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Logit Dynamics for Strategic Games

Mixing time and Metastability

Tutor
prof. **Vincenzo Auletta**

Candidato
Diodato Ferraioli

Coordinatore
prof. **Giuseppe Persiano**

Abstract

A *complex system* is generally defined as a system emerging from the interaction of several and different components, each one with their properties and their goals, usually subject to external influences. Nowadays, complex systems are ubiquitous and they are found in many research areas: examples can be found in Economy (e.g., markets), Physics (e.g., ideal gases, spin systems), Biology (e.g., evolution of life) and Computer Science (e.g., Internet and social networks). Modeling complex systems, understanding how they evolve and predicting the future status of a complex system are major research endeavors.

Historically, physicists, economists, sociologists and biologists have separately studied complex systems, developing their own tools that, however, often are not suitable for being adopted in different areas. Recently, the close relation between phenomena in different research areas has been highlighted. Hence, the aim is to have a powerful tool that is able to give us insight both about Nature and about Society, an universal language spoken both in natural and in social sciences, a modern code of nature. In a recent book [16], Tom Siegfried pointed out *game theory* as such a powerful tool, able to embrace complex systems in Economics [3, 4, 5], Biology [13], Physics [8], Computer Science [10, 11], Sociology [12] and many other disciplines.

Game theory deals with selfish agents or *players*, each with a set of possible actions or *strategies*. An agent chooses a strategy evaluating her utility or *payoff* that does not depend only on agent's own strategy, but also on the strategies played by the other players. The way players update their strategies in response to changes generated by other players defines the *dynamics* of the game and describes how the game evolves. If the game eventually reaches a fixed point, i.e., a state stable under the dynamics considered, then it is said that the game is in an *equilibrium*, through which we can make predictions about the future status of a game.

The classical game theory approach assumes that players have *complete knowledge* about the game and they are always able to select the strategy that maximizes their utility: in this *rational* setting, the evolution of a system is modeled by *best response dynamics* and predictions can be done by looking at well-known *Nash equilibrium*. Another approach is followed by *learning dynamics*: here, players are supposed to "learn" how to play in the next rounds by analyzing the history of previous plays.

By examining the features and the drawbacks of these dynamics, we can detect the basic requirements to model the evolution of complex systems and to predict their future status. Usually, in these systems, environmental factors can influence the way each agent selects her own strategy: for example, the temperature and the pressure play a fundamental role in the dynamics of particle systems, whereas the limited computational power is the main influence in computer and social settings. Moreover, as already pointed by Harsanyi and Selten [9], the complete knowledge assumption can fail due to limited information about external factors that could influence the game (e.g., if it will rain tomorrow), or about the attitude of other players (if they are risk taking), or about the amount of knowledge available to other players.

Equilibria are usually used to make predictions about the future status of a game: for this reason, we like that an equilibrium always exists and that the game converges to it. Moreover, in case that multiple equilibria exist, we like to know which equilibrium will be selected, otherwise we could make wrong predictions. Finally, if the dynamics takes too long time to reach its fixed point, then this equilibrium cannot be taken to describe the state of the players, unless we are willing to wait super-polynomially long transient time.

Thus we would like to have dynamics that models bounded rationality and induces an equilibrium that always exists, it is unique and is quickly reached. *Logit dynamics*, introduced by Blume [6], models a noisy-rational behavior in a clean and tractable way. In the logit dynamics for a game, at each time step, a player is randomly selected for strategy update and the update is performed with respect to an *inverse noise* parameter β (that represents the degree of rationality or knowledge) and of the state of the system, that is the strategies currently played by the players. Intuitively, a low value of β represents the situation where players choose their strategies "nearly at random" because they are

subject to strong noise or they have very limited knowledge of the game; instead, an high value of β represents the situation where players "almost surely" play the best response, that is, they pick the strategies yielding high payoff with higher probability. This model is similar to the one used by physicists to describe particle systems, where the behavior of each particle is influenced by temperature: here, low temperature means high rationality and high temperature means low rationality. It is well known [6] that this dynamics defines an ergodic finite *Markov chain* over the set of strategy profiles of the game, and thus it is known that a *stationary distribution* always exists, it is unique and the chain converges to such distribution, independently of the starting profile.

Since the logit dynamics models bounded rationality in a clean and tractable way, several works have been devoted to this subject. Early works about this dynamics have focused about *long-term behavior* of the dynamics: Blume [6] showed that, for 2×2 *coordination games* and *potential games*, the long-term behavior of the system is concentrated in a specific Nash equilibrium; Alós-Ferrer and Netzer [1] gave a general characterization of long term behavior of logit dynamics for wider classes of games. A lot of works have been devoted to evaluating the time that the dynamics takes to reach specific Nash equilibria of a game, called *hitting time*: Ellison [7] considered logit dynamics for *graphical coordination games* on cliques and rings; Peyton Young [15] extended this work for more general families of graphs; Montanari and Saberi [14] gave the exact graph theoretic property of the underlying interaction network that characterizes the hitting time in graphical coordination games; Asadpour and Saberi [2] studied the hitting time for a class of *congestion games*.

Our approach is different: indeed, our first contribution is to propose the stationary distribution of the logit dynamics Markov chain as a new equilibrium concept in game theory. Our new solution concept, sometimes called *logit equilibrium*, always exists, it is unique and the game converges to it from any starting point. Instead, previous works only take in account the classical equilibrium concept of Nash equilibrium, that it is known to not satisfying all the requested properties. Moreover, the approach of previous works forces to consider only specific values of the rationality parameter, whereas we are interested to analyze the behavior of the system for each value of β.

In order to validate the logit equilibrium concept we follow two different lines of research: from one hand we evaluate the performance of a system when it reaches this equilibrium; on the other hand we look for bounds to the time that the dynamics takes to reach this equilibrium, namely the *mixing time*. This approach is trained on some simple but interesting games, such as 2×2 coordination games, congestion games and two team games (i.e., games where every player has the same utility).

Then, we give bounds to the convergence time of the logit dynamics for very interesting classes of games, such as potential games, *games with dominant strategies* and graphical coordination games. Specifically, we prove a twofold behavior of the mixing time: there are games for which it exponentially depends on β, whereas for other games there exists a function independent of β such that the mixing time is always bounded by this function. Unfortunately, we show also that there are games where the mixing time can be exponential in the number of players.

When the mixing is slow, in order to describe the future status of the system through the logit equilibrium, we need to wait a long transient phase. But in this case, it is natural to ask if we can make predictions about the future status of the game even if the equilibrium has not been reached yet. In order to answer this question we introduce the concept of *metastable distribution*, a probability distribution such that the dynamics quickly reaches it and spends a lot of time therein: we show that there are graphical coordination games where there are some distributions such that for almost every starting profile the logit dynamics rapidly converges to one of these distributions and remains close to it for an huge number of steps. In this way, even if the logit equilibrium is no longer a meaningful description of the future status of a game, the metastable distributions resort the predictive power of the logit dynamics.

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