, \ \ x\delta B \text{ in } X \Rightarrow f(x)\delta' f(B) \text{ in } Y$$

.

It is interesting to notice that this formulation is a direct formulation, in the sense that from conditions on the domain we obtain conditions on the range. Instead, in General Topology, we usually adopt inverse definitions, such as, for example, the one involving the inverse image of any open set in the range.
So, by these arguments, we have that:

- topological spaces are based on ”nearness between points and sets”
- continuous functions are those that preserve this kind of relation

## 2.2 Proximity relations

Now we want to deal with nearness between two sets. To do this we generalize the axioms obtained before by replacing the generic point $x$ with a subset $A$. Furthermore it is added a simmetry axiom that is not needed in the Kuratowski closure axioms.

**Definition 2.1.** *Let $X$ be a nonempty set. A* ***Lodato proximity*** *$\delta$ is a relation on the power set of $X$ that satisfies the following conditions.*

*For all subset $A, B, C$ of $X$:*

*P0)* $A\delta B \Rightarrow B\delta A$

*P1)* $A\delta B \Rightarrow A \neq \varnothing$ *and* $B \neq \varnothing$

*P2)* $A \cap B \neq \varnothing \Rightarrow A\delta B$

*P3)* $A\delta(B \cup C) \Leftrightarrow A\delta B$ *or* $A\delta C$

*P4)* $A\delta B$ *and* $\{b\}\delta C$ *for each* $b \in B \;\Rightarrow A\delta C$

*Further $\delta$ is separated if*

*P5)* $\{x\}\delta\{y\} \Rightarrow x = y$

When we write $A\delta B$, we read ”$A$ is near to $B$”, while when we write $A \not\delta B$ we read ”$A$ is far from $B$”. A basic proximity is one that satisfies $P0) - P3)$. *Lodato proximity* or *LO-proximity* is one of the simplest proximities. As we have done before, we can associate a topology to the space $(X, \delta)$ by considering as closed sets the ones that coincide with their own closure, where for a subset $A$ we have

$$clA = \{x \in X : x\delta A\}$$

Beacause of the symmetry axiom $(P0)$ we have that the topology $\tau(\delta)$ satisfies the following property

$$(*) \; x \in cl\{y\} \Leftrightarrow y \in cl\{x\}.$$

Spaces that satisfy this property are called *weakly regular*. Now it is natural to ask if a topological space $(X,\tau)$ that is weakly regular can be endowed with a compatible LO-proximity, so with a LO-proximity whose associated topology is the starting one. In relation to it, we have the following theorem.

**Theorem 2.1.** *Every topological space that is weakly regular has a compatible LO-proximity given by:*

$$A\delta_0 B \Leftrightarrow clA \cap clB \neq \emptyset.$$

*(Fine LO-proximity $\delta_0$)*

#### 2.2.1 Other examples

• Given a $T_1$ space we can consider the following compatible LO-proximity:

$$A\delta_A B \Leftrightarrow A\delta_0 B \text{ or both } clA \text{ and } clB \text{ are non-compact.}$$

This is called **Alexandroff LO-proximity**

• In a Tychonoff space we can consider the **Functionally indistinguishable or Čech proximity** $\delta_F$:

$$A \not\delta_F B \Leftrightarrow \text{there is a continuous function } f: X \to [0,1] : f(A) = 0,\ f(B) = 1.$$

• Given a metric space $(X,d)$ we can define a compatible LO-proximity by using the *gap* between two sets $A, B$:

$$d(A,B) = inf\{d(a,b) : a \in A, b \in B\}.$$

The gap is equal to infinity if $A$ or $B$ are empty.

By this we can define the metric proximity that is a compatible LO-proximity:

$$A\delta_m B \Leftrightarrow d(A,B) = 0.$$

## 2.3 Strong inclusions

We now introduce a different approach to proximities: **proximal neighbourhoods**.

We say that a subset $B$ is a *$p$-neighbourhood* or *$\delta$-neighbourhood* of a subset $A$ if and only if $A$ is remote from the complement of $B$, that is $A \not\delta X \setminus B$. In this case we say that $A$ is strongly contained in $B$, and we write $A \ll_\delta B$. This definition produces a binary relation over $\mathscr{P}(X)$ and we refer to it as the *strong-inclusion induced by the proximity $\delta$*.

It is possible to express the axioms for a basic proximity by using strong inclusions in the following way.

For all subsets $A, B, C$ of $X$:

S0) $A \ll B \Leftrightarrow X \setminus B \ll X \setminus A$,

S1) $X \ll X$,

S2) $A \ll B \Rightarrow A \subset B$,

S3) $A \subset B \ll C \subset D \Rightarrow A \ll D$,

S4) $A \ll B,\ A \ll C \Rightarrow A \ll (B \cap C)$.

Moreover the LO-axiom is expressed by:

S5) $A \ll B \Rightarrow \forall C, \text{either} A \ll C$ or $\exists x \in X \setminus C : x \ll B$.

The relations $\delta$ and $\ll_\delta$ are interdefinable.

Observe that, in a metric space $(X, d)$, for each subset $A$ we can consider its $\epsilon$-*collar* $S_\epsilon[A] = \{x \in X : d(x, A) < \epsilon\}$, $\epsilon$ being a positive real number. In this case a subset $B$ of $X$ is a $\delta_d$–neighbourhood of $A$ if and only if $B$ contains an $\epsilon$–collar of $A$.

## 2.4 Efremovič proximities

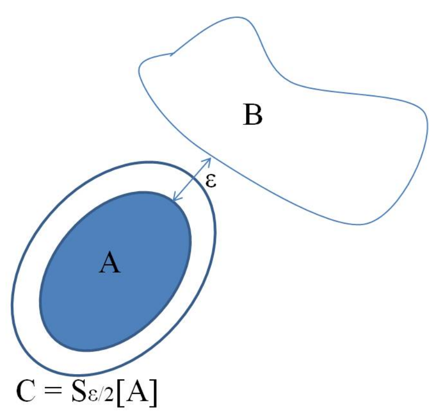
If we observe the metric proximity, we notice a significant betweenness property. This property provides a motivation to introduce a stronger kind of proximities, **Efremovič proximities** or **EF-proximities** . Usually, when one talks about proximities, refers to Efremovič proximities.

Recall that the metric proximity is defined by $A \not\delta B \Leftrightarrow d(A, B) = \epsilon > 0$. So, if we are in this situation, we can consider the set $C = S_{\frac{\epsilon}{2}}[A]$ and notice that $A \not\delta (X \setminus C)$ and $B \not\delta C$ being $d(A, C) \geq \frac{\epsilon}{2}$ and $d(X \setminus C, B) \geq \frac{\epsilon}{2}$. Hence,

$$A \not\delta B \Rightarrow A \not\delta (X \setminus C) \text{ and } C \not\delta B,$$

or using strong inclusions

$$A \ll_d X \setminus B \Rightarrow A \ll_d C = S_{\frac{\epsilon}{2}}[A] \ll_d S_\epsilon[A] \subset X \setminus B.$$



This betweenness property is called **Efremovič property**.

**Definition 2.2.** *A basic proximity $\delta$ on a nonempty set $X$ is called an **EF-proximity** iff it furthermore satisfies*

$$A \not\delta B \Rightarrow \exists C : A \not\delta (X \setminus C) \text{ and } C \not\delta B.$$

Observe that Efremovič proximities are stronger than LO-proximities, that means that every EF-proximity is a LO-proximity.

### 2.4.1 Examples

Some examples of compatible EF-proximities are the following ones.

- *Metric proximity* $\delta_d$ in a metric space $(X, d)$.
- The *Fine proximity* $\delta_0$ defined in Thm. 2.1 on a normal space is an EF-proximity. It is separated if and only if the space $X$ is $T_0$. Every $T_2$ compact space, equipped with the relative fine proximity, is a separated EF-proximity space.
- The *functionally indistinguishable proximity* (page 11) is EF on a completely regular space and it is a separated proximity if and only if the space $X$ is $T_1$. The functionally indistinguishable proximity $\delta_F$ is the finest compatible EF-proximity on a Tychonoff space. Furthermore, by the Urysohn lemma, it is the fine proximity if and only if the space $X$ is normal.
- The *Alexandroff proximity* (page 11) on a $T_2$ locally compact space is a separated EF-proximity. In a locally compact non-compact Hausdorff space, the Alexandroff proximity $\delta_A$ is the coarsest compatible EF-proximity.

### 2.4.2 Results

Now we want to recall some useful results.

**Theorem 2.2.** *Let $(X, \delta)$ be an EF-proximity space. If $A \not\delta B$, then there exists a continuous function $f : X \to [0, 1]$, where the range is equipped with the Euclidean metric, such that $f(A) = 0$ and $f(B) = 1$.*

**Theorem 2.3.** *A topological space is EF-proximisable if and only if it is completely regular.*

**Theorem 2.4.** *A compact Hausdorff space admits a unique proximity, given by the elementary proximity $A\delta_0 B$ iff $cl(A) \cap cl(B) \neq \emptyset$.*

## 2.5 Proximity functions

Let $(X, \delta)$ and $(Y, \delta')$ be two EF-proximity spaces. A function $f$ from $X$ to $Y$ is a *proximity function*, or a *proximally continuous function* iff

$$A\delta B \Rightarrow f(A)\delta' f(B).$$

A function $f$ is a *proximal isomorphism* iff it is a bijection and both $f$ and $f^{-1}$ are proximally continuous.

Obviously, proximal continuity, which preserves nearness between sets is stronger than continuity, which preserves nearness between points and sets.

$$\text{proximal continuity} \quad \Rightarrow \quad \text{continuity}$$

The converse is in general false, as we can see by the following example.

**Example 2.1.** *Let $(\mathbb{R}, \tau_e)$ be the space of real numbers endowed with the Euclidean topology. We can identify two compatible proximities: $\delta_0$, which is the fine proximity, and $\delta_m$, the metric proximity associated to the Euclidean metric. Consider now the sets $A = \{n : n \in \mathbb{N}\}$ and $B = \{n - \frac{1}{n} : n \in \mathbb{N}\}$. Then focus on the identity* *map* $i : (\mathbb{R}, \delta_m) \to (\mathbb{R}, \delta_0)$. *We can observe that this map is continuous, in fact the topology is the same, but it is not proximally continous because* $A\delta_m B$, *but* $A \not\delta_0 B$.

## 2.6 Proximities and $T_2$- compactifications

Now we want to present a one to one correspondence between the compatible proximities on a Tychonoff space and the $T_2$-compactifications of such a space.
Consider a Tychonoff space $X$ and $\gamma(X)$, a $T_2$-compactification of $X$. By Thm. 2.4 we know that there is a unique compatible proximity on $\gamma(X)$, $\delta_0$. We can restrict this proximity to the base space $X$ and we obtain:
$A, B \subset X$

$$A\delta_\gamma B \Leftrightarrow cl_{\gamma(X)}A \cap cl_{\gamma(X)}B \neq \emptyset.$$

It can be proved that $\delta_\gamma$ is a compatible EF-proximity on $X$. So in this way we have uniquely associated a proximity to each $T_2$-compactification of $X$.
Conversely, if we start from a separated EF-proximity, we can obtain a $T_2$-compactification in the following way. Consider the family $\mathscr{F} = \{f$ s.t. $f : X \to [0,1], \quad f\ \delta -$ proximal continuous$\}$, where the range $[0,1]$ is endowed with the euclidean metric proximity.
Denote the effective range of each such an $f$ by $[a_f, b_f]$. By the Tychonoff theorem we know that $\prod_{f\in\mathscr{F}}[a_f, b_f]$ is compact.
If we focus on the following function

$$x \in X \xrightarrow{e} \{f(x)\}_{f\in\mathscr{F}} \in \prod_{f\in\mathscr{F}}[a_f, b_f]$$

we can prove that it is an embedding. So $\overline{e(X)} \subset \prod_{f\in\mathscr{F}}[a_f, b_f]$ is a $T_2$-compactification of $X$. Furthermore the proximity associated to this compactification coincides with the starting one.

$$\text{proximities} \quad \leftrightarrow \quad T_2 - \text{compactifications}$$

## 2.7 Uniformities

Uniform spaces are the natural framework in which notions of uniform continuity, uniform convergence and the like are defined. They represent also an example of EF-proximity spaces. We can look at them as a framework providing generalizations of some properties holding for metric spaces.
There are different approaches to define uniform spaces. Here we will present the diagonal one.

**Definition 2.3.** *A **diagonal uniformity** on a set $X$ is a collection $\mathscr{U}$ of subsets of $X \times X$, called diagonal neighbourhoods or surroundings, which satisfy the following axioms:*

1. $U \in \mathscr{U} \Rightarrow U \supset \Delta = \{(x,x) : x \in X\}$
2. $U, V \in \mathscr{U} \Rightarrow U \cap V \in \mathscr{U}$
3. $U \in \mathscr{U}, U \subset V \Rightarrow V \in \mathscr{U}$
4. $U \in \mathscr{U} \Rightarrow \exists V \in \mathscr{U} : V \circ V \subset U$
5. $U \in \mathscr{U} \Rightarrow \exists V \in \mathscr{U} : V^{-1} = \{(x,y) : (y,x) \in V\} \subset U$

When $X$ carries such a structure, we call $(X, \mathscr{U})$ a *uniform space*. The uniform space $(X, \mathscr{U})$ is called *separated* if and only if $\bigcap\{U : U \in \mathscr{U}\} = \Delta$.
Observe that by properties $(1) - (3)$ we have that $\mathscr{U}$ is a filter on $\Delta$.

A subcollection $\mathscr{B}$ of $\mathscr{U}$ is a *base* for the uniformity $\mathscr{U}$ if and only if for each $U \in \mathscr{U}$ there exists $B \in \mathscr{B}$ such that $B \subseteq U$.

### Examples

- If $(X, d)$ is a metric space, the *metric uniformity naturally associated with* $d$, usually denoted by $\mathscr{U}_d$, admits as basic diagonal nhds the sets $V_\epsilon$, $\epsilon > 0$, where

$$V_\epsilon = \{(x,y) \in X \times X : d(x,y) < \epsilon\}$$

Observe that, if we take $d$ and $2d$, we obtain the same uniformity. So different metrics may give rise to the same uniform structure. By this we can see that a uniformity represents less structure than a metric.

- Given any set $X$, the collection $\mathscr{U}$ of all subsets of $X \times X$ which contain $\Delta$ is a uniformity on $X$ and it is called the *discrete uniformity*. A base for this uniformity is represented by the single set $\Delta$.

While a uniformity represents less structure on a set than a metric, it represents more structure than a topology. In fact, as we will show, every uniformity gives rise to a topology, but different uniformities may generate the same topology.

**Definition 2.4.** *For $x \in X$ and $U \in \mathscr{U}$, we define*

$$U[x] = \{y \in X : (x, y) \in U\}$$

*and extending it for a subset $A$ of $X$*

$$U[A] = \bigcup_{x \in A} U[x] = \{y \in X : (x, y) \in U \text{ for some } x \in A\}$$

**Theorem 2.5.** *For each $x \in X$, the collection $\mathscr{U}_x = \{U[x] : U \in \mathscr{U}\}$ forms a nhbd base at $x$, making $X$ a topological space. The same topology is produced if any base is used in place of $\mathscr{U}$. The topology is Hausdorff iff $(X, \mathscr{U})$ is separated.*

This topology is called *uniform topology $\tau_{\mathscr{U}}$ generated by $\mathscr{U}$*. If we have a topological space $(X, \tau)$ that can be endowed with a uniform structure generating the starting topology, then the space is said *uniformizable* topological space. The following result can be proved.

**Theorem 2.6.** *A topological space is uniformizable if and only if it is completely regular.*

Instead, which conditions do we need to consider if we want to know which uniformities can be obtained by metrics?

**Theorem 2.7.** *A uniformity is metrizable iff it is separating and has a countable base.*

Notice that a uniformity is metrizable if we can obtain it by the $\epsilon-$nhbds of some metric. Furthermore, if the uniformity is metrizable, so is the topology it generates. But the converse is not true: metrizability of the associated topology does not imply metrizability of the uniformity.

### 2.7.1 Interplay between proximities and uniformities

Let $(X, \mathscr{U})$ stand for a uniform space. We can define a proximity associated to $\mathscr{U}$ by setting

$$A \not\delta_{\mathscr{U}} B \Leftrightarrow \exists U \in \mathscr{U} : U[A] \cap B = \varnothing.$$

The proximity $\delta_{\mathscr{U}}$ is EF and it is separating if and only if the uniformity $\mathscr{U}$ is separating.
A uniformity and the associated proximity both induce on the base space the same topology.
Notice that a uniformity generates a proximity in a very natural way, as we have seen, but in general different uniformities may give rise to the same proximity. We can obtain a one to one correspondence if we consider *totally bounded uniformities*.

**Definition 2.5.** *A uniformity $\mathscr{U}$ is said to be totally bounded if, for each diagonal nhbd $U$ in $\mathscr{U}$, there exists a finite number of points $x_1, ..., x_n$ in $X$ such that $X = U[x_1] \cup ... \cup U[x_n]$.*

Whenever the underlying topology $\tau_{\mathscr{U}}$ is compact, then the unformity $\mathscr{U}$ is totally bounded.
It can be proved that, given a proximity $\delta$, there is a unique totally bounded unifor-

mity asoociated to $\delta$. Hence, by the observations of page 17, we have the following correspondences

EF-proximities $\leftrightarrow$ totally bounded uniformities $\leftrightarrow$ $T_2$ – compactifications

#### 2.7.2 Uniformly continuous functions

We know that a function $f : (M, d) \to (N, \rho)$ beteween metric spaces is said *uniformly continuous* iff for each $\epsilon > 0$ there exists $\delta > 0$ such that whenever $d(x, y) < \delta$, then $\rho(f(x), f(y)) < \epsilon$.
Differently from continuity, in this case we consider a property uniformly applied to pairs of points, without regard to their location.
Now we want to generalize this situation for uniform spaces.

**Definition 2.6.** *Let $(X, \mathscr{U})$ and $(Y, \mathscr{V})$ be uniform spaces. A function $f : (X, \mathscr{U}) \to (Y, \mathscr{V})$ is called* ***uniformly continuous*** *iff*

$$\forall V \in \mathscr{V}\ \exists U \in \mathscr{U} : (x, y) \in U \Rightarrow (f(x), f(y)) \in V.$$

*If $f$ is one-to-one and both $f$ and $f^{-1}$ are uniformly continuous, then we say that $f$ is a* ***uniform isomorphism****.*

We know that to each uniform space it is associated a proximity space. But which is the relation between uniform continuity and proximal continuity?

**Proposition 2.1.** *Every uniformly continuous function $f : (X, \mathscr{U}) \to (Y, \mathscr{V})$ is also a proximity function between the natural underlying EF-proximity spaces.*

By this result we have that proximity invariants of uniform spaces are also uniform invariants. So in general we have

$$\text{uniform continuity} \quad \Rightarrow \quad \text{proximal continuity.}$$

Nevertheless, in particular cases, the converse holds too.

**Proposition 2.2.** *If $\mathscr{V}$ is a totally bounded uniformity, then a function $f : (X, \mathscr{U}) \to (Y, \mathscr{V})$ is uniformly continuous if and only if it is a proximity function realtive to the natural underlying EF-proximity spaces.*

This is easily understood knowing that *a totally bounded uniformity is coarser than any uniformity that induces the same proximity*.
On the other side, recall the following result:

**Proposition 2.3.** *If a proximity class of uniformities contains a metrizable uniformity, then this achieves the maximum in the class.*

Keeping in mind this proposition, we obtain:

**Proposition 2.4.** *If a function from a metric space $(X, d)$ to a uniform space $(Y, \mathscr{V})$ is a proximity function relative to the metric proximity induced by $d$ on $X$ and the proximity $\delta_{\mathscr{V}}$ induced by $\mathscr{V}$ on $Y$, thein it is a uniformly continuous function relative to the metric uniformity induced by $d$ and $\mathscr{V}$.*

Hence we obtain that **uniform geometry and proximal geometry agree in the metric context**.

### 2.7.3 Uniform completion

The notion of completeness for uniform spaces is easily carried over from metric spaces. We have only to generalize the notion of *Cauchy sequence*.

**Definition 2.7.** *Let $(X, \mathscr{U})$ be a uniform space. A net $x_\lambda$ in $X$ is $\mathscr{U}-$Cauchy (or just Cauchy) iff for each $U \in \mathscr{U}$, there is some index $\lambda_0$ such that $(x_{\lambda_1}, x_{\lambda_2}) \in U$ whenever $\lambda_1, \lambda_2 \geq \lambda_0$.*

The following theorem, similar to the metric case, holds.

**Theorem 2.8.** *Every convergent net is Cauchy.*

**Definition 2.8.** *A uniform space $X$ is said complete iff every Cauchy net in $X$ converges.*

Furthermore, by proving that a uniform space $(X, \mathscr{U})$ is totally bounded iff each net in $X$ has a Cauchy subnet, it is easily obtained the following result.

**Theorem 2.9.** *A uniform space $(X, \mathscr{U})$ is compact iff it is complete and totally bounded.*

It can be proved that every uniform space $(X, \mathscr{U})$ can be uniformly embedded as a dense subspace of a complete uniform space $\hat{X}$ which is unique up to uniform isomorphisms. Regarding it there is an interesting characterization.

**Theorem 2.10.** *Let $(X, \mathscr{U})$ be a uniform space. Then there exists a complete Hausdorff uniform space $\hat{X}$ and a uniformly continuous mapping $i : X \to \hat{X}$ having the following property:*

*(P) Given any uniformly continuous map $f$ from $X$ into a complete Hausdorff uniform space $Y$, there is a unique uniformly continuous mapping $g : \hat{X} \to Y$ such that $f = g \circ i$.*

*If $(i_1, X_1)$ is another pair consisting of a complete Hausdorff uniform space $X_1$ and a uniformly continuous mapping $i_1 : X \to X_1$ having the property (P), then there is a unique uniform isomorphism $\phi : \hat{X} \to X_1$ such that $i_1 = \phi \circ i$.*

## 2.8 Local proximity spaces

In [40], S. Leader introduced *local proximity spaces*. They bring a very rich structure in which uniformity, proximity and boundedness have an intensive interaction. A non-empty collection $\mathscr{B}$ of non-empty subsets of a non-empty set $X$ is called a *boundedness in* $X$ if and only if it is hereditary, i.e., $A \in \mathscr{B}$ and $B \subset A$ implies $B \in \mathscr{B}$, and is closed under finite unions, i.e., $A, B \in \mathscr{B}$ implies $A \cup B \in \mathscr{B}$. The elements of a boundedness are called *bounded sets*. A boundedness of $X$ which is further a cover for $X$ is known as a *bornology* [36, 37].

A *local proximity space* $(X, \delta, \mathscr{B})$ consists of a non-empty set $X$ together with a proximity $\delta$ on $X$ and a boundedness $\mathscr{B}$ in $X$ which, in addition to the axioms $P_0)$ trough $P_3)$, (page 10) is subject to the following compatibility conditions :

$P_4)$ If $A\, \delta C$, then there is some $B \in \mathscr{B}$ such that $B \subset C$ and $A\, \delta B$.

$P_5)$ If $A \in \mathscr{B}$, $C \subset X$ and $A \ll_\delta C$ then there exists some $B \in \mathscr{B}$ such that $A \ll_\delta B \ll_\delta C$, where $\ll_\delta$ is the natural strong inclusion associated with $\delta$.

Notice that the proximity in a local proximity space is not in general Efremovič.

The prototype of local proximity spaces is the Euclidean line with the bounded subsets endowed with a suitable proximity and more generally any $T_2$ locally compact space equipped with the bornology of relatively compact subsets. The boundedness in a local proximity space is a bornology and a local family [40]. When $\delta$ is separated, then $(X, \delta, \mathscr{B})$ is said to be a *separated local proximity space*.

### 2.8.1 Local proximity spaces and local compactification

Local proximity spaces are the dual counterpart of the dense $T_2$–local compactifications of a Tychonoff space. Specifically, in [40] S. Leader embeds the underlying space $X$ of any local proximity space $(X, \delta, \mathscr{B})$, in a dense extension of $X$, $\ell(X)$, that is $T_2$ and locally compact, unique up to homeomorphisms, and completely determined from the following two properties:

$a)$ $A\, \delta B$ in $X$ if and only if their closures in $\ell(X)$ intersect.

$b)$ $B$ is bounded if and only if its closure in $\ell(X)$ is compact.

We emphasize that, by density, any point in $\ell(X)$ can be approximated with a net in $X$ whose underlying set is bounded. A local proximity, which is locally Efremovič, is not in general Efremovič. But it is possible to define a global EF-proximity which agrees with $\delta$ locally, that is wherever either of the sets involved is bounded. Namely, any local proximity space $(X, \delta, \mathscr{B})$ determines on the underlying space $X$ an EF-proximity $\delta_{\mathcal{A}}$ when declaring two subsets $A, B \subset X$ are $\delta_{\mathcal{A}}$–*close* if and only if either they are $\delta$–close or they both are unbounded. Since $\delta_{\mathcal{A}}$ can be seen as the restriction to $X$ of the Alexandroff proximity of $\ell(X)$, we refer to it as the *Alexandroff proximity of the local proximity space* $(X, \delta, \mathscr{B})$.

At this point it is interesting to know which functions on a local proximity space $(X, \delta, \mathscr{B})$ can be continuously extended to the related local compactification.
S. Leader answers to this question in the following way.

**Theorem 2.11.** *Let $(X_1, \delta_1, \mathscr{B}_1)$ and $(X_2, \delta_2, \mathscr{B}_2)$ be local proximity spaces with local compactifications $\ell(X_1)$, $\ell(X_2)$. Let f be an equicontinuous mapping of $X_1$ into $X_2$, that is:*

- $A\delta_1 B \Rightarrow f(A)\delta_2 f(B)$,
- $B \in \mathscr{B}_1 \Rightarrow f(B) \in \mathscr{B}_2$.

*Then there exists a unique continuous mapping $\hat{f} : \ell(X_1) \to \ell(X_2)$ such that it extends $f$.*
*Conversely, if $X_2$ is Hausdorff and if $\hat{f}$ is a continuous mapping of $\ell(X_1)$ into $\ell(X_2)$ such that $(\hat{f} \circ \pi_1)(X_1) \subseteq \pi_2(X_2)$ (where $\pi_i : X_i \to \ell(X_i)$ are the projections), then the mapping $f = {\pi_2}^{-1} \circ \hat{f} \circ \pi_1$ of $X_1$ into $X_2$ is equicontinuous.*

## 2.9 Hit and far-miss topologies

In many branches of mathematics and applications one has to deal with families of sets such as closed sets, compact sets and so on. So it is necessary to assign topologies to the space of subsets, called hyperspace, of a given topological space $(X, \tau)$. One example is given by hit and miss topologies such as the Vietoris topology and the Fell topology. They are the join of a part called hit part and another one called miss part.

Vietoris topology

Let $X$ be an Hausdorff space. The *Vietoris topology* on $CL(X)$, the hyperspace of all non-empty closed sets of $X$, has as subbase all sets of the form

- $V^- = \{E \in CL(X) : E \cap V \neq \varnothing\}$, where $V$ is an open subset of $X$,
- $W^+ = \{C \in CL(X) : C \subset W\}$, where $W$ is an open subset of $X$.

The topology $\tau_{V^-}$ generated by the sets of the first form is called **hit part** because, in some sense, the closed sets in this family hit the open sets $V$. Insted, the topology $\tau_{V^+}$ generated by the sets of the second form is called **miss part**, because the closed sets here miss the closed sets of the form $X \setminus W$.
The Vietoris topology is the join of the two part: $\tau_V = \tau_{V^-} \vee \tau_{V^+}$. It represents the prototype of hit and miss topologies.
The Vietoris topology was modified by Fell. He left the hit part unchanged and in the miss part, instead of taking all open sets $W$, he took only open subsets with compact complement.
Fell topology: $\tau_F = \tau_{V^-} \vee \tau_{F^+}$

It is possible to consider several generalizations. For example, instead of taking open subsets with compact complement, for the miss part we can look at subsets running in a family of closed sets $\mathscr{B}$. So we define the *hit and miss topology on*

$CL(X)$ *associated with* $\mathscr{B}$ as the topology generated by the join of the hit sets $A^-$, where $A$ runs over all open subsets of $X$, with the miss sets $A^+$, where $A$ is once again an open subset of $X$, but more, whose complement runs in $\mathscr{B}$. Another kind of generalization concerns the substitution of the inclusion present in the miss part with a strong inclusion associated to a proximity. Namely, when the space $X$ carries a proximity $\delta$, then a proximity variation of the miss part can be displayed by replacing the miss sets with *far-miss sets* $A^{++} := \{ E \in CL(X) : E \ll_\delta A \}$.
Also in this case we can consider $A$ with the complement running in a family $\mathscr{B}$ of closed subsets of $X$. Then the *hit and far-miss topology* , $\tau_{\delta,\mathscr{B}}$, *associated with* $\mathscr{B}$ is generated by the join of the hit sets $A^-$, where $A$ is open, with far-miss sets $A^{++}$, where the complement of $A$ is in $\mathscr{B}$.
Fell topology can be considered as well an example of hit and far-miss topology. In fact, in any proximity, when a compact set is contained in an open set, it is also strongly contained.
The interest in the topic goes through many years and it is supported by the possibility to apply these ideas to Convex Analysis, Optimization, Image Processing, Economics...Moreover in [47], S. Naimpally obtains a very interesting result. He shows that all known hypertopologies are hit and miss.

# 3 Bornological convergences and local proximity spaces

In this section we present some rusults collected in a recent paper [27] written jointly with Professor A. Di Concilio. In the first part, by using uniformity and bornology we display a procedure as exhaustive method of generating local proximity spaces in a very natural way. Then we identify an appropriate topology for the hyperspace of all non-empty closed subsets of a local proximity space and we study some remarkable properties. Finally we focus on a particular kind of metric spaces, $\omega_\mu$−metric spaces, and their associated Attouch-Wets topologies.

## 3.1 A constructive procedure

Here we show how to construct a local proximity space starting from a uniform space and a bornology and, conversely, we show that this is an exhaustive precedure, that is every local proximity space can be obtained in such a way.

Let $(X, \mathscr{U})$ be a uniform space, $\mathscr{B}$ a bornology of $X$. Recall that a family $\mathscr{B}$ is *stable under small enlargements with respect to a uniformity* $\mathscr{U}$ if for each $B \in \mathscr{B}$ there is $U \in \mathscr{U}$ so that $U[B]$, the $U$−enlargement of $B$, belongs again to $\mathscr{B}$.
After denoting as $\delta_{\mathscr{U}}$ the natural proximity associated with $\mathscr{U}$, we introduce over $\mathscr{P}(X)$ the following proximity $\delta_{\mathscr{U},\mathscr{B}}$ :

$$A, B \subset X, \ \ A\, \delta_{\mathscr{U},\mathscr{B}}\, B \ \text{ if and only if there exists } \ C \in \mathscr{B} \text{ such that } A \cap C\ \delta_{\mathscr{U}}\ B \cap C.$$

Of course, when $A\ \delta_{\mathscr{U},\mathscr{B}}\ B$, then $A \delta_{\mathscr{U}} B$. The converse is not in general true. But, in two special cases the converse is true as well. In fact, we can show that:

**Theorem 3.1.** *Let $(X, \mathscr{U})$ be a uniform space and $\mathscr{B}$ a bornology of $X$. Then $\delta_{\mathscr{U}}$ and $\delta_{\mathscr{U},\mathscr{B}}$ match on any pair of subsets of $X$ when they are both bounded, or when one of them is bounded and, furthermore, $\mathscr{B}$ is stable under small enlargements.*

**Proof.** First, we assume $A, B \subset X$ are both bounded and $A\ \delta_{\mathscr{U}} B$. Then, for each $U \in \mathscr{U}$, it happens that $U[A] \cap B \neq \varnothing$. So, we can choose two points $x_U \in A$, $y_U \in B$ with $(x_U, y_U) \in U$, so determining two subsets $A_1 = \{x_U : U \in \mathscr{U}\}$ in $A$ and $B_1 = \{y_U : U \in \mathscr{U}\}$ in $B$. Of course, $A_1, B_1$ are both bounded. Hence, their union $C = A_1 \cup B_1$ is in turn bounded. From $A_1\ \delta_{\mathscr{U}}$−close to $B_1$, it follows that $A \cap C$ is $\delta_{\mathscr{U}}$-close to $B \cap C$, and , from definition, that just means $A\ \delta_{\mathscr{U},\mathscr{B}}$−close to $B$. Next, we assume $\mathscr{B}$ stable under small enlargements and $A \subset X$ bounded. We choose $V \in \mathscr{U}$ so that $V[A]$ is also bounded. Then, by following a similar procedure, but now by limiting to the diagonal nbhds $U \in \mathscr{U}, U \subset V$, which are a base for $\mathscr{U}$, we obtain the result just in the same way.

□

**Theorem 3.2.** *Let $X$ be a Tychonoff space, $\mathscr{U}$ a uniformity compatible with $X$, $\delta_{\mathscr{U}}$ its natural proximity and $\mathscr{B}$ a bornology of $X$. Then, the triple $(X, \delta_{\mathscr{U},\mathscr{B}}, \mathscr{B})$ is a local proximity space on the space $X$, for which the local proximity $\delta_{\mathscr{U},\mathscr{B}}$ agrees with $\delta_{\mathscr{U}}$ when restricted to any bounded set, if and only if $\mathscr{B}$ is stable under small enlargements.*

**Proof.** As part of definitions of proximity and bornology, it is easily seen that $\delta_{\mathscr{U},\mathscr{B}}$ satisfies axioms $P_0)$ trough $P_4)$ (pages 10, 24). Now, by using essentially the previous result, we show that the local Efremovič property, axiom $P_5)$, holds. Let $A \in \mathscr{B}$, $U \in \mathscr{U}$ and $U[A] \in \mathscr{B}$. To say that $A \ll_{\delta_{\mathscr{U}}} B$ means there is a diagonal nbhd $V \in \mathscr{U}$ so that $V[A] \subset B$. But then there is also a diagonal nbhd $W \in \mathscr{U}$ with $W \circ W \subset U \cap V$. Thus, definitively, $A \ll_{\delta_{\mathscr{U}}} W[A] \ll_{\delta_{\mathscr{U}}} B$, where obviously $W[A]$ is bounded. By the previous theorem, the result is achieved. Conversely, if $A \ll_{\delta_{\mathscr{U}}} B$, then also $A \ll_{\delta_{\mathscr{U},\mathscr{B}}} B$. From axiom $P_5)$ there are bounded sets $C$ and $D$ so that $A \ll_{\delta_{\mathscr{U},\mathscr{B}}} C \ll_{\delta_{\mathscr{U},\mathscr{B}}} D \ll_{\delta_{\mathscr{U},\mathscr{B}}} B$. But, then $A \ll_{\delta_{\mathscr{U}}} C \ll_{\delta_{\mathscr{U}}} D \subset B$ and this yields that $U[A] \ll_{\delta_{\mathscr{U}}} C \subset D$ for some diagonal nbhd $U \in \mathscr{U}$. We can conclude by remarking that $U[A]$ is bounded. □

Here we use the term *nearness-boundedness preserving map* to indicate the equicontinuous map of S. Leader (see thm. 2.11 ).

The underlying space $X$ of any local proximity space $(X, \delta, \mathscr{B})$ is carrier of two proximally equivalent uniformities. The former, $\mathscr{U}_{\mathcal{A}}$, is generated by the collection, $Eq(X, \mathbb{R})$, of all nearness-boundedness preserving maps with bounded support from $X$ to the reals. The latter, $\mathscr{U}^*_{\mathcal{A}}$, is generated by the collection, $Eq(X, [0,1])$, of all nearness-boundedness preserving maps from $X$ to the unit interval $[0,1]$ which have a bounded support.

**Theorem 3.3.** *Let $(X, \delta, \mathscr{B})$ stand for a local proximity space. Then, both the uniformities $\mathscr{U}_{\mathcal{A}}$, $\mathscr{U}^*_{\mathcal{A}}$ on $X$ induce the Alexandroff proximity $\delta_A$. Furthermore, the boundedness $\mathscr{B}$ is stable under small $\mathscr{U}^*_{\mathcal{A}}$–enlargements.*

**Proof.** The uniformity $\mathscr{U}_{\mathcal{A}}$ is generated by the collection, $Eq(X, \mathbb{R})$, of all nearness-boundedness preserving maps with bounded support from $X$ to the reals. Indeed, these are the only ones which continuously extend to $\ell(X)$. Moreover, recall that $Eq(X, \mathbb{R}) = C(\ell(X), \mathbb{R})$. Hence, $\ell(X)$, equipped with the weak uniformity generated by all real-valued continuous functions on $\ell(X)$, is the uniform completion of $X$ when carrying $\mathscr{U}_{\mathcal{A}}$. The uniformity $\mathscr{U}^*_{\mathcal{A}}$ is generated by the collection $Eq(X, [0,1])$ of all nearness-boundedness preserving maps from $X$ to the unit interval $[0,1]$ which have a bounded support. These are the only ones which continuously extend to the one-point compactification $\gamma(X) = \ell(X) \cup \{\infty\}$ of $\ell(X)$. Just for that, $\gamma(X)$ is the uniform completion of $X$ when carrying $\mathscr{U}^*_{\mathcal{A}}$. By definition, the Alexandroff proximity $\delta_A$ is the localization to $X$ of the Alexandroff proximity of $\ell(X)$, which in turn is the localization to $\ell(X)$ of the unique proximity of $\gamma(X)$. Finally, axiom $P_5$) (p. 24) can be read as $\mathscr{B}$ to be stable under small enlargements with respect to $\mathscr{U}^*_{\mathcal{A}}$. □

Next, we prove that the above constructive procedure is exhaustive as well.

**Theorem 3.4.** *Any local proximity space $(X, \delta, \mathscr{B})$ can be constructed by applying the above described procedure to the Alexandroff uniformity $\mathscr{U}_A^*$ and $\mathscr{B}$, or, equivalently, to the uniformity $\mathscr{U}_A$ and $\mathscr{B}$.*

**Proof.** Recall that the two uniformities $\mathscr{U}_A^*$, $\mathscr{U}_A$, both induce the Alexandroff proximity $\delta_A$ and two subsets of $X$ are $\delta_A-$close if and only if they are $\delta-$close or both are unbounded. We have to show that the local proximity associated with $\mathscr{U}_A^*$ and $\mathscr{B}$, following the procedure described above, agrees with $\delta$. If there is $C \in \mathscr{B}$ so that $A \cap C \ \delta_A \ B \cap C$, then $A \ \delta B$. Conversely, consider $A\delta B$, then $Cl_{\ell(X)}A$ and $Cl_{\ell(X)}B$ intersect in a point $x$ in $\ell(X)$. Of course, the point $x$ can be approximated with a bounded net $\{a_\lambda\}$ extracted from $A$ and a bounded net $\{b_\mu\}$ extracted from $B$. So, if $C$ is the union of their underlying sets, it follows that $A \cap C$ and $B \cap C$ are $\delta-$close and the result is established. □

- *Examples*

Let $(X, \mathscr{U})$ be a separated uniform space. The collection $TB$ of totally bounded subsets form a bornology which is stable under small enlargements if and only if the uniform completion of $X$ is locally compact. Indeed, the closure in the uniform completion of $(X, \mathscr{U})$ of a subset of $X$ is compact if and only if it is totally bounded. Moreover, the closures of two subsets of $X$ intersect in its uniform completion if and only if they contain two adjacent Cauchy nets. Now, if we say that two sets $A, B$ are $\delta-$close if and only if they contain two adjacent Cauchy nets, then the triple $(X, \delta, TB)$ is a local proximity space if and only if its uniform completion is also locally compact.

Inside this case there is an interesting one related to the really natural *boundedness* in a uniform space $(X, \mathscr{U})$. A subset $A$ of $X$ is *Bourbaki-bounded or finitely chainable with respect to* $\mathscr{U}$ when for each $U \in \mathscr{U}$ there exist a positive integer number $n$ and a finite set $F$ of $X$ so that $A \subset U^n[F]$. It is very well-known that the Bourbaki-boundedness characterizes as totally boundedness in the weak uniformity generated by all real-valued uniformly continuous functions [63]. Equivalently, the

Bourbaki-bounded sets are those ones on which every real-valued uniformly continuous function is bounded. Any totally bounded subset is Bourbaki-bounded and, when the uniformity is metric, then any Bourbaki-bounded is in turn bounded. The converse is not in general true. It follows that the bornology of Bourbaki-bounded subsets of a separated uniform space is stable under small enlargements if and only if its uniform completion is locally compact.

## 3.2 Natural structures on $CL(X)$

Let $(X, \delta, \mathscr{B})$ be a local proximity space. Apparently, we have two natural different ways to topologize the hyperspace $CL(X)$. The first option results from the dense embedding of $X$ in $\ell(X)$, while the second one comes from joining together proximity and bornology in a hit and far-miss topology. We will show they match in just one case.

In the perspective to reduce the general case to the locally compact one, we define the *local Fell topology,* $\tau_{loc,F}$, by saying that:

If $\{A_\lambda\}_{\lambda\in\Lambda}$ stands for a net in $CL(X),\ A \in CL(X)$, and $\{Cl_{\ell(X)}A_\lambda\}_{\lambda\in\Lambda},\ Cl_{\ell(X)}A$ are their closures in $\ell(X)$, then:

$$\{A_\lambda\} \overset{\tau_{loc,F}}{\longrightarrow} A \quad \text{if and only if} \quad \{Cl_{\ell(X)}A_\lambda\} \overset{\tau_F}{\longrightarrow} Cl_{\ell(X)}A$$

where $\tau_F$ denotes the Fell topology on $CL(\ell(X))$. We underline that the $\tau_{loc,F}$-limit $A$ of any net $\{A_\lambda\}$ is unique and its closure $Cl_{\ell(X)}A$ is the Kuratowski limit of $\{Cl_{\ell(X)}A_\lambda\}$. In fact, in general the Kuratowski convergence is not topological; but, if the underlying space is $T_2$ and locally compact, then it is and the associated topology is the Fell topology. .

Next, it is very natural indeed to consider the hit and far-miss topology $\tau_{\delta,\mathscr{B}}$ induced by $\delta$ and $\mathscr{B}$. Thanks to the result in theorem 3.1, it easily seen that $\tau_{\delta,\mathscr{B}}$ agrees with the hit and far-miss topology associated with $\delta_A$ and $\mathscr{B}$. Accordingly

to the previous observation, we can make use of the results obtained by G. Beer in [12] by adapting them to local proximity spaces and we obtain that the hit and far-miss topology $\tau_{\delta,\mathscr{B}}$ on $CL(X)$ is induced by the weak uniformity generated by a particular collection of infimal value functionals.
Recall that a topology $\tau$ on a set $Y$ is expressible as a weak topology determined by a family of real-valued functions $\mathscr{F}$ if and only if it is uniformizable. In fact, if $\mathscr{F}$ determines $\tau$, then a subbase for a compatible uniformity $\mathscr{U}(\mathscr{F})$ is represented by all sets of the form

$$\{(x,y) : |f(x) - f(y)| < \epsilon\}$$

where $f \in \mathscr{F}$ and $\epsilon > 0$. Conversely, if $\tau$ admits a compatible uniformity, then $\tau$ is completely regular and we can take as family $\mathscr{F}$ either $C(Y,\mathbb{R})$ or $C(Y,[0,1])$.

Let $f$ be a function from a space $X$ to the reals. As usual, the *sublevel set of $f$ of height $\epsilon$*, where $\epsilon$ is a positive real number, is defined as $\text{sblv}(f;\epsilon) := \{x \in X : f(x) \leq \epsilon\}$.

**Lemma 3.1.** *Let $(X,\delta,\mathscr{B})$ stand for a local proximity space. If $f$ is a real-valued, nearness-boundedness preserving map, bounded, with bounded support on $X$ and $\inf f < \alpha < \beta < \sup f$, then there is a set $B$ in $\mathscr{B}$ so that sblv$(f;\alpha) \subset B \subset$ sblv$(f;\beta)$.*

This lemma simply derives from Lemma 4.4.1 of [12] and from the observations of the previuos section on $\mathscr{U}_A^*$ . As a consequence, we obtain the following result.

**Theorem 3.5.** *Let $(X,\delta,\mathscr{B})$ be a local proximity space. The hit and far-miss topology $\tau_{\delta,\mathscr{B}}$ on $CL(X)$ is induced by the weak uniformity generated by the collection of infimal value functionals of all nearness-boundedness preserving maps from $X$ to $[0,1]$ with bounded support.*

**Proof.** First, for each $A \subset X$ denote as $m_f(A) := inf\{f(x) : x \in A\}$. Let $\{A_\lambda\}$ stand for a net in $CL(X)$ convergent in $\tau_{\delta,\mathscr{B}}$ to $A \in CL(X)$. Observe that to say that the hit and far-miss topology $\tau_{\delta,\mathscr{B}}$ on $CL(X)$ is induced by the weak uniformity generated by the collection of infimal value functionals of the maps in $Eq(X,[0,1])$ is equivalent to say that $\{A_\lambda\} \xrightarrow{\tau_{\delta,\mathscr{B}}} A$ if and only if $\{m_f(A_\lambda)\} \to m_f(A)$ for all $f \in Eq(X,[0,1])$ in the Euclidean topology.

So assume that for some $f \in Eq(X,[0,1])$, the net $\{m_f(A_\lambda)\}$ is not convergent to $m_f(A)$. Two options are possible. With the former, there are two real numbers $\alpha, \beta$ with $m_f(A_\lambda) < \alpha < \beta < m_f(A)$, frequently. In this case, by the previous lemma there is a bounded set which interposes between sblv$(f;\alpha)$ and sblv$(f;\beta)$. So, definitively, $A$ is far from sblv$(f;\alpha)$, which is in turn a bounded, while $A_\lambda$ do intersect it frequently, which is a contradiction if we consider the far-miss part of the convergence. The latter, when $m_f(A) < \alpha < m_f(A_\lambda)$, frequently. In this last case, it happens that $A$ intersects the open set $f^{-1}([0,\alpha[)$, while $A_\lambda$ do not, frequently. Now we obtain a contradiction by considering the hit part of the convergence.

Conversely, let begin from the far-miss part. Whenever a closed $A$ is far from a bounded set $B$, then there is a function $f \in Eq(X,[0,1])$ with bounded support so that $f(B) = 0$ and $f(A) = 1$. Consequently, $m_f(A) = 1$ and $m_f(A_\lambda)$ is in $[\frac{1}{2},1]$, residually. But, $0$ is far from $[\frac{1}{2},1]$. Thus, since $f$ preserves nearness and boundedness, then $B \subset f^{-1}(\{0\})$ is far from $A_\lambda$, residually. Next, focus on the hit part and suppose that $A$ shares with an open set $U$ a point $x$. Take $f \in Eq(X,[0,1])$, with bounded support so that $f(x) = 0$ and $f(X \setminus U) = 1$. Hence, $m_f(A) = 0$ and $m_f(A_\lambda)$ is close to $0$, residually. It follows that $A_\lambda$ must contain points on which $f$ cannot assume $1$ as value. Therefore, $A_\lambda$ shares some point with $U$, residually.

□

### 3.2.1 Matching

**Theorem 3.6.** *Let $(X, \delta, \mathscr{B})$ be a local proximity space. The local Fell topology $\tau_{loc,F}$ on $CL(X)$ agrees with the hit and far miss topology $\tau_{\delta,\mathscr{B}}$ associated with the proximity $\delta$ and the bornology $\mathscr{B}$.*

**Proof.** Let $A_\lambda, \lambda \in \Lambda$ and $A$ belong to $CL(X)$. We show that:

$$\{A_\lambda\}_{\lambda\in\Lambda} \stackrel{\tau_{loc,F}}{\longrightarrow} A \text{ if and only if } \{A_\lambda\}_{\lambda\in\Lambda} \stackrel{\tau_{\delta,\mathscr{B}}}{\longrightarrow} A.$$

Start by assuming $\{A_\lambda\}_{\lambda\in\Lambda} \stackrel{\tau_{loc,F}}{\longrightarrow} A$ and show that $\{A_\lambda\}_{\lambda\in\Lambda} \stackrel{\tau_{\delta,\mathscr{B}}}{\longrightarrow} A$. Consider first the hit part. Let $H$ be an open set in $X$ with $A \cap H \neq \varnothing$ and $a \in A \cap H$. If $H$ is the trace on $X$ of a set $\hat{H}$ open in $\ell(X)$, then, from $\hat{H} \cap Cl_{\ell(X)}A \neq \varnothing$, it follows that $\hat{H} \cap Cl_{\ell(X)}A_\lambda \neq \varnothing$. Hence, $\hat{H} \cap A_\lambda \neq \varnothing$, eventually. In conclusion, also $H \cap A_\lambda \neq \varnothing$, eventually. Next, assume $A$ is far from a bounded set $B$. That is, $Cl_{\ell(X)}A \cap Cl_{\ell(X)}B = \varnothing$. But, $Cl_{\ell(X)}B$ is compact. So, $Cl_{\ell(X)}A_\lambda \cap Cl_{\ell(X)}B = \varnothing$, eventually. And this makes $A_\lambda$ far from $B$, eventually.

Conversely, if $Cl_{\ell(X)}A$ intersects an open set $\hat{H}$ of $\ell(X)$, then $A$ does intersect $H = \hat{H} \cap X$, the open trace on $X$ of $\hat{H}$. Consequently, $A_\lambda$ does intersect $H$, eventually. And that yields $Cl_{\ell(X)}A_\lambda \cap \hat{H} \neq \varnothing$, eventually. Next, if $K$ is a compact set in $\ell(X)$ and $Cl_{\ell(X)}A_\lambda$ shares a common point $\hat{a}_\lambda$ with $K$ frequently, then the net $\{\hat{a}_\lambda\}$ does accumulate to a point $\hat{a}$ in $K$ and the point $\hat{a}$ does belong to $Cl_{\ell(X)}A$. If not, there would be an open nbhd $W$ of $\hat{a}$ in $\ell(X)$ with compact closure sharing no point in common with $Cl_{\ell(X)}A$. Since any point $\hat{a}_\lambda$ can be approximated by a net of points extracted from $A_\lambda$, the nbhd $W$, and hence $W \cap X$ as well, should intersect $A_\lambda$, frequently. But, $B = W \cap X$ is a bounded set whose $Cl_{\ell(X)}B \subset Cl_{\ell(X)}W$ could not intersect $Cl_{\ell(X)}A$. Thus, $A$ and $B$ should be far in $\delta$. And, by hypothesis, it should follow $A_\lambda$ is far from $B$, eventually. A contradiction. □

## 3.3 Uniform bornological convergences

Uniform bornological convergences are a mixture of uniformity and bornology. Let $(X, \mathscr{U})$ be a uniform space and $\mathscr{B}$ a family of subsets of $X$.
The *upper uniform bornological convergence associated with $\mathscr{U}$ and $\mathscr{B}$*, which we denote as $\mathscr{U}_{\mathscr{B}}^{+}$, is defined as follows:

$\{A_\lambda\} \overset{\mathscr{U}_{\mathscr{B}}^{+}}{\to} A$ iff for each $U \in \mathscr{U}$ and each $B \in \mathscr{B}$, then $A_\lambda \cap B \subset U[A]$, residually.

Next, the *lower uniform bornological convergence associated with $\mathscr{U}$ and $\mathscr{B}$*, which we denote as $\mathscr{U}_{\mathscr{B}}^{-}$, is defined as follows:

$\{A_\lambda\} \overset{\mathscr{U}_{\mathscr{B}}^{-}}{\to} A$ iff for each $U \in \mathscr{U}$ and each $B \in \mathscr{B}$, then $A \cap B \subset U[A_\lambda]$, residually.

Finally, the *two-sided uniform bornological convergence associated to $\mathscr{U}$ and $\mathscr{B}$*, in short, *uniform bornological convergence*, which we denote as $\mathscr{U}_{\mathscr{B}}$, is the join of the upper and lower uniform bornological convergences related to $\mathscr{U}$ and $\mathscr{B}$.

It represents a generalization of the Hausdorff convergence. In this case we concentrate the attention only on traces on bounded sets.
The prototype of uniform bornological convergence is the Attouch-Wets or bounded Hausdorff convergence where the uniformity is metrizable and the bornology is done by all metrically bounded sets [12]. In the case, the bornology is clearly stable under small metric enlargements. A metric bornological convergence is the natural one associated with a metric uniformity when $\mathscr{B}$ is a bornology stable under small enlargements.

Besides the trivial bornology $\mathscr{P}(X)$ and the bornology of metrically bounded sets, there are some other bornologies of particular interest: for example $\mathscr{K}(X)$, the bornology of nonempty subsets of $X$ that have compact closure. In [12] it is shown the following relevant result which allows us to consider convergence of nets of sets in the Fell topology as a kind of bornological convergence in a Hausdorff uniform space.

**Theorem 3.7.** *Let $(X,\mathscr{U})$ be a Hausdorff uniform space. Let $A \in 2^X = CL(X) \cup \varnothing$, and let $\{A_\lambda\}_{\lambda\in\Lambda}$ be a net in $2^X$. Then $\{A_\lambda\}_{\lambda\in\Lambda} \xrightarrow{Fell} A$ if and only if for each $U \in \mathscr{U}$ and $K \in \mathscr{K}(X)$, there exists $\lambda_0 \in \Lambda$ such that for each $\lambda \geq \lambda_0$, we have both $A_\lambda \cap K \subset U[A]$ and $A \cap K \subset U[A_\lambda]$.*

In the seminal paper [42] Lechicki, Levi and Spakowski studied a family of uniform bornological convergences, *ante litteram*, including Attouch-Wets, Fell and Hausdorff metric topologies. We prove the uniform counterpart of the metric case by essentially miming the proofs performed in that by really using only uniform features of a metric [19].

More synthetically, we introduce diagonal nbhds in $\mathscr{P}(X)$ and look for conditions in which they form a base of a diagonal uniformity. If, for each $S \in \mathscr{B}$ and $U \in \mathscr{U}$, we put:

$$[S,U] := \{(A,B) \in CL(X) \times CL(X) : A \cap S \subset U[B] \text{ and } B \cap S \subset U[A]\ \}.$$

Indeed, we have that:

$$\{A_\lambda\} \overset{\mathscr{U}_\mathscr{B}}{\to} A \qquad \text{if and only if} \quad A_\lambda \in [S,U][A] \text{ residually for each } [S,U].$$

Evidently, any $[S,U]$ contains the diagonal $\Delta = \{(A,A) : A \in CL(X)\}$ and is symmetric. Any intersection $[S_1,U_1] \cap [S_2,U_2]$ contains $[S_1 \cup S_2, U_1 \cap U_2]$. But, in general, the composition law does not hold. Furthermore this kind of convergence is not always topological. We show that these facts are connected with stability under small enlargements of the bornology.

**Theorem 3.8.** *Let $\mathscr{B}$ a bornology in a uniform space $(X, \mathscr{U})$. Then the following are equivalent:*

*1) $\mathscr{B}$ is stable under small enlargements.*

*2) $\mathbb{B} = \{[S, U] : S \in \mathscr{B}, U \in \mathscr{U}\}$, is a base for a diagonal uniformity on $\mathscr{P}(X)$.*

*3) $\mathscr{U}_{\mathscr{B}}$–convergence is a topological convergence over $\mathscr{P}(X)$.*

**Proof.** $1) \Rightarrow 2)$. It is enough to show that the composition law holds. Given $[S, U]$, choose $W \in \mathscr{U}$ so that $W \circ W \subset U$ and also $W[S] \in \mathscr{B}$. Then, it easily seen that $[W[S], W] \circ [W[S], W] \subset [S, U]$. 2) ⇒ 3) is trivial. Finally, 3) ⇒ $1)$. Assume there is $B \in \mathscr{B}$ so that $U[B] \notin \mathscr{B}$ for each $U \in \mathscr{U}$. Let $[B, U]$ and $[S, V]$ both in $\mathbb{B}$ so that $[S, V][\varnothing] \subset [B, U][\varnothing]$. Since, for each $U \in \mathscr{U}$, $U[B] \nsubseteq S$, for each $U \in \mathscr{U}$ we can find a point $x_U$ in $U[B]$ but not in $S$. We claim that the set $A = \{x_U : U \in \mathscr{U}\}$ belongs to $[S, V][\varnothing]$ but no element of $\mathbb{B}$ containing it can be contained in the starting nbhd $[B, U][\varnothing]$, in contrast with the assumption. Given any symmetric diagonal nbhd $W \in \mathscr{U}$ and choosen $x \in B$ so that $x \in W[x_W]$, then it follows that $A \cup \{x\}$ belongs to $[S_1, W][A]$, whatever is $S_1 \in \mathscr{B}$, but not to $[B, U][\varnothing]$, since the point $x$ is in $B$.

□

The next result makes the local Fell topology on the hyperspace $CL(X)$ of a local proximity space $(X, \delta, \mathscr{B})$ as the most natural one we can associate with it. We just proved that the local Fell topology agrees with the hit and far-miss topology associated with the proximity $\delta$ and the bornology $\mathscr{B}$, or the same with $\delta_{\mathcal{A}}$ and bornology $\mathscr{B}$, which we will show in its turn to agree with the topology of two-sided uniform bornological convergence associated with the Alexandroff uniformity and bornology $\mathscr{B}$.

**Theorem 3.9.** *Let $(X, \mathscr{U})$ be a uniform space, $\mathscr{B}$ a bornology on $X$ stable under small enlargements and $(X, \delta_{\mathscr{U},\mathscr{B}}, \mathscr{B})$ the relative local proximity space. Then, the two-sided uniform bornological convergence associated with $\mathscr{U}$ and $\mathscr{B}$ implies the convergence in the local Fell topology $\tau_{loc,F}$.*

**Proof.** Let $\ell(X)$ be the natural $T_2$ local compactification of $X$ associated with $(X, \delta_{\mathscr{U},\mathscr{B}}, \mathscr{B})$. Assume that $\{A_\lambda\}$ is $\mathscr{U}_{\mathscr{B}}$–convergent to $A$. If $K$ is a compact set in $\ell(X)$ and $Cl_{\ell(X)}A_\lambda$ shares a common point $a_\lambda$ with $K$ frequently, then the net $\{a_\lambda\}$ does accumulate to a point $a$ in $K$ and the point $a$ does belong to $Cl_{\ell(X)}A$. If not, there would be an open nbhd $W$ of $a$ in $\ell(X)$, with compact closure sharing no point in common with $Cl_{\ell(X)}A$. But, since any point $a_\lambda$ can be approximated by a net of points extracted from $A_\lambda$, the nbhd $W$, and hence $W \cap X$ as well, should intersect $A_\lambda$, frequently. But, $B = W \cap X$ is a bounded set whose $Cl_{\ell(X)}B \subset Cl_{\ell(X)}W$ could not intersect $Cl_{\ell(X)}A$. Thus, $A$ and $B$ should be far in $\delta_{\mathscr{U}}$. So, there would be a diagonal nbhd $U \in \mathscr{U}$ for which $U[A] \cap B = \varnothing$. But, at the same time by hypothesis, $A_\lambda \cap B (\neq \varnothing)$ has to be contained in $U[A]$, frequently. A contradiction.

Suppose now $Cl_{\ell(X)}A \cap W \neq \varnothing$, where $W$ is an open set in $\ell(X)$. Let $a \in Cl_{\ell(X)}A \cap W$. Then $W$ contains a compact nbhd $K$ of $a$. So $a$ is approximatable by a net of points extracted from $B = A \cap int_{\ell(X)}K$, which is bounded set of $X$. Let $b$ a point in $B$ and $U[b] \subset int_{\ell(X)}K \cap X$ for some symmetric $U \in \mathscr{U}$. By hypothesis, $B$ must be contained residually in $U[A_\lambda]$. Thus $b$ has to be contained in $U[a_\lambda]$ for some $a_\lambda \in A_\lambda$. That, by symmetry, is the same as $a_\lambda \in U[b]$, which is contained in the starting open $W$.

□

**Theorem 3.10.** *Let $(X, \delta, \mathscr{B})$ a local proximity space and $\mathscr{U}^*_{\mathcal{A}}$ the relative Alexandroff uniformity. Then, the local Fell topology, $\tau_{loc,F}$ is the topology of the two-sided uniform bornological convergence associated with $\mathscr{U}^*_{\mathcal{A}}$ and $\mathscr{B}$.*

**Proof.** One implication is given by the previuos theorem. In fact, by Thm 3.4, we know that every local proximity space can be constructed starting from the relative uniform space $(X, \mathscr{U}_A^*)$. About the other implication, assume $\{A_\lambda\}$ to be $\tau_{loc,F}$–convergent, but not $\mathscr{U}_{A,\mathscr{B}}^*$–convergent, to $A$. Two options are possible. Examine first the case in which there exists a diagonal nbhd $U \in \mathscr{U}_A^*$, viewed as the trace on $X$ of an open diagonal nbhd $\overline{U}$ of the Alexandroff uniformity on $\ell(X)$ and a bounded $B$ so that $A_\lambda \cap B$ does not belong frequently to $U[A]$. Following as proof strategy that one already tested in previous theorems, we show that $A_\lambda$ cannot converge to $A$ in the miss part of $\tau_{loc,F}$. In the further case in which there exists a diagonal nbhd $U \in \mathscr{U}_A^*$, viewed again as the trace on $X$ of an open diagonal nbhd $\overline{U}$ of the Alexandroff uniformity on $\ell(X)$ and a bounded $B$ so that $A \cap B$ does not belong frequently to $U[A_\lambda]$, again following similar considerations, we can prove that $A_\lambda$ does not converge to $A$ in the hit part of $\tau_{loc,F}$.

□

## 3.4 Comparison

By the following theorem we want to point out some relations between uniform bornological convergences relative to different uniformities and bornologies.

**Theorem 3.11.** *Let $X$ be a Tychonoff space (not a single point ), $\mathscr{U}, \mathscr{V}$ two compatible uniformities on $X$ and $\mathscr{A}, \mathscr{B}$ two bornologies of $X$. When the $\mathscr{U}_{\mathscr{A}}$–convergence implies the $\mathscr{V}_{\mathscr{B}}$–convergence it happens that $\mathscr{B} \subseteq \mathscr{A}$ and $\delta_{\mathscr{U}}$ is finer than $\delta_{\mathscr{V}}$ on $\mathscr{B}$.*

**Proof.** Suppose there is $B$ in $\mathscr{B}$ not belonging to $\mathscr{A}$, thus not contained in any $A$ in $\mathscr{A}$. For each $A \in \mathscr{A}$ choose a point $x_A$ in $B$ but not in $A$. For each $x \in X$ the net $\{\{x_A, x\}\}_{A \in \mathscr{A}}$, where $\mathscr{A}$ is directed by inclusion, is $\mathscr{U}_{\mathscr{A}}$–convergent to the singleton $\{x\}$, hence $\mathscr{V}_{\mathscr{B}}$–convergent to $\{x\}$. That makes the net $\{x_A\}_{A \in \mathscr{A}}$ in $B$ to converge to any point $x \in X$. But, $X$ is not reduced to a single point. Moreover, $X$

is $T_2$, and that, assuring the uniqueness of the limit, yields a contradiction. Next, assume $i : (X, \mathscr{U}) \to (X, \mathscr{V})$ not to be a proximity function when restricted to some $B \in \mathscr{B}$. This means there exist two subsets $H, K$ in $B$ close in $\mathscr{U}$ but far in $\mathscr{V}$. For each $U \in \mathscr{U}$ it happens that $U[H] \cap K \neq \varnothing$, while, in constrast, it happens $V[H] \cap K = \varnothing$ for some $V \in \mathscr{V}$. Pick in each $U[H] \cap K$ a point $x_U$. The net $\{\{H \cup \{x_U\}\}_{U \in \mathscr{U}}$, where $\mathscr{U}$ is directed by the reverse inclusion, is $\mathscr{U}_{\mathscr{A}}$–convergent but not $\mathscr{V}_{\mathscr{B}}$–convergent to $H$.

□

In the light of the previous local proximity results, we look for necessary and sufficient conditions of uniform nature for two different uniform bornological convergences to match. Since the metric case [19] is essentially based on two facts, the former: *any metrizable uniformity is the finest one in its proximity class*,or in other words [3] is *total;* the latter: *the bornology of metrically bounded sets is stable under small metric enlargements,* we identify the key properties on one hand for uniformities to be total when localized on bounded sets and, on the other hand, for bornologies to be stable under small enlargements. We emphasize that any proximity function from a total uniform space $(X, \mathscr{U})$ towards any proximity space is uniformly continuous. The described key properties led us to focus on a special class of uniformities: those with a linearly ordered base. We will return to this topic in the following section.

# 4 $\Omega_\mu$–metrizable spaces

In this section we talk about a sort of generalization of metric spaces. The interest in this topic is born looking for necessary and sufficient conditions of uniform nature for two different uniform bornological convergences to match. We need stability of the bornologies under small enlargements and totality of the uniformities when localized on bounded sets, that is the existence of finest uniformities in the relative proximity classes. In [3] it is shown that every uniform structure with a linearly ordered base is total. We say that a uniform space has a *linearly ordered base* when there exists a base $\mathfrak{B}$ such that, if we consider the reverse inclusion between its elements ($U_i < U_j$ iff $U_i \supset U_j$ for $U_i, U_j \in \mathfrak{B}$ ), we have a total order. .

In [62], F. W. Stevenson and W.J. Thron prove that separated uniform spaces with linearly ordered bases are exactly the so called $\omega_\mu$–metrizable spaces. They are spaces whose topology is generated by a metric with values in a linearly ordered abelian $(A, <)$ group with character $\omega_\mu$, where $\omega_\mu$ denotes the $\mu$–th infinite initial ordinal number. A group $A$ is said to have character $\omega_\mu$ if there exists a decreasing $\omega_\mu$–sequence converging to $0$ in the order topology on $A$.

An $\omega_\mu$*–metric* on a set $X$ is a function $\rho$ from $X \times X$ to $(A, <)$ such that:

i) $\rho(x, y) > 0,\ \rho(x, y) = 0$ iff $x = y$,

ii) $\rho(x, y) = \rho(y, x)$,

iii) $\rho(x, y) \le \rho(x, z) + \rho(z, y)$ , $\qquad \forall x, y, z \in X$

The space $(X, \rho)$ is called $\omega_\mu$–metric space. R. Sikorski defined and investigated these spaces in an extensive work [61]. Other researchers who have studied $\omega_\mu$–metric spaces include L.W. Cohen and C. Goffman [23], H.C. Reichel [50, 56], P. Nyikos [50].

Every $\omega_\mu$–metrizable space is an $\omega_\mu$*–additive space*, that is a topological space which satisfies the condition that for any family of open sets, $\mathfrak{F}$, of power less than

$\aleph_\mu$, it follows that the intersection of all the elements of the family is again an open set. Clearly every topological space is an $\omega_0$–additive space. Moreover it is well known (see [39]) that a completely regular space is $\omega_\mu$–additive if and only if it admits an $\omega_\mu$–uniformity. We say that a uniformity $\mathscr{U}$ is an $\omega_\mu$–uniformity and $(X, \mathscr{U})$ is an $\omega_\mu$*–uniform space* if the intersection of less than $\omega_\mu$ many entourages is an entourage.

It is remarkable that the class of $\omega_0$ spaces coincides with the class of metrizable spaces even when the group $A$ is different from $\mathbb{R}$. As it is possible to notice the theory of non-metrizable linearly uniformizable spaces is a generalization of the metrizable case but it has also various particular features which don't have analogous for metrizable spaces. For example, in general a space that is $\omega_\mu$–totally bounded and $\omega_\mu$–complete is not $\omega_\mu$–compact; moreover the Hausdorff uniformity on the hyperspace of closed sets of a complete $\omega_\mu$ metric space is not necessarily complete.

**Hyperreals and $\omega_\mu$–metric spaces** An interesting example of $\omega_\mu$–metric space that is not $\omega_0$ is the space of *hyperreals* or *nonstandard reals*, $\mathbb{R}^*$. They represent an extension of the real numbers that contain elements (called infinite) greater than any real number and elements (called infinitesimal) smaller than any real number.

One of the possible constructions for the hyperreals is the following one. We can take $\mathbb{R}^\mathbb{N}$ and quotient it by a non principal ultrafilter $F$.

$$r \equiv s, \quad r, s \in \mathbb{R}^\mathbb{N} :\Leftrightarrow \{n \in \mathbb{N} : r(n) = s(n)\} \in F.$$

We denote

$$\frac{\mathbb{R}^\mathbb{N}}{\equiv} = \mathbb{R}^* = \{[r] : r \in \mathbb{R}^\mathbb{N}\}.$$

Furthermore, by the ultrafilter $F$ it is possible to define an order relation.

$$[r] < [s] \Leftrightarrow \{n \in \mathbb{N} : r(n) < s(n)\} \in F.$$

The hyperreals $\mathbb{R}^*$ form a non Archimedean ordered field containing the reals as subfield. Regarding the metric structure, they do not form a standard metric space but, by virtue of their order, it is possible to define a generalized metric, an $\omega_\mu$–metric.

$$\rho([r],[s]) = |[r] - [s]|$$

where $|| : \mathbb{R}^* \to (\mathbb{R}^*)^+$ is the absolute value in $\mathbb{R}^*$.

It is easy to see that the character of $\mathbb{R}^*$ is greater than $\omega_0$. In fact, suppose that $\{x_n\}$ is a sequence in $\mathbb{R}^*$. We can prove that it cannot converge to $0$, that is the class of the zero sequence. To this purpose consider the point represented by the class of $y$, where

$$\begin{aligned}
&y(1) = |x_1(1)|;\\
&y(2) = \min\{|x_1(2)|, |x_2(2)|\};\\
&\vdots\\
&y(m) = \min\{|x_1(m)|, ..., |x_m(m)|\};\\
&\vdots
\end{aligned}$$

Now we have that for all $n \in \mathbb{N}$, $\quad y < x_n$, in fact the set $\{m \in \mathbb{N} : y(m) < x_n(m)\}$ excludes only a finite number of indices.

This proves that the character of $\mathbb{R}^*$ is greater than $\omega_0$.

## 4.1 Definitions and properties

Here we define some concepts on an $\omega_\mu$–additive space $(X,\tau)$ that are analogous to the classic ones. The idea in defining these concepts is that the words *an enumerable sequence, a finite set, an enumerable set* should be replaced by *an $\omega_\mu$–sequence, a set of potency less than $\aleph_\mu$, a set of power $\aleph_\mu$*, respectively.

From now on we suppose that $\omega_\mu$ is a regular ordinal.

Observe that it can be shown that, in an $\omega_\mu$–additive space, every subset of power less than $\aleph_\mu$ is isolated and closed. So if a set $Y$ is dense in an $\omega_\mu$–additive space of power $\geq \aleph_\mu$, then $Y$ is also of power $\geq \aleph_\mu$. By these considerations we understand that the question whether there exists an enumerable dense subset in an $\omega_\mu$–additive with $\mu > 0$ has no topological sense. More generally, the question whether there exists a dense subset of power $< \aleph_\mu$ in an $\omega_\mu$–additive space with $\mu > 0$ has no topological sense. For this reason we need to modify this and some other concepts.

Let $(X, \tau)$ be an $\omega_\mu$– additive space. We say that:

- $(X, \tau)$ is $\omega_\mu$–*countable* iff it has a base of power $\aleph_\mu$,
- $(X, \tau)$ is $\omega_\mu$–*separable* iff there exists an everywhere dense subset $Y$ of $X$ of power less or equal to $\aleph_\mu$,
- $(X, \tau)$ is $\omega_\mu$–*compact* iff every $\omega_\mu$–sequence in $X$ has a convergent $\omega_\mu$–subsequence.

Furthermore, for an $\omega_\mu$–metric space $(X, \rho)$ we say that:

- $(X, \rho)$ is $\omega_\mu$–*totally bounded* iff for any $a \in A,\ a > 0$, there exists a subset $Y$ of $X$ of power less than $\aleph_\mu$ such that $\bigcup_{y \in Y} S_a(y) = X$, where $S_a(y) = \{z \in X\ :\ \rho(y, z) < a\}$,
- an $\omega_\mu$–sequence $\{x_\alpha\}$ is a *Cauchy* $\omega_\mu$–*sequence* iff for all $a \in A, a > 0$ there exists $\overline{\alpha} < \omega_\mu$ such that, if $\beta, \gamma > \overline{\alpha}$, then $\rho(x_\beta, x_\gamma) < a$ ,
- $(X, \rho)$ is $\omega_\mu$–*complete* iff every Cauchy $\omega_\mu$–sequence converges.

We now list some useful theorems.

If $\rho$ is an $\omega_\mu$–metric on $X$ then:

**Theorem 4.1.** | *Every regular $\omega_\mu$–metric space with $\mu > 0$ is 0–dimensional.*

Recall that a topological space is called $0$–dimensional if, for every open set $G$, and for every element $x \in G$, there exists a set $H$ which is simultaneously open and closed and such that $x \in H \subset G$.

**Theorem 4.2.** *Every $\omega_\mu$–metric space is paracompact.*

**Theorem 4.3.** *Every $\omega_\mu$–metric space is an $\omega_\mu$–additive normal space.*

**Theorem 4.4.** *$(X, \tau_\rho)$ is $\omega_\mu$–separable iff it is $\omega_\mu$–countable.*

**Theorem 4.5.** *If $(X, \rho)$ is $\omega_\mu$–totally bounded, then $(X, \tau_\rho)$ is $\omega_\mu$–separable.*

**Theorem 4.6.** *The following three statements are equivalent on $(X, \tau_\rho)$:*

*i) Every $\omega_\mu$–sequence in $X$ has a convergent $\omega_\mu$–subsequence in $X$,*

*ii) Every open cover of $X$ of power $\aleph_\mu$ has a subcover of power $< \aleph_\mu$,*

*iii) Every open cover of $X$ has a subcover of power $< \aleph_\mu$.*

**Theorem 4.7.** *$(X, \rho)$ is $\omega_\mu$–complete iff $(X, \mathscr{U}_\rho)$ is complete in the uniform sense.*

**Theorem 4.8.** *If $(X, \tau_\rho)$ is $\omega_\mu$–compact then $(X, \rho)$ is $\omega_\mu$–complete and $\omega_\mu$–totally bounded.*

The converse of this last theorem holds for $\mu = 0$, but it is not true in general. However, in [4] the authors proved that it is true if we consider a strongly inaccessible cardinal that is also weakly compact.

Another kind of generalization involves Urysohn's metrization theorem. One may ask whether it is possible to obtain a similar result for $\mu > 0$. The answer is affirmative.

**Theorem 4.9.** *Every regular $\omega_\mu$–additive space that is $\omega_\mu$–countable is $\omega_\mu$–metrizable.*

This is possible by considering the space $\mathscr{D}_\mu$ of all $\omega_\mu$–sequences whose elements are the numbers $0$ and $1$. We can put on it the metric $\sigma$, where $\sigma(x,y) = 0$ iff $x = y$ and $\sigma(x,y) = \frac{1}{\xi_0}$ if $x \neq y$, with $\frac{1}{\xi_0} \in W_\mu$ and $\xi_0$ is the first ordinal where the sequences $x$ and $y$ differ. Here $W_\mu$ is the least algebraic field containing the set of all ordinals $\xi < \omega_\mu$.
It is easy to see that $\mathscr{D}_\mu$ is an $\omega_\mu$–metric space. Sikorski showed that $(\mathscr{D}_\mu, \sigma)$ is $\omega_\mu$–complete for all $\mu$, while it is $\omega_\mu$–totally bounded for all $\omega_\mu$ that are inaccessible cardinals. In [44] it is shown that this space is not $\omega_\mu$–compact for the first inaccesible cardinal $\omega_\mu$, so it represents a counterexample for the converse of thm. 4.8.
We can put on $\mathscr{D}_\mu$ also a simpler $\omega_\mu$ metric having the same properties. It is defined as follows. Let $\rho_\mu : \mathscr{D}_\mu \times \mathscr{D}_\mu \to \mathscr{D}_\mu$ be such that $\rho_\mu(x,y) = (0,0,...)$ if $x = y$ and $\rho_\mu(x,y) = 1_\alpha$ if $x \neq y$, where $\alpha$ is the least ordinal at which the sequences $x$ and $y$ differ and $1_\alpha$ is the $\omega_\mu$–sequence which is $1$ in the $\alpha$–th coordinate and $0$ elsewhere. The set $\mathscr{D}_\mu$ is a subset of the ordered abelian group $(J_\mu, +, <)$, where $J_\mu$ is the family of all $\omega_\mu$–sequences of integers, + is coordinatewise addition, and < is the lexicographic order. This metric, too, makes $\mathscr{D}_\mu$ an $\omega_\mu$–metric space which is $\omega_\mu$–complete for all $\mu$, $\omega_\mu$–totally bounded for all $\omega_\mu$ that are inaccessible cardinals and not $\omega_\mu$–compact for the first inaccesible cardinal $\omega_\mu$.
We can consider also a subspace of $\mathscr{D}_\mu$, $\mathscr{D}^0_\mu$. It is the set of all sequences $\{a_\eta\} \in \mathscr{D}_\mu$ such that the equality $a_\eta = 1$ holds only for a finite number of ordinals $\eta < \omega_\mu$. It can be shown that $\mathscr{D}^0_\mu$ is a dense in itself, compact, $\omega_\mu$–metric space.

We have said that it is possible to prove thm. 4.9 by introducing the space $\mathscr{D}_\mu$. More precisely the following result is used.

**Theorem 4.10.** *Every regular $\omega_\mu$–additive space $X$ that is $\omega_\mu$–countable is homeomorphic to a subset of $\mathscr{D}_\mu$.*

By thm. 4.9 R. Sikorski generalizes Urysohn's metrization theorem. However it provides only a sufficient condition for metrizability. Instead Shu-Tang [59] obtains a necessary and sufficient condition with the following result.

**Theorem 4.11.** *If $(X,\tau)$ is a regular $\omega_\mu$–additive topological space, then $(X,\tau)$ is $\omega_\mu$–metrizable iff there exists a $\aleph_\mu$ basis for $\tau$.*

In [62] the authors give a metrization theorem for uniform spaces. A useful result to this purpose is the following.

**Lemma 4.1.** *Let $(X,\mathscr{U})$ be a uniform space with a linearly ordered base and $\aleph_\mu$ be the least power of such a base. Then there exists an equivalent well ordered base of power $\aleph_\mu$.*

By this it is obtained the following result which connects $\omega_\mu$–metrizable spaces and uniform spaces with linearly ordered base.

**Theorem 4.12.** *A separated uniform space $(X,\mathscr{U})$ is $\omega_\mu$–metrizable iff $(X,\mathscr{U})$ has a linearly ordered base and $\aleph_\mu$ is the least power of such a base.*

The most interesting implication is the one that states that if there is a linearly ordered base, the space is $\omega_\mu$–metrizable. It is proved by considering the gage of pseudo-metrics $\{d_\alpha\}$ associated to the uniformity and then by constructing an $\omega_\mu$–metric in the following way: $\rho : X \times X \to J_\mu$, $[\rho(x,y)](\alpha) = 0$ if $d_\alpha(x,y) = 0$, otherwise $[\rho(x,y)](\alpha) = 1$. Actually the exact range of $\rho$ is $\mathscr{D}_\mu$. It is shown that $\rho$ is an $\omega_\mu$–metric and finally that the uniformity associated to $\rho$ coincides with the starting one.

**Remarks on completeness** The importance of $\mathscr{D}_\mu$ lies in the fact that, in particular, this space is *order complete*. This property allows us to define notions like distance of a point from a subset, diameter of a set and so on.

Usually order completeness is identified with Cauchy completeness, probably because in the space of reals $\mathbb{R}$ they coincides. However, in general, one is stronger than the other. Recall the following definitions.

**Definition 4.1.** *An ordered set* $(X,<)$ *is* Dedekind complete *(or* order complete*) iff every subset of* $X$ *having an upper bound has a least upper bound.*

**Definition 4.2.** *A metric space* $(X,d)$ *is* Cauchy complete *iff every Cauchy sequence in* $X$ *converges in* $X$.

The equivalence of these two notions holds if we are in an ordered field that is also Archimedean. It can be proved that a Dedekind complete ordered field is also Archimedean and, hence, Dedekind completeness implies Cauchy completeness. While the converse, in general, is not true.

## 4.2 Hyperspace convergences

### 4.2.1 Uniform bornological convergences on $\omega_\mu$–metric spaces

In the previous section we talked about uniform bornological convergences. Looking at the metric case to compare uniform bornological convergences , we identify two basic facts:*any metrizable uniformity is the finest one in its proximity class*,or in other words [3] is *total;* the latter: *the bornology of metrically bounded sets is stable under small metric enlargements*. Hence the key properties are: totality of uniformities when localized on bounded sets and, on the other hand, stability under small enlargements for bornologies. We emphasize that any proximity function from a total uniform space $(X,\mathscr{U})$ towards any proximity space is uniformly continuous.

In [3], the authors proved the following interesting result.

**Theorem 4.13.** | *Every uniform structure with a linearly ordered base is total.*

This theorem is based on the following relevant generalization of Efremovič Lemma.

**Lemma 4.2.** *Let $(X, \mathscr{U})$ be a uniform space with linearly ordered base and $U, W \in \mathscr{U}$ such that $W^4 \subset U$. Moreover let $\{x_\alpha\}_{\alpha \in A}, \{y_\alpha\}_{\alpha \in A}$ be some generalized sequences with a linearly ordered index set $A$, and assume that $(x_\alpha, y_\alpha) \notin U$ for $a \in A$. Then there exists a cofinal subset $\Gamma \subset A$ such that $(x_\beta, y_\gamma) \notin W$ whenever $\beta$ and $\gamma$ both belong to $\Gamma$.*

By this Lemma it is also possible to prove a result on uniform bornological convergences introduced in the previous section. First we give the following definitions.

**Definition 4.3.** *Let $(X, \mathscr{U})$ and $(Y, \mathscr{V})$ be uniform spaces and let $B$ be a subset of $X$. We say that the function $f : X \to Y$ is* strongly uniformly continuous *on $B$ if for all $V \in \mathscr{V}$ there exists $U \in \mathscr{U}$ so that if $\{x, y\} \cap B \neq \varnothing$ and $(x, y) \in U$, then $(f(x), f(y)) \in V$.*
*If $\mathscr{B}$ is a family of subsets of $X$, we say that $f$ is* strongly uniformly continuous on *$\mathscr{B}$ if it is strongly uniformly continuous on each member of $\mathscr{B}$.*
*If $\mathscr{U}$ and $\mathscr{V}$ are two uniformities on $X$, we say that $\mathscr{U}$ is* uniformly stronger *than $\mathscr{V}$ on $\mathscr{B}$ if the identity map $id : (X, \mathscr{U}) \to (X, \mathscr{V})$ is strongly uniformly continuous on $\mathscr{B}$.*

Observe that strong uniform continuity on a family stable under small enlargements coincides with uniform continuity.

**Theorem 4.14.** *Let $X$ be a Tychonoff space (not a singleton), $\mathscr{A}$, $\mathscr{B}$ two bornologies of $X$ and $\mathscr{U}$, $\mathscr{V}$ two compatible uniformities on $X$. Whenever $\mathscr{U}$ has a linearly ordered base, then the following are equivalent:*

1) *The $\mathscr{U}_{\mathscr{A}}$–convergence implies the $\mathscr{V}_{\mathscr{B}}$–convergence on $CL(X)$;*

2) *$\mathscr{B} \subseteq \mathscr{A}$ and $\mathscr{U}$ is uniformly stronger than $\mathscr{V}$ on $\mathscr{B}$;*

3) *For each $B \in \mathscr{B}$ and each $V \in \mathscr{V}$ there is $A \in \mathscr{A}$ and $U \in \mathscr{U}$ so that $[A, U] \subset [B, V]$.*

**Proof.** $1) \Rightarrow 2)$ First observe that the property $\mathscr{B} \subseteq \mathscr{A}$ has been established in theorem 3.11. Next, suppose $\{U_\mu : \mu \in M\}$ is a symmetric linearly ordered base for $\mathscr{U}$, $B \in \mathscr{B}$ and $V \in \mathscr{V}$. We have to show that there exists $\mu \in M$ so that if $\{x, y\} \cap B \neq \varnothing$ and $(x, y) \in U_\mu$ then $(x, y) \in V$. If it were not so, then for each $\mu \in M$ there would be a couple $(x_\mu, y_\mu) \in U_\mu$ with $y_\mu \in B$ and $(x_\mu, y_\mu) \notin V$. Choosen in $\mathscr{V}$ a symmetric diagonal nbhd $W$ with $W^4 \subset V$, then, thanks to Lemma 4.2, it would be possible to find a cofinal subset $\Lambda$ of $M$ so that $(x_\lambda, y_\mu) \notin W$ for each $\lambda, \mu \in \Lambda$. Consequently, after putting for each $\mu \in \Lambda$,

$$A_\mu := \{ x_\mu : \mu \in \Lambda \} \cup \{ y_\lambda : \lambda \in \Lambda \text{ and } \mu < \lambda \}$$

it would be easily seen that the net $\{A_\mu\}_{\mu \in \Lambda}$, done by closed subsets of $X$, should converge in $\mathscr{U}_{\mathscr{A}}$ to the closed set $A := \{ x_\mu : \mu \in \Lambda\}$. But, since no $A_\mu$ could belong to $[B, W][A]$, then $\{A_\mu\}_{\mu \in \Lambda}$ should not converge to $A$ in $\mathscr{V}_{\mathscr{B}}$, a contradiction. $2) \Rightarrow 3)$ For each $B \in \mathscr{B}$ and any $V \in \mathscr{V}$ it is enough to take $U_\mu$ accordingly to the hypothesis. So doing yields $[B, U_\mu] \subset [B, V]$. And the result follows. $3) \Rightarrow 1)$ is trivial. □

**Corollary 4.1.** *Let $X$ be a Tychonoff space (not a singleton), $\mathscr{U}, \mathscr{V}$ two compatible uniformities both with a linearly ordered base and $\mathscr{A}, \mathscr{B}$ two bornologies of $X$. Then, the uniform bornological convergences on $CL(X)$, $\mathscr{U}_{\mathscr{A}}$ and $\mathscr{V}_{\mathscr{B}}$, coincide if and only if $\mathscr{A}$ and $\mathscr{B}$ agree and, furthermore, $\mathscr{U}, \mathscr{V}$ are strongly uniformly equivalent on any bounded set.*

Now, it appears very natural to introduce the Attouch-Wets convergence on $CL(X)$ relative to an $\omega_\mu$–metric space $(X, \rho)$ and the collection of the $\rho$–bounded sets definable in the usual way (see theorem 4.12). Since the collection of $\rho$–bounded subsets is stable under small $\rho$–enlargements, the Attouch-Wets convergence is a uniformizable topology, see theorem 3.8. Moreover:

**Proposition 4.1.** *Whenever $(X, \rho)$ is an $\omega_\mu$–metrizable space and the bornology $\mathscr{B}_\rho$ of $\rho$–bounded sets has a base of power less than or equal to $\aleph_\mu$, then the relative Attouch-Wets topology is $\omega_\mu$–metrizable.*

**Proof.** If $\mathbb{U} = \{U\}$ is a linearly ordered base for the uniformity associated with $\rho$ of power $\aleph_\mu$ and $\mathbb{B} = \{B\}$ is a base for $\mathscr{B}_\rho$ of power less than or equal to $\aleph_\mu$, then it is very easily seen that the family $\{[B, U]\}$, where $B$ runs over $\mathbb{B}$, ordered by the usual inclusion, and $U$ runs over $\mathbb{U}$, ordered by the reverse inclusion, is a linearly ordered base of power $\aleph_\mu$.

□

Since the collection of $\rho$–bounded sets of any $\omega_\mu$–metric is a bornology stable under small $\rho$–enlargements, summarizing theorems 3.6, 3.10 and corollary 4.1, we have that:

**Theorem 4.15.** *Let $X$ stand for an $\omega_\mu$–metrizable space, $\rho_1, \rho_2$ two compatible $\omega_\mu$–metrics on $X$ and $\mathscr{B}_1$, $\mathscr{B}_2$ the bornologies of their bounded sets, respectively. Then the following are equivalent:*

*$a)$ The Attouch-Wets topologies relative to $\rho_1$ and $\rho_2$ match.*

*$b)$ $\mathscr{B}_1$ and $\mathscr{B}_2$ agree and $\rho_1, \rho_2$ are proximally equivalent on any bounded set.*

*$c)$ The hit and far-miss topology associated with the natural proximity of $\rho_1$ and $\mathscr{B}_1$ agrees with the hit and far-miss topology associated with the natural proximity of $\rho_2$ and $\mathscr{B}_2$.*

**Proof.** $a) \Rightarrow b)$ It follows from corollary 4.1 and from observing that, as in this case where uniformities are total, strong uniform continuity, uniform continuity and proximal continuity are the same on a bornology when stable under small enlargements. $b) \Rightarrow c)$ By the constructive procedure we introduced (see page 29), any $\omega_\mu$–metric space $(X, \rho)$ comes associated with the local proximity space $(X, \delta_{\mathscr{U}\rho,\mathscr{B}_\rho}, \mathscr{B}_\rho)$. The relative hit and bounded far-miss topology agrees with the two-sided uniform bornological convergence induced by $\mathscr{B}_\rho$ and the uniformity $\mathscr{U}^*_{\mathcal{A}}$ of all nearness-boundedness preserving maps from $X$ to $[0,1]$ which have a bounded support. So, the result follows by applying theorems 3.6, 3.10. Finally, $c) \Rightarrow a)$. When we look at hit and bounded far-miss topologies as bornological topologies, then the comparison of these last ones yields that the bornologies of bounded sets coincide and the localizations over bounded sets of the identity map are proximity isomorphisms.

### 4.2.2 Hausdorff hypertopologies

Here we present some results contained in an article written jointly with Professor A. Di Concilio, [28]. We deal with $\omega_\mu$–metrics with values in linearly ordered Dedekind complete abelian groups. This allow us to define concepts as distance between sets, diameter of a set and to define the Hausdorff metric on closed sets.

Let $(X, d_\mu)$ an $\omega_\mu$–metric space with $d_\mu : X \times X \to G$, where $G$ is a *Dedekind complete* totally ordered abelian group with character $\omega_\mu$. Let $CL(X)$ be the set of all non-empty closed sets of $X$. In [62], F.W. Stevenson and W.J. Thron define the Hausdorff distance between two members of $CL(X)$ in the following way:

$$\hat{d}_\mu(A, B) = glb\{\alpha \in G : A \subset S_\alpha[B], B \subset S_\alpha[A]\},$$

where $S_\alpha[A] = \{x \in X : d_\mu(x, A) = glb\{d_\mu(x, y) : y \in A\} < \alpha\}$.

It has been proved that $(CL(X), \hat{d}_\mu)$ is an $\omega_\mu$–metric space.

We want to compare Hausdorff hypertopologies related to different compatible metrics on a same space. Denote by $\tau_{H_{d_\mu}}$ the Hausdorff topology on $CL(X)$ related to the $\omega_\mu$–metric $d_\mu$ on $X$.

Now, by the same procedure used in the classical case [12], we obtain the following result.

**Theorem 4.16.** *Let $(X, d_\mu)$ an $\omega_\mu$–metric space and $\rho_\mu$ another compatible $\omega_\mu$–metric. Suppose $d_\mu : X \times X \to G$, $\rho_\mu : X \times X \to F$, where $G$ and $F$ are Dedekind complete totally ordered abelian groups with character $\omega_\mu$. Then $\tau_{H_{d_\mu}} = \tau_{H_{\rho_\mu}}$ on $CL(X)$ if and only if $d_\mu$ and $\rho_\mu$ are uniformly equivalent.*

**Proof.** First suppose that the $\omega_\mu$–metrics are uniformly equivalent. We want to prove that the related Hausdorff hypertopologies on $CL(X)$ agree. As a matter of fact, in this case we can prove not only that the hypertopologies agree, but also uniform equivalence of the induced Hausdorff distances. In fact, fixed any $\alpha$ in $F$ and supposed that $\hat{\rho}_\mu(A,B) = \gamma < \alpha$ for $A, B \in CL(X)$, by the uniform equivalence of the $\omega_\mu$–metrics we can find $\delta \in G$ such that, for each non-empty subset $C$ of $X$,

$$S^\delta_{d_\mu}[C] \subseteq S^\gamma_{\rho_\mu}[C].$$

So now if we take $A, B$ such that $\hat{d}_\mu(A,B) < \delta$, we obtain $\hat{\rho}_\mu(A,B) < \alpha$. For the converse it is the same.

Now suppose that $id : (X, d_\mu) \to (X, \rho_\mu)$ is not bi-uniformly continuous. Then there exists $\epsilon \in F$ and two $\omega_\mu$–sequences $\{x_\delta\}_{\delta<\omega_\mu}$ and $\{y_\delta\}_{\delta<\omega_\mu}$ such that, for all $\delta \in G$, $d_\mu(x_\delta, y_\delta) < \delta$ but $\rho_\mu(x_\delta, y_\delta) \geq \epsilon$. Observe that in $G$ we can consider an $\omega_\mu$–sequence converging to $0$. Now take $W \in \mathscr{U}(\rho_\mu)$ such that $W^4 \subseteq S^\epsilon_{\rho_\mu}$. By lemma 4.2 there exists a cofinal subset $\Lambda$ in the index set such that $(x_\beta, y_\gamma) \notin W$ for all $\beta, \gamma \in \Lambda$. Define $A_\gamma = \{x_\delta,\ \delta \in \Lambda\} \cup \{y_\delta,\ \delta > \gamma,\ \delta \in \Lambda\}$, $\gamma \in \Lambda$, and $A = \{x_\delta,\ \delta \in \Lambda\}$. Observe that the $\omega_\mu$–sequences $\{x_\delta\}, \{y_\delta\}$ cannot have cluster points because the $\omega_\mu$–metrics are compatible with the same topology. Hence $A$ and each $A_\gamma$ are closed. Now we can see that $\hat{d}_\mu(A_\gamma, A) \to 0$ while $\hat{\rho}_\mu(A_\gamma, A) \not\to 0$. In fact, $(x_\beta, y_\delta) \notin W,\ \ \forall \beta, \delta \in \Lambda$ and so, if we take $S^\alpha_{\hat{\rho}_\mu} \subseteq W$ with $\alpha \in F$, it is impossible to find an index $\overline{\gamma}$ such that $A_\gamma \subset S^\alpha_{\hat{\rho}_\mu}[A]$ for all $\gamma > \overline{\gamma}$. It follows that $\tau_{H_{d_\mu}} \neq \tau_{H_{\rho_\mu}}$.

□

We can give also a further formulation of the previous result. It is the following.

**Theorem 4.17.** *Two uniformities with linearly ordered base give rise to the same Hausdorff hypertopology on $CL(X)$ if and only if they coincide.*

Observe that the previous result can be obtained also as a consequence of corollary 4.1.

### 4.2.3 Hausdorff convergence vs Kuratowski convergence

Now we want to present some analogues of classical theorems involving Hausdorff and Kuratowski hyperconvergences. Recall the following facts.

**Definition 4.4.** *Let $(X, \tau)$ be a Hausdorff topological space, and let $\{A_\lambda\}_{\lambda\in\Lambda}$ be a net of subsets of $X$. A point $x_0$ is said a* limit point *of $\{A_\lambda\}_{\lambda\in\Lambda}$ if each nhbd of $x_0$ intersects $A_\lambda$ for all $\lambda$ in some residual subset of $\Lambda$. A point $x_1$ is said a* cluster point *of $\{A_\lambda\}_{\lambda\in\Lambda}$ if each nhbd of $x_1$ intersects $A_\lambda$ for all $\lambda$ in some cofinal subset of $\Lambda$.*

The set of all limit points of the net $\{A_\lambda\}_{\lambda\in\Lambda}$ is denoted by $LiA_\lambda$ and it is called *lower limit*, while the set of all cluster points of $\{A_\lambda\}_{\lambda\in\Lambda}$ is denoted by $LsA_\lambda$ and it is called *upper limit*. Moreover we say that $\{A_\lambda\}_{\lambda\in\Lambda}$ is *Kuratowski convergent* to $A$ if and only if $A = LiA_\lambda = LsA_\lambda$. Observe that $LiA_\lambda \subset LsA_\lambda$, so to verify the convergence it suffices to prove that $A \subset LiA_\lambda$ and $LsA_\lambda \subset A$. In general, Kuratowski convergence is not topological. But it is if and only if the space $X$ is $T_2$ and locally compact. In this case Kuratowski convergence matches with the Fell convergence, while generally the former implies the latter.

From now on let $(X, d_\mu)$ an $\omega_\mu$–metric space with $d_\mu : X \times X \to G$, where $G$ is a *Dedekind complete* totally oredered abelian group with character $\omega_\mu$.

**Theorem 4.18.** *Let $(X, d_\mu)$ be an $\omega_\mu$–metric space. Then, in $CL(X)$,*

$$\{A_\lambda\}_{\lambda<\omega_\mu} \xrightarrow{H} A \Rightarrow \{A_\lambda\}_{\lambda<\omega_\mu} \xrightarrow{K} A$$

**Proof.** We want to show that $A \subseteq Li\{A_\lambda\}$ and $Ls\{A_\lambda\} \subseteq A$. Let begin from $A \subseteq Li\{A_\lambda\}$. Fix $x \in A$ and a nbhd of $x$, $U_x$. We have to prove that $U_x \cap A_\lambda \neq \varnothing$

eventually. We can find $\alpha \in G$ such that $S_\alpha[x] \subseteq U_x$. By the Hausdorff convergence we know that $A \subseteq S_\alpha[A_\lambda]$ eventually. So there exists $a_\lambda \in A_\lambda : d_\mu(x, a_\lambda) < \alpha$ eventually, and it means that $a_\lambda \in S_\alpha[x] \subseteq U_x$ eventually. Hence $U_x \cap A_\lambda \neq \varnothing$ eventually.

That $Ls\{A_\lambda\} = A$ is already known by a result of J.R.Isbell in [39].

□

It is possible to prove the converse of this theorem if we consider the hyperspace $K_\mu(X)$ of non-empty $\omega_\mu$–compact subsets of $X$ and we add a compactness condition on $A \cup (\bigcup_{\lambda<\omega_\mu} A_\lambda)$. First of all we show the following fact.

**Lemma 4.3.** *Let $(X, d_\mu)$ be an $\omega_\mu$–metric space. Then*

$$\{A_\lambda\}_{\lambda<\omega_\mu} \xrightarrow{H} A \textit{ and } A, A_\lambda \textit{ are } \omega_\mu\textit{–compacts } \forall \lambda < \omega_\mu \Rightarrow A \cup (\bigcup_{\lambda<\omega_\mu} A_\lambda) \textit{ is } \omega_\mu\textit{–compact}$$

**Proof.** Pick $x_\lambda \in A_\lambda$ for $\lambda < \omega_\mu$. If the net $\{x_\lambda\}$ is contained in a non cofinal subnet of $\{A_\lambda\}$ then, by the regularity of $\omega_\mu$, there must be an $\omega_\mu$–subsequence of $x_\lambda$ in some $A_{\overline{\lambda}}$. So, by compactness of $A_{\overline{\lambda}}$, we can find a convergent $\omega_\mu$–subsequence of $\{x_\lambda\}$. Observe that union of less than $\aleph_\mu$ $\omega_\mu$–compact sets is $\omega_\mu$–compact. Instead, suppose that the net $\{x_\lambda\}$ is contained in a cofinal subnet of $\{A_\lambda\}$. By the hypothesis we know that $\hat{d}_\mu(A_\lambda, A) \to 0$. So we can find an $\omega_\mu$–sequence $\{a_\lambda\}_{\lambda<\omega_\mu} \subset A$ such that $d_\mu(x_\lambda, a_\lambda) \to 0$. Being $A$ compact, it is possible to find a convergent $\omega_\mu$–subsequence of $\{a_\lambda\}$. By the adjacency of the $\omega_\mu$–sequences $\{x_\lambda\}$ and $\{a_\lambda\}$ we obtain that $\{x_\lambda\}$ has a convergent $\omega_\mu$–subsequence, too. So $A \cup (\bigcup_{\lambda<\omega_\mu} A_\lambda)$ is $\omega_\mu$–compact.

□

**Theorem 4.19.** *Let $(X, d_\mu)$ be an $\omega_\mu$–metric space. Then, in $K_\mu(X)$,*

$$\{A_\lambda\}_{\lambda<\omega_\mu} \xrightarrow{H} A \Leftrightarrow \{A_\lambda\}_{\lambda<\omega_\mu} \xrightarrow{K} A \text{ and } A \cup (\bigcup_{\lambda<\omega_\mu} A_\lambda) \text{ is } \omega_\mu\text{–compact}$$

**Proof.** " $\Rightarrow$ ". This implication is achieved by Thm. 4.18 and Lemma 4.3.
" $\Leftarrow$ ". We know that $\{A_\lambda\}_{\lambda<\omega_\mu} \xrightarrow{K} A$. So, being $A \subseteq Li\{A_\lambda\}$, for all $a \in A$ and for all $\delta \in G$, there exists $a_\lambda \in A_\lambda$ such that $d_\mu(a, a_\lambda) < \delta$ eventually. Hence for all $S_\delta,\ \delta \in G,\ \ A \subseteq S_\delta[A_\lambda]$ eventually. Now suppose by contradiction that there exists $\gamma \in G$ such that $A_\lambda \nsubseteq S_\gamma[A]$ cofinally. So there is an $\omega_\mu$–sequence $\{a_{\lambda_h}\} \in A_{\lambda_h}$ such that $d_\mu(a_{\lambda_h}, a) \geq \delta$ for all $a \in A$. But $A \cup (\bigcup_{\lambda<\omega_\mu} A_\lambda)$ is $\omega_\mu$–compact, so there exists an $\omega_\mu$–subsequence converging to a point $x$. Observe that $x \in Ls\{A_\lambda\}$ that, by Kuratowski convergence, coincides with $A$. Hence $d_\mu(a_{\lambda_k}, x) \to 0$. A contradiction.

□

# 5 Atsuji spaces

*Atsuji spaces* play an important role above all because they allow us to deal with a very nice structure when we concentrate on the most significant part of the space, that is the derived set. Moreover, we know that each continuous function between metric or uniform spaces is uniformly continuous on compact sets. It is possible to have an analogous property on a larger class of topological spaces, Atsuji spaces.
The class of uc spaces is located between the class of the complete metric spaces and that of the compact ones. In this section we want to recall some basic facts about Atsuji spaces and to present some results on this topic obtained jointly with Professor A. Di Concilio, [29].

## 5.1 Definitions

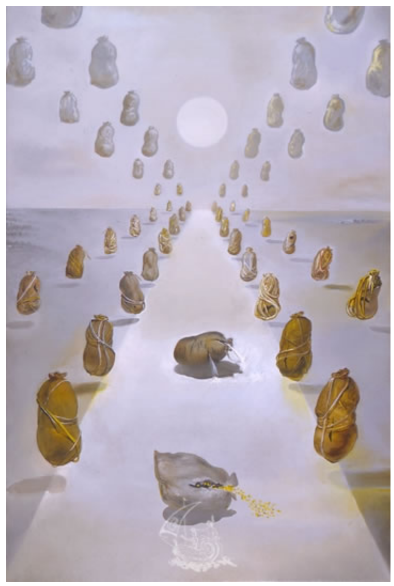
**Definition 5.1.** *A metric space $(X,d)$ is an* Atsuji space *(or also UC space ) iff each real-valued continuous function on $X$ is uniformly continuous.*

The versatility of metric *uc-ness* is witnessed by a very long list of papers containing various, a priori far from each other formulations. *Atsuji spaces* play an important role because they allow us to deal with a very nice structure when we concentrate on the most significant part of the space, that is the derived set. In fact one of the many characterizations is the following one.

- The derived set $X'$is compact and $X \setminus S_\epsilon[X']$ is uniformly discrete for each $\epsilon > 0$,where $S_\epsilon[X']$ is the $\epsilon$-collar of $X'$.

*This property emphasizes the fact that compact spaces and uniformly dicrete spaces can be considered as generators for the class of Atsuji spaces.*
Furthermore the metric *uc-ness* is characterized as the uniform normality, any two disjoint nonempty closed sets have a positive distance apart. This property was then

reformulated as a relationship between hypertopologies on the hyperspace $CL(X)$, of all non-empty closed subsets of $X$: the Vietoris topology is weaker than the Hausdorff topology induced by the Hausdorff metric. Another interesting topological formulation is the following one: each Hausdorff quotient of $X$ is (pseudo) metrizable.



”The path of the enigma”, S. Dalı’

Some other metric formulations are the following ones:

- Each pseudo-Cauchy sequence in $X$ with distinct terms has a cluster point
- Each open cover of $X$ has a Lebesgue number

We recall:

**Definition 5.2.** *Given a metric space $(X, d)$ and an open cover $\mathcal{E} = \{E_i\}_{i \in I}$ of $X$, we say that $\mathcal{E}$ has a* ***Lebesgue number*** *if and only if there exists a real number $r > 0$ such that for each $x \in X$ there exists an index $i \in I$ for which $B(x, r) \subseteq E_i$.*

We also recall what is a *pseudo-Cauchy* sequence.

**Definition 5.3.** *A sequence $x_n$ in a metric space $(X, d)$ is called* ***pseudo-Cauchy*** *if for each $\epsilon > 0$ and $\nu \in \mathbb{N}$, there exist distinct indices i and j exceeding $\nu$ for which $d(x_i, x_j) < \epsilon$.*

Intuitively, a *pseudo-Cauchy* sequence is a sequence in which pairs of terms are arbitrary close frequently. Let us observe that a metric space is *complete* when each Cauchy sequence with distinct terms has a cluster point. In analogy with this fact, a metric space is an *Atsuji space* when each *pseudo-Cauchy* sequence with distinct terms has a cluster point.
Furthermore, while any subsequence of a Cauchy sequence is itself a Cauchy sequence, for pseudo-Cauchy sequence this is not true.

**Remark 5.1.** *Observe that Atsuji spaces are located between complete spaces and compact ones and it is easy to find examples of spaces that are complete but not Atsuji, or Atsuji but not compact. In the first case we can consider $\mathbb{R}$, while in the second one we can look at $\mathbb{N}$. In fact $\mathbb{N}$ is Atsuji because $\mathbb{N}' = \emptyset$ and the space is uniformly discrete.*

$$compactness \Rightarrow Atsujiness \Rightarrow completeness$$

Different formulations have led to several generalizations, some of them in terms of localization and some other in weakening conditions. For example *boundedly uc-ness* is defined as follows: any real-valued continuous function is uniformly continuous on bounded sets. The role played in the uc-metric spaces by compact metric spaces is now played by boundedly compact metric spaces, i.e. metric spaces in which all closed balls are compact.

## 5.2 Remetrization

It may happen that a metric space $X$ is not *Atsuji* but it can be remetrised to obtain such a kind of space leaving unchanged the topology. We call these spaces *Nagata spaces.* So:

**Definition 5.4.** *A metrizable space $X$ is a* ***Nagata space*** *if and only if it admits a topologically compatible metric that is* Atsuji*.*

But for which metrizable spaces does it hold? There is the following result, [13, 46].

**Proposition 5.1.** *Let $X$ be a metrizable space. $X'$ is compact iff $X$ is a* Nagata *space.*

In fact, if $X'$ is compact and $d$ is a compatible metric for $X$, it can be shown that

$$\rho(x,y) = \begin{cases} 0 & \text{if } x = y \\ d(x,y) + max\{d(x,X'), d(y,X')\} & \text{if } x \neq y \end{cases}$$

is a compatible metric that makes $X$ an *Atsuji space*.

## 5.3 Atsuji extensions

Now we want to study when a metrizable space $X$ admits a dense *Atsuji extension* in which it topologically embeds.

**Remark 5.2.** *Observe that, if we want to construct a dense extension in which $X$ is isometrically embedded, we obtain just the metric completion of $X$. In fact, an Atsuji space is also complete; then we have a complete dense extension of $X$ in which $X$ is isometrically embedded. But because the metric completion is unique up to isometries, the space obtained is just* the *metric completion of $X$.*

By the previous remark, it is clear the reason why we study the case of *Atsuji* extensions in which the space $X$ is topologically embedded.

**Definition 5.5.** *We say that a metrizable space $X$ is **Atsuji extendable** if and only if it admits a topological dense extension that is an* Atsuji space*.*

We have the following result.

**Theorem 5.1.** *Let $X$ be a metrizable space. $X$ is* Atsuji extendable *if and only if $X'$ is separable.*

Before proving this theorem we want to recall some results. The first one is due to the Urysohn's metrization theorem.

**Theorem 5.2.** *A metrizable space admits a metric totally bounded which is compatible if and only if the space is separable.*

Recall also the following result about extensions of metrics [64].

**Theorem 5.3.** *(Hausdorff) If $X$ is any metrizable space, $A$ is a closed subspace of $X$, and $\rho$ is a compatible metric on $A$, then $\rho$ can be extended to a compatible metric on $X$.*

*Proof.* (*of Thm. 5.1*)
"$\Rightarrow$" Suppose that $X$ admits a dense topological extension $Y$ that is an *Atsuji space*. So there is a metric d on $Y$, compatible with the topology on $X$, that makes $Y'$ compact and then totally bounded. But $X' \subseteq Y'$ implies that $X'$ is totally bounded in a compatible metric. So $X'$ is separable by theorem 5.2.
"$\Leftarrow$" Assume that $X'$ is separable. Hence there exists a compatible metric $\rho$ on $X'$ that makes it totally bounded. But $X'$ is a closed set in $X$, so by the theorem 5.3 we can extend the metric $\rho$ on the whole space $X$ and we obtain a metric $\overline{\rho}$ that coincides with $\rho$ on $X'$. Now we can construct the metric completion of $(X, \overline{\rho})$ and we obtain $(\hat{X}, \hat{\rho})$ such that $\hat{\rho}$ coincides with $\overline{\rho}$ on $X$.

We now consider the space

$$Y = Cl_{\hat{X}}(X') \cup I(X)$$

where $Cl_{\hat{X}}(X')$ is the closure of $X'$ in $\hat{X}$, and $I(X)$ is the set of isolated points of $X$.

Observe that $Y$ is contained in $\hat{X}$. We will reach our aim if we prove that $X$ is dense in $Y$ and $Y'$ is compact (Prop. 5.1).

First we want to prove the density, that is $cl_Y(X) = Y$. One inclusion ($\subseteq$) is obvious. For the other ($\supseteq$) we can distinguish two cases, $y \in I(X)$ or $y \in cl_{\hat{X}}(X')$. If $y \in I(X)$ the inclusion is clear. If $y \in cl_{\hat{X}}(X')$, we can observe that $cl_{\hat{X}}(X') = cl_Y(X')$ in particular because, if $y \in cl_{\hat{X}}(X')$, surely $y \in Y$ and it is an accumulation point for $X'$, so $y \in cl_Y(X')$.

Now we prove that $Y'$ is compact. To do this we prove that $Y' = cl_{\hat{X}}(X')$. In fact, in this case, we will have $Y'$ totally bounded because $X'$ is totally bounded in the metric $\hat{\rho}$, and so also $cl_{\hat{X}}(X')$. But $Y'$ is also complete because $Y' \subseteq \hat{X}$, and so we will conclude that $Y'$ is compact.

To verify that $Y' = cl_{\hat{X}}(X')$ we observe that obviously $Y' \subseteq Y = Cl_{\hat{X}}(X') \cup I(X)$ but, if $z \in Y'$, it can't be in $I(X)$. In fact, if $z \in Y'$, then

$$\forall S_\epsilon(z) \qquad S_\epsilon(z) \cap Y \setminus \{z\} \neq \varnothing$$

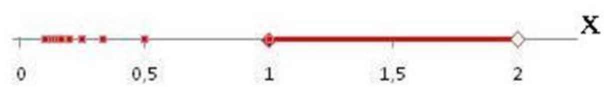
where $S_\epsilon(z)$ is a nhood in the metric $\hat{\rho}$.

But if $z$ is also in $I(X)$, then there exists $\overline{\epsilon}$ such that $S_{\overline{\epsilon}}(z) \cap Y \setminus \{z\}$ does not contain further elements of $X$. So it has to contain elements of $cl_{\hat{X}}(X')$, but it is absurd because in this case $S_{\overline{\epsilon}}(z)$would contain also elements of $X$.

So $Y' \subseteq cl_{\hat{X}}(X')$. The reverse is easy. □

The previous proof is a constructive one, so we know how to present an *Atsuji extension* of a given space $X$. But we can ask if the found extension is really different from the metric completion of $X$. We show an example.

**Example 5.1.** *Let us consider the space* $X = \left\{\frac{1}{n}\right\} \cup [1,2[$, *with* $n \in \mathbb{N}$, *endowed with the euclidean metric* $d_e$. *In this case* $X' = [1,2[$. *It is separable, so* $X$ *is Atsuji extendable.*




*The metric completion is*

$$\hat{X} = \{0\} \cup \left\{\frac{1}{n}\right\} \cup [1,2]$$

*while the Atsuji extension that we can construct following the previous procedure is*

$$Y = cl_{\hat{X}}(X') \cup I(X) = \left\{\frac{1}{n}\right\} \cup [1,2].$$

*It is a Nagata space considered with the euclidean metric, and an Atsuji space if it is endowed with the metric* $\rho$, *relating to* $d_e$, *exhibited in prop. 5.1.*
*So the metric completion and the Atsuji extension are actually different.*

## 5.4 Extension of functions

Now, given an *Atsuji extendable* space $X$ and a continuous function $f : X \to \mathbb{R}$ , the question is: " Under which assumptions can we continuously extend $f$ to the *Atsuji extension* $Y$ of $X$?".
To give an answer to this question we need the following results.

**Theorem 5.4.** *[Tietze's extension theorem] $X$ is normal if and only if whenever $A$ is a closed subset of $X$ and $f : A \to \mathbb{R}$ is continuous, there is a continuous extension of $f$ to all of $X$: i.e. there is a continuous map $F : X \to \mathbb{R}$ such that $F|A = f$.*

**Proposition 5.2.** *A uniformly continuous function $f$ on a subset $A$ of a metrizable space $X$ to a complete metrizable space can be extendable to a uniformly continuous function $\overline{f}$ on $clA$.*

By these results it is quite simple to prove the following one.

**Proposition 5.3.** *Given an Atsuji extendable space $X$ and a continuous function $f : X \to \mathbb{R}$, we can continuously extend $f$ on the Atsuji extension of $X$, $Y$, if and only if $f$ is uniformly continuous on $X'$.*

**Proof.** " $\Leftarrow$ " Let us suppose that $f$ is u.c. on $X'$. By the Prop.5.2 we can continuously extend $f$ on $cl_{\hat{X}}(X')$. But $cl_{\hat{X}}(X')$ is a closed set of $Y$ and we can use the *Tietze 's extension theorem* to obtain $f$ continuous on $Y$.
" $\Rightarrow$ " Consider $f$ continuous on $Y$. Then $f$ is continuous on $cl_{\hat{X}}(X')$ that is compact. So $f$ is uniformly continuous on $X'$.

We point out that, in the all procedure of extension of the space and of functions, the attention is always focused on $X'$. So $X'$ represents the heart and we have only to know what happens on it. If it has the right properties, we will be able to construct our extensions.

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Now we can consider the set

$$\mathcal{A} = \{f : X \to \mathbb{R} | f \text{ is continuous on } X \text{ and u.c. on } X'\}$$

and ask what is its algebraic structure.
Observe that:

1) $f, g \in \mathcal{A} \Rightarrow f + g \in \mathcal{A}$ **(closure for the sum)**

2) $\alpha \in \mathbb{R}, f \in \mathcal{A} \Rightarrow \alpha f \in \mathcal{A}$ **(closure for the scalar product)**

3) $f, g \in \mathcal{A} \Rightarrow fg \in \mathcal{A}$ **(closure for the product)**

Observe that,with our assumptions, $f(X')$ and $g(X')$ are bounded in $\mathbb{R}$. In fact, if $f$ is u.c. on $X'$, it can be extended continuously to the *Atsuji extension* of $X$,$Y$, by a function $\hat{f}$. But $Y'$ is compact because $Y$ is *Atsuji*, so $\hat{f}(Y')$ is bounded in $\mathbb{R}$. Hence, by

$$f(X') = \hat{f}(X') \subseteq \hat{f}(Y')$$

we have $f(X')$ bounded in $\mathbb{R}$. The same holds for $g(X')$.

4) $\forall f, g, h \in \mathcal{A} \quad (f + g) + h = f + (g + h)$ and $(fg)h = f(gh)$ **(associativity for the sum and the product)**

5) $\forall f, g \in \mathcal{A} \quad f + g = g + f$ and $fg = gf$ **(commutativity for the sum and the product)**

6) $\exists e \in \mathcal{A} : \forall f \in \mathcal{A} \quad e + f = f + e = f$ **(identity element for the sum)**

7) $\exists u \in \mathcal{A} : \forall f \in \mathcal{A} \quad uf = fu = f$ **(identity element for the product)**

8) $\forall f \in \mathcal{A} \quad \exists g \in \mathcal{A} : f + g = g + f = e$ **(inverse element for the sum)**

9) $\forall f, g, h \in \mathcal{A}\ f(g + h) = fg + fh$ **(distributivity of product over sum)**

By the previous properties we can conclude that $(\mathcal{A}, +, \cdot\,)$ is a **unitary commutative ring**. Moreover, if we indicate by '$*$' the product of a real number and a function in $\mathcal{A}$, we have that $(\mathcal{A}, +, \cdot\,, *)$ is a **unitary commutative algebra on** $\mathbb{R}$.

Now we want to point out what is about composition of functions.

- Suppose we have

  $f : X \to Y$, with $Y$ complete metric space, $f$ continuous on $X$ and u.c. on $X'$

  $g : Y \to Z$, with $Z$ complete metric space, $f$ continuous on $Y$ and u.c. on $Y'$.

  Then $g \circ f : X \to Z$ is a continuous function on $X$, but is it u.c. on $X'$?

Obviously, if $f(X') \subseteq Y'$, then $g \circ f$ is u.c. on $X'$. But when does it happen? Suppose $x$ is in $X'$. So for all $I_x$, nhood of $x$, $I_x \setminus \{x\} \cap X \neq \varnothing$. Now, $f(x) \in Y'$? Let $U_{f(x)}$ be any nhood of $f(x)$. Because $f$ is continuous, there exists $I_x \ni x$ such that $f(I_x) \subseteq U_{f(x)}$. But, if $f$ is injective, $I_x \setminus \{x\} \cap X \neq \varnothing$ implies $f(I_x) \setminus \{f(x)\} \cap f(X) \neq \varnothing$. Hence, by

$$f(I_x) \setminus \{f(x)\} \cap f(X) \subseteq U_{f(x)} \setminus \{f(x)\} \cap Y$$

we have that, if f is injective, $g \circ f$ is u.c. on $X'$.

# 6 Appendix

Working with Professor J.F. Peters we were interested in finding some new forms of proximity. On one side, starting from a Lodato proximity, we wanted to distinguish between a weaker form of farness and a stronger one: for this reason we introduced the concept of *strong farness*. On the other side we wanted to define a kind of nearness related to pair of sets with at least non-empty intersection: to this purpose we introduced *strong nearness*. Related to this concepts we introduced some new kinds of hit-and-miss hypertopologies, new concepts of continuity and connectedness. This research is still a work in progress. So here we want only to present some ideas.

Let $X$ be a nonempty set and $\delta$ be a *Lodato proximity* on $\mathscr{P}(X)$.

**Definition 6.1.** *We say that $A$ and $B$ are $\delta-$strongly far and we write $A \underset{\mathbb{W}}{\not\delta} B$ if and only if $A \not\delta B$ and there exists a subset $C$ of $X$ such that $A \not\delta X \setminus C$ and $C \not\delta B$, that is the Efremovič property holds on $A$ and $B$.*

Observe that $A \not\delta B$ does not imply $A \underset{\mathbb{W}}{\not\delta} B$. In fact, when the proximity $\delta$ is not an *EF-proximity* we can find a pair of subsets, $A$ and $B$, such that $A \not\delta B$ but there isn't any subset $C$ of $X$ with $A \not\delta X \setminus C$ and $C \not\delta B$.

We know that, if we consider a *local proximity space*, the proximity is Efremovič only when restricted to the elements of the boundedness. The definition of strong farness aims to generalize this concept. Actually there is a perspective change.

**Example 6.1.** *Consider the* Alexandroff proximity*: $A\delta_A B \Leftrightarrow A\delta_0 B$ or both $clA$ and $clB$ are non-compact. We know that in a $T_1$ topological space this is a compatible Lodato proximity that is not Efremovič if the space is not locally compact. Suppose that $X$ is a non-locally compact $T_4$ space. In this case, if we take two far subsets $A$ and $B$ that are relatively compact, i.e. their closures are compact, they* *are also strongly far. In fact, if they are far, they have empty intersection. Being $X$ a normal space, we can find two open sets $C$ and $D$ such that $cl(A) \subset C$, $cl(B) \subset D$ and $C \cap D = \varnothing$. Hence $A \not\delta_A D$ and $X \setminus D \not\delta_A B$ because more at least one set for each pair has compact closure. So the family of pairs of sets that are strongly far contains relatively compact sets and it is different from $\mathscr{P}(X) \times \mathscr{P}(X)$ being the proximity not Efremovič.*

By this new proximity we define special hit-and-miss hypertopologies on CL$(X)$: *hit and strongly far-miss* hypertopologies, $\tau_{\mathbb{W}}$. These are topologies having as subbase the sets of the form:

- $V^- = \{E \in$ CL$(X) : E \cap V \neq \varnothing\}$, where $V$ is an open subset of $X$,
- $A_{\mathbb{W}} = \{ \ E \in$ CL$(X) : E \not\delta^{\delta}_{\mathbb{W}} X \setminus A \ \}$, where $A$ is an open subset of $X$

We prove that this topology is not comparable with the hit and far miss topology associated to $\delta$, the Lodato proximity related to $\not\delta^{\delta}_{\mathbb{W}}$.

Furthermore we generalize $\tau_{\mathbb{W}}$ by considering as miss sets $A_{\mathbb{W}}$ those generated by open sets whose complements run is special families $\mathscr{B}$. To the purpose of studying the Hausdorffness of these we refer to suitable definitions of local families.

Following the analogous line of the previous idea, we want to stress the concept of nearness by a relation giving information about pairs of subsets with at least non-empty intersection. Actually the purpose is to generalize the property of having non-empty intersection for the *interiors* of pairs of subsets of a topological space. For this reason we introduce some axioms defining the so called *strong proximities.* Even if the name contains the word *proximity*, they don't satisfy analogous axioms. In particular for proximities holds that if two subsets have non-empty intersection, they are near. This is not in general the case for strong proximities.

Having defined this framework, the natural continuation is to look at mappings that preserve such a kind of structure. We call these *strongly proximal continuous* mappings. In particular we focus on mappings that are strongly proximal continuous on special families of subsets.

Moreover, looking at connectedness and its properties it appears quite natural trying to generalize this concept using strong proximities. Actually we obtain a strengthening of the standard concept.

Recall the following property, [64].

**Theorem 6.1.** *If $X = \bigcup_{n=1}^{\infty} X_n$ where each $X_n$ is connected and $X_{n-1} \cap X_n \neq \emptyset$ for each $n \geq 2$, then $X$ is connected.*

We define a new kind of connectedness by substituting non-empty intersection with strong nearness. This new kind of connectedness is in general stronger than the standard one. By using this concept we obtain generalizations of standard results.

Strong farness and strong nearness lead us to consider in particular two fields of application. The first one is related to *descriptive proximities*; the second one concerns *Voronoï regions.*

• The theory of descriptive nearness [51] is usually adopted when dealing with subsets that share some common properties even being not spatially close. We talk about *non-abstract points* when points have locations and features that can be measured. The mentioned theory is particularly relevant when we want to focus on some of these aspects. For example, if we take a picture element $x$ in a digital image, we can consider graylevel intensity, colour, shape or texture of $x$. We can define an $n$ real valued probe function $\Phi : X \to \mathbb{R}^n$, where $\Phi(x) = (\phi_1(x), .., \phi_n(x))$ and each $\phi_i$ represents the measurement of a particular feature. So $\Phi(x)$ is a feature vector containing numbers representing feature values extracted from $x$. $\Phi(x)$ is also called *description* of $x$. Descriptively near sets contain points with matching descriptions.

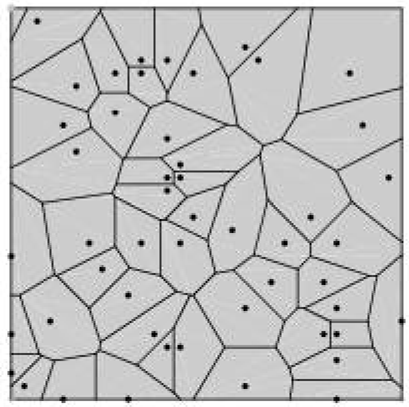

**Figure 1:** Voronoï diagram

• A *Voronoï diagram* represents a tessellation of the plane by convex polygons. It is generated by $n$ points and each polygon contains exactly one of these points. In each region there are points that are closer to its generating point than to any other. *Voronoï diagrams* were introduced by *René Descartes* (1667) looking at influence regions of stars. They were studied also by Dirichlet (1850) and Voronoï (1907), who extended the study to higher dimensions. An interesting use of *Voronoï* diagrams was done by the British physician John Snow in 1854 to show that most of the people who died in the Soho cholera epidemic lived closer to the infected Broad Street pump than to any other water pump.

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# Subject Index

# Acknowledgements

I would like to thank the following persons for their role in the completion of this thesis.

- My adviser Professor A. Di Concilio. She guided me with knowledge and humanity. She is a great Professor and a great woman.
- Professors J.F. Peters and S. Ramanna, charming persons who, in particular, made me enjoy my work experience in Canada.
- My boyfriend, Vincenzo, who takes infinity in the finite of every day.
- My parents, that have donated me a big treasure by their faith and support.
- My brothers, that are fundamental notes of my melody.
- My friends, in particular Albachiara, Federica and Serena, that with words, smiles and gestures are my special fellow travellers.