MV-ALGEBRAS, GROTHENDIECK TOPOSES AND APPLICATIONS

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Abstract

This thesis is a contribution to the research program 'toposes as bridges' introduced in [12], which aims at developing the unifying potential of the notion of Grothendieck topos as a means for relating different mathematical theories to each other through topos-theoretic invariants. The general methodology outlined therein is applied here to study already existing categorical equivalences of particular interest arising in the field of many-valued logics and also to produce new ones. The original content of the disseration is contained in [22], [21] and [23].

Grothendieck toposes

The notion of *topos* was introduced by A. Grothendieck in the early 1960s in his reformulation of sheaf theory for algebraic geometry. He considered sheaves not only on topological spaces but on *sites*, i.e., categories endowed with a so-called Grothendieck topology. He defined (Grothendieck) toposes as categories which are equivalent to a category of sheaves on a site. Since many classical properties of topological spaces can be naturally formulated as properties of the associated categories of sheaves, Grothendieck toposes can be regarded as 'generalized spaces'.

Later, W. Lawvere and M. Tierney realized that toposes can also be considered as 'generalized mathematical universes' where one can reproduce most of the familiar constructions that one is used to perform among sets, like products, coproducts, and so on. In fact, Grothendieck toposes are rich enough in terms of categorical structure to make it possible to consider models of any kind of first-order theory inside them.

At the end of the seventies, the Montréal school of categorical logic, notably including M. Makkai, G. Reyes and A. Joyal, introduced the concept of *classifying topos* of a geometric theory (i.e., a theory over a first-order signature whose axioms are sequents that involve formulas built from atomic ones by only using finitary conjunctions, infinitary disjunctions and existential quantifications). They added in this way a third viewpoint on toposes to the already mentioned ones. Indeed, they proved that every geometric theory $\mathbb{T}$ has a unique, up to categorical equivalence, classifying topos $\mathcal{E}_{\mathbb{T}}$, that is a Grothendieck topos containing a *universal model* $U_{\mathbb{T}}$ of $\mathbb{T}$, universal in the sense that any other model of $\mathbb{T}$ in any other Grothendieck topos $\mathcal{E}$ is, up to isomorphism, the image of this model under (the inverse image of) a unique morphism of toposes from $\mathcal{E}$ to $\mathcal{E}_{\mathbb{T}}$. Vice versa, every Grothendieck topos can be regarded as the classifying topos of a geometric theory. It is possible that two distinct mathematical theories have the same, up to categorical equivalence, classifying topos; in this case we say that the theories are *Morita-equivalent*. Thus, Grothendieck toposes can not only be regarded as generalized spaces or generalized universes, but also as theories, considered up to Morita-equivalence.

This third incarnation of the notion of topos became the basis of the methodology 'toposes as bridges' introduced by O. Caramello in [12] and developed throughout the last years. The existence of different representations of the same Grothendieck topos, given for instance by different sites of definition or by Morita-equivalent theories, allows to transfer information and results from one representation to the other by using topos-theoretic invariants on that topos as translating 'machines'.

The power of this technique lies in the fact that a given topos-theoretic invariant can manifest itself in completely different ways in terms of different sites of definition for the same topos. One can then establish by means of these site characterizations logical relationships

$$\begin{tikzcd} & \mathcal{E}_{\mathbb{T}} \simeq \mathcal{E}_{\mathbb{T}'} \arrow[dl, dash, dashed, bend right] \arrow[dr, dash, dashed, bend left] & \\ \mathbb{T} & & \mathbb{T}' \end{tikzcd}$$

or equivalences between completely different-looking properties or constructions pertaining to different sites. A remarkable example of the application of this technique is the topos-theoretic interpretation of Fraïssé's construction in Model theory established in [18].

Topos theory has already been successfully applied in the context of many-valued logics for establishing sheaf representations for notable classes of MV-algebras, for instance in the work of E. J. Dubuc and Y. Poveda ([31]) and J. L. Castiglione, M. Menni and W. J. Botero ([24]). Further sheaf representations were established by A. Filipoiu and G. Georgescu ([33]), and A. R. Ferraioli and A. Lettieri ([32]).

The innovation of this thesis is that we use topos-theoretic methods in order to obtain new results and conceptual insights, of both logical and algebraic nature, on central topics in the field of MV-algebras, which are not visible with classical methods. We obtain these new results by investigating the classifying toposes of notable theories of MV-algebras and by applying the bridge technique to Morita-equivalences between such theories and suitable theories of lattice-ordered abelian groups.

Many-valued logics and MV-algebras

Motivated by the fact that classical logic cannot describe situations that admit more than two outcomes, in 1920 J. Łukasiewicz introduced a three-valued logic by adding to the traditional truth values 0 and 1, interpreted as "absolute false" and "absolute true", a third degree of truth between them. Later, he presented further generalizations with n truth values (or even a countable or a continuous number of them).

The class of MV-algebras was introduced in 1958 by C. C. Chung (cf. [25] and [26]) in order to provide an algebraic semantics for Łukasiewicz multi-valued propositional logic. As this logic is a generalization of classical logic, MV-algebras are a generalization of boolean algebras (these can be characterized as the idempotent MV-algebras).

After their introduction in the context of algebraic logic, MV-algebras became objects of independent interest and many applications in different areas of Mathematics were found. The most notable ones are in functional analysis (cf. [40]), in the theory of lattice-ordered abelian groups (cf. [40] and [29]) and in the field of generalized probability theory (cf. Chapters 1 and 10 of [42] for a general overview).

Several equivalences between categories of MV-algebras and categories of lattice-ordered abelian groups (ℓ-groups, for short) can be found in the literature, the most important ones being the following:

- *Mundici's equivalence* (cf. [40]) between the whole category of MV-algebras and the category of ℓ-groups with strong unit;
- *Di Nola-Lettieri's equivalence* (cf. [29]) between the category of perfect MV-algebras (i.e., MV-algebras generated by their radical) and the whole category of ℓ-groups.

We observe that these categorical equivalences can be seen as equivalences between categories of set-based models of certain geometric theories and we prove that these theories are Morita-equivalent, i.e., there is a categorical equivalence between their categories of models inside any Grothendieck topos $\mathcal{E}$, naturally in $\mathcal{E}$.

In this way we obtain:

- a Morita-equivelence between the theory $\mathbb{MV}$ of MV-algebras and the theory $\mathbb{L}_u$ of ℓ-groups with strong unit;
- a Morita-equivalence between the theory $\mathbb{P}$ of perfect MV-algebras and the theory $\mathbb{L}$ of ℓ-groups.

We then show that the Morita-equivalence arising from Di Nola-Lettieri's equivalence is just one of a whole class of Morita-equivalences that we establish between theories of local MV-algebras in proper varieties of MV-algebras and appropriate extensions of the theory of ℓ-groups.

Consequences of the Morita-equivalence between $\mathbb{MV}$ and $\mathbb{L}_u$

An immediate consequence of the Morita-equivalence arising from Mundici's equivalence is the fact that the (infinitary) theory of ℓ-groups with strong unit is of presheaf type. This arises from the process of transferring the invariant property of being a presheaf topos across the Morita-equivalence. Recall that a theory is of presheaf type if its classifying topos is equivalent to a topos of presheaves. Every finitary algebraic theory, and more generally, every cartesian theory, is of presheaf type; thus, this property is transferred from the theory of MV-algebras to $\mathbb{L}_u$. We are interested in theories of presheaf type since they enjoy many remarkable properties that do not hold for any geometric theory.

$$\begin{tikzcd}
 & \mathcal{E}_{\mathbb{MV}} \simeq \mathcal{E}_{\mathbb{L}_u} & \\
\mathbb{MV} \arrow[ur, dash, dashed, bend left] & & \mathbb{L}_u \arrow[ul, dash, dashed, bend right]
\end{tikzcd}$$

Changing the invariant considered at the level of the classifying topos gives rise to further results. For instance, the invariant given by the property to be a subtopos induces, by the Duality Theorem of [11] (which establishes a bijection between the subtoposes of the classifying topos of a given geometric theory and the quotients of this theory), a bijection between the quotients of the theory $\mathbb{MV}$ and those of the theory $\mathbb{L}_u$. It is worth to stress that this result cannot be deduced from Mundici's equivalence. Recall that a quotient of a theory is an extension over the same signature obtained by adding new axioms. Starting from a quotient of $\mathbb{MV}$, we get the corresponding quotient of $\mathbb{L}_u$ by translating every axiom in the language of ℓ-groups with strong unit by using the interpretation from the theory $\mathbb{MV}$ to the theory $\mathbb{L}_u$. However, as proved in the same section, there is no interpretation in the converse direction that would make trivial the bijection between the quotients. If we consider now the invariant property of objects of toposes to be irreducible we get a logical characterization of the finitely presentable ℓ-groups with strong unit. They are the ℓ-groups with strong unit corresponding to the finitely presented MV-algebras under Mundici's equivalence. Specifically, we show that such groups can be characterized as the finitely presented pointed ℓ-groups $\mathcal{G}$ with a distinguishing element v which is a strong unit for $\mathcal{G}$, or, equivalently, as the ℓ-groups presented by a formula which is irreducible with respect to the theory of ℓ-groups with strong unit. This last result is used to describe a method for obtaining an axiomatization of the quotient of $\mathbb{MV}$ corresponding to a given quotient of the theory $\mathbb{L}_u$. Lastly, we establish a form of compactness and completeness for $\mathbb{L}_u$, obtained from the invariant properties of the classifying topos of $\mathbb{MV}$ (whence of $\mathbb{L}_u$) to have a compact terminal object and to have enough points.

Finally, as a particular instance of this Morita-equivalence, we obtain a sheaf-theoretic version of Mundici's equivalence valid for any topological space X, naturally in X.

Consequences of the Morita-equivalence between $\mathbb{P}$ and $\mathbb{L}$ and of the study of the classifying topos of $\mathbb{P}$

As in the case of Mundici's equivalence, the Morita-equivalence arising from Di Nola-Lettieri's equivalence involves an algebraic theory, namely the theory $\mathbb{L}$ of ℓ-groups. Thus, the property to be of presheaf type is transferred to the coherent theory $\mathbb{P}$ of perfect MV-algebras. Whilst the two theories are not classically bi-interpretable, further applications of the bridge technique lead to three different levels of bi-interpretability between particular classes of formulas: irreducible formulas, geometric sentences and imaginaries.

$$\begin{tikzcd} & \mathcal{E}_{\mathbb{P}} \simeq \mathcal{E}_{\mathbb{L}} \arrow[dl, dash, dashed, bend right] \arrow[dr, dash, dashed, bend left] & \\ \mathbb{P} & & \mathbb{L} \end{tikzcd}$$

Irreducible formulas for the theory $\mathbb{P}$ are the ones that present the finitely presentable perfect MV-algebras, that is the algebras which correspond to the finitely presented ℓ-groups via Di Nola-Lettieri's equivalence. They constitute the analogue for the theory $\mathbb{P}$ of cartesian formulas in the theory of MV-algebras. Indeed, even though the category $\mathbb{P}$-mod(**Set**) is not a variety, it is generated by its finitely presentable objects since the theory $\mathbb{P}$ is of presheaf type classified by the topos [f.p.$\mathbb{P}$-mod(**Set**), **Set**]. We also establish a bi-interpretability between the theory of lattice-ordered abelian groups and a cartesian theory $\mathbb{M}$ axiomatizing the positive cones of these groups, which we use to obtain a simpler reformulation of Di Nola-Lettieri's equivalence and to describe the partial bi-interpretations between $\mathbb{L}$ and $\mathbb{P}$. This bi-interpretation between $\mathbb{M}$ and $\mathbb{L}$ provides in particular an alternative description of the Grothendieck group associated with a model $\mathcal{M}$ of $\mathbb{M}$ as a subset, rather than a quotient as in the classical definition, of the product $\mathcal{M} \times \mathcal{M}$.

Next, we study in detail the classifying topos of the theory of perfect MV-algebras, representing it as a subtopos of the classifying topos of the algebraic theory axiomatizing the variety generated by Chang's MV-algebra. This investigation sheds light on the relationship between these two theories, notably leading to a representation theorem for finitely generated (resp. finitely presented) algebras in Chang's variety as finite products of finitely generated (resp. finitely presented) perfect MV-algebras. It is worth to note that this result, unlike most of the representation theorems available in the literature, is fully constructive. Among the other insights, we mention a characterization of the perfect MV-algebras which correspond to finitely presented lattice-ordered abelian groups via Di Nola-Lettieri's equivalence as the finitely presented objects of Chang's variety which are perfect MV-algebras, and the property that the theory axiomatizing Chang's variety proves all the cartesian sequents (in particular, all the algebraic identities) which are valid in all perfect MV-algebras.

We then revisit the representation theorem obtained through the analysis of the classifying topos of $\mathbb{P}$ from the point of view of subdirect products of perfect MV-algebras, obtaining a concrete proof of it. We also show that every MV-algebra in Chang's variety is a weak subdirect product of perfect MV-algebras. These results have close ties with the existing literature on weak boolean products of MV-algebras. Moreover, we generalize to the setting of MV-algebras in Chang's variety the Lindenbaum-Tarski characterization of boolean algebras which are isomorphic to powersets as the complete atomic boolean algebras, obtaining an intrinsic characterization of the MV-algebras in Chang's variety which are arbitrary products of perfect MV-algebras. These results show that Chang's variety constitutes a particularly natural MV-algebraic setting extending the variety of boolean algebras.

Finally, we transfer the above-mentioned representation theorems for the MV-algebras in Chang's variety in terms of perfect MV-algebras into the context of ℓ-groups with strong unit and, generalizing results in [2], we show that a theory of pointed perfect MV-algebras is Morita-equivalent to the theory of lattice-ordered abelian groups with a distinguished strong unit (whence to that of MV-algebras).

Morita-equivalences for local MV-algebras in proper varieties of MV-algebras

In light of the fact that the class of perfect MV-algebras is the intersection of the class of local MV-algebras with a specific proper variety of MV-algebras, namely Chang's variety, it is natural to wonder what happens if we replace this variety with an arbitrary variety of MV-algebras. We prove that 'globally', i.e., considering the intersection with the whole variety of MV-algebras, the theory of local MV-algebras is not of presheaf type, while if we restrict to any proper subvariety V, the theory of local MV-algebras, indicated with the symbol $\mathbb{L}oc_V$, is of presheaf type. Furthermore, we show that these theories are Morita-equivalent to suitable theories expanding the theory of ℓ-groups. More specifically, if $V = V(\{S_i\}_{i\in I}, \{S_j^\omega\}_{j\in J})$ (for finite subsets $I, J \subseteq \mathbb{N}$) we have a theory $\mathbb{G}_{(I,J)}$ which is Morita-equivalent to the theory $\mathbb{L}oc_V$ and which is written over the signature obtained from that of ℓ-groups by adding a constant symbol and propositional predicates corresponding to the elements of I and J. The categories of set-based models of these theories are not in general algebraic as in the case of perfect MV-algebras; however, we characterize the varieties V for which we have algebraicity as precisely those which can be generated by a single chain. All the Morita-equivalences contained in this new class are non-trivial, i.e., they do not arise from bi-interpretations, as we prove.

$$\begin{tikzcd} & \mathcal{E}_{\mathbb{L}oc_V} \simeq \mathcal{E}_{\mathbb{G}_{(I,J)}} \arrow[dl, dash, dashed] \arrow[dr, dash, dashed] & \\ \mathbb{L}oc_V & & \mathbb{G}_{(I,J)} \end{tikzcd}$$

Topos-theoretic methods are used here to obtain both logical and algebraic results. Specifically, we present two (non-constructively) equivalent axiomatizations for the theory of local MV-algebras in an arbitrary proper subvariety V, and we study the Grothendieck topologies associated with them as quotients of the algebraic theory $\mathbb{T}_V$ axiomatizing V. The subcanonicity of the Grothendieck topology associated with the first axiomatization ensures that the cartesianization of the theory of local MV-algebras in V is the theory $\mathbb{T}_V$. It is worth to note that this result does *not* arise from a representation theorem of the algebras in V as subdirect products or global sections of sheaves of models of the theory of local MV-algebras in V, something that would make this trivial. To verify the provability of a cartesian sequent in the theory $\mathbb{T}_V$, we are thus reduced to checking it in the theory of local MV-algebras in V. Using this, we easily prove that the radical of every MV-algebra in V is defined by an equation, which we use to present the second axiomatization. This latter axiomatization has the notable property that the associated Grothendieck topology is rigid. This allows us to conclude that the theory of local MV-algebras in V is of presheaf type. The equivalence of the two axiomatizations and the consequent equality of the associated Grothendieck topologies yields in particular a representation result of every finitely presented MV-algebra in V as a finite product of local MV-algebras. This generalizes the representation result obtained for the finitely presented MV-algebras in Chang's variety as finite products of perfect MV-algebras.

Strictly related to the theory of local MV-algebras is the theory of simple (in the sense of universal algebra) MV-algebras; indeed, an MV-algebra $\mathcal{A}$ is local if and only if the quotient

$\mathcal{A}/\mathrm{Rad}(\mathcal{A})$ is a simple MV-algebra. This theory shares many properties with the theory of local MV-algebras: globally it is not of presheaf type but it has this property if we restrict to an arbitrary proper subvariety. On the other hand, while the theory of simple MV-algebras of finite rank is of presheaf type (as it coincides with the geometric theory of finite chains), the theory of local MV-algebras of finite rank is not, as we prove.

* * *

Summarizing, in this thesis we use topos-theoretic techniques to study Morita-equivalences obtained by 'lifting' categorical equivalences which are already known in the literature of MV-algebras and also to establish new ones. This shows that, as it was already argued in [12], topos theory is indeed a powerful tool for discovering new equivalences in Mathematics, as well as for investigating known ones.

The main themes addressed in this thesis are the following:

- theories of presheaf type;
- Morita-equivalences and bi-interpretations;
- MV-algebras and lattice-ordered abelian groups;
- representation results for classes of MV-algebras;
- cartesianizations for quotients of $\mathbb{MV}$.

A particular attention is posed on the constructiveness of the results; we indicate with the symbol * the points where we use the axiom of choice.

Structure of the thesis

The thesis is organized in five chapters.

Chapter 1. In this chapter we recall the most important notions and results on topos theory. We mostly focus on the technique of 'toposes as bridge' that we apply throughout the thesis and on notions of classifying topos and of theory of presheaf type.

Chapter 2. In this chapter we introduce the classes of MV-algebras that are studied in the thesis, namely perfect, local and simple MV-algebras. Morever, we establish some preliminary results on the respective quotients of $\mathbb{MV}$. For instance, we prove that the theory of local MV-algebras and the theory of simple MV-algebras are not of presheaf type. Further, we introduce two equivalent axiomatizations for the theory of perfect MV-algebras and we show that the radical of every MV-algebra in Chang's variety is definable by an equation. This result is necessary for defining the radical of a model of the theory of perfect MV-algebras in an arbitrary Grothendieck topos as the classical definition of the radical is not constructive. We also derive the fact that the radical cannot be defined by a geometric formula in the whole class of MV-algebras as a consequence of the fact that the class of semisimple MV-algebras cannot be axiomatized in a geometric way.

Chapter 3. In this chapter we show that the theory of MV-algebras is Morita-equivalent to (but not bi-interpretable with) to that of lattice-ordered abelian groups with strong unit. This generalizes the well-known equivalence between the categories of set-based models of the two theories established by Mundici, and allows to transfer properties and results across them by using the methods of topos theory. We discuss several applications, including a sheaf theoretic version of Mundici's equivalence and a bijective correspondence between the geometric theory extensions of the two theories.

Chapter 4. We establish, generalizing Di Nola and Lettieri's categorical equivalence, a Morita-equivalence between the theory of lattice-ordered abelian groups and that of perfect MV-algebras. Further, after observing that the two theories are not bi-interpretable in the classical sense, we identify, by considering appropriate topos-theoretic invariants on their common classifying topos, three levels of bi-interpretability holding for particular classes of formulas: irreducible formulas, geometric sentences and imaginaries. Lastly, by investigating the classifying topos of the theory of perfect MV-algebras, we obtain various results on its syntax and semantics also in relation to the cartesian theory of the variety generated by Chang's MV-algebra, including a concrete representation for the finitely generated models of the latter theory as finite products of perfect MV-algebras. Among the results established on the way, we mention a Morita-equivalence between the theory of lattice-ordered abelian groups and that of cancellative lattice-ordered abelian monoids with bottom element.

Chapter 5. In this chapter we study quotients of the geometric theory of local MV-algebras, in particular those which axiomatize the class of local MV-algebras in a proper subvariety. We show that each of these quotients is a theory of presheaf type which is Morita-equivalent to an expansion of the theory of lattice-ordered abelian groups. Di Nola-Lettieri's equivalence is recovered from the Morita-equivalence for the quotient axiomatizing the local MV-algebras in Chang's variety, that is the perfect MV-algebras. We establish along the way a number of results of independent interest, including a constructive treatment of the radical for local MV-algebras in a fixed proper variety of MV-algebras and a representation theorem of the finitely presentable algebras in such a variety as finite products of local MV-algebras.

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