Dissipation, mixing and two level system in quantum field theory

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To my family

Abstract

In this dissertation we discuss several aspects of a two level system (qubit) in the context of quantum mechanics and quantum field theory. The presence of geometrical phases in the evolution of a qubit state is shown. We study geometric structures, which are correlated to an unitary time evolution and its interesting gauge structure. They can be very useful in quantum computational processes.

We illustrate the quantum field theoretical formulation of boson mixed fields, and oscillation formulas for neutral and charged fields are found. We show that the space for the mixed fields is unitary inequivalent to the state space where the unmixed field are defined, and we also derive the structure of the currents and charges for the charged mixed fields.

Phenomenological aspects of meson mixing in the presence of the decay are discussed. In particular, we show that the the effective Hamiltonian is non-Hermitian and non-normal in the Wigner-Weisskopf approximation and we use the biorthonormal basis formalism to diagonalize such an Hamiltonian. Finally, the presence of CP and CPT violations in meson mixing is shown.

Acknowledgments

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My unlimited thanks to my parents and my brother Massimo, for all unconditional support and neverending encouragement and I would like dedicate this dissertation to them and to Rosa.

Angelo

Contents

Introduction

Quantum field theory is presented as an extension of quantum mechanics to the relativistic domain. Sometimes it is referred to as "second quantization". Of course, the reasons for that come from the historical developments in the formulation of the quantum theory of elementary particle physics and solid state physics. However, a closer view to the formalism of QFT shows that it is not necessarily related with the relativistic domain and it is not simply a "second" quantization recipe subsequent the quantization procedure in QM. For example, the QFT formalism is widely used, with great success, in condensed matter physics, e.g. in the formulation of superconductivity, of ferromagnetism, etc., where typically one does not refer to the relativistic domain. On the other hand, in dealing with fermion fields one cannot rely on the quantization scheme adopted in QM for boson creation and annihilation operators.

Quantum field theory (QFT) is quite different from quantum mechanics (QM), due to the well known von Neumann theorem, which characterizes in a crucial way the structure of QM [1, 2]. In QM the von Neumann theorem states that for systems with a finite number of degrees of freedom all the representations of the canonical commutation relations are unitarily equivalent. This means that they are physically equivalent; namely, thevonneumann representations of the ccr are related by unitary operators and, as well known, physical observables are invariant under the action of unitary operators. Their value is therefore the same independently of the representation one choses to work in. Such a choice is thus completely arbitrary and does not affect the physics one is going to describe. The situation is quite different in QFT where the von Neumann theorem does not hold. Indeed, the hypothesis of finite number of degrees of freedom on which the theorem rests is not satisfied since fields involve by definition infinitely many degrees of freedom. As a consequence, infinitely many unitarily inequivalent representations of the ccr are allowed to exist [3, 4, 5]. The existence of ui representations is thus a characterizing feature of QFT and a full series of physically relevant consequences follows.

This Dissertation is organized as follows.

Part I is devoted to describing the unitary evolution of two level systems. In particular, Chapter 1 contains a basic introduction to geometric phases and their applications to the physical world: an adiabatic derivation of Berry phase and its generalization to non–adiabatic one. In Chapter 2 we study the characterisation and time evolution of a two level system and we show that it presents a Berry–like and an Anandan–Aharonov phase. We obtain the general expression of covariant derivative operator and it is associated to the free energy. The interesting phenomenon of birefringence can be shown by gauge invariance of the time evolution of a two level

system. Aharonov–Anandan phase is related to the distance in projective Hilbert space between the two independent quantum states.

In Part II we consider the non–unitary evolution of two level systems. Chapter 3 is devoted to describing the evolution of open systems in quantum theories: the formalism of Kossakowski–Lindblad equation. The damped harmonic oscillator is discussed as an simple model of open system and it was given a quantum field theory description in term of a suitable free energy operator. The boson mixing in quantum field theory is presented in Chapter 4, where we analyze the structure of currents for mixed fields and we derive the exact oscillation formula for complex and neutral boson fields. In Chapter 5 we study the meson mixing from the phenomenological point of view and the fundamental aspect of their CP and CPT violations in the Wigner–Weisskopf approximation.

Part I

Two level system: unitary evolution

CHAPTER 1

Geometric phases

In this chapter, we introduce a fascinating feature of quantum mechanics. In particular, we discuss the fundamental concept of geometric phase arising from time evolution of a quantum mechanical state.

Geometric phases have been successfully investigated over the last decades and they can be associated with a time evolution generator, which has a parametric dependence. We observe a gauge–like structure relative to such systems, and which may become manifest and characterizing for the physical system.

The aim of this chapter is to give a concise introduction into the mathematically sound theory of geometic phase in quantum mechanics. In what follows we will show the emergence of geometric phases in adiabatic evolution and their generalizations to generic time evolution.

1.1. Introduction

From his studies on interference effects of polarised light beams, Pancharatnam [6] introduced the important concept of geometric phase in quantum theory. This geometrical phenomenon is well known in classical geometry, indeed Hannay [7] found an analogue of the geometric phase for classical systems.

Moreover, Simon showed that the adiabatic evolution taken into account by Berry can be seen as the so-called parallel transport of a vector state along a curve in the parameter space. The Berry phase can have a elegant mathematical interpretation as the holonomy of a suitable connection in the appropriate fibre bundle.

In order to introduce such a fundamental concept, we can consider an intuitive classic example [8], in which we take into account the parallel transport of a vector around a closed loop on a smooth sphere. For simplicity, we can consider a unit vector, which stays tangential to the geodesic at all times. It starts form the north pole and is pointing in the direction of a meridian. Then you move the object keeping it always parallel to its initial direction down the meridian until you reach the equator and then move it parallel along the equator till another meridian which keeps an angle of γ with the original one.

Then you move the vector back to the north pole along the new meridian again keeping it always parallel. When you reach the north pole you observe that the vector has turned around an angle γ, because it points in another direction. Actually, it is linked to the intrinsic curvature of the sphere and it can demonstrate that the rotation angle γ is proportional to the integral of the curvature on the surface inside the closed loop. No such phenomenon would appear if vectors are parallel–transported along a flat manifold.

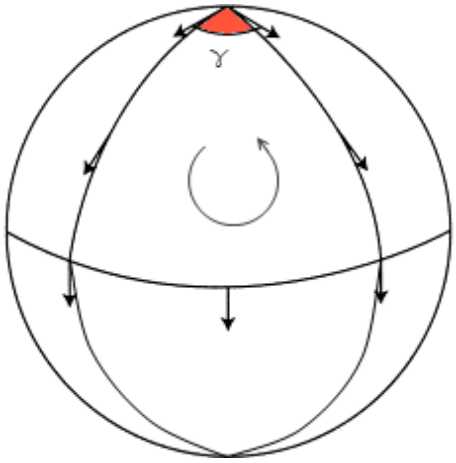

Figure 1.1. The parallel–transported vectors on the three–dimensional sphere do not point in same direction because a non–null angle raises between them.

Such an example shows how the parallel transport of a vector along a closed curve generates an interesting phenomenon, which is called a holonomy, i.e. some variables, which describe the system, do not return to their initial value. Such an effect was already known to Gauss and can be described by the so called Hannay angles [9]. This concept plays a fundamental part in a variety of physical contexts [10, 11, 12] These are all classical examples where a geometrical angles arises although the system returns to its starting point. Nearly the same situation occurs in quantum physics. Here a system picks up a geometrical phase which can be identified with a Hannay angle in the classical limit.

In the Pancharatnam paper [6] he defines the phase difference of two nonorthogonal states of polarization. Two states are in phase if the intensity of the superposed state reaches a maximum. The phase difference between two beams can be specified as the phase change which has to be applied to one beam in order to maximize the intensity of their superposition. It turns out that this phase has also geometrical properties. One can think of this as the earliest appearance of a geometrical phase definition in literature. This was pointed out by Ramaseshan and Nityananda [13] in 1986. With this concept one is able to define geometrical phases also for evolutions that are not limited to the cyclic condition, as Samuel and Bhandari [14] showed. The Pancharatnam phase has already been confirmed in many experiments. Independently, in molecular physics some aspects of geometric phases were discussed by several authors [15, 16]. However it was Berry [17] who first realised that the geometric phase is a generic feature of quantum mechanics. His approach to the Abelian geometric phase was restricted to cyclic and adiabatic evolution of non-degenerated pure quantum states, where the phase depends on the geometry of the path the Hamiltonian traces out in parameter space. Subsequently Wilczek and Zee [18] pointed out that adiabatic transport of a degenerate set of eigenstates is associated with a non-Abelian geometric phase A non-abelian Wilczek and Zee phase is a natural generalization of the Berry phase for systems described by an Hamiltinian with degenerate spectra. Aharonov and Anandan [19] discovered the

geometric phase for non-adiabatic evolutions where the phase depends on the geometry of the path in the state space. Later, Samuel and Bhandari [14] introduced the notion of non-cyclic geometric phases.

1.2. Berry phase

A considerable understanding of the formal description of quantum mechanics has been achieved after Berry's discovery of a geometric feature related to the motion of a quantum system. In 1984 Berry printed a paper [17] in which he discovered the geometric phase as a generic feature of quantum mechanics. Berry considered time evolution of a quantum state with Hamiltonian $H[z(t)]$ depending on adiabatically and cyclic changing parameters $z(t)$, and was able to show that a part of the total phase acquired by the systems state vector during one period of $z(t)$ depends only on the geometric properties of the curve $z(t)$ in the parameter space. As a matter of fact, he found that an additional phase factor occurs in contrast to the well known dynamical phase factor. Berry points out the geometrical character of this phase which is not negligible because of its nonintegrable character. Berry shows that this was not correct because the phase is gauge invariant and therefore can not be gauged away. Since this much work has been done on this issue and the so called Berry phase is now well established, theoretically as well as experimentally. In next section we could provide a derivation of Berry phase in an adiabatic time evolution.

1.2.1. Derivation in adiabatic transformation

The adiabatic evolution of a quantum state vector $|\psi(t)\rangle$ is governed by a parameter dependent Hamiltonian $\hat{\mathcal{H}}[z(t)]$, which is depends on time exclusively via the parameter $z_t \equiv z(t)$ and we suppose that for any z_t the Hamiltonian $\hat{\mathcal{H}}(z_t)$ has a purely discrete spectrum:

$$\hat{\mathcal{H}}(z_t)|n(z_t)\rangle = E_n(z_t)|n(z_t)\rangle \tag{1.1}$$

with $\langle n(z_t)|m(z_t)\rangle = \delta_{nm}$ and the state vector is an eigenstate of the Hamiltonian at time t and that the parameters $z(t)$ share varied along a closed path Γ in parameter space. Let us assume that the nth eigenvalue $E_n(z_t)$ is nondegenerate and the $\hat{\mathcal{P}}_n(z_t) = |n(z_t)\rangle\langle n(z_t)|$ is the corresponding one-dimensional projector onto the nth eigenspace $H_n(z)$, which we write $\hat{\mathcal{H}}_n(z_t) \equiv \{\beta|n(z_t)\rangle|\beta \in C\}$. The eigenvectors $|n(z_t)\rangle$ are not uniquely defined by Eq. (1.1). One may arbitrarily change its phase

$$|n(z_t)\rangle \to e^{i\beta_n(z_t)}|n(z_t)\rangle \tag{1.2}$$

where $\beta_n(z_t) \in R$. Obviously, the phase transformation does not change $\hat{\mathcal{P}}_n(z_t)$ Due to the adiabatic theorem, $\psi(t)$ stays in nth eigenspace of $\hat{\mathcal{H}}(t)$ during the adiabatic evolution, indeed if the time evolution is slow enough then the state vector remains an eigenstate of the Hamiltonian for all time $t > 0$. Therefore, if the evolution is cyclic, i.e. a curve Γ is closed ($z(0) = z(t)$), then $|n(0)\rangle$ and $\psi(T)$ both belong to $\hat{\mathcal{H}}_n(z)$ and hence they may differ only by a phase factor, that will have an additional component that has a purely geometric origin. It depends upon the geometry of manifold M and the circuit Γ itself. $|\psi(t)\rangle$ and $|n(z_t)\rangle$ differ by a time-dependent phase factor:

$$|\psi(T)\rangle = e^{i\gamma(T)}|\psi(0)\rangle \tag{1.3}$$

where $|\psi(0)\rangle$ represents an eingenstate of Hamiltonian $\hat{\mathcal{H}}(t)$, with

$$\gamma(T) = -\frac{1}{h}\int_0^T E_n(t)dt + \gamma_n(\Gamma) \tag{1.4}$$

after substitution of Eq. (1.3) in the Schrödinger equation and we can eliminate exponential terms by applying $\langle\psi(t)|$ on left side

$$\gamma(T) = -\frac{1}{2}\int_0^T E_n(t)dt + i\int_0^T \langle n(z_t)|\frac{d}{dt}|n(z_t)\rangle dt \tag{1.5}$$

Thus, the total phase has to be supplement by the following geometric quantity $\gamma_n(\Gamma)$, named Berry phase

$$\gamma_n(\Gamma) = i\int_0^T \langle n(z_t)|\frac{d}{dt}|n(z_t)\rangle dt \tag{1.6}$$

and corresponding to cyclic adiabatic evolution along Γ.

1.3. Generalization: non-cylic phase

From the original definition of geometric phase given by Berry, many generalizations have been proposed. In particular, it was soon recognized that there is no reason for the evolution to be adiabatic. In fact, the adiabatic condition is never exactly fulfilled in real processes. Furthermore, there are other generalisations of Berry phase to complex valued geometric phases [20] for non-Hermitian Hamiltonians and to off-diagonal geometric phases [21], and also such as the Hannay angle, [9], and the geometric phase for non-linear fields [22]. The Hannay angle has also been generalised to the non-adiabatic case [23]. In addiction, it is possible to define and measure Berry phases for arbitrary open paths, provided that the evolved state at the end of the path is not orthogonal to the initial one. One may consider cyclic evolutions that are not restricted by an adiabatic condition. This means that we need no parameter space to describe the cyclic evolution of the Hamiltonian but only the projective Hilbert space where the system traces out closed curves. Berry phase is then a special case of this so called Aharonov–Anandan phase. This is also the reason why Berry tried to remove the adiabatic condition by calculating adiabatic correction terms [24]. Soon after the work of Aharonov and Anandan several other generalizations occurred which are not treated in this work.

1.3.1. Aharonov–Anandan phase

In the original Berry paper the Berry phase was studied within the framework of adiabatic approximation. However, as it was noticed by Aharonov and Anandan [19], one can relax the assumption about the validity of adiabatic approximation still retaining non-trivial phase factor called afterwards the Aharonov–Anandan phase. Aharonov and Anandan realised that the notion of geometric phase is independent of the adiabatic theorem. Even if the Hamiltonian is unknown and we only know the path of the state, the total phase change of a state vector after a cyclic evolution can be decomposed into a dynamical part, expressed in terms of the expectation value of the Hamiltonian, and a geometric part. Consider a non-adiabatic evolution of a state vector solution of the Schrödinger equation

$$i\hbar\frac{d}{dt}\psi(t) = \hat{\mathcal{H}}\psi(t) \tag{1.7}$$

while defines a trajectory $t \to \psi(t)$ in the Hilbert space $\mathscr{H}$.

If the initial state vector $\psi(0) \in \mathscr{S}(H) \equiv \{\psi \in \mathscr{H} | \langle\psi|\psi\rangle = 1\}$ then the solution $\psi(t)$ remains in $\mathscr{S}(\mathscr{H})$ for any t. Such a trajectory on $\mathscr{S}(\mathscr{H})$ projects onto a trajectory in the quantum phase space. This defines a solution to the von Neumann equation

$$i\hbar\frac{d}{dt}\hat{\mathcal{P}}(t) = [\hat{\mathcal{H}}, \hat{\mathcal{P}}(t)] \tag{1.8}$$

Suppose that a trajectory $\hat{\mathcal{P}}(t) \equiv |\psi(t)\rangle\langle\psi(t)|$ is closed. We call such an evolution cyclic. We stress that we do not make any assumption about the Hamiltonian $\hat{\mathcal{H}}$ of the system. It is not even important whether or not it depends on time. Since $\psi(t)$ and $\psi(0)$ define the same physical state they may differ by a phase factor only.

$$\psi(T) = e^{i\phi}\psi(0), \tag{1.9}$$

for some $\phi \in [0, 2\pi)$. Our task in this section is to find the phase shift ϕ knowing the system Hamiltonian $\hat{\mathcal{H}}$ and a closed trajectory $\hat{\mathcal{P}}(t)$. First of all, let us note that we may make certain changes to $\hat{\mathcal{H}}$ without affecting $\hat{\mathcal{P}}(t)$. It is evident from the commutator structure of von Neumann equation that the following transformation:

$$\hat{\mathcal{H}}_f \; = \; \hat{\mathcal{H}} \; + \; f(t) \tag{1.10}$$

where f(t) is any real function of time, leaves the solution invariant. The corresponding solution to the Schröndiger equation changes as follows

$$\psi_f(t) = e^{i\phi}\psi(0) \tag{1.11}$$

with

$$\phi_f = \phi - \frac{1}{\hbar}\int_0^T f(t)dt \tag{1.12}$$

Therefore by performing a trivial change of the Hamiltonian we may change the corresponding phase ϕ completely arbitrarily. However, as was shown by Aharonov and Anandan, the total phase is $\phi = \phi_{dyn} + \phi_{geo}$ and the geometric part ϕ_{geo} is invariant under the transformation (1.10) and depends only on the closed curve $\hat{\mathcal{P}}(t)$ in quantum phase space. To see this let us take a function $f = f(t)$. Note that $f(t)$ then satisfies

$$\phi = \frac{1}{\hbar}\int_0^T f(t)dt \tag{1.13}$$

where ϕ is defined in Eq. (1.9). The new function $\psi(t)$ solves the Schrödinger equation

$$i\hbar\frac{d}{dt}\psi(t) = (\hat{\mathcal{H}} + f(t))\psi(t)) \tag{1.14}$$

and hence, taking a scalar product with $\psi(t)$ and integrating over time from 0 to T one obtains

$$\int_0^T \langle\psi(t)|\frac{d}{dt}\psi(t)\rangle dt = \frac{1}{\hbar}\int_0^T \langle\psi(t)|\hat{\mathcal{H}}|\psi(t)\rangle dt + \frac{1}{\hbar}\int_0^T f(t)dt \tag{1.15}$$

Therefore, using (5.88), we find the following formula for the phase shift ϕ

$$\phi = \int_0^T \langle\psi(t)|\frac{d}{dt}\psi(t)\rangle dt - \frac{1}{\hbar}\int_0^T \langle\psi(t)|\hat{\mathcal{H}}|\psi(t)\rangle dt \tag{1.16}$$

1.4. Experiments and Applications

In this section, we shortly report some experimental applications on the interpretation, characterization and application of geometric phases especially in the context of the quantum information. In any case, there are many different physical applications and approaches for the geometrical phase in the physical world.

1.4.1. Aharonov–Bohm effect

In 1959 a famous experiment was performed by Yakir Aharonov and David Bohm [25]. They showed the presence of a phase shift acquired by electron beams due to their different paths in space surrounded by a electromagnetic vector potential.

The electromagnetic vector potential $\hat{\mathbf{A}}$ has no physical significance in quantum theory, but it is useful to represent the electric and magnetic fields. Futhermore, the Schrödinger equation for charged particles is expressed also in terms of the vector $\hat{\mathbf{A}}$ and Aharonov and Bohm pointed out that the potential $\hat{\mathbf{A}}$ have a significance in quantum mechanics. the gauge determines the phase of the wave function. This phase is not itself measurable, but what is significant is the interference between different paths between the same points. The interference dependes on the integral of the vector potential around the loop made up of the two paths or equivalently on the magnetic flux passing through a surface bounded by the loop. This is essentially the Aharonov–Bohm effect. The schematic experiment configuration for such a physical process is shown in Fig. (1.2)

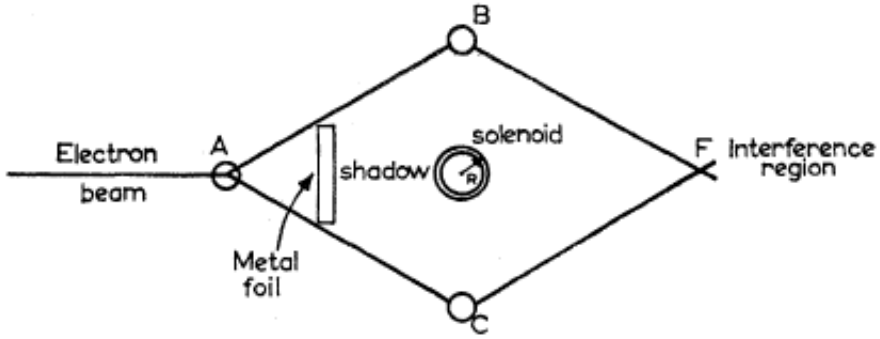

Figure 1.2. The reduced scheme of Aharonov–Bohm experimental setup (see ref. [25])

An electron beam is split into two coherent beams. They pass on opposite sides of the solenoid and then interfere. Although there are not magnetic fields outside the solenoid (in the region in which the charged particles move) a relative phase shift between the two waves can be observed as an interference pattern. The corresponding phase shift $\Delta\phi$ is given by the following formula

$$\Delta\phi = \frac{q\Phi_0}{\hbar c} = \frac{q}{\hbar c}\oint \hat{\mathbf{A}}\cdot\mathbf{dl} \tag{1.17}$$

where the integral is carried out along a closed curve formed by the union of the two paths. Although the magnetic field vanishes everywhere outside the solenoid,

the vector potential $\hat{\mathcal{A}}$ cannot vanish there. This is because the loop integral of $\hat{\mathcal{A}}$ around the solenoid is equal to the magnetic flux

$$\Phi_0 = \int_\Sigma B \cdot dS \tag{1.18}$$

through the solenoid. Note that the interference pattern is invariant under

$$\Phi_0 \to \Phi_0 + n\frac{hc}{q} \tag{1.19}$$

since $\Delta\phi \to \Delta\phi - 2\pi n$ It is clear that the AB phase shift $\Delta\phi$ may be interpreted as a geometric Berry phase that a charged particle accumulates by circling around a solenoid carrying a non-zero magnetic flux. The common feature is that both phases are non-integrable, i.e. independent of the initial and final value of the integrand. But whereas the geometric phase is local due to dependence of local changes of the physical state, the topological phase is non-local in the sense that it cannot be defined at a point in space but only as a closed integral enclosing a magnetic flux or not, i.e. solely dependent upon a topological structure. This is the reason why there is non-cyclic geometric phases, but no non-cyclic topological phases [26]. It has been shown that one can formulate the non-cyclic geometric phase in terms of a gauge-invariant reference section [27, 28] which also shows that the geometric phase is local. Aharonov and Bohm also described a phase effect for a charged particle due to an electric field, known as electric AB phase (EAB).

1.4.2. Geometric phase: experimental setups

Several experimental verifications have been performed to detect the geometric phases in physical world as the measurements of the adiabatic geometric phase for neutron spin [29], photons [30], nuclear magnetic resonance (NMR) [31], and nuclear quadrupole resonance (NQR) [32]. Also the adiabatic geometric phase has been observed for two entangled nuclear spin systems in NMR [33]. The non-adiabatic geometric phase has been measured in NMR [34]. The adiabatic geometric phase for a classical chemical oscillator [35] has been measured, as well as for molecular systems [36]. Furthermore, the off-diagonal geometric phase has been verified in neutron interferometry [37]. Interferometry is a technique familiar to all physicists and can be carried out with any wave phenomenon. An incident beam of particles with wavelength λ is split and then recombined, forming an interference pattern. This interference pattern is sensitive to any change in the effective path length of one (or both) leg(s) of the interferometer, and changing this "optical" path length in a controlled manner allows the experimenter to probe the perturbing interaction with extraordinary precision. One of first experiments was perform with photons, because they are rather easy to generate and manipulate in lab.

An experimental evidence of geometric phase was found by Chiao, Wu and Tomita [38, 39]. In their experiment, it was considered the spin of photons. In particular, they considered the elicity of photons, which is the projection of the photon's spin vector along the direction in which it is travelling, it can be easily turned by changing the direction of travel in a coiled optical fibre. Chiao, Wu and Tomita considered a linearly polarized photon propagating in the direction of a wave vector $K(t)$ by sending a light along an optical fibre. The optical fibre is coiled such that the initial and final directions of extremities of the fibre coincide. we can introduce for convenience a suitable basis $\{|\xi_1, k\rangle, |\xi_2, k\rangle\}$ on a plane orthogonal

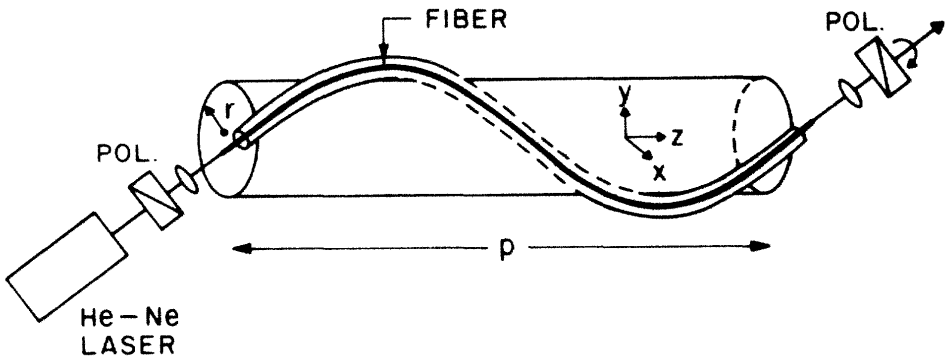

Figure 1.3. Photon experimental setup (see ref. [39])

to tangential vector of optical fibre. We consider now linearly polarized light which is a superposition of the helicity eigenstates

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|\xi_1, k\rangle + |\xi_2, k\rangle\right) \tag{1.20}$$

the initial linear polarization is a superposition of two polarized waves. The helicity eigenstates acquire a geometric phase after passage through the optical fibre. The final polarization state is then given by

$$|\psi(T)\rangle = \frac{1}{\sqrt{2}}\left(e^{-\frac{i}{\hbar}E_1 T + i\Omega}|\xi_1, k\rangle + e^{-\frac{i}{\hbar}E_2 T - i\Omega}|\xi_2, k\rangle\right) \tag{1.21}$$

After propagation through the fibre at a time T each eigenstates picks up a dynamical and a geometrical phase factor.

$$|\xi_\sigma, k\rangle \to e^{-\frac{i}{\hbar}E_\sigma T - i\Omega}|\xi_\sigma, k\rangle \tag{1.22}$$

where $\sigma = 1, 2$, thus the geometric phase that appears for circularly polarized photons corresponds to rotation of the linear polarization vector $|\psi(t)\rangle$ by angle Ω.

Neutrons are an another useful particles to use for diplaying the geometric phase in physical world. They obey to Fermi–Dirac statistics and they are sensitive to the four "basic" interactions. Thus, neutron interferometry provides a powerful tool for investigations and testing fundamental physics concepts, especially in the field of geometric phases. Indeed, the existence of the neutron interferometer stress the wave–particle duality of quantum mechanics. the spinor rotation of a spin–1/2 particle can be described by assuming two bases namely "up" and "down" spin eigenstates, and by assigning appropriate phase shifts due to the magnetic field. In the spatial case the two–dimensional Hilbert space is spanned by the two possible paths in the interferometer. It has been experimentally verified that a geometric phase for cyclic , as well as non–cyclic evolutions , can be induced. In the case of spinor evolution, where the geometric phase is generated in spin subspace, the spinor rotations are carried out independently in each sub-beam due to the macroscopic separation of the partial beams in the interferometer. Geometric phase effects are observed when the two sub-beams are recombined at the third plate of the interferometer followed by a spin analysis. In 1996 an interesting interferometer experiment with neutrons was performed by Hasegawa, Zawisky, Rauch and Ioffe

[40]. They use a two loop neutron interferometer which consists of loop A, where the geometric phase is generated, and loop B, which is a reference beam for the measurement of the phase. Various geometric phases can be generated by different combinations of a phase shifters (PS I) and an absorber. Another phase shifter (PS II) in Loop B allows to measure the geometrical phase shift. In this experiment it is possible to get rid of the dynamical phase by a certain choice of the experimental setup. It is useful to compare the interferometer with a spin–1/2 system. The

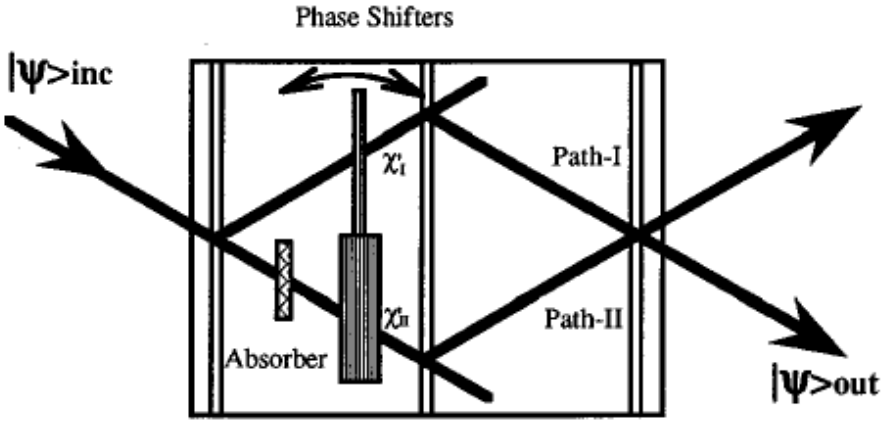

Figure 1.4. Neutron interferometry setup (see ref. [40])

two basis states "up" and "down" are identified with the two possible paths in the interferometer. In each path the neutron gets a certain phase shift χ_i. The recombined beam is said to be in phase with the initial one if the total phase shift is an integer multiple of 2π

$$\chi_I - \chi_{II} = 2\pi n \tag{1.23}$$

This gives the cyclicity condition for the system. The dynamical and the geometrical phase are defined in total analogy to the spin–1/2 case. We get for the dynamical phase

$$\phi_{dyn}(T) = -\frac{1}{\hbar}\int_0^T \langle\psi(t)|\hat{\mathcal{H}}|\psi(t)\rangle dt = \frac{1}{1+T}\left[\chi_I + T\chi_{II}\right] \tag{1.24}$$

The geometrical phase is given by $\beta(T) = \phi_{tot}(T) - \psi_{dyn}(T)$ where ϕ is the total phase shift during the cyclic evolution. To observe only the geometrical phase one has to set the change of the dynamical phase to zero. This is assured by the following condition

$$\Delta\chi_I - T\Delta\chi_{II} = 0 \tag{1.25}$$

where $\Delta\chi_\sigma \quad (\sigma = \{I, II\})$ stands for the change of phase in the σ path and T is the transmission probability of the absorber in path II. Then the observed total phase shift is equal to the geometrical phase shift.

Berry phases can be conveniently demonstrated in an NMR experiment [41] by working in a rotating frame. Zanardi and Rasetti [42] were the first to point out

that Berry phases could be used for enabling quantum computation. Geometric phases are proper candidates for realizing low noise quantum computing devices. Because of the dependence only on the net area traced out in phase space the geometric phase is an ideal construction for fault-tolerant quantum computation.

In the experiments a conditional geometric phase is applied, which means that the state of one spin determines the geometric phase acquired by the other spin. But the adiabatic geometric phase has also several drawbacks. First one is limited in time by the adiabatic condition which has to be removed to take full advantage of the short coherence time of the quantum computer. The second point is that one wants to get rid of the dynamical phase. This is done by using the so called spin-echo technique where the adiabatic evolution is applied twice in reversed direction. But here additional errors can be produced if the second path is not exactly the reversed of the first path and therefore the dynamical phase does not exactly cancel out.

1.4.3. Holonomic quantum computation

Quantum computers are more powerful computational tools than Turing machines and they can perform tasks which seem intractable for classical computers. Entanglement and quantum coherence are the main ingredients for quantum information processing and any quantum computation can be build out of simple operations involving only one or more quantum bits. From the Feynman idea [43], a first scheme of a universal quantum computer and related quantum algorithm were suggested by Deutsch in his famous paper [44]. In 1994 Shor [45] proposed an important quantum algorithm, which can factor large numbers more faster than any classical algorithm. Later, another quantum algorithm, discovered by Grover [46], makes it possible to search in an unsorted database for an element and is faster when compared to the classical algorithms by the square root of the number of elements in the database.

Despite the impressive progress in quantum computation, quantum information is extremely fragile, due to inevitable interactions between the system and its environment and it is important to maintain the coherence in the system when we increase the number of computational units. These interactions are main practical obstacle and they cause the system to lose part of its quantum nature, a process called decoherence. Deutsch [47] suggested the error correction method to overcome the impurities in quantum systems [48]. This idea has been step by step improved, e.g. by Palma et. al. who introduced decoherence free subspaces [49], and in particular by Shor [50] who proposed a quantum analog of classical error correcting codes, which makes possible to use redundant encoding of qubits to correct quantum gate errors due to interaction with the environment. Another method to reach coherent quantum computing is to use geometric or topological quantum computation. In other words, the fundamental topic is represented by the experimental realisation of the basic constituents of quantum information processing devices, namely fault-tolerant quantum logic gates.

The idea is to exploit this inherent robustness provided by the topological properties of some quantum systems as a means of constructing built-in fault–tolerant quantum logic gates. Various strategies have been proposed to reach this goal, some of them making use of purely geometric evolutions, i.e. non-Abelian holonomies [51, 52]. Others make use of hybrid strategies that combine together geometrical

and dynamical evolutions [33], and others yet use more topological structures to design quantum memories [53]. A quantum computer processes qubits by quantum gates. Such operators represent unitary trasformations that act in Hilbert space. The concept of topological invariance arises naturally in the study of fault tolerance. Topological properties are those that remain invariant when we smoothly deform a system, and a fault-tolerant quantum gate is one whose action on protected information remains invariant (or nearly so) when we deform the implementation of the gate by adding noise. To achieve fault tolerant quantum computation we can use the geometric phase to implement quantum gates. Such phases are, as mentioned before, fault tolerant to state space area preserving operations.

The advantages of using geometrical evolution are several. First of all, there is no dynamical phase in the evolution. This is because we are using degenerate states to encode information so that the dynamical phase is the same for both states. Also, all the errors stemming from the dynamical phase are automatically eliminated. Secondly, the states being degenerate do not suffer from any bit flip errors between the states. So, the evolution is protected against these errors as well. Thirdly, the size of the error depends on the area covered and is therefore immune to random noise in the first order in the driving of the evolution. This is because the area is preserved under such a noise as formally proven by DeChiara end Palma [54]. Also, by tuning the parameters of the driving field it may be possible to make the phase independent of the area to a large extent and make it dependent only on a singular topological feature - such as in the Aharonov–Bohm effect where the flux can be confined to a small area - and this would then make the phase resistant under very general errors.

CHAPTER 2

Two level system

2.1. Introduction

In Nature a wide range of physical systems can be modeled by using a simple two level system. Indeed, the properties of these systems can be represented analytically without any kind of approximation in a two-dimensional space. Such systems are not only of academic interest, actually many interesting modern developments in quantum theories and applications to the physical world involve systems, which are described by two independent quantum states.

The classic example is the intrinsic spin of an subatomic particle, a purely quantum mechanical observable that can take on only "up" and "down" values. For instance we can take into account the general description of two state systems using the ammonia molecule and there are several applications to nuclear magnetic resonance, the ammonia maser, neutrino oscillations, and the physics of strange particles (kaons).

Another significant case of two-level system is displayed by the photon, which is the quanta of electromagnetic field. It can have only two orthogonal polarizations. The photons differ from the spin-$\frac{1}{2}$ particles because they are massless bosons with spin-1. In this chapter we will report some results of references [55, 56]

2.2. General formalism in quantum mechanics

In quantum information, the elementary unit is called the "quantum bit" or qubit and corrispondig to the bit, which is defined in classical information. A qubit can be modelled with an atom, nuclear spin, or a polarised photon and it lives in the smallest nontrivial Hilbert space, in which we may introduce an orthonormal basis for a two-dimensional vector space as $\{|0\rangle, |1\rangle\}$. In such a space, the most general normalized state can be written as

$$a|0\rangle + b|1\rangle, \tag{2.1}$$

where a and b in general are complex numbers and the global phase is physically irrelevant. When we perform a measurement on the qubit, the measurement irrevocably disturbs the state, except in the cases $a = 0$ and $b = 0$. We can consider a generic generator of phase transformation:

$$\hat{\mathcal{H}} = \omega_1|0\rangle\langle 0| + \omega_2|1\rangle\langle 1| \ . \tag{2.2}$$

The states $|0\rangle$ and $|1\rangle$ are eigenstates of $\hat{\mathcal{H}}$ associated to the eigenvalues ω_1 and ω_2, respectively:

$$\hat{\mathcal{H}}|0\rangle = \omega_1|0\rangle \ , \qquad \hat{\mathcal{H}}|1\rangle = \omega_2|1\rangle \ . \tag{2.3}$$

with non-degenerates eigenvalues ω_1 and ω_2 that can denote the values of a quantum number (energy, or charge, or spin, etc.) characterizing the states $|0\rangle$ and $|1\rangle$, respectively. In the following, we shall consider them as the energy eigenvalues in natural units ($\hbar = c = 1$). At time $t = 0$, we perfom a convenient rotation in the plane $\{|0\rangle, |1\rangle\}$, so we obtain the superpositions:

$$|\phi(0)\rangle = \cos\theta\,|0\rangle + \sin\theta\,|1\rangle\,, \tag{2.4}$$
$$|\psi(0)\rangle = -\sin\theta\,|0\rangle + \cos\theta\,|1\rangle\,. \tag{2.5}$$

The orthonormality conditions between the states $|\psi\rangle$ and $|\phi\rangle$ require that the complex parameters a and b have to satisfy the relations

$$\begin{array}{l} |a|^2 + |b|^2 = 1 \quad , \\ (a^*)b - a(b^*) = 0 \quad . \end{array} \tag{2.6}$$

In other word, we set the coefficients $a = e^{i\gamma_1}\cos\theta$ and $b = e^{i\gamma_2}\sin\theta$ and the difference of phases is $\Delta\gamma \equiv \gamma_1 - \gamma_2 = n\pi, \quad n = 0, 1, 2....$

In general, in the preparation process we have a limited control on the fluctuations of the a and b coefficients (the initialization problem [57, 58, 59]). However, in some cases, such as in nuclear magnetic resonance and electron spin resonance systems, a good precision may be reached in the control of the initialization problem [60]. Since t denotes the time parameter, $\hat{\mathcal{H}}$ plays the role of the Hamiltonian operator and the time evolution operator is $e^{-it\hat{\mathcal{H}}}$. Applying it to the states Eqs. (2.4)(2.5) we obtain

$$|\phi(t)\rangle = e^{-i\hat{\mathcal{H}}t}|\phi(0)\rangle = e^{-i\omega_1 t}(\cos\theta|0\rangle + e^{-i(\omega_2-\omega_1)t}\sin\theta|1\rangle)\,, \tag{2.7}$$
$$|\psi(t)\rangle = e^{-i\hat{\mathcal{H}}t}|\psi(0)\rangle = e^{-i\omega_1 t}(-\sin\theta|0\rangle + e^{-i(\omega_2-\omega_1)t}\cos\theta|1\rangle)\,. \tag{2.8}$$

with $\langle\phi(t)|\psi(t)\rangle = 0$ and $\langle\psi(t)|\psi(t)\rangle = 1$, $\langle\phi(t)|\phi(t)\rangle = 1$, for all t. We note that the states $|\phi(t)\rangle$ and $|\psi(t)\rangle$, for all t and $\omega_1 \neq \omega_2$, are not eigenstates of Hamiltonian. If we consider the matrix element of the Hamiltonian on new basis $\{|\psi(t)\rangle, |\phi(t)\rangle\}$ then we have

$$\langle\phi(t)|\,\hat{\mathcal{H}}\,|\phi(t)\rangle = \omega_1\,\cos^2\theta + \omega_2\,\sin^2\theta = \omega_{\phi\phi}\,, \tag{2.9}$$

$$\langle\psi(t)|\,\hat{\mathcal{H}}\,|\psi(t)\rangle = \omega_1\,\sin^2\theta + \omega_2\,\cos^2\theta = \omega_{\psi\psi}\,, \tag{2.10}$$

$$\langle\psi(t)|\,\hat{\mathcal{H}}\,|\phi(t)\rangle = \langle\phi(t)|H|\psi(t)\rangle = \frac{1}{2}(\omega_2 - \omega_1)\,\sin 2\theta = \omega_{\phi\psi}\,, \tag{2.11}$$

and $\omega_{\phi\psi} = \omega_{\psi\phi}$. Notice that the matrix elements of $\hat{\mathcal{H}}$ in these equations are time-independent. We can rewrite above relations as

$$\omega_{\phi\phi} = \omega_1 + \delta\omega_{12}\sin^2\theta, \tag{2.12}$$
$$\omega_{\psi\psi} = \omega_1 + \delta\omega_{12}\cos^2\theta, \tag{2.13}$$
$$\omega_{\phi\psi} = \delta\omega_{12}\sin\theta\cos\theta. \tag{2.14}$$

and we also have

$$tg2\theta = \frac{2\omega_{\phi\psi}}{\delta\omega_{\phi\psi}}\,. \tag{2.15}$$

where we introduce the notation

$$\delta\omega_{\phi\psi} \equiv \omega_{\psi\psi} - \omega_{\phi\phi} \tag{2.16}$$
$$\delta\omega_{12} \equiv \omega_2 - \omega_1. \tag{2.17}$$

We remark that the variance in the phase $\Delta\omega$ is given by the off-diagonal matrix element of $\hat{\mathcal{H}}$, Eq. (2.11). Indeed, we find that

$$\Delta\,\omega^2 = \langle\xi(t)|\hat{\mathcal{H}}^2|\xi(t)\rangle - \langle\xi(t)|\hat{\mathcal{H}}|\xi(t)\rangle^2 = \omega_{\phi\psi}^2\,, \tag{2.18}$$

with $\xi = \{\phi,\,\psi\}$. We also remark that as far as $\gamma_1(t) = \gamma_2(t)$ (i.e. $\omega_1 = \omega_2$) the operator $\hat{\mathcal{H}}$ is invariant under the rotation by θ in the plane $\{|0\rangle, |1\rangle\}$ (cylindrical symmetry under the $U(1)$ phase transformation: $\hat{\mathcal{H}} \to \hat{\mathcal{H}}' = \hat{\mathcal{H}}$)

$$|0\rangle \to |\phi(0)\rangle \;=\; \cos\theta\,|0\rangle \;+\; \sin\theta\,|1\rangle\;, \tag{2.19}$$

$$|1\rangle \to |\psi(0)\rangle \;=\; -\sin\theta\,|0\rangle \;+\; \cos\theta\,|1\rangle\;, \tag{2.20}$$

and for $\gamma_1(t) = \gamma_2(t)$ these rotated states, $|\phi(0)\rangle$ and $|\psi(0)\rangle$, are still eigenstates of H.

However, such a cylindrical symmetry is broken due to any fluctuation in the phases making them different at any point of the quantum computation circuit where $\omega_2(t) - \omega_1(t) \neq 0$. In such cases, $\Delta\omega \neq 0$ and nonvanishing off-diagonal matrix elements of $\hat{\mathcal{H}}$, Eq. (2.11), appear. Indeed, by using Eqs. (2.19), (2.20) and Eq. (2.2), we obtain for any t:

$$\hat{\mathcal{H}} = \omega_{\phi\phi}|\phi(t)\rangle\langle\phi(t)| + \omega_{\psi\psi}|\psi(t)\rangle\langle\psi(t)| + \omega_{\phi\psi}(|\phi(t)\rangle\langle\psi(t)| + |\psi(t)\rangle\langle\phi(t)|)\;, \tag{2.21}$$

which immediately gives Eqs. (2.9)–(2.11) and shows that, when the degeneracy in the phases is removed (i.e. $\delta\omega_{12} \neq 0$), $\hat{\mathcal{H}}$ acquires the $\omega_{\phi\psi}$ "mixed term" responsible for "oscillations" between the states $|\phi(t)\rangle$ and $|\psi(t)\rangle$ (Same situation occurs in the mixing of neutrinos and in general of particles with different masses [61]).

These terms are known also to control the (linear) entropy associated to the mixed states, whose density matrix is $\cos^2\theta\,|0\rangle\langle 0| + \sin^2\theta\,|1\rangle\langle 1|$ and whose probability of being in the state $|0\rangle$ is $\cos^2\theta$ and of being in the state $|1\rangle$ is $\sin^2\theta$.

2.2.1. Geometric representation: Bloch sphere

In quantum mechanics, a very useful caratterization of a quantum mechanical systems is the formalism of density operator. A quantum state can be described by a density operator $\hat{\rho} = |\chi\rangle\langle\chi|$ that acts on the Hilbert space and not contain any information about the global phase associating with the relative state. It enables to obtain all the physical predictions that can be calculated from $|\chi\rangle$.

Thus, we can rewrite the Eqs. (2.4) and (2.5) as

$$\hat{\rho}_\xi = |\xi\rangle\langle\xi| \tag{2.22}$$

where $|\xi\rangle = \{|\psi\rangle, |\phi\rangle\}$, the ρ_ξ operator is one-dimensional projector of pure state $|\xi\rangle$. the density operator is represented in $\{|0\rangle, |1\rangle\}$ basis by a matrix called the density matrix, whose elements are $\rho = \langle\sigma|\,\hat{\rho}_\xi\,|\tau\rangle$, $\sigma, \tau \in \{0, 1\}$. In our case, the density matrix has four elements and can be expanded in the basis of Pauli matrices $\{I, \sigma_1, \sigma_2, \sigma_3\}$, so ρ_ξ will be expressed as

$$\rho_\xi = \frac{1}{2}(I + \Sigma_i\, a_i\sigma_i) \tag{2.23}$$

where $\{a_i\}$ are the components of the Bloch vector of the quantum state. Moreover, The eigenvalues of ρ_ξ matrix are given by

$$\frac{1}{2}(1 \pm |a|) \tag{2.24}$$

and Det $[\rho_\xi] = \frac{1}{4}(1 + |a|^2)$. Where $|a| \leq 1$, because the density operators must be positive-semidefinite. The whole set of density matricies has a natural correspon-

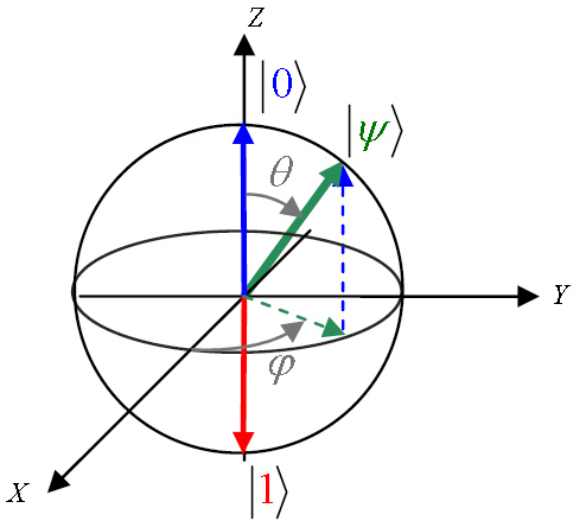

Figure 2.1. The Bloch sphere

dence with the points of the unit three-dimensional solid sphere, which is generally called the Bloch sphere. In fact, It can represent density matricies of spin–1/2, which are also described as

$$\frac{1}{2}\begin{pmatrix} 1+a & b+ic \\ b-ic & 1-a \end{pmatrix} \tag{2.25}$$

The boundary of the Bloch sphere contains the density matrices with vanishing determinant, which stand for pure states because they are formed by simple projectors. Since Tr $[\rho_\xi] = 1$, these density matrices must have eigenvalues 0 and 1. The pure state of a single qubit is of the form $|\psi(\theta, \phi)\rangle$ and can be envisioned as a spin pointing in the (θ, ϕ) direction. Indeed using the property $(\hat{\mathbf{n}} \cdot \sigma) = 1$ where $\hat{\mathbf{n}}$ is a unit vector, we can easily verify that the pure-state density matrix

$$\rho(\hat{\mathbf{n}}) \quad = \quad \frac{1}{2}(1 + \hat{\mathbf{n}} \cdot \sigma) \tag{2.26}$$

satises the property $(\hat{\mathbf{n}}\sigma)\rho(\hat{\mathbf{n}}) = \rho(\hat{\mathbf{n}})(\hat{\mathbf{n}}\sigma) = \rho(\hat{\mathbf{n}})$ and, therefore is the projector $\rho(\hat{\mathbf{n}}) = |\psi(\hat{\mathbf{n}})\rangle\langle\psi(\hat{\mathbf{n}})|$, that is, $\hat{\mathbf{n}}$ is the direction along which the spin is pointing up. we may compute directly that

$$\rho(\hat{\mathbf{n}}) \quad = \quad \frac{1}{2}(1 + \hat{\mathbf{n}} \cdot \sigma) \tag{2.27}$$

where $\hat{\mathbf{n}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. One good property of the Bloch parametrization of the pure states is that while $|\psi(\theta, \phi)\rangle$ has an arbitrary overall phase that has no physical significance, there is no phase ambiguity in the density matrix $\rho(\hat{\mathbf{n}}) = |\psi(\hat{\mathbf{n}})\rangle\langle\psi(\hat{\mathbf{n}})|$ and all the parameters in ρ have a physical meaning.

2.3. Global phase in time evolution

The quantum state of a system is only determined up to a phase and they are defined as equivalence classes of the vectors in Hilbert space $\mathscr{H}$, which are called the projective Hilbert space $\mathscr{P}$. When a system evolves in time it is described by the time dependent Schrödinger equation. The solution of this differential equation is given by a phase factor times the initial state. This phase factor is called the dynamical one, because it comes from the dynamics of the system. As we have seen in the first chapter, also other additional phase factors can occur. The total phase a system, that is gained during its evolution, will be the sum of the dynamical and the geometrical phase.

2.3.1. Dynamical phase

The phase difference between the states at two times t_1 and t_2 called the total phase, is given by

$$\phi_t = \arg\langle\psi(t_1)|\psi(t_2)\rangle \tag{2.28}$$

the dynamical phase can be defined as

$$\phi_d = -\frac{1}{\hbar}\int_{t_1}^{t_2}\langle\psi(t)|H(t)|\psi(t)\rangle dt \tag{2.29}$$

If the Hamiltonian is time-independent and $|\psi(0)\rangle$ is an eigenvalue E_n then the dynamical phase is the same the total phase

$$\phi_t = \phi_d = \frac{E_n(t_1 - t_2)}{\hbar} \tag{2.30}$$

The state $|\xi(t)\rangle$, apart from a phase factor, reproduces the original state at time $t = 0$ after a period $T = \frac{2\pi}{\omega_2 - \omega_1}$:

$$|\xi(T)\rangle = e^{i\varphi}|\xi(0)\rangle\ , \qquad \varphi = -\frac{2\pi\omega_1}{\omega_2 - \omega_1}\,, \tag{2.31}$$

where it is essential that $\omega_1 \neq \omega_2$. One thus recognizes that such a time evolution does contain a purely geometric part, i.e. the Berry–like phase. The total phase acquired during an evolution of a quantum system generally consists of two components: the usual dynamical phase ϕ_d and the geometric phase ϕ_g. The dynamical phase, which depends on the dynamical properties, such as energy or time, is given by ϕ_d.

2.3.2. Geometric phase

As we have shown in previous sections the quantum state exhibits an additional phase, when it performs a cyclic evolution and in particular it presents a purely geometric phase. The peculiarity of the geometric phase lies in the fact that it does not depend on the dynamics of the system, but purely on the evolution path of the state. We observe that it is an easy matter to compute the Berry-like phase. Indeed, one immediately gets the geometrical phase β_ϕ and β_ψ by adopting the

standard method [17]:

$$\beta_\phi \;=\; \varphi + \int_0^T \langle\phi(t)|\, i\partial_t\, |\phi(t)\rangle\, dt \;=\; 2\pi \sin^2\theta\,. \tag{2.32}$$

$$\beta_\psi \;=\; \varphi + \int_0^T \langle\psi(t)|\, i\partial_t\, |\psi(t)\rangle\, dt \;=\; 2\pi \cos^2\theta\,. \tag{2.33}$$

that phases are independent from the eigenvalues ω_i's, $i = 1, 2$ and depend only on the θ-parameter The sum of both phases $\beta_\psi + \beta_\phi = 2\pi$ for any θ, so we obtain (2.31) as

$$|\xi(T)\rangle = e^{i\beta_\xi} e^{-i\omega_{\phi\phi}T} |\xi(0)\rangle\,. \tag{2.34}$$

The meaning of Eqs. (2.32)-(2.34) can be better understood by noticing that, for any t,

$$\langle\phi(0)|\phi(t)\rangle \;=\; e^{-i\omega_1 t}\cos^2\theta + e^{-i\omega_2 t}\sin^2\theta\,. \tag{2.35}$$

Thus, as an effect of the non vanishing difference $\delta\omega_{12} \neq 0$ of the phases, the components $|0\rangle$ and $|1\rangle$ evolve with different "weights" and the state $|\phi(t)\rangle$ "rotates" as shown by Eq. (2.35). This is similar to what happens in the context of particle mixing (see ref. [62]). In general, for $t = T + \tau$, we have

$$\begin{aligned}\langle\phi(0)|\phi(t)\rangle &= e^{i\varphi}\,\langle\phi(0)|\phi(\tau)\rangle \\ &= e^{i2\pi\sin^2\theta} e^{-i\omega_{\phi\phi}T}\left(e^{-i\omega_1\tau}\cos^2\theta + e^{-i\omega_2\tau}\sin^2\theta\right)\,.\end{aligned} \tag{2.36}$$

Also notice that

$$\langle\psi(0)|\phi(t)\rangle \;=\; \frac{1}{2}\, e^{i\varphi} e^{-i\omega_1\tau}\, \sin 2\theta \left(e^{-i(\omega_2-\omega_1)\tau} - 1\right), \qquad \text{for } t = T + \tau\,, \tag{2.37}$$

which is zero only at $t = T$. Eq. (2.37) expresses the fact that $|\phi(t)\rangle$ "oscillates", getting a component of the $|\psi(0)\rangle$ state, besides getting the Berry-like phase. At $t = T$, $|\phi(t)\rangle$ and $|\psi(0)\rangle$ are again each other orthogonal states.

If we consider a generalization of Eq. (2.32) to $n-$cycles, we can rewritten it in the following form

$$\beta_\phi^{(n)} \;=\; \int_0^{nT} \langle\phi(t)|\, i\partial_t - \omega_1\, |\phi(t)\rangle\, dt \;=\; 2\pi\, n\, \sin^2\theta\,, \tag{2.38}$$

and Eq. (2.36) becomes

$$\langle\phi(0)|\phi(t)\rangle \;=\; e^{in\varphi}\,\langle\phi(0)|\phi(\tau)\rangle\,, \qquad \text{for } \; t = nT + \tau\,. \tag{2.39}$$

Similarly, in Eq. (2.37) one obtains the phase $e^{in\varphi}$ instead of $e^{i\varphi}$. Eq. (2.38) shows that the Berry-like phase acts as a "counter" of $|\phi(t)\rangle$ oscillations, adding up $2\pi\,\sin^2\theta$ to the phase of the $|\phi(t)\rangle$ state after each complete oscillation and in similar way it adds up $2\pi\,\cos^2\theta$ to the phase of the $|\psi(t)\rangle$ state.

2.4. Gauge structure in time evolution

2.4.1. Local gauge transformation

The generalized phases $\beta_{\{\phi,\psi\}}^{(n)}$ in Eq. (2.38) can be rewritten as

$$\begin{aligned}\beta_\phi^{(n)} &= \int_0^{nT} \langle\phi(t)|\, U^{-1}(t)\, i\partial_t \left(U(t)\, |\phi(t)\rangle\right) dt \\ &= \int_0^{nT} \langle\tilde\phi(t)|\, i\partial_t|\tilde\phi(t)\rangle\, dt \;=\; 2\pi\, n\, \sin^2\theta\,,\end{aligned} \tag{2.40}$$

$$\begin{aligned}\beta_{\psi}^{(n)} &= \int_0^{nT} \langle\psi(t)|\, U^{-1}(t)\, i\partial_t \left(U(t)\,|\psi(t)\rangle\right) dt \\ &= \int_0^{nT} \langle\tilde{\psi}(t)|\, i\partial_t|\tilde{\psi}(t)\rangle\, dt \;=\; 2\pi\, n\, \cos^2\theta\,, \end{aligned} \tag{2.41}$$

with a suitable function

$$U(t) = e^{-if(t)}\;,\; f(t) = f(0) - \omega_1. \tag{2.42}$$

the quantity $f(0)$ is an arbitrary constant, and

$$\begin{aligned}|\tilde{\phi}(t)\rangle &\equiv U(t)|\phi(t)\rangle \\ &= e^{-if(0)}\left(\cos\theta\,|0\rangle \;+\; e^{-i(\omega_2-\omega_1)t}\;\sin\theta\,|1\rangle\right). \end{aligned} \tag{2.43}$$

Moreover, we can regard

$$|\phi(t)\rangle \to U(t)|\phi(t)\rangle = |\tilde{\phi}(t)\rangle \tag{2.44}$$

as a local (in time) gauge transformation of the state $|\phi(t)\rangle$. In contrast with the state $|\phi(t)\rangle$, the gauge transformed state $|\tilde{\phi}(t)\rangle$ is not "tilted" in its time evolution:

$$\langle\tilde{\phi}(0)|\tilde{\phi}(t)\rangle \;=\; \langle\tilde{\phi}(0)|\tilde{\phi}(\tau)\rangle\,, \qquad\qquad \text{for}\;\; t = nT + \tau\,, \tag{2.45}$$

which has to be compared with Eq. (2.39). From Eq. (2.43) we see that time evolution only affects the $|1\rangle$ component of the state $|\tilde{\phi}(t)\rangle$.

The gauge transformation acts as a "filter" freezing out time evolution of the $|0\rangle$ state component, so that we have

$$\begin{aligned} i\partial_t|\tilde{\phi}(t)\rangle &= (\omega_2-\omega_1)e^{-if(0)}e^{-i(\omega_2-\omega_1)t}\sin\theta|1\rangle \\ &= (H-\omega_1)e^{-if(0)}\left(\cos\theta\,|0\rangle \;+\; e^{-i(\omega_2-\omega_1)t}\;\sin\theta\,|1\rangle\right) \\ &= (H-\omega_1)|\tilde{\phi}(t)\rangle\,, \end{aligned} \tag{2.46}$$

namely

$$-\,i(\partial_t + iH)|\tilde{\phi}(t)\rangle = \omega_1|\tilde{\phi}(t)\rangle\,. \tag{2.47}$$

Eq. (2.41) actually provides an alternative way for defining the Berry–like phase [17], which makes use of the state $|\tilde{\phi}(t)\rangle$ given in Eq. (2.43). Eq. (2.41) directly gives us the geometric phase because the quantity $i\langle\tilde{\phi}(t)|(i\partial_t|\tilde{\phi}(t)\rangle\, dt$ is the overlap of $|\tilde{\phi}(t)\rangle$ with its "parallel transported" $(i\partial_t|\tilde{\phi}(t)\rangle\, dt)$ at $t + dt$. Similar results can be obtain if we consider $|\psi\rangle$ state.

2.4.2. Covariant derivative

Now, we will try to recast the Schrödinger equations in covariant form by introducing a suitable gauge potential. By applying the Eq. (2.21) to the states $|\phi(t)\rangle$ and $|\psi(t)\rangle$, we obtain the following equations

$$\hat{\mathcal{H}}|\phi(t)\rangle = \omega_{\phi\phi}\,|\phi(t)\rangle \;+\; \omega_{\phi\psi}\;|\psi(t)\rangle \tag{2.48}$$

$$\hat{\mathcal{H}}|\psi(t)\rangle = \omega_{\psi\psi}\,|\psi(t)\rangle \;+\; \omega_{\phi\psi}\;|\phi(t)\rangle. \tag{2.49}$$

On the other hand, since $\hat{\mathcal{H}}|\phi(t)\rangle = i\,\partial_t\,|\phi(t)\rangle$ and $\hat{\mathcal{H}}|\psi(t)\rangle = i\,\partial_t\,|\psi(t)\rangle$ (cf. Eqs. (2.7), (2.8) and (2.2)), we get new form of the evolution equations

$$i\,\partial_t\,|\xi(t)\rangle \;=\; \omega_d\,|\xi(t)\rangle \;+\; \omega_{\phi\psi}\,\sigma_1\,|\xi(t)\rangle\,, \tag{2.50}$$

where $|\xi(t)\rangle = (|\phi(t)\rangle, |\psi(t)\rangle)^T$, $\omega_d = diag(\omega_{\phi\phi}, \omega_{\psi\psi})$ and σ_1 denotes the Pauli matrix. By using the notation

$$g \equiv \tan 2\theta = \frac{2\omega_{\phi\psi}}{\delta\omega_{\phi\psi}}, \;\; \omega_{\phi\psi} = \frac{1}{2} g\, \delta\omega_{\phi\psi} \tag{2.51}$$

We also put $A_0 = A_0^{(1)}\, \sigma_1 = \frac{1}{2}\, \delta\omega_{\phi\psi}\, \sigma_1$. Then we may write

$$D_t = \partial_t + i\, \omega_{\phi\psi}\, \sigma_1 = \partial_t + i\, g\, A_0^{(1)}\, \sigma_1\,, \tag{2.52}$$

which acts as the covariant derivative, where g and $A_0^{(1)}$ play the role of the coupling constant and the (non-abelian) gauge field, respectively (a similar situation occurs in the different context of neutrino mixing, see ref.[63]). The motion equations (2.50) now can be written as

$$iD_t\, |\xi(t)\rangle = \omega_d\, |\xi(t)\rangle\,. \tag{2.53}$$

It is easy to show that

$$iD'_t\, |\xi'(t)\rangle = \omega_d\, |\xi'(t)\rangle\,, \tag{2.54}$$

with

$$D'_t = \partial_t + i\, g\, (A_0^{(1)}\, \sigma_1 + \partial_t\, \lambda(t)\, \sigma_1), \tag{2.55}$$

$$|\xi'(t)\rangle = e^{-ig\, \lambda(t)\, \sigma_1} |\xi(t)\rangle\,, \tag{2.56}$$

so that, defining $U(t) \equiv e^{-ig\, \lambda(t)\, \sigma_1}$, it is

$$U(t)\, (iD_t\, |\xi(t)\rangle) = iD'_t\, U(t)\, |\xi(t)\rangle \tag{2.57}$$

and

$$g\, A_0^{(1)'}\, \sigma_1 = U(t)\, g\, A_0^{(1)}\, \sigma_1\, U^{-1}(t) + i\, (\partial_t\, U(t))\, U^{-1}(t)\,, \tag{2.58}$$

as it should be indeed for a gauge field transformation (see Eq. (2.55)). We can express the above result by saying that the time evolution of the vector doublet $|\xi(t)\rangle$ (our two level system or qubit) is controlled by its coupling with a non-abelian gauge field background so to preserve the invariance of the dynamics against local in time gauge transformations (phase fluctuations). We also note that since the only non-vanishing component of A_μ is A_0 and this is a constant ($A_0 \equiv \frac{1}{2}\, \delta\omega_{\phi\psi}\, \sigma_1$), the field strength $F_{\mu\nu}$ is identically zero. This is a feature which, for example, occurs in the case where the gauge potential is a pure gauge (with non-singular gauge functions).

2.5. Other aspects of time evolution

2.5.1. Free energy and Fubini-Study metric

As we have seen in previous section, the Schrödinger equation can be rewritten in a futher form, Eq. (2.53), where we used an non-abelian gauge potential $\hat{\mathcal{A}}$, but such a reformulation is more useful because it offers a new thermodynamical point of view. In particular, we report Eqs. (2.50) as

$$(\hat{\mathcal{H}} - \omega_{\phi\psi}\, \sigma_1)\, |\xi(t)\rangle = \omega_d\, |\xi(t)\rangle\,, \tag{2.59}$$

with the covariant derivative denoted by $\hat{\mathcal{H}} - \omega_{\phi\psi}\, \sigma_1$. Then the operator

$$\hat{\mathcal{F}} = (\hat{\mathcal{H}} - \omega_{\phi\psi}\, \sigma_1) \tag{2.60}$$

may be interpreted as the free energy operator, provided that one identifies the term $\omega_{\phi\psi}\,\sigma_1 = gA_0$ with the entropy term $T\hat{\mathcal{S}}$ in the traditional free energy expression, where the "temperature" is $T = g$ and the entropy $\hat{\mathcal{S}} = A_0$. We thus see that time evolution is controlled by the free energy (2.60) where the gauge field plays the role of the entropy. In terms of the states $|\phi(t)\rangle$ and $|\psi(t)\rangle$, the term $T\hat{\mathcal{S}}$ is written as:

$$T\hat{\mathcal{S}} = \omega_{\phi\psi}(|\phi(t)\rangle\langle\psi(t)| + |\psi(t)\rangle\langle\phi(t)|)\ . \tag{2.61}$$

In order to better understand this feature, it is convenient to consider the geometric phases associated to the system evolution.

Another geometric invariant is the Anandan–Aharonov phase discussed in ref. [64] (cf. Chapter 1). It has the advantage to be well defined also for systems with non-cyclic evolution in the following form

$$s = 2\int \Delta\omega(t)dt \tag{2.62}$$

where $\Delta\omega(t)$ is the variance given by

$$\Delta\,\omega^2 = \Delta\,\omega^2_{\phi\phi} = \Delta\,\omega^2_{\psi\psi} = \langle\xi(t)|\hat{\mathcal{H}}^2|\xi(t)\rangle - \langle\xi(t)|\hat{\mathcal{H}}|\xi(t)\rangle^2 = \Delta\omega^2_{\phi\psi} = \omega^2_{\phi\psi}\ , \tag{2.63}$$

with $\xi = \{\phi,\,\psi\}$. The relation between the entropy and the geometric invariant s is obtained by considering that

$$\begin{aligned}\int \langle\xi(t)|TS\sigma_1|\xi(t)\rangle\ dt &= \int \langle\xi(t)|g\,A_0^{(1)}|\xi(t)\rangle\ dt \\ &= 2\int \omega_{\phi\psi}\,dt = s.\end{aligned} \tag{2.64}$$

It is interesting to note that the relation between TS and the variance of the energy $\Delta\,\omega = \omega_{\phi\psi}$ is through the non-diagonal elements of $\hat{\mathcal{H}}$, namely it is proportional to the energy gap, $\omega_2 - \omega_1$, between the two levels (cf. Eq. (2.63) and (2.11)).

We also recognize that the integrand $\langle\xi(t)|g\,A_0^{(1)}|\xi(t)\rangle$ in Eq. (2.64) is related to the adiabatic connection [18] emerging in the study of the non–abelian holonomy (generalized Berry phase) [65]. Moreover, one can also show [61] that these connections are related with the parallel transport of the vectors in the parameter space, as well known [65]. The invariant s in Eq. (2.64) can be interpreted in terms of the distance between states in the Hilbert space. We consider the evolution of quantum states $|\xi(t)\rangle = \{|\phi(t)\rangle, |\psi(t)\rangle\}$, which is controlled by the Schrödinger equation

$$i\ \partial_t|\xi(t)\rangle = \hat{\mathcal{H}}|\xi(t)\rangle\,, \tag{2.65}$$

Expanding the state $|\xi(t+dt)\rangle$ up to the second order in dt , we obtain

$$\langle\xi(t)|\xi(t+dt)\rangle = 1 - idt\,\langle\xi(t)|\hat{\mathcal{H}}|\xi(t)\rangle - \frac{dt^2}{2}\,\langle\xi(t)|\hat{\mathcal{H}}^2|\xi(t)\rangle + O(dt^3)\,,$$

and

$$|\langle\xi(t)|\xi(t+dt)\rangle|^2 = 1 - dt^2\frac{\omega_{12}^2}{4}\sin^2 2\theta\ +\ O(dt^3)\,, \tag{2.66}$$

where we have used Eqs.(2.9), (2.10) and (2.63) and

$$\langle\phi(t)|\,\hat{\mathcal{H}}^2\,|\phi(t)\rangle = \omega_1^2\,\cos^2\theta\ +\ \omega_2^2\,\sin^2\theta\,, \tag{2.67}$$

$$\langle\psi(t)|\,\hat{\mathcal{H}}^2\,|\psi(t)\rangle = \omega_2^2\,\cos^2\theta\ +\ \omega_1^2\,\sin^2\theta\,. \tag{2.68}$$

We also have

$$|\langle\phi(t)|\psi(t+dt)\rangle|^2 = |\langle\psi(t)|\phi(t+dt)\rangle|^2 = dt^2\Delta\omega^2_{\phi\psi} + O(dt^3)\,. \tag{2.69}$$

The Fubini–Study metric [64] is defined as

$$ds^2 \;=\; 2\,g_{\mu\nu}\,dZ^\mu\,d\bar{Z}^\nu \;=\; 4\,(1 \,-\, |\langle\xi(t)|\xi(t+dt)\rangle|^2)\,, \tag{2.70}$$

where Z^μ are coordinates in the projective Hilbert space $\mathscr{P}$, which is the set of rays of the Hilbert space $\mathscr{H}$. From Eqs. (2.66), (2.69) and (2.70), we get the infinitesimal geodetic distance between the points $\Pi(|\phi(t)\rangle)$ and $\Pi(|\phi(t+dt)\rangle)$ in the space $\mathscr{P}$

$$ds \;=\; 2\,\Delta\omega_{\xi\xi}\,dt \;=\; \delta\omega_{12}\,\sin 2\theta\,dt\,. \tag{2.71}$$

The rate of change of this distance is

$$\frac{ds}{dt} = \delta\omega_{12}\,\sin 2\theta = 2\,\omega_{\phi\psi}, \tag{2.72}$$

with $\delta\omega_{12} \equiv \omega_2 - \omega_1 \neq 0$. In the case of the above two level or qubit states, the Fubini–Study metric coincides with the usual metric on a sphere of unitary radius: $ds^2 = d\Theta^2 + \sin^2\Theta\,d\varphi^2$, with $\Theta = 2\,\theta$ (θ = mixing angle) and $\Theta \in [0,\pi]$. Since θ is constant, we have $ds = \sin 2\theta\,d\varphi$ and, by comparison with Eq. (2.71), $d\varphi = \delta\omega_{12}\,dt$. We thus obtain

$$s = \int \sin 2\theta\,d\varphi \;=\; 2\int \omega_{\phi\psi}\,dt\,, \tag{2.73}$$

which is the Anandan–Aharonov invariant (cf. Eq. (2.64)). Thus, the Anandan–Aharonov invariant s represents the distance between evolution states, as measured by the Fubini–Study metric, in the projective Hilbert space $\mathscr{P}$.

2.5.2. Birefringence effect of the non-abelian gauge field

We now show that the time evolution described above can be interpreted in terms of a birefringence phenomenon (the analogy with birefringence has been considered for the case of neutrino mixing [66]). Let us assume now that the states $|0\rangle$ and $|1\rangle$ are degenerate states, namely their time evolution "in the vacuum" is given by

$$|\sigma(t)\rangle = e^{-i\hat{\mathcal{H}}'t}|\sigma\rangle = e^{-i\omega t}|\sigma\rangle, \tag{2.74}$$

where $\sigma = \{0,1\}$ and $\omega = 2\,\pi\,\nu$. The propagation speed "in the vacuum" is $v_0 = \lambda\,\nu$.

Suppose then that the propagation occurs in a medium presenting different refraction indexes, n_1 and n_2 for $|0\rangle$ and $|1\rangle$, respectively, i.e. where the propagation over a given path of length ℓ occurs in different times, t_1 and t_2 for $|0\rangle$ and $|1\rangle$, respectively:

$$t_j \;=\; \frac{\ell}{v_j} \;=\; \frac{\ell\,n_j}{v_0} = t\,n_j \qquad\quad j = 1,2; \tag{2.75}$$

where v_1 and v_2 are the propagation speeds in the medium for $|0\rangle$ and $|1\rangle$, respectively, and $t = \frac{\ell}{v_0}$. Time evolution is then described by the phase factors $e^{-i\omega\,t_1} = e^{-i\omega_1 t}$ and $e^{-i\omega\,t_2} = e^{-i\omega_2 t}$ for the two states, respectively, where $\omega\,t_i = \omega\frac{\ell}{v_0}\,n_i = 2\,\pi\,\nu\,t\,n_i = 2\,\pi\,\nu_i\,t = \omega_i\,t$, $i = 1,2$, has been used, together with $\lambda_i\,\nu = v_i$, $\lambda_i\,\nu_i = v_0$ and $n_i = \frac{v_0}{v_i} = \frac{\nu_i}{\nu}$.

If we now consider the mixed states $|\phi\rangle$ and $|\psi\rangle$ given by Eqs. (2.4) and (2.5), we have

$$\begin{pmatrix} |\phi(t)\rangle \\ |\psi(t)\rangle \end{pmatrix} = e^{-i\omega_1 t} \begin{pmatrix} \cos\theta & e^{-i(\omega_2-\omega_1)t}\sin\theta \\ -\sin\theta & e^{-i(\omega_2-\omega_1)t}\cos\theta \end{pmatrix} \begin{pmatrix} |0\rangle \\ |1\rangle \end{pmatrix}, \tag{2.76}$$

which is the time evolution generated by $\hat{\mathcal{H}}$ given by Eq. (2.2) with $\omega_1 \neq \omega_2$ (cf. Eqs. (2.7) and (2.8)).

In conclusion, Eq. (2.76) shows that, provided that $\omega_1 \neq \omega_2$, for $\theta \neq \frac{\pi}{4} + \frac{n\,\pi}{2}$, the effect of time evolution through the refractive medium is equivalent to the effect of the background gauge field

$$A_0^{(1)} \;=\; \frac{1}{2}(\omega_2 - \omega_1)\;\cos 2\theta \;=\; \frac{1}{2}\omega(n_2 - n_1)\;\cos 2\theta, \tag{2.77}$$

which indeed disappears when propagation occurs in the vacuum, $n_1 = n_2 = n_0 = 1$ (i.e. $\omega_1 = \omega = \omega_2$).

Part II

Two level system: non-unitary evolution

CHAPTER 3

Dissipative systems

In the previous chapter we have seen that inequivalent representation in QFT play a relevant dynamical role. We further analyze their role in the present chapter by focusing our attention on the physical process of dissipation, in the context of quantum theories. We will examine some useful formalisms and prototypes to achieve a good introduction to that significant phenomena.

3.1. Introduction

The study of open quantum systems represents one of the most interesting issue of quantum theories. Irreversibility is a fundamental aspect of the dissipative processes and in fact such physical processes can evolve only in a preferred direction in time. Hence, the description of such systems can be obtained in the framework of quantum dynamical semigroups [67, 68] Lindblad [69] suggested the general form of the generators of such semigroups as independently, it was done by Gorini, Kossakowski and Sudarshan. That formalism has been studied for the damped harmonic oscillator [70, 71] and applied to various physical phenomena.

For instance, the dissipative systems are most useful in high energy physics and in early universe, as well as in many body theories, in phase transition phenomena and in many practical applications of quantum field theory at non-zero temperature and damping of collective modes in deep inelastic collisions in nuclear physics [72]. In a Hamiltonian system various physical properties of the system remain constant in time and it can be conceived conservative. Therefore in such a system the initial conditions are remembered during a process, so that the information is not lost but merely reformulated. Most real systems are however not isolated, but interact with their surroundings. In such a system energy is dissipated and information gets lost, and the system is called dissipative. Unlike Hamiltonian systems, for a dissipative system the phase space volume does not remain constant as the system develops in time. Even if most systems in the world exhibit dissipation, many are nevertheless nearly conservative, like for instance the solar system. The long-term behaviour of a dissipative system can be largely independent of how the system is started up. The possibility of exhibiting the gauge theory structure and topologically non-trivial features underlying dissipative phenomena is very appealing and has a wide range of possible applications. In this chapter we study the problem of the canonical quantization of the damped harmonic oscillator in the operator algebra. It has been intensively studied [73, 74, 75, 76, 77] from the results of Bateman and it has been shown that the canonical quantization of the damped oscillator can be properly obtained by doubling the phase-space degrees of freedom in the framework of quantum field theory [73]. The space of the states has been shown to split

into unitarily inequivalent representations of the canonical commutation relations, and the non-unitary character of the irreversible time evolution is expressed as tunneling among such inequivalent representations. The vacuum has been found to have the structure of SU(1,1) time-dependent coherent state and its statistical and thermodynamical properties have been recognized to be the ones of the thermal vacuum in Thermo Field Dynamics [77].

3.2. Dissipation in quantum mechanics

3.2.1. The Kossakowski–Lindblad master equation

In quantum mechanics the unitary evolution of quantum state $|\psi_u(t)\rangle$ can be described via the Schrödinger equation for density operator $\hat{\rho}_u(t) = |\psi_u(t)\rangle\langle\psi_u(t)|$

$$\frac{d}{dt}\hat{\rho}_u(t) = \frac{1}{i\hbar}\left[\hat{\mathcal{H}}, \hat{\rho}_u(t)\right] \tag{3.1}$$

and in unitary case, one introduces the time operator

$$\hat{U}(t_2, t_1) = \exp\{-i\hat{\mathcal{H}}(t_2 - t_1)\}, \tag{3.2}$$

which draws the time evolution form t_1 to t_2 and it is an unitary operator $\hat{U}(t_2, t_1) = \hat{U}^\dagger(t_2, t_1)$. In general, the evolution of an open system is quite different from isolated one. In fact a generalization of Schrödinger equation is not feasible and it was derivated an extension, called Lindblad equation [69]. Now we consider the evolution of an open system $|\psi_n\rangle$ over a short time interval δt under two physical condition: the timescale T_S over the system changes should be greater than δt and δt should also be large compared with the time T_E over which the environment 'forgets' its information about the system. The main idea is to look for a suitable quantum operation such that $\hat{\rho}_n(t) = |\psi_n(t)\rangle\langle\psi_n(t)|$ should be altered only to order δt,

$$\hat{\rho}_n(\delta t) = \sum_k \hat{\Pi}_k \hat{\rho}_n(0)\hat{\Pi}_k^\dagger = \hat{\rho}_n(0) + \mathrm{O}(\delta t). \tag{3.3}$$

Thus it follows that one of the Kraus operators, $\hat{\Pi}_0$ say, must be $\hat{1} + \mathrm{O}(\delta t)$, and the others must be $\mathrm{O}(\sqrt{\delta t})$. Thus, let us write

$$\hat{\Pi}_0 = \hat{1} + \left(\hat{\mathcal{W}} + \frac{1}{i\hbar}\hat{\mathcal{H}}\right)\delta t, \tag{3.4}$$

$$\hat{\Pi}_k = \hat{L}_k\sqrt{\delta t}, \quad k \geq 1, \tag{3.5}$$

where $\hat{\mathcal{W}}$ and $\hat{\mathcal{H}}$ are Hermitian operators, but are otherwise arbitrary at this stage; the operators $\hat{L}_k$ are also arbitrary and are known as Lindblad operators However, the normalization condition on the Kraus operators requires $\sum_k \hat{\Pi}_k^\dagger\hat{\Pi}_k = \hat{1}$, thus

$$\hat{\mathcal{W}} = -\frac{1}{2}\sum_k \hat{L}_k^\dagger\hat{L}_k, \tag{3.6}$$

and therefore

$$\begin{aligned}\hat{\rho}_n(\delta t) &= \left[\hat{1}+\left(\hat{\mathcal{W}}+\frac{1}{i\hbar}\hat{\mathcal{H}}\right)\delta t\right]\hat{\rho}_n(0)\left[\hat{1}+\left(\hat{\mathcal{W}}-\frac{1}{i\hbar}\hat{\mathcal{H}}\right)\delta t\right]+\delta t\sum_k \hat{L}_k\hat{\rho}_n(0)\hat{L}_k^\dagger \\ &= \hat{\rho}_n(0)-\left\{\frac{i}{\hbar}\left[\hat{\mathcal{H}},\hat{\rho}_n(0)\right]+\sum_k\left[L_k\hat{\rho}_n(0)L_k^\dagger-\frac{1}{2}\left\{\hat{\rho}_n(0),\hat{L}_k^\dagger\hat{L}_k\right\}\right]\right\}\delta t+\hat{\rho}_n(0)(\delta t)^2,\end{aligned} \tag{3.7}$$

where $\left\{\hat{A},\hat{B}\right\}$ represents the anti–commutator $\hat{A}\hat{B}+\hat{B}\hat{A}$. Taking the limit $\delta t\to 0$ we obtain the Kossakowski–Lindblad master equation:

$$\frac{d}{dt}\hat{\rho}_n=\frac{1}{i\hbar}\left[\hat{\mathcal{H}},\hat{\rho}_n(0)\right]+\frac{1}{2}\left\{\hat{\rho}_n(0),\hat{L}_k^\dagger\hat{L}_k\right\}. \tag{3.8}$$

Such an equation can be considered the most general kind of markovian master equation describing non–unitary evolution of the density matrix ρ that is trace preserving and completely positive for any initial condition. Note that $\hat{\mathcal{H}}$ is not necessarily equal to $\hat{\mathcal{H}}^\dagger$. It may also incorporate effective unitary dynamics arising from the system-environment interaction.

3.3. Damped harmonic oscillator

As a prototype of dissipative system, we can consider a simple damped harmonic oscillator (dho) [78, 79, 80]. It can be described by a classical equation

$$M\ddot{x}+R\dot{x}+\kappa x=0 \quad , \tag{3.9}$$

In order to perform the canonical quantization, we require to double the phase-space dimensions in order to consider an isolated system. Thus, we introduce a new equation, which describes an effective degree of freedom for the heat bath.

$$M\ddot{y}-R\dot{y}+\kappa y=0 \quad , \tag{3.10}$$

which is the time reversed ($R\to -R$) of Eq.(3.9). Dissipation enters our considerations if there is a coupling to a thermal reservoir yielding a mechanical resistance. The Lagrangian for the global system is (see [78] - [76, 81])

$$\mathscr{L}=M\dot{x}\dot{y}+\frac{1}{2}R(x\dot{y}-\dot{x}y)-\kappa xy\ . \tag{3.11}$$

The system described by (3.11) is sometimes called Bateman's dual system [76, 81]. In the present case our system has been assumed to be coupled with a thermal source and it has been necessary to obtain a closed system by including the reservoir. Eq. (3.11) is indeed the closed system Lagrangian. The canonical momenta are given by

$$p_x \equiv \frac{\partial\mathscr{L}}{\partial\dot{x}}=M\dot{y}-\frac{1}{2}Ry \tag{3.12}$$

$$p_y \equiv \frac{\partial\mathscr{L}}{\partial\dot{y}}=M\dot{x}+\frac{1}{2}Rx. \tag{3.13}$$

Canonical quantization is performed by introducing the commutators

$$[x,p_x]=i\,\hbar=[y,p_y] \ , \tag{3.14}$$

$$[x,y]=0=[p_x,p_y] \ , \tag{3.15}$$

with the following hamiltonian

$$\hat{\mathcal{H}} = \frac{1}{m} p_x p_y + \frac{1}{2m}\gamma(y p_y - x p_x) + \left(\kappa - \frac{\gamma^2}{4m}\right) xy \tag{3.16}$$

and the corresponding sets of annihilation operators

$$a \equiv \left(\frac{1}{2\hbar\Omega}\right)^{\frac{1}{2}} \left(\frac{p_x}{\sqrt{M}} - i\sqrt{M}\Omega x\right), \tag{3.17}$$

$$b \equiv \left(\frac{1}{2\hbar\Omega}\right)^{\frac{1}{2}} \left(\frac{p_y}{\sqrt{M}} - i\sqrt{M}\Omega y\right), \tag{3.18}$$

where we may set the frequency of the two oscillators Eq. (3.9) and Eq. (3.10) as

$$\Omega \equiv \left[\frac{1}{M}\left(\kappa - \frac{R^2}{4M}\right)\right]^{\frac{1}{2}}. \tag{3.19}$$

In case of non overdamping, we have $\Omega \in \mathbb{R}$ and $\kappa > \frac{R^2}{4M}$. By using the canonical linear transformations

$$A \equiv \frac{1}{\sqrt{2}}(a+b), \tag{3.20}$$

$$B \equiv \frac{1}{\sqrt{2}}(a-b), \tag{3.21}$$

the quantum Hamiltonian $\hat{\mathcal{H}}$ can be recast [78, 79] in a new form

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \hat{\mathcal{H}}_I \quad , \tag{3.22}$$

$$\hat{\mathcal{H}}_0 = \hbar\Omega(A^\dagger A - B^\dagger B) \quad , \quad \hat{\mathcal{H}}_I = i\hbar\Gamma(A^\dagger B^\dagger - AB) \quad , \tag{3.23}$$

where the decay constant is $\Gamma \equiv \frac{R}{2M}$ and we observe that the states generated by $B^\dagger$ represent the sink where the energy dissipated by the quantum damped oscillator flows: the B-oscillator represents the reservoir or heat bath coupled to the A-oscillator.

The dynamical group structure associated with the system of coupled quantum oscillators is that of $SU(1,1)$. The two mode realization of the algebra $su(1,1)$ is indeed generated by

$$J_+ = A^\dagger B^\dagger, \quad J_- = J_+^\dagger = AB, \quad J_3 = \frac{1}{2}(A^\dagger A + B^\dagger B + 1), \tag{3.24}$$

$$[J_+, J_-] = -2J_3, \quad [J_3, J_\pm] = \pm J_\pm. \tag{3.25}$$

The Casimir operator $\mathscr{C}$ is defined as

$$\mathscr{C}^2 \equiv \frac{1}{4} + J_3^2 - \frac{1}{2}\left(J_+ J - + J_- J_+\right) \tag{3.26}$$

$$= \frac{1}{4}(A^\dagger A - B^\dagger B)^2. \tag{3.27}$$

3.3.1. QFT description of damped harmonic oscillator

The time evolution of the vacuum

$$|0\rangle \equiv |0\rangle \otimes |0\rangle \,, \tag{3.28}$$

$$(A \otimes 1)|0\rangle \otimes |0\rangle \equiv A\,|0\rangle = 0; \tag{3.29}$$

$$(1 \otimes B)|0\rangle \otimes |0\rangle \equiv B\,|0\rangle = 0, \tag{3.30}$$

is controlled by $\hat{\mathcal{H}}_I$

$$|0(t)\rangle = \exp\left(-it\frac{\hat{\mathcal{H}}}{\hbar}\right)|0(t)\rangle = \exp\left(-it\frac{\hat{\mathcal{H}}_I}{\hbar}\right)|0(t)\rangle$$

$$= \frac{1}{\cosh(\Gamma t)}\exp\left(\tanh(\Gamma t)A^\dagger B^\dagger\right)|0(t)\rangle \quad , \tag{3.31}$$

$$\langle 0(t)|0(t)\rangle = 1 \quad \forall t \,, \tag{3.32}$$

$$\lim_{t\to\infty}\langle 0(t)|0(t)\rangle \propto \lim_{t\to\infty}\exp(-t\Gamma) = 0 \quad . \tag{3.33}$$

Notice that once one sets the initial condition of positiveness for the eigenvalues of $\hat{\mathcal{H}}_0$, such a condition is preserved by the time evolution since $\hat{\mathcal{H}}_0$ is the Casimir operator. In other words, there is no danger of dealing with energy spectrum unbounded from below. Time evolution for annihilation operators is given by

$$A(t) = \mathrm{e}^{-i\frac{t}{\hbar}\hat{\mathcal{H}}_I} A\ \mathrm{e}^{i\frac{t}{\hbar}\hat{\mathcal{H}}_I} = A\cosh(\Gamma t) - B^\dagger \sinh(\Gamma t) \,, \tag{3.34}$$

$$B(t) = \mathrm{e}^{-i\frac{t}{\hbar}\hat{\mathcal{H}}_I} B\ \mathrm{e}^{i\frac{t}{\hbar}\hat{\mathcal{H}}_I} = B\cosh(\Gamma t) - A^\dagger \sinh(\Gamma t) \tag{3.35}$$

Eqs. (3.34) and (3.35) are Bogolubov transformations: they are canonical transformations preserving the ccr. Eq. (3.33) expresses the instability (decay) of the vacuum under the evolution operator $\exp\left(-it\frac{\hat{\mathcal{H}}_I}{\hbar}\right)$. In other words, time evolution leads out of the Hilbert space of the states. This means that the QM framework is not suitable for the canonical quantization of the damped harmonic oscillator. A way out from such a difficulty is provided by QFT [78]: the proper way to perform the canonical quantization of the dho turns out to be working in the framework of QFT. In fact, for many degrees of freedom the time evolution operator $\hat{\mathcal{U}}(t)$ and the vacuum are formally (at finite volume) given by

$$\hat{\mathcal{U}}(t) = \prod_\kappa \exp\left(\Gamma_\kappa t\left(A_\kappa^\dagger B_\kappa^\dagger - A_\kappa B_\kappa\right)\right), \tag{3.36}$$

$$|0(t)\rangle = \prod_\kappa \frac{1}{\cosh(\Gamma_\kappa t)}\exp\left(\tanh(\Gamma_\kappa t)A_\kappa^\dagger B_\kappa^\dagger\right)|0\rangle \quad , \tag{3.37}$$

with $\langle 0(t)|0(t)\rangle = 1,\ \forall t$. Using the continuous limit relation $\sum_\kappa \mapsto \frac{V}{(2\pi)^3}\int d^3\kappa$, in the infinite-volume limit and $\int d^3\kappa\ \Gamma_\kappa$ finite and positive we obtain

$$\langle 0(t)|0\rangle \to 0 \text{ as } V\to\infty\ \forall\ t \,, \tag{3.38}$$

and in general, $\langle 0(t)|0(t')\rangle \to 0$ as $V\to\infty\ \forall\ t$ and $t',\ t'\neq t$. At each time t a representation $\{|0(t)\rangle\}$ of the ccr is defined and turns out to be unitary inequivalent (ui) to any other representation $\{|0(t')\rangle,\ \forall t'\neq t\}$ in the infinite volume limit. In such a way the quantum damped harmonic oscillator evolves in time through

unitary inequivalent representations of ccr (tunneling). The vacuum $|0(t)\rangle$ is a two-mode time dependent generalized coherent state [82, 83] and we have

$$\hat{\mathcal{N}}_{A_\kappa}(t) = \langle 0(t)|A_\kappa^\dagger A_\kappa|0(t)\rangle = \sinh^2\Gamma t \ , \tag{3.39}$$

The Bogolubov transformations, Eqs. (3.34) and (3.35) can be implemented for every κ as inner automorphism for the algebra $su(1,1)_\kappa$. At each time t one has a copy $\{A_\kappa(t), A_\kappa^\dagger(t), B_\kappa(t), B_\kappa^\dagger(t)\,;\,|0(t)\rangle\,|\,\forall\kappa\}$ of the original algebra induced by the time evolution operator which can thus be thought of as a generator of the group of automorphisms of $\bigoplus_\kappa su(1,1)_\kappa$ parameterized by time t (we have a realization of the operator algebra at each time t, which can be implemented by Gel'fand-Naimark-Segal construction in the C*-algebra formalism [3, 84]). Notice that the various copies become unitarily inequivalent in the infinite-volume limit, as shown by Eqs. (3.38): the space of the states splits into ui representations of the ccr each one labeled by time parameter t. As usual, one works at finite volume and only at the end of the computations the limit $V \to \infty$ is performed.

3.3.2. Thermodinamics and entropy in time evolution

Finally, in Refs. 2 and 3 it has been shown that the representation $|0(t)\rangle$ is equivalent to the TFD representation $|0(\beta(t))\rangle$, thus recognizing the relation between the dho states and the finite temperature states. It is useful [78] to introduce the functional $\hat{\mathcal{F}}_A$ for the A-modes

$$\hat{\mathcal{F}}_A \equiv \langle 0(t)|\Big(\hat{\mathcal{H}}_A - \frac{1}{\beta}\hat{\mathcal{S}}_A\Big)|0(t)\rangle \quad , \tag{3.40}$$

where β is a non-zero c-number, $\hat{\mathcal{H}}_A$ is the part of $\hat{\mathcal{H}}_0$ relative to A- modes only, namely

$$\hat{\mathcal{H}}_A = \sum_\kappa \hbar\Omega_\kappa A_\kappa^\dagger A_\kappa, \tag{3.41}$$

and the $\hat{\mathcal{S}}_A$ is given by

$$\hat{\mathcal{S}}_A \equiv -\sum_\kappa \Big\{A_\kappa^\dagger A_\kappa \ln\sinh^2\big(\Gamma_\kappa t\big) - A_\kappa A_\kappa^\dagger \ln\cosh^2\big(\Gamma_\kappa t\big)\Big\} \quad . \tag{3.42}$$

One then considers the extremal condition

$$\frac{\partial\hat{\mathcal{F}}_A}{\partial\vartheta_\kappa} = 0 \quad \forall\kappa \ , \ \vartheta_\kappa \equiv \Gamma_\kappa t \tag{3.43}$$

to be satisfied in each representation, and using the definition $E_\kappa \equiv \hbar\Omega_\kappa$, one finds

$$\hat{\mathcal{N}}_{A_\kappa}(t) = \sinh^2\big(\Gamma_\kappa t\big) = \frac{1}{\mathrm{e}^{\beta(t)E_\kappa} - 1} \quad , \tag{3.44}$$

which is the Bose distribution for A_κ at time t, provided $\beta(t)$ is the (time-dependent) inverse temperature. Inspection of Eqs. (3.40) and (3.42) then suggests that $\hat{\mathcal{F}}_A$ and $\hat{\mathcal{S}}_A$ can be interpreted as the free energy and the entropy, respectively. $\{|0(t)\rangle\}$ is thus recognized to be a representation of the ccr at finite temperature. Use of Eq. (3.42) shows that

$$\frac{\partial}{\partial t}|0(t)\rangle = -\left(\frac{1}{2}\frac{\partial\hat{\mathcal{S}}}{\partial t}\right)|0(t)\rangle \quad . \tag{3.45}$$

One thus see that $i\left(\frac{1}{2}\hbar\frac{\partial\hat{\mathcal{S}}}{\partial t}\right)$ is the generator of time translations, namely time evolution is controlled by the entropy variations [85]. It is remarkable that the same dynamical variable $\hat{\mathcal{S}}$ whose expectation value is formally the entropy also controls time evolution: damping (or, more generally, dissipation) implies indeed the choice of a privileged direction in time evolution (arrow of time) with a consequent breaking of time-reversal invariance. One may also show that

$$d\hat{\mathcal{F}}_A = dE_A - \frac{1}{\beta}d\hat{\mathcal{S}}_A = 0 \ , \tag{3.46}$$

which expresses the first principle of thermodynamics for a system coupled with environment at constant temperature and in absence of mechanical work. As usual, one may define heat as $dQ = \frac{1}{\beta}dS$ and see that the change in time $d\hat{\mathcal{N}}_A$ of particles condensed in the vacuum turns out into heat dissipation dQ:

$$dE_A = \sum_\kappa \hbar\Omega_\kappa \dot{\mathcal{N}}_{A_\kappa}(t)dt = \frac{1}{\beta}dS = dQ \ . \tag{3.47}$$

Here $\dot{\hat{\mathcal{N}}}_{A_\kappa}$ denotes the time derivative of $\hat{\mathcal{N}}_{A_\kappa}$.

CHAPTER 4

Boson Mixing in quantum field theory

4.1. Introduction

A rich non–perturbative vacuum structure has been discovered to be associated with the mixing of fermion fields in the context of Quantum Field Theory [86, 87]. The careful study of such a structure [88] has led to the determination of the exact QFT formula for neutrino oscillations [61, 89, 90], exhibiting new features with respect to the usual quantum mechanical Pontecorvo formula [91]. Actually, it turns out [92, 93] that the non–trivial nature of the mixing transformations manifests itself also in the case of the mixing of boson fields. In the framework of the QFT analysis of Refs. [94], a study of the meson mixing and oscillations has been carried out in Ref.[95], where modifications to the usual oscillation formulas, connected with the vacuum structure, have been presented.

4.2. Mixing transformations of neutral bosons

We consider the mixing of two spin zero neutral boson fields [93, 96]. For instance we could consider the $\eta - \eta'$ mixing, which is one of the most interesting systems due to the large mass difference of the mixed components. Now we introduce the following lagrangian [92, 94, 95] to describe such systems

$$\begin{aligned} \mathscr{L}_N(x) &= \partial_\mu \Phi_f^T(x)\, \partial^\mu \Phi_f(x) \,-\, \Phi_f^T(x)\, \mathbf{M}\, \Phi_f(x) && (4.1) \\ &= \partial_\mu \Phi_m^T(x)\, \partial^\mu \Phi_m(x) \,-\, \Phi_m^T(x)\, \mathbf{M}_d\, \Phi_m(x) && (4.2) \end{aligned}$$

were the mass matrix $\mathbf{M} = \begin{pmatrix} m_A^2 & m_{AB}^2 \\ m_{BA}^2 & m_B^2 \end{pmatrix}$. the flavor fields $\Phi_f^T = (\phi_A, \phi_B)$ are associated to the free fields $\Phi_m^T = (\phi_1, \phi_2)$. They have to satisfy the canonical commutation relations:

$$[\phi_i(x), \pi_j(y)]_{x_0=y_0} = i\,\delta_{ij}\,\delta^3(\mathbf{x}-\mathbf{y})\ , \tag{4.3}$$

The mass matrix $\mathbf{M}_d = \begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix}$ is tranformed by a suitable rotation in mass field space:

$$\phi_A(x) = \phi_1(x)\ \cos\theta + \phi_2(x)\ \sin\theta \tag{4.4}$$

$$\phi_B(x) = -\phi_1(x)\ \sin\theta + \phi_2(x)\ \cos\theta \tag{4.5}$$

and a similar one for the conjugate momenta $\pi_i = \partial_0 \phi_i$. The Fourier expansion of free fields ϕ_i and the conjugate momenta $\pi_i = \partial_0 \phi_i$ are given by

$$\phi_i(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{2\omega_{k,i}}} \left(a_{\mathbf{k},i}\ e^{-i\omega_{k,i}t} + a^\dagger_{-\mathbf{k},i}\ e^{i\omega_{k,i}t} \right) e^{i\mathbf{kx}} \tag{4.6}$$

$$\pi_i(x) = -i \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \sqrt{\frac{\omega_{k,i}}{2}} \left(a_{\mathbf{k},i}\ e^{-i\omega_{k,i}t} - a^\dagger_{-\mathbf{k},i}\ e^{i\omega_{k,i}t} \right) e^{i\mathbf{kx}}, \tag{4.7}$$

with $i = 1, 2$ and $\omega_{k,i} = \sqrt{\mathbf{k}^2 + m_i^2}$ are the energies of the fields. By using the unitary operator $\hat{G}_\theta(t)$, we can rewrite the Eqs.(4.4),(4.5) as:

$$\phi_{A,B}(x) = \hat{G}_\theta^{-1}(t)\ \phi_{1,2}(x)\ \hat{G}_\theta(t) \tag{4.8}$$

$$\pi_{A,B}(x) = \hat{G}_\theta^{-1}(t)\ \pi_{1,2}(x)\ \hat{G}_\theta(t) \tag{4.9}$$

where the operator $\hat{G}_\theta(t)$ is the generator of the mixing transformations Eqs. (4.4),(4.5):

$$\hat{G}_\theta(t) = \exp\left[-i\ \theta \int d^3\mathbf{x}\, (\pi_1(x)\phi_2(x) - \pi_2(x)\phi_1(x)) \right], \tag{4.10}$$

at finite volume, the $\hat{G}_\theta(t)$ is an unitary operator $\hat{G}_\theta^{-1}(t) = \hat{G}_{-\theta}(t) = \hat{G}_\theta^\dagger(t)$, which can be written as

$$\hat{G}_\theta(t) = \exp\{\theta \hat{S}(t)\} \tag{4.11}$$

where $\hat{S}(t)$ is defined as

$$\begin{aligned} \hat{S}(t) = \int d^3\mathbf{k} \Big(& U^*_{\mathbf{k}}(t)\ a^\dagger_{\mathbf{k},1} a_{\mathbf{k},2} - V^*_{\mathbf{k}}(t)\ a_{-\mathbf{k},1} a_{\mathbf{k},2} \\ & + V_{\mathbf{k}}(t)\ a^\dagger_{\mathbf{k},2} a^\dagger_{-\mathbf{k},1} - U_{\mathbf{k}}(t)\ a^\dagger_{\mathbf{k},2} a_{\mathbf{k},1} \Big) \end{aligned} \tag{4.12}$$

The flavor fields can be expanded as:

$$\phi_\sigma(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{2\omega_{k,j}}} \left(a_{\mathbf{k},\sigma}(t)\ e^{-i\omega_{k,j}t} + a^\dagger_{-\mathbf{k},\sigma}(t)\ e^{i\omega_{k,j}t} \right) e^{i\mathbf{k}\cdot\mathbf{x}} \tag{4.13}$$

$$\pi_\sigma(x) = -i \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \sqrt{\frac{\omega_{k,j}}{2}} \left(a_{\mathbf{k},\sigma}(t)\ e^{-i\omega_{k,j}t} - a^\dagger_{-\mathbf{k},\sigma}(t)\ e^{i\omega_{k,j}t} \right) e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{4.14}$$

with $(\sigma, j) = \{(A,1),(B,2)\}$ and the flavor annihilation operators are given by:

$$a_{\mathbf{k},A}(t) \equiv G_\theta^{-1}(t)\, a_{\mathbf{k},1}\, \hat{G}_\theta(t) = \cos\theta\ a_{\mathbf{k},1} + \sin\theta \left(U^*_{\mathbf{k}}(t)\ a_{\mathbf{k},2} + V_{\mathbf{k}}(t)\, a^\dagger_{-\mathbf{k},2} \right), \tag{4.15}$$

$$a_{\mathbf{k},B}(t) \equiv G_\theta^{-1}(t)\ a_{\mathbf{k},2}\ \hat{G}_\theta(t) = \cos\theta\ a_{\mathbf{k},2} - \sin\theta \left(U_{\mathbf{k}}(t)\ a_{\mathbf{k},1} - V_{\mathbf{k}}(t)\ a^\dagger_{-\mathbf{k},1} \right). \tag{4.16}$$

where the quantities $U_{\mathbf{k}}(t)$ and $V_{\mathbf{k}}(t)$ are Bogoliubov coefficients given by

$$U_{\mathbf{k}}(t) \equiv \frac{1}{2} \left(\sqrt{\frac{\omega_{k,1}}{\omega_{k,2}}} + \sqrt{\frac{\omega_{k,2}}{\omega_{k,1}}} \right) e^{i(\omega_{k,2} - \omega_{k,1})t} \tag{4.17}$$

$$V_{\mathbf{k}}(t) \equiv \frac{1}{2} \left(\sqrt{\frac{\omega_{k,1}}{\omega_{k,2}}} - \sqrt{\frac{\omega_{k,2}}{\omega_{k,1}}} \right) e^{i(\omega_{k,1} + \omega_{k,2})t} \tag{4.18}$$

They satisfy the following relation that is valid for the boson case:

$$|U_{\mathbf{k}}|^2 - |V_{\mathbf{k}}|^2 = 1\,, \tag{4.19}$$

The $\hat{G}_\theta(t)$ generator induces an $SU(2)$ coherent state structure on the vacuum $|0\rangle_{1,2}$ for the fields $\phi_i(x)$ [83], which is defined as $a_{\mathbf{k},i}|0\rangle_{1,2} = 0,\ i = 1,2$. The flavor vacuum is the state $|0(\theta,t)\rangle_{A,B}$ for neutral bosons

$$|0(\theta,t)\rangle_{A,B} \equiv \hat{G}_\theta^{-1}(t)\,|0\rangle_{1,2}\,, \tag{4.20}$$

and the state for a mixed particle with "flavor" A and momentum $\mathbf{k}$ may be defined as:

$$|a_{\mathbf{k},A}(t)\rangle \equiv a^\dagger_{\mathbf{k},A}(t)|0(t)\rangle_{A,B} \tag{4.21}$$

$$= \hat{G}_\theta^{-1}(t)a^\dagger_{\mathbf{k},1}|0\rangle_{1,2} \tag{4.22}$$

In the following we work in the Heisenberg picture, flavor states and vacuum will be assumed at reference time $t = 0$. We may also set

$$|a_{\mathbf{k},A}\rangle \equiv |a_{\mathbf{k},A}(0)\rangle, \tag{4.23}$$

and consider the momentum operator, defined as the diagonal space part of the energy-momentum tensor [97]:

$$P^j = \int d^3\mathbf{x}\partial^0\phi(x)\partial^j(x)\phi - g^{0j}\left[\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2\right]. \quad j = \{1,2,3\} \tag{4.24}$$

For the free fields ϕ_i we get:

$$\hat{\mathbf{P}}_i = \int d^3\mathbf{x}\,\pi_i(x)\nabla\phi_i(x) = \int d^3\mathbf{k}\,\frac{\mathbf{k}}{2}\left(a^\dagger_{\mathbf{k},i}a_{\mathbf{k},i} - a^\dagger_{-\mathbf{k},i}a_{-\mathbf{k},i}\right), \tag{4.25}$$

with $i = 1,2$. In a similar way we can define the momentum operator for mixed fields:

$$\hat{\mathbf{P}}_\sigma(t) = \int d^3\mathbf{x}\ [\pi_\sigma(x)\nabla\phi_\sigma(x)]$$

$$= \int d^3\mathbf{k}\,\frac{\mathbf{k}}{2}\left[a^\dagger_{\mathbf{k},\sigma}(t)a_{\mathbf{k},\sigma}(t) - a^\dagger_{-\mathbf{k},\sigma}(t)a_{-\mathbf{k},\sigma}(t)\right], \tag{4.26}$$

were $\sigma = A,B$. The two operators are obviously related: $\hat{\mathbf{P}}_\sigma(t) = \hat{G}_\theta^{-1}(t)\,\hat{\mathbf{P}}_i\,\hat{G}_\theta(t)$. Note that the total momentum is conserved in time since commutes with the generator of mixing transformations (at any time):

$$\hat{\mathbf{P}}_A(t) + \hat{\mathbf{P}}_B(t) = \hat{\mathbf{P}}_1 + \hat{\mathbf{P}}_2 \equiv \hat{\mathbf{P}} \tag{4.27}$$

$$\left[\hat{\mathbf{P}}\,,\hat{G}_\theta(t)\right] = 0 \quad , \quad \left[\hat{\mathbf{P}}\,,H\right] = 0\,. \tag{4.28}$$

Thus in the mixing of neutral fields, the momentum operator plays an analogous role to that of the charge for charged fields. We will present it in the next section.

4.2.1. Oscillation formulas for neutral bosons

We now consider the expectation values of the momentum operator for flavor fields on the flavor state $|a_{\mathbf{k},\varrho}\rangle$ with definite momentum $\mathbf{k}$. Obviously, this is an eigenstate of $\hat{\mathbf{P}}_\varrho(t)$ at time $t = 0$:

$$\hat{\mathbf{P}}_\varrho(0)\,|a_{\mathbf{k},A}\rangle = \mathbf{k}\,|a_{\mathbf{k},\varrho}\rangle\,, \tag{4.29}$$

which follows from $\hat{\mathbf{P}}_1 \left| a_{\mathbf{k},1} \right\rangle = \mathbf{k} \left| a_{\mathbf{k},1} \right\rangle$ by application of $G_\theta^{-1}(0)$. At time $t \neq 0$, the expectation value of the momentum (normalized to the initial value) gives:

$$\mathscr{P}_\sigma^\varrho(t) \equiv \frac{\langle a_{\mathbf{k},\varrho} | \hat{\mathbf{P}}_\sigma(t) | a_{\mathbf{k},\varrho} \rangle}{\langle a_{\mathbf{k},\varrho} | \hat{\mathbf{P}}_\sigma(0) | a_{\mathbf{k},\varrho} \rangle} = \left| \left[a_{\mathbf{k},\sigma}(t), a^\dagger_{\mathbf{k},\varrho}(0) \right] \right|^2 - \left| \left[a^\dagger_{-\mathbf{k},\sigma}(t), a^\dagger_{\mathbf{k},\varrho}(0) \right] \right|^2, \tag{4.30}$$

with $\sigma = \{A, B\}$, which is of the same form as the expression one obtains for the charged field. One can explicitly check that the (flavor) vacuum expectation value of the momentum operator $\hat{\mathbf{P}}_\sigma(t)$ does vanish at all times:

$$_{A,B}\langle 0 | \hat{\mathbf{P}}_\sigma(t) | 0 \rangle_{A,B} = 0 \quad , \qquad \sigma = A, B \tag{4.31}$$

which can be understood intuitively by realizing that the flavor vacuum $|0\rangle_{A,B}$ does not carry momentum since it is a condensate of pairs carrying zero total momentum (like the BCS ground state, for example). The explicit calculation of the oscillating quantities $\mathscr{P}^\varrho_{\mathbf{k},\sigma}(t)$ gives:

$$\mathscr{P}^\varrho_{\mathbf{k},A}(t) = 1 - \sin^2(2\theta) \left[|U_\mathbf{k}|^2 \, \sin^2 \left(\frac{\omega_{k,2} - \omega_{k,1}}{2} t \right) - |V_\mathbf{k}|^2 \, \sin^2 \left(\frac{\omega_{k,2} + \omega_{k,1}}{2} t \right) \right] \tag{4.32}$$

$$\mathscr{P}^\varrho_{\mathbf{k},B}(t) = \sin^2(2\theta) \left[|U_\mathbf{k}|^2 \, \sin^2 \left(\frac{\omega_{k,2} - \omega_{k,1}}{2} t \right) - |V_\mathbf{k}|^2 \, \sin^2 \left(\frac{\omega_{k,2} + \omega_{k,1}}{2} t \right) \right]. \tag{4.33}$$

in complete agreement with the charged field case [92]. The Eqs. (4.64), (4.65) are the flavor oscillation formulas for the neutral mesons, such as $\eta - \eta'$, $\phi - \omega$. By definition of the momentum operator, the Eqs.(4.64), (4.65) are the relative population densities of flavor particles in the beam.

4.3. Mixing transformations of charged bosons

The observed boson oscillations always involve particles with zero electrical charge. In the case of $K^0 - \bar{K}^0$, $B^0 - \bar{B}^0$, $D^0 - \bar{D}^0$, what oscillate are some other quantum numbers such as the strangeness and the isospin. Therefore, in the study of boson mixing, for these particles, we can consider [98] complex fields. The charge in question is some "flavor charge" (e.g. the strangeness) and thus the complex fields are "flavor charged" fields, referred to as "flavor fields" for simplicity. We introduce the following lagrangian:

$$\mathscr{L}_C(x) = \partial_\mu \Phi_f^\dagger(x) \, \partial^\mu \Phi_f(x) - \Phi_f^\dagger(x) \, \mathbf{M} \, \Phi_f(x) \tag{4.34}$$

with $\Phi_f^T = (\phi_A, \phi_B)$ and the mass matrix $\mathbf{M} = \begin{pmatrix} m_A^2 & m_{AB}^2 \\ m_{BA}^2 & m_B^2 \end{pmatrix}$. The mass matrix $\mathbf{M}_d = \begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix}$. is tranformed by a suitable rotation in mass field space: We define the mixing relations as:

$$\begin{aligned} \phi_A(x) &= \phi_1(x) \, \cos\theta + \phi_2(x) \, \sin\theta \\ \phi_B(x) &= -\phi_1(x) \, \sin\theta + \phi_2(x) \, \cos\theta \end{aligned} \tag{4.35}$$

where generically we denote the mixed fields with suffixes A and B. The fields $\phi_i(x)$ $(i = 1, 2)$, are free complex fields with definite masses, which diagonalize the Lagrangian Eq. (4.34) in the following form

$$\mathscr{L}_C(x) \;=\; \partial_\mu \Phi^\dagger_m(x)\, \partial^\mu \Phi_m(x) \;-\; \Phi^\dagger_m(x)\, \mathbf{M}_d\, \Phi_m(x) \tag{4.36}$$

where $\mathbf{M}_d = diag(m_1^2, m_2^2)$. Their conjugate momenta are $\pi_i(x) = \partial_0 \phi_i^\dagger(x)$ and the commutation relations are the usual ones:

$$[\phi_i(x), \pi_j(y)]_{t=t'} \;=\; i\delta^3(\mathbf{x}-\mathbf{y})\,\delta_{ij} \tag{4.37}$$

$$\left[\phi_i^\dagger(x), \pi_j^\dagger(y)\right]_{t=t'} \;=\; i\delta^3(\mathbf{x}-\mathbf{y})\,\delta_{ij}, \qquad i,j = 1,2\,. \tag{4.38}$$

with the other equal–time commutators vanishing. The Fourier expansions of fields and momenta are:

$$\phi_i(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{2\omega_{k,i}}} \left(a_{\mathbf{k},i}\, e^{-i\omega_{k,i}t} + b^\dagger_{-\mathbf{k},i}\, e^{i\omega_{k,i}t} \right) e^{i\mathbf{k}\cdot\mathbf{x}} \tag{4.39}$$

$$\pi_i(x) = i \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \sqrt{\frac{\omega_{k,i}}{2}} \left(a^\dagger_{\mathbf{k},i}\, e^{i\omega_{k,i}t} - b_{-\mathbf{k},i}\, e^{-i\omega_{k,i}t} \right) e^{i\mathbf{k}\cdot\mathbf{x}}\,, \tag{4.40}$$

where $\omega_{k,i} = \sqrt{\mathbf{k}^2 + m_i^2}$ and $[a_{\mathbf{k},i}, a^\dagger_{\mathbf{p},j}] = [b_{\mathbf{k},i}, b^\dagger_{\mathbf{p},j}] = \delta^3(\mathbf{k}-\mathbf{p})\delta_{ij}$, with $i,j = 1,2$ and the other commutators vanishing.

The flavor fields and the corresponding momenta can be written in terms of mass ones and are given by [92]:

$$\phi_{\{A,B\}}(x) = G_\theta^{-1}(t)\; \phi_{\{1,2\}}(x)\; G_\theta(t) \tag{4.41}$$

$$\pi_{\{A,B\}}(x) = G_\theta^{-1}(t)\; \pi_{\{1,2\}}(x)\; G_\theta(t) \tag{4.42}$$

The operator $G_\theta(t)$ realizes the mixing transformations expressed in Eq.(4.35):

$$G_\theta(t) = \exp\{-i\,\theta \int d^3\mathbf{x} \pi_1(x)\phi_2(x) - \phi_1^\dagger(x)\pi_2^\dagger(x) \\ -\pi_2(x)\phi_1(x) + \phi_2^\dagger(x)\pi_1^\dagger(x)\,\}, \tag{4.43}$$

in the limit of a finite volume, the $G_\theta(t)$ is an unitary operator $G_\theta^{-1}(t) = G_{-\theta}(t) = G_\theta^\dagger(t)$. if we consider the vacuum $|0\rangle_{1,2}$ for the fields $\phi_{1,2}(x)$: $a_{\mathbf{k},i}|0\rangle_{1,2} = 0,\ i = 1,2$ then we can show that the generator of the mixing transformations induces an $SU(2)$ coherent state structure on $|0\rangle_{1,2}$ [83]:

$$|0(\theta,t)\rangle_{_{A,B}} \equiv G_\theta^{-1}(t)\; |0\rangle_{1,2}\,. \tag{4.44}$$

From now on we will refer to the state $|0(\theta,t)\rangle_{_{A,B}}$ as to the "flavor" vacuum for bosons. The suffixes A and B label the flavor charge content of the state.

We can define annihilation operators for the vacuum $|0(t)\rangle_{_{A,B}}$ as

$$a_{\mathbf{k},\{A,B\}}(\theta,t) \;\equiv\; G_\theta^{-1}(t)\; a_{\mathbf{k},\{1,2\}}\; G_\theta(t), \tag{4.45}$$

$$b_{-\mathbf{k},\{A,B\}}(\theta,t) \;\equiv\; G_\theta^{-1}(t)\; b_{-\mathbf{k},\{1,2\}}\; G_\theta(t), \tag{4.46}$$

with $a_{\mathbf{k},A}(\theta,t)|0(t)\rangle_{A,B} = 0$. For simplicity we will use the notation $a_{\mathbf{k},A}(t) \equiv a_{\mathbf{k},A}(\theta,t)$. Explicitly, we have:

$$a_{\mathbf{k},A}(t) = \cos\theta\, a_{\mathbf{k},1} + \sin\theta \left(U^*_{\mathbf{k}}(t)\, a_{\mathbf{k},2} + V_{\mathbf{k}}(t)\, b^\dagger_{-\mathbf{k},2}\right), \quad (4.47)$$

$$a_{\mathbf{k},B}(t) = \cos\theta\, a_{\mathbf{k},2} - \sin\theta \left(U_{\mathbf{k}}(t)\, a_{\mathbf{k},1} - V_{\mathbf{k}}(t)\, b^\dagger_{-\mathbf{k},1}\right), \quad (4.48)$$

$$b_{-\mathbf{k},A}(t) = \cos\theta\, b_{-\mathbf{k},1} + \sin\theta \left(U^*_{\mathbf{k}}(t)\, b_{-\mathbf{k},2} + V_{\mathbf{k}}(t)\, a^\dagger_{\mathbf{k},2}\right), \quad (4.49)$$

$$b_{-\mathbf{k},B}(t) = \cos\theta\, b_{-\mathbf{k},2} - \sin\theta \left(U_{\mathbf{k}}(t)\, b_{-\mathbf{k},1} - V_{\mathbf{k}}(t)\, a^\dagger_{\mathbf{k},1}\right). \quad (4.50)$$

As for the case of the fermion mixing, the structure of the generator Eq.(4.43) is recognized to be the one of a rotation combined with a Bogoliubov transformation. In Eqs.(4.47)-(4.50) the Bogoliubov coefficients are defined as

$$U_{\mathbf{k}}(t) \equiv \frac{1}{2}\left(\sqrt{\frac{\omega_{k,1}}{\omega_{k,2}}} + \sqrt{\frac{\omega_{k,2}}{\omega_{k,1}}}\right) e^{i(\omega_{k,2}-\omega_{k,1})t}, \quad (4.51)$$

$$V_{\mathbf{k}}(t) \equiv \frac{1}{2}\left(\sqrt{\frac{\omega_{k,1}}{\omega_{k,2}}} - \sqrt{\frac{\omega_{k,2}}{\omega_{k,1}}}\right) e^{i(\omega_{k,1}+\omega_{k,2})t} \quad (4.52)$$

and satisfy the relation

$$|U_{\mathbf{k}}|^2 - |V_{\mathbf{k}}|^2 = 1\,, \quad (4.53)$$

At equal times, these operators satisfy the canonical commutation relations. In their expressions the Bogoliubov transformation part is evidently characterized by the terms with the U and V coefficients. The condensation density of the flavor vacuum is given for any t by

$${}_{A,B}\langle 0(t)|a^\dagger_{\mathbf{k},i}a_{\mathbf{k},i}|0(t)\rangle_{A,B} = {}_{A,B}\langle 0(t)|b^\dagger_{-\mathbf{k},i}b_{-\mathbf{k},i}|0(t)\rangle_{A,B} = \sin^2\theta\,|V_{\mathbf{k}}|^2, \quad (4.54)$$

with $i = 1, 2$.

4.3.1. Oscillation formulas for charged bosons

The Lagrangian $\mathscr{L}_C$ is invarian under the global $U(1)$ phase transformation and we have the conservation of the Noether charge $Q = \int I^0(x)\, d^3x$, which is the total charge of the system. We now perform the $SU(2)$ transformation on the flavor doublet Φ_f

$$\Phi_f = e^{i\alpha_j\tau_j}\Phi_f \quad (4.55)$$

and we obtain the following current by substitution the above equation in the Lagrangian $\mathscr{L}_C$

$$J^\mu_{j,f}(x) = i\Phi^\dagger_{j,f}(x)\tau_j\partial^\mu\Phi_{j,f}(x), \quad j = 1,2,3 \quad (4.56)$$

The relative charges $\hat{\mathcal{Q}}_{j,f}(t) = \int d^3x J^0_{j,f}(0)$ close the $SU(2)$ algebra at each time t and the Casimir operator is $\hat{\mathcal{C}}_f$. Due to the mixing terms in the mass matrix, $\hat{\mathcal{Q}}_{3,f}(t)$ is time–dependent and we may set

$$\hat{\mathcal{Q}}_A(t) \equiv \frac{1}{2}\hat{\mathcal{Q}} + \hat{\mathcal{Q}}_{3,f}(t)\,, \quad (4.57)$$

$$\hat{\mathcal{Q}}_A(t) \equiv \frac{1}{2}\hat{\mathcal{Q}} - \hat{\mathcal{Q}}_{3,f}(t)\,, \quad (4.58)$$

with $\hat{\mathcal{Q}}_A(t)+\hat{\mathcal{Q}}_B(t)=\hat{\mathcal{Q}}$. We can recast in term of annihilator and creator operators

$$\hat{\mathcal{Q}}_\sigma(t)=\int d^3k\left(a^\dagger_{k,\sigma}(t)a_{k,\sigma}(t)-b^\dagger_{k,\sigma}(t)b_{k,\sigma}(t)\right)\quad \sigma=A,B \tag{4.59}$$

The observable quantities are the expectation values of the flavor charges on the flavor states: the oscillation formulas thus obtained do not depend on the arbitrary mass parameters. In the following, we will work in the Heisenberg picture: this is particularly convenient in the present context since special care has to be taken with the time dependence of flavor states [61, 89]. Let us now define the state of the a_A particle as

$$|a_{\mathbf{k},A}\rangle_{A,B}\equiv a^\dagger_{\mathbf{k},A}(0)|0\rangle_{A,B} \tag{4.60}$$

and consider the expectation values of the flavor charges Eq.(4.59) on it (analogous results follow if one considers $|a_{\mathbf{k},B}\rangle_{A,B}$). We obtain:

$$\begin{aligned}\mathscr{Q}^A_{\mathbf{k},\sigma}(t) &\equiv {}_{A,B}\langle a_{\mathbf{k},A}|\hat{\mathcal{Q}}_\sigma(t)|a_{\mathbf{k},A}\rangle_{A,B}\\ &= \left|\left[a_{\mathbf{k},\sigma}(t),a^\dagger_{\mathbf{k},A}(0)\right]\right|^2-\left|\left[b^\dagger_{-\mathbf{k},\sigma}(t),a^\dagger_{\mathbf{k},A}(0)\right]\right|^2,\end{aligned} \tag{4.61}$$

with $\sigma=A,B$. We also have ${}_{A,B}\langle 0|\hat{\mathcal{Q}}_{\mathbf{k},\sigma}(t)|0\rangle_{A,B}=0$ and $\hat{\mathcal{Q}}^A_{\mathbf{k},A}(t)+\hat{\mathcal{Q}}^A_{\mathbf{k},B}(t)=\hat{1}$.

A straightforward direct calculation shows that the above quantities do not depend on μ_A and μ_B, i.e.:

$${}_{A,B}\langle a_{\mathbf{k},A}|\hat{\mathcal{Q}}_{\mathbf{k},\sigma}(t)|a_{\mathbf{k},A}\rangle_{A,B}={}_{A,B}\langle a_{\mathbf{k},A}|\hat{\mathcal{Q}}_{\mathbf{k},\sigma}(t)|a_{\mathbf{k},A}\rangle_{A,B},\quad \sigma=\{A,B\}, \tag{4.62}$$

and similar one for the expectation values on $|a_{\mathbf{k},B}\rangle_{A,B}$. Eq.(4.62) is a central result of this chapter [92]: it confirms that the only physically relevant quantities are the above expectation values of flavor charges. Note that expectation values of the number operator, of the kind

$${}_{A,B}\langle a_{\mathbf{k},A}|\hat{\mathcal{N}}_\sigma(t)|a_{\mathbf{k},A}\rangle_{A,B}=\left|\left[a_{\mathbf{k},\sigma}(t),a^\dagger_{\mathbf{k},A}(0)\right]\right|^2 \tag{4.63}$$

and similar ones, do indeed depend on the arbitrary mass parameters, although the flavor states are properly defined (i.e. on the flavor Hilbert space). The cancellation of these parameters happens only when considering the combination of squared modula of commutators of the form Eq.(4.61) One may think it could make sense to take the expectation value of the flavor charges on states defined on the mass Hilbert space. A direct calculation however shows that this is not the case and these expectation values depends on the mass parameters: the conclusion is that one must use the flavor Hilbert space. A similar cancellation occurs for fermions [90] with the sum of the squared modula of anticommutators. Finally, the explicit

calculation [92] gives

$$\begin{aligned}\mathcal{Q}^A_{\mathbf{k},A}(t) &= \left|\left[a_{\mathbf{k},A}(t), a^\dagger_{\mathbf{k},A}(0)\right]\right|^2 - \left|\left[b^\dagger_{-\mathbf{k},A}(t), a^\dagger_{\mathbf{k},A}(0)\right]\right|^2 \\ &= 1 - \sin^2(2\theta)\left[|U_\mathbf{k}|^2\ \sin^2\left(\frac{\omega_{k,2}-\omega_{k,1}}{2}t\right) - |V_\mathbf{k}|^2\ \sin^2\left(\frac{\omega_{k,2}+\omega_{k,1}}{2}t\right)\right], \end{aligned} \tag{4.64}$$

$$\begin{aligned}\mathcal{Q}^A_{\mathbf{k},B}(t) &= \left|\left[a_{\mathbf{k},B}(t), a^\dagger_{\mathbf{k},A}(0)\right]\right|^2 - \left|\left[b^\dagger_{-\mathbf{k},B}(t), a^\dagger_{\mathbf{k},A}(0)\right]\right|^2 \\ &= \sin^2(2\theta)\left[|U_\mathbf{k}|^2\ \sin^2\left(\frac{\omega_{k,2}-\omega_{k,1}}{2}t\right) - |V_\mathbf{k}|^2\ \sin^2\left(\frac{\omega_{k,2}+\omega_{k,1}}{2}t\right)\right] \end{aligned} \tag{4.65}$$

Notice the negative sign in front of the $|V_\mathbf{k}|^2$ terms in these formulas, in contrast with the fermion case [61, 89, 90]: the boson flavor charge can assume also negative values. This fact points to the statistical nature of the phenomenon: it means that when dealing with mixed fields, one intrinsically deals with a many–particle system, i.e. a genuine field theory phenomenon. This situation has a strong analogy with Thermal Field Theory (i.e. QFT at finite temperature) [5], where quasi–particle states are ill defined and only statistical averages make sense.

The above formulas are obviously different from the usual quantum mechanical oscillation formulas, which however are recovered in the relativistic limit (i.e. for $|\mathbf{k}|^2 \gg \frac{m_1^2+m_2^2}{2}$). Apart from the extra oscillating term (the one proportional to $|V_\mathbf{k}|^2$) and the momentum dependent amplitudes, the QFT formulas carry the remarkable information about the statistics of the oscillating particles: for bosons and fermions the amplitudes (Bogoliubov coefficients) are drastically different according to the two different statistics ($|U_\mathbf{k}|$ and $|V_\mathbf{k}|$ are circular functions in the fermion case and hyperbolic functions in the boson case). This fact also fits with the above mentioned statistical nature of the oscillation phenomenon in QFT. Note also that our treatment is essentially non–perturbative [92].

4.4. Remark on Fourier expansion of flavor fields

Above we have expanded the mixed fields $\phi_{A,B}$ in the same basis as the free fields $\phi_{1,2}$. However, this is not the most general possibility [88]. Indeed, one could as well expand the flavor fields in a basis of fields with arbitrary masses. Of course, these arbitrary mass parameters should not appear in the physically observable quantities. A consistent treatment of the flavor oscillation for bosons in QFT can be given which does not exhibit the above pathological dependence on arbitrary parameters [92].

CHAPTER 5

Phenomenology of strange meson particles

We consider the quantum mechanics formulation of meson mixing and obtain the oscillation formula in the presence of the decay. We analyze phenomenological aspects of this effect.

5.1. Introduction

Quark mixing and meson mixing are widely studied and verified [99]. However, many features of the physics of mixing are still obscure, for example the issue related to its origin in the context of Standard Model and the related problem of the generation of masses. The problem of the boson mixing is known since 1955 when Gell-Mann and Pais predicted the existence of two neutral kaons [100]: K^0 of strangeness $S = 1$ and $\bar{K}^0$ of strangeness $S = -1$.

These are particle and antiparticle, and are connected by the process of charge conjugation, which involves a reversal of values of I_3 and a change of strangeness $\Delta S = 2$. Strong interactions conserve I_3 and S, so that as far as production is concerned, the separate neutral-kaon eigenstates are $K^0 - \bar{K}^0$. Now suppose K^0 and $\bar{K}^0$ particles propagate through empty space. Since both are neutral, both can decay to pions by weak interaction, with $\Delta S = 1$. Thus, mixing can occur via (virtual) intermediate pion states:

$$\begin{aligned} K^0 &\leftrightarrows 2\pi \leftrightarrows \bar{K}^0 \\ K^0 &\leftrightarrows 3\pi \leftrightarrows \bar{K}^0 \end{aligned} \tag{5.1}$$

These transitions are $\Delta S = 2$ and thus second order weak interactions. Although extremely weak, this implies that if one has a pure K^0-state at $t = 0$, at any later time $t > 0$ one will have a superposition of both K^0 and $\bar{K}^0$, so that the state can be written

$$|K(t)\rangle = \alpha(t)|K^0\rangle + \beta(t)|\bar{K}^0\rangle. \tag{5.2}$$

The phenomena has been explained by realizing that what we observe is the mixture of two mass and mean life eigenstates K_S and K_L expressed by

$$\begin{aligned} |K_S\rangle &= \frac{1}{\sqrt{2(1+|\varepsilon|^2)}}[(1+\varepsilon)|K^0\rangle + (1-\varepsilon)|\bar{K}^0\rangle] = \frac{1}{\sqrt{(1+|\varepsilon|^2)}}(|K_1\rangle + \varepsilon|K_2\rangle), \\ |K_L\rangle &= \frac{1}{\sqrt{2(1+|\varepsilon|^2)}}[(1+\varepsilon)|K^0\rangle - (1-\varepsilon)|\bar{K}^0\rangle] = \frac{1}{\sqrt{(1+|\varepsilon|^2)}}(|K_2\rangle + \varepsilon|K_1\rangle), \end{aligned} \tag{5.3}$$

where ε is a small, complex parameter responsible for CP symmetry breaking and K_1, K_2 are CP eigenstates:

$$\begin{aligned} |K_1\rangle &= \frac{1}{\sqrt{2}}(|K^0\rangle + |\bar{K}^0\rangle) \qquad CP|K_1\rangle = |K_1\rangle, \\ |K_2\rangle &= \frac{1}{\sqrt{2}}(|K^0\rangle - |\bar{K}^0\rangle) \qquad CP|K_2\rangle = -|K_2\rangle \end{aligned} \tag{5.4}$$

with the convention

$$\begin{aligned} CP|K^0\rangle &= |\bar{K}^0\rangle \\ CP|\bar{K}^0\rangle &= |K^0\rangle. \end{aligned} \tag{5.5}$$

Unlike K^0 and $\bar{K}^0$, distinguished by their mode of production (the K^0 can be produced by nonstrange particles in association with a hyperon and $\bar{K}^0$ can be produced only in association with a kaon or antihyperon, of strangeness $S = 1$), K_S and K_L are distinguished by their mode of decay. Consider 2π and 3π decay modes. Since pions have no spin, angular momentum conservation requires that the two pions resulting from $K^0 \longrightarrow 2\pi$ decay carry relative angular momentum equal to the spin of the kaon. A neutral 2π state with specified angular momentum l is an eigenstate of C with eigenvalue $C = (-1)^l$, since the action of C is just to exchange the two pions.

Being K^0 a spinless particle, Gell-Mann e Pais concluded that only the component K_1, CP eigenstate with eigenvalue 1, would be capable of 2π decay; while K_2 would only decay in 3π state with $CP = -1$. But, in 1964 Christenson, Cronin, Fitch and Turlay [101] demonstrate that the K_2 state could also decay to $\pi^+\pi^-$ with a branching ratio of order 10^{-3}. Then there is a CP violation in the K^0 decay and the physical component of K^0 are K_S (short lived component) and K_L (long lived component), where K_S consists principally of a $CP = +1$ amplitude, but with a little $CP = -1$, and K_L vice versa. Experimentally the mean lives of K_S and K_L are $\tau_S = (0.8934 \mp 0.0008) \times 10^{-10}$ sec, $\tau_L = (5.17 \mp 0.04) \times 10^{-8}$ sec respectively.

An important phenomenon is the K^0 regeneration [102]. Suppose we produce a pure K^0 beam and let it travel in vacuo for the order of $100K_S$ mean lives, so that all the K_S component has decayed and we are left with K_L only. Now let the K_L beam traverse a slab of material and interact. Immediately, the strong interactions will pick out the strangeness $S = +1$ and $S = -1$ components of the beam.

Thus, of the original K^0 beam intensity, about 50% has disappeared by K_S decay. The remainder, K_L upon traversing a slab where its nuclear interaction can be observed, should consist of 50% K^0 and 50% $\bar{K}^0$.

The K^0 and $\bar{K}^0$ must be absorbed differently; K^0 particles can only undergo elastic and charge exchange scattering, while $\bar{K}^0$ particles can also undergo strangeness exchange giving hyperons:

$$K^0 + p \to K^+ + n \tag{5.6}$$

$$K^0 + n \to K^0 + n \tag{5.7}$$

and

$$\bar{K}^0 + p \to \begin{cases} \Lambda^0 + \pi^+ \\ \Sigma^+ + \pi^0 \\ p + K^+ + K^- \end{cases} \tag{5.8}$$

$$\bar{K}^0 + n \to \Lambda^0 + \pi^0. \tag{5.9}$$

With more strong channels open, the $\bar{K}^0$ is therefore absorbed more strongly than K^0. After emerging from the slab, we shall therefore have a K^0 amplitude $f|K^0\rangle$ and a $\bar{K}^0$ amplitude $\bar{f}|\bar{K}^0\rangle$, where $\bar{f} < f < 1$. If we neglect the CP symmetry breaking, the emergent beam will be

$$\begin{aligned}\frac{1}{2}(f|K^0\rangle - \bar{f}|\bar{K}^0\rangle) &= \frac{f+\bar{f}}{2\sqrt{2}}(|K^0\rangle - |\bar{K}^0\rangle) + \frac{f-\bar{f}}{2\sqrt{2}}(|K^0\rangle + |\bar{K}^0\rangle) \\ &= \frac{1}{2}(f+\bar{f})|K_L\rangle + \frac{1}{2}(f-\bar{f})|K_S\rangle. \end{aligned} \tag{5.10}$$

Since $f \neq \bar{f}$, it follows that some of the K_S state has been regenerated. The main prediction of the particle mixture hypothesis is the possibility of observing the $K_S - K_L$ interference. If we make the assumption of exponential decay, each component will have a time dependence of the form $e^{-(\Gamma_i/2\hbar + iE_i/\hbar)t}$ where E_i is the total energy of the particle $i = L, S$ and $\Gamma_i = \hbar/\tau_i$ is the width of the state, τ_i being the mean life in the frame in which the energy E_i is defined.

Set $\hbar = c = 1$, and measure all times in the rest frame, so that τ_i is the proper lifetime and $E_i = m_i$, the particle rest mass, then the time dependence became $e^{-(\Gamma_i/2 + im_i)t}$. The neutral $K^0 - \bar{K}^0$ boson system is not the only one where the quantum mechanical mass mixing can be considered. We can expect to observe the same phenomenon in other neutral boson systems: $D^0 - \bar{D}^0$, $B^0 - \bar{B}^0$ and $\eta - \eta'$. Generally, flavor oscillations of particles can occur when states produced and detected in a given experiment, are superpositions of two or more eigenstates with different masses. The mesons oscillations has been used to place stringent constraints on physics beyond the Standard Model. It has been shown that the geometric phases for mixed meson systems could permit the direct determination of the CP violating parameter.

5.2. Symmetry violations in meson mixing

5.2.1. The Wigner–Weisskopf approximation

The time evolution of neutral bosons, like kaons, can be described by the Wigner-Weisskopf approximations [103]. The time evolution is thus determined by the Schrodinger equation

$$i\frac{d}{dt}|\Phi(t)\rangle = H|\Phi(t)\rangle, \quad \Phi = \{M^0, \overline{M}^0\} \tag{5.11}$$

where the effective Hamiltonian $H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}$ of the system is non-Hermitian and can be written as $H = M - i\frac{\Gamma}{2}$ with M and Γ Hermitian matrices.

The biorthonormal basis formalism [104, 105] will be used to describe the time evolution of mixed mesons in the presence of CP violation. In particular, the general discussion of Ref. [105] is applied to the particular case of meson mixing. Let $\lambda_j = m_j - i\Gamma_j/2$, with $j = L, H$, be the eigenvalues of Hamiltonian H with $|M_j\rangle$ the corresponding eigenvectors (L denotes the light mass state and H the heavy mass state. For K mesons, usually the mass eigenstates are defined according to their lifetimes: K_S is the short lived and K_L is the long lived. In this system K_L is the heavier state):

$$H|M_j\rangle = \lambda_j|M_j\rangle\,, \tag{5.12}$$

We denote with ε_j and $|\widetilde{M}_j\rangle$, $(j = a, b)$ the eigenvalues and the eigenvectors of $H^\dagger$: $H^\dagger|\widetilde{M}_j\rangle = \varepsilon_j|\widetilde{M}_j\rangle$. Such an equation can be recast in the form

$$\langle\widetilde{M}_j|H = \langle\widetilde{M}_j|\varepsilon_j^* . \tag{5.13}$$

By projecting Eq.(5.13) on the state $|M_j\rangle$ we have

$$\langle\widetilde{M}_j|H|M_j\rangle = \langle\widetilde{M}_j|\varepsilon_j^*|M_j\rangle = \langle\widetilde{M}_j|\lambda_j|M_j\rangle , \tag{5.14}$$

then $\varepsilon_j^* = \lambda_j$. Moreover we have $\langle\widetilde{M}_i|H|M_j\rangle = \langle\widetilde{M}_i|\varepsilon_i^*|M_j\rangle = \langle\widetilde{M}_i|\lambda_j|M_j\rangle$, hence: $(\lambda_j - \varepsilon_i^*)\langle\widetilde{M}_j|M_i\rangle = 0$. This last relation together with Eqs.(5.14) implies

$$\langle\widetilde{M}_j|M_i\rangle = \langle M_i|\widetilde{M}_j\rangle = \delta_{ij} , \tag{5.15}$$

(in contrast to $\langle M_j|M_{i\neq j}\rangle \neq 0$). Thus the eigenvalues of H are the complex conjugate of those of $H^\dagger$ and the relative eigenvectors are biorthogonal each other. Note also that the state vector $|\psi(t)\rangle$ describing the neutral boson system (without its decay products) can be expressed as $|\psi(t)\rangle = \sum_{j=1,2} a_j(t)|M_j\rangle = \sum_{j=1,2} \widetilde{a}_j(t)|\widetilde{M}_j\rangle$, with $a_j(t) = \langle\widetilde{M}_j|\psi(t)\rangle$ and $\widetilde{a}_j(t) = \langle M_j|\psi(t)\rangle$, i.e. $|\psi(t)\rangle = \sum_{j=1,2}|M_j\rangle\langle\widetilde{M}_j|\psi(t)\rangle = \sum_{j=1,2}|\widetilde{M}_j\rangle\langle M_j|\psi(t)\rangle$. This equation implies the completeness relations

$$\sum_j |M_j\rangle\langle\widetilde{M}_j| = \sum_j |\widetilde{M}_j\rangle\langle M_j| = 1 . \tag{5.16}$$

Summarizing, since H is non-Hermitian and non-normal, the conjugate states $\langle\widetilde{M}_j|^\dagger \equiv |\widetilde{M}_j\rangle$ are not isomorphic to $|M_j\rangle$, i.e. $|\widetilde{M}_j\rangle \neq |M_j\rangle$ and similar for $|M_j\rangle^\dagger \equiv \langle M_j|$: $\langle M_j| \neq \langle\widetilde{M}_j|$. The right and left eigenvectors of the Hamiltonian H in Eq.(5.11) are $(p_L, q_L)^T$, $(p_H, -q_H)^T$ and $\frac{1}{q_L p_H + q_H p_L}(q_H, p_H)$, $\frac{1}{q_L p_H + q_H p_L}(q_L, -p_L)$ respectively, and also introduce

$$\frac{q}{p} = \sqrt{\frac{q_L q_H}{p_L p_H}} = \sqrt{\frac{H_{21}}{H_{12}}} . \tag{5.17}$$

The existence of a complete biorthonormal set of eigenvector of H implies that H is diagonalizable. Indeed this is equivalent to say that the matrix V exists such that $V^{-1}HV = diag(\lambda_L, \lambda_H)$ with

$$V = \begin{pmatrix} p_L & p_H \\ q_L & -q_H \end{pmatrix} , \qquad V^{-1} = \frac{1}{\text{Det }[V]} \begin{pmatrix} q_H & p_H \\ q_L & -p_L \end{pmatrix} . \tag{5.18}$$

The mass eigenstates $|M_L\rangle$ and $|M_H\rangle$ are then written in terms of $|M^0\rangle$, $|\bar{M}^0\rangle$ as

$$|M_L\rangle = p_L\,|M^0\rangle + q_L\,|\bar{M}^0\rangle , \tag{5.19}$$

$$|M_H\rangle = p_H\,|M^0\rangle - q_H\,|\bar{M}^0\rangle , \tag{5.20}$$

and, in a similar way, $\langle\widetilde{M}_L|$ and $\langle\widetilde{M}_H|$ are expressed as

$$\langle\widetilde{M}_L| = \frac{1}{q_L p_H + q_H p_L}\left[q_H\,\langle\widetilde{M}^0| + p_H\,\langle\widetilde{\bar{M}}^0|\right] , \tag{5.21}$$

$$\langle\widetilde{M}_H| = \frac{1}{q_L p_H + q_H p_L}\left[q_L\,\langle\widetilde{M}^0| - p_L\,\langle\widetilde{\bar{M}}^0|\right] . \tag{5.22}$$

Since the time evolution operator associated with H: $U(t) = e^{-iHt}$ is not unitary, then we introduce the time evolution operator of $H^\dagger$: $\widetilde{U}(t) = e^{-iH^\dagger t}$ such that

$U\widetilde{U}^\dagger = \widetilde{U}^\dagger U = 1$. The spectral form of the Hamiltonian and of the operators $U(t)$ and $\widetilde{U}(t)$ are then given by

$$H = \sum_j \lambda_j |M_j\rangle\langle\widetilde{M}_j| \,,$$

$$U(t) = \sum_j e^{-i\lambda_j} |M_j\rangle\langle\widetilde{M}_j| \,, \qquad \widetilde{U}(t) = \sum_j e^{-i\lambda_j^*} |\widetilde{M}_j\rangle\langle M_j| \,, \tag{5.23}$$

respectively. Thus, the expressions of the states $|M^0(t)\rangle$ and $|\bar{M}^0(t)\rangle$ in terms of $|M_L\rangle$ and $|M_H\rangle$ are obtained by Eqs.(5.19)-(5.23):

$$|M^0(t)\rangle = \frac{1}{q_L p_H + q_H p_L} \left[q_H \, |M_L\rangle \, e^{-i\lambda_L t} + q_L \, |M_H\rangle \, e^{-i\lambda_H t} \right] , \tag{5.24}$$

$$|\bar{M}^0(t)\rangle = \frac{1}{q_L p_H + q_H p_L} \left[p_H \, |M_L\rangle \, e^{-i\lambda_L t} - p_L \, |M_H\rangle \, e^{-i\lambda_H t} \right] , \tag{5.25}$$

$$\langle\widetilde{M}^0(t)| = p_L \, \langle\widetilde{M}_L| \, e^{i\lambda_L t} + p_H \, \langle\widetilde{M}_H| \, e^{i\lambda_H t} \,, \tag{5.26}$$

$$\langle\widetilde{\bar{M}^0}(t)| = q_L \, \langle\widetilde{M}_L| \, e^{i\lambda_L t} - q_H \, \langle\widetilde{M}_H| \, e^{i\lambda_H t} \,. \tag{5.27}$$

The states in Eqs.(5.24)–(5.27) are the correct ones to be used to derive the expressions of the asymmetries describing the violations of CP and CPT invariance. The constrains on H imposed by CP and T invariance suggest to adopt the following CP and T violation parameter:

$$\varepsilon = \frac{|p_H/q_H| - |q_L/p_L|}{|p_H/q_H| + |q_L/p_L|} = \frac{|p/q| - |q/p|}{|p/q| + |q/p|} = \frac{|H_{12}| - |H_{21}|}{|H_{12}| + |H_{21}|} \,. \tag{5.28}$$

Moreover, the CPT invariance imposes the equality of the diagonal elements of the Hamiltonian H in Eq.(5.11): $H_{11} = H_{22}$, thus such invariance can be tested by checking that the difference $H_{22} - H_{11}$ is equal to zero. The CPT violation can be described conveniently by the phase convention independent quantity

$$z = \frac{\frac{q_L}{p_L} - \frac{q_H}{p_H}}{\frac{q_L}{p_L} + \frac{q_H}{p_H}} = \frac{(H_{22} - H_{11})}{\lambda_L - \lambda_H} \,. \tag{5.29}$$

Note that in the case of CPT invariance: $p/q = p_L/q_L = p_H/q_H$ and $z = 0$. By using Eq.(5.29), the states (5.24)-(5.27) can be also expressed as

$$|M^0(t)\rangle = \frac{1}{2p} \left[\sqrt{1-z} \, |M_L\rangle \, e^{-i\lambda_L t} + \sqrt{1+z} \, |M_H\rangle \, e^{-i\lambda_H t} \right] , \tag{5.30}$$

$$|\bar{M}^0(t)\rangle = \frac{1}{2q} \left[\sqrt{1+z} \, |M_L\rangle \, e^{-i\lambda_L t} - \sqrt{1-z} \, |M_H\rangle \, e^{-i\lambda_H t} \right] , \tag{5.31}$$

$$\langle\widetilde{M^0}(t)| = p \left[\sqrt{1-z} \, \langle\widetilde{M_L}| \, e^{i\lambda_L t} + \sqrt{1+z} \, \langle\widetilde{M_H}| \, e^{i\lambda_H t} \right] , \tag{5.32}$$

$$\langle\widetilde{\bar{M}^0}(t)| = q \left[\sqrt{1+z} \, \langle\widetilde{M_L}| \, e^{i\lambda_L t} - \sqrt{1-z} \, \langle\widetilde{M_H}| \, e^{i\lambda_H t} \right] , \tag{5.33}$$

5.2.2. CP violation in meson mixing

The violation of time-reversal invariance can be reveled by the comparison between the probability of transition from $\bar{M}^0$ to M^0: $P_{\bar{M}^0 \to M^0}$ and the probability of transition from M^0 to $\bar{M}^0$: $P_{M^0 \to \bar{M}^0}$ in the asymmetry:

$$A_T(\Delta t) = \frac{P_{\bar{M}^0 \to M^0}(\Delta t) - P_{M^0 \to \bar{M}^0}(\Delta t)}{P_{\bar{M}^0 \to M^0}(\Delta t) + P_{M^0 \to \bar{M}^0}(\Delta t)} \tag{5.34}$$

with $\Delta t = t_f - t_i$ denoting the time interval between the initial t_i and the final time t_f. The transition amplitudes $A_{\bar{M}^0 \to M^0}(\Delta t)$ and $A_{M^0 \to \bar{M}^0}(\Delta t)$ are given respectively by

$$A_{\bar{M}^0 \to M^0}(\Delta t) = \langle \widetilde{M^0}(t_f) | \bar{M}^0(t_i) \rangle \; = \; \frac{1}{2} \frac{q}{p} \sqrt{1 - z^2} \left(e^{-i\lambda_L \Delta t} - e^{-i\lambda_H \Delta t} \right) , \quad (5.35)$$

$$A_{M^0 \to \bar{M}^0}(\Delta t) = \langle \widetilde{\bar{M}^0}(t_f) | M^0(t_i) \rangle \; = \; \frac{1}{2} \frac{p}{q} \sqrt{1 - z^2} \left(e^{-i\lambda_L \Delta t} - e^{-i\lambda_H \Delta t} \right) . \quad (5.36)$$

To derive the results in Eqs.(5.35)–(5.36) the unitary operator $|M_L\rangle\langle\widetilde{M_L}| + |M_H\rangle\langle\widetilde{M_H}| = 1$ has been introduced on the right side of the operator $e^{-iH\Delta t}$. The corresponding transition probabilities are then

$$P_{\bar{M}^0 \to M^0}(\Delta t) = \frac{1}{2} \left| \frac{q}{p} \right|^2 \left| \sqrt{1 - z^2} \right|^2 e^{-\frac{\Gamma}{2} \Delta t} \left[\cosh \left(\frac{\Delta\Gamma \Delta t}{2} \right) - \cos(\Delta m \Delta t) \right] , \quad (5.37)$$

$$P_{M^0 \to \bar{M}^0}(\Delta t) = \frac{1}{2} \left| \frac{p}{q} \right|^2 \left| \sqrt{1 - z^2} \right|^2 e^{-\frac{\Gamma}{2} \Delta t} \left[\cosh \left(\frac{\Delta\Gamma \Delta t}{2} \right) - \cos(\Delta m \Delta t) \right] , \quad (5.38)$$

where $\Delta m = m_H - m_L$, $\Delta\Gamma = \Gamma_H - \Gamma_L$ and $\Gamma = \Gamma_L + \Gamma_H$; in the following it is also introduced m to denote $m = m_L + m_H$. Note that the sign of $\Delta\Gamma$ is not yet established for B and B_s mesons, $\Delta\Gamma < 0$ for K mesons and $\Delta\Gamma > 0$ for D mesons. The asymmetry A_T is time independent and it is given by

$$A_T \; = \; \frac{1 - \left| \frac{q}{p} \right|^4}{1 + \left| \frac{q}{p} \right|^4} . \quad (5.39)$$

A value different from zero of the quantity in Eq(5.39) indicates a direct T violation independent from CPT violation.

5.2.3. CPT violation in meson mixing

On the other hand, the violation of CPT invariance can be reveled by the comparison between the probability of transition from $\bar{M}^0$ to $\bar{M}^0$: $P_{\bar{M}^0 \to \bar{M}^0}$ and the probability of transition from M^0 to $\bar{M}^0$: $P_{M^0 \to M^0}$ in the asymmetry:

$$A_{CPT}(\Delta t) \; = \; \frac{P_{M^0 \to M^0}(\Delta t) - P_{\bar{M}^0 \to \bar{M}^0}(\Delta t)}{P_{M^0 \to M^0}(\Delta t) + P_{\bar{M}^0 \to \bar{M}^0}(\Delta t)} . \quad (5.40)$$

The transition amplitudes $A_{M^0 \to M^0}(\Delta t)$ and $A_{\bar{M}^0 \to \bar{M}^0}(\Delta t)$ are given respectively by

$$A_{M^0 \to M^0}(\Delta t) = \langle \widetilde{M^0}(t_f) | M^0(t_i) \rangle \; = \; \left(\frac{1 + z}{2} \right) e^{-i\lambda_H \Delta t} + \left(\frac{1 - z}{2} \right) e^{-i\lambda_L \Delta t} , \quad (5.41)$$

$$A_{\bar{M}^0 \to \bar{M}^0}(\Delta t) = \langle \widetilde{\bar{M}^0}(t_f) | \bar{M}^0(t_i) \rangle \; = \; \left(\frac{1 - z}{2} \right) e^{-i\lambda_H \Delta t} + \left(\frac{1 + z}{2} \right) e^{-i\lambda_L \Delta t} , \quad (5.42)$$

where again the relation $|M_L\rangle\widetilde{\langle M_L|} + |M_H\rangle\widetilde{\langle M_H|} = 1$ has been introduced on the right side of $e^{-iH\Delta t}$. The corresponding transition probabilities are then

$$\begin{aligned} P_{M^0\to M^0}(\Delta t) &= e^{-\frac{\Gamma}{2}\Delta t}\Big[\left(\frac{1+|z|^2}{2}\right)\cosh\left(\frac{\Delta\Gamma\Delta t}{2}\right) \\ &\quad -\Re z\sinh\left(\frac{\Delta\Gamma\Delta t}{2}\right)+\left(\frac{1-|z|^2}{2}\right)\cos(\Delta m\Delta t)+\Im z\sin(\Delta m\Delta t)\Big], \end{aligned} \tag{5.43}$$

$$\begin{aligned} P_{\bar{M}^0\to\bar{M}^0}(\Delta t) &= e^{-\frac{\Gamma}{2}\Delta t}\Big[\left(\frac{1+|z|^2}{2}\right)\cosh\left(\frac{\Delta\Gamma\Delta t}{2}\right) \\ &\quad +\Re z\sinh\left(\frac{\Delta\Gamma\Delta t}{2}\right)+\left(\frac{1-|z|^2}{2}\right)\cos(\Delta m\Delta t)-\Im z\sin(\Delta m\Delta t)\Big]. \end{aligned} \tag{5.44}$$

The asymmetry A_{CPT} is thus given by

$$A_{CPT}(\Delta t) = \frac{-2\,\Re z\,\sinh\left(\frac{\Delta\Gamma\Delta t}{2}\right)+2\,\Im z\,\sin(\Delta m\Delta t)}{(1+|z|^2)\,\cosh\left(\frac{\Delta\Gamma\Delta t}{2}\right)+(1-|z|^2)\,\cos(\Delta m\Delta t)}. \tag{5.45}$$

Omitting second order terms in z and making the approximation $\sinh\left(\frac{\Delta\Gamma\Delta t}{2}\right) \simeq \frac{\Delta\Gamma\Delta t}{2}$ which is valid in the range $|\Delta t| < 15ps$ used in the experimental analysis [106], [107], one has

$$A_{CPT}(\Delta t) \simeq \frac{-\Re z\,\Delta\Gamma\Delta t+2\,\Im z\,\sin(\Delta m\Delta t)}{\cosh\left(\frac{\Delta\Gamma\Delta t}{2}\right)+\cos(\Delta m\Delta t)}, \tag{5.46}$$

which coincide with Eq.(6) of Ref.[107]. In the case of CPT invariance, $z = 0$ and $A_{CPT} = 0$.

Conclusion

In conclusion, we have shown the presence of geometric phases in the evolution of a two level system and studied its gauge structure. We have computed the covariant derivative and pointed out that it acts as the free energy with the gauge field acting as the entropy. In such a picture time evolution is thus controlled by the free energy. When applied to a qubit state, these results may be of interest in quantum computing studies. we have shown that time evolution of a two level system or qubit is controlled by a covariant derivative accounting for the coupling of the state with a (non-abelian) gauge field background so to preserve the invariance of the dynamics against local in time gauge transformations. We have shown that the effect of the gauge field background can be depicted as the effect of a birefringence phenomenon, the gauge field background acting as the analogous of the refractive medium. We have also shown that the covariant derivative plays the role of the free energy with the gauge field acting as the entropy. In such a picture time evolution is controlled by the free energy. Finally, the relation of our result with the geometric phase and the so-called adiabatic connection has been pointed out. We have also shown that the distance in the projective Hilbert space between two level or qubit evolution states is measured by the Fubini–Study metric in terms of the Anandan–Aharonov geometric invariant.

We have studied the structure of the currents and charges for the charged mixed fields and we have shown that the effective Hamiltonian of mixed neutral meson systems is non-Hermitian and non-normal and it can be studied in the Wigner–Weisskopf approximation. The asymmetries describing the T and CPT violations are computed using the biorthonormal basis formalism.

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