Abstract

Fractional derivative of the Riemann zeta function

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In this work of thesis, the Riemann zeta function was studied by using an unconventional approach. The reason for choosing this approach was to explore the many applications that the Riemann zeta has not only in pure mathematics, but also in tangential fields of theoretical physics and engineering. The use of fractional calculus allowed the computation of the α-order fractional derivative $\zeta^{(\alpha)}$. The biggest difficulty was represented by the fractional differentiation in the complex plane. In particular, two generalizations of the fractional derivative (Caputo derivative and Grünwald-Letnikov derivative) to the complex field were used in this thesis.

The first chapter includes several preliminaries on the analytics number theory and on fractional calculus. In the second chapter the computation of $\zeta^{(\alpha)}$ is given together with its convergence. $\zeta^{(\alpha)}$ is expressed as a complex series and represents a fractional generalization to the integer derivative of ζ. In fact, by replacing in the right hand side the fractional exponent α with the integer exponent k, $\zeta^{(\alpha)}$ becomes $\zeta^{(k)}$. Some properties of this derivative were obtained in order to show its chaotic decay to zero and several links with the analytic number theory.

The third chapter presents the computation of the functional equation together with some simplified versions, in accordance with the classical theory of the Riemann zeta function. Since the Caputo-Ortigueira fractional derivative does not satisfy the generalized Leibniz rule, the generalization of the Grünwald-Letnikov fractional derivative to complex plane must be introduced. The desired functional equation was obtained by starting from the asymmetric from of the functional equation of ζ. Further properties relating to this equation are proposed and comprehensively discussed in this chapter. Generalizations of this fractional derivative were obtained by introducing the series of Dirichlet, the Hurwitz zeta function and the Lerch zeta function. By using the generalization of the Grünwald-Letnikov fractional derivative, generalizations of the functional

equations associated to $\zeta^{(\alpha)}$ are given. In particular, the functional equation associated to the fractional derivative of the zeta Lerch have supplied new results, and new horizons for research seem to open in the fractional calculus functional. Additionally, an integral representation of $\zeta^{(\alpha)}$, in terms of numbers of Bernoulli is also presented. All of the aforementioned results are in agreement with the classical theory of Riemann zeta function.

In the fourth chapter, the link between $\zeta^{(\alpha)}$ and the distribution of prime numbers is discussed by using the Euler products. The logarithmic fractional derivative of the Riemann ζ function provides a partial result in this direction. The introduction of the zeta function Dirichlet and the computation of its fractional derivative have given a better knowledge of $\zeta^{(\alpha)}$ in the critical strip $0<\mathrm{Re}(s)<1$. The convergence half-plane of $\eta^{(\alpha)}$ is given by $\mathrm{Re}\ s > \alpha$, hence $\zeta^{(\alpha)}$ and $\eta^{(\alpha)}$ suggest the strip $\alpha < \mathrm{Re}\ s < 1+\alpha$ as a fractional counterpart of the critical strip. This result shows that there is a clear link between this function and the distribution of prime numbers.

The fifth chapter provides an application of two signal processing networks associated to $\eta^{(\alpha)}$. The spectral properties of both $\zeta^{(\alpha)}$ and $\eta^{(\alpha)}$ are given and the symmetry is shown.

The fractional derivative of the Riemann ζ function seems to have many promising applications in pure and applied mathematics. For instance, an example is presented based on the knowledge that complex functions can be studied in a suitable function space in order to solve a given problem. This is represented by the Hilbert spaces of entire functions, in which de Branges had linked the Riemann hypothesis with a positivity condition on these particular function spaces. It is assumed that by taking into account the interest that the fractional calculus has had in recent years, $\zeta^{(\alpha)}$ has the potential to bring interesting results in fractional Hilbert spaces.