Abstract

This manuscript is devoted to the study of the qualitative behaviour of the solutions of evolution equations arising from elliptic and parabolic problems on unbounded domains with unbounded coefficients. In particular, we deal with the elliptic operator of the form

$$A = \operatorname{div}(Q\nabla) + F \cdot \nabla - V,$$

where the matrix $Q(x) = (q_{ij}(x))$ is symmetric and uniformly elliptic and the coefficients q_{ij}, F and V are typically unbounded functions.

Since the classical semigroup theory does not apply in case of unbounded coefficients, in Chapter 1 we illustrate how to construct the minimal semigroup $T(\cdot)$ associated with A in $C_b(\mathbb{R}^d)$. It provides a solution of the corresponding parabolic Cauchy problem

$$\begin{cases} \partial_t u(t,x) = Au(t,x), & t > 0,\, x \in \mathbb{R}^d, \\ u(0,x) = f(x), & x \in \mathbb{R}^d, \end{cases}$$

for $f \in C_b(\mathbb{R}^d)$, that is given through an integral kernel p as follows

$$T(t)f(x) = \int_{\mathbb{R}^d} p(t,x,y)f(y)\,dy.$$

Moreover, such solution is unique if a Lyapunov function exists. Since an explicit formula is in general not available, it is interesting to look for pointwise estimates for the integral kernel p.

In Chapter 2 we consider a Schrödinger type operator in divergence form, namely the operator A when $F = 0$. We prove first that the minimal realization $A_{\min}$ of A in $L^2(\mathbb{R}^d)$ with unbounded coefficients generates a symmetric sub-Markovian and ultracontractive semigroup on $L^2(\mathbb{R}^d)$ which coincides on $L^2(\mathbb{R}^d) \cap C_b(\mathbb{R}^d)$ with the minimal semigroup generated by a realization of A on $C_b(\mathbb{R}^d)$. Moreover, using time dependent Lyapunov functions, we show pointwise upper bounds for the heat kernel of A. We then improve such estimates and deduce some spectral properties of $A_{\min}$ in concrete examples, such as in the case of polynomial and exponential diffusion and potential coefficients.

Chapter 3 deals with the whole operator A. With appropriate modifications, similar kernel estimates described above are valid for this operator. In addition, we prove global Sobolev regularity and pointwise upper bounds for the gradient of p. We finally apply such estimates in case of polynomial coefficients.