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MV-algebras, Semirings and their Applications

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Contents

Introduction

The thesis is divided in two parts: the first part regards MV-semirings, involutive semirings and semimodules over them with particular attention to injective and projective semimodules; the second part of the thesis is focused on the tropical semiring and has the purpose to characterize the sets which arise as images of retractions that are nonexpansive with respect to a hemi-norm which plays a key role in tropical geometry.

Semirings and semimodules, and their applications, arise in various branches of Mathematics, Computer Science, Physics, as well as in many other areas of modern science (see, for instance, [30]). MV-algebras arose in the literature as the algebraic semantics of Łukasiewicz propositional logic, one of the longest-known many-valued logic. A connection between MV-algebras and a special category of additively idempotent semirings (called *MV-semirings* or *Łukasiewicz semirings*) was first observed in [18]. On the one hand, every MV-algebra has two *semiring reducts* isomorphic to each other by the involutive unary operation * of MV-algebras (see, e.g., [21, Proposition 4.8]); on the other hand, the category of MV-semirings defined in [21] is isomorphic to the one of MV-algebras. The term equivalence between MV-algebras and MV-semirings allows us to import results and techniques of semiring and semimodule theory in the study of MV-algebras as well to use properties and theorems regarding MV-algebras in the study of semimodules over MV-semirings.

Indeed, as the theory of modules is an essential chapter of ring theory, so the theory of semimodules is a crucial aspect in semiring theory and two of the most important objects in semimodule theory are projective and injective semimodules.

Although, in general, describing the structure of projective and injective semimodules seems to be a quite difficult task, we shall give a criterion for injectivity of semimodules over additively idempotent semirings which we shall use to describe the structure of injective semimodules over MV-semirings with an atomic Boolean center, i. e. the boolean elements of the MV-semiring form an atomic lattice.

In the first three sections we shall provide all necessary notions, facts and

examples on semirings and MV-algebras. The aforementioned term equivalence between MV-algebras and MV-semirings shall be recalled as well as the one between MV-semirings and coupled semirings originally presented in [18].
In Section 1.4 we shall discuss congruences and ideals in MV-semirings. Contrarily to what happens for MV-algebras congruences and ideals are not in bijection in MV-semirings and while congruence simple MV-semirings coincide with simple MV-algebras, the only ideal simple MV-semiring is the boolean semifield $\{0, 1\}$.

In Chapter 2 we shall investigate the semiring of polynomials in the variable x over the boolean semifield $\{0, 1\}$ evaluated on the standard MV-semiring $([0, 1], \vee, \odot, 0, 1)$ and we shall denote such semiring as $\mathbb{B}[x]$. In particular, we shall prove that it coincides with the semiring of nondecreasing convex McNaughton functions over $[0, 1]$ with pointwise operations and that it is also isomorphic to the semiring $(\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$.

In Chapter 3 we shall present a term equivalence between involutive residuated lattices and a special class of semirings called *involutive semirings* which can be seen as a generalization of the one between MV-algebras and MV-semirings since the variety of involutive residuated lattices strictly contains the one of MV-algebras. The semiring perspective helps us find a necessary and sufficient condition for the interval $[0, 1]$ to be a subalgebra of an involutive residuated lattice. This result is meaningful because with 0 and 1 we don't mean the top and bottom elements of the involutive residuated lattice: 0 is an additional constant and 1 is the neutral element of the multiplication. In particular, we show that $[0, 1]$ is a subalgebra of an involutive residuated lattice if and only if 0 is a multiplicatively idempotent element. This chapter is based on a joint paper with Peter Jipsen titled "Injective and projective semimodules over involutive semirings" which is currently revised and resubmitted.

Chapter 4 is about injective and projective semimodules. First, we shall provide all the basic notions and results about semimodules and then we shall give the definitions of injective and projective semimodules. While for modules over rings, we have various equivalent definitions of injective and projective modules (see [59]), unfortunately this is not the case for semimodules over semirings because the different definitions of injective and projective semimodules analogous to the ones for modules don't lead to the same class of semimodules. Through the thesis we shall always refer to the categorical definition of injective and projective semimodules (i. e. injective and projective objects in the category of semimodules over a given semiring). One of the main results presented in Chapter 4 is the following criterion for injectivity of semimodules over additively idempotent semirings, i. e.

semirings for which the sum is idempotent:

Theorem. *Let S be an additively idempotent semiring and M a left S-semimodule. Then M is injective if and only if there exists a set X such that M is a retract of the left S-semimodule $Hom_{\mathbb{B}}(S, \mathbb{B})^X$, where $\mathbb{B}$ is the Boolean semifield.*

The boolean semifield $\mathbb{B}$ is the two-element semiring $\{0, 1\}$ and

$$Hom_{\mathbb{B}}(S, \mathbb{B})^X$$

is the set of morphisms from S to $\mathbb{B}$ seen as B-semimodules or, equivalently, join-semilattices.

We shall also prove that the same criterion can be restated in terms of ideals of join-semilattices.

Regarding projective semimodules, it is well-known that in any variety of algebras the projective objects are the retracts of free objects. In the category of semimodules over a semiring S, the free object over a set X is the set of functions from X to S with finite support, i. e. $S^{(X)} = \{f : X \to S \mid f(x) = 0$ for all but finitely many $x \in X\}$ ([21]). So, we obtain the following characterization of projective semimodules.

Theorem. *Let S be a semiring. An S-semimodule P is projective if and only if it is a retract of the semimodule $S^{(X)}$ for some set X.*

The results of this chapter come both from the aforementioned joint paper with Peter Jipsen and from the paper "On injectivity of semimodules over additively idempotent division semirings and chain MV-semirings" (joint work with Antonio Di Nola, Giacomo Lenzi and Tran Giang Nam, published in 2019).

In Chapter 5, using the aforementioned criteria, we characterize self-injective MV-semirings with an atomic Boolean center. In the following theorem we denote with $A^{\vee\odot}$ one of the two isomorphic semiring reducts of the MV-algebra A; in particular the semiring $A^{\vee\odot}$ has the set A as its universe and the MV-algebra operations $\vee$ and $\odot$ as its semiring operations.

Theorem. *For any MV-algebra A with an atomic Boolean center, the following conditions are equivalent:*

(1) The semiring $A^{\vee\odot}$ is self-injective;

(2) All finitely generated projective $A^{\vee\odot}$-semimodules are injective;

(3) All cyclic (generated by one element) projective $A^{\vee\odot}$-semimodules are injective;

(4) A is a complete MV-algebra.

We also give a description of (finitely generated) injective semimodules over finite MV-semirings. In particular we have that over a finite MV-semiring finitely generated projective and injective semimodules coincide.

Finally, we show that complete Boolean algebras are precisely the MV-semirings in which every principal ideal is an injective semimodule. This chapter is based on the aforementioned joint paper with Di Nola et al.

Then, in Chapter 6, analogously to what we do for MV-semirings, we investigate injective and projective semimodules over involutive semirings and we shall generalize some of the results regarding semimodules over MV-semirings. Indeed, many of the results regarding MV-semimodules can be proven taking into account only the involutive property of MV-semirings and not their isomorphism to MV-algebras. So, it was natural to generalize these results to a broader class of semirings (which are involutive) and that contains MV-semirings as special cases. We show for example that, also for finite commutative involutive semirings, finitely generated injective and projective semimodules coincide. This result could make us wonder if, removing the hypothesis of involution, the coincidence of finitely generated injective and projective still holds. The answer is no and we shall provide a counterexample. It leads us to observe that, even if the involution appears only in the semiring and it doesn't affect at all the structure of the semimodule, it still plays a fundamental role in the study of injective and projective semimodules.
In Section 6.1 we shall recall the definition of strong MV-semimodules and generalize it to semimodules over involutive semirings. Furthermore, we shall prove that strong semimodules coincide with faithful semimodules, where the notion of faithful semimodule is a generalization of the one of faithful module. The interesting thing is that, despite the fact that strong semimodules can be defined only for particular semirings such as MV or involutive semirings, they coincide with faithful semimodules which, on the contrary, can be defined on every semiring. This chapter is based on the aforementioned joint work with Peter Jipsen.

The second part of the thesis is devoted to a special class of semimodules over the zero-free semiring $\mathbb{R}$ called *ambitropical cones* and their applications to game theory. They differ from the standard definition of semimodules since they have a lattice structure. In particular we shall prove that they coincide with the retractions of a class of maps called *Shapley operators* which are operators that describe the evolution of the value function of a zero-sum game as a function of the horizon.

Before giving the definition of Shapley operators we shall briefly explain the class of games to which they apply. We shall consider two-player zero-sum repeated games with a finite state space $[n] = \{1, \dots, n\}$. We shall denote the two players MIN and MAX and their sets of actions dependent on the state i, respectively, A_i and B_i. Let $a \in A_i$ and $b \in B_i$ be the actions selected by the two players, then r_i^{ab} represents the transition payment (that player MIN pays to player MAX) and the vector $(P_{ij}^{ab})_{j\in[n]}$ represents the transition probability (from a state to another state). Note that both these quantities depend on the current state of the game i and on the actions of the two players.

For this class of games the value vector (of $\mathbb{R}^n$), if it exists, satisfies the following recursive formula:

$$v^0 = 0 \qquad v^k = T(v^{k-1})$$

where T is an operator from $\mathbb{R}^n$ to $\mathbb{R}^n$ whose ith coordinate is given by

$$T_i(x) = \min_{a\in A_i} \max_{b\in B_{i,a}} \left(r_i^{ab} + \sum_{j\in[n]} P_{ij}^{ab} x_j \right) \tag{1}$$

The vector $v^k = (v_i^k)_{i\in[n]} = T^k(0)$, where T^k denotes the kth of T, gives the value of the game in horizon k, as a function of the initial state i. A central problem in the theory of zero-sum game is to characterize the mean payoff per time unit, i.e., the vector $\lim_{k\to\infty} T^k(0)/k$. This is studied by means of the ergodic eigenproblem, which consists in finding $u \in \mathbb{R}^n$, the "ergodic eigenvector", and $\lambda \in \mathbb{R}$, the "ergodic eigenvalue", such that

$$T(u) = \lambda e_n + u$$

where e_n denotes the vector of $\mathbb{R}^n$ whose entries are identically 1. Then, the mean payoff vector does exist, and is of the form λe_n, meaning that the mean payoff is equal to λ, for all choices of the initial state.Therefore, it is of interest to characterize the set of eigenvectors of T. By replacing T by $-\lambda e_n + T$, this is equivalent to characterizing the fixed point sets of operators of the form 1.

Since the operators for repeated games of the form of 1 coincide with monotone and additively homogeneous operators from $\mathbb{R}^n$ to $\mathbb{R}^n$ ([40]), we can investigate fixed point sets of these operators that we shall call "abstract" Shapley operators. This motivates the following definition:

Definition. A *Shapley operator* is a map $T : \mathbb{R}^n \to \mathbb{R}^n$ that satisfy

1. $x \leqslant y$ implies $T(x) \leqslant T(y)$;
2. $T(\lambda e_n + x) = \lambda e_n + T(x)$,

The interest in *Shapley retractions* is also motivated by reasons related to the concept of convexity.

Indeed, let us consider the following result from convex analysis:

Theorem. *A subset $C \subset \mathbb{R}^n$ is closed and convex if and only if it is the image of a nonexpansive retraction in a Euclidean norm.*

In the light of Theorem 7.1 we can investigate a generalized notion of convexity. Indeed, we can study the sets which arise as images of retractions that are nonexpansive with respect not only to Euclidean norms but also to other families of norms or hemi-norms.

We recall that a *hemi-norm* is a function f from a real vector space X to $\mathbb{R}$ such that: it is subadditive, i. e. $f(x+y) \leqslant f(x) + f(y)$ $\forall x, y \in X$ and positively homogeneous, i. e. $f(\alpha x) = \alpha f(x)$ $\forall \alpha \in \mathbb{R}, \alpha \geqslant 0$, $\forall x \in X$.

We will be especially interested in the hemi-norm

$$\mathbf{t}(x) := \max_{i \in [n]} x_i$$

Shapley operators are linked to the hemi-norm $\mathbf{t}$ through the following result ([32])

Proposition. *T is a Shapley operator if and only if it is nonexpansive in the "$\mathbf{t}$" hemi-norm:*

$$\mathbf{t}(T(x) - T(y)) \leqslant \mathbf{t}(x - y) \ .$$

So, the sets which arise as images of retractions that are nonexpansive with respect to the hemi-norm $\mathbf{t}$ coincide with Shapley retracts (images of idempotent Shapley operators).

One of the main results of Chapter 7 shows that these sets are closed (with respect to Euclidean topology) ambitropical cones, i. e. additive cones which are also lattices wrt the componentwise order on $\mathbb{R}^n$.

In the first two sections of 7 we shall provide motivation for our work, basic definitions and results.
Section 7.3 contains definitions and results regarding Shapley operators.
In Section 7.4 we have the two main results of the chapter. The first one characterizes Shapley retracts in terms of lattice properties (indeed, an ambitropical cone is closed if and only if it is a conditionally complete lattice):

Theorem. *Let E be a subset of $\mathbb{R}^n$. The following assertions are equivalent:*

1. *E is a closed ambitropical cone;*
2. *E is a Shapley retract of $\mathbb{R}^n$;*
3. *E is the fixed point set of an idempotent Shapley operator T.*

The second one gives a geometric characterization of Shapley retracts in terms of sets of existence of best tropical co-approximation; where a *set of* *existence of best co-approximation* is a subset E of a Banach space $(X, \|\cdot\|)$ such that, for all $z \in X$, the set

$$B_E^{\|\cdot\|}(z) := \{x \in X \mid \|y - x\| \leqslant \|y - z\|,\ \forall y \in E\}$$

contains an element of E.

In the following theorem we denote by $P_E^{\max}$ the operator from $\mathbb{R}^n$ to $\mathbb{R}^n$ defined by

$$P_E^{\max}(x) := \sup\{y \in E^{\max} \mid y \leqslant x\}$$

where E is a nonempty subset of $\mathbb{R}^n$ and $E^{\max}$ is the set of elements of the form

$$\sup\{\lambda_f + f \mid f \in E\} \tag{2}$$

where $\lambda_f + f$ denote the vector of $(\lambda_f + f_i)_{i\in[n]}$ and $\lambda_f \in \mathbb{R}_{\max}$ are such that the family of elements $(\lambda_f + f)_{f\in E}$ is bounded from above and the λ_f are not identically $-\infty$. The operator $P_E^{\min}$ and the set $E^{\min}$ are defined analogously.

Theorem. *Let E be a subset of $\mathbb{R}^n$. The following assertions are equivalent:*

1. *E is a Shapley retract of $\mathbb{R}^n$;*
2. *E is a set of existence of best tropical co-approximation;*
3. *for all $z \in \mathbb{R}^n$, $[P_E^{\max}(z), P_E^{\min}(z)] \cap E \neq \emptyset$;*
4. *$P_E^{\min}(z) \in E$ holds for all $z \in E^{\max}$;*
5. *$P_E^{\max}(z) \in E$ holds for all $z \in E^{\min}$;*
6. *E is the fixed point set of the operator $\bar{Q}_E^{+} = P_E^{\max} \circ P_E^{\min}$;*
7. *E is is the fixed point set of the operator $\bar{Q}_E^{-} = P_E^{\min} \circ P_E^{\max}$.*

In Section 7.5, we show that several canonical classes of sets in tropical geometry are special cases of ambitropical cones, that can be characterized through Shapley operators by suitable strengthening of the previous results.

Chapter 8 contains a generalization of the two previous theorems to order preserving maps over conditionally complete lattices. The second part of the thesis (Chapter 7 and Chapter 8) is based on a joint work with Stéphane Gaubert and Marianne Akian which is going to be submitted soon.

Appendix A is the Proof of 7.1; appendix B consists of an example of an ambitropical polyhedron.

Chapter 1

MV-semirings

In this chapter we shall provide all the necessary notions about MV-semirings and MV-algebras and we shall recall the well-known categorical isomorphism between the two aforementioned categories. For insights about MV-algebras and MV-semirings we refer the reader respectively to [11] and [21], whereas for all the definitions and notions about arbitrary semirings we refer to [30]. For all the basic notions about category theory we refer the reader to [48]:

1.1 Basic notions

Definition 1.1 (Semiring). A *semiring* S is an algebra $(S, +, \cdot, 0, 1)$ of type (2,2,0,0) such that:

1. $(S, +, 0)$ is a commutative monoid;
2. $(S, \cdot, 1)$ is a monoid;
3. $x \cdot (y+z) = x \cdot y + x \cdot z$ and $(y+z) \cdot x = y \cdot x + z \cdot x$, for every $x, y, z \in S$.
4. $0 \cdot s = 0 = s \cdot 0$, for each $s \in S$.

Examples 1.1 (Semirings).

1. *we can think of semirings as unitary rings without an additive inverse. Obviously, rings are special examples of semirings;*
2. *bounded distributive lattices are semirings;*
3. *starting from the ring of real numbers* $(\mathbb{R}, +, \cdot, 0, 1)$ *we can construct a semiring by taking only the nonnegative elements of* $\mathbb{R}$*, in a similar way we can obtain semirings starting from* $\mathbb{Q}$ *or* $\mathbb{Z}$*;*
4. *more generally, if we have a ring* $(R, +, \cdot, 0, 1)$ *with a partial order satisfying for all* $a, b \in R$*: if* $0 \leqslant a$ *and* $0 \leqslant b$ *then* $0 \leqslant a + b$ *and* $0 \leqslant a \cdot b$ *we can obtain a semiring from* R *by taking its "positive part", i. e. the subset* $\{r \in R \mid 0 \leqslant r\}$*;*

5. *a prominent example of semiring is given by* max tropical semiring $\mathbb{R}_{\max} := (\mathbb{R} \cup \{-\infty\}, \max, +, -\infty, 0)$*, it is obtained by extended real numbers substituting* $+$ *by* max *and* $\cdot$ *by* $+$*. Similarly we can define the* min tropical semiring $\mathbb{R}_{\min} := (\mathbb{R} \cup \{+\infty\}, \min, +, +\infty, 0)$*, tropical semirings have various applications and they form the basis of tropical geometry (see [47]);*

6. *another important example of semiring is the* boolean semiring $\mathbb{B} := (\{0,1\}, +, \cdot, 0, 1)$*; in this semiring it holds* $1 + 1 = 1$*.*

Definition 1.2. A semiring S is *additively idempotent* if for every $s \in S$, $s + s = s$.
A *semifield* is a semiring in which all non-zero elements have a multiplicative inverse.
A semiring S is *commutative* if $(S, \cdot)$ is a commutative monoid.

Examples 1.2.

1. *Both the tropical semirings and the boolean semiring are additively idempotent semifields;*

2. *a bounded distributive lattice* $(L, \vee, \wedge, 0, 1)$ *is an additively idempotent semiring but it is a semifield if and only if* $L = \{0, 1\}$*.*

Definition 1.3. For any additively idempotent semiring S, there exists a natural order given by

$$s \leqslant t \iff s + t = t;$$

with $s, t \in S$. In this case $+$ is denoted by $\vee$ and $(S, \vee)$ is a join-semilattice.

Furthermore, it is possible to define a natural order on any semiring $(S, +, \cdot, 0, 1)$ in the following way:

$$a \leqslant b \iff b = a + c \textit{ for some } c \in S$$

Note that the two orders coincide on additively idempotent semirings.

Definition 1.4 (MV-semiring)**.** Let $S = (S, \vee, \cdot, 0, 1)$ be a commutative, additively idempotent semiring. S is a *MV-semiring* iff there exists a map $^* : S \to S$, called *negation*, such that:

1. $a \cdot b = 0$ iff $b \leqslant a^*$;

2. $a \vee b = (a^* \cdot (a^* \cdot b)^*)^*$.

From now on we shall include the negation symbol in the signature of the MV-semiring, denoting S with $(S, \vee, \cdot, 0, 1, ^*)$.

Examples 1.3. *1. The interval* $[0, 1]$ *where the operations* $\vee$, $\odot$ *and* $*$ *are defined respectively by* $x \vee y = \max\{x, y\}$, $x \odot y = \max\{x + y - 1, 0\}$ *and* $x^* = 1 - x$ *for all* $x, y \in [0,\ 1]$, *this MV-semirings is called "the standar MV-semiring";*

2. let $n \geqslant 2$ *be an integer, consider the subset*

$$\boldsymbol{L}_n = \{0, 1/(n-1), \cdots, (n-2)/(n-1)\}$$

of rationals with the operations $\vee$, $\cdot$*and* $*$ *defined as the restriction of these operations defined in* $[0, 1]$*. These are examples of MV-semirings.*

Remark 1.1. Note that for an arbitrary additively idempotent semiring 1 is not in general the top element with respect to the natural order of the additively idempotent semring, whereas 1 is the top element for any MV-semiring; indeed

$$1 = 1 \vee 1 = (1^* \cdot (1^*)^*)^* = 0*$$

since $a^* \cdot (a^*)^* = 0 \iff (a^*)^* \leqslant (a^*)^*$ and

$$a \leqslant 0^* \iff 0 \cdot a = 0.$$

Semiring and MV-semiring morphism are defined in the standard way, for the sake of completeness we shall recall the definitions.

Definition 1.5 (Semiring morphism)**.** Let R and S be semirings. A *semiring morphism* between R and S is a map $f : R \to S$ such that:

1. $f(r + r') = f(r) + f(r')$;

2. $f(r \cdot r') = f(r) \cdot f(r')$;

3. $f(0) = 0$ and $f(1) = 1$;

for all $r, r' \in R$.

Definition 1.6 (MV-semiring morphism)**.** Let R and S be MV-semirings. A *MV-semiring morphism* between R and S is a map $f : R \to S$ such that f is a semiring morphism and $f(r^*) = f(r)^*$, for every $r \in R$.

We indicate by *MVS* the category whose objects are MV-semirings and whose arrows are MV-semiring morphisms.

Definition 1.7 (MV-algebra)**.** An *MV-algebra* is an algebra $(A, \oplus, ^*, 0)$ of type (2,1,0) such that, for every $x, y \in A$ we have:

1. $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;

2. $x \oplus y = y \oplus x$;

3. $x \oplus 0 = x$;
4. $(x^*)^* = x$;
5. $x \oplus 0^* = 0^*$;
6. $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$.

Examples 1.4. *1. The interval* $[0,1]$ *is a prominent example of MV-algebra, called the standard MV-algebra. The operations* $\oplus$, $*$ *are defined respectively by* $x \oplus y = \min\{x+y, 1\}$ *and* $x^* = 1 - x$ *for all* $x, y \in [0,\, 1]$*;*

2. let $n \geqslant 2$ *be an integer, consider the subset*

$$\boldsymbol{L}_n = \{0, 1/(n-1), \cdots, (n-2)/(n-1), 1\}$$

of rationals with the operations $\oplus$ *and* $*$ *defined as the restriction of the standard MV-algebra of these operations. These are examples of MV-algebras;*

3. for any Boolean algebra $(B, \vee, \wedge, ', 0, 1)$*, the structure* $(B, \vee, ', 0)$ *is an MV-algebra.*

Definition 1.8. On every MV-algebra A, it is possible to define another constant $1 := 0^*$ and the operation $x \odot y := (x^* \oplus y^*)^*$. From now on we shall include these in the signature of the MV-algebra, denoting A with $(A, \oplus, \odot, ^*, 0, 1)$.

Remark 1.2. Note that $x \oplus y = (x^* \odot y^*)^*$.

Definition 1.9. For any MV-algebra A, there exists a natural order given by:

$$x \leqslant y \iff x^* \oplus y = 1,$$

for every $x, y \in A$.

The natural order determines a structure of bounded distributive lattice on A, with 0 and 1 respectively bottom and top element and the operations of *sup* and *inf* defined by:

$$x \vee y := (x \odot y^*) \oplus y$$

and

$$x \wedge y := (x^* \vee y^*)^*.$$

Definition 1.10 (MV-algebras morphisms)**.** Let A and B be MV-algebras. A *MV-algebra morphism* between A and B is a map $f : A \to B$ such that:

1. $f(x \oplus y) = f(x) \oplus f(y)$;

2. $f(x^*) = f(x)^*$;
3. $f(0) = 0$;

for all $x, y \in A$.

We indicate by MV the category whose objects are MV-algebras and whose arrows are MV-algebra morphisms.

Definition 1.11. An MV-algebra A is called an *MV-chain* if the natural order of A is total; and the MV-algebra A is called *complete* if it is complete as a lattice.

Examples 1.5. *The MV-algebras* $[0,1]$ *and* $\boldsymbol{L}_n$ *for any* $n \geqslant 2$ *are examples of complete MV-chains.*

Definition 1.12 (Partially ordered abelian groups)**.** A *partially ordered abelian group* is an abelian group $(G, +, -, 0)$ endowed with a partial order relation $\leqslant$ that is compatible with addition; i. e.

$$x \leqslant y \Longrightarrow t + x \leqslant t + y$$

for all $x, y, t \in G$.

A *lattice-ordered abelian group* (l - *group* for short) is a partially ordered abelian group in which the partial order relation defines a lattice structure.

Proposition 1.1. *In any* l - group *we have*

$$t + (x \vee y) = (t + x) \vee (t + y)$$

and

$$t + (x \wedge y) = (t + x) \wedge (t + y)$$

Definition 1.13. A *strong order unit* in a lattice-ordered abelian group G is an element $0 \leqslant u \in G$ such that for any $x \in G$, there exists a positive integer n such that $x \leqslant nu$.

In [49] Mundici constructed a categorical equivalence between the category $\mathcal{MV}$ of MV-algebras with MV-algebra homomorphisms and the category $\mathcal{L}_u$ of lattice-ordered Abelian groups with a distinguished strong order unit whose morphisms are lattice-ordered group homomorphisms which preserve the distinguished strong unit. The two functors of the equivalence are usually denoted by $\Gamma : \mathcal{L}_u \longrightarrow \mathcal{MV}$ and $\Xi : \mathcal{MV} \longrightarrow \mathcal{L}_u$; while the former is very easy to present and shall be recalled hereafter, the latter requires more work and the details of its construction are not really relevant to our discussion.

Let $G = (G, +, -, \leqslant, \vee, \wedge, 0, u)$ be a lattice-ordered Abelian group with a distinguished strong order unit u. Then the MV-algebra $\Gamma(G, u)$ is

$$([0,\ u] := \{x \in G \mid 0 \leqslant x \leqslant u\},\ \oplus,\ ^*,\ 0)$$

with $x \oplus y = (x + y) \wedge u$ and $x^* = u - x$ for all $x, y \in [0,\ u]$. In this case, the operation $\odot$ in $\Gamma(G, u)$ is defined by $x \odot y = (x^* \oplus y^*)^* = u - (2u - x - y) \wedge u$ for all $x, y \in [0,\ u]$.

In the light of categorical equivalence between MV-algebras and lattice-ordered abelian groups with a distinguished strong order unit we can review some of the main examples of MV-algebras.

Examples 1.6 (MV-algebras)**.** *1. Let $\mathbb{R}$ be the additive groups of reals with the natural order. Then $\Gamma(\mathbb{R}, 1) = [0,\ 1]$ is the standard MV-algebra. In the standard MV-algebra the order relation (and therefore the lattice structure) is the usual one of real numbers; the operations $\oplus$, $*$ and $\odot$ are defined respectively by $x \oplus y = \min\{x+y, 1\}$, $x^* = 1 - x$ and $x \odot y = \max\{x + y - 1, 0\}$ for all $x, y \in [0,\ 1]$.*

2. Let $\mathbb{Z}$ be the additive group of integers and $n \geqslant 2$ an integer. Consider the subgroup $\mathbb{Z}\frac{1}{n-1} = \{\frac{z}{n-1} \mid z \in \mathbb{Z}\}$ of the additive group of rationals with denominator $n - 1$ with the natural order. Then

$$\boldsymbol{L}_n := \Gamma\Big(\mathbb{Z}\frac{1}{n-1}, 1\Big) = \{0, 1/(n-1), \cdots, (n-2)/(n-1), 1\}$$

yields an MV-chain with the operations defined as the restriction of the standard MV-algebra of these operations.

1.2 MV-algebras and MV-semirings

Proposition 1.2. *Let A be an MV-algebra. Then $A^{\vee\odot} = (A, \vee, \odot, 0, 1)$ and $A^{\wedge\oplus} = (A, \wedge, \oplus, 1, 0)$ are semirings and the involution $^* : A \to A$ is an isomorphism between them. In particular $A^{\vee\odot}$ and $A^{\wedge\oplus}$ are MV-semirings with negation *.*

Proposition 1.3. *If $(A, +, \cdot, 0, 1,^*)$ is an MV-semiring, the structure $(A, \oplus, \cdot,^*, 0, 1)$ with, for all $x, y \in A$*

$$x \oplus y = (x^* \cdot y^*)^*$$

is an MV-algebra.

Proposition 1.4. *Let A and B be two MV-algebras and $h : A \to B$ a map. Then h is an MV-algebras morphism if and only if it is an MV-semirings morphism.*

Proof. It comes from the fact that the two varieties are term-equivalent. □

The two previous propositions shows that MV-algebras and MV-semirings are two term-equivalent varieties. Since the morphisms of the two categories coincide it is straightforward to show that the two corresponding categories are isomorphic, so we have

Theorem 1.1. *MVS and MV are isomorphic categories.*

1.3 Coupled semirings

In this section, to complete the picture of term-equivalences between MV-algebras and classes of semirings, we shall present the term-equivalence between MV-algebras and coupled semirings, for insights we refer the reader to [18].

Definition 1.14. A semiring $(S, +, \cdot, 0, 1)$ is *lattice-ordered* iff it has also a lattice-structure such that

1. $a + b = a \vee b$;
2. $a \cdot b \leqslant a \wedge b$;

for all $a, b \in S$.

A semiring $(S, +, \cdot, 0, 1)$ is *dual lattice-ordered* iff it has also a lattice-structure such that

1. $a + b = a \wedge b$;
2. $a \cdot b \geqslant a \vee b$;

for all $a, b \in S$.

Lattice-ordered semirings and dual-lattice ordered semirings are additively idempotent. In the following we shall use the name *lc-semiring* for a lattice ordered commutative semiring and *dual lc-semiring* for a dual-lattice ordered commutative semiring.

Definition 1.15. A *coupled semiring* S is a triple (S_1, S_2, α) such that

1. $S_1 = (A, \vee, \cdot, 0, 1)$ and $S_2 = (A, \wedge', \cdot', 0', 1')$ are respectively a *lc-semiring* and a *dual lc-semiring*;
2. $0' = 1$ and $1' = 0$;
3. $\alpha : A \to A$ is a semiring isomorphism from S_1 onto S_2;
4. $\alpha(\alpha(x)) = x$, for every $x \in A$;
5. $x \vee y = x \cdot' (\alpha(x) \cdot y)$.

Definition 1.16 (Coupled semirings morphisms)**.** Let (S_1, S_2, α) and (S_1', S_2', α') be coupled semirings. Then, they will be of the form

$$(S_1, S_2, \alpha) = ((A, \vee_A, \cdot_A, 0_A, 1_A), (A, \wedge_A', \cdot_A', 0_A', 1_A'), \alpha)$$

and

$$(S_1', S_2', \alpha') = ((B, \vee_B, \cdot_B, 0_B, 1_B), (B, \wedge_B', \cdot_B', 0_B', 1_B'), \alpha').$$

A *coupled semiring morphism* between (S_1, S_2, α) and (S_1, S_2, α) is a semirings morphism $f : A \to B$ such that

1. $f(a \vee_A a') = f(a) \vee_B f(a')$;
2. $f(a \wedge_A' a') = f(a) \wedge_B' f(a')$;
3. $f(a \cdot_A a') = f(a) \cdot_B f(a')$;
4. $f(a \cdot_A' a') = f(a) \cdot_B' f(a')$;
5. $f(0_A) = 0_B$;
6. $f(1_A) = 1_B$;
7. $f(0_A') = 0_B'$;
8. $f(1_A') = 1_B'$;
9. $f \circ \alpha = \alpha' \circ f$.

We indicate by CS the category whose objects are coupled semirings and whose arrows are coupled semirings morphisms.

Proposition 1.5. *Let $A = (S_1, S_2, \alpha)$ be a coupled semiring, where $S_1 = (A, \vee, \cdot, 0, 1)$ and $S_2 = (A, \wedge', \cdot', 0', 1')$. Then $(A, \cdot', \cdot, \alpha, 0, 1)$ is an MV-algebra.*

Note that in definition 1.15 it is not stated that $(A, \vee, \wedge')$ is a lattice, i. e. $\vee$ and $\wedge$ could come from two different lattice structures. We shall now prove that this is not the case and that $(A, \vee, \wedge')$ is a lattice using proposition 1.5.

Proposition 1.6. *Let $(S_1, S_2, \alpha) = ((A, \vee_A, \cdot_A, 0_A, 1_A), (A, \wedge_A', \cdot_A', 0_A', 1_A'), \alpha)$ be a coupled semiring, then $(A, \vee, \wedge')$ is a lattice.*

Proof. Let A be the universe of a coupled semiring(S_1, S_2, α). We shall denote by $\vee_{MV}$ and $\wedge_{MV}$ the lattice operation of the MV-algebra obtained from the coupled semiring. Since in the coupled semiring $x \vee y = x \cdot' (\alpha(x) \cdot y)$ by definition, a moment of reflection shows us that $x \vee y = x \vee_{MV} y$. Since

α is a semiring isomorphism from S_1 to S_2 and from S_2 to S_1 (note that α is the inverse of itself) we have that

$$\alpha(x \wedge' y) = \alpha(x) \vee \alpha(y)$$

and applying α again we obtain

$$x \wedge' y = \alpha(\alpha(x) \vee \alpha(y))$$

By [18, Lemma 3.4] we know that $\alpha(\alpha(x) \vee \alpha(y)) = x \cdot (\alpha(x) \cdot' y)$ and it is easy to see that $x \cdot (\alpha(x) \cdot' y) = x \wedge_{MV} y$.

So, since $\vee$ and $\wedge'$ represent the two lattice operation of the MV-algebra obtained from the coupled semiring we have that $(A, \vee, \wedge')$ is a lattice.

$\square$

Proposition 1.7. *Let $(A, \oplus, \odot, ^*, 0, 1)$ be an MV-algebra. Then, $S_A^{\vee} = (A, \vee, \odot, 0, 1)$ and $S_A^{\wedge} = (A, \wedge, \oplus, 1, 0)$ form a coupled semiring $(S_A^{\vee}, S_A^{\wedge}, ^*)$.*

Remark 1.3. Similar to remark 1.4, we have that if A and B are two MV-algebras and $h : A \to B$ is a map between them, then h is an MV-algebras morphism if and only if it is a coupled semirings morphism.

The two previous propositions show that MV-algebras and coupled semirings are two term-equivalent varieties. Since the morphisms of the two categories coincide it is straightforward to show that the two corresponding categories are isomorphic, so we have

Theorem 1.2. *CS and MV are isomorphic categories.*

The isomorphism between coupled semirings and MV-algebras can be revisited in the light of [28] and [24]; where it is proven that MV-algebras are isomorphic to a full subcategory of $\ominus$-algebras. $\ominus$-algebras are particular double quasioperator algebras; where these last ones are expansions of distributive lattices. The double quasioperator algebras approach to MV-algebras is motivated by the intention of proving a suitable extension of Priestley duality for MV-algebras. For the definitions and facts regarding double quasioperator alegbras and $\ominus$-algebras we refer to [28] and [24]. We shall revisit Theorem 1.2 by introducing particular double quasioperator algebras that we shall call $\odot - \alpha$ algebras.

Definition 1.17 ($\odot - \alpha$ algebras)**.** A $\odot - \alpha$ algebra is a bounded distributive lattice $(A, \vee, \wedge, \odot, \alpha, 0, 1)$ with a binary operation $\odot$ and a unary operation α satisfying:

1. $\odot$ is a double quasioperator of type (1,1). That is, for all a, b, $c \in A$,

$$(a \wedge b) \odot c = (a \odot c) \wedge (b \odot c) \text{ and } (a \vee b) \odot c = (a \vee c) \odot (b \vee c)$$

$$a \odot (b \wedge c) = (a \odot b) \wedge (a \odot c) \text{ and } a \odot (b \vee c) = (a \vee b) \odot (a \vee c)$$

2. α is a double quasioperator of type *op*. That is, for all a, $b \in A$,

$$\alpha(a \vee b) = \alpha(a) \wedge \alpha(b) \text{ and } \alpha(a \wedge b) = \alpha(a) \vee \alpha(b)$$

In the previous definition we stated that the double quasioperator $\odot$ preserves both meets and joins in both coordinates and that the double quasioperator α interchanges meets and joins. Furthermore, on any $\odot - \alpha$ algbera it is possible to define another operation $\oplus$ in the following way:

$$a \oplus b = \alpha(\alpha(a) \odot \alpha(b))$$

for every $a, b \in A$.

Proposition 1.8. *Let $(A, \vee, \wedge, \odot, \alpha, 0, 1)$ be a $\odot - \alpha$ algebra. The following are equivalent:*

1. *$(A, \oplus, \alpha, 0)$ is an MV-algebra.*
2. *For all $a, b \in A$,*

$$a \odot 1 = 1 \odot a = a; \tag{1.1}$$

$$\alpha(\alpha(a)) = a; \tag{1.2}$$

$$a \vee b = a \oplus (\alpha(a) \odot b). \tag{1.3}$$

Proof. We shall proof that $\odot - \alpha$ algebras for which 1.1, 1.2 and 1.3 hold are term equivalent to coupled semirings and the proposition shall follow from Theorem 1.2.

In particular we shall prove that, if $(A, \vee, \wedge, \odot, \alpha, 0, 1)$ is a $\odot - \alpha$ algebra, then

$$((A, \vee, \odot, 0, 1), (A, \wedge, \oplus, \alpha(0), \alpha(1)), \alpha)$$

is a coupled semiring.

Since $a \odot 1 = 1 \odot a = 1$, we have that $a \odot b = a \odot (b \wedge 1) = (a \odot b) \wedge (a \odot 1) = (a \odot b) \wedge a$ which implies $a \odot b \leqslant a$; in the same way we obtain that $a \odot b \leqslant b$ and then $a \odot b \leqslant a \wedge b$ proving that $(A, \vee, \odot, 0, 1)$ is an *lc-semiring.* Since α interchanges meets and joins we have that $x \leqslant y \iff \alpha(x) \geqslant \alpha(y)$ and we easily obtain that $(A, \wedge, \oplus, 1, 0)$ is a *dual lc-semiring.* It is easy to see that α is a semiring isomorphism between $(A, \vee, \odot, 0, 1)$ and $(A, \wedge, \oplus, \alpha(0), \alpha(1))$, we shall only prove that α is injective. Suppose $\alpha(a) = \alpha(b)$ for some $a, b \in A$; then

$$\alpha(a \vee b) = \alpha(a) \wedge \alpha(b) = \alpha(a)$$

so $\alpha(a \vee b) = \alpha(a)$ which implies $a \vee b = a$. In the same way we can obtain $a \vee b = b$ and consequently $a = b$. To prove that $\alpha(0)$ is equal to 1, we observe that $\alpha(0) = \alpha(0 \wedge a) = \alpha(0) \wedge \alpha(a)$ and so $\alpha(0) \geqslant \alpha(a)$ for every $a \in A$ so $\alpha(0)$ is the top element of A, hence 1 and so $\alpha(1) = \alpha(\alpha(0)) = 0$.

Viceversa, given a coupled semiring $((A, \vee, \cdot, 0, 1), (A, \wedge, \cdot', 0', 1'), \alpha)$, we have that $(A, \vee, \wedge, \cdot, \alpha, 0, 1)$ is a $\odot - \alpha$ algebra. First of all, by proposition 1.6 we have that $(A, \vee, \wedge, 0, 1)$ is a bounded distributive lattice; indeed the join and meet coincide with the lattice operations on the corresponding MV-algebra and this implies the distributivity of A. Since $(A, \vee, \cdot, 0, 1)$ is an *lc-semiring*, we have that $a = a \cdot 1 \leqslant a \wedge 1$ which implies $a \leqslant 1$ for any $a \in A$, hence 1 is the top element of the lattice. The fact that $\cdot$ preserves joins in both coordinates come from the definition of semiring. We shall prove that

$$(a \wedge b) \cdot c = (a \cdot c) \wedge (b \cdot c)$$

First of all, since α is a semiring ismorphism, we have that $\alpha(a \cdot b) = \alpha(a) \cdot' \alpha(b)$ which implies $a \cdot b = \alpha(\alpha(a) \cdot' \alpha(b))$.

Recalling that α is a semiring isomorphism and that $\cdot'$ distributes over join on both sides, we have:

$$(a \wedge b) \cdot c = \alpha(\alpha(a \wedge b) \cdot' \alpha(c)) = \alpha[(\alpha(a) \vee \alpha(b)) \cdot' \alpha(c)] =$$

$$\alpha[(\alpha(a) \cdot' \alpha(c)) \vee (\alpha(b) \cdot' \alpha(c))] = \alpha[\alpha(a \cdot c) \vee \alpha(b \cdot c)] = (a \cdot c) \wedge (b \cdot c)$$

The proof of $a \cdot (b \wedge c) = (a \cdot b) \wedge (a \cdot c)$ is similar.

1.1 and 1.2 are obviously satisfied by the coupled semiring. In order to prove that 1.3 holds it is sufficient to observe that, since α is an idempotent semiring isomorphism $a \cdot' b = \alpha(\alpha(a) \cdot \alpha(b))$ and the proof is complete. □

1.4 Congruences and Ideals in MV-semirings

The notion of congruence comes from universal algebra. For the sake of convenience we shall write the definition for the case of MV-semirings.

Definition 1.18 (Congruence on MV-semirings)**.** Given an MV-semiring $(A, \vee, \cdot, 0, 1, ^*)$ we define a *congruence* $\equiv$ on it as an equivalence relation such that the following hold for every $a, b, a_1, a_2, \in A$:

1. If $a \equiv b$ and $a_1 \equiv b_1$ then $a \vee a_1 \equiv b \vee b_1$ and $a \cdot a_1 \equiv b \cdot b_1$;

2. If $a \equiv b$ then $a^* \equiv b^*$.

The notion of ideal instead doesn't come from universal algebra, so to define ideals in MV-semirings we shall use the notion of ideal of an arbitrary semiring.

Definition 1.19 (Left ideal)**.** Let $(S, +, \cdot, 0, 1)$ be a semiring. A *left ideal* I of the semiring S is a subset of S satysfing the following conditions:

1. If $a, b \in I$, then $a + b \in I$;

2. If $a \in I$ and $s \in S$, then $sa \in I$.

Definition 1.20. A *right ideal* of S is defined in the analogous manner and an *ideal* of S is a subset which is both a left ideal and a right ideal. An ideal I is *proper* if $I \neq S$.

Remark 1.4. Note that for commutative semirings we can simply talk about ideals since right and left ideals coincide.
Note that $0 \in I$ for every ideal I.

Note that, since the set $Id(S)$ of ideals of a semiring S is closed by intersection we can define in a standard way the ideal generated by a subset $M \subseteq S$, in particular

Example 1.1. Let S be a commutative semiring and x an element of S. Then the ideal generated by x is $Sx = \{sx \mid s \in S\}$.

Definition 1.21. The semiring ideals generated by a single element are called *principal ideals.*

We shall now specialize the definition of ideals to the case of MV-semirings.

Definition 1.22 (Ideals in MV-semirings)**.** Given an MV-semiring $(A, \vee, \cdot, 0, 1, ^*)$ we define an *ideal* of it as a subset I of A such that the following hold:

1. For every $i, j \in I$, $i \vee j \in I$;

2. $a \cdot i \in I$, for every $a \in A$ and $i \in I$;

Remark 1.5. Since MV-semirings are commutative we shall consider simply ideals instead of left and right ideals.

Definition 1.23 (Ideal-simplicity)**.** An MV-semiring $(A, \vee, \cdot, 0, 1, ^*)$ is *ideal-simple* if its only proper ideal is $\{0\}$ (the trivial one).

Examples 1.7. *Note that the only ideal-simple MV-semiring is the commutative semiring* $(\{0, 1\}, \vee, \cdot, 0, 1, ^*)$ *that consists of only two elements, (in this semiring we have that* $1 \vee 1 = 1$, $1^* = 0$ *and* $0^* = 1$*). Otherwise, let* x *be an element of an MV-semiring such that* $0 < x < 1$*, then the downward closure of* x *is a proper, non trivial ideal of the MV-semiring.*

For completeness we also recall the definition of an ideal of an MV-algebra.

Definition 1.24 (Ideal of an MV-algebra)**.** An *ideal* of an MV-algebra A is a subset I of A satisfying the following conditions:

1. $0 \in I$;
2. if $x \in I$, $y \in A$ and $y \leqslant x$ then $y \in I$;
3. if $x \in I$ and $y \in I$ then $x \oplus y \in I$.

Definition 1.25 (Congruence-simplicity)**.** An MV-semiring $(A, \vee, \cdot, 0, 1, ^*)$ is *congruence-simple* if its only congruence are the diagonal one and the total one.

Examples 1.8. *The MV-semiring* $\boldsymbol{L}_3 = \left\{0, \frac{1}{2}, 1\right\}$ *is congruence simple. Let denote with* $\triangle_{\boldsymbol{L}_3}$ *and* $\nabla_{\boldsymbol{L}_3}$ *respectively the diagonal and total congruence on* $\boldsymbol{L}_3$ *and suppose that* θ *is a congruence different from both of them.*
We recall that the diagonal congruence is given by the set of ordered pairs of identical elements and the total congruence is given by the set of all possible ordered pairs, i.e. $\triangle_{\boldsymbol{L}_3} = \{(a, a) \mid a \in \boldsymbol{L}_3\}$ *and* $\nabla_{\boldsymbol{L}_3} = \{(a, b) \mid a, b \in \boldsymbol{L}_3\}$*. The diagonal congruence is contained in any other congruence and any congruence is contained in the total one.*
Now suppose that θ *is a congruence on* $\boldsymbol{L}_3$ *different both from the diagonal and the total. So, it contains* $\triangle_{\boldsymbol{L}_3}$ *and at least one other couple of different elements of the semiring but it doesn't contain all the ordered pairs. Suppose that* $\left(0, \frac{1}{2}\right) \in \theta$ *then* $\left(0^*, \frac{1}{2}^*\right) = \left(1, \frac{1}{2}\right) \in \theta$ *but this implies* $(0,1) \in \theta$ *and the congruence is total. The same reasoning applies starting from* $\left(1, \frac{1}{2}\right) \in \theta$*. Suppose now that* $(0, 1) \in \theta$*, then* $(0,1) \vee \left(\frac{1}{2}, \frac{1}{2}\right) = \left(0 \vee \frac{1}{2}, 1 \vee \frac{1}{2}\right) = \left(\frac{1}{2}, 1\right) \in \theta$ *and we can apply the previous argument.*

Proposition 1.9. *Let* $(A, \vee, \cdot, 0, 1, ^*)$ *be an MV-semiring. The natural order of an MV-semiring is the same as the order of the MV-algebra associated to it. In particular, we have that* $a \leqslant b$ *iff* $\exists z \in A$ *such that* $a = b \cdot z$*.*

Proof. Let $(A, \oplus, \odot, ^*, 0, 1)$ be an MV-algebra and $a, b \in A$. From [11, Lemma 1.1.2] we have that $a \leqslant b$ if and only if there is an element $z \in A$ such that $a \oplus z = b$ and from [11, Lemma 1.1.4] we have that $a \leqslant b \iff b^* \leqslant a^*$. So, $a \leqslant b \iff b^* \leqslant a^* \iff b^* \oplus z = a^*$ for some $z \in A$. Now $b^* \oplus z = a^* \iff b^* \oplus (z^*)^* = a^* \iff (b^* \oplus (z^*)^*)^* = a \iff b \odot z^* = a$ and the proposition is proved. □

Definition 1.26 (Principal ideals)**.** Let $(A, \vee, \cdot, 0, 1, ^*)$ be an MV-semiring and $a \in A$. A *principal ideal* of A is an ideal of the form $A^{\vee \cdot}a = \{x \in A | x \leqslant A\}$ (the principal ideal generated by the element a).

As regards additively idempotent semirings it is known that every ideal induces a congruence relation on the semiring. Indeed, let $(S, +, \cdot, 0, 1)$ be an additively idempotent commutative semiring and let I be one of its ideals. Then the relation $\equiv_I$ defined by $a \equiv_I b$ iff exists $i \in I$ such that $a+i = b+i$ is a congruence on S such that $I \subseteq [0]_{\equiv_I}$.

Instead, if we consider an MV-semiring A and an ideal I of A, the relation $\equiv_I$ defined above doesn't necessarily define a congruence. For example, let us consider the standard MV-semiring $([0,1], \vee, \cdot, 0, 1, ^*)$ and its ideal $I = [0, 1/2]$. We have that $0 \equiv_I 1/2$ since $0 \vee 1/2 = 1/2 \vee 0$ but $1 = 0^*$ is not in relation with $1/2 = 1/2^*$.

It is easy to see that a congruence on an MV-algebra A is also a congruence on the MV-semiring reduct $A^{\vee\cdot}$ and that a congruence on an MV-semiring is also a congruence on the MV-algebra associated to it. So there is a one-to-one correspondence between congruences on MV-algebras and congruences on their reduct MV-semirings but, contrarily to what happened in MV-algebras there is no bijection between ideals and congruences over MV-semirings. Indeed, the fact that $\mathbf{L}_3$ is congruence simple as an MV-semiring could be also obtained observing that it is ideal simple as an MV-algebra. For insights about ideals in MV-semirings and in MV-algebras we refer the reader to [22].

A semiring is usually defined simple when it is both congruence-simple and ideal-simple. We prefer to follow universal algebra and since congruences can be defined in a universal algebra approach we shall call simple those MV-semirings which are congruence simple. In this way, from the previous remark, we have a correspondence between simple MV-algebras and simple MV-semirings.

Chapter 2

The polynomial semiring $\mathbb{B}[x]$

2.1 The polynomial semiring $\mathbb{B}[x]$

In this section we shall investigate the semiring of polynomials in one variable over the over the boolean semifield $\{0, 1\}$ that we shall denote $\mathbb{B}[x]$. Since $\mathbb{B}$ is a subsemiring of the standard MV-semiring $([0, 1], \vee, \odot, 0, 1)$, it makes sense to evaluate the elements of $\mathbb{B}[x]$ on the standard MV-semiring; we shall denote the set of these evaluations with $\mathbb{B}_{[0,1]}[x]$. Starting from the representation of the free MV-algebra over one variable in terms of McNaughton functions we shall prove that the set $\mathbb{B}_{[0,1]}[x]$ can be described in terms of a suitable subset of McNaughton functions over one variable. In particular, we shall prove (Theorems 2.1 and 2.3) that it is a semiring that coincides with the semiring of non-decreasing convex McNaughton functions over $[0, 1]$ with pointwise operations $\vee$ and $\odot$ and that it is also isomorphic to the semiring $(\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$.

Analogously to the definition of polynomials over a ring, we can give the definition over polynomials over semirings. We shall focus our attention on polynomials over semiring in one variable.

Definition 2.1 (Polynomials over semirings)**.** Let $(S, +, \cdot, 0, 1)$ be a semiring, and x a variable.

The semiring of polynomials over S in the variable x is given by the set

$$\left\{ \sum_{i=0}^{n} s_i x^i \mid n \in \mathbb{N}, s_i \in S \quad \forall i \in \{0, 1, \dots, n\} \right\}$$

Evaluating the semiring of polynomials $\mathbb{B}[x]$ over the standard MV-semiring $([0, 1], \vee, \odot, 0, 1)$ we obtain the set of functions from $[0, 1]$ to $[0, 1]$ of the form:

$$\bigvee_{i \in I} x^i$$

where
$x^i = \underbrace{x \odot \cdots \odot x}_{i \quad times}$ and I is a finite subset of the natural numbers plus the constant functions 0 and 1.

Recalling that on the MV-semiring $[0,1]$ we have that $x \odot x \leqslant x$, the set of evaluations $\mathbb{B}_{[0,1]}[x]$ amounts to the set of functions from $[0,1]$ to $[0,1]$ of the form $x^{\min I}$ for some finite subset of the natural numbers I plus the constant functions 0 and 1 (we are implicitly identifying the functions that coincide on any value of $[0,1]$) .

So, $\mathbb{B}_{[0,1]}[x] = \{x^n \mid n \in \mathbb{N}\} \cup \{0,1\}$.

We can now show the first theorem of the chapter.

Theorem 2.1. *On the set $\mathbb{B}_{[0,1]}[x]$ it is possible to define a structure of additively idempotent and commutative semiring and it is isomorphic to the semiring $(\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$.*

Proof. Recall that the set $\mathbb{B}_{[0,1]}[x]$ is given by all the functions $x^n : [0,1] \to [0,1]$ for all $n \in \mathbb{N}$, $n \geqslant 1$, plus the constant function 0 and 1.

We define

$$x^n \vee x^m = x^{\min\{n,m\}}$$

and

$$x^n \odot x^m = x^{n+m}.$$

for all $n, m \in \mathbb{N}_{\geqslant 1}$. The 0 and the 1 are given by the corresponding constant function and we have that $1 \vee x^n = 1$, for all natural $n \geqslant 1$.

To prove that the semiring $(\mathbb{B}_{[0,1]}[x], \vee, \odot, 0, 1)$ is isomorphic to $(\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$ it is sufficient to consider the semiring isomorphism

$$\Phi : \mathbb{B}_{[0,1]}[x] \to (\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$$

given by

$$\Phi(x^n) = n$$

for any $n \in \mathbb{N}$, $n \leqslant 1$ and

$$\Phi(0) = \infty, \quad \Phi(1) = 0.$$

□

Definition 2.2. Let $n \geqslant 1$ be an integer. Then a function

$$f : [0,1]^n \to [0,1]$$

is called a *McNaughton function* over $[0,1]^n$ iff it satisfies the following conditions:

1. f is continuous with respect to the natural topology of $[0,1]^n$;

2. there are linear polynomials $p_1, \dots, p_k$ with integer coefficients,

$$p_i(x_0, \dots, x_{n-1}) = b_i + m_{i0}x_0 + \cdots + m_{i(n-1)}x_{n-1},$$

   ($b_i, m_{it} \in \mathbb{Z}$) such that for each point $\mathbf{y} = (y_0, \dots, y_{n-1}) \in [0,1]^n$ there is an index $j \in \{1, \dots, k\}$ with $f(\mathbf{y}) = p_j(\mathbf{y})$.

We shall call the linear polynomials associated to a function f the "linear components" of the function.

Theorem 2.2. *(Theorem 9.1.5 [11]) Let n be an integer. The free MV-algebra over n generators $Free_n$ coincides with the MV-algebra of McNaughton functions over $[0,1]^n$ with the pointwise operations $\vee$ and $\odot$.*

Since any function $x^n : [0,1] \to [0,1]$ and the constant functions 0 and 1 are elements of the free MV-algebra over one variable, they are McNaughton functions. So, it is clear that the set $\mathbb{B}_{[0,1]}[x]$ is a subset of the set of McNaughton functions over one variable, in particular we can wonder if the semiring $\mathbb{B}_{[0,1]}[x]$ can be fully described by a suitable subset of McNaughton functions over one variable.

It is straightforward that the functions x^n for any $n \in \mathbb{N}_{\geqslant 1}$ and the constant functions 0 and 1 are all convex non-decreasing McNaughton functions of one variable so the semiring $\mathbb{B}_{[0,1]}[x]$ is contained in the set of convex non-decreasing McNaughton functions of one variable. We can now wonder if it the reverse inclusion is true.

We shall answer the question positively by describing all the convex non-decreasing McNaughton functions of one variable in terms of their linear components.

Lemma 2.1. *If a non-decreasing convex McNaughton function has only one linear component it is equal to* 0, 1 *or* x.

Proof. Let f be a non-decreasing convex McNaughton function from $[0,1]$ to $[0,1]$ such that $f(x) = ax + b$ for some $a, b \in \mathbb{Z}$ and for all $x \in [0,1]$. First of all observe that $a \geqslant 0$, since the function is non-decreasing. If $a = 0$ the only two possibilities are $f = 0$ or $f = 1$. Let us consider the case $a > 0$;

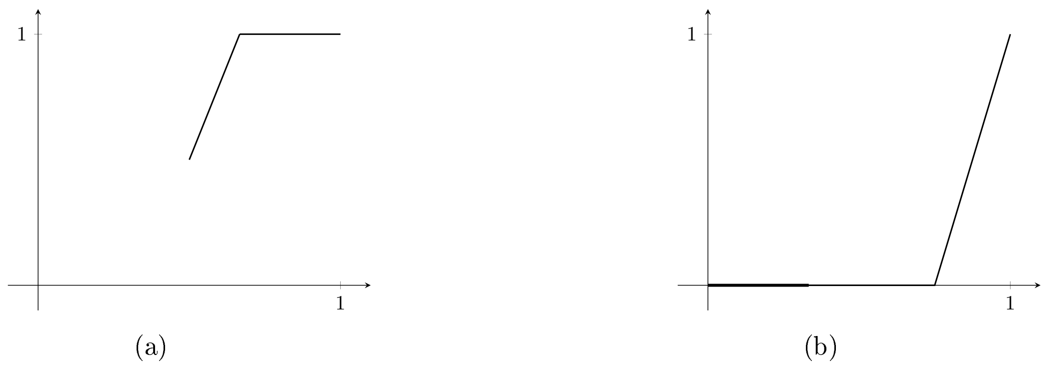


Figure 2.1

since a and b are integers we have that the value $f(1)$ must be an integer number and, since the function is not constant and non-decreasing, the only possibility is $f(1) = 1$ which implies $b = 1 - a$ anf $f(x) = ax + (1 - a)$ for some $a \in \mathbb{N}$. Suppose $a \geqslant 2$ then $f(0) = 1 - a < 0$ which is absurd. So, if $a > 0$, then $a = 1$ and $f(x) = x$. □

Lemma 2.2. *Let f be a non-decreasing convex McNaughton function with two linear components; then, it is equal to* $\max\{0, nx + 1 - n\}_{n\in\mathbb{N}, n\geqslant 1}$ *for some $n \in \mathbb{N}$, $n > 0$.*

Proof. By hypothesis we have that

$$f(x) = \begin{cases} ax + b \text{ } \textit{if } x \in [0, z] \\ cx + d \text{ } \textit{if } x \in [z, 1] \end{cases}$$

for some $a \in \mathbb{N}$ (every component must have a positive slope since the function is non-decreasing), $b \in \mathbb{Z}$ and for some $z \in (0, 1)$.

Claim 2.1. None of the linear components can be the constant 1.

If $ax + b = 1$ for every $x \in [0, z]$, then $cx + d = 1$ for every $x \in [z, 1]$, since the function is non-decreasing; but in this case we would have that f has only one linear component ($f = 1$). If $cx + d = 1$ for every $x \in [z, 1]$, we would have a configuration similar to the one represented in figure 2.1a, and we can see that in this case the function is not convex.

Claim 2.2. It is not possible that both a and c are strictly greater than 0.

Suppose that a and c are strictly greater than 0; then, we have to consider the two possibilities $a < c$ or $a > c$. If $a > c$; we would have an arrangement similar to the one in figure 2.2a, but in this case the function is not convex. Suppose now that $a < c$ (see figure 2.2b). As we observed before, $a1 + b$ is an integer number (a and b are integers) and it is $\geqslant 1$ ($ax + b$ is increasing since $a > 0$ by hypothesis). Since $cx + d$ is strictly greater than $ax + b$ for any $x \in (z, 1]$ we have that $cz_1 + d = 1$ for some $z_1 \in (z, 1)$ but in this case

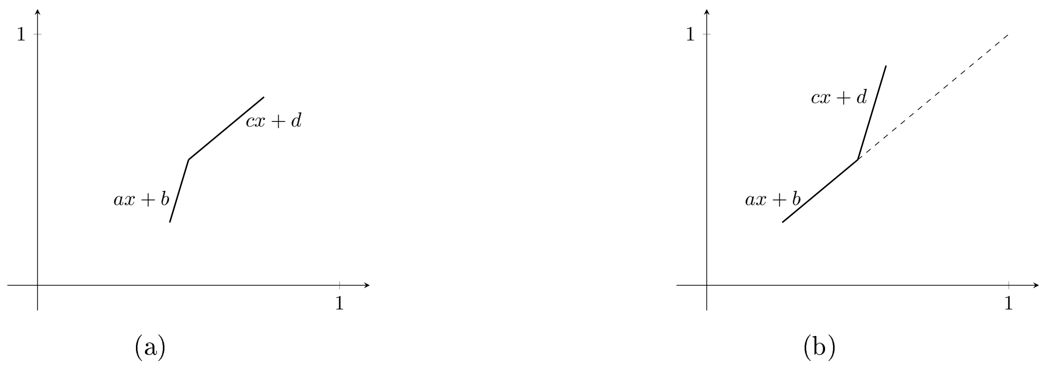


Figure 2.2

we would have that $f(x) = 1$ for every $x \in [z_1, 1]$ i. e. f has the constant 1 as the third linear component which is absurd.

So, it can be concluded that if a convex non-decreasing McNaughton functions has two linear components, one of them is the constant 0 and the other one is equal to $ax + b$ for some $a \in \mathbb{N}_{>0}$ and $b \in \mathbb{Z}$. In particular we have that $b = 1 - a$ (recall that $f(1) = 1$) and the point in which the two linear components join is $\frac{a-1}{a}$; so

$$f(x) = \max\{0, ax + 1 - a\}_{a \in \mathbb{N}, a \geqslant 1}$$

□

Lemma 2.3. *A non-decreasing convex McNaughton function cannot have more than two linear components.*

Proof. Let be $\{f_1, \ldots, f_k\}$ with $k > 2$ the k linear components of a non-decreasing convex McNaughton function f. In the light of Lemma 2.1, it is easy to see that none of the linear components can be equal to the constant 1. Furthermore, since the function is non-decreasing, there can be at most one linear component equal to the constant function 0 and it would be f_1. A reasoning similar to the one of Lemma 2.2 leads to the conclusion that there cannot be two consecutive linear components f_i and f_{i+1} with strictly positive slopes. □

Summarizing, we have

Proposition 2.1. *Every non-decreasing convex McNaughton functions has either one linear component (and in this case it is equal to* 0, 1 *or* x*) or two linear components and in this case it is equal to* $\max\{0, nx+1-n\}$ *for some* $n \in \mathbb{N}$, $n \geqslant 2$*. So the set non-decreasing convex McNaughton functions is* $\{0, 1\} \cup \{\max\{0, nx + 1 - n\}\}_{n \in \mathbb{N}, n \geqslant 1}$.

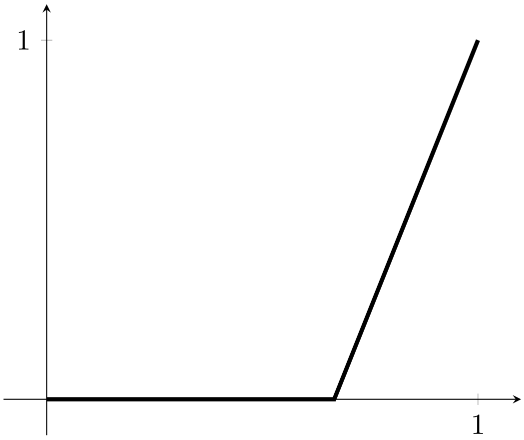


Figure 2.3

See figure 2.3 for a representation of the non-decreasing convex McNaughton functions of one variable corresponding to $n = 3$.

We can now present the main result of the chapter.

Theorem 2.3. *The set of non-decreasing convex McNaughton functions over* $[0, 1]$ *with the pointwise operations* $\vee$ *and* $\odot$ *of the standard MV-semiring and the constant functions* 0 *and* 1 *is an additively idempotent and commutative semiring that coincides with* $(\mathbb{B}_{[0,1]}[x], \vee, \odot, 0, 1)$.

Remark 2.1. Observe that we denote with the same symbols $\vee$ and $\odot$ different operations.

As regards McNaughton functions $\vee$ and $\odot$ denote the pointwise operations of the standard MV-semring, i. e.

$$(f \vee g)(x) = f(x) \vee g(x) = \max\{f(x), g(x)\}$$

and

$$(f \odot g)(x) = f(x) \odot g(x) = \max\{0, f(x) + g(x) - 1\}$$

whereas for the semiring $\mathbb{B}_{[0,1]}[x]$ the operations are the ones defined in Theorem 2.1. Even if in the context of Theorem 2.3 we shall prove that these operations coincide this remark is necessary in order to avoid confusion.

Proof. In order to prove that the above semiring coincide with $\mathbb{B}_{[0,1]}[x]$ we have to prove (by induction) that

$$x^n = \max\{0, nx + 1 - n\}$$

for all the integers $n \geqslant 1$ and for all $x \in [0, 1]$.

We have that $x = \max(0, x)$ so the basic step is proved.

Assume $x^n = \max\{0, nx + 1 - n\}$ and we shall prove that $x^{n+1} = \max\{0, (n+1)x + 1 - n - 1\} = \max\{0, nx + x - n\}$.

$$x^{n+1} = x^n \odot x = \max\{0, nx+1-n\} \odot x = \max\{0, \max\{0, nx+1-n\}+x-1\}$$

Suppose $\max\{0, nx + x - n\} = nx + x - n$, so $0 \leqslant nx + x - n$, then $0 \leqslant nx+1-n$ $(x \leqslant 1)$ and $\max\{0, \max\{0, nx+1-n\}+x-1\} = \max\{0, nx+1-n+x-1\} = \max\{0, nx - n + x\} = nx + x - n$.

Now suppose $\max\{0, nx + x - n\} = 0$, so $nx + x - n \leqslant 0$. We have now two cases:

1. $nx+1-n \leqslant 0$, then $\max\{0, \max\{0, nx+1-n\}+x-1\} = \max\{0, x-1\} = 0$;

2. $nx+1-n \geqslant 0$, then $\max\{0, \max\{0, nx+1-n\}+x-1\} = \max\{0, nx+1-n+x-1\} = \max\{0, nx - n + x\} = 0$.

In both cases we have that $\max\{0, nx + x - n\} = \max\{0, \max\{0, nx + 1 - n\} + x - 1\}$.

We have now to prove that the McNaughton functions operations and the operations of $\mathbb{B}_{[0,1]}[x]$ coincide.

Let m, n be two natural numbers different from 0 and x be an arbitrary point of $[0, 1]$; then

$$\max(\max(0, nx + 1 - n), \max(0, mx + 1 - m)) =$$

$$= \max(0, \min\{m, n\}x + 1 - \min\{m, n\}) = x^{\min\{n,m\}} = x^n \vee x^m$$

and

$$\max(0, \max(0, nx + 1 - n) + \max(0, mx + 1 - m) - 1) =$$

$$= \begin{cases} 0 \ \textit{if} \ x \leqslant \frac{m+n-1}{m+n} \\ (m+n)x + 1 - (m+n) \ \textit{if} \ x > \frac{m+n-1}{m+n} \end{cases} =$$

$$= \max(0, (m+n)x + 1 - (m+n)) = x^{m+n} = x^m \odot x^n$$

□

This result leads to a new interpretation of theorem 2.2 in terms of slopes. Indeed, we can think of the function Φ as the function which sends every function to the slope of its non-constant linear component.

2.2 The polynomial semiring $\mathbb{B}'[x]$

We have a mirror situation considering the semiring of polynomials in the variable x over the boolean semifield $\{0,\ 1\}$ evaluated on the MV-semiring $([0,1],\wedge,\oplus,0,1)$ and I shall denote such semiring of evaluations as $\mathbb{B}'_{[0,1]}[x]$.

The elements of $\mathbb{B}'_{[0,1]}[x]$ will be functions from $[0,1]$ to $[0,1]$ of the following form:

$$\bigwedge_{i\in I} ix$$

where
$ix = \underbrace{x\oplus\cdots\oplus x}_{i \quad times}$ and I is a finite subset of the natural numbers.

Recalling that in the standard MV-semiring $([0,1],\wedge,\oplus,1,0)$ $x\leqslant x\oplus x$ the set of evaluations $\mathbb{B}'_{[0,1]}[x]$ amounts to the set of functions from $[0,1]$ to $[0,1]$ of the form $(\min I)x$ for some finite subset of the natural numbers I plus the constant functions 0 and 1 (we are again implicitly identifying the functions that coincide on any value of $[0,1]$) .

So, $\mathbb{B}'_{[0,1]}[x] = \{nx \mid n\in\mathbb{N}\}\cup\{0,1\}$.

It is easy to see that the map

$$*:\mathbb{B}_{[0,1]}[x]\to\mathbb{B}'_{[0,1]}[x]$$

that sends x^n to nx, 0 to 1 and 1 to 0 is a semiring isomorphism.
From this it follows that

- the semirings $\mathbb{B}'_{[0,1]}[x]$ and $(\mathbb{N}\cup\{\infty\},\min,+,\infty,0)$ are isomorphic;
- the semiring $\mathbb{B}'_{[0,1]}[x]$ coincides with the semiring of non-decreasing concave McNaughton functions over $[0,1]$ with the pointwise operations $\wedge$ and $\oplus$ of the MV-semiring $[0,1]^{\wedge\oplus}$;

indeed the isomorphism $*$ sends non-decreasing convex to non-decreasing concave McNaughton functions.

See 2.4 for a representation of the non-decreasing concave McNaughton function $3x$.

We shall conclude this chapter leaving some open questions:

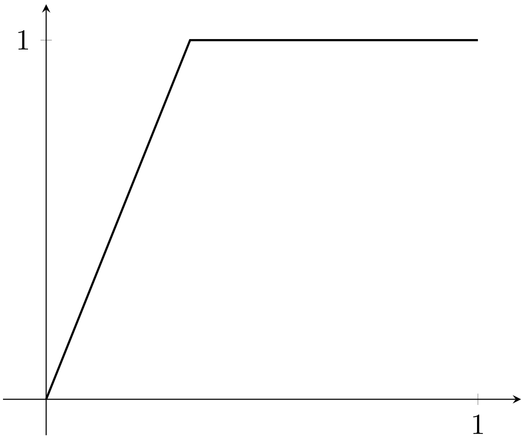


Figure 2.4

Questions Let n be an integer greater than 1. Is it true that the semiring of the evaluations of polynomials in the variable x over the boolean semifield $\{0,\ 1\}$ on the standard MV-semiring $([0,1]^n, \vee, \odot, 0, 1)$ coincides with the semiring of non-decreasing convex McNaughton functions over $[0,1]^n$ with the pointwise operations $\vee$ and $\odot$?

Does then exist a semiring isomorphism between the semiring of non-decreasing convex McNaughton functions over $[0,1]^n$ and a numerical semiring similar to the one we have for the one variable case?

As regards the first, we already have one inclusion indeed all the functions of the form

$$\bigvee_{i=0}^{m} x_1^{j_{1i}} \ldots x_n^{j_{ni}}$$

where m and j_{ki} are integers for any $k \in \{1, \ldots, n\}$ and $i \in \{0, \ldots, m\}$

are non-decreasing convex McNaughton functions over $[0,1]^n$ (the operation $\vee$ and $\odot$ are both non-decreasing and the sup of a collection of convex functions is convex) but whether the reverse inclusion holds or not is a question still open.

Chapter 3

Involutive semirings

In this chapter, we shall present a categorical isomorphism between involutive residuated lattices and a special class of semirings called *involutive semirings*. That isomorphim can be seen as a generalization of the one between MV-algebras and MV-semirings since the variety of involutive residuated lattices strictly contains the one of MV-algebras. Since arbitrary involutive residuated lattices are not required to be bounded we shall first consider zero-free semirings. The results contained in this chapter are part of a joint work with Peter Jipsen ([36]). For all the definitions and results about involutive residuated lattices we refer to [25].

3.1 Basic notions

Definition 3.1. A *0-free semiring* is an algebra $(A, +, \cdot, 1)$ of type (2, 2, 0) such that

1. $(A, +)$ is a commutative semigroup,
2. $(A, \cdot, 1)$ is a monoid and
3. $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ for all $a, b, c \in A$

Example 3.1. We can obtain zero-free semirings from the tropical semifields by dropping $-\infty$ or $+\infty$ respectively, for example
$(\mathbb{R}, \max, +, 0)$ is a zero-free semiring that shall play an important role in the last chapter of the thesis.

Definition 3.2. An *involutive semiring* is an algebra $(A, \vee, \cdot, 1, \sim, -)$ of type (2, 2, 0, 1, 1) such that

1. $(A, \vee, \cdot, 1)$ is a 0-free additively idempotent semiring and
2. $x \leqslant y \iff x \cdot {\sim} y \leqslant -1 \iff -y \cdot x \leqslant -1$ for all $x, y \in A$.

Definition 3.3 (Join-semilattice)**.** A *join-semilattice* is an algebraic structure $(M, \vee)$ consisting of a set M with a binary operation $\vee$ such that the following conditions hold for all $x, y, z \in M$:

1. $x \vee (y \vee z) = (x \vee y) \vee z$;
2. $x \vee y = y \vee x$;
3. $x \vee x = x$.

If the symbol $\wedge$ replaces $\vee$ in the definition just given, the structure is called a *meet-semilattice.*

Definition 3.4. Let $(M, \vee)$ be a join-semilattice, M has a *bottom element* 0 if $x \vee 0 = x$ for all $x \in M$.

Observe that any additively idempotent semiring has an underlying structure of join-semilattice. The element -1 is denoted by 0, although it need not be the bottom element of the join-semilattice.

Definition 3.5 (Involutive semirings morphisms)**.** Let A and B be involutive semirings. A *semiring morphism* between A and B is a map $f : A \to B$ such that:

1. $f(x \vee y) = f(x) \vee f(y)$;
2. $f(x \cdot y) = f(x) \cdot f(y)$;
3. $f(1) = 1$;
4. $f({\sim}x) = {\sim}f(x)$;
5. $f(-x) = -f(x)$.

for all $x, y \in A$.

Definition 3.6 (Residuated lattice)**.** A *residuated lattice* is an algebra $(A, \vee, \wedge, \cdot, 1, \backslash, /)$ of type (2, 2, 0, 2, 2) such that

1. $(A, \vee, \wedge)$ is a lattice,
2. $(A, \cdot, 1)$ is a monoid and
3. (res) $\quad xy \leqslant z \iff x \leqslant z/y \iff y \leqslant x\backslash z \quad$ holds for all $x, y, z \in A$.

Definition 3.7. A *pointed residuated lattice* $(A, \vee, \wedge, \cdot, 1, \backslash, /, 0)$ is a residuated lattice with an additional constant 0. Note that this constant needs not be the least element of the lattice.

Definition 3.8. On a pointed residuated lattice we can define two order-reversing operations $\sim, -$ as ${\sim}x = x\backslash 0$ and $-x = 0/x$. They are called respectively *left* and *right linear negation.*

Definition 3.9 (Involutive residuated lattice)**.** An *involutive residuated lattice* is a pointed residuated lattice that satisfies $\sim\!-x = x = -\!\sim\! x$ for all $x \in A$.

Definition 3.10 (Involutive residuated lattices morphisms)**.** Let A and B be involutive residuated lattices. A *involutive residuated lattices morphism* between A and B is a map $f : A \to B$ such that:

1. $f(x \vee y) = f(x) \vee f(y)$;
2. $f(x \wedge y) = f(x) \wedge f(y)$;
3. $f(x \cdot y) = f(x) \cdot f(y)$;
4. $f(x \backslash y) = f(x) \backslash f(y)$;
5. $f(x/y) = f(x)/f(y)$;
6. $f(1) = 1$;
7. $f(0) = 0$.

for all $x, y \in A$.

Remark 3.1. If $\cdot$ is commutative then $x\backslash y = y/x$, hence $\sim x = -x$.

Examples 3.1. *1. The variety of involutive residuated lattices strictly contains the variety of MV-algebras so, every MV-algebra is an example of involutive residuated lattice. In particular, an* MV-algebra *is a* 1-*bounded (i. e.* 1 *is the top element), commutative involutive residuated lattice that satisfies* $x \vee y = (x/y)\backslash x$.

2. The chain in the picture is an example of an involutive residuated lattice which is not an MV-algebra. Indeed there are only two examples of 4-element MV-algebras: the chain $\boldsymbol{L}_4$ *and the product of chains* $\boldsymbol{L}_2 \times \boldsymbol{L}_2$*. The chain in the picture can't be* $\boldsymbol{L}_4$ *because the only multiplicatively idempotent elements of* $\boldsymbol{L}_4$ *are* 0 *and* 1.

$$\begin{array}{cl} 1 & \\ | & \\ b & \quad b^2 = b \\ | & \\ a & \quad a^2 = 0 \\ | & \\ 0 & \end{array}$$

Remark 3.2. Note that residuated lattices which has the multiplicative unit 1 as the greatest element are usually called *integral*. However, since the term "integral" has a different meaning in semiring theory, we preferred to substitute "integral" with "1-bounded" in order to avoid confusion.

The residuation equivalences (res) can be replaced by four identities, hence involutive residuated lattices form a variety, denoted by InRL.

It is well known (see [25], Lemma 3.17 and Lemma 3.18) that $\backslash, /, 0$ can be expressed by the linear negations and the monoid operation:

$$x\backslash y = \sim((-y)x), \quad x/y = -(y(\sim x)) \text{ and } 0 = \sim 1 = -1.$$

Consequently, we have that an involutive residuated lattice

$$(A, \vee, \wedge, \cdot, 1, \backslash, /, 0)$$

can be equivalently represented trough the signature $(A, \vee, \wedge, \cdot, 1, \sim, -)$ (we shall use this signature from now on to denote involutive residuated lattices). It is straightforward that involutive residuated lattices morphisms remain the same.

Since residuation also implies that $\cdot$ distributes over $\vee$ (see [25, Lemma 2.6]), hence $(A, \vee, \cdot, 1)$ is a 0-free idempotent semiring, i. e. the 0-free semiring reduct of the involutive residuated lattice.

3.2 Involutive residuated lattices and semirings

Below we show that the main result of the chapter, i. e. the term equivalence between involutive residuated lattices and involutive semirings. The general term-equivalence was first shown between generalized coupled semirings and involutive residuated lattices in [35].

Lemma 3.1. *Let $(A, \vee, \cdot, 1, \sim, -)$ be an involutive semiring, then $\sim - x = x = - \sim x$.*

Proof.

Claim 3.1. $\sim - x \leqslant x$

Proof. To prove the inequality $\sim - x \leqslant x$, note that $-x \leqslant y \iff -x \cdot \sim y \leqslant 0 \iff \sim y \leqslant x$, these equivalences are obtained applying property (2) of the definition 3.2 and recalling that $0 = -1$.

We can substitute y by $-x$ in the first and last terms of the previous equivalence and we get $-x \leqslant -x \iff \sim - x \leqslant x$, so we obtain $\sim - x \leqslant x$. □

Claim 3.2. $x \leqslant \sim - x$

Proof. Substituting x by $\sim - x$ in the inequality obtained in claim 3.1 we get $\sim - \sim - x \leqslant \sim - x \leqslant x$. This is equivalent to $-x \leqslant - \sim - x$ (recall that $-x \leqslant y \iff \sim y \leqslant x$).

Similarly, in $-x \leqslant y \iff {\sim}y \leqslant x$ we can now substitute x by ${\sim}y$ obtaining $-{\sim}y \leqslant y$, and replacing y by $-x$ we have $-{\sim}{-x} \leqslant -x$, hence the identity $-x = -{\sim}{-x}$ holds.

Now $x \leqslant x$ implies $-x \cdot x \leqslant 0$, hence $-0 \cdot (-x \cdot x) \leqslant 0$ (applying property (2) of the definition 3.2 and recalling that $0 = -1$). From the identity $-x = -{\sim}{-x}$ it follows that $-{\sim}{-0} \cdot (-{\sim}{-x} \cdot x) \leqslant 0$, which implies $-{\sim}{-x} \cdot x \leqslant {\sim}{-0} \leqslant 0$ (by property (2) of the definition 3.2) and therefore $x \leqslant {\sim}{-x}$ (also, by property (2)). So we have shown that the identity ${\sim}{-x} = x$ holds, and $-{\sim}x = x$ is proved similarly. □

□

Theorem 3.1. *Involutive residuated lattices are term-equivalent to involutive semirings.*

Proof. Let $(A, \vee, \wedge, \cdot, 1, \sim, -)$ be an involutive residuated lattice, we want to prove that its 0 - free semiring reduct $(A, \vee, \cdot, 1, \sim, -)$ is an involutive semiring.

Claim 3.3. $(A, \vee, \cdot, 1, \sim, -)$ is an involutive semiring

Proof. In particular we have to prove the condition (2) of the definition 3.2. We shall now prove $x \leqslant y \iff x \cdot {\sim}y \leqslant -1$. Since A is an involutive residuated lattice we know that $y = -{\sim}y$, so $x \leqslant y$ is equivalent to $x \leqslant -{\sim}y$. By the definition 3.8 we have that $-{\sim}y = 0/{\sim}y$, so $x \leqslant y \iff x \leqslant -{\sim}y \iff x \leqslant 0/{\sim}y$. The last is equivalent to $x \cdot {\sim}y \leqslant 0$ by (res) but since $0 = -1$ this is equivalent to $x \cdot {\sim}y \leqslant -1$. Through this chain of equivalences we obtained that $x \leqslant y \iff x \cdot {\sim}y \leqslant -1$. The equivalence $x \leqslant y \iff -y \cdot x \leqslant -1$ is proved similarly, showing that any involutive residuated lattice is an involutive semiring. □

Conversely, let $(A, \vee, \cdot, 1, \sim, -)$ be an involutive semiring. To show that this is an involutive residuated lattice we have to define $\wedge$ in a suitable way. So, we define $x \wedge y = {\sim}(-x \vee -y)$. We have to prove that with this definition of $\wedge$ $(A, \vee, \wedge)$ is actually a lattice and that (res) holds.

Claim 3.4. $\sim$ and $-$ are order-reversing

Proof. Next, observe that $x \leqslant y \iff -y \cdot x \leqslant 0 \iff -y \cdot \sim - x \leqslant 0 \iff -y \leqslant -x$ (applying property (2) and Lemma 6.1). A similar calculation for $\sim$ shows that the unary operations in an involutive semiring are order-reversing inverses of each other.

□

Since $(A, \vee)$ is a join-semilattice, the operation $\wedge$ that we defined inherits from $\vee$ the property of commutativity, associativity and idempotence. To show that $(A, \vee, \wedge)$ is a lattice we only need to prove the absorption laws.

Claim 3.5. Absorption laws

Proof. We now prove the absorption laws: $x \wedge (x \vee y) = \sim(-x \vee -(x \vee y))$ (by the definition of $\wedge$). We have that $x \leqslant x \vee y$ and, since $-$ is order-reversing we will have $-(x \vee y) \leqslant -x$, which implies $-x \vee -(x \vee y) = -x$. So, $x \wedge (x \vee y) = \sim(-x \vee -(x \vee y)) = \sim -x = x$. Similarly $x \vee (x \wedge y) = x \vee \sim(-x \vee -y) = x$ since $-x \leqslant -x \vee -y$ implies $\sim(-x \vee -y) \leqslant x$. Hence $(A, \vee, \wedge)$ is a lattice.

□

Claim 3.6. Proof of (res)

Proof. Finally we prove that if the residuals are defined as $x\backslash z = \sim(-z \cdot x)$ and $z/y = -(y \cdot \sim z)$ then (res) holds:

$$y \leqslant \sim(-z \cdot x) \Leftrightarrow -z \cdot x \leqslant -y \Leftrightarrow$$

(applying $-$ to both sides and recalling that $-$ is order-reversing)

$$\Leftrightarrow -z \cdot x \cdot y \leqslant -1 \Leftrightarrow$$

(applying (2) and the identity $\sim - y = y$)

$$\Leftrightarrow -z \leqslant -(x \cdot y) \Leftrightarrow$$

(writing $x \cdot y$ as $\sim - (x \cdot y)$ and then applying (2))

$$\Leftrightarrow xy \leqslant z.$$

($-$ is order-reversing). The second equivalence of (res) is proved similarly, hence any involutive semiring determines an involutive residuated lattice. □

□

Remark 3.3. Similar to remark 1.4, we have that if A and B are two involutive residuated lattices and $h : A \to B$ is a map between them, then h is an involutive residuated lattices morphism if and only if it is an involutive semirings morphism.

The two previous propositions show that involutive residuated lattices and involutive semirings are two term-equivalent varieties. Since the morphisms of the two categories coincide it is straightforward to show that the two corresponding categories are isomorphic, so we have

Theorem 3.2. *Involutive residuated lattices and involutive semirings are isomorphic categories.*

In the next result we shall see how the semiring perspective may help us answer an open question about involutive residuated lattices. In particular we shall find a necessary and sufficient condition in order for the interval $[0,1] = \{a \mid 0 \leqslant a \leqslant 1\}$ to be a subalgebra of an involutive residuated lattice.

Theorem 3.3. *In any involutive semiring (equivalently involutive residuated lattice) the interval* $[0,1]$ *is a subalgebra if and only if* 0 *is a multiplicative idempotent element, i. e.,* $0 \cdot 0 = 0$.

Proof. ($\Rightarrow$) Assume $[0,1]$ is a subalgebra of an involutive semiring. Since any subalgebra of an involutive semiring can't be empty (it must contain at least the element 1), then $0 \leqslant 1$. We also have that $0 \leqslant 1 \iff 0 \cdot 0 \leqslant 0$ (applying (2) of def. 3.2 and recalling that $\sim 1 = -1 = 0$). Since 0 belongs to the subalgebra $[0,1]$, then $0 \cdot 0$ must belong to the subalgebra too (it is closed under the operations). So, $0 \cdot 0 \in [0,1]$ implies the reverse inequality $0 \leqslant 0 \cdot 0$, hence we obtain $0 \cdot 0 = 0$.
($\Leftarrow$) Conversely, assume $0 \cdot 0 = 0$. The equivalence $0 \cdot 0 \leqslant 0 \iff 0 \leqslant 1$ shows that the interval $[0,1]$ is not empty and that 0 and 1 belong to it. The interval is certainly closed under joins.

The closure under $\sim$ follows from

$$0 \leqslant a \leqslant 1 \quad \iff \quad 0 = \sim 1 \leqslant \sim a \leqslant \sim 0 = 1.$$

The argument for closure under $-$ is the same. As regards the closure under multiplication we have that if $a, b \in [0,1]$ then $ab \leqslant a, b$ (indeed $ab \vee a = a \cdot (b \vee 1) = a$) and in particular $ab \leqslant 1$. Observe that if $a \leqslant b$ and $c \leqslant d$, we have that $ac \leqslant bd$ (indeed $ac \vee ad = a(c \vee d) = ad$ and $ad \vee bd = (a \vee b)d = bd$, so $0 \leqslant a, b$ implies $0 \cdot 0 \leqslant ab$. Since we assume $0 \cdot 0 = 0$ we have that $0 \leqslant ab$, hence $a, b \in [0,1]$. ☐

In order to study semimodules over involutive semirings we have to restrict our attention to those involutive semirings for which 0 is the bottom element or equivalently the additive identity. So, 1 is the top and from (res) , it follows that $x0 = 0 = 0x$. For these reasons such semirings can be defined as follows:

Definition 3.11. An *1 - bounded involutive semiring* is an algebra $(A, \vee, \cdot, 0, \sim, -)$ of type (2, 2, 0, 0, 1, 1) such that

1. $(A, \vee, \cdot, 0, 1)$ is an additively idempotent semiring and
2. $x \leqslant y \iff x \cdot \sim y = 0 \iff -y \cdot x = 0.$

Note that in this case 1 can be defined as $\sim 0 = -0$. These semirings are bounded from below by 0 and from above by 1 and so shall be the corresponding involutive residuated lattices. Therefore, the previous isomorphisms still holds between 1-bounded involutive semirings and 1-bounded involutive residuated lattices. Semirimodules over 1-bounded involutive semirings shall be the main topic of Chapter 6.

The next diagram shows the relevant isomorphisms between special classes of semirings and the algebraic structures considered in this thesis; the vertical lines read upwards indicate the relation "being a subvariety of".

$$\begin{array}{ccc} InRL & \longleftrightarrow & InS \\ \Big| & & \Big| \\ 1-bInRL & \longleftrightarrow & 1-bInS \\ \Big| & & \Big| \\ MV & \longleftrightarrow & MVS \end{array}$$

$InRL$ = involutive residuated lattices;
$1-bInRL$ = 1-bounded involutive residuated lattices;
MV = MV-algebras;
InS = involutive semirings;
$1-bInS$ = 1-bounded involutive semirings;
MVS = MV-semirings.

Chapter 4

Injective and projective semimodules

The two categorical isomorphisms (Theorem 1.1 and the 1-bounded version of Theorem 3.1) presented in the previous chapters allow us to import some results and technique of semiring theory in the study of MV-algebras and involutive residuated lattices. One of the main topic in semiring theory is semimodule theory, in particular we focused on the characterization of injective and projective semimodules over the two aforementioned classes of semirings. First of all, we shall provide some definitions and results for injective and projective semimodules over additively idempotent semirings. For the definitions and basic results about semimodules we refer to [30], whereas for all the definitions regarding lattice theory we refer to [31].

4.1 Basic notions

Definition 4.1 (Semimodule)**.** Let S be a semiring. A *(left) S-semimodule* is a commutative monoid $(M, +, 0)$ with a scalar multiplication $\cdot : (a, x) \in S \times M \to a \cdot x \in M$, such that the following conditions hold for all $a, b \in S$ and $x, y \in M$:

1. $(ab) \cdot x = a \cdot (b \cdot x)$;
2. $a \cdot (x + y) = (a \cdot x) + (a \cdot y)$;
3. $(a + b) \cdot x = (a \cdot x) + (b \cdot x)$;
4. $0_S \cdot x = 0_M = a \cdot 0_M$;
5. $1 \cdot x = x$.

The definition and properties of right S-semimodules are completely analogous. From now on, we will refer generically to semimodules without specifying left or right and we will use the notations of left semimodules.

Example 4.1. Let $S = (S, +, \cdot, 0, 1)$ be a semiring.

1. Its additive monoid reduct $(S, +, 0)$ is a semimodule over S where the scalar multiplication is the semiring multiplication;

2. if n is an integer $(S^n, +, 0)$ is a semimodule over S with $+$ and the scalar multiplication defined coordinate-wise;

3. Let x be an element of S, then the principal ideal Sx is a semimodule over S.

Definition 4.2 (Subsemimodule)**.** Let S be a semiring and $M = (M, +, 0)$ an S-semimodule. An *S-subsemimodule* N of M is a subset of M for which the following conditions hold for all $a, b \in N$ and $s \in S$:

1. $a + b \in N$;

2. $0 \in N$;

3. $s \cdot a \in N$.

Definition 4.3. Since the intersection of any family of S-subsemimodules of a S-semimodule M is still a subsemimodule of M we can define the S-subsemimodule generated by an arbitrary subset X of M as the intersection of all the S-sebsemimodules containing X. We will denote this object with $\langle X \rangle$. It is obvious that $\langle X \rangle = S \cdot X = \{\sum_{i=1}^{n} a_i \cdot x_i | a_i \in S, x_i \in X, n \in \mathbb{N}\}$, i. e. the set of all the linear combinations of elements of X.

Definition 4.4. An S-semimodule is called *cyclic* if it is generated by a single element v, such a semimodule shall be denoted by $S \cdot v$.

Example 4.2.

1. Let x be an element of S, then the principal ideal generated by x, $Sx = \{sx \mid s \in S\}$, with $x \in S$ is a cyclic semimodule;

2. the additive monoid reduct of a semiring $(S, +, 0)$ is the cyclic semimodule generated by the element 1 of the semiring.

The product of a family of semimodules and the power of a semimodule are defined in the standard way.

Recall that any additively idempotent semiring has an underlying structure of join-semilattice and, in this case, we write $\vee$ for the semiring addition.

Proposition 4.1. *If* $(M, +, 0)$ *is a semimodule over an additively idempotent semiring, then it is a join-semilattice with bottom element* 0*.*

Proof. The only thing we need to show is that the operation $+$ is idempotent. Let $x \in M$, we have $x = 1x = (1 \vee 1)x = 1x + 1x = x + x$ which completes the proof. $\square$

Remark 4.1. Note that in this case we have a partial order relation $\leqslant$ called *the natural order* on M defined for any $m, m' \in M$ by $m \leqslant m' \iff m + m' = m'$.

From now on, for semimodules over additively idempotent semirings (so, in particular, for MV-semimodules) we shall use the notation $(M, \vee, 0)$ instead of the one with $+$.

Definition 4.5. Let S be a semiring and M, N two left S-semimodules. A map $f : M \to N$ is an *S-semimodule homomorphism* if $f(x+y) = f(x)+f(y)$ for any $x, y \in M$, and $f(a \cdot x) = a \cdot f(x)$, for all $a \in S$ and $x \in M$.

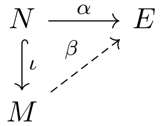
Definition 4.6. Given a semiring S, define the category ${}_S M$ of left S-semimodules whose object are left S-semimodules and whose arrows are the homomorphisms defined above.

4.2 Injective semimodules

In this section we shall provide all the necessary definitions and notions about injective and projective semimodules. As regards modules over rings we have various equivalent definitions of injective and projective modules (see [59]), unfortunately this is not the case for semimodules over semirings because the different definitions of injective and projective semimodules analogous to the ones for modules don't lead to the same class of semimodules. For insights over the different definitions of injective and projective semimodules and relations between them we refer to [2] and [3]. Through the thesis we shall always refer to the categorical definition of injective and projective semimodules (i. e. injective and projective objects in the category of semimodules over a given semiring). The results contained in this and the next section are part of the two papers [19] and [36].

Definition 4.7. Let S be a semiring. A left S-semimodule E is *injective* if, given a left S-semimodule M and a subsemimodule N, any S-homomorphism from N to E can be extended to an S-homomorphism from M to E.

The definition is equivalent to say that, given ι the inclusion of N in M, for any $\alpha : N \to E$ homomorphism, there exists β such that the following diagram commutes.



Definition 4.8. A semiring S is called *self-injective* if the regular S-semimodule S is injective.

We shall now provide a sufficient and necessary condition for the injectivity of semimodules over additively idempotent semirings. First of all, we shall recall the definition of retract in a category.

Definition 4.9 (Retract)**.** An object A in a category is called a *retract* of an object B if there are morphisms $i : A \to B$ and $r : B \to A$ such that $r \circ i = id_A$.

Recalling that in any category injectivity is preserved by retraction we have that

Lemma 4.1. *For any semiring S, any retract of an injective left S-semimodule is injective.*

Definition 4.10. Let $\pi : T \longrightarrow S$ be a semiring homomorphism. S is canonically a left T-semimodule where the scalar multiplication is defined by $t \cdot s = \pi(t)s$ for all $t \in T$ and $s \in S$. Let M be a left T-semimodule. Ler $Hom_T(S, M)$ denote the set of T-semimodule morphisms from S to T. Then, $Hom_T(S, M)$ is a left S-semimodule with respect to componentwise addition and scalar multiplication given by: $(s'\alpha)(s) = \alpha(ss')$ for all $\alpha \in Hom_T(S, M)$ and $s, s' \in S$.

We recall that with $\mathbb{B}$ we mean the two-element semifield $(\{0, 1\}, +, \cdot, 0, 1)$.

Lemma 4.2. *([56, Lemma 1]) $\mathbb{B}$ is an injective semimodule over itself.*

The Lemma is proved in [56], we report here below the proof for the reader's convenience.

Proof. Let N be a subsemimodule of a $\mathbb{B}$-semimodule M and let $\phi : N \to B$ be a $\mathbb{B}$-semimodule morhpism. Let $\psi : M \to B$ be the function defined by

$$\psi(c) = \begin{cases} 0 \ \textit{if } \phi(a + c) = 0 \textit{ for some } a \in M \\ 1 \ \textit{otherwise} \end{cases}$$

We claim that ψ is a $\mathbb{B}$-semimodule morphism.

Let c_1, c_2 be elements of M. If $\psi(c_1+c_2) = 0$, then there exists an element $a \in M$ such that $\phi(c_1+c_2+a) = 0$; then, $\phi(c_1+(c_2+a)) = 0 = \phi(c_2+(c_1+a))$ and so $\psi(c_1) = 0 = \psi(c_2)$. Hence $\psi(c_1+c_2) = \psi(c_1)+\psi(c_2)$. If $\psi(c_1+c_2) = 1$ then we must have $\psi(c_1) = 1$ or $\psi(c_2) = 1$ for otherwise if $\psi(c_1) = \psi(c_2) = 0$ there would exist a_1, $a_2 \in M$ such that $\phi(c_1 + a_1) = 0 = \phi(c_2 + a_2)$ and so $\phi(c_1+c_2+a_1+a_2) = 0$, implying that $\psi(c_1+c_2) = 0$, which is a contradiction. Therefore, in any case $\psi(c_1 + c_2) = \psi(c_1) + \psi(c_2)$. □

Lemma 4.3. *For any additively idempotent semiring S, the left S-semimodule $Hom_{\mathbb{B}}(S, \mathbb{B})$ is injective.*

Proof. Let S be an additively idempotent semiring. We then have that $\mathbb{B}$ may be considered as a subsemiring of S; hence, every left S-semimodule is also a $\mathbb{B}$-semimodule. Let N be a subsemimodule of a left S-semimodule M and let $\alpha : N \longrightarrow Hom_{\mathbb{B}}(S, \mathbb{B})$ be an S-homomorphism. Notice that N is also a $\mathbb{B}$-subsemimodule of M. Define a map $\theta : N \longrightarrow \mathbb{B}$ by setting $\theta(n) = (\alpha(n))(1)$ for all $n \in N$. Then θ is a $\mathbb{B}$-homomorphism. Indeed, if $n, n' \in N$ then $\theta(n + n') = (\alpha(n + n'))(1) = (\alpha(n) + \alpha(n'))(1) = (\alpha(n))(1) + (\alpha(n'))(1) = \theta(n) + \theta(n')$.

By Lemma 4.2 $\mathbb{B}$ is an injective $\mathbb{B}$-semimodule, and so there exists a $\mathbb{B}$-homomorphism $\phi : M \longrightarrow \mathbb{B}$ such that $\phi\theta = id_N$. Define a map $\beta : M \longrightarrow Hom_{\mathbb{B}}(S, \mathbb{B})$ by setting $(\beta(m))(s) = \phi(sm)$ for all $m \in M$ and $s \in S$. We show that β is an S-homomorphism. Indeed, for all $m_1, m_2 \in M$ and $s_1, s_2 \in S$ we have

$$(\beta(s_1m_1 + s_2m_2))(s) = \phi(s(s_1m_1 + s_2m_2)) = \phi(s(s_1m_1) + s(s_2m_2)) =$$

$$= \phi(s(s_1m_1)) + \phi(s(s_2m_2)) = \phi((ss_1)m_1)) + \phi((ss_2)m_2)) =$$

$$= (\beta(m_1))(ss_1) + (\beta(m_2))(ss_2) = (s_1\beta(m_1)(s) + (s_2\beta(m_2)(s) =$$

$$= (s_1\beta(m_1) + s_2\beta(m_2))(s)$$

for all $s \in S$. This implies that β is an S-homomorphism. Furthermore, β extends α since for each $n \in N$ and $s \in S$ we have $(\beta(n))(s) = \phi(sn) = \theta(sn) = (\alpha(sn))(1) = (s\alpha(n))(1) = (\alpha(n))(s)$. Thus, $Hom_{\mathbb{B}}(S, \mathbb{B})$ is an injective left S-semimodule, finishing the proof. □

We are now able to give a necessary and sufficient condition for the injectivity of semimodules over additively idempotent semirings. The argument of the following theorem is based on the proof of [38, Theorem 4.2].

Theorem 4.1. *Let S be an additively idempotent semiring and M a left S-semimodule. Then, M is injective if and only if there exists a set X such that M is a retract of the left S-semimodule $Hom_{\mathbb{B}}(S, \mathbb{B})^X$, where $\mathbb{B}$ is the Boolean semifield.*

Proof. ($\Longrightarrow$). Let M be an injective left S-semimodule. We then have that $Hom_{\mathbb{B}}(Hom_{\mathbb{B}}(M, \mathbb{B}), \mathbb{B})$ is a left S-semimodule, where the scalar multiplication defined by: $(s \cdot \beta)(\alpha) = \beta(\alpha \cdot s)$ for all $\beta \in Hom_{\mathbb{B}}(Hom_{\mathbb{B}}(M, \mathbb{B}), \mathbb{B})$, $\alpha \in Hom_{\mathbb{B}}(M, \mathbb{B})$ and $s \in S$. Note that $Hom_{\mathbb{B}}(M, \mathbb{B})$ is a right S-semimodule, where the scalar multiplication defined by $(\alpha \cdot s)(m) = \alpha(sm)$ for all $\alpha \in Hom_{\mathbb{B}}(M, \mathbb{B})$, $s \in S$ and $m \in M$.

Claim 4.1. The map $\phi : M \longrightarrow Hom_{\mathbb{B}}(Hom_{\mathbb{B}}(M, \mathbb{B}), \mathbb{B})$, defined by $\phi(m)(f) = f(m)$ for all $m \in M$ and $f \in Hom_{\mathbb{B}}(M, \mathbb{B})$, is an S-homomorphism

Proof. Indeed, for all $s, s' \in S$ and $m, m' \in M$ we have $(\phi(sm + s'm'))(f) = f(sm + s'm') = f(sm) + f(s'm') = (f \cdot s)(m) + (f \cdot s')(m') = (\phi(m))(f \cdot s) + (\phi(m'))(f \cdot s') = (s \cdot \phi(m))(f) + (s' \cdot \phi(m'))(f) = (s \cdot \phi(m) + s' \cdot \phi(m'))(f)$ for all $f \in Hom_{\mathbb{B}}(M, \mathbb{B})$, thus the claim is proved. □

Claim 4.2. ϕ is injective

Proof. Indeed, we first note that since S is an additively idempotent semiring, the semimodule M is additively idempotent. By Remark 4.1, the monoid $(M, +, 0)$ is a join-semilattice with the partial ordering $\leqslant$ on M defined for any two elements $m, m' \in M$ by $m \leqslant m'$ if $m + m' = m'$.

Let $m, m' \in M$ such that $m \neq m'$. We then must have that $m \nleqslant m'$ or $m' \nleqslant m$. Without loss of generality we may assume that $m \nleqslant m'$. We consider the $\mathbb{B}$-homomorphism $f : M \longrightarrow \mathbb{B}$, defined by: for all $x \in M$,

$$f(x) = \begin{cases} 0 & \text{if } x \leqslant m', \\ 1 & \text{otherwise} \end{cases} .$$

We then have that

$$\phi(m)(f) = f(m) = 1 \neq 0 = f(m') = \phi(m')(f),$$

that means, $\phi(m) \neq \phi(m')$. This implies that ϕ is injective. □

For the right S-semimodule $Hom_{\mathbb{B}}(M, \mathbb{B})$, by [30, Proposition 17.11], there exists a surjective S-homomorphism $\theta : \bigoplus_{x \in X} S_x \longrightarrow Hom_{\mathbb{B}}(M, \mathbb{B})$ from a free right S-semimodule $\bigoplus_{x \in X} S_x$, $S_x \cong S$ as right S-semimodules for all $x \in X$, where X is any set of generators for the right S-semimodule $Hom_{\mathbb{B}}(M, \mathbb{B})$. This surjection induces an injective S-homomorphism

$$\theta^* : Hom_{\mathbb{B}}(Hom_{\mathbb{B}}(M, \mathbb{B}), \mathbb{B}) \longrightarrow Hom_{\mathbb{B}}(\bigoplus_{x \in X} S_x, \mathbb{B}),$$

defined by $\theta^*(\beta) = \beta\theta$, for all $\beta \in Hom_{\mathbb{B}}(Hom_{\mathbb{B}}(M, \mathbb{B}), \mathbb{B})$. Therefore, we obtain an injective S-homomorphism $\theta^*\phi : M \longrightarrow Hom_{\mathbb{B}}(\bigoplus_{x \in X} S_x, \mathbb{B})$.

Consider for $x \in X$ the natural injection $\iota_x : S_x \longrightarrow \bigoplus_{x \in X} S_x$. We then have an S-isomorphism

$$Hom_{\mathbb{B}}(\bigoplus_{x \in X} S_x, \mathbb{B}) \longrightarrow \prod_{x \in X} Hom_{\mathbb{B}}(S_x, \mathbb{B}),$$

by the map $f \longmapsto (f\iota_x)_{x \in X}$, and so we have an injective S-homomorphism $\mu : M \longrightarrow Hom_{\mathbb{B}}(S, \mathbb{B})^X$. Since M is injective, there exists an S-homomorphism $\eta : Hom_{\mathbb{B}}(S, \mathbb{B})^X \longrightarrow M$ such that $\eta\mu = id_M$; that means, M is a retract of the left S-semimodule $Hom_{\mathbb{B}}(S, \mathbb{B})^X$.

($\Longleftarrow$). By Lemma 4.3, the left S-semimodule $Hom_{\mathbb{B}}(S, \mathbb{B})$ is injective, and so $Hom_{\mathbb{B}}(S, \mathbb{B})^X$ is also an injective left S-semimodule, by [30, Proposition 17.23 (1)]. Then, by Lemma 4.1, we immediately get that M is injective, finishing our proof. □

The previous theorem can be restated in terms of ideals of the join-semilattice reduct of the semiring, we shall now provide the definition.

Definition 4.11. An *ideal* of a join-semilattice $(S, \vee, 0)$ is a subset I of S such that

1. if $a, b \in I$, then $a \vee b \in I$;

2. if $a \in I$, $b \in S$ and $b \leqslant a$, then $b \in I$.

Remark 4.2. For an additively idempotent semiring S, an element $s \in S$ and an ideal $I \subseteq S$, define scalar multiplication by $s \cdot I = \{x \in S \mid xs \in I\}$. Then $s \cdot I$ is also an ideal of S, and it is straightforward to check that $(Id(S), \cap, I)$ is an S-semimodule (ordered by reverse inclusion). Recall also that for a semiring homomorphism $f : S \to B$, $Ker(f) = \{x \in A \mid f(x) = 0\}$, and this is a member of $Id(S)$.

Proposition 4.2. *Let S be an additively idempotent semiring. Then $Hom_{\mathbb{B}}(S, \mathbb{B})$ and $Id(S)$ are isomorphic as S-semimodules.*

Proof. As noted above, Ker is a map from $Hom_{\mathbb{B}}(S, \mathbb{B})$ to $Id(S)$, and since a function $f : S \to \mathbb{B}$ is determined by the preimage of $\{0\}$, the map Ker is a bijection. For $f, g \in Hom_{\mathbb{B}}(S, \mathbb{B})$ and $s \in S$ we have

$$Ker(f \vee g) = \{x \in S \mid (f \vee g)(x) = 0\} = Ker(f) \cap Ker(g) \text{ and}$$

$$Ker(s \cdot f) = \{x \in S \mid (s \cdot f)(x) = 0\} = \{x \in S \mid f(xs) = 0\},$$

which agrees with $s \cdot Ker(f) = \{x \in S \mid xs \in Ker(f)\}$. □

With this result we can restate Theorem 4.1.

Corollary 4.1. *Let S be an additively idempotent semiring and M an S-semimodule. Then, M is injective if and only if M is a retract of $Id(S)^X$ for some set X.*

As an application of Corollary 4.1, we obtain a necessary condition for injectivity of a semimodule over an additively idempotent semiring.

Definition 4.12. A *frame* is a complete lattice L satisfying the distributivity law

$$(\bigvee A) \wedge b = \bigvee \{a \wedge b \mid a \in A\}$$

[Coframe]([52]) A *coframe* is a complete lattice L satisfying the distributivity law

$$(\bigwedge A) \vee b = \bigwedge \{a \vee b \mid a \in A\}$$

Definition 4.13. A join-semilattice $(S, \vee, 0)$ is *join-distributive* if for any a, b_0 and b_1 elements of S such that $a \leqslant b_0 \vee b_1$, then there exist a_0, $a_1 \in S$ such that $a_0 \leqslant b_0$, $a_1 \leqslant b_1$ and $a = a_0 \vee a_1$.

The following result is well known from lattice theory.

Lemma 4.4. *Let S be a join-semilattice. Then the lattice of ideals*

$$(Id(S), \cap, \wedge)$$

ordered by reverse inclusion, is complete. If S is join-distributive, then $Id(S)$ is a coframe (i. e. $J \cap \bigwedge_{i\in I} J_{i\in I} = \bigwedge_{i\in I}(J \cap J_{i\in I})$, for any $J, J_i \in Id(S)$).

Proof.

Claim 4.3. $Id(S)$ is complete

Proof. For the completeness it is sufficient to observe that for any $J_i \in Id(S)$ we have that $\bigvee_{i\in I} J_i = \bigcap_{i\in I} J_i$ and since the set of ideals of a lattice is closed under arbitrary intersections we have that the lattice is complete. □

Claim 4.4. $Id(S)$ is a coframe

Proof. For the second part, observe that

$$\bigwedge_{i\in I} J_i = \{\vee_{k=1}^{n} a_{i_k} \mid a_{i_k} \in J_{i_k}, \{i_1, \ldots, i_n\} \subseteq I, n \in \mathbb{N}\}$$

This set is obviously closed under joins, to see that it is downward closed consider an element $x \leqslant a \in \bigwedge_{i\in I} J_i$, then $a = a_{i_1} \vee \cdots \vee a_{i_n}$ where $a_{i_k} \in J_{i_k}$, for some $J_{i_k} \in Id(S)$ for every $k = 1, \ldots, n$. Then, since S is join-distributive, we have that there exist elements $a'_{i_1}, \ldots, a'_{i_1}$ such that $a'_{i_k} \leqslant a_{i_k}$ for every $k \in \{1, \ldots, n\}$ and $x = a'_{i_1} \vee \cdots \vee a'_{i_n}$. Since any ideal J_{i_k} is downward closed we have that $a'_{i_k} \in J_{i_k}$ for every $k \in \{1, \ldots, n\}$, so $x \in \bigwedge_{i\in I} J_i$. It is now straightforward to see that $J \cap (\bigwedge_{i\in I} J_i) = \bigwedge_{i\in I}(J \cap J_i)$ and the proof is complete. □

□

Lemma 4.5. *Let $(B, \vee, 0)$ and $(M, \vee, 0)$ be two semimodules over an additively idempotent semiring S and suppose that M is a retract of B. If B is a coframe then M is also a coframe.*

Proof. Let $\alpha : M \to B$ and $\beta : B \to M$ be the two homomorphisms which determine the retraction. We prove that M is a complete semimodule and that $\bigwedge_{i\in I} m_i = \beta(\bigwedge_{i\in I} \alpha(m_i))$.

Claim 4.5. M is complete

Proof. Indeed, we first note that

$$m_i = \beta\alpha(m_i) \geqslant \beta\alpha(\bigwedge_{i\in I} m_i)$$

for all $i \in I$. If $m' \in M$ and $m_i \geqslant m'$ for all $i \in I$, then we have that $\alpha(m_i) \geqslant \alpha(m')$ for all $i \in I$, and hence $\bigwedge_{i\in I} \alpha(m_i) \geqslant \alpha(m')$. This implies that $\beta(\bigwedge_{i\in I} \alpha(m_i)) \geqslant \beta(\alpha(m')) = m'$. Therefore, $\bigwedge_{i\in I} m_i$ exists in M and is equal to $\beta(\bigwedge_{i\in I} \alpha(m_i))$. So, M is a complete.

□

Claim 4.6. M satisfies is a coframe

Proof. Since B is a coframe, we have

$$\begin{aligned} m \vee \textstyle\bigwedge_{i\in I} m_i &= \beta(\alpha(m)) \vee \beta(\textstyle\bigwedge_{i\in I} \alpha(m_i)) = \beta(\alpha(m) \vee \textstyle\bigwedge_{i\in I} \alpha(m_i)) \\ &= \beta(\textstyle\bigwedge_{i\in I}(\alpha(m) \vee \alpha(m_i))) = \beta(\textstyle\bigwedge_{i\in I} \alpha(m \vee m_i)) \\ &= \textstyle\bigwedge_{i\in I}(m \vee m_i), \end{aligned}$$

so, M is a coframe and the statement is proved. □

□

Theorem 4.2. *Let S be an additively idempotent semiring whose join-semilattice reduct is join-distributive and M an injective semimodule over S. Then M is a coframe.*

Proof. We know that M is injective if and only if it is a retract of $Id(S)^X$ for some set X. Since $Id(S)$ is a coframe by theorem 4.4 and this property is preserved by retraction, we have that M is a coframe. □

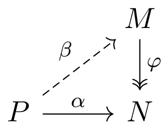
Remark 4.3. Note that the previous theorem can be useful as a “negative” criterion to establish if a semimodule is injective. For example, if the semimodule is not complete as a lattice it can’t be injective.

It seems useful to compare the previous result with the one of Bruns and Lakser (Theorem 1, [8]) which states that injective meet-semilattices are precisely frames. If we consider the boolean semifield $\mathbb{B}$, by the mirror version of Bruns and Lakser’s result we have that injective semimodules over $\mathbb{B}$ are precisely coframes (semimodules over $\mathbb{B}$ being join-semilattices). Instead, given an arbitrary semiring S and an arbitrary coframe M, we don’t know if M is an injective semimodule over S since Theorem 4.2 don’t provide a necessary and sufficient condition for a semimodule to be injective; whether the converse implication of Theorem 4.2 holds is indeed an interesting open problem to address.

4.3 Projective semimodules

Definition 4.14. Let S be a semiring. An S-semimodule P is *projective* if the following condition holds: if $\varphi : M \longrightarrow N$ is a surjective S-homomorphism of S-semimodules and if $\alpha : P \longrightarrow N$ is an S-homomorphism then there exists an S-homomorphism $\beta : P \longrightarrow M$ satisfying $\varphi\beta = \alpha$.

The definition is equivalent to say that, given any surjective homomorphism φ and any homomorphism α, there exists a homomorphism β such that the following diagram commutes.



It is well-known that in any variety of algebras the projective objects are the retracts of free objects. In the category of semimodules over a semiring A, the free object over a set X is $S^{(X)} = \{f : X \to S \mid f(x) = 0 \text{ for all but finitely many } x \in X\}$ ([21]). So, we obtain the following characterization of projective semimodules.

Theorem 4.3. *Let S be a semiring. An S-semimodule P is projective if and only if it is a retract of the semimodule $S^{(X)}$ for some set X.*

Chapter 5

Semimodules over MV-semirings

In this chapter, we give a criterion for self-injectivity of an MV-semiring with an atomic Boolean center, and give a complete description of (finitely generated) injective semimodules over a finite MV-semiring. Also, we show (Proposition 5.3) that complete Boolean algebras are precisely the MV-semirings in which every principal ideal is injective. The results contained in this section come from the paper [19].

Since the two semiring reducts of an MV-algebra A, $A^{\vee\odot} := (A, \vee, \odot, 0, 1)$ and $A^{\wedge\oplus} := (A, \wedge, \oplus, 1, 0)$ are isomorphic to each other by the involutive unary operation * of A we are allowed to limit our attention to one of these *two semiring reducts* of A; therefore, whenever not otherwise specified, we will refer only to $A^{\vee\odot}$, all the results holding also for $A^{\wedge\oplus}$ up to the application of *.

As an application of Corollary 4.1, we obtain a necessary condition for injectivity of semimodules over MV-semirings.

Proposition 5.1. *Let A be an MV-algebra and M an injective $A^{\vee\odot}$-semimodule. Then M is a coframe.*

Proof. By [11, Propositions 1.1.5 and 1.5.1], the natural order determines a structure of distributive lattice on the semiring $A^{\vee\odot}$, which implies that the join-semilattice reduct of A is join-distributive. So, M is a coframe. □

In [23, Theorem 4] Fofanova showed that a semimodule over a Boolean algebra is injective if and only if it is a coframe. It is also well-known (see, e.g., [49, Corollary 1.5.5]) that Boolean algebras are precisely the MV-algebras satisfying the additional equation $x \oplus x = x$. In the light of these results and Theorem 4.2, it is natural to pose the following question.

Problem 1. In the previous proposition we proved that every injective semimodule over an MV-semiring is a coframe so we could wonder if the converse is true. Is it true that every semimodule which is a coframe over an MV-semiring is injective?

Definition 5.1 (Atom)**.** Let A be a partially ordered set with least element 0 and a an element of A. Then a is called an *atom* if $0 < a$ and there is no $x \in A$ such that $0 < x < a$.

Definition 5.2. An element a of an MV-algebra A is called *idempotent* if $a \oplus a = a$.

Remark 5.1. Note that we use the term "idempotent" in different contexts and with different meanings. As regards semirings, we have two notions: additively idempotent (the operation + is idempotent and it is denoted by $\vee$) and multiplicatively idempotent (the operation $\cdot$ is idempotent). In this thesis we are not considering multiplicatively idempotent semirings but sometimes we deal with multiplicative idempotent elements of them. When we use the term "idempotent" in the context of MV-algebras, as in the previous definition, we mean that the operation $\oplus$ is idempotent.

Definition 5.3. The set $\mathbf{B(A)}$ of all idempotent elements of an MV-algebra A is a Boolean algebra, usually called the *Boolean center* of the MV-algebra A.

Definition 5.4. An *atom* of an MV-algebra A is an atom of the lattice A for the natural order.

We say that A is *atomic* if for each nonzero element $x \in A$ there exists an atom $a \in A$ with $a \leqslant x$.

In the following theorem we provide criteria for an MV-semiring with an atomic Boolean center to be self-injective, which solves a part of Problem 1. To do this, we need the following useful lemmas.

Lemma 5.1. *(1) For each integer $n \geqslant 2$, $\boldsymbol{L}_n^{\vee\odot} \cong Hom_{\mathbb{B}}(\boldsymbol{L}_n^{\vee\odot}, \mathbb{B})$ as $\boldsymbol{L}_n^{\vee\odot}$-semimodules. Consequently, $\boldsymbol{L}_n^{\vee\odot}$ is a self-injective semiring.*

(2) $[0,\,1]^{\vee\odot}$ is a self-injective semiring.

Proof. Let $G = (G, +, -, \leqslant, \vee, \wedge, 0, u)$ be a lattice-ordered Abelian group with a distinguished strong order unit u. To avoid confusion we denote the additive operation in the $\Gamma(G,u)^{\vee\odot}$-semimodule $Hom_{\mathbb{B}}(\Gamma(G,u)^{\vee\odot}, \mathbb{B})$ by the notation $\boxplus$.

For each $x \in \Gamma(G,u)^{\vee\odot}$, we define $f_x \in Hom_{\mathbb{B}}(\Gamma(G,u)^{\vee\odot}, \mathbb{B})$ as follows:

$$f_x(t) = \begin{cases} 0 & \text{if } 0 \leqslant t \leqslant x, \\ 1 & \text{otherwise.} \end{cases}$$

Claim 5.1. The map $\phi : \Gamma(G,u)^{\vee\odot} \longrightarrow Hom_{\mathbb{B}}(\Gamma(G,u)^{\vee\odot}, \mathbb{B})$, defined by $\phi(x) = f_{(u-x)}$ for all $x \in \Gamma(G,u)^{\vee\odot}$, is an injective $\Gamma(G,u)^{\vee\odot}$-homomorphism

Proof. Indeed, let $x, y \in \Gamma(G,u)^{\vee\odot}$. We have that for all $t \in \Gamma(G,u)^{\vee\odot}$,

$$\begin{aligned} t \leqslant u - x \;\&\; t \leqslant u - y &\iff x \leqslant u - t \;\&\; y \leqslant u - t \\ &\iff x \vee y \leqslant u - t \iff t \leqslant u - (x \vee y), \end{aligned}$$

so $f_{(u-(x\vee y))} = f_{(u-x)} \boxplus f_{(u-y)}$, that means, $\phi(x \vee y) = \phi(x) \boxplus \phi(y)$.

Let $x, y \in \Gamma(G,u)^{\vee\odot}$. We then have that $x \odot y = u - (2u - x - y) \wedge u$ and $\phi(y \odot x) = f_{(2u-x-y)\wedge u}$. On the other hand, for any $t \in \Gamma(G,u)^{\vee\odot}$, we have that

$$(y\phi(x))(t) = (yf_{(u-x)})(t) = f_{(u-x)}(y \odot t) = f_{(u-x)}(u - (2u - y - t) \wedge u)$$

and

$$\begin{aligned} u - (2u - y - t) \wedge u \leqslant u - x &\iff x \leqslant (2u - y - t) \wedge u \\ &\iff x \leqslant 2u - y - t \qquad (\textit{since } x \leqslant u) \\ &\iff t \leqslant 2u - x - y \\ &\iff t \leqslant (2u - x - y) \wedge u \quad (\textit{since } t \leqslant u), \end{aligned}$$

so $(y\phi(x))(t) = 0$ if and only if $f_{(2u-x-y)\wedge u}(t) = 0$. This implies that $y\phi(x) = f_{(2u-x-y)\wedge u} = \phi(y \odot x)$. Therefore, ϕ is a $\Gamma(G,u)^{\vee\odot}$-homomorphism. Similarly to the proof of [19, Theorem 3.12], we get that ϕ is injective, proving the claim. □

(1) Consider the case that $G = \mathbb{Z}\frac{1}{n-1}$ and $\mathbf{L}_n := \Gamma(\mathbb{Z}\frac{1}{n-1}, 1)$. Take any $f \in Hom_{\mathbb{B}}(\mathbf{L}_n^{\vee\odot}, \mathbb{B})$. Since $Ker(f)$ is a lower subset of the lattice $\mathbf{L}_n^{\vee\odot}$ for the natural order, $Ker(f) = \{x \in \mathbf{L}_n^{\vee\odot} \mid 0 \leqslant x \leqslant i/(n-1)\}$ for some $0 \leqslant i \leqslant n-1$, so $f = f_{i/(n-1)}$. We then have that $\phi((n-i-1)/(n-1)) = f_{i/(n-1)} = f$, and so ϕ is surjective. Therefore, ϕ is an isomorphism of $\mathbf{L}_n^{\vee\odot}$-semimodules, giving the statement (1).

(2) Consider the case that $G = \mathbb{R}$ and $\Gamma(\mathbb{R}, 1) = [0, 1]$. We shall now prove that the map $\theta : Hom_{\mathbb{B}}([0, 1]^{\vee\odot}, \mathbb{B}) \longrightarrow [0, 1]^{\vee\odot}$ defined by $\theta(f) = 1 - \bigvee_{t\in Ker(f)} t$ for all $f \in Hom_{\mathbb{B}}([0, 1]^{\vee\odot}, \mathbb{B})$, is a $[0, 1]^{\vee\odot}$-homomorphism.

Indeed, let f and $g \in Hom_{\mathbb{B}}([0, 1]^{\vee\odot}, \mathbb{B})$. Then, for any $t \in [0, 1]^{\vee\odot}$, $(f \boxplus g)(t) = 0 \iff f(t) \vee g(t) = 0 \iff f(t) = 0 = g(t)$, and so $Ker(f \boxplus g) = Ker(f) \cap Ker(g)$.

Let $x := \bigvee_{t\in Ker(f)} t$ and $y := \bigvee_{t\in Ker(g)} t$. Since $[0, 1]^{\vee\odot}$ is an MV-chain, one of the two lower subsets $Ker(f)$ and $Ker(g)$ is included in the other, so $\bigvee\{t \in \Gamma(G)^{\vee\odot} \mid t \in Ker(f) \cap Ker(g)\} = x \wedge y$. Also, since $\Gamma(G)^{\vee\odot}$ is an MV-chain, $1 - (x \wedge y) = (1 - x) \vee (1 - y)$. This implies that

$$\theta(f \boxplus g) = 1 - (x \wedge y) = (1 - x) \vee (1 - y) = \theta(f) \vee \theta(y).$$

Let $y \in [0,\, 1]^{\vee\odot}$ and $f \in Hom_{\mathbb{B}}([0,\, 1]^{\vee\odot}, \mathbb{B})$. Then, for each $t \in [0,\, 1]^{\vee\odot}$, we have that

$$(yf)(t) = f(y \odot t) = f(1 - (2 - y - t) \wedge 1) = f((t + y - 1) \vee 0).$$

We show that

$$\bigvee_{t \in Ker(yf)} t = (1 + x - y) \wedge 1,$$

where $x := \bigvee_{t \in Ker(f)} t$.

Claim 5.2. $\bigvee_{t \in Ker(yf)} t \leqslant (1 + x - y) \wedge 1$

Proof. Indeed, for any $t \in [0,\, 1]^{\vee\odot}$ with $(yf)(t) = 0$, we have that $f((t + y - 1) \vee 0) = 0$, so $(t + y - 1) \vee 0 \leqslant x$. We also note that $(t + y - 1) \vee 0 \leqslant x \Longrightarrow t + y - 1 \leqslant x \Longrightarrow t \leqslant 1 + x - y \Longrightarrow t \leqslant (1 + x - y) \wedge 1$ (since $t \leqslant 1$). Therefore, $\bigvee_{t \in Ker(yf)} t \leqslant (1 + x - y) \wedge 1$. □

Claim 5.3. $\bigvee_{t \in Ker(yf)} t \geqslant (1 + x - y) \wedge 1$

Proof. Take any $a \in [0,\, 1]^{\vee\odot}$ with $a < (1 + x - y) \wedge 1$. We have that $a < 1 + x - y$, that is, $a + y - 1 < x$. If $x \in Ker(f)$ then $(a + y - 1) \vee 0 \leqslant x$, so $(yf)(a) = f((a + y - 1) \vee 0) = 0$, since $Ker(f)$ is a lower subset of the lattice $[0,\, 1]^{\vee\odot}$. Otherwise, we have that $0 < x$, and so $(a + y - 1) \vee 0 < x$. Then, since $x = \bigvee_{t \in Ker(f)} t$, there exists $t \in Ker(f)$ such that $(a + y - 1) \vee 0 < t$, which shows that $(yf)(a) = f((a + y - 1) \vee 0) = 0$, since $Ker(f)$ is a lower subset of $[0,\, 1]^{\vee\odot}$. In any case we have that $a \in Ker(yf)$, and so $\bigvee_{t \in Ker(yf)} t = (1 + x - y) \wedge 1$. □

From this observation, we get that

$$\theta(yf) = 1 - (1 + x - y) \wedge 1 = y \odot (1 - x) = y \odot \theta(f).$$

Therefore, θ is a $[0,\, 1]^{\vee\odot}$-homomorphism, proving the claim. Furthermore, for any $x \in [0,\, 1]^{\vee\odot}$, we have that $\theta\phi(x) = \theta(f_{1-x}) = 1 - (1 - x) = x = id_{[0,\,1]^{\vee\odot}}(x)$, that is, $\theta\phi = id_{[0,\,1]^{\vee\odot}}$, and so $[0,\, 1]^{\vee\odot}$ is a retract of the $[0,\, 1]^{\vee\odot}$-semimodule $Hom_{\mathbb{B}}([0,\, 1]^{\vee\odot}, \mathbb{B})$. Then, by Theorem 4.1, we get that $[0,\, 1]^{\vee\odot}$ is a self-injective semiring, giving the statement (2), thus the proof is complete. □

The following lemma is an analog of [41, Corollary 3.11B] for our semiring setting.

Lemma 5.2. *(cf. [41, Corollary 3.11B]) Let* $S = \prod_{i \in I} S_i$ *be a direct product of semirings* S_i*. Then* S *is left self-injective if and only if each* S_i *is left self-injective.*

Proof. We first note that every left S_i-semimodule may be viewed as a left S-semimodule via the natural projection $S \longrightarrow S_i$. This provides that for each $i \in I$ the left S_i-semimodule S_i is viewed as a left S-semimodule. We then have that $S \cong \prod_{i\in I} S_i$ as left S-semimodules. By the dual of [30, Proposition 17.19], S is left self-injective if and only if each S_i is injective left S-semimodule.

Claim 5.4. S_i is injective left S-semimodule if and only if S_i is left self-injective

Proof. Indeed, suppose S_i is injective left S-semimodule. Let $f : A \longrightarrow B$ be an injective S_i-homomorphism and $g : A \longrightarrow S_i$ an S_i-homomorphism. Then, by the above note, f and g may be viewed as S-homomorphisms. Since S_i is injective left S-semimodule, there exists an S-homomorphism $h : B \longrightarrow S_i$ such that $hf = g$, giving that S_i is left self-injective.

Conversely, suppose S_i is left self-injective. Let $f : A \longrightarrow B$ be an injective S-homomorphism and $g : A \longrightarrow S_i$ an S-homomorphism. Write $S = S_i \times S_i^c$, where $S_i^c = \prod_{j\in I, j\neq i} S_j$. We then have that $A = S_iA \oplus S_i^cA$ and $A = S_iB \oplus S_i^cB$. By f and g are S-homomorphisms, we have that $f(S_iA) = S_if(A) \subseteq S_iB$, $f(S_i^cA) = S_i^cf(A) \subseteq S_i^cB$ and $g(S_i^cA) = S_i^cg(A) \subseteq S_i^cS_i = 0$. Since S_i is left self-injective, there exists an S_i-homomorphism $h : S_i^cB \longrightarrow S_i$ such that $h \circ g|_{S_iB} = f|_{S_iA}$. We extend h to $h' : B \longrightarrow S_i$ by taking $h'|_{S_i^cB}$. We then get that $h'f = g$. This implies that S_i is an injective left S-semimodule, proving the claim. □

From these observations we immediately get that S is left self-injective if and only if each S_i is left self-injective. □

Theorem 5.1. *For any MV-algebra A with an atomic Boolean center, the following conditions are equivalent:*

(1) The semiring $A^{\vee\odot}$ is self-injective;

(2) All finitely generated projective $A^{\vee\odot}$-semimodules are injective;

(3) All cyclic projective $A^{\vee\odot}$-semimodules are injective;

(4) A is a complete MV-algebra.

Proof. (1)$\Longrightarrow$(2). Suppose $A^{\vee\odot}$ is a self-injective semiring and M is a finitely generated projective $A^{\vee\odot}$-semimodule. Then, by [30, Proposition 17.16], M is a retract of a free $A^{\vee\odot}$-semimodule $(A^{\vee\odot})^X$ with a finite set X. Since $A^{\vee\odot}$ is a self-injective semiring and by [30, Proposition 17.23 (1)], $(A^{\vee\odot})^X$ is an injective $A^{\vee\odot}$-semimodule, so M is also an injective $A^{\vee\odot}$-semimodule, by Lemma 4.1.

(2)$\Longrightarrow$(3). Since every cyclic semimodule is finitely generated, the statement is obvious.

(3)$\Longrightarrow$(4). Since $A^{\vee\odot}$ is a cyclic projective semimodule over itself, and by hypothesis (3), $A^{\vee\odot}$ is a self-injective semiring. From this and Proposition 5.1, we get that A is a complete lattice for the natural order, so A is a complete MV-algebra.

(4)$\Longrightarrow$(1). Since A is a complete MV-algebra with an atomic Boolean center $\mathbf{B}(A)$, A is a direct product of complete MV-chains, by [11, Theorem 6.8.1]. Also, by [11, Theorem 6.8.5], every complete MV-chain is either a finite MV-chain or isomorphic to the standard MV-algebra. From these observations, Lemmas 5.1 and 5.2, we immediately get the statement, finishing the proof. □

Remark 5.2. Note that the previous theorem is the corrected version of [19, Theorem 4.7] since it contains a mistake in the proof of the implication (4)$\Longrightarrow$(1).

In [58] Wilding, Johnson and Kambites introduced exact semirings, defined in terms of a Hahn-Banach-type separation property on semimodules arising in the tropical case from the phenomenon of tropical matrix duality (see, e.g., [14], [13], [15] and [33]).

We write $M_{m\times n}(S)$ for the additive monoid of m row, n column matrices with entries in a semiring S, where $m, n \in \mathbb{N}$. Matrix addition is the component-wise addition and it is an internal operation for fixed $m, n \in \mathbb{N}$. Instead matrix multiplication AB is defined if $A \in M_{m\times n}(S)$ and $B \in M_{n\times r}(S)$ for some $m, n, r \in \mathbb{N}$. In this case the matrix $AB \in M_{m\times r}(S)$ so matrix multiplication is not an internal operation for fixed $m, n \in \mathbb{N}$. Matrix multiplication behaves in the usual ways: where defined it is associative and distributes over matrix addition.

For each $A \in M_{m\times n}(S)$ has an associated *row space* $Row(A) = \{x \in M_{1\times n}(S) \mid x = uA$ for some $u \in M_{1\times m}(S)\}$ and an associated *column space* $Col(A) = \{y \in M_{m\times 1}(S) \mid y = Au$ for some $u \in M_{n\times 1}(S)\}$.

Definition 5.5 ([58, Definition 3.1]). A semiring S is *exact* if for every matrix $A \in M_{m\times n}(S)$, (i) any matrix $x \in M_{1\times n}(S) \setminus Row(A)$ there exist t and $u \in M_{n\times 1}(S)$ satisfying $At = Au$, but $xt \neq xu$; and (ii) any matrix $y \in M_{m\times 1}(S) \setminus Col(A)$ there exist v and $w \in M_{1\times m}(S)$ satisfying $vA = wA$, but $vy \neq wy$.

Definition 5.6. A left S-semimodule M is *FP-injective* if every S-homomorphism $f : X \longrightarrow M$ from a finitely generated left subsemimodule X of a free S-semimodule F can be extended to F. A semiring S is *left (resp. right) FP-injective* if the regular left (resp. right) S-semimodule S is FP-injective. The semiring S is called *FP-injective* if S is both left and right FP-injective.

In [37, Lemma 3.1], Johnson and Nam noted that a semiring S is exact if and only if it is FP-injective. In [53] Shitov proved the interesting result that a semifield S is exact if and only if S is either a field or an additively idempotent semifield. As immediate corollary of Theorem 5.1, we get the following result, which provides us with many examples of exact semirings.

Corollary 5.1. *Every complete MV-semiring with an atomic Boolean center is an exact semiring.*

Proof. Let S be a complete MV-semiring with an atomic Boolean center. By Theorem 5.1, S is a self-injective semiring, and so it is FP-injective. Then, by [37, Lemma 3.1], we get that S is an exact semiring, finishing the proof. □

The following theorem gives the structure of injective semimodules over finite MV-semirings.

Theorem 5.2. *Let A be a finite MV-algebra and M a $A^{\vee\odot}$-semimodule. Then M is injective if and only if there exists a set X such that M is a retract of the $A^{\vee\odot}$-semimodule $(A^{\vee\odot})^X$.*

Proof. By Theorem 4.3, an $A^{\vee\odot}$-semimodule M is injective if and only if there exists a set X such that M is a retract of the $A^{\vee\odot}$-semimodule $Hom_{\mathbb{B}}(A^{\vee\odot}, \mathbb{B})^X$. By [11, Proposition 3.6.5], $A \cong \mathbf{L}_{n_1} \times \cdots \times \mathbf{L}_{n_d}$ (as MV-algebras) for some integers $2 \leqslant n_1 \leqslant \cdots \leqslant n_d$, and so the semiring $A^{\vee\odot}$ is isomorphic to $\prod_{i=1}^{d} \mathbf{L}_{n_i}^{\vee\odot}$. For each $1 \leqslant i \leqslant d$, $\mathbf{L}_{n_i}^{\vee\odot}$ may be viewed as an $A_{n_i}^{\vee\odot}$-semimodule via the natural projection $A_{n_i}^{\vee\odot} \longrightarrow \mathbf{L}_{n_i}^{\vee\odot}$. We then have that $A^{\vee\odot} \cong \prod_{i=1}^{d} \mathbf{L}_{n_i}^{\vee\odot}$ as $A^{\vee\odot}$-semimodules, and

$$Hom_{\mathbb{B}}(A^{\vee\odot}, \mathbb{B}) \cong Hom_{\mathbb{B}}(\prod_{i=1}^{d} \mathbf{L}_{n_i}^{\vee\odot}, \mathbb{B}) \cong \prod_{i=1}^{d} Hom_{\mathbb{B}}(\mathbf{L}_{n_i}^{\vee\odot}, \mathbb{B})$$

as $A^{\vee\odot}$-semimodules. By Lemma 4.5 (1), we get that $Hom_{\mathbb{B}}(\mathbf{L}_{n_i}^{\vee\odot}, \mathbb{B}) \cong \mathbf{L}_{n_i}^{\vee\odot}$ as $A^{\vee\odot}$-semimodules, and so

$$Hom_{\mathbb{B}}(A^{\vee\odot}, \mathbb{B}) \cong \prod_{i=1}^{d} Hom_{\mathbb{B}}(\mathbf{L}_{n_i}^{\vee\odot}, \mathbb{B}) \cong \prod_{i=1}^{d} \mathbf{L}_{n_i}^{\vee\odot} \cong A^{\vee\odot}$$

as $A^{\vee\odot}$-semimodules. This implies that $Hom_{\mathbb{B}}(A^{\vee\odot}, \mathbb{B})^X \cong (A^{\vee\odot})^X$ as $A^{\vee\odot}$-semimodules, so the statement is proved, finishing the proof. □

The following theorem provides us with the structure of finitely generated injective semimodules over finite MV-semirings.

Theorem 5.3. *Let A be a finite MV-algebra and M a finitely generated $A^{\vee\odot}$-semimodule. Then the following statements are equivalent:*

(1) M is injective;

(2) M is FP-injective;

(3) M is a retract of a $A^{\vee\odot}$-semimodule $(A^{\vee\odot})^X$ for some finite set X;

(4) M is projective.

Proof. (1)$\Longrightarrow$(2). It is obvious.

(2)$\Longrightarrow$(3). As in the proof of Theorem 3.3, there always exists an injective $A^{\vee\odot}$-homomorphism $\mu : M \longrightarrow Hom_{\mathbb{B}}(A^{\vee\odot}, \mathbb{B})^X$, where X is any set of generators for the $A^{\vee\odot}$-semimodule $Hom_{\mathbb{B}}(A^{\vee\odot}, \mathbb{B})$. Since $A^{\vee\odot}$ is finite, the $A^{\vee\odot}$-semimodule $Hom_{\mathbb{B}}(A^{\vee\odot}, \mathbb{B})$ is finitely generated, so we can pick X which is a finite set. Similar to the proof of Theorem 5.2 we have that $Hom_{\mathbb{B}}(A^{\vee\odot}, \mathbb{B})^X \cong (A^{\vee\odot})^X$ as $A^{\vee\odot}$-semimodules. Therefore, we get an injective $A^{\vee\odot}$-homomorphism $\mu : M \longrightarrow (A^{\vee\odot})^X$. Since M is both finitely generated and FP-injective, there exists a surjective $A^{\vee\odot}$-homomorphism $\theta : (A^{\vee\odot})^X \longrightarrow M$ such that $\theta\mu = id_M$; that means, M is a retract of the $A^{\vee\odot}$-semimodule $(A^{\vee\odot})^X$.

(3)$\Longrightarrow$(4). Since X is finite, $A^{\vee\odot}$-semimodule $(A^{\vee\odot})^X$ is free, so the statement follows from [30, Proposition 17.16].

(4)$\Longrightarrow$(1). Since M is both finitely generated and projective, M is a retract of a free $A^{\vee\odot}$-semimodule F with a finite basis set X, by [30, Proposition 17.16]. Now applying Theorem 5.2, we get the statement, finishing the proof. □

As shown in [34, Corollary 3.2], every semimodule can be represented, in a canonical way, as a colimit of its cyclic subsemimodules. This observation motivates the study of semirings over which any semimodule is a colimit of cyclic semimodules possessing some special properties (see, e.g. [1] and [34]). Thus, it is quite natural to present complete characterizations of MV-semirings in terms of the injectivity of cyclic semimodules. Notice that as a corollary of [1, Theorem 4.6], we obtain that all cyclic S-semimodules over an MV-semiring S are injective if and only if S is a finite Boolean algebra. The following result shows that complete Boolean algebras are precisely the MV-algebras in which every principal ideal is injective. Before doing so, we need the following notion and simple fact. A semiring S is called *von Neumann regular* if for any $x \in S$ there exists $y \in S$ such that $x = xyx$.

Lemma 5.3. *If S is a semiring in which every principal left ideal is injective then S is left self-injective and von Neumann regular.*

Proof. Let S be a semiring in which every principal left ideal is injective. Since S is a principal left ideal of S generated by 1, S is left self-injective, by the hypothesis. Take any $x \in S$. Then, by the hypothesis, Sx is an injective left S-semimodule, and so there exists an A-homomorphism $f : S \longrightarrow Sx$ such that $f|_{Sx} = id_{Sx}$. It implies that $x = f(x) = f(x.1) = xf(1)$. Since $f(1) \in Sx$, there exists $y \in S$ such that $f(1) = yx$, and so $x = xyx$. Thus, S is von Neumann regular, finishing the proof. □

Definition 5.7. A semiring S is *multiplicatively idempotent* if $x \cdot x = x$ for every $x \in S$.

Proposition 5.2. *Let A be an MV-semiring. Then A is von Neumann regular if and only if A is multiplicatively idempotent.*

Proof. ($\Rightarrow$) We know that $a \cdot a \leqslant a$, for any $a \in A$. Suppose that A is von Neumann regular, so $a \vee (a \cdot a) = (a \cdot b \cdot a) \vee (a \cdot 1 \cdot a) = a \cdot (b \vee 1) \cdot a = a \cdot a$. Since $a \cdot a \leqslant a$, we have that $a \cdot a = a$.

($\Leftarrow$) Suppose A is multiplicatively idempotent, then $a = a \cdot a = a \cdot 1 \cdot a$. □

Proposition 5.3. *For every MV-algebra A, the following statements are equivalent:*

(1) Every principal ideal of $A^{\vee\odot}$ is injective;

(2) $A^{\vee\odot}$ is a self-injective and von Neumann regular semiring;

(3) A is a complete Boolean algebra.

Proof. (1)$\Longrightarrow$(2). It follows from Lemma 5.3.

(2)$\Longrightarrow$(3). Since $A^{\vee\odot}$ is a self-injective semiring and by Theorem 4.2, the lattice A is complete. Since $A^{\vee\odot}$ is a von Neumann regular semiring, by the previous proposition we have that it is multiplicatively idempotent. Thus A is a complete Boolean algebra.

(3)$\Longrightarrow$(1). Suppose A is a complete Boolean algebra. Then, by [11, Theorem 1.5.3], the semiring $A^{\vee\odot}$ is also a complete Boolean algebra. By [23, Corollary 2], $A^{\vee\odot}$ is a self-injective semiring. Take any $a \in A$. We have that $a \odot a = a$. Define two $A^{\vee\odot}$-homomorphisms $\alpha : A^{\vee\odot}a \longrightarrow A^{\vee\odot}$ and $\beta : A^{\vee\odot} \longrightarrow A^{\vee\odot}a$ by setting $\alpha(b \odot a) = b \odot a$ and $\beta(b) = b \odot a$ for all $b \in A$. It is obvious that $\beta\alpha = id_{A^{\vee\odot}a}$; that means, $A^{\vee\odot}a$ is a retract of the $A^{\vee\odot}$-semimodule $A^{\vee\odot}$. Since $A^{\vee\odot}$ is self-injective and by Lemma 4.1, $A^{\vee\odot}a$ is an injective $A^{\vee\odot}$-semimodule, and so statement (1) is proved, finishing the proof. □

As was mentioned above, Boolean algebras are precisely the MV-algebras satisfying the additional equation $x \oplus x = x$; that means, Boolean algebras form a subvariety of the variety of MV-algebras which is generated by $\mathbf{L}_2$. In the light of this remark and Lemma 5.3, we end this chapter by posing the following problem.

Problem 2. We showed that the condition that all the principal ideals of an MV-semiring are injective as semimodules characterizes complete Boolan algebras among MV-algebras. So, we can wonder: could one describe other subvarieties of the variety of MV-algebras (as, for example the one generated by $\mathbf{L}_n$) in terms of the injectivity and projectivity of semimodules?

Chapter 6

Semimodules over involutive semirings

In this chapter we shall restrict our attention to 1-bounded involutive semirings, since all the definitions and notions about semimodules are in reference to semirings with zero. We shall present some generalizations of the results about injective and projective semimodules over MV-semirings that are shown in the previous chapter. Indeed, we shall show (Theorem 6.2) that the involution of a semiring plays a crucial role in the characterization of injective semimodules over it, for example an analogous of Theorem 5.2 shall be proved without using MV-algebra representation theorems in the more general context of 1-bounded involutive semirings. In particular, we prove that involution is a necessary and sufficient condition in order for projective and injective semimodules to coincide. Indeed, we shall provide an example (Example 6.1) of a non involutive semiring and a finitely generated semimodule over it which is projective but not injective whereas this can't happen for semirings with an involution. The results contained in this section come from the preprint [36].

The theorem 5.3 shows that over a finite MV-semiring finitely generated injective and projective semimodules coincide, we shall see that this is still true for finite 1-bounded involutive semirings. As an intermediate step, we shall prove the following fact which we state as a theorem since it is interesting in itself.

Theorem 6.1. *Let A be a finite 1-bounded pointed residuated join-semilattice. Then A is an involutive semiring if and only if A and $Id(A)$ are isomorphic as A-semimodules via the map $\Phi(a) = \downarrow -a$.*

Proof.

Claim 6.1 ($\Rightarrow$).

Proof. By Proposition 4.2 we can consider $Id(A)$ in place of $Hom_{\mathbb{B}}(A, \mathbb{B})$. First, assume A is a finite 1-bounded involutive semiring and define a map $\Phi : A \to Id(A)$ by $\Phi(a) = \downarrow\!-a = \{x \in A \mid x \leqslant -a\}$, where $-a = 0/a$. Since every ideal of a finite join-semilattice is principal, and since $-$ is a bijection, this map is also bijective. It is order-preserving since $-$ is order-reversing and $Id(A)$ is ordered by reverse inclusion, hence $\Phi(a \vee b) = \Phi(a) \cap \Phi(b)$. The following calculation shows that Φ preserves scalar multiplication:

$$b \cdot \Phi(a) = \{x \in A \mid xb \leqslant -a\} = \{x \in A \mid xba \leqslant 0\} =$$

$$= \{x \in A \mid x \leqslant -(ba)\} = \Phi(ba).$$

□

Claim 6.2 ($\Leftarrow$).

Proof. Conversely, assume A is a finite 1-bounded residuated join-semilattice, and A, $Id(A)$ are isomorphic as A-semimodules via the map $\Phi(a) = \downarrow\!-a$, where $-a = 0/a$ and 0 is the bottom element of A. Let $f(a) = \bigvee \Phi(a) = -a$. Since A and $Id(A)$ are assumed to be isomorphic, f is a bijection. From residuation it follows that $x \leqslant 0/y \iff xy \leqslant 0 \iff y \leqslant x\backslash 0$, hence $-, \sim$ form a Galois connection, hence $-\sim$ and $\sim-$ are closure operators and $-\sim-x = -x$. Since $f(x) = -x$ is a bijection, we get $\sim-x = x$ and $-\sim x = x$, so A is an involutive semiring by Theorem 3.1. □

□

The previous theorem together with Corollary 4.1 gives the following result.

Corollary 6.1. *Let A be a finite* 1*-bounded involutive semiring and M a semimodule over A. Then M is injective if and only if it is a retract of A^X for some set X.*

Theorem 6.2. *Let A be a finite* 1*-bounded involutive semiring and M a finitely generated A-semimodule. Then, M is injective if and only if it is projective.*

Proof. Since $A \cong Hom_{\mathbb{B}}(A, \mathbb{B})$ as A-semimodules, we have that retracts of A^X for some finite set X (projective semimodules) are exactly the retracts of $Hom_{\mathbb{B}}(A, \mathbb{B})^X$ (injective semimodules). □

We can wonder in which cases injective and projective semimodules coincide and in particular if we can weaken the hypothesis of involution of the semiring assumed in the above theorem. The answer is no and we shall provide an example.

Example 6.1. Consider the three-element idempotent semiring $A = \{0, a, 1\}$ with $0 < a < 1$ and $a \cdot a = a$, then injective and projective semimodules over this semiring don't coincide. First of all observe that $Id(A) = \{0, \downarrow a, A\}$. We know that A is a projective semimodule over itself. We shall now prove that A can't be injective. Suppose that A is self-injective, so it should be a retract of $Id(A)^n$ for some finite $n \in \mathbb{N}$ since A is finitely generated. In this case, we should have an A- semimodule morphism $\Phi : Id(A)^n \to A$ such that $Im(\Phi) = A$, if $\Phi(\{0\}, \{0\}, \ldots, \{0\})$ is a or 0, then $|Im(\Phi)| \leqslant 2$ (Φ is order-preserving), in particular $Im(\Phi) \neq A$, so $\Phi(\{0\}, \{0\}, \ldots, \{0\}) = 1$, but in this case

$$1 = \Phi(\{0\}, \ldots, \{0\}) = \Phi(a \cdot (\{0\}, \ldots, \{0\})) = a \cdot \Phi(\{0\}, \ldots, \{0\}) = a \cdot 1 = a$$

which is absurd.

We shall now show that also proposition 5.3 can be generalized in the variety of 1-bounded involutive semirings. To do this, we shall need the following proposition that involves Heyting algebras and whose definition we recall here below.

Definition 6.1 (Heyting algebras). A Heyting algebra $(H, \vee, \wedge, \to, 0, 1)$ is an algebra of type $(2, 2, 2, 0, 0)$ that satisfies:

- $(H, \vee, \wedge)$ is a distributive lattice;
- $x \wedge 0 = 0$ and $x \vee 1 = 1$;
- $x \to x = 1$;
- $(x \to y) \wedge y = y$; $x \wedge (x \to y) = x \wedge y$;
- $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$; $(x \vee y) \to z = (x \to z) \wedge (y \to z)$.

In particular, a Heyting algebras is a 1-bounded, commutative, multiplicatively idempotent and distributive residuated lattice with $x \cdot y = x \wedge y$ and $x \backslash y = y / x = x \to y$. The constant 0 is the bottom element (so, multiplicatively absorbing) and the negation $\neg$ is defined by $\neg x = x \to 0$; Boolean algebras are the subclass of Heyting algebras that satisfy: $\neg\neg x = x$.

Proposition 6.1. *A 1-bounded involutive semiring A is multiplicatively idempotent if and only if A is a Boolean algebra.*

Proof. From $xx = x \leqslant 1$ it follows that $x \cdot y = x \wedge y$, so the semiring is commutative and, in particular, $-x = {\sim} x$. Defining $x \to y$ as ${\sim}((-y) \cdot x) = -((-y) \cdot x)$, we obtain that $(A, \vee, \wedge, \to, 0, 1)$ is a Heyting algebra. We have $\neg x$ defined as $x \to 0$ and so $\neg\neg x = (x \to 0) \to 0 = -(1 \cdot (-x)) - (-x) = x$. Therefore the Heyting algebra is a Boolean algebra. □

Theorem 6.3. *For a* 1*-bounded involutive semiring* A*, the following statements are equivalent:*

1. *Every principal semiring ideal* Aa *of* A *is injective as a semimodule;*

2. A *is a self-injective von Neumann regular semiring;*

3. A *is a complete Boolean algebra.*

Proof. (1)⇒(2). By lemma 5.3

(2)⇒(3). Since A is a self-injective semiring and applying Theorem 4.2, the lattice A is complete.

From proposition 5.2 we know that A is multiplicatively idempotent and consequently, by the previous proposition, a Boolean algebra.

(3)⇒(1). Verbatim from proposition 5.3. □

We can generalize the definitions of von Neumann regular and multiplicatively idempotent semirings.

Definition 6.2. Let A be a 1-bounded involutive semiring. For a $n \in \mathbb{N}$, a semiring A is *n-von Neumann regular* if for every $a \in A$, there exists $b \in A$ such that $a^n = a^n \cdot b \cdot a^n$.

Definition 6.3. A semiring A is *n-potent* if $a^n = a^{n+1}$ for every $a \in A$.

Proposition 6.2. *Let* A *be a 1-bounded multiplicatively idempotent semiring and* $n \in \mathbb{N}$*. Then* A *is n-von Neumann regular if and only if* A *is n-potent.*

Proof. By proposition 5.2 we already have the result for $n = 1$.

Now let $n \in \mathbb{N}$, $n > 1$. From the two implications proved above, we have that A is n-von Neumann regular iff $a^{2n} = a^n$. Since $a^{2n} \leqslant a^{n+1} \leqslant a^n$, this implies $a^{n+1} = a^n$. Obviously $a^n = a^{n+1}$ implies $a^{2n} = a^n$ for any $a \in A$. □

Theorem 6.4. *Let* A *be a 1-bounded semiring. Then, for a fixed* $n \in \mathbb{N}$*, the following statements are equivalent:*

1. *for every* $a \in A$ *the cyclic semimodule generated by* a^n *is injective as a semimodule on* A*;*

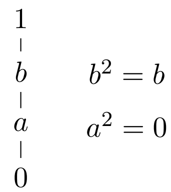
2. A *is self-injective and n-potent.*

Proof. (1) ⇒ (2) Obviously A is self-injective since it is generated by 1^n. If $A \cdot a^n$ is injective, then exists a A - homomorphism $f : A \to A \cdot a^n$ such that $f|_{A \cdot a^n} = id_{A \cdot a^n}$. It implies that $a^n = f(a^n) = f(a^n \cdot 1) = a^n \cdot f(1)$. Since $f(1) \in A \cdot a^n$, we have that exists an element $b \in A$ such that $f(1) = b \cdot a^n$, so $a^n = a^n \cdot b \cdot a^n$.

We then get that A is n-von Neumann regular and for a previous remark $a^n = a^{n+1}$, for every $a \in A$.

(2) ⇒ (1) Define $\alpha : A \cdot a^n \to A$ by $\alpha(b \cdot a^n) = b \cdot a^n$ and $\beta : A \to A \cdot a^n$ by $\beta(b) = b \cdot a^n$. We then have that $\beta\alpha(b \cdot a^n) = b \cdot a^{2n}$. Since $a^n = a^{n+1}$ implies $a^{2n} = a^n$ and consequently $b \cdot a^{2n} = b \cdot a^n$, we have that $\beta\alpha = id_{A \cdot a^n}$. So, $A \cdot a^n$ is a retract of A which is self-injective. This implies that $A \cdot a^n$ is injective too. □

Example 6.2. Let us consider any MV-chain $\mathbf{L}_{n+1}$ for $n \geqslant 1$, we know that this chains are self-injective and n-potent but we also know that $a^n = 0$ for every $a \in \mathbf{L}_{n+1}$, $a \neq 1$ so the cyclic semimodules generated by a^n are only the semiring itself and $\{0\}$, therefore the previous theorem isn't interesting in this context.
On the contrary, we can usefully apply it in the context of finite involutive linearly-ordered 1-bounded involutive semiring. As an example, consider the finite commutative involutive linearly-ordered 1-bounded involutive semiring $C = \{0, a, b, 1\}$ where $0 < a < b < 1$, $a \cdot a = 0$ and $b \cdot b = b$.



It is easy to see that C is 2-potent and we know that C is self-injective since it is a projective C-semimodule and therefore injective by Theorem 6.2. Hence all the cyclic semimodules of the form Cc^n for some $c \in C$ are injective and projective. In particular we have that the semimodules $\{0\}$, Cb and C are injective and, using Theorem 6.2 again, also projective.

6.1 Strong semimodules

The definition of strong semimodules was given in the paper [21] for semimodules over MV-semirings

Definition 6.4 (Strong MV-semimodules)**.** Let A be an MV-semiring with negation * and M an A-semimodule. M is said to be a *strong semimodule* if for all $a, b \in A$ we have that:
$a \cdot x = b \cdot x$ for all $x \in M$ implies $a^* \cdot x = b^* \cdot x$ for all $x \in M$.

This definition can be extended to semimodules over involutive semirings as follows

Definition 6.5. A semimodule M over an involutive semiring A is *strong* if for all $a, b \in A$

$$\forall m \in M\ (a \cdot m = b \cdot m) \implies \forall m \in M\ (-a \cdot m = -b \cdot m \text{ and } {\sim}a \cdot m = {\sim}b \cdot m).$$

We shall see that the definition of strong semimodule is related to the one of faithful semimodule.

Definition 6.6. A semiring A is called *nilpotent* if for every $a \in A$, $a \neq 1$, there exists a $n \in \mathbb{N}$ such that $a^n = 0$.

The notion of strong semimodule is not a standard definition of semimodule theory since it is requires an additional unary operation besides the semiring operations. The interesting thing is that, under the assumption of the nilpotency of the semiring, strong semimodules coincide with faithful semimodules and that these last ones can be, instead, defined over an arbitrary semiring.

Definition 6.7. An A-semimodule M is *faithful* if the action of each $a \neq 0$ in A on M is nontrivial, i. e. $a \cdot x \neq 0$ for some $x \in M$.

Theorem 6.5. *Let A be a nilpotent 1-bounded involutive semiring and M a nontrivial A-semimodule. Then M is a strong semimodule if and only if M is faithful.*

Proof.

Claim 6.3 ($\Leftarrow$).

Proof. Note that for any $a \in A$ we have that $a \cdot ({\sim}a) = (-a) \cdot a = 0$. Suppose M is faithful and let $a \cdot x = b \cdot x$, for all $x \in M$. Then we have $0 = ((-a)a) \cdot x = ((-a)b) \cdot x$ for all $x \in M$ and also $((-b)a) \cdot x = 0$ for all $x \in M$. Since M is faithful we have $(-a)b = (-b)a = 0$, which implies respectively that $b \leqslant a$ and $a \leqslant b$. Consequently we have $a = b$ and obviously $-a \cdot x = -b \cdot x$ and ${\sim}a \cdot x = {\sim}b \cdot x$ for all $x \in M$. ☐

Claim 6.4 ($\Rightarrow$).

Proof. Viceversa, suppose M is strong and that $a \cdot x = 0 = 0 \cdot x$ for all $x \in M$, for some $0 \neq a$ in A. Then we have $-a \cdot x = -0 \cdot x = 1 \cdot x = x$ for all $x \in M$, which implies $(-a)^n \cdot x = x$ for all $x \in M$ and $n \in \mathbb{N}$. But, since A is nilpotent, we have that $-a = 1$ and so $a = 0$, which contradicts the hypothesis. ☐

☐

Examples 6.1.

- *Any 1-bounded involutive semiring A is a strong semimodule over itself (where the action is the semiring multiplication).*

  Indeed $a \cdot x = b \cdot x$ for every $x \in A$ implies, in particular that $a = a \cdot 1 = b \cdot 1 = b$ and from this it follows that $-a = -b$ and ${\sim}a = {\sim}b$.

- *Consider the standard MV-semiring* $[0, 1]$ *and the real interval* $[0, 1/2]$ *as a semimodule over it. In this case the semimodule operation is the* max *and the action is given by the MV-algebra multiplication. It is easy to see that* $[0, 1/2]$ *is not a strong semimodule, indeed*

$$\frac{1}{2} \odot x = \frac{1}{3} \odot x = 0 \text{ for every } x \in [0, 1/2]$$

but
$\frac{1}{2}^* \odot x = \frac{1}{2} \odot x = 0$ *for every* $x \in [0, 1/2]$ *and* $\frac{1}{3}^* \odot x = \frac{2}{3} \odot x \neq 0$ *for every* $\frac{1}{3} < x \leqslant \frac{1}{2}$.

We shall conclude the first part of the thesis with some considerations regarding the results we have presented so far.

Characterizing injective and projective semimodules is one of the main question of semimodule theory and hard to address when we refer to semimodules over arbitrary semirings. Restricting our attention to additively idempotent semirings we obtained some interesting results in this direction and considering in particular MV-semirings among additively idempotent semirings we obtained a criterion for self-injectivity of an MV-semiring with an atomic Boolean center, and we gave a complete description of (finitely generated) injective semimodules over a finite MV-semiring. This last description had been later generalized in the context of involutive semirings. Summing up, it seems that considering semirings with additional properties could lead to interesting results regarding injective and projective semimodules. This can be seen as a starting point to investigate semimodules over particular semirings which hopefully shall be proved to be isomoprhic to some interesting algebraic structures and, in particular, to the algebraic semantics of some non-classical logic.

On the other hand, Proposition 5.3 (and its generalization to involutive semirings), in which it is shown that complete Boolean algebras are precisely the MV-semirings in which every principal ideal is injective, gives the intuition that subvarieties of semirings can be characterized in terms of properties of semimodules over them. In particular, the property that every principal ideal is injective as a semimodule characterizes boolean algebras among both MV-semrings and involutive semrings. So, we could wonder if there exists, for example, an assertion regarding semimodules that characterizes precisely MV-algebras among involutive semirings.

So, it seems that semimodule theory could represent a promising yet unexplored tool in the study of algebraic structures which are isomorphic to classes of semirings.

Chapter 7

Ambitropical convexity

The tropical semifield, $\mathbb{R}_{\max}$, is the set $\mathbb{R}\cup\{-\infty\}$ equipped with the addition $(x, y) \mapsto x \vee y := \max(x, y)$ and with the multiplication $(x, y) \mapsto x + y$. The min-plus version of the tropical semifield, $\mathbb{R}_{\min}$ is the set $\mathbb{R}\cup\{+\infty\}$ equipped with the addition $(x, y) \mapsto x \wedge y := \min(x, y)$ and with the multiplication $(x, y) \mapsto x + y$. The semifields $\mathbb{R}_{\max}$ and $\mathbb{R}_{\min}$ are isomorphic.
A simple example of semimodule over $\mathbb{R}_{\max}$ is the n-fold Cartesian product of $\mathbb{R}_{\max}$, $(\mathbb{R}_{\max})^n$ in which the internal law is $(x, y) \mapsto x \vee y := (x_i \vee y_i)_{i\in[n]}$, (where $[n] = \{1, 2, \ldots, n\}$), for $x, y \in (\mathbb{R}_{\max})^n$, and the action of $\mathbb{R}_{\max}$ on $(\mathbb{R}_{\max})^n$ is defined by $(\lambda, x) \mapsto (\lambda + x_i)_{i\in[n]}$, for $\lambda \in \mathbb{R}_{\max}$ and $x \in (\mathbb{R}_{\max})^n$.

We shall now consider the zero-free reduct of the semiring $(\mathbb{R}_{\max})^n$, i. e. $(\mathbb{R}_{\max})^n$ without $\{-\infty\}$. From now on with $\mathbb{R}^n$ we shall denote the zero-free tropical semifield, for any n integer. In this chapter we shall investigate a class of objects called *ambitropical cones*. They are a special class of semimodules over the zero-free semiring $\mathbb{R}$ since they have a lattice structure. In particular we shall prove (Theorem 7.4) that they coincide with the retractions of a class of maps called *Shapley operators*. The interest in *Shapley retractions* is motivated by reasons related to game theory and tropical geometry, in particular to the concept of convexity.
The definitions and results of this chapter come from a paper in preparation with Stephane Gaubert and Marianne Akian. In the first section we shall discuss the motivation of our work. For all the definitions and results regarding tropical geometry and its applications we refer the reader to [47].

7.1 Motivation

Definition 7.1 (Nonexpansive map). We say that a map $T : \mathbb{R}^n \to \mathbb{R}^n$ is *nonexpansive* wrt a norm $\|\cdot\|$ if

$$\|T(x) - T(y)\| \leqslant \|x - y\| \ .$$

Definition 7.2 (Nonexpansive retraction)**.** *Nonexpansive retractions* are nonexpansive maps $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $T^2 = T$.

Definition 7.3 (Euclidean norm)**.** A *Euclidean norm* is a norm arising from a scalar product on a vector space.

The present work is inspired by the following observation from convex analysis.

Theorem 7.1. *A subset $C \subset \mathbb{R}^n$ is closed and convex if and only if it is the image of a nonexpansive retraction in a Euclidean norm.*

Indeed, if C is closed and convex, the map T which associates to $x \in \mathbb{R}^n$ the unique point $y \in C$ which minimizes the Euclidean distance between x and y is nonexpansive wrt this Euclidean distance, see [29, Th. 3.6]. The converse implication in 7.1 is less known, it holds more generally in any strictly convex Banach space, we provide the proof in appendix. Theorem 7.1 raises the issue of studying the sets which arise as images of retractions that are nonexpansive with respect to other families of norms or hemi-norms, thinking these norms will give raise to new structures of convexity.
As regards the link between non-expansive maps and non-linear Perron-Frobenius Theory we refer the reader to [44]and [43].

Definition 7.4 (Hemi-norm)**.** ([50, 27]) A *hemi-norm* is a function f from a real vector space X to $\mathbb{R}$ such that:

1. $f(x + y) \leqslant f(x) + f(y)\ \forall x, y \in X$ (subadditive);
2. $f(\alpha x) = \alpha f(x)\ \forall \alpha \in \mathbb{R}, \alpha \geqslant 0,\ \forall x \in X$.

We will be especially interested in the "top" hemi-norm.

Definition 7.5 ("Top" hemi-norm)**.** The "top" hemi-norm is the map from $\mathbb{R}^n$ to $\mathbb{R}$ defined by

$$\mathbf{t}(x) := \max_{i \in [n]} x_i$$

Hemi-norms are sometimes called weak Minkowski norms [50, 27]. This map is also a special case of the "Funk metric" studied in Hilbert's geometry [50, 55]. The hemi-norm $\mathbf{t}$ is asymmetric since it doesn't satisfy $\mathbf{t}(x) = \mathbf{t}(-x)$ but it can be symmetrized in two ways:

$$x \mapsto \mathbf{t}(x) \vee \mathbf{t}(-x) = \|x\|_\infty := \max_{i \in [n]} |x_i|$$

yields the sup-norm, whereas

$$x \mapsto \mathbf{t}(x) + \mathbf{t}(-x) = \|x\|_H := \max_{i \in [n]} x_i - \min_{j \in [n]} x_j$$

yields a map called *Hopf oscillation* or *Hilbert's semi-norm.* The $\mathbf{t}$ hemi-norm appears in game theory. In this context, one considers *Shapley operators*, i.e. dynamic programming operators for zero-sum games with state space $[n]$, without discount described in the introduction.

Definition 7.6. We denote by $\leqslant$ the partial order of $(\mathbb{R} \cup \{\pm\infty\})^n$, and by e_n the vector of $\mathbb{R}^n$ whose entries are identically 1. We also denote by $\vee$ the supremum of vectors of $(\mathbb{R} \cup \{\pm\infty\})^n$, so that $x \vee y$ is the entrywise maximum of the vectors x and y. Similarly, $\wedge$ denote the infimum of vectors.

Definition 7.7 (Shapley operators)**.** Shapley operators are maps $T : \mathbb{R}^n \to \mathbb{R}^n$ that satisfy

1. $x \leqslant y$ implies $T(x) \leqslant T(y)$;

2. $T(\lambda e_n + x) = \lambda e_n + T(x)$,

where e_n denotes the vector of $\mathbb{R}^n$ whose entries are identically 1.

We shall refer to an (abstract) Shapley operator for a map T satisfying these two axioms. Nonexpansiveness properties of Shapley operators play a fundamental role in the theory of zero-sum games [17]. In particular, it was observed in [32] that T is a Shapley operator if and only if it is nonexpansive in the "$\mathbf{t}$" hemi-norm:

$$\mathbf{t}(T(x) - T(y)) \leqslant \mathbf{t}(x - y) \ .$$

Maps which are nonexpansive in the "$\mathbf{t}$" hemi-norm are called *topical functions* ([45]). Shapley operators, the $\mathbf{t}$ hemi-norm, and Hilbert's semi-norm play a key role in tropical geometry [4], in particular, these weak norms provide the tropical analogous of classical Euclidean structure, the projection in the sense of Hilbert's semi-norm being an analogous of the Euclidean projection.

Definition 7.8. We shall say that T is a *Shapley retraction* if T is a Shapley operator and $T = T^2$. The image of a Shapley retraction T is a *Shapley retract.*

In view of the latter remarks, a Shapley retraction is precisely a nonexpansive retraction in the $\mathbf{t}$ weak norm. In view of 7.1, we may ask the following question:

Question 7.1. *How can we characterize the sets which arise as images of nonexpansive retractions with respect to the* $\mathbf{t}$ *weak-norm?*

7.2 Basic notions

Definition 7.9. A *tropical cone* of $\mathbb{R}^n$ is a subset C of $\mathbb{R}^n$ such that

1. $x, y \in C \implies x \vee y \in C$
2. $x \in C, \lambda \in \mathbb{R} \implies \lambda e_n + x \in C$.

A *dual tropical cone* is defined similarly, by requiring that (2) holds and, that $x, y \in C \implies x \wedge y \in C$, instead of (1).

Definition 7.10 (Root system)**.** Let E be a finite-dimensional Euclidean vector space. A *root system* Φ in E is a finte set of non-zero vectors (called *roots*) that satisfy the following conditions:

1. The roots span E;
2. the only scalar multiples of a root $\alpha \in \Phi$ that belong to Φ are α itself and $-\alpha$;
3. for every root $\alpha \in \Phi$, the set Φ is closed under reflection through the hyperplane perpendicular to α;
4. if α and β are roots in Φ, then the projection of β onto the line through α is an integer or half-integer multiple of α.

Example 7.1. An important class of tropical cones and dual tropical cones consists of alcoved polyhedra. The latter were introduced in [42]: in general, an acolved polyhedron associated to a root system is a polyhedron whose facets have normals that are proportional to vectors of this root system. Here, the root system is A_n, the collection of vectors $\{e_i - e_j \mid i, j \in [n], i \neq j\}$. Note that here the vector e_i and e_j are the vector of the canonical basis of $\mathbb{R}^n$ and not the vectors whose entries are identically 1.

Definition 7.11. An *alcoved polyhedron* [42] is a polyhedron of the form

$$\mathcal{A}(M) = \{x \in \mathbb{R}^n \mid x_i \geqslant M_{ij} + x_j, \quad \forall 1 \leqslant i, j \leqslant n\} \tag{7.1}$$

for some matrix $M = (M_{ij}) \in (\mathbb{R}_{\max})^{n\times n}$.

Definition 7.12. *Order polyhedra* are remarkable examples of alcoved polyhedra. These are of the form $\{x \in \mathbb{R}^n \mid x_i \geqslant x_j \text{ if } (i, j) \in E\}$ where $E \subset [n]\times[n]$ is a partial order relation on the set $[n]$.

Definition 7.13. Intersection of order polyhedra with the hypercube $[0, 1]^n$ are known as *order polytopes*, they were studied by Stanley [54].

Definition 7.14 (Operations on matrices)**.** We shall denote by $\vee$ the tropical addition of matrices, so that, for all $A, B \in (\mathbb{R}_{\max})^{m\times n}$, $(A_{ij}) \vee (B_{ij}) := (A_{ij} \vee B_{ij}) \in (\mathbb{R}_{\max})^{m\times n}$. The tropical multiplication of matrices will be denoted by concatenation, i.e, for $A \in (\mathbb{R}_{\max})^{m\times n}$ and $B \in (\mathbb{R}_{\max})^{n\times p}$, $AB \in (\mathbb{R}_{\max})^{m\times p}$ is the matrix with (i, j)-entry $\vee_{k\in[n]}(A_{ik} + B_{kj})$. Then, when $m = n$, for all r, the *rth tropical power* of A is denoted by $A^r := A \cdots A$ (A is repeated r times).

There are well known relations between alcoved polyhedra and operations of metric closures which we next recall.

Definition 7.15. The *tropical Kleene star* of a matrix M is defined by

$$M^* = I \vee M \vee M^2 \vee \cdots \ ,$$

This supremum may be infinite, i.e., in general $(M^*)_{ij}$ may take the value $+\infty$.

Definition 7.16. A *directed graph* (or *digraph*) is an ordered pair $G = (V, A)$ where V is the set of the *vertices* of the graph and A is a set of ordered pairs of vertices, called *edges*.

A graph is *weighted* if a number (the weight) is associated to each edge.

A *path* of a graph is a finite or infinite sequence of edges which join a sequence of vertices; a *circuit* is a path for which the initial and the final vertices coincide.

The *weight of a path* is the sum of the weights of the traversed edges.

Recall that to a matrix $M \in (\mathbb{R}_{\max})^{n\times n}$ it is possible to associate a weighted digraph in the following way. The set of nodes is $[n]$ and the set of edges is given by $\{(i, j) \in [n]^2 \mid M_{ij} > -\infty\}$; an edge $i \to j$ has weight M_{ij}.

From Prop 2.2 of [5] we have that $(M^n)_{ij}$ yields the maximal weight of a path of length n from i to j, and $(M^*)_{ij}$ yields the supremum of the weights of paths from i to j, of arbitrary length. We have $(M^*)_{ij} < +\infty$ for all i, j if and only if there is no circuit with positive weight in the digraph associated to M.

Definition 7.17. The *critical circuits* of the matrix M^* are the circuits in the digraph associated to M^* with weight 0.

Definition 7.18. The union of the critical circuits consitutes the *critical digraph.*

The following result is a special case of the characterization of the eigenspace in max-plus spectral theory, see e.g. [6, 5]. We provide details for completeness.

Lemma 7.1. *The polyhedron $\mathcal{A}(M)$ is non-empty if and only there is no circuit of positive weight in the digraph associated to M. Then,*

$$M^* = M^0 \vee \cdots \vee M^{n-1} \tag{7.2}$$

where n is the number of nodes of the graph associated to M, and

$$\mathcal{A}(M) = \{M^* y \mid y \in \mathbb{R}^n\} = \{x \in \mathbb{R}^n \mid M^* x \leqslant x\} = \{x \in \mathbb{R}^n \mid M^* x = x\} \ . \tag{7.3}$$

Proof.

Claim 7.1. The polyhedron $\mathcal{A}(M)$ is non-empty if and only there is no circuit of positive weight in the digraph associated to M

Proof. If $x \in \mathcal{A}(M)$, summing the relations $x_{i_2} \geqslant M_{i_2 i_1} + x_{i_1}$, $x_{i_3} \geqslant M_{i_3 i_2} + x_{i_2}$, $\dots x_{i_1} \geqslant M_{i_1 i_k} + x_{i_k}$ in 7.1 and cancelling $x_{i_1} + \cdots + x_{i_k}$, we deduce that $0 \geqslant M_{i_1 i_k} + \cdots + M_{i_3 i_2} + M_{i_2 i_1}$, which shows that the circuit $(i_k \to \cdots \to i_2 \to i_1 \to i_k)$ has a nonpositive weight. The same argument can be applied to any other circuit in the diagraph associated to M and so we obtain that there is no circuit of positive weight. If there is no circuit of positive weight, the maximum of the weight of a path from i to j is achieved by a path of length at most $n-1$, which gives (7.2). This formula implies in particular that no entry of M^* is equal to $+\infty$. Note also that the diagonal entries of M^* are equal to 0. We deduce that the vector $x := M^* e$, where concatenation denotes the tropical matrix-vector product, is finite. Since $MM^* = \vee_{k \geqslant 1} M^k \leqslant M^* = \vee_{k \geqslant 0} M^k$, we deduce that $Mx \leqslant x$, and so, $x \in \mathcal{A}(M)$. □

This shows the first part of the proposition.

We now show the second part.

Claim 7.2. The following assertions are equivalent

i. $Mx \leqslant x$

ii. $M^* x \leqslant x$

iii. $M^* x = x$

Proof. *(i)⇒(ii)* Indeed, the inequality $Mx \leqslant x$ implies $M^k x \leqslant x$ for all k, and taking the sup we get $M^* x \leqslant x$, hence the implication.

(ii)⇒(iii) Since $M^* \geqslant I$.

(iii)⇒(i) Since $M^* \geqslant M$.

It follows that $\mathcal{A}(M) = \{x \in \mathbb{R}^n \mid M^*x \leqslant x\} = \{x \in \mathbb{R}^n \mid M^*x = x\} \subset \{M^*y \mid y \in \mathbb{R}^n\}$. Moreover, since $MM^* \leqslant M^*$, and from the fact that M^*y is finite for all $y \in \mathbb{R}^n$, because there is no circuit of positive weight in $G(M)$, $\{M^*y \mid y \in \mathbb{R}^n\} \subset \mathcal{A}(M)$. □

□

Definition 7.19. A *topological semimodule* over a semiring S is a semimodule with a topology such that the semimodule operation $+ : M \times M \to M$ and the semiring multiplications $\{f_s\}_{s\in S} : M \to M$ are continuos with respcect to the topology of M, where on $M \times M$ we consider the Tychonoff topology induced by the topology of M.

We equip $\mathbb{R}_{\max}$ with the topology defined by the metric

$$(a, b) \mapsto |e^a - e^b|$$

The semimodule $(\mathbb{R}_{\max})^n$, equipped with the topology of the metric $d_\infty(x, y) = \max_{i,j\in[n]} |e^{x_i} - e^{x_j}|$ is a topological semimodule. Observe that the induced topology on $\mathbb{R}^n \subset (\mathbb{R}_{\max})^n$ is the Euclidean topology. Dual considerations apply to $(\mathbb{R}_{\min})^n$. Note that here we indicate with "e" the mathematical constant known as Euler's number.

Definition 7.20. Given a subset $C \subset \mathbb{R}^n$, we define the *lower closure* of $\mathrm{clo}^{\downarrow} C \subset (\mathbb{R}_{\max})^n$ to be the set of limits of nonincreasing sequences of elements of C. Similarly, we define the *upper closure* $\mathrm{clo}^{\uparrow} C$ to be the set of limits of nondecreasing sequences of elements of C.

Example 7.2. For instance, if $C = \{x \in \mathbb{R}^2 \mid |x_1 - x_2| \leqslant 1\}$, $\mathrm{clo}^{\downarrow} C = C \cup \{(-\infty, -\infty)\}$, whereas if $C = \{x \in \mathbb{R}^2 \mid x_1 \geqslant x_2\}$, $\mathrm{clo}^{\downarrow} C = \{x \in (\mathbb{R}_{\max})^2 \mid x_1 \geqslant x_2\}$.

Definition 7.21. The *support* of a vector $y \in (\mathbb{R}_{\max})^n$ is $\operatorname{supp} y := \{i \in [n] \mid y_i > -\infty\}$.

The following proposition shows that closed tropical cones are in one-to-one correspondence with closed tropical semimodules that contain vectors with finite entries.

Proposition 7.1. *The map $C \mapsto V := \mathrm{clo}^{\downarrow} C$ establishes a bijective correspondence between the nonempty closed tropical cones $C \subset \mathbb{R}^n$ and the closed subsemimodules V of $(\mathbb{R}_{\max})^n$ such that $V \cap \mathbb{R}^n \neq \emptyset$. The inverse map is given by $V \mapsto V \cap \mathbb{R}^n$.* □

The dual property, concerning $\mathrm{clo}^{\uparrow} C$ and $(\mathbb{R}_{\min})^n$ instead of $\mathrm{clo}^{\downarrow} C$ and $(\mathbb{R}_{\max})^n$, also holds.

Proof.

Claim 7.3. Any closed tropical cone contained in $\mathbb{R}^n$ is a closed subsemimodule of $(\mathbb{R}_{\max})^n$.

Proof. Suppose $C \subset \mathbb{R}^n$ is a closed tropical cone. Since $(\mathbb{R}_{\max})^n$ is a topological semimodule, $\mathrm{clo}^{\downarrow} C$ is a semimodule over $\mathbb{R}_{\max}$. We next show that $\mathrm{clo}^{\downarrow} C$ is closed. Let x^k denote a sequence of elements of $\mathrm{clo}^{\downarrow} C$ converging to an element $x \in (\mathbb{R}_{\max})^n$. Let ϵ_k denote an arbitrary sequence of positive numbers decreasing to 0. Let $y^k := x^k + \epsilon_k e \in \mathrm{clo}^{\downarrow} C$. Observe that $\eta_k := d_\infty(x^k, y^k) = (e^{\epsilon_k} - 1) \max_{i\in[n]} e^{x_i^k} \to 0$ as $k \to \infty$. Moreover, by definition of $\mathrm{clo}^{\downarrow} C$, we can find $z^k \in C$ such that $Y^k \leqslant z^k$ and $d_\infty(y^k, z^k) \leqslant \eta_k$. It follows that the sequence z^k also converges to x. We claim that there is a subsequence z^{n_k} of z^k that is nonincreasing. Indeed, suppose by induction that $z^{n_1} \geqslant \cdots \geqslant z^{n_k}$ have already been selected. Let $I := \mathrm{supp}\, x$. Since $z^l \to x$ as $l \to \infty$, we get $z_i^l \to x_i \in \mathbb{R}$ for $i \in [n]$, and $z_i^l \to -\infty$ for $i \in [n] \setminus I$. However, by construction, $z^{n_k} \geqslant \epsilon_{n_k} e + x$, and so $z_i^{n_k} \geqslant \epsilon_{n_k} + x_i$ for all $i \in I$. We deduce that there is an index l such that $z^l \leqslant z^{n_k}$, and we set $n_{k+1} := l$. This shows that $x \in \mathrm{clo}^{\downarrow} C$, and so, $\mathrm{clo}^{\downarrow} C$ is closed. □

Claim 7.4. Reverse inclusion

Proof. Conversely, suppose that V is a closed subsemimodule of $(\mathbb{R}_{\max})^n$. Then, it is immediate to see that $V \cap \mathbb{R}^n$ is a closed tropical cone. □

Claim 7.5. The correspondence is bijective

Proof. It remains to show that the correspondence is bijective. We have trivially $\mathrm{clo}^{\downarrow} C \cap \mathbb{R}^n = C$ for all nonempty closed tropical cones $C \subset \mathbb{R}^n$. Conversely, if V is a closed tropical semimodule of $(\mathbb{R}_{\max})^n$ such that $V \cap \mathbb{R}^n$ is nonempty, we must show that $V = \mathrm{clo}^{\downarrow}(V \cap \mathbb{R}^n)$. Let $x \in V$, and let us choose an arbitrary element $y \in V \cap \mathbb{R}^n$. Then, the path $\lambda \mapsto \gamma(\lambda) := x \vee (\lambda e + y)$, defined for $\lambda \in [-\infty, 0]$ is such that $\gamma(\lambda) \in V \cap \mathbb{R}^n$ for all $\lambda > -\infty$, and $\inf_{\lambda > -\infty} \gamma(\lambda) = x$. It follows that $x \in \mathrm{clo}^{\downarrow}(V \cap \mathbb{R}^n)$, hence, $V \subset \mathrm{clo}^{\downarrow}(V \cap \mathbb{R}^n)$. The other inclusion is immediate. □

□

Definition 7.22. Recall that an element u of a tropical subsemimodule $A \subset (\mathbb{R}_{\max})^n$ is an *extreme generator* of A if $u = v \vee w$ with $v, w \in A$ implies that $u = v$ or $u = w$.

Definition 7.23. A *tropical linear combination* of elements of A is a vector of the form $\vee_{i\in I}(\lambda_i + a_i)$ where $(\lambda_i)_{i\in I} \subset \mathbb{R}_{\max}$ and $(a_i)_{i\in I} \subset A$ are finite families.

Definition 7.24. We say that $G \subset A$ is a *tropical generating set* if every element of A is a tropical linear combination of a family of elements of G.

Definition 7.25. We say also that two vectors are *tropically proportional* if they differ by an additive constant.

The next result summarizes results from [26, 10]; it shows that a closed tropical subsemimodule of $(\mathbb{R}_{\max})^n$ is generated by its extreme rays.

Theorem 7.2 (See Theorem 3.1 in [26] or Theorem 14 in [10])**.** *Suppose that A is a closed tropical subsemimodule of $(\mathbb{R}_{\max})^n$. Then, every element of A is a tropical linear combination of at most n extreme generators of A. Moreover, these extreme generators are characterized as follows. For all $i \in [n]$, let $A_i := \{x \in A \mid x_i = 0\}$, and let $\operatorname{Min} A_i$ denote the set of minimal elements of A_i. Then, every extreme generator of A is tropically proportional to an element of $\cup_{i\in[n]} \operatorname{Min} A_i$.*

This is reminiscent of the classical Carathéodory theorem for closed convex pointed cones that we shall recall. Note that the following theorem comes from classical geometry so, only in this case, $\mathbb{R}$ is the classical ring of real numbers.

Theorem 7.3 (Caratheodory theorem (classical geometry))**.** *If a point $x \in \mathbb{R}^d$ lies in the convex hull of a set P, then x can be written as the convex combination of at most $d+1$ points in P.*

7.3 Shapley operators

Definition 7.26 (Order additive cone)**.** An *ordered additive cone* is a partially ordered set $(E, \leqslant)$, equipped with an action of $(\mathbb{R}, +)$, $(\lambda, x) \mapsto \lambda + x$, which is such that:

1. $x \leqslant y \implies \lambda + x \leqslant \lambda + y$,
2. $\lambda \leqslant \mu \implies \lambda + x \leqslant \mu + x$, for all $x, y \in E$ and $\lambda, \mu \in \mathbb{R}$.

Definition 7.27. A *additive cone of* $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$, equipped with the order induced by the standard partial order on $\mathbb{R}^n$, and with the action $(\lambda, x) \mapsto \lambda e_n + x$.

Note that an additive cone of $\mathbb{R}^n$ which is closed by taking suprema is a semimodule over the zero-free semiring $\mathbb{R}$. Indeed it is straightforward to see that it respects all the properties of the definition of semimodule except, obviously, the one involving the zero of the semiring. However we are interested in the study of additive cones of $\mathbb{R}^n$ considering a broader class of functions than the one of morphisms between semimodules. Indeed when dealing with tropical semimodules, the canonical choice of maps to consider consists of tropically linear maps (maps that commute with the supremum and with the addition of a constant, i.e., morphisms of semimodules) but since, for the reasons explained in the first section, we are interested in considering *Shapley operators* and we ignore the lattice structure of $\mathbb{R}^n$ and consider it as just a partially ordered set.

The following observation, made in [32, Prop. 1.1], shows that Shapley operators are characterized by a nonexpansiveness property. Recall that $\mathbf{t}(x) := \max_{i\in[n]} x_i$ denotes the "top" hemi-norm of a vector $x \in \mathbb{R}^n$. It will also be convenient to use the notation $\mathbf{b}(x) := -\mathbf{t}(-x) = \min_{i\in[n]} x_i$.

Proposition 7.2 ([32])**.** *Let $T : \mathbb{R}^n \to \mathbb{R}^n$. The following assertions are equivalent:*

i. T is a Shapley operator;

ii. $\mathbf{t}(T(x) - T(y)) \leqslant \mathbf{t}(x - y)$ for all $x, y \in \mathbb{R}^n$;

iii. $\mathbf{b}(T(x) - T(y)) \geqslant \mathbf{b}(x - y)$ for all $x, y \in \mathbb{R}^n$.

This fact will be used systematically and so we provide the proof for completeness. We shall prove the equivalence between i and ii, the equivalence between i and iii is straightforward.

Proof.

Claim 7.6 (i ⇒ ii).

Proof. Observe first that $x \leqslant y + \mathbf{t}(x - y)e_n$ holds for all $x, y \in \mathbb{R}^n$. Hence, if T is a Shapley operator, we deduce that $T(x) \leqslant T(y) + \mathbf{t}(x - y)e_n$, and so $\mathbf{t}(T(x) - T(y)) \leqslant \mathbf{t}(x - y)$, showing the implication. □

Claim 7.7 (ii ⇒ i).

Proof. Conversely, suppose that *(ii)* holds. Consider $y = x + \lambda e_n$ with $\lambda \in \mathbb{R}$. Then, $\mathbf{t}(T(x) - T(y)) \leqslant \mathbf{t}(x - y) = -\lambda$, and $\mathbf{t}(T(y) - T(x)) \leqslant \mathbf{t}(y - x) = \lambda$. Hence, $T(x) - T(y) \leqslant -\lambda e_n$, and $T(y) - T(x) \leqslant \lambda e_n$, implying that $T(y) = T(x) + \lambda e_n$. Thus, property (2) of 7.7 holds.

Suppose now that $x \leqslant y$. Then $\mathbf{t}(T(x) - T(y)) \leqslant \mathbf{t}(x - y) \leqslant 0$, showing that $T(x) \leqslant T(y)$. Thus, property (1) of 7.7 holds. □

□

Proposition 7.3. *A Shapley operator $T : \mathbb{R}^n \to \mathbb{R}^n$ admits a unique continuous extension $T_- : (\mathbb{R}_{\max})^n \to (\mathbb{R}_{\max})^n$, given by*

$$T_-(x) = \inf\{T(y) \mid y \geqslant x,\ y \in \mathbb{R}^n\} \enspace .$$

Similarly, T has a unique continuous extension $(\mathbb{R}_{\min})^n \to (\mathbb{R}_{\min})^n$, given by

$$T_+(x) = \sup\{T(y) \mid y \leqslant x,\ y \in \mathbb{R}^n\} \enspace .$$

Proof. Theorem 3.10 of [9] shows that any order preserving homogeneous map f from $\mathbb{R}^n_{>0}$ to $\mathbb{R}^n_{>0}$ has a unique continuous extension from $\mathbb{R}^n_{\geqslant 0}$ to $\mathbb{R}^n_{\geqslant 0}$, given by $f(x) = \inf\{f(y) \mid y \geqslant x,\ y \in \mathbb{R}^n_{>0}\}$. The statement of the proposition is gotten by applying this theorem to the maps $f(x) = \exp T \log(x)$ and $f(x) = \exp(-T(-\log x))$, the operations log and exp being understood entrywise. □

Remark 7.1. 7.3 provides only one-sided extensions of T, either to $(\mathbb{R} \cup \{-\infty\})^n$ or to $(\mathbb{R} \cup \{+\infty\})^n$. A Shapley operator defined on $\mathbb{R}^n$ generally does not extend canonically to $(\mathbb{R} \cup \{\pm\infty\})^n$ (consider $n = 2$ and $T(x) = ((x_1 + x_2)/2)e$).

Definition 7.28. A Shapley operator is said to be *tropically linear* if $T(x \vee y) = T(x) \vee T(y)$ holds for all $x, y \in \mathbb{R}^n$.
It is said to be *dually tropically linear* if $T(x \wedge y) = T(x) \wedge T(y)$.

7.4 Ambitropical Cones

Definition 7.29 (Conditionally complete lattice)**.** A lattice L is said *conditionally complete* if every nonempty subset of L that has an upper bound has a join (a least upper bound), and if every nonempty subset of L that has a lower bound has a meet (a greatest lower bound).

Definition 7.30. An *ambitropical cone* is a non-empty subset C of $\mathbb{R}^n$ such that:

1. C is a lattice in the induced order;
2. for all $x \in C$ and $\lambda \in \mathbb{R}$, $\lambda e_n + x \in C$.

Definition 7.31. An *ambitropical polyhedron* is an ambitropical cone that is a finite union of alcoved polyhedra.

Thus, a subset $C \subset \mathbb{R}^n$ is an ambitropical polyhedron if it is an ambitropical cone that can be written as a finite union

$$C = \bigcup_{k \in [K]} \mathcal{A}(M_k)$$

for some matrices $M_1, \dots, M_K \in (\mathbb{R}_{\max})^{n \times n}$. Observe that an ambitropical polyhedron is closed.

In the second appendix we shall present a graphic example of an ambitropical polyhedron.

In this section we shall prove the main results of the chapter, i. e. the characterization of Shapley retracts of $\mathbb{R}^n$ in terms both of lattice theoretic properties and geometric properties.

Lemma 7.2. *Let C be an ambitropical cone. A subset of $\mathbb{R}^n$ is bounded from above by an element of $\mathbb{R}^n$ if and only if it is bounded from above by an element of C. The dual statement holds for subsets bounded from below.*

Proof. ($\Rightarrow$) Let X be a subset of $\mathbb{R}^n$ bounded by an element of $\mathbb{R}^n$, i. e. there exists $u \in \mathbb{R}^n$ such that $x \leqslant u$ holds for all $x \in X$. Let y be an arbitrary element of the ambitropical cone C and consider the element $z := y + \mathbf{t}(u - y)e_n$. Since C is a cone and $y \in C$ we have that $z \in C$, therefore $u \leqslant z$ which implies $x \leqslant z$ for all $x \in X$ and the implication follows.
($\Leftarrow$) the converse implication is obvious.
The dual statement holds for subsets bounded from below considering the hemi-norm $\mathbf{b}$. □

We shall denote by $\sup^C X$ the supremum of a non-empty subset X of C that has an upper bound when this supremum exists. Similarly, we shall denote by $\inf^C X$ the infimum of a non-empty subset X of C that has a lower bound when this infimum exists. We shall also use the infix notation $\vee^C, \wedge^C$ for the lattice laws of $(C, \leqslant)$, i.e.,

$$x \vee^C y = \inf{}^C\{z \in C \mid z \geqslant x,\ z \geqslant y\}, \quad x \wedge^C y = \sup{}^C\{z \in C \mid z \leqslant x,\ z \leqslant y\},$$

for all $x, y \in C$. It is essential to note that the lattice laws of C may differ from the lattice laws $\vee$ and $\wedge$ of $(\mathbb{R}^n, \leqslant)$.

Proposition 7.4. *An ambitropical cone is a conditionally complete lattice if and only if it is closed in the Euclidean topology.*

Proof.

Claim 7.8 ($\Leftarrow$).

Proof. Suppose that the ambitropical cone C is closed in the Euclidean topology, and let $X \subset C$ be a nonempty set bounded from above by some element $y \in C$. Let $\mathcal{P}_f(X)$ denote the set of nonempty finite subsets of X. For all $F \in \mathcal{P}_f(X)$, let $u_F := \sup^C F$. Then, $(u_F)_{F \in \mathcal{P}_f(X)}$ is a nondecreasing net of elements of C, bounded from above by y. Since C is closed in the Euclidean topology, the limit of a net of elements of C belongs to C, and so $u := \lim_F u_F \in C$. By construction, $u \geqslant x$ holds for all $x \in X$. Moreover, if $z \in C$ is an upper bound of X, we get $z \geqslant u_F$ for all $F \in \mathcal{P}_f(X)$, and so $z \geqslant u$. This shows that u is the least upper bound of X. A dual argument works for greatest lower bounds. Hence, C is a conditionally complete lattice. □

Claim 7.9 ($\Rightarrow$).

Proof. Conversely, suppose that C is a conditionally complete lattice. Observe that for every bounded sequence (x_k) of elements of C, the following "liminf" and "limsup" constructions both define elements that belong to C:

$$\operatorname{limsup}^C_{k\to\infty} x_k := \inf^C_{k\geqslant 1}\sup^C_{\ell\geqslant k} x_\ell, \qquad \operatorname{liminf}^C_{k\to\infty} x_k := \sup^C_{k\geqslant 1}\inf^C_{\ell\geqslant k} x_\ell \ .$$

We shall use the fact that $\operatorname{limsup}^C_{k\to\infty} x_k \geqslant \operatorname{liminf}^C_{k\to\infty} x_k$. This inequality, which is standard when $C=\mathbb{R}^n$, is still valid in general. Indeed, for all $k,m\geqslant 1$, we have $\sup^C_{\ell\geqslant k} x_\ell \geqslant \inf^C_{\ell\geqslant m} x_\ell$, and so

$$\sup^C_{\ell\geqslant k} x_\ell \geqslant \sup^C_{m\geqslant 1}\inf_{\ell\geqslant m} x_\ell = \operatorname{liminf}^C_{r\to\infty} x_r .$$

Hence,

$$\operatorname{limsup}^C_{k\to\infty} x_k = \inf^C_{k\geqslant 1}\sup^C_{\ell\geqslant k} x_\ell \geqslant \operatorname{liminf}^C_{r\to\infty} x_r \ . \tag{7.4}$$

Suppose that the sequence $(x_k)_{k\geqslant 1}$ of elements of C converges to $x\in\mathbb{R}^n$. Then, for all $\epsilon>0$, there exists an index m such that $\|x_\ell - x\|\leqslant\epsilon$ for all $\ell\geqslant m$. In particular, $x_\ell \leqslant x_m + \|x_\ell - x_m\| e \leqslant x_m + 2\epsilon e$. We deduce that $\operatorname{limsup}^C_{\ell\to\infty} x_\ell \leqslant x_m + 2\epsilon e \leqslant x + 3\epsilon e$. Since the latter inequality holds for all $\epsilon>0$, we deduce that $\operatorname{limsup}^C_{\ell\to\infty} x_\ell \leqslant x$. A dual argument shows that $\operatorname{liminf}^C_{\ell\to\infty} x_\ell \geqslant x$. Using (7.4), we conclude that $x = \operatorname{limsup}^C_{\ell\to\infty} x_\ell = \operatorname{liminf}^C_{\ell\to\infty} x_\ell \in C$, showing that C is closed in the Euclidean topology. □

□

Remark 7.2. In the sequel, when writing that an ambitropical is *closed*, we shall always refer to the Euclidean topology.

Definition 7.32. We define, for all closed ambitropical cones C, and for all $x\in\mathbb{R}^n$, the following "projection" maps:

$$Q^-_C(x) := \sup{}^C\{y\in C \mid y\leqslant x\} \ , \qquad Q^+_C(x) := \inf{}^C\{y\in C\mid y\geqslant x\} \ . \tag{7.5}$$

We shall denote by $\operatorname{Im} f := \{f(x)\mid x\in X\}$ the *image* or *range* of a map $f: X\to Y$.

Proposition 7.5. *Suppose that C is a closed ambitropical cone of $\mathbb{R}^n$. Then, Q^-_C is an idempotent Shapley operator, i.e., $(Q^-_C)^2 = Q^-_C$, and the range of Q^-_C is C. The same is true for Q^+_C.*

Proof. Since C is conditionally complete, for all $x\in\mathbb{R}^n$, $Q^-_C(x)$ is well defined and $Q^-_C(x)\in C$. Moreover, Q^-_C trivially fixes C, implying that $\operatorname{Im} Q^-_C = C$ and $(Q^-_C)^2 = Q^-_C$. We also have, for all $x\in\mathbb{R}^n$ and $\lambda\in\mathbb{R}$, $Q^-_C(\lambda e_n + x) = \sup^C\{y\in C\mid y\leqslant \lambda e_n + x\} = \sup^C\{y\in C\mid -\lambda e_n + y\leqslant x\} = \sup^C\{\lambda e_n + z\in C\mid z\leqslant x\} = \lambda e_n + Q^-_C(x)$. The operator Q^-_C is trivially order preserving, hence it is a Shapley operator. Dual arguments applies to Q^+_C. □

If C is a closed tropical cone, then C is closed under tropical linear combinations, and it is also closed under taking the supremum of nondecreasing sequences, it follows that the supremum $\sup^C$ relative to C coincides with the supremum law sup of $\mathbb{R}^n$. We deduce the following result.

Lemma 7.3. *If C is a closed tropical cone, then $Q_C^- \leqslant I$. Similarly, if C is a closed dual tropical cone, then $Q_C^+ \geqslant I$.*

Proof. If C is a closed tropical cone, then for all $x \in \mathbb{R}^n$, $Q_C^-(x) = \sup^C\{y \in C \mid y \leqslant x\} = \sup\{y \in C \mid y \leqslant x\} \leqslant x$. The proof of the property concerning Q_C^+ is dual. □

Consider the set of additive cones of $\mathbb{R}^n$. First of all, we can consider additive cones of $\mathbb{R}^n$ closed by taking suprema which are, as already observed, semimodules over the zero - free semiring $\mathbb{R}$ (it is shown in [46] and [13] that appropriate tropical analogous of Hilbert's spaces are obtained by considering spaces that are closed by taking suprema).
If we consider order additive cones of $\mathbb{R}^n$ which are conditionally complete as lattices and that have the zero (in this case $-\infty$) we obtain conditionally complete semimodules over $(\mathbb{R}_{\max})^n$, (this reminds of the notion of b-complete idempotent space in [46]).
Allowing inconditional suprema leads to the notion of complete semimodules over $(\mathbb{R}_{\max})^n$ [13].

Hence, we shall perform a (one sided) conditional completion which allows us to construct from an arbitrary nonempty subset of $\mathbb{R}^n$ a closed tropical cone and then, by taking the lower closure of it, a closed subsemimodule of $(\mathbb{R}_{\max})^n$.

If E is a nonempty subset of $\mathbb{R}^n$, we shall denote by $E^{\max}$ the set of tropical linear combinations of *infinite* families of elements of E, i.e., the set of elements of the form

$$\sup\{\lambda_f e_n + f \mid f \in E\} \tag{7.6}$$

where the $\lambda_f \in \mathbb{R}_{\max}$ such that the family of elements $(\lambda_f e_n + f)_{f\in E}$ is bounded from above and the λ_f are not identically $-\infty$. Up to the adjunction of a bottom element, the set $E^{\max}$ is the b-complete idempotent space generated by E in the sense of [46].

We shall also need to consider the $\mathbb{R}_{\max}$-semimodule obtained by taking the *lower closure* of $E^{\max}$, a notion already introduced in 7.20:

$$\bar{E}^{\max} := \mathrm{clo}^{\downarrow} E^{\max} \quad .$$

Similarly, we shall denote by $E^{\min}$ the set of elements of the form $\inf\{\lambda_f e_n + f \mid f \in E\}$ where the $\lambda_f \in \mathbb{R}_{\min}$ are such that the family of elements

$(\lambda_f e_n + f)_{f\in E}$ is bounded from below, and the λ_f are not identically $+\infty$.. We also set $\bar{E}^{\min} := \mathrm{clo}^{\uparrow} E^{\min}$.

Proposition 7.6. *Let $C \subset \mathbb{R}^n$ be an additive cone. Then, the following statements are equivalent:*

1. *C is closed;*
2. *C is stable by limits of bounded nondecreasing sequences;*
3. *C is stable by limits of bounded nonincreasing sequences.*

Proof.

Claim 7.10 (1⇒ 2).

Proof. trivial. □

Claim 7.11 (2)⇒ 3).

Proof. Let x_k be a bounded nonincreasing sequence of elements of C converging to $x \in \mathbb{R}^n$. Consider the sequence $y_k := x_k - 2\|x - x_k\|_\infty e \in C$. We have $y_k \leqslant -\|x - x_k\|_\infty e + x$. Moreover, y_k also converges to x. It follows that for all k, we can find an index $l \geqslant k$ such that $y_l \geqslant y_k$. Hence, we can construct a nondecreasing subsequence y_{n_k} converging to x. Applying (2), we conclude that $x \in C$. □

Claim 7.12 (3⇒1).

Proof. Suppose x_k is a sequence of elements of C converging to $x \in \mathbb{R}^n$. Consider now $y_k := x_k + 2\|x - x_k\|_\infty e$. Then, arguing as in the proof of the previous implication, we deduce that we can construct a nonincreasing subsequence y_{n_k} still converging to x. Applying (3), we conclude that $x \in C$. □

□

Corollary 7.1. *Let E denote a non-empty subset of $\mathbb{R}^n$. Then, $E^{\max}$ is a closed tropical cone. Similarly, $E^{\min}$ is a closed dual tropical cone.*

Proof. By definition, $E^{\max}$ is a tropical cone. Let us consider a bounded nondecreasing sequence $x_k \in E^{\max}$. We can write $x_k = \sup\{\lambda_f^k e_n + f \mid f \in E\}$ where for each k, the family $(\lambda_f^k)_{f\in F}$ is not identically $-\infty$. Let $x := \lim_k x_k = \sup_k x_k \in \mathbb{R}^n$. From $\lambda_f^k + f \leqslant x_k \leqslant x$, we deduce that $\lambda_f^k \leqslant \mathbf{b}(x - f)$. So, the sequence $(\lambda_f)_{k\geqslant 1}$ is bounded from above. It follows that $\lambda_f := \sup\{\lambda_f^k \mid k \geqslant 1\} < +\infty$. Moreover, using the associativity of the supremum operation, we get $x = \sup x_k = \sup\{\lambda_f e_n + f \mid f \in E\} \in E^{\max}$. Hence, $E^{\max}$ is stable under limits of nondecreasing sequences. It follows from 7.6 that $E^{\max}$ is closed in the Euclidean topology. A dual argument applies to $E^{\min}$.

□

Corollary 7.2. *Let E denote a non-empty subset of $\mathbb{R}^n$. Then, $\bar{E}^{\max}$ is a closed subsemimodule of $(\mathbb{R}_{\max})^n$. Similarly, $\bar{E}^{\min}$ is a closed subsemimodule of $(\mathbb{R}_{\min})^n$*

Proof. By definition, $E^{\max}$ is a tropical cone, and by 7.1, it is a closed subset of $\mathbb{R}^n$. Then, it follows from 7.1 that $\bar{E}^{\max} = \mathrm{clo}^{\downarrow} E^{\max}$ is a closed subsemimodule of $(\mathbb{R}_{\max})^n$. □

Definition 7.33. For all nonempty subsets E of $\mathbb{R}^n$, and for all $x \in \mathbb{R}^n$, we define

$$P_E^{\max}(x) := \sup\{y \in E^{\max} \mid y \leqslant x\} \ , \qquad P_E^{\min}(x) := \inf\{y \in E^{\min} \mid y \geqslant x\} \ .$$

This is a specialization of the notion of projectors Q_C^- and Q_C^+ to $C = E^{\max}$ or $C = E^{\min}$, introduced in (8.1). Indeed, if $C = E^{\max}$, the operation $\sup^C$ coincides with the ordinary supremum sup of $\mathbb{R}^n$. The dual property holds for $C = E^{\min}$.

The next proposition tabulates elementary properties of these projectors.

Proposition 7.7. *Let E be a nonempty subset of $\mathbb{R}^n$. Then, $P_E^{\max}$ and $P_E^{\min}$ are Shapley operators from $\mathbb{R}^n \to \mathbb{R}^n$ such that:*

$$P_E^{\max} \leqslant I, \qquad P_E^{\min} \geqslant I \ . \tag{7.7}$$

$$\operatorname{Im} P_E^{\max} = E^{\max}, \qquad \operatorname{Im} P_E^{\min} = E^{\min} \tag{7.8}$$

and

$$P_E^{\max} = (P_E^{\max})^2, \qquad P_E^{\min} = (P_E^{\min})^2 \ . \tag{7.9}$$

Furthermore, $P_E^{\max}$ and $P_E^{\min}$ fix E.

Proof. The inequalities (7.7) follow from 7.3. By definition, $P_E^{\max}$ fixes $E^{\max}$, and $P_E^{\max}(\mathbb{R}^n) \subset E^{\max}$, so $P_E^{\max} = (P_E^{\max})^2$. The same property holds for $P^{\min}$, showing (7.9). (7.8) follows from the fact that $P_E^{\max}$ fixes $E^{\max}$ and that $P_E^{\min}$ fixes $E^{\min}$. □

The maps $\bar{Q}_E^-$ and $\bar{Q}_E^+$ defined in the next proposition will play a key role.

Proposition 7.8. *Let E be a nonempty subset of $\mathbb{R}^n$. Then, the maps*

$$\bar{Q}_E^- := P_E^{\min} \circ P_E^{\max}, \quad \textit{and} \quad \bar{Q}_E^+ := P_E^{\max} \circ P_E^{\min}$$

are such that

i. $\bar{Q}^-_E$; $\bar{Q}^+_E$ *are Shapley operators;*

ii. $\bar{Q}^-_E$ *and* $\bar{Q}^+_E$ *fix* E*;*

iii. $(\bar{Q}^-_E)^2 = \bar{Q}^-_E$*;*

iv. $(\bar{Q}^+_E)^2 = \bar{Q}^+_E$*;*

v. $\bar{Q}^+_E \circ \bar{Q}^-_E \circ \bar{Q}^+_E = \bar{Q}^+_E$*;*

vi. $\bar{Q}^-_E \circ \bar{Q}^+_E \circ \bar{Q}^-_E = \bar{Q}^-_E$*;*

vii. $\bar{Q}^+_E \leqslant \bar{Q}^-_E \circ \bar{Q}^+_E$

viii. $\bar{Q}^-_E \geqslant \bar{Q}^+_E \circ \bar{Q}^-_E$.

Proof. (i). We showed in 7.7 that $P_E^{\min}$ and $P_E^{\max}$ are both Shapley operators and the collection of Shapley operators is stable by composition.

(ii). This follows from the fact that $P_E^{\min}$ and $P_E^{\max}$ both fix E (7.7).

(iii). Using the second inequality in 7.7, and the first equality in 7.9, we get $(\bar{Q}^-_E)^2 = P_E^{\min} \circ P_E^{\max} \circ P_E^{\min} \circ P_E^{\max} \geqslant P_E^{\min} \circ P_E^{\max} \circ P_E^{\max} = P_E^{\min} \circ P_E^{\max}$. Using now the first inequality in 7.7, and the second equality in 7.9, we get $P_E^{\min} \circ P_E^{\max} \circ P_E^{\min} \circ P_E^{\max} \leqslant P_E^{\min} \circ P_E^{\min} \circ P_E^{\max} = P_E^{\min} \circ P_E^{\max}$, showing that $(\bar{Q}^-_E)^2 = \bar{Q}^-_E$.

(iv). Argument dual to *(iii)*.

(v). Using 7.9, we get $\bar{Q}^+_E \circ \bar{Q}^-_E \circ \bar{Q}^+_E = P_E^{\max} \circ P_E^{\min} \circ P_E^{\min} \circ P_E^{\max} \circ P_E^{\max} \circ P_E^{\min} = P_E^{\max} \circ P_E^{\min} \circ P_E^{\max} \circ P_E^{\min} = (\bar{Q}^+_E)^2 = \bar{Q}^+_E$, by *(iv)*.

(vi). Argument dual to *(v)*.

(vii). We have that $\bar{Q}^-_E \circ \bar{Q}^+_E = P_E^{\min} \circ P_E^{\max} \circ P_E^{\max} \circ P_E^{\min} = P_E^{\min} \circ P_E^{\max} \circ P_E^{\min} \geqslant P_E^{\max} \circ P_E^{\min}(z) = \bar{Q}^+_E$, using $P_E^{\min} \geqslant I$.

(viii). Argument dual to *(vii)*. □

The next proposition motivates the introduction of the operators $\bar{Q}^-$ and $\bar{Q}^+$. It shows that when C is a closed ambitropical cone, Q^-_C is obtained by composing the projection on the tropical cone $C^{\max}$ with the projection on the dual tropical cone $C^{\min}$.

Proposition 7.9. *For all closed ambitropical cones* C*, we have*

$$Q^-_C = \bar{Q}^-_C \quad \textit{and} \quad Q^+_C = \bar{Q}^+_C$$

Proof. Suppose C is a closed ambitropical cone.

Claim 7.13. For all z in $\mathbb{R}^n$,

$$P_C^{\max}(z) = \sup\{x \in C \mid x \leqslant z\} \ . \tag{7.10}$$

Proof. Indeed, $P_C^{\max}(z) = \sup\{x \in C^{\max} \mid x \leqslant z\} \geqslant \sup\{x \in C \mid x \leqslant z\}$. However, an element $u \leqslant z$ of $C^{\max}$ can be written as $u = \vee_{y \in C}(\lambda_y e_n + y)$ with $\lambda_y \in \mathbb{R}_{\max}$, and $\lambda_y e_n + y \leqslant z$. So, $\lambda_y e_n + y \leqslant \sup\{x \in C \mid x \leqslant z\}$, and so, $u \leqslant \sup\{x \in C \mid x \leqslant z\}$, which entails (8.2). Dually, $P_C^{\min}(z) = \inf\{x \in C \mid x \geqslant z\}$. □

Using (8.2), and the dual property, we get

$$\begin{aligned}\bar{Q}_C^-(x) = P_C^{\min} \circ P_C^{\max}(x) &= \inf\{y \in C \mid y \geqslant P_C^{\max}(x)\} \\ &= \inf\{y \in C \mid y \geqslant \sup\{z \in C \mid z \leqslant x\}\} \\ &= \inf\{y \in C \mid (z \leqslant x, z \in C) \implies z \leqslant y\} \\ &= \sup{}^C\{z \in C \mid z \leqslant y\} = Q_C^-(x) \ .\end{aligned}$$

The proof that $\bar{Q}_C^+ = Q_C^+$ is dual. □

We shall now present one of the main results of the chapter. It states that Shapley retracts of $\mathbb{R}^n$ coincide with closed ambitropical cones.

Theorem 7.4. *Let E be a subset of $\mathbb{R}^n$. The following assertions are equivalent*

1. *E is a closed ambitropical cone;*
2. *E is a Shapley retract of $\mathbb{R}^n$;*
3. *E is the fixed point set of a Shapley operator T.*

Proof. *(1) ⇒ (2).* If E is a closed ambitropical cone, then, by 7.5, E is the image of Q_E^- and Q_E^- is an idempotent Shapley operator.

(2) ⇒ (3) Choose T as the idempotent Shapley operator which gives rise to the retraction in 2.

(3) ⇒ (1). Since$E$ is the fixed point set of a Shapley operator T, E is a closed cone. We shall now show that it is also a lattice in the induced order. Let $x, y \in E$ and we claim that $x \vee_E y = \lim_{n\to\infty} T^n(x \vee_{\mathbb{R}^n} y)$. We have that $x, y \leqslant x \vee_{\mathbb{R}^n} y$, so $x = T(x) \leqslant T(x \vee_{\mathbb{R}^n} y)$ and $y = T(y) \leqslant T(x \vee_{\mathbb{R}^n} y)$. Applying again T to these inequalities and passing to the limit, we obtain that $\lim_{n\to\infty} T^n(x \vee_{\mathbb{R}^n} y)$ is an upper bound of x and y. Let $z \in E$ such that $x, y \leqslant z$. Then $x \vee_{\mathbb{R}^n} y \leqslant z$ and $T(x \vee_{\mathbb{R}^n} y) \leqslant T(z) = z$; applying again T to this inequality and passing to the limit, we obtain that $\lim_{n\to\infty} T^n(x \vee_{\mathbb{R}^n} y) \leqslant z$. We have now to prove that $\lim_{n\to\infty} T^n(x \vee_{\mathbb{R}^n} y) \in E$. $T(\lim_{n\to\infty} T^n(x \vee_{\mathbb{R}^n} y)) = \lim_{n\to\infty} T^n(x \vee_{\mathbb{R}^n} y)$ by definition. So, $\lim_{n\to\infty} T^n(x \vee_{\mathbb{R}^n} y)$ is a fixed point of T and it belongs to E. The case of inf is dual. □

In order to state the theorem that characterizes Shapley retracts in terms of best co-approximation and projections $P_E^{\max}$ and $P_E^{\min}$, we shall now provide some definitions.

We take inspiration from a notion introduced by Papini and Singer in the theory of best approximation in Banach spaces [51].

Definition 7.34. If E is a subset of a Banach space $(X, \|\cdot\|)$, E is said to be a *set of existence of best co-approximation* if, for all $z \in X$, the set

$$B_E^{\|\cdot\|}(z) := \{x \in X \mid \|y - x\| \leqslant \|y - z\|, \ \forall y \in E\}$$

contains an element of E.

It is immediate that if E is a nonexpansive retract of X, then E is a set of existence of best co-approximation. The converse is known to hold in L^p spaces with $1 \leqslant p < \infty$, see [57] and the references therein. Here we shall be interested in Shapley retracts of $\mathbb{R}^n$. By 7.2, these are precisely the images of $\mathbb{R}^n$ by idempotent maps that are nonexpansive in the "top" hemi-norm $\mathbf{t}x = \max_{i\in[n]} x_i$. We shall now introduce the following analogous of the set $B_E^{\|\cdot\|}(z)$.

Definition 7.35. Let E be a subset of $\mathbb{R}^n$. For any $y, z \in \mathbb{R}^n$ we define $B(y, z) = \{x \in \mathbb{R}^n \mid y + \mathbf{b}(z - y)e_n \leqslant x \leqslant y + \mathbf{t}(z - y)e_n\}$ where $\mathbf{t}(x) = \max_{i\in[n]} x_i$ and $\mathbf{b}(x) = \min_{i\in[n]} x_i$ for any $x \in \mathbb{R}^n$. Then, we define $B_E(z) = \cap_{y\in E} B(y, z)$.

Definition 7.36. Let E be a subset of $\mathbb{R}^n$, we say that E is a *set of existence of best tropical co-approximation* if $B_E(z) \cap E \neq \emptyset$ for every $z \in \mathbb{R}^n$.

Lemma 7.4. *We have* $B_E(z) = \{x \in \mathbb{R}^n \mid P_E^{\max}(z) \leqslant x \leqslant P_E^{\min}(z)\}$.

Proof. We shall prove that $\sup_{y\in E}(\mathbf{b}(z - y)e_n + y) = P_E^{\max}(z) = \sup\{x \in E^{\max} \mid x \leqslant z\}$. Let $x \in E^{\max}$, $x = \sup_{y\in E} \lambda e_n + y$. Suppose $x \leqslant z$, then we have that $\lambda e_n \leqslant \mathbf{b}(x - y)e_n \leqslant \mathbf{b}(z - y)e_n$. In a similar way, we see that $P_E^{\min}(z) = \inf_{y\in E} \mathbf{t}(z - y)e_n + y$. □

Lemma 7.5. *If C is closed and ambitropical, we have that $Q_C^-(x) \leqslant Q_C^+(x)$ for any $x \in \mathbb{R}^n$.*

Proof. Let $z \in C$ be such that $x \geqslant z$. Then, $y \in C$ and $y \geqslant x$ implies $z \leqslant y$. In particular $z \leqslant \inf_C\{y \in C \mid y \geqslant x\} = Q_C^+(x)$. Since this holds for any $z \in C$ such that $x \geqslant z$, we have that $Q_C^-(x) = \sup_C\{z \in C \mid z \leqslant x\} \leqslant Q_C^+$. □

Lemma 7.6. *Let E be a subset of $\mathbb{R}^n$. If E is a set of existence of best tropical co-approximation, then $\bar{B}_E(z) := [\bar{Q}_E^-(z), \bar{Q}_E^+(z)] \cap E \neq \emptyset$ for any $z \in \mathbb{R}^n$.*

Proof. We know that, if E is a set of best tropical co-approximation then for any $z \in \mathbb{R}^n$, there exists $u \in E$ such that $P_E^{\max}(z) \leqslant u \leqslant P_E^{\min}(z)$. Composing the first inequality by $P_E^{\min}$, and composing the second inequality by $P_E^{\max}$, we get $\bar{Q}_E^-(z) = P_E^{\min} \circ P_E^{\max}(z) \leqslant P_E^{\min}(u) = u$, and $u = P_E^{\max}(u) \leqslant P_E^{\max} \circ P_E^{\min}(z) = \bar{Q}_E^+(z)$. □

Theorem 7.5. *Let E be a subset of $\mathbb{R}^n$. The following assertions are equivalent*

1. *E is a Shapley retract of $\mathbb{R}^n$;*
2. *E is a set of existence of best tropical co-approximation;*
3. *for all $z \in \mathbb{R}^n$, $[P_E^{\max}(z), P_E^{\min}(z)] \cap E \neq \emptyset$;*
4. *$P_E^{\min}(z) \in E$ holds for all $z \in E^{\max}$;*
5. *$P_E^{\max}(z) \in E$ holds for all $z \in E^{\min}$;*
6. *E is the fixed point set of the operator $\bar{Q}_E^+ = P_E^{\max} \circ P_E^{\min}$;*
7. *E is is the fixed point set of the operator $\bar{Q}_E^- = P_E^{\min} \circ P_E^{\max}$.*

Proof. $1 \Rightarrow 2$. Suppose $E = P(\mathbb{R}^n)$ where $P = P^2$ is a Shapley operator. Then, for all $y \in E$, and for all $z \in \mathbb{R}^n$, $\mathbf{t}(P(z) - y) = \mathbf{t}(P(z) - P(y)) \leqslant \mathbf{t}(z - y)$, i.e., $P(z) \leqslant y + \mathbf{t}(z - y)e_n$, and dually, $P(z) \geqslant y + \mathbf{b}(z - y)e_n$, showing that $P(z) \in B(z) \cap E$.

$2 \Rightarrow 3$. This follows from 7.4.

$3 \Rightarrow 4$. As a preliminary result we prove that

$$\forall z \in \mathbb{R}^n,\ [P_E^{\max}(z), P_E^{\min}(z)] \cap E \neq \emptyset \Rightarrow \forall z \in E^{\max}, [z, P_E^{\min}(z)] \cap E \neq \emptyset \tag{7.11}$$

Let $z \in E^{\max}$, then $[P_E^{\max}(z), P_E^{\min}(z)] = [z, P_E^{\min}(z)]$.

Now, the condition $[z, P_E^{\min}(z)] \cap E \neq \emptyset$ is equivalent to: $\exists u \in E$ such that $z \leqslant u \leqslant P_E^{\min}(z)$. However, $P_E^{\min}(z)$ is the minimal element $v \in E^{\min}$ such that $v \geqslant z$, it follows that $P_E^{\min}(z) \leqslant u$, and so $P_E^{\min}(z) = u \in E$.

$4 \Rightarrow 5$. By hypothesis, we have that for any $z \in E^{\max}$, $P_E^{\min}(z) \in E$, so in particular $[z; P_E^{\min}(z)] \cap E \neq \emptyset$. Consider an arbitrary $z \in \mathbb{R}^n$. Then, we have that $P_E^{\max}(z) \in E^{\max}$ and, consequently $[P_E^{\max}(z), P_E^{\min}(P_E^{\max}(z))] \cap E \neq \emptyset$. Recalling that $[P_E^{\max}(z), P_E^{\min}(P_E^{\max}(z))] \subseteq [P_E^{\max}(z), P_E^{\min}(z)]$ since $P_E^{\max} \leqslant I$, we have that for any $z \in \mathbb{R}^n$, $[P_E^{\max}(z); P_E^{\min}(z)] \cap E \neq \emptyset$. In particular, let $z \in E^{\min}$, then we have that $[P_E^{\max}(z), z] \cap E \neq \emptyset$ and by the dual argument of the previous implication we obtain that $P_E^{\max}(z) \in E$.

$5 \Rightarrow 6$. We will denote with $Fix(\bar{Q}_E^+)$ the fixed points set of $\bar{Q}_E^+$. Since any element of E is fixed by $\bar{Q}_E^+$, $E \subseteq Fix(\bar{Q}_E^+)$. We shall now prove the other inclusion. Let $z \in \mathbb{R}^n$ such that $P_E^{\max}(P_E^{\min}(z)) = z$. Since $P_E^{\min}(z) \in E^{\min}$, $z = P_E^{\max}(P_E^{\min}(z)) \in E$.

$6 \Rightarrow 7$. The fact that E is the fixed points set of $\bar{Q}_E^+$ implies that for any $y \in E^{\min}$, $P_E^{\max}(y) \in E$. As in the previous implication we know that $E \subseteq Fix(\bar{Q}_E^-)$ and we shall now prove the other inclusion. Let $z \in \mathbb{R}^n$ such that $P_E^{\min}(P_E^{\max}(z)) = z$, so $z \in E^{\min}$ an, by hypothesis $P_E^{\max}(z) \in E$. Since E is fixed by $P_E^{\min}$, we have that $z = P_E^{\min}(P_E^{\max}(z)) \in E$.

$7 \Rightarrow 1$. This follows from 7.8, *(iii)*. □

Corollary 7.3. *Let E be a non-empty subset of $\mathbb{R}^n$, included in a closed ambitropical cone F. Then, $E^{\max} \cap E^{\min} \subseteq F$.*

Proof. The set $E^{\max} \cap E^{\min}$ is fixed both by $P_E^{\max}$ and $P_E^{\min}$. If $E \subseteq F$, the fixed point set of $P_E^{\max}$ is included in the fixed points set of $P_F^{\max}$. The same is true for F. So, $E^{\max} \cap E^{\min}$ is included in the fixed points set of $P_F^{\min} \circ P_F^{\max}$, which by 7.5,*(7)*, coincides with F. □

Corollary 7.4. *Suppose C is a closed ambitropical cone. Then $C = C^{\max} \cap C^{\min}$.*

Proof. The inclusion $C^{\max} \cap C^{\min} \subseteq C$ follows from 7.3. The other inclusion is trivial. □

If $E \subset \mathbb{R}^n$, the set $E^{\max} \cap E^{\min}$ is generally not an ambitropical cone (it can be disconnected). Moreover, the intersection of ambitropical sets is generally not ambitropical, so the notion of ambitropical hull of a set E cannot be defined in the naïve manner, as the intersection of ambitropical sets containing E. However, the next results show that there is a proper notion of ambitropical hull.

Corollary 7.5. *For each nonempty subset $E \subset \mathbb{R}^n$, the sets $\operatorname{Im} \bar{Q}_E^-$ and $\operatorname{Im} \bar{Q}_E^+$ are closed ambitropical cones containing E that are isomorphic.*

Proof. If $E \subset \mathbb{R}^n$ is non-empty, the operator $\bar{Q}_E^-$ maps $\mathbb{R}^n$ to $\mathbb{R}^n$, and it follows from its definition that it satisfies the axioms of Shapley operators (7.7). Moreover, we have seen in 7.8, *(iii)* that $\bar{Q}_E^-$ is idempotent. It follows that $\operatorname{Im} \bar{Q}_E^- = \bar{Q}_E^-(\mathbb{R}^n)$ is a Shapley retract, and so $\operatorname{Im} \bar{Q}_E^-$ is ambitropical. Moreover, $\operatorname{Im} \bar{Q}_E^- \supseteq E$. By duality, the same is true for $\operatorname{Im} \bar{Q}_E^+$. □

7.5 Special classes of ambitropical cones

In this section, we show that several canonical classes of sets in tropical geometry are special cases of ambitropical cones, that can be characterized through Shapley operators by suitable reinforcements of 7.4.

The simplest examples of ambitropical cones consist of alcoved polyhedra, discussed in the second section. Actually, the following result shows that alcoved polyhedra are characterized by the property of being sublattices of $\mathbb{R}^n$. Observe that all properties but one do not assume polyhedrality, polyhedrality comes as a consequence in a perhaps unexpected way.

Proposition 7.10. *Let $C \subset \mathbb{R}^n$. The following statements are equivalent:*

1. *C is an alcoved polyhedron;*

2. *C is a closed tropical cone and a closed dual tropical cone,*

3. *C is a closed ambitropical cone in which the infimum and supremum laws coincide with the ones of $\mathbb{R}^n$.*

4. *There is a tropically linear Shapley operator T such that $C = \{x \in \mathbb{R}^n \mid T(x) \leqslant x\}$.*

5. *There is a dually tropically linear Shapley operator P such that $C = \{-x \in \mathbb{R}^n \mid P(x) \geqslant x\}$.*

Proof of 7.10.

Claim 7.14. 1, 2 and 3 are equivalent

Proof. ((1) implies (2)) An alcoved polyhedron is stable by pointwise supremum and pointwise infimum of vectors.
((2) implies (3)) Trivially.

((3) implies (1)) Suppose now that (3) holds. Then, by 7.4, C is a conditionally complete lattice, and the lattice operations of C are the pointwise supremum and pointwise infimum of vectors. Define, for all $i, j \in [n]$,

$$M_{ij} := \sup\{\lambda \mid v_i \geqslant \lambda + v_j, \qquad \forall v \in C\}$$

The latter set is noempty, it is closed and bounded from above, so the supremum is achieved. In particular, we have $M_{ij} \in \mathbb{R}_{\max}$. By construction, $C \subset \mathcal{A}(M)$, and $M_{ij} = M^*_{ij}$. Observe also that the inequality $v_i \geqslant \lambda e_n + v_j$, is equivalent to $w_i \geqslant \lambda$ where $w := v - v_j\delta_j \in C$ is such that $w_j = 0$, denoting by $\delta_j = (0, \dots, 0, 1, 0, \dots 0)$ the j-th vector of the canonical basis of $\mathbb{R}^n$.

It follows that:

$$M_{ij} = \inf C_{ij}, \text{ where } C_{ij} := \{v_i \mid v \in C_j\} \text{ and } C_j := \{v \in C \mid v_j = 0\} \ .$$

Denoting by u^j the jth column of the matrix M, we deduce that

$$u^j = \inf C_j \in \operatorname{clo}^{\downarrow} C \ .$$

Define, $A_j := \{v \in \operatorname{clo}^{\downarrow} C \mid v_j = 0\}$. Since C is a conditionally complete lattice, the set $A_j \subset \operatorname{clo}^{\downarrow} C$ is stable by taking infima. Hence, the set $\operatorname{Min} A_j$ consists of a single point, u^j. By 7.2, every element of C is a tropical linear combination of vectors u^j. This implies that $\mathcal{A}(M) = \{M^*y \mid y \in \mathbb{R}^n\} = C$. □

Claim 7.15. (4) is equivalent to the first three conditions

Proof. ((1) implies (4)) If $C = \mathcal{A}(M)$ is an alcoved polyhedron, it follows from (7.3) that $C = \{x \in \mathbb{R}^n \mid T(x) \leqslant x\}$ where $T(x) = M^*x$ is a Shapley operator. So (7.3) implies (4).

((4) implies (3)) Conversely, if $C = \{x \in \mathbb{R}^n \mid T(x) \leqslant x\}$ for some tropically linear Shapley operator, then, for all $x, y \in C$, since T is order preserving, $T(x \wedge y) \leqslant T(x) \wedge T(y) \leqslant x \wedge y$, and since T is tropically linear, $T(x \vee y) = T(x) \vee T(y) \leqslant x \vee y$, showing that $x \wedge y$ and $x \vee y$ belong to C. Moreover, C is closed, since T is continuous (in fact, T is sup-norm nonexpansive). □

Claim 7.16. (4) and (5) are equivalent.

Proof. It is sufficient to observe that given a tropically linear (dually tropically linear) Shapley operator T, we can define a dually tropically linear (tropically linear) Shapley operator P by posing

$$P(x) = -T(-x)$$

and observing that

$$x \leqslant P(x) \iff T(-x) \leqslant -x$$

□

□

Tropical cones, and dual tropical cones, are also remarkable ambitropical cones.

Proposition 7.11. *Let $C \subset \mathbb{R}^n$. The following statements are equivalent:*

1. *C is a closed tropical cone;*
2. *C is a closed ambitropical cone in which the supremum law coincides with the one of $\mathbb{R}^n$;*
3. *there is a Shapley operator T such that $C = \{x \in \mathbb{R}^n \mid T(x) \geqslant x\}$.*

Proof. 1⇒2. Suppose that C is a closed tropical cone, and let X denote a non-empty subset of C bounded from above by an element of $\mathbb{R}^n$. Then, for all finite subsets $F \in \mathcal{P}_f(X)$, $\sup F$ belongs to C, because C is stable by supremum, and $\sup X = \lim_{F \in \mathcal{P}_f(X)} \sup F \in C$ because C is closed. It follows that X has a supremum in C which coincides with its supremum in $\mathbb{R}^n$. Suppose now that X is bounded from below by an element z of $\mathbb{R}^n$. Consider $Y := \{y \in C \mid y \leqslant x, \forall x \in C\}$. Then, Y is non-empty and it is bounded from above. It follows from the previous observation that $\sup Y$ is the supremum of Y, in C. Moreover, $\sup Y$ is precisely the infimum of Y in C, showing that C is ambitropical.

2⇒3. Since C is a closed ambitropical cone, then, it is the fixed points set of $z \mapsto Q^-_C(z) = \sup\{x \in C \mid x \leqslant z\}$, and $Q^-_C \leqslant I$. So, $C = \{z \in \mathbb{R}^n \mid Q^-_C(z) = z\} = \{z \in \mathbb{R}^n \mid Q^-_C(z) \geqslant z\}$.

3⇒1. Suppose that $C = \{x \in \mathbb{R}^n \mid T(x) \geqslant x\}$. Since T is continuous, C is closed. Moreover, since T is order preserving, for all $x, y \in C$, $T(x \vee y) \geqslant T(x) \vee T(y) \geqslant x \vee y$. Since T commutes with the addition of constant vectors, we conclude that C is a closed tropical cone.

□

We state the following dual version of 7.11.

Proposition 7.12. *Let $C \subset \mathbb{R}^n$. The following statements are equivalent:*

1. *C is a closed dual tropical cone;*
2. *C is an ambitropical cone in which the infimum law coincides with the one of $\mathbb{R}^n$;*
3. *there is a Shapley operator T such that $C = \{x \in \mathbb{R}^n \mid T(x) \leqslant x\}$.* □

To describe a fourth special class of ambitropical cones, a definition is in order.

Definition 7.37. An ambitropical cone C is *homogeneous* if for all $\alpha > 0$ and for all $x \in C$, $\alpha x \in C$. A Shapley operator $T : \mathbb{R}^n \to \mathbb{R}^n$ is *homogeneous* if $T(\alpha x) = \alpha T(x)$ holds for all $\alpha > 0$ and for all $x \in \mathbb{R}^n$.

Remark 7.3. For all $\alpha > 0$, the self-map $x \mapsto \alpha x$ of $\mathbb{R}_{\max}$ is natural from the tropical perspective. Indeed, it is a semifield automorphism, which is the tropical analogous of the Frobenius map [16].

Proposition 7.13. *Let $C \subset \mathbb{R}^n$. The following statements are equivalent:*

1. *C is a closed homogeneous ambitropical cone;*
2. *there is an idempotent homogeneous Shapley operator whose fixed points set is C;*
3. *there is an homogeneous Shapley operator whose fixed points set is C.*

Proof. (1⇒2) If C is a closed homogeneous ambitropical cone, we know from 7.4 that C is the range of the idempotent Shapley operator $\bar{Q}^-_C = Q^-_C$. Observe that, for all $\alpha > 0$ and $x \in \mathbb{R}^n$, $Q^-_C(\alpha x) = \sup^C\{y \in C \mid y \leqslant \alpha x\} = \sup^C\{\alpha\alpha^{-1}y \mid y \in C,\ \alpha^{-1}y \leqslant x\} = \sup^C\{\alpha z \mid z \in C,\ z \leqslant x\} = \alpha Q^-_C(x)$ since $y \mapsto \alpha^{-1}y$ is a bijection from C to C, which is order preserving and whose inverse also preserves the order. This shows the implication.
(2⇒3) The implication is immediate.
(3⇒1) If C is the fixed points set of an homogeneous Shapley operator T, then, we know from 7.4 that C is an ambitropical cone, and it follows from the homogeneity of T that C is homogeneous. This shows the implication. □

Chapter 8

A generalization to conditionally complete lattices

It is instructive to observe that the main result of the previous chapter has an analogous which applies to order preserving maps over conditionally complete lattices.

In this section L is a conditionally complete lattice.

Definition 8.1. We say that a non-empty subset $E \subset L$ is *archimedean* if every nonempty subset X of E that is bounded from above by an element of L, is bounded from above by an element of E, and if every nonempty subset X of E that is bounded from below by an element of L is bounded from below by an element of E.

We observed in 7.2 that this property is automatically satisfied when $L = \mathbb{R}^n$ and E is an ambitropical cone. A special case, which we shall encounter in the sequel, arises when L is a complete lattice. Then, any subset $E \subset L$ which contains the top and bottom elements of L is trivially archimedean.

Definition 8.2. If E is a non-empty subset of L, we now define $E^{\max}$ to be the set of elements of the form $\sup F$ where F is a non-empty subset of E bounded from above. The set $E^{\min}$ is defined in the dual way.

Remark 8.1. Observe that this is similar to (7.6), under the replacement of $\mathbb{R}_{\max}$ by the Boolean semifield.

Definition 8.3. Recall that a family $(u_f)_{f\in F}$ of elements of a partially ordered set is *filtered* if for all $f_1, f_2 \in F$, there exists $f_3 \in F$ such that $u_{f_3} \leqslant u_{f_1}$ and $u_{f_3} \leqslant u_{f_2}$; *directed* families are defined in a dual manner.

Definition 8.4. We now define $\bar{E}^{\max} := \operatorname{clo}^{\downarrow} E^{\max}$ as the set of infima of filtered families of elements of $E^{\max}$, and $\bar{E}^{\min} := \operatorname{clo}^{\uparrow} E^{\min}$ as the set of suprema of directed families of elements of $E^{\min}$.

Taking into account these new definitions of $E^{\max}$ and $E^{\min}$ we can define

Definition 8.5. For any archimedean subset E of a conditionally complete lattice L, and for all $x \in L$, we define

$$P_E^{\max}(x) := \sup\{y \in E^{\max} \mid y \leqslant x\} \ , \qquad P_E^{\min}(x) := \inf\{y \in E^{\min} \mid y \geqslant x\} \ .$$

and

Definition 8.6.

$$\bar{Q}_E^- := P_E^{\min} \circ P_E^{\max}, \quad \text{and} \quad \bar{Q}_E^+ := P_E^{\max} \circ P_E^{\min}$$

The operators $P_E^{\max}$, $P_E^{\min}$, $\bar{Q}_E^-$ and $\bar{Q}_E^+$ satisfy the same properties of propositions 7.7 and 7.8.

Remark 8.2. In order to define Q_E^- and Q_E^+ as in the previous chapter we have to restrict ourselves to the special case in which E is a conditionally complete lattice in the induced order of L.

Definition 8.7. Let E be an archimedean subset E of a conditionally complete lattice L which is a conditionally complete lattice in the induced order of L, then for all $x \in L$, we can define the following "projection" maps:

$$Q_E^-(x) := \sup{}^E\{y \in E \mid y \leqslant x\} \ , \qquad Q_E^+(x) := \inf{}^E\{y \in E \mid y \geqslant x\} \ . \tag{8.1}$$

The following is an analogous of proposition 7.5 and the proof is similar

Proposition 8.1. *Suppose that L is a conditionally complete lattice and E is an archimedean subset of L which is a conditionally complete lattice in the induced order of L, then Then, Q_E^- is an idempotent order preserving map, i.e., $(Q_E^-)^2 = Q_E^-$, and the range of Q_E^- is E. The same is true for Q_E^+.*

We can now prove the analogous of proposition 7.9

Proposition 8.2. *For all archimedean subsets E of a conditionally complete lattice L which are conditionally complete lattices in the induced order of L, we have*

$$Q_E^- = \bar{Q}_E^- \quad \textit{and} \quad Q_E^+ = \bar{Q}_E^+$$

Proof. Suppose E is archimedean. Observe first that for all z in L,

$$P_E^{\max}(z) = \sup\{x \in E \mid x \leqslant z\} \ . \tag{8.2}$$

Indeed, $P_E^{\max}(z) = \sup\{x \in E^{\max} \mid x \leqslant z\} \geqslant \sup\{x \in E \mid x \leqslant z\}$.

However, an element $u \leqslant z$ of $E^{\max}$ is an element of the form $u = \sup F$ with F a nonempty subset of E bounded from above. Since $u \leqslant z$ it follows that $F \leqslant z$, i. e. any element of F is less or equal than z. So, $u = \sup F \leqslant \sup\{x \in E \mid x \leqslant z\}$.

Using (8.2), and the dual property, we get

$$\begin{aligned}\bar{Q}_E^-(x) = P_E^{\min} \circ P_E^{\max}(x) &= \inf\{y \in E \mid y \geqslant P_E^{\max}(x)\} \\ &= \inf\{y \in E \mid y \geqslant \sup\{z \in E \mid z \leqslant x\}\} \\ &= \inf\{y \in E \mid (z \leqslant x, z \in E) \implies z \leqslant y\} \\ &= \sup{}^C\{z \in E \mid z \leqslant y\} = Q_E^-(x) \ .\end{aligned}$$

The proof that $\bar{Q}_E^+ = Q_E^+$ is dual. □

Theorem 8.1. *Let L be a conditionally complete lattice and let E be an archimedean non-empty subset of L. The following statements are equivalent:*

1. *E is a conditionally complete lattice in the induced order of L;*
2. *$E = P(L)$ for some order-preserving map $P : L \to L$ such that $P = P^2$, i.e. E is an order-preserving retract of L;*
3. *E is the fixed points set of an order-preserving map $T : L \to L$.*

The proof is similar to the one of 7.4.

Proof. *(i)* ⇒ *(2)* E is the image of Q_E^- and Q_E^- is an idempotent order-preserving map.

(2) ⇒ *(3)* Choose $T = P$.

(3)⇒*(i)*. Let X be a family of elements of E bounded by some element $z \in E$, in particular $z \geqslant \sup_L X$ and $T(z) = z$. Since T is order-preserving we have that $z = T(z) \geqslant T(\sup X) \geqslant \sup T(X) = \sup X$. Setting $u := \sup X$, we get $T(u) \geqslant u$. Then, we define inductively, for all ordinal α, $T^\alpha(u)$ as follows: $T^0(u) = u$, $T^\beta(u) = T(T^\alpha(u))$ if $\beta = \alpha + 1$ is a successor ordinal, and $T^\beta(u) = \lim_{\alpha \uparrow \beta, \alpha < \beta} T^\alpha(u)$ if β is a limit ordinal. There must exist an ordinal α such that $T^{\alpha+1}(u) = T^\alpha(u)$. Then, $T^\alpha(\sup X)$ is an element of E and it is precisely $\sup^E X$. □

All the properties of 7.5, but the one concerning best coapproximation, carry over to the setting of conditionally complete lattices. The proof is similar to the one of 7.5, we shall only show the implications that are different. The proof is similar to the one of Theorem 7.4. We only provide the arguments which need to be adapted.

Theorem 8.2. *Let L be a conditionally complete lattice and let E be an archimedean non-empty subset of L. The following statements are equivalent:*

i. $E = P(L)$ *for some order-preserving map* $P : L \to L$ *such that* $P = P^2$*, i.e.* E *is an order-preserving retract of* L*;*

ii. for all $z \in L$*,* $[P_E^{\max}(z), P_E^{\min}(z)] \cap E \neq \emptyset$*;*

iii. $P_E^{\min}(z) \in E$ *holds for all* $z \in E^{\max}$*;*

iv. $P_E^{\max}(z) \in E$ *holds for all* $z \in E^{\min}$*;*

v. E *is the fixed point set of* $\bar{Q}_E^{+} = P_E^{\max} \circ P_E^{\min}$*;*

vi. E *is the fixed point set of* $\bar{Q}_E^{-} = P_E^{\min} \circ P_E^{\max}$*.*

Proof. *(i)* $\Rightarrow$ *(ii)* By 8.1 we have that E is a conditionally complete lattice in the induced order of L and so E is the image of $\bar{Q}_E^{-}(x)$. Since $P_E^{\max} \leqslant I$ and $P_E^{\min} \geqslant I$, we have that for all $z \in L$ $P_E^{\max}(z) \leqslant P_E^{\min}(P_E^{\max}(z)) \leqslant P_E^{\min}(z)$, so $P_E^{\min}(P_E^{\max}(z)) \in [P_E^{\max}(z), P_E^{\min}(z)] \cap E$.

□

Appendices

Appendix A

Proof of 7.1

As said at the beginning of Chapter 7, we only need to prove the "if" part. If $T = T^2$ is idempotent and nonexpansive in a Euclidean norm $\| \cdot \|$ and if $C = T(\mathbb{R}^n)$, then $C = \{x \in \mathbb{R}^n \mid x = T(x)\}$ is closed. It remains to check that C is convex. We have, for all $x, y \in C$, $0 < \alpha < 1$ and $z = \alpha x + (1-\alpha)y$ $\|x - T(z)\| + \|T(z) - y)\| = \|T(x) - T(z)\| + \|T(z) - T(y)\| \leqslant \|x - z\| + \|z - y\| = \|x - y\|$. Observe that the euclidean geodesic between x and y is unique and that it is precisely the segment $[x, y]$ (this is valid more generally in any Banach space with a strictly convex norm). Then, the inequality $\|x - T(z)\| + \|T(z) - y)\| \leqslant \|x - y\|$ implies that $T(z)$ lies on this geodesic. So, T sends $[x, y]$ to itself.

In particular, $T(z) = \beta x + (1 - \beta)y$ for some $0 \leqslant \beta \leqslant 1$. Since $(1 - \beta)\|y - x\| = \|T(z) - x\| = \|T(z) - T(x)\| \leqslant \|z - x\| = (1 - \alpha)\|y - x\|$, we deduce that $1 - \beta \leqslant 1 - \alpha$. By exchanging the roles of x and y, we deduce that $\beta \leqslant \alpha$. Hence, $\alpha = \beta$, showing that $T(z) = z$. Hence, $z \in C$, which shows 7.1.

Appendix B

An example of ambitropical polyhedron

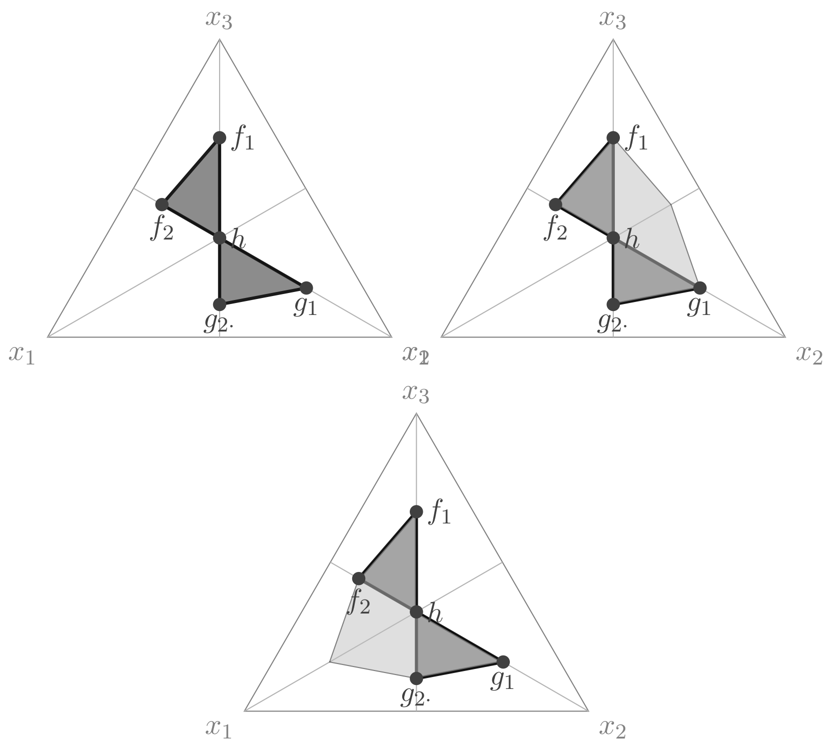


Figure B.1: An ambitropical polyhedron E consisting of two alcoved polyhedra (left). The tropical polyhedral cones $E^{\max}$ (right) and $E^{\min}$ (center).

Example B.1. Consider the ambitropical polytope E in Figure B.1. We have that in this case

$$Q_E^- \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (x \wedge y \wedge (1+z)) \vee (x \wedge (1+y) \wedge z) \vee (y \wedge z \wedge (1+x)) \\ y \wedge (1+x) \wedge (1+z) \\ z \wedge (1+y) \wedge (1+x) \end{pmatrix}$$

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