Maximum principles, entire solutions and removable singularities of fully nonlinear second order equations

Giulio Galise

Abstract

This PhD thesis is devoted to some qualitative aspect of viscosity solutions of nonlinear second order elliptic partial differential equations of the form

$$F(x, u(x), Du(x), D^2u(x)) = f(x), \tag{1}$$

where $F : \Omega \subseteq \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n \mapsto \mathbb{R}$ and $f : \Omega \subseteq \mathbb{R}^n \mapsto \mathbb{R}$ are prescribed functions, Ω is a domain, $\mathcal{S}^n$ is the linear space of symmetric $n \times n$ real matrices and u is the real valued unknown function. The gradient Du and the Hessian matrix D^2u do not have a classical meaning, but they are understood in a weak sense.

The notion of viscosity solution firstly appeared in the early 80s in the works of M.G. Crandall, L.C. Evans, P.L. Lions [2]-[3]-[4] relatively to the first order Hamilton-Jacobi equations and afterwards extended by Jensen [6] to the second order case.

Many other authors, among whom Cabré, Caffarelli, Hishii, Świech, Trudinger, have contributed to the development of this theory showing existence, uniqueness, regularity, approximation and stability results so emphasizing the flexibility and the usefulness of viscosity solutions to handle nonlinear elliptic (and parabolic) problems. Moreover the range of applications is plentiful: optimal control, differential game, front propagation, Mathematical Finance are only some of the research fields in which viscosity theory applies.

In this spirit Chapter 1 is a brief introduction to viscosity solutions. First we deal with the continuous case (F and f in (1) are assumed to be continuous functions) and present some results: comparison principle, Perron method, stability and half-relaxed limits, Jensen approximation. In the last section we consider L^p-viscosity solutions, a suitable notion of viscosity solutions to treat equations with measurable dependence on x.

Chapter 2 is concerned with the existence and uniqueness of entire solutions (i.e. solutions in the whole space $\mathbb{R}^n$) of (1). The operator F is uniformly elliptic and $F(\cdot, r, \cdot, \cdot)$ satisfies a superlinear monotonicity assumption. No growth condition at infinity is assumed. Such results are known [1], in distributional sense, for the equation

$$\Delta u - |u|^{d-1}u = f(x), \quad f \in L^1_{\text{loc}}(\mathbb{R}^n),\, d > 1 \tag{2}$$

and [5] for

$$F(D^2u) - |u|^{d-1}u = f(x), \quad f \in L^n_{\text{loc}}(\mathbb{R}^n),\, d > 1 \tag{3}$$

in the L^n-viscosity case. Our aim is to consider a larger class of equations than (3), allowing the dependence on x, on the gradient Du and going below the exponent n. The technique used to prove the existence of entire solutions allows us to solve the Dirichlet boundary problem in regular domains (even unbounded). In the last section a non-existence result is proved for the equation

$$F(x, Du, D^2u) - e^u = 0 \quad \text{in } \mathbb{R}^n.$$

Blow-up solutions are the content of Chapter 3. We extend to the nonlinear viscosity setting some known results of [7]-[8] concerning the Laplace operator.
Chapter 4 is devoted to the Extended Maximum Principle (EMP). Every subsolution on the uniform elliptic equation $F(D^2u) = 0$ in a bounded domain Ω satisfies the condition

$$\limsup_{x \to \partial\Omega} u(x) \le 0 \Rightarrow u \le 0 \quad \text{in } \Omega$$

in view of maximum principle. In other words the sign of u on the boundary $\partial\Omega$ propagate inside Ω. We show that the boundary condition can be weakened without altering the validity of the maximum principle: any bounded subsolution u of $F(D^2u) = 0$ is non-positive in Ω assuming $u \le 0$ on $\partial\Omega$ up to a set E of null α-Riesz capacity (for a suitable α). So the set of zero capacity can be ignored in the maximum principle. The key point to establish this result is to construct a supersolution of a maximal equation which blows up on E and is finite outside E. The α-Riesz potential works in this case.
Next we generalize such results to the class of uniformly elliptic equations depending on the gradient variable $F(Du, D^2u) = 0$. As application of EMP we present a removable singularities result: every bounded viscosity solution of $F(Du, D^2u) = f(x)$ in $\Omega\backslash E$, $E \Subset \Omega$, can be extended to a solution of the same equation in all Ω if E has zero α-Riesz capacity.
Finally a larger class of pure second order degenerate elliptic operator is considered by showing that EMP continues to work in this case.

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