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Orbital motion, periastron advance and galaxy rotation curves beyond General Relativity

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Abstract

The Extended Theories of Gravity (ETG) have become one of the most investigated theoretical proposals among the alternative explanations for the observed flatness of galaxy rotation curves, related to the dark matter problem, as well as for the accelerated expansion of the universe, related to the dark energy problem. The reason lies in the fact that ETG's can provide predictions consistent with the observational surveys without implicating invisible matter. In this framework, these phenomena are explained as a physical manifestation of extra-curvature terms of the geometry of the Universe. In the first part, we focus on this class of theories which are a curvature-based extension of GR. Higher order scalar curvature invariants are included in the Einstein-Hilbert action giving rise to the Higher Order Theories and corresponding field equations. For the extension of GR, we consider the Scalar-Tensor-Fourth-Order Gravity (STFOG) in metric formalism as a representative general class for the ETG, obtained from combination of Fourth Order Gravity plus a coupled scalar field. The NonCommutative Spectral Gravity is a special case of STFOG. Other Higher Order Theories, like the $f(R)$-gravity models, are sub-classes of it.

In this scenario to analyse the orbital motion of interacting objects constituting astrophysical gravitating systems, like the Solar System or galaxies, is a very important issue for making new predictions and testing theories. We discuss the fundamentals of the physical regime given by the Weak Field limit, Newtonian and Post-Newtonian limits, and their corresponding expansions as a mathematical procedure to solve the field equations in STFOG. This makes it possible to deal with the problems of motion for a system of many particles and reproduce many physical configurations. We solve the linearized field equations of STFOG stemming from the weak field limit, and this is done in the Standard Post-Newtonian gauge, which is the suitable choice for the purpose. Then we find the space-time metric and the potentials connected to each metric component that give rise to the gravitational field. Their behaviour presents a modification to the Newtonian potential induced by the Yukawa-like potential terms (5th force) of the type $V(r) = \alpha\frac{e^{-\beta r}}{r}$. Finally, in the context of the STFOG, we determine the relativistic Lagrangian leading to the equations of orbital motion for a system of N-body and involving the Post-Newtonian fields. This allows

to find out the equations governing the dynamics of a generic N-body system, like those in the Solar System, binary systems (when $N = 2$) or possibly the S-stars cluster around Sagittarius A*, thus providing a theoretical reference for Relativistic Celestial Mechanics beyond General Relativity and the possibility to study realistic astrophysical models and gravitational tests.

In the second part, we expose the problem of anomalistic precession and deal with the analysis of the periastron shift. We consider the Adkins & MacDonell integrals and making use of the data coming from the precession of planets, we deduce constraints on the parameters of the STFOG, therefore also of Non-Commutative Spectral Gravity (NCSG) (as particular case), including a study for the Quintessence Field (deformation of the Schwarzschild geometry induced by a dark energy) related to a power-law potential. We show that the periastron shift of planets allows us to improve the bounds on the range of interaction β by several orders of magnitude. Then we develop a new resolution method for the determination of the periastron advance by relying on the epicyclic perturbation, which includes also the Post-Newtonian contributions and can be applied to theories beyond GR like the ETG, or models within, without the necessity of numerical integration. Using it, we obtain the final results and then deduce the full analytic expressions for the advance relative to the examined ETG. We carry out the preceding analysis once more, and further improvements on the bounds are achieved.

In the last part, by resorting to the Newtonian limit, we provide the theoretical galaxy rotation curves in the context of the $f(R)$-theory, the more general STFOG and the a NonCommutative Spectral Gravity. Therefore, the first analysis of galaxy rotation curves in NCSG is conducted. Through the parametric fits with observed data, we derive direct predictions on the physical parameters (total mass and mass-to-light ratio) for an unexplored sample of spiral galaxies of the THINGS catalogue. Good reproductions are obtained for these theories as well as numerical predictions on the physical parameters characterizing a galaxy. The predictions are directly comparable with the observations. We compare the numerical outcomes for the metric $f(R)$-theory with those of the Palatini formalism and, in the end, we make a comparison of the results relative to the examined ETG with the observed astronomical estimations.

> If I have seen further than others, it is because I have stood on the shoulders of giants.

\- Isaac Newton

> A roof shingle by itself does not kill a man. It produces this effect only through the acquired velocity, that is, the man is killed by space and time.

\- G. W. F. Hegel, Encyclopedia of Philosophical Sciences in Compendium.

> The scientist does not study nature because it is useful to do so. He studies it because he takes pleasure in it, and he takes pleasure in it because it is beautiful. If nature were not beautiful it would not be worth knowing, and life would not be worth living.

\- Henri Poincaré, Science and Method.

> There is no royal road to science, and only those who do not fear the fatiguing climb of steep paths have a chance of reaching its luminous summits.

\- Karl Marx, Das Kapital.

> The claim that absolute space and time exist ”independently of any external object” seems strange to Newton because he often emphasises the fact that he intends to investigate only what is real, that is, what can be detected by observation. His motto, defined and concise, is ’hypothesis non fingo’ (I do not formulate hypotheses). However, what exists "independently of any external object" is not observable and is not a real fact. We are evidently dealing here with a case in which preconceived ideas have been applied to the objective world, whose veracity has not been completely examined.

\- Max Born, La sintesi Einsteiniana.

> The author has made every effort to present the basic ideas in the clearest and simplest form possible. To achieve maximum clarity, it seemed inevitable to me to repeat the same concept several times, without caring for the elegance of the exposition. I scrupulously followed the precept of the brilliant theoretical physicist Ludwig Boltzmann, according to whom the problems of elegance should be left to tailors and shoemakers.

\- Albert Einstein, Uber die spezielle und allgemeine Relativitatstheorie.

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Introduction

0.1 The Dark Universe

A large number of different and independent astrophysical surveys carried out on galactic, extra-galactic and cosmological scales, highlights a possible abundance of non-baryonic dark matter with respect to the amount of baryonic matter present in the Universe. The first indications of the existence of dark matter were due to J.H. Oort and F. Zwicky. In 1932, Oort inferred the velocities of the stars near the plane of the Milky Way by means of the observation of their Doppler shift, and noticed that the stellar velocities around the Milky Way centre could be obtained only by a central gravitational force of the galactic system much stronger than that of the Solar System. Therefore, a large amount of matter that originating this stronger gravitational pull had to be invoked to find an explanation. Anyway, it became clear very soon that all the luminous mass contained in our galaxy was not enough to yield such a force and reproduce such stellar velocities. Furthermore, in 1933, by examining velocity dispersions of galaxies in the Coma Berenices cluster and applying the Virial Theorem, Zwicky analogously discovered that the only visible matter was not able to give rise to a sufficient gravitational pull to keep the cluster stably bounded, and other not-detected mass was necessary to explain the velocities of the objects inside the cluster. Zwicky called it *dunkle materie* (dark matter), and it must be underlined that both Oort and Zwicky carried out their studies by starting from Newtonian gravity. For a few years the problem was lightly underestimated, because it was quite believed that such dark matter could be constituted by baryonic ordinary massive particles, which is why it was thought that the real problem could be how to detect this non-visible matter at extra-galactic scales, from the practical point of view. However, since 1978 further seminal works concerning the rotation curves of galaxies by V.C. Rubin, W.K. Ford, and N. Thonnard have ultimately shown that the problem was really serious because those similar phenomena were observed on galactic scales as well. The gravitational effects associated to this dark matter were effective at galactic scales, but not only this: it was inferred that baryonic luminous matter did not at all correspond to the required mass needed to give rise to those rotation curves, beyond any reasonable doubt. By that moment, it became clear this invisible matter should be composed of an unknown exotic matter different from the baryonic ordinary one, of which stars, gas, dusts and anything else are made. The issue of dark matter could no longer be underestimated. These fundamental investigations gave rise to one of the most important enigmas of modern physics: the *missing mass problem*. Since then, even more intensive research on dark matter followed. [1; 2; 8; 9; 10].

Besides this, there is another fundamental component of the Universe which is considered related to the accelerated expansion of the Universe: the so-called dark energy. In fact in 1998 two independent projects, the Supernova Cosmology Project and the High-Z Supernova Search Team, showed that the Universe is expanding with a positive acceleration by using distant type Ia supernovae to measure the acceleration. This means that the velocity at which a distant galaxy recedes from the observer is growing over time and such an effect is attributed to a sort of constant repulsive force acting at cosmological scales, and whose origin at the moment is thought to be the *cosmological constant* viewed as an intrinsic dark energy (or *vacuum energy*) of space [3; 4]. If we also take into account the discovery of the Cosmic Microwave Background (CMB) and the successful predictions of the standard cosmology on the abundance of light elements, the peak positions of the CMB acoustic spectrum and baryon acoustic oscillations, the problem is further compounded by the fact that dark matter and ordinary baryonic matter (as well as neutrinos and photons) contribute only for the 25% and 5% to the dynamics of the Universe, respectively [5].

0.1.1 The ΛCDM model and its shortcomings

All these phenomenologies of the Dark Universe which were found out during the last decades, quickly led to shape the modern Standard Model of Cosmology, which is commonly referred to as ΛCDM-model, which is based on the following five assumptions:

1. correctness of General Relativity's field equation, theory constructed on Local Lorentz Invariance, universality of free fall and conservation of energy-momentum;
2. the matter fields are given by fluids of dust and radiation;
3. the space is homogenous and isotropic on large cosmological scales;
4. 3 spatial dimensions below the electro-weak scales;

5. the Standard Model of particles;

In order to match with observational evidence, these assumptions necessitate the presence of the dark matter and dark energy components, respectively. In other words, it is possible to achieve a working model for the description of the Universe by incorporating the cosmological constant Λ (which accounts for the dark energy) in Einstein's field equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} + \Lambda g_{\mu\nu}$$

and, then, supposing that the total amount of matter is constituted by a visible baryonic percentage (originated by primordial nucleosynthesis) and a much larger percentage of supposed invisible nonbaryonic particles, which dominate both at galactic and extra-galactic scales. In particular, the Friedmann-Robertson-Walker-Lemaitre metric

$$ds^2 = c^2dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$$

provides the usual set equations of the Standard Model of Cosmology, which are

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3}, \tag{1}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3\frac{p}{c^2}\right) + \frac{\Lambda c^2}{3}, \tag{2}$$

$$\dot{\rho} = -\left(\frac{\dot{a}}{a}\right)\left(\rho + \frac{p}{c^2}\right), \tag{3}$$

$$p = w\rho. \tag{4}$$

where we remind $a(t)$ is the scale factor of the geometry related to the observed redshift z by $a(t_{em}) = 1/(1+z)$[1], k is the spatial curvature of the Universe[2], ρ and p mass-energy density of the fluid in the co-moving frame and pressure respectively, w a constant of the equation of state having a specific value depending on the kind of fluid, $G = 6.674 \times 10^{-11}\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2} \pm 2.2 \times 10^{-5}$ the gravitational constant[3]. It is also possible to introduce the Hubble parameter $H(t) \equiv \dot{a}/a$ for describing the expansion rate and consider critical density

$$\rho_c = \frac{3H_0^2}{8\pi G} = 1.8784 \times 10^{-26}\,h^2\,\mathrm{kg\,m}^{-3}$$

with $H_0 = v/d = 64.7 \pm 0.5$ km s^{-1} Mpc^{-1} the proper Hubble constant[4] and $h \equiv H_0/(100\ \mathrm{km\,s}^{-1}\,\mathrm{Mpc}^{-1})$ the dimensionless reduced Hubble constant; ρ_c is the useful

[1] t_{em} is the time of light emission.

[2] $k = 1$ is the case of a closed Universe, $k = 0$ flat Universe, $k = -1$ open Universe.

[3] Such experimental value of G is recommended by *2018 CODATA recommended values*.

[4] v and r are the recession velocities (redshifts) and distances of a sample of objects, such as supernova host galaxies.

present-day density value for which the Universe is flat ($k = 0$) and obtained assuming the $\Lambda = 0$. Then, after defining the present-day density parameter Ω_x as the dimensionless ratio

$$\Omega_x \equiv \frac{\rho_x(t_0)}{\rho_c} = \frac{8\pi G\rho_x(t_0)}{3H_0^2}$$

where t_0 is the time of the present-day, ρ_x represents the different species of contribution to the dynamics of the Universe. Denoting with ρ_b, ρ_{CDM}, ρ_r, ρ_k, ρ_Λ, the baryonic matter, cold dark matter, radiation, spatial curvature and dark energy densities respectively, thanks to this parametrisation, the first Friedmann equation can be written in the following effective form

$$\left(\frac{H(a)}{H_0}\right) = (\Omega_b + \Omega_{CDM})a_0/a^{-3} + \Omega_r a_0/a^{-4} + \Omega_k a_0/a^{-2} + \Omega_\Lambda a_0/a^{-3(1+w)}.$$

If the Friedmann equation, written in function of the density parameters normalized to the critical density ρ_c, is considered to the present day values, then immediately follows

$$\Omega_b + \Omega_{CDM} + \Omega_r + \Omega_k + \Omega_\Lambda = 1$$

which is also known as the concordance model equation. According to the hypothesis of this model and consistently with the predictions of Big Bang nucleosynthesis, combinations of cosmological and astrophysical measurements (e.g. supernovae), together with those concerning CMB anisotropies from large to small scales and galaxy clustering from the Sloan Digital Sky Survey (SDSS), as well as an extensive survey conducted by the Planck Collaboration [6] in 2018, generally show that the normalized density parameters seem to fit well with the overall data only for $\Omega_k = 0.001 \pm 0.002$ (meaning substantially a flat Universe), $\Omega_r \simeq 2.47 \times 10^{-5}$ (the Universe is dominated by matter and dark energy), the dark energy has a pressure fluid parameter $w = -1.03 \pm 0.04$, and specifically we get

$$\Omega_m = \Omega_b + \Omega_{CDM} = 0.315 \pm 0.007, \qquad \Omega_\Lambda = 0.6847 \pm 0.0073.$$

Here Ω_m is the normalized density parameter associated with the total amount of fluid matter that combines baryonic dark matter ($\Omega_b h^2 = 0.0224 \pm 0.0001$) and non-baryonic dark matter ($\Omega_{CDM} h^2 = 0.120 \pm 0.001$) [5; 6]. The ΛCDM-model is at the moment the simplest model capable of reproducing the major observed properties of the Universe, which are the large-scale structure in the distribution of galaxies, the abundances of hydrogen, helium, and lithium, the existence and the structure of the CMB, the accelerating cosmological expansion observed by studying the redshift of spectral absorption or emission lines in the light from distant galaxies and the time dilation in the light decay of Type Ia Supernovae luminosity curves. Alongside dark energy, there is supposed to be the fundamental presence of dark matter in galaxies and clusters. Indeed, the ΛCDM-model is composed of two parts: dark (or vacuum) energy and cold dark matter.

The tenets of the *Cold Dark Matter (CDM) model* are the following:

- dark matter must be constituted by invisible exotic non-baryonic particles, as we can infer by the nucleosynthesis processes of baryonic matter (protons, neutrons, electrons, etc.) in the early Universe.
- dark matter particles interact with ordinary visible matter particles only through gravitational force and possibly the weak force, but not with through other fundamental forces;
- the velocity of the exotic particles constituting dark matter is non-relativistic, i.e. it is very low compared to the speed of light at the epoch of radiation-matter equivalence and it cannot cool by radiating photons, in this way it sufficiently moves slow to collect around galaxies[5];
- it forms collisionless particle fluids around the galaxies, usually collected in spherical halos.

Starting with these hypotheses, it is possible to elaborate dark-matter profiles by means of N-body simulations and in order to study the galaxies formation and clustering in a Universe dominated by cold dark matter as well. One of the most popular general profiles simulating dark matter halos is the *Navarro-Frenk-White (NFW) profile*

$$\rho(r) = \frac{\rho_0}{\frac{r}{R_s}\left(1 + \frac{r}{R_s}\right)^2},$$

where ρ_0 and the scale radius R_S are parameters varying from halo to halo of the different galaxies (add some other information). Another successful one is the *Burkert profile*

$$\rho(r) = \frac{\rho_0 r_c^3}{(r + r_c)(r^2 + r_c^2)},$$

which, from the very beginning, became widespread because it was able to get better fits of the observed galaxy rotation curves than the NFW profile. It was conceived as a best-fitting law density consistently describing also the observed curves of dwarf galaxies with a pseudo-isothermal halo's behaviour; differently from other dark matter profiles, it

[5]For instance, neutrinos are not baryons but however they move at relativistic velocities, therefore too fast for collecting around galaxies.

has also a central core characterized by two structural parameters: the central density ρ_0 and the core radius r_c.

However, despite a relevant number of successful predictions maturated over the years (among which, for example, there are those regarding the statistics of weak gravitational lensing of galaxies, where galaxies themselves and their supposed dark matter halos are employed as lenses), the ΛCDM-model has revealed a growing number of shortcomings. Furthermore, more recent outcomes, accompanied by new re-analysed measurements as well as new sophisticated simulations, are again questioning what appeared to be confirmed physical hypotheses and acquired results. We may summarise at least some of these major tensions in the following list.

- Unexplained smallness of the cosmological constant, which has a registered positive value $\Lambda = 1.1056 \times 10^{-52}\text{m}^{-2}$ with observed density $\rho_\Lambda = 10^{-48}\,\text{GeV}^4$, and a great discrepancy of about 120 orders of magnitude with the estimated value $\rho_{Pl} = 10^{72}\,\text{GeV}^4$ of vacuum energy density from Quantum Field Theory (QFT) up to the Planck scale, where it is supposed that the action of the zero-point fluctuations plays the role of the dark energy contribution to the dynamics of the Universe[6].

- A possible violation of the homogeneity principle, in fact several large-scale structures such as the Clowes-Campusano and U1.11 Large Quasar Group (580 Mpc and 780 Mpc lengths, respectively), the massive superstructure named Hercules–Corona Borealis Great Wall (with estimated length between 2000 and 3000 Mpc) or the more recently discovered Giant Arc of galaxies and clusters (about 1000 Mpc length), appear to be in contrast with the expectation that the spatial distribution of galaxies can be considered statistically homogeneous if averaged over scales $260/h$ Mpc [16], therefore challenging such principle.

- A possible violation of isotropy by virtue of a hemispheric bias in the CMB with respect to the average temperature and larger variations in the degree of density perturbations; such anisotropies in the CMB result in being statistically relevant and now must seriously be taken into account [7] and, furthermore, anomalies have been encountered by testing the cosmological isotropy through new studies on Type Ia supernovae [21] as well as on the distribution of galaxies and clusters [22].

- Starting from the CDM model, N-body simulations reveal that the density profiles of

[6] In QFT, the empty space is defined by the vacuum state composed of a collection of quantum fields, which exhibit fluctuations in their ground state arising from the zero-point energy of the space.

dark matter halos are much more peaked than it should be with respect to the observed distribution in galaxies. In other words, galaxies are observed to have cores such that their density flattens out at the centre, this clear evidence emerges by investigating their rotation curves [23] and it is known as the *cusp halo problem*.

- Another really important problem regards the velocity of galactic bars, because the presence of a massive dark-matter halo surrounding the entire galaxy should slow down its bars by virtue of the dynamical friction with the halo itself [27]. This problem is called the *galaxy bar problem*.

- N-body simulations, in the context of the CDM model, show a discrepancy between the estimated number and the observed number of small dwark galaxies around galaxies of larger sizes (for example, the Milky Way itself). In particular, on the basis of these simulations the number of dwarf galaxies should be much higher and, to make matters worse, it was also found out that their orbits result to be localised on a thin planar structure around the Milky Way and Andromeda galaxy, while the CDM model predict that they should generally have a random distribution around them [24; 29]. The issue itself of the much higher number of predicted dwarf galaxies could also be related with other problems of the CDM model at sub-galactic scales, such as the excessive amount of dark matter predicted in the innermost regions of the galaxies which is inconsistent with what is expected by many observation [30].

- The ΛCDM-model is generally founded on the correctness of the Strong Equivalence Principle, but there is a possibility of a violation of such principle. In fact, an effect which is not consistent with the tidal effects supported by the model and therefore in contrast to what is expected was recently observed. This was realized by analysing data from the SPARC catalogue and then detecting an external gravitational field effect close to rotationally supported galaxies [28]. Whether this would be confirmed, such an outcome should entail a violation of the Strong Equivalence Principle.

- Last but not least: the ΛCDM-model does not expect the existence of the Tully-Fisher and the Faber-Jackson relations, as well as the curious tight correlation between baryons and dark matter in galaxies consisting in a universal acceleration scale $a_0 \simeq 1.2 \times 10^{-10}\,\mathrm{m\,s^{-2}}$, because in general ordinary baryonic matter should not know how dark matter behaves.

0.1.2 Theoretical Approaches to the problem

Until now, many theoretical explanations based on new theories and models have been proposed to understand the nature of the dark universe. Referring more specifically to the delicate dark matter issue, these theoretical attempts must be able to reproduce the dynamics and predicted observables in agreement with well-established experimental and observational measures [17; 18; 20]. Several approaches have been taken into consideration and, in particular, if we also refer to the ΛCDM model, it is thought that the missing matter is constituted by non-baryonic exotic particles like WIMPs (Weakly Interacting Massive Particles) [25; 4], particles predicted by SuperSymmetric theories, or simply by a kind of unknown matter. For these reasons, we may define this kind of approach as the *paradigm of particles*.

But it is not the only viable manner to face the problem; in fact another significant possibility is provided by the extensions (or modifications) of our current reference gravity field theory, i.e. General Relativity. We could denominate this approach *the geometrical paradigm*. This name is due to the fact that the phenomenologies of the Dark Universe could be physical manifestations of extra-curvature contributions to the space-time geometry not predicted by General Relativity, which also involve that the laws of gravity should act in a diverse manner for large-scale structure and, therefore, they could have a different description with respect to what is experienced at Solar System scales.

Actually, modifications and extensions of GR started from the very beginning: since from 1919 H. Weyl and A. Eddington tried to generalize the Einstein-Hilbert action by relaxing the hypothesis $\mathcal{L} = R$ and introducing higher order invariant terms in the Lagrangian density. They explored and studied these additive higher order curvature invariant terms because motivated by the will to enhance further the mathematical arrangement of GR, discover new interesting features underlying the theory, reveal some observational discrepancies with it, better understand why it worked so well, and what higher order scalar invariants imply from a physical point of view, reveal some observational discrepancies with it, better understand why it worked so well. The first form of the theory was investigated in Refs. [69; 82; 62]. In the following decades, several authors explored the interesting but non-trivial mathematical properties and physical meanings of this approach. At present day, the purpose to explore and possibly understand the nature of dark matter and dark energy gradually led to a renewed interest to the framework of Extended Theories of Gravity (ETG), which are based on the extension of the Lagrangian density in the Einstein-Hilbert action to higher order terms of the Ricci invariants and possibly including a scalar field [48]

$$\mathcal{L} = f(R, R^2, R_{\alpha\beta}R^{\alpha\beta}, \Box R, \phi). \tag{5}$$

Looking at (5), we remark two main features: the geometry can couple non-minimally to some scalar field so that we get a scalar-tensor gravity, and then derivatives of the metric components of order higher than second can appear giving place to higher order theories. There is also to consider that effective actions coming from quantum fields in curved space-times produce mixed higher order/scalar-tensor gravity. This is possible by taking in account quantum corrections which add to the effective Lagrangian of the gravitational field a combination of minimally or non-minimally coupled scalar fields and higher order terms, in other words, by combining coupled terms between scalar fields and geometry. Such theories are also related to an Einstein's gravity plus one or multiple coupled scalar field by means of conformal transformations [69; 81; 82].

Among these, the $f(R)$-theory represents also a possible explanation of the cosmological inflation [49], and the circumstance of the possibility of explaining galactic rotation curves began to spread because their Newtonian limit conducted to potentials in the analytical form required to reproduce the right profile without invoking a non-baryonic invisible component of matter [48; 65; 109; 111]. The subsequent astrophysical discoveries previously discussed, along with the necessities of elaborating a unified theory of all fundamental interactions of nature, revealed that the extension of the action was also a well motivated operation, sometimes even necessary, to realise first trials of a Theory of Everything. In addition to the recent fundamental discoveries concerning the evidence of the dark components apparently dominating the Universe of which we detect the gravitational effects, we should also take into account the stability of certain astrophysical objects that would be considered unstable as such as some neutron stars [124; 123]. Whether these reasons were not enough, in cosmology, where we have discovered the presence of a Dark Energy component leading to an accelerated expansion of the Universe by means of the distance luminosity measures of type Ia Supernovae, the observational measures coming from the in-depth studies of the CMB revealed an initial inflationary phase for the Universe and, so, these further problems of modern physics constitute other fundamental physical motivations to extend the Lagrangian and consider a more general function. In fact, these theories have acquired a relevant interest because, among other things, they exhibit inflationary behaviour in a very natural way [48; 49; 84], allowing in this way to overcome the anomalies and the failures of the Standard Model of Cosmology. Looking at astrophysical problems, we should emphasize the nontrivial result that they are able to reproduce the rotation curves and some general dynamical properties of galaxies without using the required amount of dark matter necessary to explain why the rotation curves are not the ones predicted with respect to Newtonian gravity. Besides this, dark matter has not yet been directly detected.

All these facts lead us to take seriously the idea that it is possible that R is simply the lower-order term of a series containing arbitrarily higher power terms of invariants constructed with curvature tensor and its contractions. Among the different possibilities, one of the most studied ETG is the $f(R)$-theory [84; 124], where $f(R)$ is an arbitrary analytical function of the Ricci scalar; one can introduce in the Einstein-Hilbert action a general function of R in the form of an infinite series of curvature powers instead of the only linear invariant curvature scalar R, meaning

$$S = \int_{\mathcal{V}} \sqrt{-g} f(R)\, d^4x, \tag{6}$$

with the function $f(R)$ expressed in the following analytical form

$$f(R) = R + a_2 R^2 + a_3 R^3 + ... + a_n R^n, \tag{7}$$

with a_2, a_3, a_n constants yielding appropriate length dimensions and R^n denoting the n-th power of the invariant curvature scalar R. Other possibilities are discussed in [18; 20; 210; 211]. We note that these terms also occur in the action field of Kaluza-Klein theories or string theories, since corrections to the action of Einstein-Hilbert in the form (7) are predicted when one uses methods of dimensional reduction. In particular, it should be stressed that in the low-energy limit of the Lagrangian of these theories, extra-curvature terms appear when extra spatial dimensions are compactified (for more details, see [63]. The constants $a_2, a_3, ..., a_n$ represent powers of the string length λ_s which is a parameter of the theory. Whether we consider terms induced by quantum corrections of the constants a_2, a_3, a_n yet representing the powers of the parameter λ linked to the coupling constant $\mathcal{X}$ by the relation $\lambda \sim \hbar c \mathcal{X}$, this shows how all the corrections are cancelled in classic limit $\hbar \to 0$.

0.2 Program of the Thesis

The program of the thesis work is organized as follows:

1. In Chapter 1, we introduce the argument by starting from a brief summary of General Relativity and important aspects of the theory. Hence, we enunciate the Lovelock Theorem as a relevant tool to better infer how it is possible to go beyond General Relativity and different properties a theory may have to realize such a purpose. We motivate physically and more deeply why curvature based extensions and Extended Theories of Gravity could be one the most plausible approaches.

2. In Chapter 2, we then present a general class of ETG, the Scalar-Tensor-Fourth-Order Gravity (STFOG), its particular classes, the special case of Non-Commutative Spectral Geometry (NCSG) as well as the Quintessence field model (GR + Dark Energy) and the corresponding actions and field equations for each theory in metric formalism. This part is also devoted to outline relevant highlights, properties of the theories. We first begin from the most general case of ETG, the Scalar-Tensor-Fourth-Order Gravity, then we specialize to the cases of their main classes and arrive to Non-Commutative Spectral Geometry and Quintessence fields.

3. In Chapter 3, we discuss the weak field limit as well as the Newtonian and Post-Newtonian approximations, needed for the objectives of the thesis and how to apply it in order to find solutions of the field equations. It is presented the geodesic principle as leading technique to deduce the equations of motion for an isolated system of N bodies in our relativistic framework, and motivations by which the weak field expansion is sufficient to the aim. We then solve the linearized field equations in the *Standard Post-Newtonian gauge* for STFOG and NCSG. After this, we treat more specifically solutions in spherically symmetric metric, passing from isotropic to spherically symmetric solutions as well as showing what further method to determine solutions by fixing the spherical symmetry in the starting metric. At the end of the Chapter, we focus on the determination of the relativistic Lagrangian equations descending from the field solutions, i.e. the relativistic STFOG equations of orbital motion for the N-body system of interacting particles in the most general case. The general result solves the problem of finding the set of differential equations corresponding to a given astrophysical configuration and opens further possibilities for the Relativistic Celestial Mechanics beyond General Relativity, allowing to realize models valid for numerous real astrophysical configurations, for example, such as the

Solar System or any kind of N-body stellar system as well as binary systems.

4. In Chapter 4, we analyze the classical dynamical tests in the context of ETG, i.e. the periastron precession, the light deflection and the Shapiro time delay. Regarding the first test, we first proceed to determine the periastron advance by employing the Adkins & MacDonnel's method and then find the results on the new constraints to the theories. Later we elaborate a new method for the determination of the periastron advance, valid for any theory and model, based on the generalization of orbital epicyclic perturbation and then, through the final formula, we obtain the new improved results on the constraints of the theories. The tests are performed in the the Solar System or the S-stars around Sagittarius A, i.e. the Black Hole at the galactic centre of the Milky Way.

5. In Chapter 5, we achieve the galaxy rotation curves formulae in the framework of $f(R)$-theory, the Scalar-Tensor-Fourth-Order Gravity and Non-Commutative Spectral Geometry. Thus, concerning the NCSG a first ever analysis is carried out. We develop parametric fits from which to infer the numerical values of total baryonic mass, core radius and mass-to-light ratio. To this end, we consider a point-like source and a spherically symmetric metric. We apply the Newtonian limit and present the theoretical velocity curve of galaxies by assuming circular motion and adopting a suitable profile for the distribution of matter. We perform the fits of the velocity curves considering the observational data for an unexplored sample of six spiral galaxies. By using the HI Nearby Galaxy Survey catalogue [219; 220], we reproduce the observed curves as well as we infer the values of the physical properties of galaxies. Provided the numerical predictions and which are their order of magnitude, we then carry out the analysis again by relaxing an assumption on the matter profile, and we get predictions on the galactic properties in agreement with astronomical expectations for most galaxies of the sample. The analysis is performed by using data coming from the HI Nearby Galaxy Survey and the THINGS catalogue[7]

6. In Chapter 6, we draw our conclusions.

[7]in literature such an analysis have been also performed by using the SPARC (Spitzer Photometry & Accurate Rotation Curves) catalogue (for example, see [218]).

Chapter 1

Beyond General Relativity

In all the approaches aiming to deal with the fundamental astrophysical and cosmological issues, General Relativity is the starting point and so it is especially for the geometrical paradigm that will be treated here, consisting of its extensions through different extra-curvature terms. Hence, we briefly summarize the basics of Eintein's theory.

1.1 A Summary of Einstein's General Relativity

Up to now, General Relativity (GR) is still the most baseline and successful gravity field theory more than a century after its first formulation by A. Einstein (1915) and many experimental verifications which arrived year after year. It contains Newtonian gravity in the particular case of a small velocity of bodies with respect to the speed of light. As early as 1915, Einstein was able to explain the famous anomalous Mercury's perihelion precession found out by Urbain Le Verrier (1867) and this definitely convinced him that he had the right gravitational field equations in the hand [59; 60; 61]. But the first clear proof arrived in 1919, when Arthur Eddington measured the shift of a star's position on the celestial sphere, which was very close to the Sun during the eclipse of that year: the shift of the position was due to the light deflection predicted by the theory and calculated by Einstein himself [69; 54; 58]. In the following years until today, General Relativity obtained a long series of successes grounded on verified consequences implied by the theory and experimentally tested. Important examples are: the gravitational redshift, the gravitomagnetic frame-dragging precession known as Lense-Thirring effect [52; 53], the relativistic collapse of supermassive stars into a Black Hole [53; 54], the emission of gravitational waves by a Binary Pulsar stars system [26] and their direct detection announced in February 2016 [53], the first direct observation of a Black Hole at the centre of the galaxy

M87 published in April 2019 as well as the one investigated at the centre of the Milky Way, namely Sagittarius A*, in May 2022 [31; 32].

1.1.1 Relativity, Invariance and Equivalence Principles

The theory of General Relativity is based on the following basic principles:

- *Relativity Principle*: all reference frames are equivalent for the formulation of the fundamental laws of physics. The laws of physics are the same in all inertial and non-inertial reference frames.

- *Local Lorentz Invariance*: physical quantities are invariant under Lorentz transformations. The outcome of any local non-gravitational experiment is independent of the velocity of the freely falling laboratory reference frame.

- *Local Position Invariance*: the outcome of any local non-gravitational experiment is independent of position and time of the laboratory reference frame.

- *Weak Equivalence Principle*: the test particles undergo the same acceleration. Their motion under gravitational forces is the same and does not depend on their weights, internal structures or compositions. It is the principle associated with the universality of free-fall and is represented by the equality

  $$m_i = m_g$$

  between inertial and gravitational mass. This principle is valid for bodies which are not self-gravitating. We can evaluate the self-gravitating nature of a body through the *parameter*

  $$\varsigma = \frac{GMm_g^2}{r}\frac{1}{m_i c^2},$$

  where G is the gravitational constant, c the speed of light and r the size of the body. For the equality $m_i = m_g$, ς becomes

  $$\varsigma = \frac{GM}{c^2 r}.$$

  Whether $\sigma \ll 1$ the self-gravity is safely negligible. In its most general version, the Principle holds also for bodies whose self-gravity is relevant, and it is called *Gravitational Weak Equivalence Principle*. In this case, self-gravity does not affect the motion of the body and the universality of free-fall comprises self-gravitating objects.

- *Einstein's Equivalence Principle*: the outcome of any non-gravitational experiment is not affected by the presence of gravitational field at any point of space-time. It is the core of any metric theory of gravity. Following the *Clifford M. Will*'s scheme, this principle is given by the composition of the Local Lorentz Invariance and Local Position Invariance along with the Weak Equivalence Principle [80].

- *Strong Equivalence Principle*: the outcome of all physical experiments, both gravitational and non-gravitational, are not affected by the presence of a gravitational field at any point of space-time. Hence, the motion of a small test-body is independent of its internal composition and it depends only on its initial conditions (position and velocity) in space-time. It necessarily implies that $G = 6.674 \times 10^{-11}\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$ is always constant. In the *Will*'s scheme, this principle is given by the composition of the Local Lorentz Invariance and Local Position Invariance along with the Gravitational Weak Equivalence Principle [80].

- *General Covariance Principle*: the laws of physics are the same in any reference frame and are represented by tensorial equations that never change with respect to a change of the coordinate system. Fields of any rank are expressed by tensors.

- *Causality Principle*: each point of space-time admits a universally valid notion of past, present, and future. In other words, the causal connection must be preserved.

By virtue of the fact that the effects of gravitational interaction are locally indistinguishable from those of an accelerated system, so that the gravitational effects can be locally cancelled by simply applying an appropriate acceleration. The Einstein's Equivalence Principle can be reformulated in a shorter manner: in a local space-time region the gravitational force can always be locally cancelled[1]. It must be stressed that the Equivalence principles establish an equivalence among gravitational and inertial forces. For Refs. about the classification of the Equivalence Principles, see e.g. [53; 79; 80; 101].

[1]An infinitesimal neighbourhood of a given point of space-time.

1.1.2 The Connection between Gravitation, Geometry and Dynamics

In GR the space-time is a 4-dimensional differentiable manifold $\mathcal{M}$ equipped with a metric tensor $g_{\mu\nu}$, which describes its structure; the causal structure of space-time describes the causal relations between two points, and these relations are defined by regular curves that link them. The quadratic line element which express the distance between a pair of points is

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu, \tag{1.1}$$

where $g_{\mu\nu}(x)$ is the metric tensor of the manifold $\mathcal{M}$. The Einstein's Equivalence Principle requires that the motion of a particle subject to a gravitational force is given by the geodesics equation. This is equivalent to the request of stationarity of the action

$$\delta \int_a^b ds = 0. \tag{1.2}$$

when the variational principle is applied. The line element is $ds = \sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} d\lambda$, with λ a parameter along the wordline $x^\alpha(\lambda)$. Eq. (1.1) implies $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 1$ where $\dot{x}^\mu$ is the 4-velocity [2], and it identifies the Lagrangian $\mathcal{L}$ of the action 1.2. By applying the variation of the action, we get

$$\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0, \tag{1.3}$$

where

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\lambda}(g_{\lambda\mu,\nu} + g_{\nu\lambda,\mu} - g_{\mu\nu,\lambda}) \tag{1.4}$$

are the affine connections, which play the role of the gravitational forces. The affine connections (here Levi-Civita) are directly related to the metric $g_{\mu\nu}$ identifying the gravitational potential and its first derivative. One can quickly realise that the motion of the particles is thus determined by the causal structure of space-time and the equation linking its causal structure to dynamics is the geodesic equation. Geodesics are the extremal curves (shortest and largest lines connecting two points) defined on the manifold. In GR, geodesics also coincide with auto-parallel curves, which are the curves of the shortest length between a pair of two points sufficiently close to each other[3]. Therefore, the particles moving in a light cone follow the geodesics defined on the space-time manifold $\mathcal{M}$ and thus their

[2] The dot indicates the differentiation with respect to λ.

[3] If the points are not close to each other manifold, this is not generally true. In fact, among all curves connecting a pair of points on a manifold, the geodesics are generally those that keep the length functional *stationary*. This is the only requirement for a curve to be a geodesic.

motion is also governed by the Euler-Lagrangian equation. In fact, it is easy to demonstrate the equivalence between the geodesics equation and the Euler-Lagrange equation

$$\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{x}^{\alpha}} - \frac{\partial L}{\partial x^{\alpha}} = 0, \tag{1.5}$$

by considering Eq. (1.5) with $L = \frac{1}{2} g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}$ and perform the calculation. Eq. (1.5) is a system of second-order ordinary differential equations for the dynamics of a particle in a gravitational field. Since geodesics are defined independently of the coordinate system, and the same is valid for the geodesics equation, this law applies to any arbitrary reference frame. In Special Relativity (SR) the space-time structure is represented by a Minkowski flat metric. Hence SR is valid only in the absence of a gravitational field, i.e. in the absence of matter-energy deforming the space-time manifold. SR properly means absence of gravity and the inertial motion of a body that is not subject to external forces moves with uniform rectilinear motion. Thus, if we consider a reference frame located in a space-time region in which SR is valid, the equation describing a uniform rectilinear motion is the geodesic one. In flat space-time the geodesic is a straight line, while in the curved space-time of GR the geodesic becomes a curve line. In other words, this generalization involves that geodesics represent the equivalent of the inertial motion in curved space-time and so the elements of the metric tensor in the kinetic term of the Lagrangian are sufficient to determine the equations of motion of the objects and their dynamics. In the perspective of General Relativity, the gravitational force is no longer a real force in the classical meaning, but a local manifestation of the geometrical structure of the space-time curved by the presence matter (more generally the mass-energy), thus representing the gravitational source.

As outlined before, the underlying idea to Einstein's theory is that the geometry of the Universe is determined by the distribution of celestial bodies[4]. General Relativity is relativistic field theory of gravity, where space-time is interpreted as a four-dimensional manifold $\mathcal{M}$ equipped with a metric tensor $g_{\mu\nu}$ representing the fundamental fields from which the dynamics derive, and whose curvature originates from the mass-energy. The essential geometrical concepts required for its formulation are the metric $g_{\mu\nu}$, the affine connections $\Gamma^{\alpha}_{\mu\nu}$ associated to the covariant differentiation and and the curvature of the space-time manifold $\mathcal{M}$. Another important concept is the covariant derivative ∇_{α}: it is the generalization of the classical directional derivative of vector fields on a arbitrary Riemaniann or pseudo-Riemaniann manifold, transforming covariantly under a general coordinate transformation, and it is related to the parallel transport of a vector field (or tensor field) along a curve of the manifold itself. Since $g_{\mu\nu}$ defines for each point the scalar

[4]For the first time conceived by B. Riemann, at least in its primordial concepts.

products and encodes the variation of the vector modulus, while $\Gamma^\alpha_{\mu\nu}$ defines the covariant differential encoding the variation of a vector in direction and modulus due to its transport from one point to another, they are generally independent and both needed to represent the geometry of space-time. In particular the affine connection, as a fundamental mathematical object on the manifold, can generally be split into symmetric and anti-symmetric parts in the two lower indices respectively

$$\Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{(\mu\nu)} + \Gamma^\alpha_{[\mu\nu]}.$$

It is important to remark that GR is assumed to be torsion-free, i.e. $\Gamma^\alpha_{[\mu\nu]} = 0$. Furthermore, it assumes that the metric is preserved, namely $\nabla^\alpha g_{\mu\nu} = 0$, thus establishing a direct relation between the affine connection and the metric (partial derivatives of the metric). When this is the case, the affine connection is then called Levi-Civita connection and they are given by the Christoffel symbol. The only symmetric part of the affine connection is sufficient to identify the structure of the manifold. These two assumptions ($\nabla^\alpha g_{\mu\nu} = 0$, $\Gamma^\alpha_{[\mu\nu]} = 0$) are just motivated by the physical consistency with the character of the gravitational pulls between the bodies in the Universe [5]. In this case, the metric is the only field required for a complete knowledge (distances and parallel transport) of the space-time geometry. In synthesis, GR is a *metric* theory of gravity and therefore we say that the theory is elaborated by following a *metric formalism*. When a relativistic theory of gravity is treated by considering the $g_{\mu\nu}$ and $\Gamma^\alpha_{\mu\nu}$ as independent fields, the space-time manifold possess a metric-affine structure. For instance, this is precisely the approach of the *Palatini formalism* to GR and it returns the same results of the metric formalism. GR is the unique theory where metric and Palatini formalism are equivalent.

The components of the metric tensor represent the gravitational potentials, while the affine connections play the role of field forces. The curvature of the space-time manifold is described by the Riemann tensor[6]

$$R^\alpha{}_{\beta\mu\nu} = \frac{\partial \Gamma^\alpha{}_{\beta\nu}}{\partial x^\mu} - \frac{\partial \Gamma^\alpha{}_{\beta\mu}}{\partial x^\nu} + \Gamma^\lambda{}_{\beta\nu}\Gamma^\alpha{}_{\mu\lambda} - \Gamma^\lambda{}_{\beta\mu}\Gamma^\alpha{}_{\nu\lambda}, \tag{1.6}$$

from which the Ricci curvature tensor derives by contracting the first and third index of Riemann tensor

$$R_{\mu\nu} = R^\alpha_{\mu\alpha\nu},$$

[5]These assumptions could not hold at atomic and sub-atomic scale, where the particles dynamical system is governed by Quantum Mechanics. Attempts to unify GR with QM as Supersymmetric theories, go in this direction.

[6]For details see [20].

thereby we can write

$$R_{\mu\nu} = \frac{\partial \Gamma^{\rho}_{\mu\rho}}{\partial x^{\nu}} - \frac{\Gamma^{\rho}_{\mu\nu}}{\partial x^{\sigma}} + \Gamma^{\sigma}_{\mu\rho}\Gamma^{\rho}_{\sigma\nu} - \Gamma^{\sigma}_{\mu\nu}\Gamma^{\rho}_{\nu\rho}, \tag{1.7}$$

and the Ricci scalar is

$$R = g^{\mu\nu} R_{\mu\nu}. \tag{1.8}$$

1.1.3 Action and Field Equations

The equations describing the gravitational field are written in tensor form to obey the principle of covariance and such as to provide the Newtonian solution for weak fields and low velocities with respect to those of light. In the Lagrangian formulation of theory, they are deduced by the application of the variational principle.

The action for the fields is defined as the integral of a Lagrangian density

$$S = \int_{\mathcal{V}} \sqrt{-g}\, \mathcal{L}\, d^4x\,, \tag{1.9}$$

with

$$\mathcal{L} = \mathcal{X}\mathcal{L}_m + R. \tag{1.10}$$

Here g is the determinant of the metric tensor $g_{\mu\nu}$ and the invariant 4-dimensional volume is given by $\sqrt{-g}d^4x$. The total action can be see as the sum of two terms

$$S = S_m + S_{EH} \tag{1.11}$$

The first term in (1.11) represents the contribution of matter fields and the second represents the Einstein-Hilbert action. Therefore the Lagrangian density (1.11) is based on the fundamental hypothesis that the invariant curvature contribution is simply expressed by the linear function of the Ricci scalar $\mathcal{L} = R$ (the trace of the Ricci tensor $R_{\mu\nu}$, i.e. $R = g^{\mu\nu}R_{\mu\nu}$)[7]. In order to find the gravitational field equation, we apply the variational principle to the action (1.11) with respect to the inverse metric $g^{\mu\nu}$ is zero, with the condition that variations of the metric are null on the border of the four-dimensional volume where $\mathcal{L}$ is defined, namely $\delta g_{\mu\nu}\big|_{\partial\mathcal{V}} = 0$[8]. By imposing the stationarity of the action with respect to

[7] The $\mathcal{X}$ constant has the dimension $[\mathcal{X} = E^{-1}L$ because the action's dimension is $[S] = EL$, and Ricci scalar has the dimension $[R] = L^{-2}$. E indicates the energy, L the length.

[8] It is important to specify that the variational principle is not well placed with the only condition $\delta g_{\mu\nu}\big|_{\partial\mathcal{V}} = 0$ and it is necessary to impose that also the variation of the first derivatives is null on the boundary $\partial\mathcal{V}$. This is because R contains linear terms in the second derivatives of $g_{\mu\nu}$, giving rise to contributions proportional to the variation of the first derivatives $(\partial\delta g)$ of $g_{\mu\nu}$ on the boundary $\partial\mathcal{V}$ of the four-dimensional volume. In fact, these contributions do not go to zero with the only condition $\delta g_{\mu\nu}\big|_{\partial\mathcal{V}} = 0$, which only ensures that gradient of δg is equal to zero along the directions lying on the hyper-surface $\partial\mathcal{V}$, but not of the normal ones.

the variation, one obtains the Einstein's equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \mathcal{X} T_{\mu\nu} \tag{1.12}$$

with R the curvature scalar, $\mathcal{X} = 8\pi G/c^4$ the constant of the coupling's strength between matter and geometry[9]. $T_{\mu\nu}$ identifies the covariant energy-momentum tensor for a perfect fluid in dust form

$$T_{\mu\nu} = (\rho c^2 + p) u_\mu u_\nu - p g_{\mu\nu} \tag{1.13}$$

with ρc^2 the proper energy density, ρ the density at rest and p the pressure of the matter fluid. The energy-momentum tensor comes from the variation of the matter Lagrangian density, i.e. it is given by $T_{\mu\nu} = -\frac{1}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}$. The field equation can be summarized in a more compact convention by simply introducing the Einstein tensor

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \tag{1.14}$$

from which

$$G_{\mu\nu} = \mathcal{X} T_{\mu\nu}. \tag{1.15}$$

When $T_{\mu\nu} = 0$ one has Einstein's equation in vacuum and it regards the solution external to the mass-energy source and it is the analogue of the Laplace equation in Newtonian gravity. It also possible to give a different form to Einstein's equation through a multiplication by $g^{\mu\nu}$ of (1.12), yielding

$$R = -\mathcal{X} T, \tag{1.16}$$

which is the trace equation with $T = T^\rho_\rho$ trace of the energy-momentum tensor, useful both from a mathematical and physical point view as well as calculations. In particular, it allows us to rewrite the field equation in its trace-reversed form by substituting Eq. (1.16) in Eq. (1.12), one finds

$$R_{\mu\nu} = \mathcal{X} \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right). \tag{1.17}$$

The Einstein's Field Equation (EFE) connects the geometry of space-time to the distribution of matter–energy in the Universe, and it constitutes a set of non-linear second-order hyperbolic partial differential equations. To find the solutions of the EFE means to find the the space-time metric $g_{\mu\nu}$, which provide the knowledge about the gravitational field generated by the sources.

[9] Its experimental value is $\mathcal{X} = \dfrac{8\pi G}{c^4} = 2.077 \times 10^{-43}\,\mathrm{N}^{-1}$ and such coupling arises from the required consistency of weak fields with Newtonian gravity.

1.1.4 The Lovelock Theorem

In GR there are several theorems concerning its characteristics as well as its solutions[10]. One of the most important theorems, full of significance for any gravity field theory, is the Lovelock Theorem. It states:

In a 4-dimensional space-time the only second-order tensor of the field equation, preserving coordinate invariance and that we can get from a Lagrangian density depending uniquely on the metric $g_{\mu\nu}$ ($\mathcal{L}[g_{\mu\nu}]$), is

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} \tag{1.18}$$

In other words, the only symmetric divergence-free rank-2 tensor obtained uniquely by $g_{\mu\nu}$ and its derivatives up to second differential order, is the Einstein tensor plus a cosmological constant.

This means that in a 4-dimensional space-time manifold the only second-order field equation obtainable by the application of the variational principle with respect to the metric $g_{\mu\nu}$ and its derivatives, is the Einstein's equation with or without cosmological constant[11].

1.2 Open issues in GR and Mach's Principle

Concerning the issues related to the research for a Unifying Theory of all interactions, we should also remind General Relativity is a theory of gravity that follows a classical scheme, i.e. the Heisenberg's Uncertainty Principle is not applicable, and it is not a renormalizable theory unlike other field theories, such as Electrodynamics. GR does not fit with Quantum Field Theory (QFT), the development of a Quantum Gravity theory is an open issue and to which is added the fact that, in the Big Bang scenario, the Universe firstly goes through an era in which its dimensions are smaller than the Planck scale, the so called Planck era. In all of this, there is still no evidence that the gravitational field should have a quantum representation at high energies. Moreover its nature is not certain at such small scales because it is weak compared with the other fundamental interactions

[10] Like the Birkhoff's one stating that any spherically symmetric solution of the EFE in vacuum ($T_{\mu\nu 0} = 0$) must be static and asymptotically flat.

[11] Nevertheless, in a 4-dimensional space-time, it should be reminded that the Einstein–Hilbert Lagrangian density is not the unique to provide the EFE because the more general one from which we get it, is $\mathcal{L} = R - 2\Lambda + a(\epsilon^{\mu\nu\sigma\tau} R^{\alpha\beta}{}_{\mu\nu} R_{\alpha\beta\sigma\tau}) + b(R^2 - 4R^{\mu}{}_{\nu} R^{\nu}{}_{\mu} + R^{\mu\nu}{}_{\sigma\tau} R^{\sigma\tau}{}_{\mu\nu})$

and the characteristic scale under which one might observe gravitational quantum effects, is the Planck scale (i.e. 10^{-33} cm). Such a scale is currently far from any plausible experiment.

Then, GR predicts the Black Holes' singularities within their event horizon, the physical condition's point where gravity becomes so strong that space-time breaks down, although in principle it is a manifold supposed to be differentiable. In addition, we have previously discussed the most fundamental inconsistencies emerged in astrophysics and cosmology concerning the dark matter and dark energy problems and their detected effects. At this point, to better deal with these shortcomings at IR and UV scales and try to solve at least part of them, it is fair to ask if GR is really the ultimate theory of the gravitational interaction (as already happened with Newtonian gravity). Along with its predictive power and amazing successes, it however remains the starting point to be preserved for a future theory. We quickly realise that some of these issues are possibly present in GR independently of the fact it does not include quantum effects. Moreover, a priori it cannot be excluded a link among dark matter, dark energy and Quantum Gravity, which might be part of the same problem.

All these reasons justify the attempts to modify/extended General Relativity and there is a huge number of different kinds of approaches to this aim. In this regard, we should also pay attention to the Mach's Principle as well (that inspired Einstein himself while he was beginning to work on GR), according to which the inertial phenomena are due to the accelerations with respect to the distribution of all celestial bodies of the Universe. That is to say: the local laws of physics, including the gravitational force itself, are determined by the distribution of matter and its large-scale structure in the Universe and, therefore, the Newton's constant G depends on the matter distribution in the Universe. This could give rise to a possible new dynamical field in the overall Universe, implying a relation of G with such a field. For this reason, G itself is no longer constant but becomes a dynamical field. Consequently the measure of the mass of each material body, carried out by measures of the gravitation acceleration, should originate by this new kind of coupling. For years already, experimental tests about a possible variation of Newton's coupling are currently executed (see [79; 80; 53]).

1.3 How to go beyond and Curvature-based Extensions

The Lovelock Theorem constitutes a general reference to bear in mind and its consequences characterize the physical and mathematical nature of gravity field theories in metric formalism which have the purpose to go beyond General Relativity and deal with its (supposed or real) shortcomings. Therefore, in order to elaborate a viable new relativistic theory of the gravitational field, there are several possibilities and we mention the main ones:

1. to relax the hypothesis of the linear Einstein-Hilbert action by considering new higher order functions of the Ricci scalar or curvature invariants. In such approach one allows the existence of higher order derivatives of the metric in the field equations. This lead to the so-called class of Higher-Order Theories (HOG) and more generally the Extended Theories of Gravity (ETG). [81; 82; 62; 84; 87; 49; 48; 64; 65; 48; 110];

2. to introduce in the action further scalar and/or invariants built with vector or tensor fields in addition to the metric, or different from the metric. The scalar field can be minimally or non-minimally coupled. For example, this is the case of the Brans-Dicke and Scalar-Tensor theories (non-minimally coupled scalar field), the Scalar-Tensor-Vector Gravity (STVG) [39; 40; 41; 42] or the Tensor-Vector-Scalar Gravity (TeVeS), that is a relativistic version of MOND [44];

3. to consider extra-spatial dimensions of the background manifold $\mathcal{M}$ of the theory, with respect to the GR's space-time manifold ($D = 4$). Their existence ($D = 4 + d$) was the starting point of the Kaluza-Klein theory which aimed to unify electrodynamics and General Relativity. In time, this attempt gave rise to the class of Super-strings Theory and M-Theory [186; 187; 188; 189; 181; 190; 191; 192; 193; 194; 195];

4. to take in account the idea that a new theory of gravity could renounce to the Locality Principle, like Non-Local Gravity [128].

However, we remark the existence of many other theoretical approaches and models within their frameworks. For instance we just mention that, by renouncing to the assumption that space-time is torsion-free and then affine connection also contains the anti-symmetric part (related to the torsion), one gets the Einstein-Cartan-Sciama-Kibble Theory. Here, differently from GR, the presence of torsion implies that the auto-parallel curves do not necessarily coincide with the geodesics. On the other hand, other attempts to unify GR with Quantum Mechanics also go in this direction: to give up this assumption along with the metricity hypothesis ($\nabla_\alpha g_{\mu\nu} = 0$) is a requirement for SuperSymmetric theories, because such GR assumptions (torsion-free, metricity) could no longer be valid at atomic and sub-atomic particles' scale.

Here we deal with the Higher-Order Theories, with extra-curvature terms plus a possible coupled scalar field, in metric formalism, by considering a general class of ETG. As we will see later, in Newtonian and Weak Field limit, they predict Yukawa-like corrective gravitational potentials in addition to the Newtonian one, and whose contributions become increasingly relevant the larger the scale of the examined gravitating system and the range of interaction are. The forces stemming from these Yukawa-like potentials are the so-called fifth forces; they are associated to massive 2-spin gravitons that mediate the additive gravitational interactions and, in the end, conduct to a modification of the coupling's strength of the resulting force. In fact a relevant consequence of the Higher Order Theories, together with scalar-tensor theories, STVG or TeVeS just to cite a few, is that they are consistently in the spirit of the Mach's Principle because they predict that Newton's gravitational constant G can vary,thus changing the nature and the description of the gravitational pull on scales larger than Solar System. The reason is that ETG are constructed on the reliable Einstein's Equivalence Principle but not the Strong Equivalence Principle (see section (1.2.1)), that requires G to be always constant and indeed it holds only for GR. Therefore, the self-gravity of a single object can affect its motion. Although, up to now, laboratory and orbital experiments in the Solar System (which are also intended to explore the possible presence or not of fifth forces, however constraining their parameters) have obtained a good agreement with the SEP, both at shorter range and longer range (e.g. Lunar Laser Ranging Experiment and tests on Nordvedt effect) [33; 34; 35; 36; 37; 79] confirmed by further analysis concerning the Big Bang nucleosynthesis [53; 80] (according to which G cannot have varied by more than 10%), we should also bear in mind that possible violations of the SEP, at scales much larger than the Solar Systems, have already been suggested and registered by studying the SPARC data of external gravitational fields in the spatial proximities of rotationally supported galaxies [28]. This violation is in agreement with external field effects predicted by modified gravity models.

Before concluding this chapter, it is required a supplement about the *metric formalism* in ETG, which is adopted in the present work. An extended theory can be treated by means of two different formalism indeed: the metric and the Palatini's approach. In the metric formalism, the affine connection are expressed as a function of the metric and its partial derivatives. Thus, the metric is sufficient to describe the geometry of a manifold and the field equations are given by varying the action of the field with respect to the metric. In the Palatini formalism, the affine connections Γ are independent of the metric g of the space-time, and the metric structure of the space-time is decoupled from the geodesic structure. The gravitational field equations are obtained by varying the action with respect to the metric g and to the affine connection Γ, by requiring that the Ricci scalar depends on g and Γ, i.e. $\mathcal{R} = \mathcal{R}(g, \Gamma) = g^{\alpha\beta}\mathcal{R}_{\alpha\beta}(\Gamma)$, with $\mathcal{R}(\Gamma)$ the Ricci tensor of the non-metric connection. We remark that in General Relativity, these two approaches are equivalent. This is not the case for the ETG, because the two formalism yield different field equations that may therefore lead to different results. For example, in $f(R)$-gravity, the field equations in Palatini formalism leads to two metrics $g_{\mu\nu}$ and $\hat{g}_{\mu\nu} = f'(\mathcal{R})g_{\mu\nu}$ linked by a conformal transformation (the conformal factor must be non-degenerate). The geodesics can be obtained by the connections $\Gamma^{\alpha}_{\mu\nu}$, of the metric $\hat{g}_{\mu\nu}$. It must be underlined that Palatini $f(R)$-theory means that the field equations are deduced with a Lagrangian density of matter that does not depend on the non-metric connection Γ (otherwise, in case matter also depends on the connection, the theory is *metric-affine*), as well as it should be reminded that the trace equation containing the trace of stress-energy tensor is fundamental in order to manage the solutions of the field equations. In Palatini's approach, in fact, it turns out that in vacuum ($T = 0$), the traced equation yields constant solutions and therefore the field equations reduce to General Relativity plus a cosmological constant (for detailed references see [116; 117; 118; 119]).

Chapter 2

Extended Theories of Gravity

In this chapter, we show a general class of Extended Theories of Gravity (ETG) denominated Scalar-Tensor-Fourth-Order Gravity (STFOG), as well as particular theories contained within as interesting cases. The STFOG is the most general Fourth Order Gravity extending General Relativity and it also includes and a coupled scalar field providing the scalar-tensor character to the theory. Here, we use, for the Ricci tensor, the convention $R_{\mu\nu} = R^{\sigma}{}_{\mu\sigma\nu}$, whilst for the Riemann tensor we define $R^{\alpha}{}_{\beta\mu\nu} = \Gamma^{\alpha}_{\beta\nu,\mu} + \cdots$. The affine connections are the Christoffel symbols of the metric, namely $\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}g^{\mu\sigma}(g_{\alpha\sigma,\beta} + g_{\beta\sigma,\alpha} - g_{\alpha\beta,\sigma})$, and finally we adopt the signature $(+,-,-,-)$.

2.1 Scalar-Tensor-Fourth-Order Gravity (STFOG)

The action for this ETG is given by (see for example [48; 110; 111; 112])

$$S = \int d^4x\sqrt{-g}\Big[f(R, R_{\alpha\beta}R^{\alpha\beta}, \phi) + \omega(\phi)\phi_{;\alpha}\phi^{;\alpha} + \mathcal{X}\mathcal{L}_m\Big], \tag{2.1}$$

where f is a generic function of the Ricci scalar R, the invariant $R_{\alpha\beta}R^{\alpha\beta} = Y$ with $R_{\alpha\beta}$ the Ricci tensor, the scalar field ϕ, g is the determinant of metric tensor $g_{\mu\nu}$ and $\mathcal{X} = 8\pi G/c^4$. The Lagrangian density $\mathcal{L}_m$ is the minimally coupled ordinary matter Lagrangian density, $\omega(\phi)$ is a generic function of the scalar field. We now perform the variation of the action (2.1) with respect to metric $g_{\mu\nu}$, and to do this we split the calculation into three parts: the one only associated to the Ricci scalar $\mathcal{L}_f = f(R)$, another to the Ricci curvature combination's invariant $\mathcal{L}_Y = Y = R_{\alpha\beta}R^{\alpha\beta}$ and finally the one related to scalar field $\mathcal{L}_\phi = \omega(\phi)\phi_{;\alpha}\phi^{;\alpha}$. The matter fields are not coupled to ϕ, hence there is no dependence of the ordinary matter Lagrangian density $\mathcal{L}_m$ on ϕ. As in General Relativity, we quickly remind the variation of the determinant results to be

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}, \tag{2.2}$$

the Ricci scalar is $R = g^{\mu\nu}R_{\mu\nu}$ and varying R we have

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu} = R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\left(\nabla_\rho\delta\Gamma^\rho_{\nu\mu} - \nabla_\nu\delta\Gamma^\rho_{\rho\mu}\right). \tag{2.3}$$

By virtue of the fact that $\delta\Gamma^\lambda_{\mu\nu}$ transforms as a tensor, being the difference of two connections, it yields

$$\delta\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda a}\left(\nabla_\mu\delta g_{a\nu} + \nabla_\nu\delta g_{a\mu} - \nabla_a\delta g_{\mu\nu}\right), \tag{2.4}$$

and inserting it into Eq. (2.4), we have

$$R = R_{\mu\nu}\delta g^{\mu\nu} + g_{\mu\nu}\Box\delta g^{\mu\nu} - \nabla_\mu\nabla_\nu\delta g^{\mu\nu} \tag{2.5}$$

with ∇_μ covariant derivative and $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$ the d'Alembert operator. With these mathematical relations at the hands, we proceed by starting from the first action $S_f = \int\sqrt{-g}\,\mathcal{L}_f\,d^4x = f(R)$ and the variation principle gives

$$\delta S_f = \int\left(\sqrt{-g}\delta f + \delta\sqrt{-g}\,f\right)\,d^4x \tag{2.6}$$

$$= \int\left[f_R\delta R\sqrt{-g} - \frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}f\right]\,d^4x \tag{2.7}$$

$$= \int\sqrt{-g}\left[f_R(R_{\mu\nu}\delta g^{\mu\nu} + g_{\mu\nu}\Box\delta g^{\mu\nu} - \nabla_\mu\nabla_\nu\delta g^{\mu\nu}) - \frac{1}{2}g_{\mu\nu}\delta g^{\mu\nu}f\right]\,d^4x \tag{2.8}$$

Through integration by parts on the second and third terms and neglecting a pure divergence, we get

$$\delta S_f = \int\sqrt{-g}\left(f_R R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f + [g_{\mu\nu}\Box f_R - f_{R;\mu\nu}]\right)\,\delta g^{\mu\nu}\,d^4x \tag{2.9}$$

Analogously we now proceed to vary the action $S_Y = \int\sqrt{-g}\,\mathcal{L}_Y\,d^4x$ with $Y = R_{\alpha\beta}R^{\alpha\beta}$, from which

$$\delta S_Y = \delta\int\sqrt{-g}\,R_{\alpha\beta}R^{\alpha\beta}\,d^4x \tag{2.10}$$

$$= \delta\int\sqrt{-g}\,R_{\alpha\beta}g^{\alpha\rho}g^{\beta\sigma}R_{\rho\sigma}\,d^4x \tag{2.11}$$

$$= \int\sqrt{-g}\left[\left(R_\mu{}^\alpha R_{\alpha\nu} - \frac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta}\right)\delta g^{\mu\nu} + 2R^{\mu\nu}\delta R_{\mu\nu}\right]\,d^4x \tag{2.12}$$

$$= \int\sqrt{-g}\left[\left(R_\mu{}^\alpha R_{\alpha\nu} - \frac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta}\right)\delta g^{\mu\nu} + R^{\mu\nu}\left(2g^{\rho\sigma}\delta g_{\rho(\mu;\nu)\sigma} - \Box\delta g_{\mu\nu} - g^{\rho\sigma}\delta g_{\rho\sigma;\mu\nu)}\right)\right]\,d^4x, \tag{2.13}$$

then, after some computations, integrating again by parts and neglecting a pure divergence, we get

$$\delta S_Y = \int \sqrt{-g}\left(R_\mu^{\ \alpha} R_{\alpha\nu} - \frac{1}{2} g_{\mu\nu} R_{\alpha\beta} R^{\alpha\beta} - 2R^\sigma{}_{(\mu;\nu)\sigma} - \Box R^{\mu\nu} - R^{\sigma\tau}{}_{;\sigma\tau} g_{\mu\nu)}\right) \delta g^{\mu\nu}\, d^4x \quad (2.14)$$

The last step is to evaluate the variation of the action S_ϕ with respect to the metric, from which readily follows

$$\delta S_\phi = \int \sqrt{-g}\left(\omega(\phi)\phi_{;\mu}\phi_{;\nu} - \frac{\omega(\phi)\phi_{;\alpha}\phi^{;\alpha}}{2} g_{\mu\nu}\right) \delta g^{\mu\nu}\, d^4x. \quad (2.15)$$

Finally, by putting together Eqs. (2.9)-(2.14)-(2.15) and imposing the stationarity of the action $S = S_f + S_Y + S_\phi + S_m$, after the variation with respect to the metric tensor (i.e. $\frac{\delta S}{\delta g^{\mu\nu}} = 0$), we deduce the field equations

$$f_R R_{\mu\nu} - \frac{f + \omega(\phi)\phi_{;\alpha}\phi^{;\alpha}}{2} g_{\mu\nu} - f_{R;\mu\nu} + g_{\mu\nu}\Box f_R + 2 f_Y R_\mu{}^\alpha R_{\alpha\nu} + \quad (2.16)$$

$$-2[f_Y R^\alpha{}_{(\mu}]_{;\nu)\alpha} + \Box[f_Y R_{\mu\nu}] + [f_Y R_{\alpha\beta}]^{;\alpha\beta} g_{\mu\nu} + \omega(\phi)\phi_{;\mu}\phi_{;\nu} \;=\; \mathcal{X}\, T_{\mu\nu}\,,$$

where

$$T_{\mu\nu} \;=\; -\frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}} \quad (2.17)$$

is the energy-momentum tensor of matter. By varying with respect to the scalar field ϕ, one also gets the generalized Klein-Gordon equation

$$2\omega(\phi)\Box\phi + \omega_\phi(\phi)\phi_{;\alpha}\phi^{;\alpha} - f_\phi \;=\; 0, \quad (2.18)$$

while from Eq. (2.16) we easily obtain the trace equation

$$f_R R + 2 f_Y R_{\alpha\beta} R^{\alpha\beta} - 2f + 3\Box f_R + \Box f_Y R + f_Y R^{\alpha\beta}{}_{;\alpha\beta} - \omega(\phi)\phi_{;\alpha}\phi^{;\alpha} = \mathcal{X} T. \quad (2.19)$$

In Eqs. (2.16)-(2.18)-(2.19), we introduced the notations:

$$-\frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}$$

We confine ourselves to the case in which the generic function f can be expanded as follows

$$\begin{aligned} f(R, R_{\alpha\beta}R^{\alpha\beta}, \phi) \simeq \quad & f_R(0,0,\phi^{(0)})\, R + \frac{f_{RR}(0,0,\phi^{(0)})}{2} R^2 + \frac{f_{\phi\phi}(0,0,\phi^{(0)})}{2}(\phi - \phi^{(0)})^2 \\ & + f_{R\phi}(0,0,\phi^{(0)}) R\,\phi + f_Y(0,0,\phi^{(0)}) R_{\alpha\beta}R^{\alpha\beta}. \end{aligned} \quad (2.20)$$

It must noticed that all the other possible contributions in f are negligible [110; 111; 113; 112; 109], and it was also found that the field equations of the general class of STFOG have a well-posed initial value problem [85]. Concerning the matter lagrangian density $\mathcal{L}_m$, we can also evaluate the expression for $\mathcal{L}_m$. Since in metric theories the energy-momentum tensor depends only on the metric, we have

$$T_{\mu\nu} = -\frac{\delta \mathcal{L}_m}{\delta g_{\mu\nu}} + \frac{1}{2} g_{\mu\nu} \mathcal{L}_m \tag{2.21}$$

Hence by considering consider a source of mass M, regardless of its internal structure, the energy-momentum tensor is

$$T_{\mu\nu} = \rho c^2 u_\mu u_\nu, \qquad T = \rho c^2 \tag{2.22}$$

with ρ mass density and u_μ that fulfills the condition $g^{00} u_0 = 1$ and $u_i = 0$. From the definition (2.17), we can write

$$\delta \int \sqrt{-g}\, \mathcal{L}_m\, d^4x = -\delta \int \sqrt{-g}\, (\, T_{\mu\nu}\, \delta g^{\mu\nu}\,)\, d^4x = -\delta \int \sqrt{-g}\, (\, \rho c^2\, u_\mu u_\nu \delta g^{\mu\nu}\,)\, d^4x. \tag{2.23}$$

Since $\delta(\sqrt{-g}\rho c^2) = -1/2\sqrt{-g}\, \rho c^2\, u_\mu u_\nu\, \delta g^{\mu\nu} = 1/2\sqrt{-g}\, \rho c^2\, u^\mu u^\nu\, \delta g_{\mu\nu}$ and the variation of the density is $\delta \rho c^2 = (\rho c^2/2)(g_{\mu\nu} - u_\mu u_\nu)\delta g^{\mu\nu}$, we finally conclude that

$$\mathcal{L}_m = 2\rho c^2. \tag{2.24}$$

For completeness, in this kind of theory we also highlight the presence of ghost-like instabilities from the point of view of linear fluctuations. In fact, in generic fourth-order theories massive spin-2 degrees of freedom appear together with a scalar degree of freedom and the usual massless spin-2 degree of freedom from General Relativity. In such a context, the problem with ghosts (perturbative modes with negative norm) are generally related to the massive spin-2 fields [98; 99].

2.2 Classes of Fourth-Order Gravity Theories

We now discuss some notable classes of Fourth-Order Gravity theories and emphasize some outlines of greater relief. Each theory can be consider a particular class of the more general STFOG, which is the general class of ETG considered in our research works.

2.2.1 The $f(R)$-gravity

The $f(R)$-theory of gravity is one of the most relevant and studied Higher-Order Gravity and the Lagrangian density $\mathcal{L} = f(R)$ is just a function of the Ricci scalar and yield a specific model of the theory. Its widespread interest is due to the fact that they are able to explain with good simplicity a remarkable set of phenomena, among which we find the inflationary phase of the Universe, the rotation curves of galaxies and cluster of galaxy, the accelerated expansion of the Universe. Moreover the quadratic $f(R)$-theory is renormalizable, as demonstrated by K.S. Stelle [87; 161], since the quadratic Ricci scalars are sufficient to formulate a renormalizable theory of Quantum Gravity thanks to the demonstration that the quantization of matter fields in an not-quantized space–time conduct to this kind of theories and, at last, it does not suffer stability issues [100; 86] like that of the Ostrogradskij instability. Remarkably, ghost fields are absent from $f(R)$-theories since they contain only the massless spin-2 fields of General Relativity and a single scalar field, as it is evident from the existence of the conformal transformations. The $f(R)$-theories of gravity therefore do not always suffer from the same problems with ghosts as more general higher-order theories. By referring to the explicit calculations in section (2.1), here we have just to consider $S = S_f + S_m$ and after the application the variational principle with respect to $g_{\mu\nu}$ to the action

$$S = \int d^4x\sqrt{-g}\bigg[f(R) + \mathcal{X}\mathcal{L}_m\bigg], \tag{2.25}$$

one gets

$$f_R R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f - f_{R;\mu\nu} + g_{\mu\nu}\Box f_R = \mathcal{X}\,T_{\mu\nu}\,, \tag{2.26}$$

with trace equation that can readily obtained

$$3\Box f_R + Rf_R - 2f = \mathcal{X}T \tag{2.27}$$

In particular, for our aims we can safely concentrate just on the simplest version of the theory, commonly known as the Starobinsky model, which is represented by the quadratic model $\mathcal{L} = R + aR^2$. The physical motivation is that it contains the quadratic contribution of the quantum correction to General Relativity. This $f(R)$-gravity first model of the theory was investigated for the first time in Refs. [69; 82; 62], and then became relevant because A. Starobinsky showed that such an additionally quantum-motivated quadratic term could constitute a successful explanation of the cosmological inflation [49]. Up to now, the metric $f(R)$-gravity models has passed several tests carried out over the years although some issues still remain open, however, it has to be reminded that the solution in vacuum is not unique as it occurs in Einstein's theory and Birkhoff's theorem is not generally valid, but it is so only under strong restrictions on the scalar curvature, more precisely up to fourth order in perturbations in the Newtonian limit of the theory, thus implying possible time-dependent spherically symmetric solutions in dependence of the order of the perturbations [104; 105]. Moreover, if no screening mechanism is present, then the theory can involve an unbounded mass of an astrophysical object, when examining stellar structure models [120]. The field equation relative to the Starobinsky model is equivalent to those of the most general $f(R)$-theory with analytical Lagrangian density $\mathcal{L} = f(R)$. Indeed, without loss of generality, it can be developed in a Taylor series up to the second order

$$f(R) = \sum_{k=1}^{n} \frac{f^{(k)}(R_0)}{k!}(R - R_0)^k \simeq f_0 + f_0' R + f_0'' R^2 + \dots , \tag{2.28}$$

with $R_0 = 0$ [124]. The field equations at the zero-th order provides the further condition $f_0 = 0$. Then, by placing $a = f_0''/f_0'$ and $m = \sqrt{-f_0'/6f_0''}$, being $f_0' = 1$, the Starobinsky field equations are recovered. In this case, Eq. (2.27) is an effective Klein-Gordon equation for the scalar field $\varphi = \frac{(f_R - 1)}{2}$ and $m^2 = -1/6a$, represents the *mass* of scalaron field. Hence, a must be negative.

2.2.2 The $f(R, R_{\mu\nu}R^{\mu\nu})$-gravity

The quadratic corrections to General Relativity can be included in the theory also by means of general combinations of the Ricci and Riemann curvature, i.e. the Lagrangian density is not just $\mathcal{L} = f(R)$ as in the preceding case, but even of any of the three linear and quadratic contractions of the Riemann curvature tensor as follows

$$\mathcal{L} = R + a\, R^2 + b\, R_{\mu\nu} R^{\mu\nu} + c\, R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} . \tag{2.29}$$

However, thanks to the Gauss-Bonnet combination

$$\mathfrak{G} = 4R_{\alpha\beta}R^{\alpha\beta} - R^2 - R_{\alpha\beta\rho\sigma}R^{\alpha\beta\rho\sigma} \tag{2.30}$$

which does not contain non-linear terms in the second derivatives of $g_{\mu\nu}$[1], since it can be proved it is a topological invariant and then it can also be written in terms of a total divergence in the action and integrates to a boundary term that can be discarded. This entails that the Gauss-Bonnet does not contribute to the field equations and, without loss of generality, it is possible to reduce such theory to an equivalent form

$$\mathcal{L} = R + a\,R^2 + b\,R_{\mu\nu}R^{\mu\nu} \tag{2.31}$$

by redefining the parameters a and b and therefore discarding the invariant curvature $R_{\alpha\beta\rho\sigma}R^{\alpha\beta\rho\sigma}$. Hence, finally, we can just consider $S = S_f + S_Y + S_m$ and the variation of the action

$$S \,=\, \int d^4x\sqrt{-g}\Big[f(R, R_{\alpha\beta}R^{\alpha\beta}) + \mathcal{X}\mathcal{L}_m\Big], \tag{2.32}$$

with respect to metric $g_{\mu\nu}$, yields the field equation

$$f_R R_{\mu\nu} - \frac{f}{2}g_{\mu\nu} - f_{R;\mu\nu} + g_{\mu\nu}\Box f_R + 2f_Y R_{\mu}{}^{\alpha}R_{\alpha\nu} + \tag{2.33}$$

$$-2[f_Y R^{\alpha}{}_{(\mu}]_{;\nu)\alpha} + \Box[f_Y R_{\mu\nu}] + [f_Y R_{\alpha\beta}]^{;\alpha\beta}g_{\mu\nu} \,=\, \mathcal{X}\,T_{\mu\nu}\,.$$

From this field equation the trace equation reads

$$f_R R + 2f_Y R_{\alpha\beta}R^{\alpha\beta} - 2f + 3\,\Box f_R + \Box f_Y R + f_Y R^{\alpha\beta}{}_{;\alpha\beta} = \mathcal{X}T \tag{2.34}$$

2.2.3 The $f(R,\phi)$-gravity

The $f(R,\phi)$-gravity is a scalar-tensor generalization of the $f(R)$-model where in the Lagrangian density, as a function of the Ricci scalar, is introduced scalar field ϕ which can possibly be non-minimally coupled. Thereby this is the case of $S = S_f + S_\phi + S_m$ and by varying the action

$$S \,=\, \int d^4x\sqrt{-g}\Big[f(R,\phi) + \omega(\phi)\phi_{;\alpha}\phi^{;\alpha} + \mathcal{X}\mathcal{L}_m\Big], \tag{2.35}$$

with respect to metric $g_{\mu\nu}$ and ϕ,the field equation reads

$$f_R R_{\mu\nu} - \frac{f + \omega(\phi)\phi_{;\alpha}\phi^{;\alpha}}{2}g_{\mu\nu} - f_{R;\mu\nu} + g_{\mu\nu}\Box f_R + \omega(\phi)\phi_{;\mu}\phi_{;\nu} \,=\, \mathcal{X}\,T_{\mu\nu}\,,$$

$$2\omega(\phi)\Box\phi + \omega_\phi(\phi)\phi_{;\alpha}\phi^{;\alpha} - f_\phi \,=\, 0. \tag{2.36}$$

[1]There is a global cancellation between the three contributions.

Eqs. (2.36) and (2.36) can be accompanied with the trace equation

$$f_R R - 2f - \omega(\phi)\phi_{;\alpha}\phi^{;\alpha} + 3\Box f_R = \mathcal{X}T. \tag{2.37}$$

In this context, the scalar field ϕ is approximated as the Ricci scalar, i.e. $\phi \simeq \phi^{(0)}+\phi^{(2)}+... \simeq \psi$. Therefore, analogously to the $f(R)$-theory, we can develop the generic function $f(R,\phi)$ with its partial derivatives f_R, f_{RR}, f_ϕ, $f_{\phi\phi}$, $f_{\phi R}$, and $\omega(\phi)$ up to the c^{-4} order preserving, in such a way, the utmost generality and without the necessity to fix a specific theory. The Taylor expansion reads

$$\begin{aligned} f(R,\phi) \simeq \quad & f(0,\phi^{(0)}) + f_R(0,\phi^{(0)})\,R + \frac{f_{RR}(0,\phi^{(0)})}{2}\,R^2 + f_\phi(\phi-\phi^{(0)}) \\ & + \frac{f_{\phi\phi}(0,\phi^{(0)})}{2}(\phi-\phi^{(0)})^2 + f_{R\phi}(0,\phi^{(0)})R\,\phi \end{aligned} \tag{2.38}$$

From the field equations (2.36) at the zero-th order, we also infer the further two conditions $f(0,\phi^{(0)}) = 0$ and $f_\phi.(0,\phi^{(0)}) = 0$

2.3 A Special Case: Non-Commutative Spectral Geometry (NCSG)

As a special case of ETG, we want to discuss the Non-Commutative Spectral Geometry (NCSG) [131; 132]. It is a specific STFOG that should deserve attention by scientific community. Indeed among the various attempts to unify all interactions, including gravity, NCSG is one of the most interesting candidate [134; 138; 142]. It proposes that the Standard Model (SM) fields and gravity are packaged into geometry and matter on a Kaluza-Klein non-commutative space. In NCSG, geometry is composed by a two-sheeted space, made from the product of a four-dimensional compact Riemannian manifold $\mathcal{M}$ (with a fixed spin structure), describing the geometry of space-time, and a discrete non-commutative space $\mathcal{F}$, describing the internal space of the particle physics model. The SM fields and gravity enter into matter and geometry on a non-commutative space which has the product form $\mathcal{M}\times\mathcal{F}$. Such a product space is physically interpreted in the way that left- and right-handed fermions are placed on two different sheets with the Higgs fields being the gauge fields in the discrete dimensions (the Higgs can be seen as the difference (thickness) between the two sheets). The choice of a two-sheet geometry has a deep physical meaning, since such a structure accommodates the gauge symmetries of the SM, and incorporates the seeds of quantisation (see [144] for details). In the gravitational sector, in which we are interested, the action includes the coupling between the Higgs field ϕ and the Ricci curvature scalar R [134]

$$S_{\text{grav}} = \int \left(\frac{R}{2\kappa^2} + \alpha_0 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \tau_0 R R^\star - \xi_0 R |\mathbf{H}|^2 \right) \sqrt{-g}\, d^4x\,, \tag{2.39}$$

where $\kappa^2 \equiv 8\pi G/c^4$, $\mathbf{H} = (\sqrt{a f_0}/\pi)\phi$ is the Higgs field, with a a parameter related to fermion and lepton masses and lepton mixing, while $C^{\mu\nu\rho\sigma}$ is the Weyl tensor (the square of the Weyl tensor can be expressed in terms of R^2 and $R_{\mu\nu}R^{\mu\nu}$: $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} = 2R_{\mu\nu}R^{\mu\nu} - \frac{2}{3}R^2$) and $R^*R = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}R_{\alpha\beta\sigma\rho}R_{\gamma\delta}{}^{\sigma\rho}$ ($R^\star R^\star$ is the topological term related to the Euler characteristic). At unification scale (fixed by the cutoff Λ), $\alpha_0 = -3f_0/(10\pi^2)$. The NCSG model offers a framework to study several topics [145; 146; 147; 148; 149; 150; 160]). On the other hand, as remarked in sec. (2.2.1), it is also worth to note that the quadratic curvature terms in the action functional does not give rise to the emergence of negative [87] energy massive graviton modes. The higher derivative terms that are quadratic in curvature lead to [167]

$$\int \left(\frac{1}{2\eta} C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} - \frac{\omega}{3\eta}R^2 + \frac{\theta}{\eta}E \right) \sqrt{-g} d^4x;$$

$E = R^\star R^\star$ denotes the topological term which is the integrand in the Euler characteristic $\int E\sqrt{-g}d^4x = \int R^\star R^\star \sqrt{-g}d^4x$. The running of the coefficients η, ω, θ of the higher derivative terms is determined by the renormalization group equations [167]. The coefficient η goes slowly to zero in the infrared limit, so that $1/\eta = \mathcal{O}(1)$ up to scales of the order of the size of the Universe. Note that $\eta(t)$ varies by at most one order of magnitude between the Planck scale and infrared energies. All three coefficients $\eta(t), \omega(t), \theta(t)$ run to a singularity at a very high energy scale $\mathcal{O}(10^{23})$GeV (i.e., above the Planck scale). To avoid low energy constraints, the coefficients of the quadratic curvature terms $R_{\mu\nu}R^{\mu\nu}$ and R^2 should not exceed 10^{74} [167], which is indeed the case for the running of these coefficients [167]. The variation of the action (2.39) with respect to the metric tensor yields the NCSG field equations [145]

$$G^{\mu\nu} + \frac{1}{\beta^2_{NCSG}}[2\nabla_\lambda \nabla\kappa C^{\mu\nu\lambda\kappa} + C^{\mu\lambda\nu\kappa}R_{\lambda\kappa}] = \kappa^2 T^{\mu\nu}_{(\text{matter})}\,, \tag{2.40}$$

where $G^{\mu\nu}$ is the Einstein tensor, $T^{\mu\nu}$ the energy-momentum tensor of matter and $\beta^2 = 5\pi^2/(6\kappa^2 f_0)$. Thanks to the Bianchi identity $\nabla^\sigma R_{\mu\lambda\nu\sigma} = -\nabla_\lambda R_{\mu\nu} + \nabla_\mu R_{\lambda\nu}$ and $2\nabla^\sigma R_{\lambda\sigma} = \nabla_\lambda R$, the second term above becomes

$$\begin{aligned} 2\nabla^\lambda\nabla^\sigma C_{\mu\lambda\nu\sigma} + C_{\lambda\mu\sigma\nu}R^{\lambda\sigma} &= -\Box\left(R_{\mu\nu} - \frac{1}{6}g_{\mu\nu}R\right) + \frac{1}{3}\nabla_\mu\nabla_\nu R \\ &\quad -2R_{\mu\rho}R_\nu{}^\rho + \frac{2}{3}R\,R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}\left(R_{\alpha\beta}R^{\alpha\beta} - \frac{R^2}{3}\right)\,, \end{aligned} \tag{2.41}$$

where $\Box \equiv \nabla_\mu\nabla^\mu$.

2.4 Quintessence Field

An interesting possibility we wish to discuss is related to quintessence field, invoked to explain the speed-up of the present Universe [181]. Quintessence may generate a negative pressure, and, since it is diffuse everywhere in the Universe, it can be the responsible of the observed accelerated phase, as well as it is present around a massive astrophysical object deforming the space-time around it [182]. The studies of quintessential black holes are also motivated from M-theory/superstring inspired models [183; 184; 185] (see [186; 187; 188; 189; 181; 190; 191; 192; 193; 194; 195] for applications). The solution of Einstein's field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} = \mathcal{X}\, T_{\mu\nu} + \Lambda g_{\mu\nu} \tag{2.42}$$

for a static spherically symmetric quintessence surrounding a black hole in 4 dimension is given by [182; 186]

$$g_{\mu\nu} = \mathrm{diag}\left(-f(r), f^{-1}(r), r^2, r^2 \sin^2\theta\right) , \tag{2.43}$$

with

$$f(r) = 1 - \frac{2M}{r} - \frac{c}{r^{3\omega_Q+1}} , \tag{2.44}$$

where ω_Q is the adiabtic index (the parameter of equation of state), $-1 \leqslant \omega_Q \leqslant -\frac{1}{3}$, and c the quintessence parameter. The cosmological constant (ΛCMD model) follows from (4.17) and (2.44) with $\omega_Q = -1$ and $c = \Lambda/3$,

$$f(r) = 1 - \frac{2M}{r} - \frac{\Lambda r^2}{3} \quad . \tag{2.45}$$

The Quintessential potential reads $V_Q = -\frac{c}{r^{3\omega_Q+1}}$.

Chapter 3

Weak field limit in ETG: solutions and orbital motion of bodies

In this chapter, we show how it is possible to find solutions of the field equations in Extended Theories of Gravity relative to important and common physical scenarios, which satisfy certain conditions related to the strength of the overall gravitational field: the Newtonian limit, the Weak Field limit and the Post-Newtonian limit. This is a relevant point because the application of the conditions involving these approximations, is valid for a great variety of astrophysical gravitating systems such the Solar System, stellar motions of the S-stars around the Black Holes at the center of Milky Way (in general dynamics around a Black Hole), binary systems of dwarf stars or neutron stars, the motion of stars in globular clusters and galaxies. In particular, if we work with this kind of analytical approximation of the field equation legitimated by the physical situation of interest, it is possible to determine exact solutions that must also be asymptotically flat. To find solutions of the field equation in any metric theory of gravity (including GR), means to find the components of the metric tensor $g_{\mu\nu}$, in this perspective representing the gravitational potentials of the field. One might have in mind to follow the similar case of electrodynamics, whereby the field equations are linear and we can calculate the 4-potential A_μ simply by specifying the source, given by current density j_μ. However in the case of GR and ETG, not only we have second order and fourth order non-linear field equations respectively, but also an energy-momentum tensor $T_{\mu\nu}$ containing the information on metric itself. Therefore, differently from electrodynamics, the determination of the solutions must also address the fact that we must know both $g_{\mu\nu}$ and $T_{\mu\nu}$ together. However the conservation equation of the energy-momentum tensor $\nabla^\mu T_{\mu\nu} = 0$, already implied by GR for example, gives us the required information about the behaviour of the matter distribution. This enables a great deal of work to be saved by simply assuming a specific form for $T_{\mu\nu}$, because the dynamical equation of the fluid of matter is already contained by the field equation (see [58]). Therefore, in order to make these purposes real, together with approximations given by the Newtonian, Weak Field and Post-Newtonian limits, we need general methods to determine solutions of the coupled field and matter equations, which satisfy the prescription of asymptotic flatness of space-time.

Such solutions are important because in this way we can work out a theoretical model which represents an optimal description of the examined physical system. In this regard, it is frequently fundamental to analyse the motion of a system of N bodies which interact through their mutual gravitational attraction, possibly accompanied by a significant emission of gravitational radiation, thus transforming the global motion of the astrophysical system. The first fundamental studies dedicated to this problem were conducted by A. Einstein, J. Droste W. de Sitter, T. Levi-Civita, L. Infeld, B. Hoffmann, A. Eddington, H. P. Robertson, as well as by V. Fock, E. Lifshitz, L. Landau, followed by S. Chandrasekhar [11; 13; 14; 12; 45; 46; 68; 50; 51; 71; 73; 75; 76; 69; 54; 80; 53; 130]. Seminal works by V. I. Brumberg, S. Kopeikin, M. H. Soffel, T. Damour, N. Deruelle and L. Blanchet [56; 78; 57; 79; 106; 107; 108; 47] contributed to the ultimate affirmation of the novel *relativistic celestial mechanics* and its everlasting importance for modern physics. In fact, to study the dynamics of celestial bodies and apply the results to interesting astrophysical systems also allows us to elaborate a lot of tests of General Relativity, Gravity Field Theories and their underlying principle. Here we consider matter distributions that describe a system of celestial bodies where no relevant effects due to gravitational radiation should be present.

We first start to discuss the physical meaning of the Newtonian, Weak Field and Post-Newtonian limits and why it is necessary for studying the motion of bodies in the Universe. Subsequently we present the general method to find the solutions in the Post-Newtonian limit of the Einstein's equation. Starting from this, we determine the solutions for ETG's field equations by assuming the Standard Post-Newtonian gauge in the Weak Field limit, i.e. for the STFOG, its sub-classes and NCSG. Finally, by relying on the Brumberg conjecture, we explain why the Weak Field limit can be considered sufficient for our purposes and, then, we deduce the Lagrangian function for a system of N-interacting particles in ETG and relative equation of motion.

3.1 Weak Field, Newtonian and Post-Newtonian limits

In order to treat the problem of orbital motion in a relativistic framework, valid for any *metric* gravity field theory starting from General Relativity and going beyond, the first step is to identify the physical conditions and quantities methodically involved in the process. First, we realize that the physical regime occurring for many common scenarios is related to the strength of the gravitational field. Here we are interested in the analysis of non-relativistic self-gravitating systems with $v \ll c$, whereas a single particle moves in the curved space-time generated by the presence of the other particles. After this, it is possible to define the quantities with respect to which we can establish the more appropriate order of approximation of the equations of motion for the celestial bodies, given a certain configuration of the astrophysical system. Then, the process requires the deduction of the field equations and their solutions which provide the space-time, namely the field potentials of the overall gravitational interaction. In the end, by following an approach based on geodetic or variational principles, it is possible to find out the the dynamical equations governing the evolution of a the gravitating distribution of masses.

We begin by introducing the physical regimes and their corresponding conditions for studying the motion self-gravitating systems such as the Solar System, the S-stars at the centre of the Milky Way and the galaxies. We can distinguish between three kinds of approximations:

3.1.1 The Weak Field limit

This is the first fundamental limit underlying both the description of massive particle and massless particle systems. The gravitational field generated by a mass is said to be *weak* when its potential energy is much smaller than its rest mass energy, in other words if the following condition is fulfilled

$$\sigma = \frac{|\Phi|}{c^2} \ll 1, \tag{3.1}$$

where Φ is gravitational potential and c^2 the squared speed of light, and the dimensionless quantity σ is defined as the *compactness parameter*. It represents a reference measure for quantifying the effects of General Relativity near the surface of a source with mass M and radius R. For instance, the gravitational field of the Sun is weak because $\varepsilon \simeq 0.2 \times 10^{-5}$, satisfying the condition (3.1). More specifically, in the Solar System the typical values of the Newtonian gravitational potential Φ are never larger in modulus than 10^{-5}. On the other hand, the field is considered *strong* when $\sigma \sim 1$. For example, this is the case for Schwarzschild black holes for which the compactness is $\sigma = 1/2$. The other condition to be satisfied to accomplish this limit, is that that the gravitational field is independent on time, which is expressed through the stationarity of the metric tensor elements, i.e.

$$\frac{\partial g_{\mu\nu}}{\partial x^0} = 0. \tag{3.2}$$

This allows us to develop the required method for our objectives. In fact, the fundamental property of weak field generated by a massive source locally warping the space-time, allows us to assume the departure from the Minkowski space-time. Indeed, without loss of the accuracy required, we have well-founded physical reasons for which we can consider the effects on the space-time to be expressed as small perturbations to the metric tensor of the flat space-time. Therefore, in a metric gravity field theory, given the space-time

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{3.3}$$

we can express the gravitational field generated by a massive source as a small perturbation of the flat space-time, described as follows

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x), \tag{3.4}$$

where the small quantities $|h_{\mu\nu}| \ll 1$ are the metric perturbations, being $x_0 = ct$ and $\eta_{\mu\nu}$ the background Minkowskian space-time

$$\eta_{00} = 1, \qquad \eta_{0i} = 0, \qquad \eta_{ij} = -\delta_{ij}. \tag{3.5}$$

The departure from the Minkowski tensor can be written in terms of the inverse powers of the speed of light c^{-1}. Especially, if we refer to the dynamics of material particles having velocity v much smaller than c, for example celestial bodies like planets, stars, etc., it is valid the slow motion condition

$$\varepsilon \equiv \frac{v}{c}, \qquad \varepsilon \ll 1, \tag{3.6}$$

where ε denotes the *velocity parameter*. Since for a system of interacting material particles, the kinetic energy $(1/2)\overline{M}\overline{v}^2$ approximately equals the potential energy $G\overline{M}/\overline{r}$[1], it is possible identify the relation

$$\overline{v}^2 = \frac{G\overline{M}}{\overline{r}}, \tag{3.7}$$

being $\overline{M}$ and $\overline{v}$ the average values of mass enclosed in the matter distribution and velocity of the particles respectively, while $\overline{r}$ is the average value of the distance among them. Hence

[1]We remind that for the Virial Theorem $2\overline{T} + \overline{U} = 0$,

if we consider $\overline{v}$ as a function of the potential $|\Phi|$, it gives only second order contributions with respect ε, that is

$$\varepsilon^2 = \frac{v^2}{c^2} \sim \frac{|\Phi|}{c^2} \sim \frac{p}{\rho c^2} \sim O(\varepsilon^2). \tag{3.8}$$

On this basis, we can affirm that $(v/c)^4$ contributes to the fourth order $O(\varepsilon^4)$, $(v/c)^6$ to the sixth order $O(\varepsilon^6)$, as well as the definition $\varepsilon = v/c$ represents the first order $O(\varepsilon^1)$. $v|\Phi|$ contributes to third order $O(\varepsilon^3)$, and so on. Due to this kind of expansion, we are able to describe the motion of the system beyond a certain order with respect to the quantities $(v/c)^2$ and $(GM/c^2\overline{r})$.

For this reason, we can state that the method is based on a series development of the metric tensor with respect to the compactness parameter $\sigma = (GM/r)\, c^{-2}$, which for massive point particles also becomes $\varepsilon = (v/c)$, making explicit why we consider the generic $O(\varepsilon^n)$, with n integer number as the order of development. As a consequence of this approximation, for the spatial and time derivative, we have the following relations

$$\nabla \sim \frac{1}{\overline{r}}, \qquad \frac{\partial}{\partial x_0} \sim \frac{\overline{v}}{\overline{r}} \sim \mathbf{v} \cdot \nabla \sim \frac{\varepsilon}{\overline{r}}, \tag{3.9}$$

and therefore

$$\frac{|\partial/\partial x_0|}{|\nabla|} \sim O(\varepsilon^1), \tag{3.10}$$

implying that the time derivatives of the metric are smaller than the spatial derivatives. This must be considered when one performs the calculations starting from the Christoffel symbols. At this point, it is useful to highlight that not only the planetary velocities in the Solar System generally satisfy the condition $\bar{v}^2 \lesssim |\Phi|$, but also the matter pressures p experienced inside the Sun and the planets, are smaller than the matter gravitational energy density $\rho|\Phi|$ because their values are $p/\rho c^2 \sim 10^{-5}$ and $p/\rho c^2 \sim 10^{-10}$ respectively [80], meaning that $p/\rho \lesssim |\Phi|$. Other forms of energy such as thermal energy, compressional energy, radiation, etc. have small intensities and the specific energy density Π, which is the ratio of the energy density to the rest mass density, is related to $|\Phi|$ by $\Pi \lesssim |\Phi|$, where $\Pi \sim 10^{-5}$ in the Sun and $\Pi \sim 10^{-9}$ in the Earth [80].

The *weak field limit expansion* can be given as

$$\begin{aligned} h_{00} &= c^{-2} \overset{2}{g}_{00} + O(\varepsilon^4), \\ h_{0i} &= c^{-3} \overset{3}{g}_{0i} + O(\varepsilon^5), \\ h_{ij} &= c^{-2} \overset{2}{g}_{ij} + O(\varepsilon^4), \end{aligned} \tag{3.11}$$

which contains all the first non-zero terms of the development, as perturbations for each component of the metric tensor. The overset index on temporal, cross-term, and spatial components g_{00}, g_{0i}, g_{ij} respectively, indicates the order of development with respect to the parameter ε. We notice that the cross element h_{0i} only contains odd powers of the development order $O(\varepsilon^n)$ because it must change sign under time reversal, while g_{00} and g_{ij} are even under time reversal. Thus the metric tensor in the weak field limit can be summarized as

$$g_{\mu\nu} \simeq \begin{pmatrix} 1 + \overset{2}{g}_{00}(x_0, \mathbf{x}) & g_{0i}(x_0, \mathbf{x}) \\ g_{i0}(x_0, \mathbf{x}) & -\delta_{ij} + \overset{2}{g}_{ij}(x_0, \mathbf{x}) \end{pmatrix} \tag{3.12}$$

In other words, the weak field expansion means $g_{\mu\nu}$ developed up to the orders $g_{00} \sim O(\varepsilon^2)$ for the the temporal component of the metric, $g_{0i} \sim O(\varepsilon^3)$, for the cross components, and $g_{ij} \sim O(\varepsilon^2)$, for the spatial components. The g_{ij} and g_{0i} represent the first Post-Newtonian relativistic corrections to the gravitational interaction beyond Newtonian gravity. We can extend this type of expansion by affirming that an higher level of accuracy in the study of relativistic orbital motion, can be achieved developing the series up to the generic order n for $g_{\mu\nu}$ as follows: performed as $g_{00} \sim O(\varepsilon^n)$ for time terms, $g_{0i} \sim O(\varepsilon^{n+1})$ for the cross components, and $g_{ij} \sim O(\varepsilon^n)$ for spatial terms[2]. Finally, identifying the metric components as the gravitational field potentials, we have

$$g_{\mu\nu} \simeq \begin{pmatrix} 1 + 2\Phi(\mathbf{x}) & Z_i(\mathbf{x}) \\ Z_i(\mathbf{x}) & -\delta_{ij} + 2\Psi(\mathbf{x})\,\delta_{ij} \end{pmatrix} \tag{3.13}$$

and the general space-time reads

$$ds^2 = \left(1 + \frac{2}{c^2}\Phi\right) c^2 dt^2 - 2\left(\frac{Z_i}{c^3}\right) c dt dx^i - \left(1 - \frac{2}{c^2}\Psi\right)\delta_{ij} dx^i dx^j. \tag{3.14}$$

Hence the weak field approximation enables us to make the field equations linear, and the solutions will provide the linearized metric for weak gravitational fields. We notice the fundamental fact that *a priori* this limit does not rely on the low-velocity property of the bodies, but just on the weakness of the gravitational field and its stationarity. Indeed, here the slow-motion condition (3.6) is not essential, but it is just an additional assumption holding for material bodies. This makes such a physical approximation appropriate also for the description of phenomena involving the dynamics of massless particles.

[2]For example, to realize the subsequent approximation beyond the Weak Field limit in the spirit of this method, it involves $g_{00} \sim O(\varepsilon^4)$, $g_{0i} \sim O(\varepsilon^5)$ and $g_{ij} \sim O(\varepsilon^4)$.

3.1.2 The Newtonian limit

The Newtonian limit is based on the weak field limit expressed by means of the assumptions (3.1) and (3.2), plus the low-velocity condition (3.6) of the particles. In this case, the latter condition is required as well as it occurs in the Post-Newtonian limit. It is summarized by the metric tensor

$$g_{\mu\nu} \simeq \begin{pmatrix} 1 + \overset{2}{g}_{00}(x_0, \mathbf{x}) & 0 \\ 0 & -\delta_{ij} \end{pmatrix} \tag{3.15}$$

that is, $g_{\mu\nu}$ is developed up to the orders $g_{00} \sim O(\varepsilon^2)$ for the the temporal component of the metric, g_{0i} for the cross-term are accounted to be negligible, and $g_{ij} \sim O(\varepsilon^0)$ for the spatial components. Finally, identifying the metric components as the gravitational field potentials, we have

$$g_{\mu\nu} \simeq \begin{pmatrix} 1 + 2\Phi(\mathbf{x}) & 0 \\ 0 & -\delta_{ij} \end{pmatrix} \tag{3.16}$$

and the corresponding space-time is

$$ds^2 = \left(1 + \frac{2}{c^2}\Phi\right) c^2 dt^2 - \delta_{ij} dx^i dx^j . \tag{3.17}$$

In General Relativity, this approximation coincides with the classical Newtonian Mechanics and it means to consider that only the dominant 00-component of Einstein's field equation contribute to gravity, represented by the Poisson and Laplace[3] equations. Starting from this remark, we notice that Einstein's theory naturally leads to the corrective relativistic contributions represented by the fields of the spatial ij-component and cross-term $0i$-component, which therefore enrich the nature of gravitational interactions. Such relativistic contributions are neglected in Newtonian gravity, and this is a legitimate choice depending on the type of the model we are treating and the level of required accuracy. A criterion to establish when the relativistic effects are detectable is the vicinity of the bodies to the Schwarzschild radius of the main source. In the Solar System, the main source is the Sun, as it constitutes 99.9% of the total mass of the particle distribution. For instance, the analysis of the rotation curves of galaxies does not require the relativistic corrective terms induced by the gravity field theory, because they are too small to contribute to the stellar dynamics as one also considers the scale distances to which the star's motion occurs. So, they are safely negligible. However, this is not the case if one wants to conduct high-accuracy analysis regarding the relativistic motion of bodies in the Solar Systems, binary systems,

[3] When one considers the *linearized* Einstein's equation *in vacuo* $R_{00} = 0$.

or S-stars around the Sagittarius A* Black Hole at the centre of the Milky Way. At these scale distances, the effects on motion due to the relativistic terms encoded by g_{ij} and g_{0i} are relevant. This introduces the Post-Newtonian limit.

3.1.3 The Post-Newtonian limit

The Post-Newtonian limit represents the subsequent approximation after that of the Newtonian limit. It also relies on the weak field regime plus the low velocity condition. Differently from the weak field limit, here, such a condition concerning the low-velocity of the material particles (Eq. 3.6) is essential since the expansion only occurs through a series expansion with respect to the velocity parameter ε. But here, to perform the calculations, the method demands the expansion up to the first subsequent order of the velocity parameter ε after the Newtonian regime, in the metric terms g_{00}, g_{0i} and g_{ij}. That is to say, the metric tensor in the Post-Newtonian expansion is summarized as

$$g_{\mu\nu} \simeq \begin{pmatrix} 1 + \overset{2}{g}_{00}(x_0, \mathbf{x}) + \overset{4}{g}_{00}(x_0, \mathbf{x}) & g_{0i}(x_0, \mathbf{x}) \\ g_{i0}(x_0, \mathbf{x}) & -\delta_{ij} + \overset{2}{g}_{ij}(x_0, \mathbf{x}) \end{pmatrix} \tag{3.18}$$

Therefore, the expansion involves g_{00} developed up to the fourth order of velocity v^4, i.e. $g_{00} \sim O(\varepsilon^4)$, then $g_{0i} \sim O(\varepsilon^3)$ for the cross-term and $g_{ij} \sim O(\varepsilon^2)$ for the spatial terms. Hence, to generalize such a method, the expansion requires g_{00} up to $O(\varepsilon^n)$, g_{0i} to ε^{n-1} order and finally g_{ij} to ε^{n-2}. In this regard, the Newtonian limit represents the lowest order of the series development, that is, $n = 2$. The Post-Newtonian expansion requires $n = 4$ and it differs from the Weak Field expansion (3.13) uniquely for the term $\overset{4}{g}_{00}$. More precisely, Eq. (3.18) represent the *first Post-Newtonian approximation* (1PN), because we can generically add the subsequent terms of the series development up to the v^n, i.e. $O(\varepsilon^n)$, depending on the level of accuracy to be achieved. Furthermore, a loss of energy by the system of mutually attracting bodies due to the emission of radiation in the form of gravitational waves can be taken into account only arriving at the fifth order of the approximation $O(\varepsilon)$. Therefore, at weak gravitational field and 1PN approximation level, there is no loss of energy to be evaluated. In conclusion it is useful to summarize that the velocity parameter at a generic order of approximation n, as a function of the potential, can be generally written as [47]

$$\varepsilon^n = \left(\frac{v}{c}\right)^{2n} = \left(\frac{GM}{c^2 r}\right)^n \tag{3.19}$$

In the end, the values for the levels of approximation can be synthesized in Table 3.1, while the velocity parameter for each values in the Solar System are reported in Table 3.2. While the values in the Solar System are reported in Table 3.2.

PN order	Approximation	ε
$n = 0$	Newtonian	0
$n = 1$	1PN	1×10^{-4}
$n = 2$	2PN	1×10^{-2}
$n = 3$	3PN	5×10^{-2}
$n = 4$	4PN	1×10^{-1}
$n = \infty$	Exact GR	1

Table 3.1: Estimated velocity parameters $\varepsilon \equiv v/c$ for corresponding level of approximation, values for which the approximation is adequate. We notice that the 1PN can be suitably employed for studying the Solar System, or analog cases.

Planet	ε
Mercury	1.60×10^{-4}
Venus	1.18×10^{-4}
Earth	1.00×10^{-4}
Mars	0.81×10^{-4}
Jupiter	0.44×10^{-4}
Saturn	0.32×10^{-4}

Table 3.2: Velocity parameter $\varepsilon \equiv v/c$ (see Eq. (3.2)) for each binary system Sun-planet, estimated by taking the common known values for the mass of the Sun (as a dominant contribution) and the average distance of the planets from the Sun.

3.2 Geodesic Principle and Lagrangian for a System of N Bodies in Mutual Gravitational Interaction

By following the methods of the Weak Field, Newtonian, and Post-Newtonian limits, we are able to insert the expanded metrics in the field equation of a generic theory and then find the solutions providing the fields $g_{\mu\nu}$ at the corresponding same level of approximation of the gravitational field itself. More precisely, they constitute the gravitational potentials we need for the Lagrangian of the N-system expanded up to the required order ε^n, and then determination of the equations of motion governing the dynamics for a system of N-bodies. At the base of this theoretical paradigm, there is the geodesic principle.

The geodesic principle consists in the fact that the motion of a test particle occurs along the geodesics of space-time, so that the motion is governed by the equation (1.3)

$$\frac{d^2x^\alpha}{c^2d\tau^2}+\Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{cd\tau}\frac{dx^\nu}{cd\tau}=0, \tag{3.20}$$

where we recall that here $d\lambda = cd\tau$. We can re-express it as

$$\begin{aligned}\frac{d^2x^i}{dt^2} &= \left(\frac{dt}{cd\tau}\right)^{-1}\frac{d}{cd\tau}\left[\left(\frac{dt}{cd\tau}\right)\frac{dx^i}{cd\tau}\right]=\left(\frac{dt}{cd\tau}\right)^{-2}\frac{d^2x^i}{cd\tau^2}-\left(\frac{dt}{cd\tau}\right)^{-3}\frac{d^2t}{c^2d\tau^2}\frac{dx^i}{cd\tau}\\ &= -c^2\,\Gamma^i_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}+c^2\,\Gamma^0_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\frac{dx^i}{dt}\\ &= -c^2\,\Gamma^i_{00}-2c\Gamma^i_{0h}v^h-\Gamma^i_{hl}v^hv^l+\left[c\,\Gamma^0_{00}+2\Gamma^0_{0h}v^h+2c^{-1}\,\Gamma^0_{hl}v^hv^l\right]v^i\end{aligned} \tag{3.21}$$

Now we consider specific motions corresponding to the physical regimes under examination. The Newtonian limit, where the velocities of particles are treated as vanishingly small, requires to consider only the first term having $O(\varepsilon^2)$ order in the development of g_{00} and this is equivalent to retain only the first term on the second member of the previous equation[4]. Hence, in the difference between $g_{\mu\nu}$ and the Minkowski metric $\eta_{\mu\nu}$, the quantity $1-g_{00}$ is of order $(v/c)^2\sim GM/c^2\bar{r}$, providing

$$\frac{d^2x^i}{dt^2}\simeq -c^2\,\Gamma^i_{00}\simeq -\frac{c^2}{2}\frac{\partial g_{00}}{\partial x^i}, \tag{3.22}$$

so that we get

$$\frac{d^2x^i}{dt^2}\simeq -\frac{c^2}{2}\frac{\partial g_{00}}{\partial x^i}\simeq -\frac{c^2}{2}\frac{\partial}{\partial x^i}\left(1+\frac{2}{c^2}\Phi\right)\simeq -\frac{\partial\Phi}{\partial x^i}\sim O(\varepsilon^2) \tag{3.23}$$

the Newtonian approximation gives $\frac{d^2x^i}{dt^2}$ to the order $G\bar{M}/c^2\bar{r}^2$, namely to the order $\bar{v}^2/c^2r$. This is the approximation sufficient for the study concerning the galaxy rotation curves and what we need for it. In this case, the relativistic corrections to the gravitational force are not relevant.

In order to include the presence of general relativistic effects on motion, as in the case of the weak field limit, we consequently have to compute $\frac{d^2x^i}{dt^2}$ up to the order $\bar{v}^4/c^4\bar{r}^2$. Through the geodesic principle, it is also possible to find the correct Lagrangian for the Post-Newtonian equations of motion of a test particle in a given gravitational field. But

[4] i.e. the one associated to the Christoffel symbol Γ^i_{00}

being this described by the space-time, then we need the solutions in weak field limit $g_{\mu\nu}$ that provide it (Eqs. (3.14), (3.13)). If we look at equation (3.21), now the other terms have to be preserved. It should be remarked that by the terminology Post-Newtonian equations of motion, we are referring to the system of differential equations involving the relativistic potentials (i.e. the PN terms) beyond the Newtonian limit independently of kind of expansion[5] utilized to obtain it. First of all, we must deduce the Lagrangian needed to calculate the Euler-Lagrange equations for the dynamics of interacting particles, which allows us to analyze the orbital motion in N-body systems as well as the restricted case of the $(N-1)$-body system. In this way, the problem of the analysis of motion for a generic model (like the 2- and 3-body problem) is solved. Furthermore we are able to describe the multiple cases of STFOG, its sub-classes and NCSG, in just a unified theoretical framework of Celestial Mechanics, without the necessity to construct a specific model for the problem and perform computations for each theory singularly taken. We start by considering that the equations of motion are deduced from variational principle

$$-m_a c\ \delta \int \left(\frac{ds}{dt}\right) dt = 0 \tag{3.24}$$

where $ds^2 = c^2 dt^2$ this allows to find the right relativistic Lagrangian for a given body moving in the space-time produced by all the other $N-1$ material points at level of the weak field expansion for a given body, denoted with the letter a with $a = 1, 2, ..., N$, we can write

$$L_a = -m_a c^2 (g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu)^{1/2} = -m_a c^2 \left(1 + \overset{2}{g}_{00} + 2\overset{3}{g}_{0i}\frac{v_a^i}{c} + \overset{2}{g}_{ij}\frac{v_a^i v_a^j}{c^2} - \frac{v_a^2}{c^2}\right)^{1/2}. \tag{3.25}$$

Expanding as $\sqrt{(1+x)} \simeq 1 + x/2 + x^2/8$, after some algebra[6], one gets

$$L_a = \frac{1}{2} m_a v_a^2 + \frac{1}{8}\frac{m_a v_a^4}{c^2} - m_a c^2 \left(\frac{\overset{2}{g}_{00}}{2} + \overset{3}{g}_{0i}\frac{v_a^i}{c} + \frac{\overset{2}{g}_{ij}}{2}\frac{v_a^i v_a^j}{c^2} - \frac{(\overset{2}{g}_{00})^2}{8} + \frac{\overset{2}{g}_{00}}{4c^2} v_a^2\right) \tag{3.26}$$

Then, if we find the solutions of the STFOG field equation, by inserting them in L_a and considering the action on each body with respect to the interaction with the others and their dynamics, we can fully determine the searched equations of motion. These are represented by the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_a^i} - \frac{\partial L}{\partial x_a^i} = 0. \tag{3.27}$$

[5]Whether it is weak field expansion (3.13) or Post-Newtonian (3.18).

[6]Here the term $-m_a c^2$ has been discarded because not relevant.

Thereby, it is possible to achieve the most general system of differential equations of motion including the relativistic Post-Newtonian extra-potentials and formulate the relative adequate Cauchy problem for a specific initial condition. This system of equations enables us to study in a generic STFOG how particles propagate in the curved space-time generated by all bodies, useful for special configurations of astrophysical scenarios like the 3-body system.

3.2.1 The Brumberg's Conjecture

The usual technique of the Post-Newtonian expansion, expressed by Eq. (3.18), would also demand $\overset{4}{g}_{00}$. However the weak field expansion, whereby $\overset{4}{g}_{00}$ is not a starting explicit requirement, turned out to be sufficient for an effective description of the dynamics of celestial objects including relativistic potential terms, at least at a first level of precision required by astrophysical and astronomical problems. In particular, we start from the fact that the general-relativistic equations of motion, the so-defined *Einstein-Infeld-Hoffmann* equations [56; 79; 76; 80], can be derived just from the linearized (weak field) metric [75; 77]. This is precisely the content of the Brumberg's conjecture [78; 77]. In fact, by resuming this idea due to L. Infeld [75; 76], V. Brumberg argued that treating the problem through a direct application of the variational principle in place of the geodesic principle (after determining the metric $g_{\mu\nu}$), it was clearly demonstrated that specifying the term $\overset{4}{g}_{00}$ is not an essential prescription and there is no need in it. Following this kind of approach (which resulted in even a more optimised technique from the computation point of view with respect to the canonical Post-Newtonian technique), it was proved that the final Lagrangian from which one derives the EIH equations of motion [12] generally does not need an explicit $\overset{4}{g}_{00}$ [77]. As a consequence, the supposed conjecture had a positive answer. Nevertheless, it should also be emphasised that for alternative theories of gravity as well as curvature-based *metric* theories beyond General Relativity (like $f(R)$-theory or a more general STFOG), one has to pay more attention to the prescription of including also $\overset{4}{g}_{00}$ in order to have a more general model involving further Post-Newtonian potentials. In fact under certain conditions, which will be soon clarified, it occurs that the only weak field expansion may not furnish all the relativistic corrections and so we should limit ourselves to taking into account a bit more restricted model. Anyway, we will shortly see that fortunately this is not the case. It is possible to get an answer and understand how to discriminate between two different situations, by introducing a quantitative scheme providing the conditions under which the agreement with General Relativity is respected and the explicit prescription of $\overset{4}{g}_{00}$ is not necessary.

3.2.2 The Eddington-Robertson Expansion

Since from its genesis, one of the most relevant tool for the theoretical and experimental verification of a relativistic theory of gravity was the *Eddington-Robertson expansion* [53; 83; 54; 69; 70]

$$g_{00} = 1 + \frac{2GM}{c^2 r} + 2(\beta - \gamma)\left(\frac{GM}{c^2 r}\right)^2 \tag{3.28}$$

$$g_{ij} = 1 + 2\gamma\,\frac{2GM}{c^2 r} \tag{3.29}$$

It consists in the expansion of the metric for weak fields outside a non-rotating spherically symmetric matter distribution, i.e. a metric for the one-body problem, without a priori knowing the solution to the field equations. It has historically been the beginning of the *Parametrized Post-Newtonian formalism* of which became the most basic version. It was originally conceived as a tool aimed at a straightforward comparison between Newtonian gravity and Einstein's Theory in the weak field limit and soon turned out to be suitable for quantifying deviations from GR by other theories through classic tests in the Solar System as well as observational estimations of the parameters related to the tests [69; 70; 71]. The PPN parameterization of the potentials through γ and β in Eqs. (3.28) and (3.29) was introduced[7] just empirically and they indicate a reference measure for space curvature produced per unit rest mass and non-linearity in the superposition law for gravity, respectively [80; 79; 53]. They are also referred to as first and second PPN parameter, respectively. Especially the relation among the coefficient γ and the potentials is defined as follows

$$\gamma = \frac{\Psi}{\Phi}\,, \tag{3.30}$$

We highlight that γ and β coefficients are supposed to be independent from each other. Nevertheless, as pointed out by S. Weinberg [54], the verification of the perihelia precession, for $\gamma \simeq 1$ as in GR descends, must satisfy the relationship

$$\beta \simeq 2\gamma - 1\,. \tag{3.31}$$

Moreover, one can quickly note that the agreement with Schwarzschild metric necessarily implies that for both parameters we have $\gamma = 1$ and $\beta = 1$. These are the two PPN coefficient values for GR[8]. Later, when many alternative *metric* theories of gravity began to

[7] Early parameterizations were due to A. Eddington [69].

[8] While in Newtonian gravity we have $\gamma = 0$ and $\beta = 0$.

PPN parameter	Bounds	Effect	Experiment
$\|\gamma - 1\|$	$(2.1 \pm 2.3) \times 10^{-5}$	Light deflection, Time delay	Cassini tracking
$\|\beta - 1\|$	8×10^{-5}	Perihelion shift	Perihelion shift
$\|\beta - 1\|$	$(1.2 \pm 1.1) \times 10^{-4}$	Nordtvedt effect	Nordtvedt effect

Table 3.3: Numerical PPN parameters (first column) with corresponding bounds on the measure (second column), type of experiment conducted (third column) with respect to the classical test employed (fourth column).

become more widespread and studies by virtue of their interesting properties (some of which have been discussed in Chapter 1), the Eddington-Robertson expansion was systematically extended by the works of C. Will, K. Nordtvedt, K. Thorne and C. Misner [33; 80; 79; 53], that implemented a complete theoretical background within which one could compare and interpret the predictions of a *metric* theory relative to the classic Solar System tests such as the periastron shift, the Shapiro time delay, the Lunar laser ranging, the Nordtvedt effect, and in 1974 the Hulse–Taylor binary pulsar [159; 33]. The PPN formalism enables an overall assessment for the alternative proposals to General Relativity. Several theories have been downsized, or definitively ruled out, thanks to the violation of the experimental bound on the PPN parameters. To this regard, even if the PPN formalism is not appointed to this aim and does not provide a direct experimental confirmation, it can be conversely considered an indirect confirmation of Einstein's theory. We recall here the numerical bounds on the PPN-parameters with respect to the level of accuracy of astrometric and observational tests, as reported by C. Will [80]. In short words the Eddington-Robertson expansion, and the more general scheme of the the PPN formalism, allows to constrain a theory by a straightforward comparison with the experimentally known bounds on the PPN parameters, without the need of executing computations for single gravitational phenomena.

At this point, we must recall that γ and β may be generally different from the GR values in the context of alternative theories of gravity. For instance, in the scalar-tensor theory the higher order term does not vanish because the coefficient $2(\beta-\gamma)$ in Eq. (3.28) is not zero, regardless of the derivation technique[9]. The corresponding metric would no longer be compatible with a metric of General Relativity. Thus, for a complete description of the problem, the $\overset{4}{g}_{00}$ term of the canonical Post-Newtonian expansion is explicitly required by the development of the 00-component metric perturbation after the Newtonian approximation.

[9]Whether it is geodesic or variational principle.

For a complete description of the problem. The only weak field expansion could not furnish all the relativistic corrections and we should limit ourselves to consider a bit more restricted model, without including some of the Post-Newtonian potentials that are encoded within of the $\overset{4}{g}_{00}$. More specifically, if we choose the $f(R)$-theory as a reference example for the ETG, its linearized metric can lead to a PPN parameter value $\gamma = 1/2$ and, consequently, to a strict violation of the experimental bounds (see Table 3.3). Indeed, this is what actually occurs if a specific physical condition on the effective mass[10], present in the potential of a Yukawa-like gravitational interaction and expressed by $mr \ll 1$, is assumed for the calculations. Such a condition implies the smallness of the massive graviton mediating this interaction term. This aspect will become clearer later on when we will discuss the periastron advance. Under this assumption, the effective mass would have an interaction range larger than the size of the Solar System. Nevertheless, that physical condition does not appear to be fulfilled because of the fundamental presence of *screening mechanisms* underlying the $f(R)$-theory, preventing the possibility that could be ruled out [123; 48]. This characteristic was revealed also by the fact that in order to modify the gravity law so that the cosmological evolution is well reproduced, the theory must have a notable deviation from GR at the IR scales. However, it also became evident that the modifications at IR scales provoke important deviations from the predictions of General Relativity at both terrestrial and Solar System scales, i.e. for experiments including dynamical tests as the deflection of light or time delay of light. As a consequence, a mechanism suppressing this deviation has to exist if metric fourth-order gravity is taken into consideration as a working extension of General Relativity.

One of the better explanations for this mechanism are the *chameleon fields*. The idea is that in theories having a non-minimally coupled scalar field, their effective mass (of the scalar degree of freedom) depends on the energy density and curvature of the environment [212; 213; 214; 215; 216]. Hence, in relatively dense environments as the one represented by the Earth and the Solar System, the effective mass can be large (leading to $mr \gg 1$) and potentially fulfil the tight bounds on non-minimally coupled scalar degrees of freedom, while a new behaviour emerges in gradually less dense environments as it occurs on cosmological scales. Consequently, the scalar field is short-range in the Solar System and long-range at cosmological distances. The presence of such chameleon scalar fields is what makes the fourth order theories compatible with GR, terrestrial and Solar System bounds dictated by classical tests as well as able to produce effects on large-scale cosmological dynamics and, so, on accelerated expansion of the Universe. We have already seen that the compatibility

[10]It is the effective mass of the scalar degree of freedom arising from the theory.

of ETG with GR is valid only if $\gamma = 1$ and $\beta = 1$, namely, the values emerging from General Relativity. Looking at the Robertson expansion (3.28), it implies the vanishing of the second perturbation term of order $O(\varepsilon^4)$ in g_{00}, since we have $\gamma = \beta = 1$. By virtue of the chameleon mechanism, the consistency with General Relativity is thereby physically motivated for a curvature-based fourth-order gravity and, therefore, also the agreement with the GR values for the PPN parameters γ and β.

In short words, the General Relativity's PPN parameters in the Solar System $\gamma \simeq 1$ and $\beta \simeq 1$, must hold for the ETG as well. This allows that in Eq. (3.28) the higher order term vanishes by virtue of the compatibility with the PPN parameters in GR and, provided the ETG are in agreement with the GR's PPN outcomes in the Solar System, these motivations allow us to perform the weak field expansion as a sufficient method for the determination of solutions, the relativistic Lagrangian for a N-system and the corresponding equations of motions.

Since in GR the first PPN parameter is $\gamma \simeq 1$, the calculations of the next sections will include the assumption

$$\Phi \simeq \Psi. \tag{3.32}$$

In the next sections, we show the basic calculations in order to infer the linearized (weak field) equations in a generic framework of a relativistic *metric* theory. The computations regarding the Christoffel symbols, the Ricci tensor and scalar, as well as the energy-momentum tensor, are the fundamental steps through which we obtain the linearized field equations in both GR and STFOG. Then we go on by illustrating the classical resolution for Einstein's equation in the Post-Newtonian limit and, finally, we find the solution of the Scalar-Tensor-Fourth-Order Gravity in Weak Field limit. This will be achieved by introducing and fixing the Standard Post-Newtonian gauge, which is particularly adequate to this aim. As we will see, the solutions obtained in this gauge condition differ from the harmonic gauge only with respect to the g_{0i} component, but, however, it is feasible to identify a further transformation which connects them, in regard of the fact that the final outcomes have to be independent of the coordinate choice.

3.3 Deduction of the Field Equations

Due to the Equivalence Principle and the differentiability of space-time manifold, we expect that it should be possible to find out a coordinate system in which the metric tensor is nearly equal to the Minkowski one $\eta_{\mu\nu}$, the correction being expandable in powers of $(\bar{v}^2/c^2) \sim G\bar{M}/c^2\bar{r}$.

3.3.1 The Metric Tensor and the Affine Connections

In other words, one has to consider the metric developed as follows:

$$\begin{aligned} g_{00} &\simeq 1 + \overset{2}{g}_{00} + \overset{4}{g}_{00} + O(c^{-6}) \\ g_{0i} &\simeq \overset{3}{g}_{0i} + O(c^{-5}) \\ g_{ij} &\simeq -\delta_{ij} + \overset{2}{g}_{ij} + O(c^{-4}) \end{aligned} \qquad . \tag{3.33}$$

where δ_{ij} is the Kronecker delta. The controvariant form of $g_{\mu\nu}$ is

$$\begin{aligned} g^{00} &\simeq 1 + \overset{2}{g}{}^{00} + \overset{4}{g}{}^{00} + O(c^{-6}) \\ g^{0i} &\simeq \overset{3}{g}{}^{0i} + O(c^{-5}) \\ g^{ij} &\simeq -\delta_{ij} + \overset{2}{g}{}^{ij} + O(c^{-4}) \end{aligned} \qquad . \tag{3.34}$$

The inverse of the metric tensor (3.33) is defined by the equations $g^{\mu\alpha}g_{\alpha\beta} = \delta^\mu_\nu$. Thus, the relations between the components of the metric are given by

$$\begin{aligned} \overset{2}{g}{}^{00} &= -\overset{2}{g}_{00} \\ \overset{4}{g}{}^{00} &= (\overset{2}{g}_{00})^2 - \overset{4}{g}_{00} \\ \overset{3}{g}{}^{0i} &= \overset{3}{g}_{0i} \\ \overset{2}{g}{}^{ij} &= -\overset{2}{g}_{ij} \end{aligned} \qquad . \tag{3.35}$$

In evaluating $\Gamma^\alpha_{\mu\nu}$ we remind that the scale of distance and time has to be taken into account (Eqs. (3.9)-(3.10)). In fact, in section (3.1), we showed that in our systems they are respectively set by $\bar{r}$ and $\bar{r}/\bar{v}$, and so the space and time derivatives should be regarded as being of order

$$\frac{\partial}{\partial x^i} \sim \frac{1}{\bar{r}}\,, \qquad \frac{\partial}{\partial x^0} \sim \frac{\bar{v}}{\bar{r}} \sim \frac{\varepsilon}{\bar{r}} \tag{3.36}$$

Using the above approximations (3.33), (3.34) and (3.35) we have, from the definition (3.37),

$$\begin{aligned}
&\overset{3}{\Gamma}{}^0_{00} = \frac{1}{2}\overset{2}{g}_{00,0} && \overset{2}{\Gamma}{}^i_{00} = \frac{1}{2}\overset{2}{g}_{00,i} \\
&\overset{2}{\Gamma}{}^i_{jk} = \frac{1}{2}\left(\overset{2}{g}_{jk,i} - \overset{2}{g}_{ij,k} - \overset{2}{g}_{ik,j}\right) && \overset{3}{\Gamma}{}^0_{ij} = \frac{1}{2}\left(\overset{3}{g}_{0i,j} + \overset{3}{g}_{j0,i} - \overset{2}{g}_{ij,0}\right) \\
&\overset{3}{\Gamma}{}^i_{0j} = \frac{1}{2}\left(\overset{3}{g}_{0j,i} - \overset{3}{g}_{i0,j} - \overset{2}{g}_{ij,0}\right) && \overset{4}{\Gamma}{}^0_{0i} = \frac{1}{2}\left(\overset{4}{g}_{00,i} - \overset{2}{g}_{00}\overset{2}{g}_{00,i}\right) \\
&\overset{4}{\Gamma}{}^i_{00} = \frac{1}{2}\left(\overset{4}{g}_{00,i} + \overset{2}{g}_{ih}\overset{2}{g}_{00,h} - 2\overset{3}{g}_{i0,0}\right) && \overset{2}{\Gamma}{}^0_{0i} = \frac{1}{2}\overset{2}{g}_{00,i}
\end{aligned} \tag{3.37}$$

3.3.2 The Ricci and Einstein tensors

The components of the Ricci tensor (1.7) are

$$\begin{aligned}
&\overset{2}{R}_{00} = \tfrac{1}{2}\overset{2}{g}_{00,hh} \\
&\overset{4}{R}_{00} = \tfrac{1}{2}\overset{4}{g}_{00,hh} + \tfrac{1}{2}\overset{2}{g}_{hl,h}\overset{2}{g}_{00,h} + \tfrac{1}{2}\overset{2}{g}_{hl}\overset{2}{g}_{00,hl} + \tfrac{1}{2}\overset{2}{g}_{hh,00} - \tfrac{1}{4}\overset{2}{g}_{00,h}\overset{2}{g}_{00,h} - \tfrac{1}{4}\overset{2}{g}_{hh,l}\overset{2}{g}_{00,l} - \overset{3}{g}_{0h,0h} \\
&\overset{3}{R}_{0i} = \tfrac{1}{2}\overset{3}{g}_{0i,hh} - \tfrac{1}{2}\overset{2}{g}_{ih,h0} - \tfrac{1}{2}\overset{3}{g}_{h0,hi} + \tfrac{1}{2}\overset{2}{g}_{hh,0i} \\
&\overset{2}{R}_{ij} = \tfrac{1}{2}\overset{2}{g}_{ij,hh} - \tfrac{1}{2}\overset{2}{g}_{ih,hj} - \tfrac{1}{2}\overset{2}{g}_{jh,hi} - \tfrac{1}{2}\overset{2}{g}_{00,ij} + \tfrac{1}{2}\overset{2}{g}_{hh,ij}.
\end{aligned} \tag{3.38}$$

The Ricci scalar (1.8) reads

$$\begin{aligned}
&\overset{2}{R} = \overset{2}{R}_{00} - \overset{2}{R}_{hh} = \overset{2}{g}_{00,hh} - \overset{2}{g}_{ll,hh} + \overset{2}{g}_{hl,hl} \\
&\overset{4}{R} = \overset{4}{R}_{00} - \overset{2}{g}_{00}\overset{2}{R}_{00} - \overset{2}{g}_{hl}\overset{2}{R}_{hl} = \\
&= \tfrac{1}{2}\overset{4}{g}_{00,hh} + \tfrac{1}{2}\overset{2}{g}_{hl,h}\,\overset{2}{g}_{00,l} + \tfrac{1}{2}\overset{2}{g}_{hl}\,\overset{2}{g}_{00,hl} + \tfrac{1}{2}\overset{2}{g}_{hh,00} - \tfrac{1}{4}\overset{2}{g}_{00,h}\,\overset{2}{g}_{00,h} + \\
&\quad -\tfrac{1}{4}\overset{2}{g}_{hh,l}\,\overset{2}{g}_{00,l} - \overset{3}{g}_{0h,0h} - \tfrac{1}{2}\overset{2}{g}_{00}\,\overset{2}{g}_{00,hh} - \tfrac{1}{2}\overset{2}{g}_{hl}\left(\overset{2}{g}_{hl,rr} - \overset{2}{g}_{hr,rl} - \overset{2}{g}_{lr,rh} - \overset{2}{g}_{00,hl} + \overset{2}{g}_{rr,hl}\right)
\end{aligned} \tag{3.39}$$

Thus, the components of the Einstein tensor (1.14) are given by

$$
\begin{aligned}
&\overset{2}{G}_{00} = \overset{2}{R}_{00} - \tfrac{1}{2}\overset{2}{R} = \tfrac{1}{2}\overset{2}{g}_{hh,ll} + \tfrac{1}{2}\overset{2}{g}_{hl,hl} \\
&\overset{4}{G}_{00} = R^{(4)}_{00} - \tfrac{1}{2}\overset{4}{R} - \tfrac{1}{2}\overset{2}{g}_{00}\overset{2}{R} \\
&\overset{3}{G}_{0i} = \overset{3}{R}_{0i} = \tfrac{1}{2}\overset{3}{g}_{0i,hh} - \tfrac{1}{2}\overset{2}{g}_{ih,h0} - \tfrac{1}{2}\overset{3}{g}_{h0,hi} + \tfrac{1}{2}\overset{2}{g}_{hh,0i} \\
&\overset{2}{G}_{ij} = \overset{2}{R}_{ij} + \tfrac{\delta_{ij}}{2}\overset{2}{R} = \tfrac{1}{2}g_{ij,hh} - \tfrac{1}{2}\overset{2}{g}_{ih,hj} - \tfrac{1}{2}\overset{2}{g}_{jh,hi} - \tfrac{1}{2}\overset{2}{g}_{00,ij} + \tfrac{1}{2}\overset{2}{g}_{hh,ij} + \tfrac{\delta_{ij}}{2}\left(\overset{2}{g}_{00,hh} - \overset{2}{g}_{ll,hh} + \overset{2}{g}_{hl,hl}\right)
\end{aligned}
\tag{3.40}
$$

3.3.3 Gauge Transformations

Now a huge simplification can be reached by means of a transformation $x'^{\alpha} = x^{\alpha} + \xi^{\alpha}(x)$ with respect to a suitable coordinates system, and considering the tensor relation

$$
g_{\mu\nu} = \frac{\partial x'^{\alpha}}{\partial x^{\mu}}\frac{\partial x'^{\beta}}{\partial x^{\nu}} g'_{\alpha\beta} = (\eta_{\alpha}\beta + h'_{\alpha\beta})(\delta^{\alpha}_{\mu} + \xi^{\alpha}_{,\mu})(\delta^{\beta}_{\nu} + \xi^{\beta}_{,\nu}) = \eta_{\mu\nu} + h_{\mu\nu} + \xi_{\nu,\mu} + \xi_{\mu,\nu} \tag{3.41}
$$

along with $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, it implies the equation

$$
h'_{\mu\nu} = h_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} + O(h^2), \tag{3.42}
$$

that is to say a gauge transformation of the metric perturbation for an infinitesimal diffeomorphism. For our purposes aiming to establsih the Post-Newtonian equations of motion as foundation for the relativistic celestial mechanics in a curvature-based gravity field theory, the better choice is to work with the *Standard Post-Newtonian gauge*

$$
g_{0j,j} - \frac{1}{2}g_{jj,0} = O(c^{-5}), \tag{3.43}
$$

$$
g_{ij,j} - \frac{1}{2}(g_{jj} - g_{00})_{,i} = O(c^{-4}), \tag{3.44}
$$

that implies

$$
\overset{3}{g}_{0h,h} - \frac{1}{2}\overset{2}{g}_{hh,0} = 0, \tag{3.45}
$$

$$
\overset{2}{g}_{ij,j} + \frac{1}{2}\overset{2}{g}_{00,i} - \frac{1}{2}\overset{2}{g}_{jj,i} = 0. \tag{3.46}
$$

Now we can differentiate Eq. (3.45) with respect to x_0 and x^i, and get

$$\overset{3}{g}_{0h,hi} - \frac{1}{2}\overset{2}{g}_{hh,0i} = 0, \tag{3.47}$$

$$\overset{3}{g}_{0h,0h} - \frac{1}{2}\overset{2}{g}_{hh,00} = 0 \tag{3.48}$$

The gauge condition (3.47) can be used to simplify $\overset{3}{R}_{0i}$, while the other gauge condition (3.48) for $\overset{4}{R}_{00}$. Now, by differentiating Eq. (3.46) with respect to x^h

$$\overset{2}{g}_{ij,jh} + \frac{1}{2}\overset{2}{g}_{00,ih} - \frac{1}{2}\overset{2}{g}_{jj,ih} = 0 \tag{3.49}$$

through an interchange of the indexes $i \leftrightarrow h$, we have

$$\overset{2}{g}_{hj,ji} + \frac{1}{2}\overset{2}{g}_{00,hi} - \frac{1}{2}\overset{2}{g}_{jj,hi} = 0 \tag{3.50}$$

and summing the equations one gets

$$\overset{2}{g}_{00,ih} + \overset{2}{g}_{ij,jh} + \overset{2}{g}_{hj,ji} - 2g_{jj,ih} = 0 \tag{3.51}$$

This gauge condition simplifies $\overset{2}{R}_{ij}$. Alternatively another good choice, useful to these aims and common in literature, is given by the harmonic coordinates. In order to get the right gauge along with the validity of the field equations for both $h_{\mu\nu}$ and $\tilde{h}_{\mu\nu}$, the vector ξ_μ must fulfill the harmonic condition expressed by the equation $\Box\xi^\alpha = 0$, in other words the coordinate functions satisfy the d'Alembert equation. This quickly leads to

$$\Box\xi^\alpha = \left[(x^\alpha)_{,\mu\nu} - \Gamma^\lambda_{\mu\nu}(x^\alpha)_{,\lambda}\right] g^{\mu\nu} = \left(\delta^\alpha_{\mu,\nu} - \delta^\alpha_\lambda\Gamma^\lambda_{\mu\nu}\right) g^{\mu\nu} = \left(0 - \Gamma^\alpha_{\mu\nu}\right) g^{\mu\nu} = -g^{\mu\nu}\Gamma^\alpha_{\mu\nu} = 0. \tag{3.52}$$

Hence the final relation

$$g^{\mu\nu}\Gamma^\alpha_{\mu\nu} = 0. \tag{3.53}$$

represents the *harmonic gauge* (or *de Donder gauge*). Next, we have to work out the relations stemming from Eq. (3.53) for the different components. For $\alpha = 0$, we have

$$2g^{\mu\nu}\Gamma^0_{\mu\nu} = \overset{2}{g}_{00,0} - 2\overset{3}{g}_{0h,h} + \overset{2}{g}_{hh,0} = 0 \tag{3.54}$$

while for $\alpha = i$

$$2g^{\mu\nu}\Gamma^i_{\mu\nu} = \overset{2}{g}_{00,i} + 2\overset{2}{g}_{hi,h} - \overset{2}{g}_{hh,i} = 0 \tag{3.55}$$

If we now differentiate (3.54) with respect to x_0 and x^j together with (3.55) with respect to x_0 and x^j, we have

$$\overset{2}{g}_{00,00} - 2\overset{3}{g}_{0h,h0} + \overset{2}{g}_{hh,00} = 0, \tag{3.56}$$

$$\overset{2}{g}_{00,0j} - 2\overset{3}{g}_{h0,jh} + \overset{2}{g}_{hh,0j} = 0, \tag{3.57}$$

$$\overset{2}{g}_{00,0i} + 2\overset{2}{g}_{hi,0h} - \overset{2}{g}_{hh,0i} = 0. \tag{3.58}$$

$$\overset{2}{g}_{00,ij} + 2\overset{2}{g}_{hi,jh} - \overset{2}{g}_{hh,ij} = 0 \tag{3.59}$$

Now let us subtract Eq. (3.57) to Eq. (3.58) and then change sign, we obtain

$$\overset{2}{g}_{hh,0i} - \overset{2}{g}_{hi,0h} - \overset{3}{g}_{h0,hi} = 0. \tag{3.60}$$

Concerning Eq. (3.59) we can rename the indexes as $j \to i$, $i \to j$ because these are mute indexes, and thus one obtains

$$\overset{2}{g}_{00,ij} + 2\overset{2}{g}_{hj,ih} - \overset{2}{g}_{hh,ij} = 0 \tag{3.61}$$

Finally, from a combination of Eq. (3.59) and Eq. (3.61), we find

$$\overset{2}{g}_{00,ij} + \overset{2}{g}_{hi,jh} + \overset{2}{g}_{hj,ih} - \overset{2}{g}_{hh,ij} = 0 \tag{3.62}$$

3.3.4 The Ricci and Einstein tensors in Standard Post-Newtonian gauge

Therefore, we can simplify the Ricci tensor components (3.38) by using Eqs. (3.56), (3.60), (3.62), and obtain

$$\begin{aligned}
\overset{2}{R}_{00} &= \tfrac{1}{2}\triangle\overset{2}{g}_{00} \\
\overset{4}{R}_{00} &= \tfrac{1}{2}\triangle\overset{4}{g}_{00} + \tfrac{1}{2}\overset{2}{g}_{hl}\overset{2}{g}_{00,hl} + \tfrac{1}{2}\overset{2}{g}_{hl,h}\overset{2}{g}_{00,l} - \tfrac{1}{2}|\bigtriangledown\overset{2}{g}_{00}|^2 \\
\overset{3}{R}_{0i} &= \tfrac{1}{2}(\triangle\overset{3}{g}_{0i} + \Phi_{,0i}) \\
\overset{2}{R}_{ij} &= \tfrac{1}{2}\triangle\overset{2}{g}_{ij},
\end{aligned} \tag{3.63}$$

hence the Ricci scalar (3.39) now reads

$$\begin{aligned}
&\overset{2}{R} = \tfrac{1}{2}\triangle\overset{2}{g}_{00} - \tfrac{1}{2}\triangle\overset{2}{g}_{hh} \\
&\overset{4}{R} = \tfrac{1}{2}\triangle\overset{4}{g}_{00} + \tfrac{1}{2}\overset{2}{g}_{hl}\overset{2}{g}_{00,hl} + \tfrac{1}{2}\overset{2}{g}_{hl,h}\overset{2}{g}_{00,l} - \tfrac{1}{2}|\nabla\overset{2}{g}_{00}|^2 - \tfrac{1}{2}\overset{2}{g}_{00}\triangle\overset{2}{g}_{00} - \tfrac{1}{2}\overset{2}{g}_{hl}\triangle\overset{2}{g}_{hl}
\end{aligned} \tag{3.64}$$

where ∇ and $\triangle$ are the gradient and the Laplacian in flat space, respectively. In conclusion, with the Ricci tensor and scalar in hand, we can readily provide the components of the Einstein tensor (1.14)

$$\begin{aligned}
&\overset{2}{G}_{00} = \tfrac{1}{4}\triangle\overset{2}{g}_{00} + \tfrac{1}{4}\triangle\overset{2}{g}_{hh} \\
&\overset{3}{G}_{0i} = \tfrac{1}{2}(\triangle\overset{3}{g}_{0i} + \Phi_{,0i}) \\
&\overset{2}{G}_{ij} = \tfrac{1}{2}\triangle\overset{2}{g}_{ij} + \tfrac{\delta_{ij}}{4}\left[\triangle\overset{2}{g}_{00} - \triangle\overset{2}{g}_{hh}\right]
\end{aligned} \tag{3.65}$$

The remaining component $\overset{4}{G}_{00}$ can be readily obtained by combining together Eqs. $(3.63)_2$ and $(3.64)_2$.

3.3.5 The Energy-Momentum Tensor

We now deal with the part concerning the distribution of matter, that is the right-hand side of the field equations in General Relativity and Extended Theories of Gravity, that in its more general form reads

$$T_{\mu\nu} \;=\; (\rho c^2 + \Pi\rho + p)u_\mu u_\nu - p g_{\mu\nu}, \tag{3.66}$$

where Π is the internal energy[11] of the perfect fluid. However since we are interested to the motion of celestial bodies moving in the space-time generated by a system of the other gravitating $N-1$ bodies, to the aims of the resulting interactions we can neglect the the pressure p and internal energy Π of each body by virtue of the distances between them. Hence, by referring to the general definition of the energy-momentum tensor of a pressure-less perfect fluid, in order to describe the system of particles we need the tensor

$$T_{\mu\nu} \;=\; \rho(\mathbf{x})c^2\, u_\mu u_\nu \tag{3.67}$$

[11] Examples: thermal energy, radiation energy.

with $u_\sigma u^\sigma = 1$ and $\rho(\mathbf{x}) = M\delta(\mathbf{x})$, where δ is the delta function. The components forming the energy-momentum tensor are

$$T_{00} = \rho c^2 + \rho(v^2 + 2\Phi)$$

$$T_{0i} = -\rho c\, v_i \tag{3.68}$$

$$T_{ij} = \rho v_i v_j$$

Such a formalism stems from the theoretical setting of the Newtonian dynamics, which needs of the appropriate method of approximation when obtained from a more general relativistic theory. The method consists in a gravitational theory studied at the first order of perturbation in a curved space-time metric. Furthermore, keeping in mind that we are working in the context of linearized field equations, as a consequence the linear superposition principle is preserved. Thus, in this case, the 00-component of the energy-momentum tensor is simply given by the sum of the mass-energy volume density of sources

$$\overset{0}{T}_{00}(t,\mathbf{x}) = \sum_a m_a c^2\,\delta(\mathbf{x} - \mathbf{x}_a). \tag{3.69}$$

In curved space-time, the energy-momentum tensor for a system of N point particles is derived as follows

$$T^{\mu\nu}(t,\mathbf{x}) = \frac{1}{\sqrt{-g}}\sum_{a=1}^{N}\gamma_a m_a \frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\,\delta(\mathbf{x} - \mathbf{x}_a(t)) \tag{3.70}$$

with

$$\gamma_a = \frac{dt}{d\tau_a} = \left(\sqrt{g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}}\right)^{-1} = \left(\sqrt{g_{00} + \frac{g_{ij}v^i v^j}{c^2}}\right)^{-1} \simeq 1 + \phi - \frac{v_a^2}{2c^2} \tag{3.71}$$

while for the determinant we have

$$-g = 1 - \overset{2}{g}_{00} + \sum_i^3 \overset{2}{g}_{ii} = 1 - 4\Phi, \qquad \frac{1}{\sqrt{-g}} \simeq 1 + 2\Phi \tag{3.72}$$

and, therefore, the expansion of Eq. (3.70) gives the explicit form of the energy-momentum tensor

$$\overset{0}{T}{}^{00} = \textstyle\sum_a m_a c^2 \delta(\mathbf{x} - \mathbf{x}_a)$$

$$\overset{2}{T}{}^{00} = \textstyle\sum_a m_a \left(\frac{v^2}{2} + \Phi c^2\right)\delta(\mathbf{x} - \mathbf{x}_a) \tag{3.73}$$

$$\overset{1}{T}{}^{0i} = \textstyle\sum_a m_a c\, v_a^i\,\delta(\mathbf{x} - \mathbf{x}_a)$$

$$\overset{2}{T}{}^{ij} = \textstyle\sum_a m_a v^i v^j\,\delta(\mathbf{x} - \mathbf{x}_a)$$

where v_a^i is the velocity of the source. As a first application of these results, let us now take into account the simplest case, that is GR.

3.4 Field Equations and Solutions in Post-Newtonian Limit of General Relativity

We remind here that the Einstein Equation can be re-expressed as

$$R_{\mu\nu} = \mathcal{X}\left(T_{\mu\nu} - \frac{T}{2}g_{\mu\nu}\right) \tag{3.74}$$

From the interpretation of the components of the energy-momentum tensor as energy density, momentum density and momentum flux, we have T_{00}, T_{0i} and T_{ij} at the various orders

$$\begin{aligned} T_{00} &= \overset{0}{T}_{00} + \overset{2}{T}_{00} \\ T_{0i} &= \overset{1}{T}_{0i} \\ T_{ij} &= \overset{2}{T}_{ij} \end{aligned} \tag{3.75}$$

where $\overset{N}{T}_{\mu\nu}$ denotes the term in $T_{\mu\nu}$ of order $\bar{M}/\bar{r}^3\,\bar{\varepsilon}^N$. In particular, by comparing Eq. (3.75) with Eq. (3.68), we immediately identify $\overset{0}{T}_{00} = \rho c^2$ as the density of rest mass, $\overset{2}{T}_{00} = \rho\,(v^2 + 2\Phi)$ as the non-relativistic part of the energy density, as well as $\overset{1}{T}_{0i} = -\rho c\, v_i$ and $\overset{2}{T}_{ij} = \rho v_i v_j$. Thus, the required Einstein's field equations can be found in the form (3.74) by introducing the new tensor

$$S_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T \tag{3.76}$$

$GM/\bar{r}$ is of the order $\bar{v}^2$, therefore the metric (3.33) and (3.75) yield

$$\begin{aligned} S_{00} &= S_{00}^{(0)} + S_{00}^{(2)} \\ S_{0i} &= S_{0i}^{(1)} \\ S_{ij} &= S_{ij}^{(0)} \end{aligned} \tag{3.77}$$

where $\overset{N}{S}_{\mu\nu}$ denotes the term in $S_{\mu\nu}$ of order $(M/\bar{r}^3)\,\bar{\varepsilon}^N$. Thereby, we can write

$$\begin{aligned}
\overset{0}{S}_{00} &= \tfrac{1}{2}\overset{0}{T}_{00} \\
\overset{2}{S}_{00} &= \tfrac{1}{2}(\overset{2}{T}_{00} - 2\overset{2}{g}_{00}\overset{0}{T}_{00} + \overset{2}{T}_{hh}) \\
\overset{1}{S}_{0i} &= \overset{1}{T}_{0i} \\
\overset{0}{S}_{ij} &= \tfrac{1}{2}\delta_{ij}\overset{0}{T}_{00}
\end{aligned} \tag{3.78}$$

Now, by inserting Eqs. (3.63) and (3.77) in Eqs. (3.74), we notice the consistency between the field equation (in harmonic coordinates) and the expansions we are adopting. We get

$$\begin{aligned}
\overset{2}{R}_{00} &= \mathcal{X}\overset{0}{S}_{00} \\
\overset{4}{R}_{00} &= \mathcal{X}\overset{2}{S}_{00} \\
\overset{3}{R}_{0i} &= \mathcal{X}\overset{0}{S}_{0i} \\
\overset{2}{R}_{ij} &= \mathcal{X}\overset{0}{S}_{ij}
\end{aligned} \tag{3.79}$$

that is, the set of field of equations

$$\begin{aligned}
\triangle\overset{2}{g}_{00} &= \mathcal{X}\,\overset{0}{T}_{00} \\
\triangle\overset{3}{g}_{0i} &= 2\,\mathcal{X}\,\overset{1}{T}_{0i} - \Phi_{,0i} \\
\triangle\overset{2}{g}_{ij} &= \mathcal{X}\,\delta_{ij}\,\overset{0}{T}_{00} \\
\triangle\overset{4}{g}_{00} &= \mathcal{X}\left(\overset{2}{T}_{00} - 2\overset{2}{g}_{00}\overset{0}{T}_{00} + \overset{2}{T}_{hh}\right) - \overset{2}{g}_{hl}\overset{2}{g}_{00,hl} - \overset{2}{g}_{hl,h}\overset{2}{g}_{00,l} + |\bigtriangledown\overset{2}{g}_{00}|^2
\end{aligned} \tag{3.80}$$

From Eq. $(3.80)_1$, by also considering a point-like source $\rho = M\delta(\mathbf{x})$ and Eq. (3.68), we find

$$\overset{2}{g}_{00} = -\frac{\mathcal{X}}{4\pi}\int\frac{\overset{0}{T}_{00}}{|\mathbf{x}-\mathbf{x}'|}d^3\mathbf{x}' = -2G\int\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3\mathbf{x}' = -\frac{2GM}{|\mathbf{x}-\mathbf{x}'|} \equiv 2\Phi(\mathbf{x}) \tag{3.81}$$

where $\Phi(\mathbf{x})$ is the Newtonian gravitational potential

$$\Phi(\mathbf{x}) \;=\; -\frac{GM}{|\mathbf{x}-\mathbf{x}'|} \tag{3.82}$$

as we were to expect. From Eq. (3.80)$_2$, the solution is found as

$$\overset{3}{g}_{0i} \;=\; -\frac{\mathcal{X}}{2\pi}\int \frac{\overset{1}{T}_{0i}}{|\mathbf{x}-\mathbf{x}'|}d^3\mathbf{x}' - X_{,0i} = 4\int \frac{\rho(\mathbf{x}')v^i}{|\mathbf{x}-\mathbf{x}'|}d^3\mathbf{x}' - X_{,0i} = \frac{4GM}{|\mathbf{x}-\mathbf{x}'|}v_i' \;\equiv\; A_i(\mathbf{x}) - X_{,0i}. \tag{3.83}$$

Here X is the *superpotential*, which is solution for the equation $\triangle X \;=\; \Phi$ and thanks to which we have found $\overset{3}{g}_{0i}$. For a complete determination of it, we find that

$$X(\mathbf{x}) \;=\; -\frac{1}{2}GM|\mathbf{x}-\mathbf{x}'|. \tag{3.84}$$

Then, the final analytical expression for g_{0i} is

$$\overset{3}{g}_{0i} = \frac{1}{2}\frac{GM}{|\mathbf{x}-\mathbf{x}'|}\left[7v_i' + (\mathbf{v}'\cdot\mathbf{n}')\,n_i'\right] \tag{3.85}$$

with $\mathbf{n}' = (\mathbf{x}-\mathbf{x}')/|\mathbf{x}-\mathbf{x}'|$ and $n_i' = (x-x')_i/|\mathbf{x}-\mathbf{x}'|$. While from Eq. (3.80)$_3$, we determine the solutions

$$\overset{2}{g}_{ij} \;=\; -\frac{\mathcal{X}}{4\pi}\,\delta_{ij}\int \frac{\overset{0}{T}_{00}}{|\mathbf{x}-\mathbf{x}'|}d^3\mathbf{x}' \;=\; \int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3\mathbf{x}' \;=\; -\delta_{ij}\frac{2GM}{|\mathbf{x}-\mathbf{x}'|} \;=\; 2\delta_{ij}\Phi(\mathbf{x}) \tag{3.86}$$

Lastly, by taking advantage of the Poisson equation $\triangle\Phi \;=\; -4\pi G\overset{0}{T}_{00}$ together with the identity $4(\nabla\Phi)^2 = 2\triangle(\Phi^2) - 4\Phi\triangle\Phi$, we can express Eq. (3.80)$_4$ as follows

$$\triangle\left(\overset{4}{g}_{00} - 2\Phi^2\right) \;=\; \mathcal{X}\left(\overset{2}{T}_{00} + \overset{2}{T}_{hh}\right). \tag{3.87}$$

By defining the potential $2\psi = \overset{4}{g}_{00} - 2\Phi^2$ and recurring to Eqs. (3.73)$_{2,4}$, the above equation becomes

$$\triangle\psi = 4\pi GM\left[\Phi(\mathbf{x}') + \frac{3}{2}\mathbf{v}'^2\right]\delta(\mathbf{x}-\mathbf{x}'), \tag{3.88}$$

where $\Phi(\mathbf{x}') = -GM'|\mathbf{x}'-\mathbf{x}''|^{-1}$. Thus the solution for ψ reads

$$\psi(\mathbf{x}) = -\frac{GM\Phi(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} - \frac{3}{2}\frac{GM}{|\mathbf{x}-\mathbf{x}'|}v_i'^2 \tag{3.89}$$

By using the equations at second order we obtain the solution for $\overset{4}{g}_{00}$, correction at fourth order in the time-time component of the metric, that is expressed as

$$\overset{4}{g}_{00} = -2(\Phi^2 + \psi) \tag{3.90}$$

This completes the determination of the field solutions in Post-Newtonian limit of General Relativity.

3.5 STFOG Field Equations in Weak Field limit

We now study, in the weak-field approximation, models of Extended Gravity at Solar System scales. In order to perform the weak-field limit, we have to perturb Eqs. (2.16), (2.19) and (2.18) on a Minkowski background $\eta_{\mu\nu}$ [109; 110; 111]. We set

$$g_{\mu\nu} \simeq \begin{pmatrix} 1+\overset{2}{g}_{00}(x_0,\mathbf{x})+\dots & \overset{3}{g}_{0i}(x_0,\mathbf{x})+\dots \\ \overset{3}{g}_{0i}(x_0,\mathbf{x})+\dots & -\delta_{ij}+\overset{2}{g}_{ij}(x_0,\mathbf{x})+\dots \end{pmatrix}$$

$$\phi \simeq \phi^{(0)}+\phi^{(2)}+\dots = \phi^{(0)}+\varphi,$$

where $\overset{2}{g}_{00}$, $\overset{2}{g}_{ij}$, φ are proportional to the power $(v/c)^2$, while $\overset{3}{g}_{0i}$ is proportional to $(v/c)^3$. The function f, up to the $(v/c)^3$ order, can be developed as

$$f(R,R_{\alpha\beta}R^{\alpha\beta},\phi) = f_R(0,0,\phi^{(0)})\,R+\frac{f_{RR}(0,0,\phi^{(0)})}{2}\,R^2+\frac{f_{\phi\phi}(0,0,\phi^{(0)})}{2}(\phi-\phi^{(0)})^2 \\ +f_{R\phi}(0,0,\phi^{(0)})R\,\phi+f_Y(0,0,\phi^{(0)})R_{\alpha\beta}R^{\alpha\beta}, \tag{3.91}$$

while all other possible contributions in f are negligible [112]. To make the notation lighter, we now simply decide to drop the parentheses of the partial derivatives f_R, f_{RR}, $f_{\phi\phi}$, $f_{R\phi}$, f_Y. The field equations (2.16), (2.19) and (2.18) hence read

$$\begin{aligned}
&f_R\left[R_{00}-\tfrac{R}{2}\right]-f_Y\triangle R_{00}-[f_{RR}+\tfrac{f_Y}{2}]\triangle R-f_{R\phi}\triangle\varphi = \mathcal{X}\,T_{00},\\
&f_R\left[R_{ij}+\tfrac{R}{2}\delta_{ij}\right]-f_Y\triangle R_{ij}+[f_{RR}+\tfrac{f_Y}{2}]\delta_{ij}\triangle R-f_{RR}R_{,ij}\\
&\qquad -2f_Y R^{\alpha}{}_{(i,j)\alpha}-f_{R\phi}(\partial^2_{ij}-\delta_{ij}\triangle)\varphi = \mathcal{X}\,T_{ij},\\
&f_R\,R_{0i}-f_Y\triangle R_{0i}-f_{RR}R_{,0i}-2f_Y R^{\alpha}{}_{(0,i)\alpha}-f_{R\phi}\,\varphi_{,0i} = \mathcal{X}\,T_{0i},\\
&f_R\,R+[3f_{RR}+2f_Y]\triangle R+3f_{R\phi}\triangle\varphi = -\mathcal{X}\,T,\\
&2\omega(\phi^{(0)})\triangle\varphi+f_{\phi\phi}\varphi+f_{R\phi}R = 0,
\end{aligned} \tag{3.92}$$

where $\triangle$ is the Laplace operator in the flat space. For the following, which concerns the finding of the solutions, we will adopt the convention with unitary speed of light $c=1$, meaning $\mathcal{X}=8\pi G$ for the coupling constant between matter and geometry. The geometric quantities $R_{\mu\nu}$ and R are evaluated at the first order with respect to the metric potentials Φ, Ψ and Z_i. By introducing the quantities[12]

$$\begin{aligned}
{m_R}^2 &\equiv -\tfrac{f_R}{3f_{RR}+2f_Y}, \\
{m_Y}^2 &\equiv \tfrac{f_R}{f_Y}, \\
{m_\phi}^2 &\equiv -\tfrac{f_{\phi\phi}}{2\omega(\phi^{(0)})},
\end{aligned} \tag{3.93}$$

and setting $f_R = 1$, $\omega(\phi^{(0)}) = 1/2$ for simplicity[13], we get the complete set of differential equations

$$\begin{aligned}
&(\triangle - {m_Y}^2)R_{00} + \left[\tfrac{{m_Y}^2}{2} - \tfrac{{m_R}^2+2{m_Y}^2}{6{m_R}^2}\triangle\right]R + {m_Y}^2\, f_{R\phi}\,\triangle\varphi \,=\, -{m_Y}^2\mathcal{X}\,T_{00}, \\
&(\triangle - {m_Y}^2)R_{ij} + \left[\tfrac{{m_R}^2-{m_Y}^2}{3{m_R}^2}\,\partial^2_{ij} - \delta_{ij}\left(\tfrac{{m_Y}^2}{2} - \tfrac{{m_R}^2+2{m_Y}^2}{6{m_R}^2}\triangle\right)\right]R \\
&\qquad\qquad\qquad + {m_Y}^2\, f_{R\phi}\,(\partial^2_{ij} - \delta_{ij}\triangle)\varphi \,=\, -{m_Y}^2\,\mathcal{X}\,T_{ij}, \\
&(\triangle - {m_Y}^2)R_{0i} + \tfrac{{m_R}^2-{m_Y}^2}{3{m_R}^2}R_{,0i} + {m_Y}^2\, f_{R\phi}\,\varphi_{,0i} \,=\, -{m_Y}^2\mathcal{X}\,T_{0i}, \\
&(\triangle - {m_R}^2)R - 3{m_R}^2\, f_{R\phi}\,\triangle\varphi \,=\, {m_R}^2\,\mathcal{X}\,T, \\
&(\triangle - {m_\phi}^2)\varphi + f_{R\phi}\,R \,=\, 0.
\end{aligned} \tag{3.94}$$

We work with the Standard Post-Newtonian gauge and the components of the Ricci tensor in Eq. (3.94) in the weak-field limit read

$$\begin{aligned}
R_{00} &= \tfrac{1}{2}\triangle\overset{2}{g}_{00} \,=\, \triangle\Phi, \\
R_{ij} &= \tfrac{1}{2}\overset{2}{g}_{ij,hh} - \tfrac{1}{2}\overset{2}{g}_{ih,hj} - \tfrac{1}{2}\overset{2}{g}_{jh,hi} - \tfrac{1}{2}\overset{2}{g}_{00,ij} + \tfrac{1}{2}\overset{2}{g}_{hh,ij} \,=\, \triangle\Psi\,\delta_{ij} + (\Psi-\Phi)_{,ij}, \\
R_{0i} &= \frac{1}{2}\overset{3}{g}_{0i,hh} - \tfrac{1}{2}\overset{2}{g}_{ih,0h} - \tfrac{1}{4}\overset{2}{g}_{hh,0i} + \tfrac{1}{2}\overset{2}{g}_{hh,0i} \,=\, \frac{1}{2}(\triangle Z_i + \Psi_{,0i}).
\end{aligned} \tag{3.95}$$

[12] In the Newtonian and post-Newtonian limits, we can consider as Lagrangian $f(R) = a\,R + b\,R^2 + c\,R_{\alpha\beta}R^{\alpha\beta}$. Then the masses (3.93) become ${m_R}^2 = -\frac{a}{2(3b+c)}$, ${m_Y}^2 = \frac{a}{c}$. For a correct interpretation of these quantities as real masses, we have to impose $a > 0$, $b < 0$ and $0 < c < -3b$.

[13] We can define a new gravitational constant: $\mathcal{X} \to \mathcal{X}\,f_R(0,0,\phi^{(0)})$ and $f_{R\phi}(0,0,\phi^0) \to f_{R\phi}(0,0,\phi^0)\,f_R(0,0,\phi^{(0)})$.

The energy-momentum tensor $T_{\mu\nu}$ can be also expanded. For a perfect fluid, when the pressure is negligible with respect to the mass density ρ, it reads $T_{\mu\nu} = \rho\, u_\mu u_\nu$ with $u_\mu u^\mu = 1$. However, the development starts form the zeroth order, hence $T_{00} = \overset{0}{T}_{00} = \rho$, $T_{ij} = \overset{0}{T}_{ij} = 0$ and $T_{0i} = \overset{1}{T}_{0i} = -\rho\, v_i$, where ρ is the density mass and v^i is the velocity of the source. In this way, $T_{\mu\nu}$ is independent of metric potentials and the ordinary conservation condition $T^{\mu\nu}{}_{,\mu} = 0$ is satisfied. Equations (3.94) thus read

$$
\begin{aligned}
&(\triangle - {m_Y}^2)\triangle\Phi + \left[\tfrac{{m_Y}^2}{2} - \tfrac{{m_R}^2+2{m_Y}^2}{6{m_R}^2}\triangle\right]R + {m_Y}^2\, f_{R\phi}\,\triangle\varphi \,=\, -{m_Y}^2\mathcal{X}\,\rho,\\
&\left\{(\triangle - {m_Y}^2)\triangle\Psi - \left[\tfrac{{m_Y}^2}{2} - \tfrac{{m_R}^2+2{m_Y}^2}{6{m_R}^2}\triangle\right]R - {m_Y}^2\, f_{R\phi}\,\triangle\varphi\right\}\delta_{ij}\\
&\qquad\qquad +\left\{(\triangle - {m_Y}^2)(\Psi-\Phi) + \tfrac{{m_R}^2-{m_Y}^2}{3{m_R}^2}\,R + {m_Y}^2\, f_{R\phi}\,\varphi\right\}_{,ij} \,=\, 0,\\
&(\triangle - {m_Y}^2)\triangle Z_i + \left\{(\triangle - {m_Y}^2)\Psi + 2\left[\tfrac{{m_R}^2-{m_Y}^2}{3{m_R}^2}R + {m_Y}^2\, f_{R\phi}\,\varphi\right]\right\}_{,0i} \,=\, 2{m_Y}^2\mathcal{X}\,\rho\, v_i,\\
&(\triangle - {m_R}^2)R - 3{m_R}^2\, f_{R\phi}\,\triangle\varphi \,=\, {m_R}^2\,\mathcal{X}\,\rho,\\
&(\triangle - {m_\phi}^2)\varphi + f_{R\phi}\,R \,=\, 0.
\end{aligned}
\tag{3.96}
$$

3.5.1 Field Solutions for a Point-like Source

Eqs. (3.96) represent a system of linear fourth-order partial differential equations and, in order to show an efficient way to deal with the problem, we now discuss how it is possible to get a fast resolution. The strategy is to perform direct calculations following a simple approach based on linear differential operators theory. In fact, by noticing the presence of products of two linear differential operators, i.e. the Laplacian $\triangle$ (in flat space) and the operator $\triangle - m_Y^2$ associated with the Klein-Gordon equation, the method consists of a decomposition of a single fourth-order equation (where the two operators appear) into two decoupled second-order equations. In such a way the two decoupled second-order equations can be easily solved, for example, by resorting to the Green functions. In fact, it is proved that the solution of the starting fourth order equation will be given by a certain combination of the two particular solutions of the decoupled second order equations (refer to [88; 89; 90; 91; 93; 95; 92; 94; 96; 97]). Therefore, let us consider two differential equations

$$\mathcal{A}\,u_1 = f \tag{3.97}$$

$$\mathcal{B}\,u_2 = f \tag{3.98}$$

where $\mathcal{A} = (\triangle - \alpha)$, $\mathcal{B} = (\triangle - \beta)$ are the two linear symmetric elliptic[14] differential operators with $\alpha \neq \beta$ real numbers, then satisfying the relations $\mathcal{A}\,\mathcal{B} = \mathcal{B}\,\mathcal{A}$ and $\mathcal{A}\,\mathcal{A}^{-1} = \mathcal{B}\,\mathcal{B}^{-1} = I$, while $u_1 = \mathcal{A}^{-1}f$ and $u_2 = \mathcal{B}^{-1}f$ are assumed to be the particular solutions of f. Then the fourth order partial differential equation

$$\mathcal{A}\,\mathcal{B}\,u = f \tag{3.99}$$

arising from the product of the two operators, admits the particular solution $u = (\mathcal{A}\,\mathcal{B})^{-1}f$ expressed by

$$u = \frac{u_1 - u_2}{\alpha - \beta}. \tag{3.100}$$

In order to prove the relation (3.100), we first notice that α and β are not eigenvalues of the Laplacian, and thus $\mathcal{A}$ and $\mathcal{B}$ are invertible operators. Let us start from the difference $\mathcal{B} - \mathcal{A}$ that can be written as

$$\mathcal{B} - \mathcal{A} = \mathcal{B}\,\mathcal{A}\,\mathcal{A}^{-1} - \mathcal{A}\,\mathcal{B}\,\mathcal{B}^{-1} = \mathcal{B}\,\mathcal{A}\,(\mathcal{A}^{-1} - \mathcal{B}^{-1}), \tag{3.101}$$

But such a difference also gives $\mathcal{B} - \mathcal{A} = (\alpha - \beta)$, from which

$$\mathcal{B}\,\mathcal{A}\,(\mathcal{A}^{-1} - \mathcal{B}^{-1}) = \alpha - \beta \;\Rightarrow\; (\mathcal{B}\,\mathcal{A})^{-1} = \frac{\mathcal{A}^{-1} - \mathcal{B}^{-1}}{\alpha - \beta}. \tag{3.102}$$

Since the solution is properly given by $u = (\mathcal{A}\,\mathcal{B})^{-1}f$, it follows that

$$u = (\mathcal{A}\,\mathcal{B})^{-1}f = (\mathcal{B}\,\mathcal{A})^{-1}f = \frac{(\mathcal{A}^{-1} - \mathcal{B}^{-1})}{\alpha - \beta}f = \frac{\mathcal{A}^{-1}f - \mathcal{B}^{-1}f}{\alpha - \beta} \tag{3.103}$$

finally providing the relation

$$u = \frac{u_1 - u_2}{\alpha - \beta}. \tag{3.104}$$

Since the result (3.100) represents the solution for the fourth order partial differential equation given by the product of these two linear differential operators, the determination of the solution is a straightforward application of it. As a working example, whether we identify the linear differential operators with $\mathcal{A} = \triangle - m_Y^2$ and $\mathcal{B} = \triangle$, which involve $\alpha = m_Y^2$ and $\beta = 0$, we can find the solutions $u_1 = \int \mathcal{G}_1(\mathbf{x}, \mathbf{x}')f(\mathbf{x}')d\mathbf{x}'$ and $u_2 = \int \mathcal{G}_2(\mathbf{x}, x')f(\mathbf{x}')d\mathbf{x}'$ by means of the Green functions

$$\mathcal{G}_1(\mathbf{x}, \mathbf{x}') = -\frac{e^{-m_Y|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|} \quad , \quad \mathcal{G}_2(\mathbf{x}, \mathbf{x}') = -\frac{1}{4\pi|\mathbf{x}-\mathbf{x}'|}, \tag{3.105}$$

[14]The elliptic operator is a generalization of the Laplacian operator.

associated to the Klein-Gordon equation and to the Laplacian operator respectively. They both tend to zero as $|\mathbf{x}| \to 0$. Thus, the particular solution of such a fourth-order equation will be represented as

$$u = \frac{1}{m_Y^2}\left(\int \mathcal{G}_1(\mathbf{x},\mathbf{x}')f(\mathbf{x}')d^3\mathbf{x}' - \int \mathcal{G}_2(\mathbf{x},\mathbf{x}')f(\mathbf{x}')d^3\mathbf{x}'\right) \tag{3.106}$$

In the following, we apply this method.

3.5.2 Scalar Fields φ and R

Equations (3.96)$_1$ and (3.96)$_2$ constitute a coupled set and for a point-like source $\rho(\mathbf{x}) = M\,\delta(\mathbf{x})$, admit the solutions

$$\begin{aligned}
\varphi(\mathbf{x}) &= \sqrt{\tfrac{\xi}{3}}\,\tfrac{r_g}{|\mathbf{x}|}\,\tfrac{e^{-m_+|\mathbf{x}|}-e^{-m_-|\mathbf{x}|}}{\omega_+-\omega_-},\\
R(\mathbf{x}) &= -{m_R}^2\,\tfrac{r_g}{|\mathbf{x}|}\,\tfrac{(\omega_+-\eta^2)\,e^{-m_+|\mathbf{x}|}-(\omega_--\eta^2)\,e^{-m_-|\mathbf{x}|}}{\omega_+-\omega_-},
\end{aligned} \tag{3.107}$$

where r_g is the Schwarzschild radius, $\omega_\pm = \frac{1-\xi+\eta^2\pm\sqrt{(1-\xi+\eta^2)^2-4\eta^2}}{2}$, $\xi = 3f_{R\phi}^2$, the masses present in the Yukawa-like terms are $m_\pm^2 = m_R^2\,\omega_\pm$, and finally $\eta = \frac{m_\phi}{m_R}$ [112]. The parameter ξ is generally defined as $\frac{3f_{R\phi}^2}{2\,f_R\,\omega(\phi^{(0)})}$, moreover ξ and η satisfy the condition $(\eta-1)^2-\xi > 0$ [113; 109; 110; 111].

3.5.3 Gravitational Potentials Φ and Ψ

It is possible to solve (3.96)$_1$ by making use of equations (3.96)$_4$-(3.96)$_5$ and then adopting the method described above. After performing calculations, the formal solution for Φ is

$$\begin{aligned}
\Phi(\mathbf{x}) \quad &= -\frac{\mathcal{X}}{4\pi}\int d^3\mathbf{x}'\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} - \frac{1}{8\pi}\int d^3\mathbf{x}'\frac{R(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} - \left(\frac{2{m_Y}^2+{m_R}^2}{6\,m_Y^2\,m_R^2}\right)R(\mathbf{x}) + \\
&+\frac{\xi^{1/2}}{\sqrt{3}}\varphi(\mathbf{x}) + \frac{1}{4\pi m_Y^2}\int d^3\mathbf{x}'\frac{e^{-m_Y|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\Bigg[\mathcal{X}\rho(\mathbf{x}')\frac{4{m_Y}^2-{m_R}^2}{6} + \\
&+\frac{{m_Y}^2-{m_R}^2(1-\xi)}{6}R(\mathbf{x}') - \frac{{m_R}^4\eta^2}{2\sqrt{3}}\,\xi^{1/2}\,\varphi(\mathbf{x}')\Bigg].
\end{aligned}$$

In particular, the gravitational potential for a point-like source is

$$\Phi(\mathbf{x}) \;=\; -\frac{GM}{|\mathbf{x}|}\biggl[1+g(\xi,\eta)\,e^{-m_+|\mathbf{x}|}+[\frac{1}{3}-g(\xi,\eta)]\,e^{-m_-|\mathbf{x}|}-\frac{4}{3}\,e^{-m_Y|\mathbf{x}|}\biggr], \tag{3.108}$$

where

$$g(\xi,\eta) \;=\; \frac{1-\eta^2+\xi+\sqrt{\eta^4+(\xi-1)^2-2\eta^2(\xi+1)}}{6\sqrt{\eta^4+(\xi-1)^2-2\eta^2(\xi+1)}}\,.$$

It should be noted that for $f_Y \;\to\; 0$, namely $m_Y \;\to\; \infty$, one gets the same result for the gravitational potential in Ref. [112] for a $f(R,\phi)$-theory. The linearity of the field equations (3.96) ensures that the solution (3.108) is a linear combination of solutions obtained within the $f(R,\phi)$-theory and the $f(R,R_{\alpha\beta}R^{\alpha\beta})$-theory. Such a linearity is also ensured by the absence of the coupling term between the curvature invariant $Y = R_{\alpha\beta}R^{\alpha\beta}$ and the scalar field ϕ. Now it is possible to calculate Ψ by formally solving Eq. $(3.96)_2$. The solution can be obtained by setting $\{\ldots\}_{,ij} \;=\; 0$ in Eq. $(3.96)_2$, while, along with it, we also have $\{\ldots\}\delta_{ij} \;=\; 0$.

$$\Psi(\mathbf{x}) \;=\; \Phi(\mathbf{x})+\frac{{m_R}^2-{m_Y}^2}{12\pi {m_R}^2}\int d^3\mathbf{x}'\frac{e^{-m_Y|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\,R(\mathbf{x}')+\frac{{m_Y}^2\xi^{1/2}}{4\sqrt{3}\pi}\int d^3\mathbf{x}'\frac{e^{-m_Y|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\,\varphi(\mathbf{x}')\;,$$

which for a point-like source reads

$$\Psi(\mathbf{x}) \;=\; -\frac{GM}{|\mathbf{x}|}\biggl[1-g(\xi,\eta)\,e^{-m_+|\mathbf{x}|}-[1/3-g(\xi,\eta)]\,e^{-m_-|\mathbf{x}|}-\frac{2}{3}\,e^{-m_Y|\mathbf{x}|}\biggr], \tag{3.109}$$

The solutions (3.108) and (3.109) are those relative to the g_{00} and g_{ij} components respectively, and accomplishes a generalization of the results of the theory $f(R,\, R_{\alpha\beta}R^{\alpha\beta})$ [110; 111].

3.5.4 Vector Potential Z_i and Superpotential X

Concerning Eq. $(3.96)_3$, since in Eq. $(3.96)_2$ we set $\{\ldots\}_{,ij} \;=\; 0$, we notice that in our adopted Standard Post-Newtonian condition the expression $\{\ldots\}_{,0i} \;=\; 0$ provide the further term $(\triangle - m_Y^2)\Phi_{,0i}$. Therefore, the equation to solve is reduced to

$$(\triangle - {m_Y}^2)\triangle(Z_i+X_{,0i}) \;=\; 2{m_Y}^2\mathcal{X}\,\rho\,v_i, \tag{3.110}$$

where the superpotential X appears, that is related to the equation $\triangle X = \Phi$ from which it derives. By carrying out the calculations as before, using Eq. (3.100), from Eq. $(3.96)_3$ we immediately obtain the solution for Z_i that reads

$$Z_i(\mathbf{x}) \;=\; \frac{\mathcal{X}}{2\pi}\int d^3\mathbf{x}'\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,v_i'-\frac{\mathcal{X}}{2\pi}\int d^3\mathbf{x}'\frac{e^{-m_Y|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\,\rho(\mathbf{x}')\,v_i'-X_{,0i}. \tag{3.111}$$

We can represent the first two terms at the second member as the vector potential term

$$A_i(\mathbf{x}) = \frac{4GM}{|\mathbf{x}|}\, v'_i - 4GM \frac{e^{-m_Y|\mathbf{x}|}}{|\mathbf{x}|}\, v'_i, \tag{3.112}$$

that is, the first term of GR and the second term as the massive mode contribution induced by the presence of the invariant $R_{\alpha\beta}R^{\alpha\beta}$. Hence it just remains to solve the equation $\triangle X = \Phi$ for the superpotential, where Φ is given by Eq. (3.108). After a straightforward integration, we find

$$\begin{aligned} X(\mathbf{x}) \quad &= -\frac{GM}{2}|\mathbf{x}| + \frac{GM}{m_+^2|\mathbf{x}|}\left(1 - e^{m_+|\mathbf{x}|} - m_+|\mathbf{x}|\right) g(\xi,\eta) + + \frac{GM}{m_-^2|\mathbf{x}|}\left(1 - e^{m_-|\mathbf{x}|} - m_-|\mathbf{x}|\right) \\ &\times \left[\frac{1}{3} - g(\xi,\eta)\right] - \frac{4}{3}\frac{GM}{m_Y^2|\mathbf{x}|}\left(1 - e^{m_Y|\mathbf{x}|} - m_Y|\mathbf{x}|\right). \end{aligned} \tag{3.113}$$

Then the solution relative to the g_{0i} component can be rapidly summarized as

$$Z_i(\mathbf{x}) = A_i(\mathbf{x}) - X(\mathbf{x})_{,0i}\,. \tag{3.114}$$

The solution (3.111) can be rewritten as the sum of vector potential A_i and $X_{,0i}$ relative to the superpotential in General Relativity, plus the corresponding additional contributions stemming from the STFOG, thus extending the preceding GR outcomes. Furthermore, we readily realize that it is more convenient to make explicit both time and spatial partial derivatives of the superpotential X when needed for the deduction of the relativistic Lagrangian of the N-body system and thus for the equations of motion of the system.

Here we collect all the solutions found for the components of the metric tensor in terms of the potentials generated by the point-like source as follows

$$\begin{aligned}
g_{00} &= 1+\frac{2}{c^2}\Phi(\mathbf{x}) = 1-\frac{2GM}{c^2\,|\mathbf{x}|}\biggl[1+g(\xi,\eta)\,e^{-m_+|\mathbf{x}|} \\
&\qquad +[1/3-g(\xi,\eta)]\,e^{-m_-|\mathbf{x}|}-\frac{4}{3}\,e^{-m_Y|\mathbf{x}|}\biggr], \\
g_{ij} &= -(1-\frac{2}{c^2}\Psi(\mathbf{x}))\,\delta_{ij} = -\delta_{ij}-\frac{2GM}{c^2\,|\mathbf{x}|}\biggl[1-g(\xi,\eta)\,e^{-m_+|\mathbf{x}|} \\
&\qquad -[1/3-g(\xi,\eta)]\,e^{-m_-|\mathbf{x}|}-\frac{2}{3}\,e^{-m_Y|\mathbf{x}|}\biggr]\delta_{ij}, \\
g_{0i} &= \frac{Z_i}{c^3} = \frac{1}{c^3}(A_i(\mathbf{x})-X(\mathbf{x})_{,0i}) = \frac{4GM}{c^3\,|\mathbf{x}|}\,v_i'-4GM\frac{e^{-m_Y|\mathbf{x}|}}{c^3\,|\mathbf{x}|}\,v_i'+ \\
&\qquad -\frac{1}{c^3}\biggl(\frac{GM}{2}|\mathbf{x}|-\frac{GM}{m_+^2|\mathbf{x}|}\left[1-e^{m_+|\mathbf{x}|}-m_+|\mathbf{x}|\right]\,g(\xi,\eta)+ \\
&\qquad -\frac{GM}{m_-^2\,|\mathbf{x}|}\left[1-e^{m_-|\mathbf{x}|}-m_-|\mathbf{x}|\right]\left[\frac{1}{3}-g(\xi,\eta)\right]+ \\
&\qquad +\frac{4}{3}\frac{GM}{m_Y^2|\mathbf{x}|}\left[1-e^{m_Y|\mathbf{x}|}-m_Y|\mathbf{x}|\right]\biggr)_{,0i}
\end{aligned} \tag{3.115}$$

It is useful, alternatively, to provide a summarized list of the potentials in the following way

$$\Phi(\mathbf{x}) = -\frac{GM}{|\mathbf{x}|}\Bigl[1+\zeta(|\mathbf{x}|)\Bigr], \tag{3.116}$$

$$\zeta(|\mathbf{x}|) \equiv g(\xi,\eta)\,e^{-m_+|\mathbf{x}|}+\left[\frac{1}{3}-g(\xi,\eta)\right]e^{-m_-|\mathbf{x}|}-\frac{4}{3}\,e^{-m_Y|\mathbf{x}|}, \tag{3.117}$$

$$\Psi(\mathbf{x}) = -\frac{GM}{|\mathbf{x}|}\Bigl[1-\eta(|\mathbf{x}|)\Bigr], \tag{3.118}$$

$$\eta(|\mathbf{x}|) \equiv g(\xi,\eta)\,e^{-m_+|\mathbf{x}|}+\left[\frac{1}{3}-g(\xi,\eta)\right]e^{-m_-|\mathbf{x}|}+\frac{2}{3}\,e^{-m_Y|\mathbf{x}|}, \tag{3.119}$$

$$Z_i(\mathbf{x}) = A_i(\mathbf{x})-X(\mathbf{x})_{,0i} \tag{3.120}$$

$$A_i(\mathbf{x}) = \frac{4\,GM}{|\mathbf{x}|}\,v_i^{'}-\frac{4\,GMe^{-m_Y|\mathbf{x}|}}{|\mathbf{x}|}\,v_i^{'}, \tag{3.121}$$

$$X(\mathbf{x}) = -\frac{GM}{2}|\mathbf{x}|+\widetilde{X}(\mathbf{x}) \tag{3.122}$$

$$\widetilde{X}(\mathbf{x}) \equiv \frac{GM}{m_+^2|\mathbf{x}|}\Bigl(1-e^{m_+|\mathbf{x}|}-m_+|\mathbf{x}|\Bigr)\,g(\xi,\eta)+ \tag{3.123}$$

$$\frac{GM}{m_-^2|\mathbf{x}|}\Bigl(1-e^{m_-|\mathbf{x}|}-m_-|\mathbf{x}|\Bigr)\left[\frac{1}{3}-g(\xi,\eta)\right] \tag{3.124}$$

$$-\frac{4}{3}\frac{GM}{m_Y^2|\mathbf{x}|}\Bigl(1-e^{m_Y|\mathbf{x}|}-m_Y|\mathbf{x}|\Bigr) \tag{3.125}$$

$$\varphi(\mathbf{x}) = \frac{GM}{|\mathbf{x}|}\sqrt{\frac{\xi}{3}}\,\frac{2}{\omega_+-\omega_-}\left[e^{-m_+\,|\mathbf{x}|}\;-\;e^{-m_-\,|\mathbf{x}|}\right], \tag{3.126}$$

where $f_R(0,0,\phi^{(0)}) = 1$, $\omega(\phi^{(0)}) = 1/2$, and

$$g(\xi,\eta) = \frac{1-\eta^2+\xi+\sqrt{\eta^4+(\xi-1)^2-2\eta^2(\xi+1)}}{6\sqrt{\eta^4+(\xi-1)^2-2\eta^2(\xi+1)}}\,, \tag{3.127}$$

$$\xi = 3{f_{R\phi}(0,0,\phi^{(0)})}^2\,, \quad \eta = \frac{m_\phi}{m_R}\,, \tag{3.128}$$

$$m_\pm^2 = m_R^2\,\omega_\pm\,, \tag{3.129}$$

$$\omega_\pm = \frac{1-\xi+\eta^2\pm\sqrt{(1-\xi+\eta^2)^2-4\eta^2}}{2}\,, \tag{3.130}$$

$$ {m_R}^2 \doteq -\frac{f_R(0,0,\phi^{(0)})}{3f_{RR}(0,0,\phi^{(0)})+2f_Y(0,0,\phi^{(0)})}\,, \tag{3.131}$$

$$ {m_Y}^2 \doteq \frac{f_R(0,0,\phi^{(0)})}{f_Y(0,0,\phi^{(0)})}\,, \quad {m_\phi}^2 \doteq -\frac{f_{\phi\phi}(0,0,\phi^{(0)})}{2\omega(\phi^{(0)})}\,. \tag{3.132}$$

The ETG studied here and present in literature, are reported in Table 3.4 (see [113] for further details).

Table 3.4: We report different cases of Extended Theories of Gravity including a scalar field and higher-order curvature terms. The free parameters are given as effective masses with their asymptotic behavior. Here, we assume $f_R(0,\,0,\,\phi^{(0)}) = 1$, $\omega(\phi^{(0)}) = 1/2$.

Case	ETG	Parameters				
		m_R^2	m_Y^2	m_ϕ^2	m_+^2	m_-^2
A	$f(R)$	$-\frac{f_R(0)}{3f_{RR}(0)}$	∞	0	m_R^2	∞
B	$f(R, R_{\alpha\beta}R^{\alpha\beta})$	$-\frac{f(0)}{3f_{RR}(0)+2f_Y(0)}$	$\frac{f_R(0)}{f_Y(0)}$	0	m_R^2	∞
C	$f(R,\phi)+\omega(\phi)\phi_{;\alpha}\phi^{;\alpha}$	$-\frac{f_R(0)}{3f_{RR}(0)}$	∞	$-\frac{f_{\phi\phi}(0)}{2\omega(\phi^{(0)})}$	$m_R^2 w_+$	$m_R^2 w_-$
D	$f(R, R_{\alpha\beta}R^{\alpha\beta},\phi)+\omega(\phi)\phi_{;\alpha}\phi^{;\alpha}$	$-\frac{f(0)}{3f_{RR}(0)+2f_Y(0)}$	$\frac{f_R(0)}{f_Y(0)}$	$-\frac{f_{\phi\phi}(0)}{2\omega(\phi^{(0)})}$	$m_R^2 w_+$	$m_R^2 w_-$

3.5.5 Field for a Ball-like Source

In General Relativity, the exterior solution for a material point distribution coincide with the exterior solution for a generic spherically symmetric matter distribution. But for a fourth-order theory - as in this case - this is no longer valid and a sphere cannot be reduced to a point. Therefore, equivalence no longer holds and the type of distribution in the space is relevant. In other words, the Gauss theorem is satisfied only in General Relativity and Scalar-Tensor Theories, while generally it is not satisfied by the corrective terms entailed by a fourth-order gravity. If one considers a spherical mass with arbitrary density $\rho(\mathbf{x})$ and radius $\mathcal{R}$, the solution relative to the potentials Φ and Ψ, present a geometrical corrective factor on the Yukawa-like term depending on the form of the source. Therefore, in this section, we also take into consideration the fields generated by a ball-like source. Here for each term $\propto \frac{e^{-mr}}{r}$ there is a geometric factor multiplying the Yukawa term, expressed by

$$F(m\mathcal{R}) = 3\frac{m\mathcal{R}\cosh m\mathcal{R} - \sinh m\mathcal{R}}{m^3\mathcal{R}^3}. \tag{3.133}$$

If we set $x = m\mathcal{R}$ in $F(m\mathcal{R})$, when $x \ll 1$, we have $\lim_{x\to 0} F(x) = 1$ and the point-like mass solution is recovered [62; 109; 127]. In particular, regarding the solutions, one gets

$$\begin{aligned}
\Phi(\mathbf{x}) &= -\frac{GM}{|\mathbf{x}|}\Bigg[1 + g(\xi,\eta)\,F(m_+\mathcal{R})\,e^{-m_+|\mathbf{x}|} + \\
&\qquad +[\frac{1}{3} - g(\xi,\eta)]\,F(m_-\mathcal{R})\,e^{-m_-|\mathbf{x}|} - \frac{4\,F(m_Y\mathcal{R})}{3}\,e^{-m_Y|\mathbf{x}|}\Bigg], \\
\Psi(\mathbf{x}) &= -\frac{GM}{|\mathbf{x}|}\Bigg[1 - g(\xi,\eta)\,F(m_+\mathcal{R})\,e^{-m_+|\mathbf{x}|} - \\
&\qquad -[\frac{1}{3} - g(\xi,\eta)]\,F(m_-\mathcal{R})\,e^{-m_-|\mathbf{x}|} - \frac{2\,F(m_Y\mathcal{R})}{3}\,e^{-m_Y|\mathbf{x}|}\Bigg].
\end{aligned} \tag{3.134}$$

Therefore, in terms of the potentials generated by the ball source with radius $\mathcal{R}$, the components of the metric $g_{\mu\nu}$ read

$$\begin{aligned}
g_{00} &= 1 + \frac{2}{c^2}\Phi_{ball}(\mathbf{x}) = 1 - \frac{2GM}{c^2\,|\mathbf{x}|}\Bigg[1 + g(\xi,\eta)\,F(m_+R\mathcal{R})\,e^{-m_+|\mathbf{x}|} \\
&\qquad +[1/3 - g(\xi,\eta)]\,F(m_-\mathcal{R})\,e^{-m_-|\mathbf{x}|} - \frac{4\,F(m_Y\mathcal{R})}{3}\,e^{-m_Y|\mathbf{x}|}\Bigg], \\
g_{ij} &= -(1 - \frac{2}{c^2}\Psi_{ball}(\mathbf{x}))\,\delta_{ij} = -\delta_{ij} - \frac{2GM}{c^2\,|\mathbf{x}|}\Bigg[1 - g(\xi,\eta)\,F(m_+\mathcal{R})\,e^{-m_+|\mathbf{x}|} \\
&\qquad -[1/3 - g(\xi,\eta)]\,F(m_-\mathcal{R})\,e^{-m_-|\mathbf{x}|} - \frac{2\,F(m_Y\mathcal{R})}{3}\,e^{-m_Y|\mathbf{x}|}\Bigg]\delta_{ij},
\end{aligned}$$

We quickly notice the fundamental fact that the modifications induced by the Scalar-Tensor-Fourth-Order-Gravity action to the Newtonian potentials Φ and Ψ as appear in Eq. (3.134) are similar to those induced by a fifth-force through the potential of the type

$$V(r) = -\frac{GM}{r}\left(1 + \alpha e^{-r/\lambda}\right), \tag{3.135}$$

where α is a dimensionless strength parameter and λ a length scale associated to a Yukawa-like potential. As a result, later on we will apply the above analysis to the case of bodies moving in the gravitational field.

3.5.6 The Case of NonCommutative Spectral Geometry

Concerning the NonCommutative Spectral Geometry, which turns out to be a special case of STFOG (see section (2.3)), a first point that deserves to be remarked is that neglecting the non-minimal coupling between the Higgs field and the Ricci curvature, NCSG does not lead to corrections for homogeneous and isotropic cosmologies. This physical approximation enables us to analytically obtain a lower bound on f_0. By referring to the resolution presented in [114; 113] achieved in harmonic coordinates and proceeding to direct calculations in Standard Post-Newtonian gauge[15], one finds that in terms of the field potentials Φ, Ψ, A_i, the components of $g_{\mu\nu}$ are

$$\begin{aligned}
g_{00} &= 1+\frac{2}{c^2}\Phi(\mathbf{x}) = 1-\frac{2GM}{c^2\,|\mathbf{x}|}\left(1-\frac{4}{3}e^{-\beta|\mathbf{x}|}\right)\,,\\
g_{0i} &= \frac{Z_i}{c^3} = \frac{1}{c^3}(A_i(\mathbf{x})-X_{,0i}) = \frac{4GM}{c^3\,|\mathbf{x}|}\left(1-e^{-\beta|\mathbf{x}|}\right)v'_i-\frac{1}{c^3}\,X_{,0i}\,,\\
g_{ij} &= -(1-\frac{2}{c^2}\Psi(\mathbf{x}))\,\delta_{ij} = \delta_{ij}-\frac{2GM}{c^2\,|\mathbf{x}|}\left(1+\frac{5}{9}e^{-\beta|\mathbf{x}|}\right)\delta_{ij}\,,
\end{aligned} \tag{3.136}$$

where the superpotential X stems from the equation $\triangle X = \Phi$, yielding

$$X(\mathbf{x}) = -\frac{GM}{2}|\mathbf{x}|+\frac{GM}{\beta^2|\mathbf{x}|}\left(1-e^{-\beta|\mathbf{x}|}-\beta|\mathbf{x}|\right). \tag{3.137}$$

Also here we note that the modifications induced by the NCSG action to the Newtonian potentials Φ and Ψ appear to be similar to those induced by a fifth-force through a Yukawa-like potential. Especially, in the Newtonian limit, we have for the strength parameter $\alpha = (4/3)\,GM$ in Eq. (3.135).

[15]Through a suitable transformation, it is possible to pass from Standard Post-Newtonian gauge to harmonic gauge.

3.6 Spherically Symmetric Field Equations

In this section we want to show a possible alternative derivation of the Fourth Order Gravity field equations by simply assuming the weak field limit in spherically symmetric coordinate system, with a point-like source. The spherical symmetry is often important both for theoretical and practical issues concerning the modeling of astrophysical gravitating systems. In particular, as we will do later to deal with the periastron precession and the galaxy rotation curves, it allows to analyse planetary or stellar motions starting from a simpler but effective reference model just by taking advantage of the radial symmetry. To this end, we consider the case of $f(R)$-theory as working example[16] (see section (2.2)). In Refs. [48; 62; 87; 65; 110; 111] and others in literature, several derivations and resolutions with spherical symmetry are presented. In Ref. [121] are already derived the analog field equations in the general case of a star-model including pressure effects. For the $f(R)$-theory, in particular Starobinsky's quadratic model $f(R) = R + aR^2$, the field equations reads

$$(1+2aR)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(R + aR^2) + 2a(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)R = \mathcal{X}T_{\mu\nu}. \tag{3.138}$$

The trace equation is

$$\Box R - \frac{1}{6a}R = \chi T \tag{3.139}$$

where $T = T^\mu_\mu$ is the trace of the stress-energy tensor. By introducing the new scalar field $\varphi = aR$ associated to the curvature scalar, we can recast the entire system as

$$(1+2\varphi)G_{\mu\nu} + 3m^2\varphi^2 g_{\mu\nu} + 2(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)\varphi = \frac{8\pi G}{c^4}T_{\mu\nu}\,, \tag{3.140}$$

$$\Box\varphi + m^2\varphi = \frac{4\pi G}{3c^4}T\,. \tag{3.141}$$

Eq.(3.141) is an effective Klein-Gordon equation for the scalar field φ and $m^2 = -1/6a$, represents the *mass* of the scalaron field. Hence, a must be negative. The field equations at the zero-th order provides the further condition $f_0 = 0$. By solving the system of Eqs.(3.140) and (3.141), we find the solutions providing the evolution of the scalar field and the gravitational potentials related to the metric. To this aim, we consider the static time-independent spherically symmetric metric

$$ds^2 = U(r)c^2dt^2 - V(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \tag{3.142}$$

[16] As discussed in sec. [2.2.1] we remind that, for an analytical lagrangian density $\mathcal{L} = f(R)$, the field equation of the Starobinsky model is equivalent to the most general thanks to a Taylor series expansion up to the second order

$$f(R) = \sum_{k=1}^{n} \frac{f^{(k)}(R_0)}{k!}(R - R_0)^k \simeq f_0 + f'_0 R + f''_0 R^2 + \dots\,,$$

with $R_0 = 0$ [124]. By setting $a = f''_0/f'_0$ and $m = \sqrt{-f'_0/6f''_0}$, being $f'_0 = 1$, the field equation are recovered.

and for the stress-energy tensor we assume a point-like source with mass M and mass density ρ, that is

$$T_{tt} = \rho c^2 u_t u_t, \qquad T_{ij} = 0, \qquad T_{tj} = 0 \qquad T = \rho c^2 \qquad i, j = r, \theta, \phi. \tag{3.143}$$

Here $u_t u_t = U(r)$ follows from the condition $g^{tt} u_t u_t = 1$ which is satisfied by u_μ ($u_i = 0$), thereby $T_{tt} = \rho c^2 U(r)$. To solve Eqs. (3.140) and (3.141) in the weak field approximation $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with $h_{\mu\nu} \ll 1$. Specifically, comparing with Eq. (4.17) we set

$$U(r) = 1 + 2\Phi(r), \qquad V(r) = 1 - 2\Psi(r), \qquad \Phi(r) \ll 1, \Psi(r) \ll 1 \tag{3.144}$$

and require the asymptotic flatness

$$\lim_{r\to\infty} U(r) = \lim_{r\to\infty} V(r) = 1.$$

The Christoffel symbols for a spherically symmetric metric are

$$\Gamma^t_{tr} = \Gamma^t_{rt} = \frac{U'}{2U},$$

$$\Gamma^r_{tt} = \frac{U'}{2V}, \quad \Gamma^r_{rr} = \frac{V'}{2V}, \quad \Gamma^r_{\theta\theta} = \frac{r}{V}, \quad \Gamma^r_{\phi\phi} = \frac{r}{V}\sin^2\theta,$$

$$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}, \quad \Gamma^\theta_{\phi\phi} = -\cos\theta\sin\theta,$$

$$\Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{1}{r}, \quad \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\theta\phi} = \cot\theta. \tag{3.145}$$

The prime denotes the derivative with respect to r. By using the Christoffel symbols, we calculate the field equations. We first derive the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \tag{3.146}$$

and the Ricci tensor

$$R_{\mu\nu} = \Gamma^\rho_{\mu\rho,\nu} - \Gamma^\rho_{\mu\nu,\sigma} + \Gamma^\sigma_{\mu\rho}\Gamma^\rho_{\sigma\nu} - \Gamma^\sigma_{\mu\nu}\Gamma^\rho_{\nu\rho}, \tag{3.147}$$

One finds $R_{\mu\nu} = 0$ for $\mu \neq \nu$, and for $\mu = \nu$ one has

$$\begin{aligned}
R_{tt} &= \frac{U''}{2V} - \frac{U'V'}{4V^2} - \frac{(U')^2}{4UV} + \frac{1}{r}\frac{U'}{V}, \\
R_{rr} &= -\frac{U''}{2U} + \frac{(U')^2}{4U^2} + \frac{U'V'}{4UV} + \frac{V'}{rV}, \\
R_{\theta\theta} &= -\frac{rU'}{2UV} - \frac{1}{V} + \frac{rV'}{2V^2} + 1, \\
R_{\phi\phi} &= R_{\theta\theta}\sin^2\theta.
\end{aligned} \tag{3.148}$$

Then the scalar curvature is

$$R = R^{\mu}_{\mu} = g^{\mu\nu}R_{\mu\nu} = g^{tt}R_{tt} + g^{rr}R_{rr} + g^{\theta\theta}R_{\theta\theta} + g^{\phi\phi}R_{\phi\phi}, \tag{3.149}$$

which can be expressed as

$$R = \frac{U''}{UV} - \frac{U'V'}{2UV^2} - \frac{(U')^2}{2U^2V} + \frac{2U'}{rUV} - \frac{2V'}{rV^2} + \frac{2}{r^2}\left(\frac{1}{V} - 1\right). \tag{3.150}$$

The components of the Einstein tensor are

$$\begin{aligned}
G_{tt} = \quad & R_{tt} - \frac{1}{2}g_{tt}R = \\
& \frac{U''}{2V} - \frac{(U')^2}{4UV} - \frac{U'V'}{4V^2} + \frac{U'}{rV} - \frac{U''}{2V} + \frac{(U')^2}{4UV} + \frac{U'V'}{4V^2} - \frac{U'}{rV} + \frac{UV'}{rV^2} - \frac{U}{r^2}\left(\frac{1}{V} - 1\right) = \\
& U\left[\frac{V'}{rV^2} + \frac{1}{r^2}\left(1 - \frac{1}{V}\right)\right],
\end{aligned} \tag{3.151}$$

and

$$\begin{aligned}
G_{rr} = \quad & R_{rr} - \frac{1}{2}g_{rr}R = \\
& -\frac{U''}{2U} + \frac{(U')^2}{4U^2} + \frac{U'V'}{4UV} + \frac{V'}{rV} + \frac{U''}{2U} - \frac{(U')^2}{4U^2} - \frac{U'V'}{4UV} + \frac{U'}{rU} - \frac{V'}{rV} + \frac{V}{r^2}\left(\frac{1}{V} - 1\right) = \\
& V\left[\frac{U'}{rUV} + \frac{1}{r^2}\left(\frac{1}{V} - 1\right)\right].
\end{aligned} \tag{3.152}$$

Notice that for the metric tensor spherically symmetric, G_{tt} and G_{rr} are sufficient to obtain the field equations.

We now derive the D'Alembert operator. By taking in account that $\sqrt{-g} = \sqrt{UV}r^2\sin\theta$, we have

$$
\begin{aligned}
\Box\varphi &= \frac{1}{\sqrt{-g}}\partial_\alpha(\sqrt{-g}g^{\alpha\beta}\partial_\beta\varphi)\\
&= \frac{1}{\sqrt{UV}(r^2\sin\theta)}\partial_t(\sqrt{UV}r^2\sin\theta g^{tt}\partial_t\varphi) + \frac{1}{\sqrt{UV}r^2\sin\theta}\partial_r(\sqrt{UV}r^2\sin\theta g^{rr}\partial_r\varphi)\\
&= \frac{1}{\sqrt{UV}(r^2\sin\theta)}\partial_t(\sqrt{UV}r^2\sin\theta(U^{-1})\partial_t\varphi) + \frac{1}{\sqrt{UV}r^2\sin\theta}\partial_r(\sqrt{UV}r^2\sin\theta(-V^{-1})\partial_r\varphi)\\
&= \frac{r^2\sin\theta}{\sqrt{UV}(r^2\sin\theta)}\left(-\frac{\varphi'\left(VU'+UV'\right)}{2V\sqrt{UV}} + \frac{\sqrt{UV}V'\varphi'}{V^2} - \frac{\sqrt{UV}\varphi''}{V} - \frac{2\sqrt{UV}\varphi'}{rV}\right)\\
&= -\frac{U'\varphi'}{2UV} + \frac{V'\varphi'}{2V^2} - \frac{\varphi''}{V} - \frac{2\varphi'}{rV}\\
&= -\frac{1}{V}\left(\varphi'' + \frac{2}{r}\varphi' + \frac{U'}{2U}\varphi' - \frac{V'}{2V}\varphi'\right) =\\
&= -\frac{1}{V}\left[\varphi'' + \left(\frac{2}{r} + \frac{U'}{2U} - \frac{V'}{2V}\right)\varphi'\right]. \qquad (3.153)
\end{aligned}
$$

Moreover, we evaluate the term

$$\nabla_\mu\nabla_\nu\varphi = \nabla_\mu\partial_\nu\varphi = \partial_\mu\partial_\nu\varphi - \Gamma^\rho_{\mu\nu}\partial_\rho\varphi \qquad (3.154)$$

for the tt-components and the rr-components of the field equation. One gets

$$\nabla_t\partial_t\varphi = (\partial_t\partial_t\varphi - \Gamma^\rho_{tt}\partial_\rho\varphi) = \left(\partial_t\partial_t\varphi - \Gamma^t_{tt}\partial_t\varphi - \Gamma^r_{tt}\partial_r\varphi\right) = -\frac{U'}{2V}\varphi' \qquad (3.155)$$

and

$$\nabla_r\partial_r\varphi = (\partial_r\partial_r\varphi - \Gamma^\rho_{rr}\partial_\rho\varphi) = \left(\partial_r\partial_r\varphi - \Gamma^t_{rr}\partial_t\varphi - \Gamma^r_{rr}\partial_r\varphi\right) = \left(\varphi'' - \frac{V'}{2V}\varphi'\right) \qquad (3.156)$$

Putting all the pieces together in the three equations, for the tt-components of the field equation, we have

$$
\begin{aligned}
&(1+2\varphi)\left[U\left(\frac{V'}{rV^2} + \frac{1}{r^2}\left(1 - \frac{1}{V}\right)\right)\right] - 3m^2\varphi^2 U +\\
&+2\left[U\left(-\frac{\varphi''}{V} - \frac{2}{rV}\varphi' - \frac{U'}{2UV}\varphi' + \frac{V'}{2V^2}\varphi'\right) + \frac{U'}{2UV}\varphi'\right] = \frac{8\pi G}{c^2}\rho U. \qquad (3.157)
\end{aligned}
$$

For the rr-components one has

$$(1+2\varphi)\left[V\left(\frac{U'}{rUV}-\frac{1}{r^2}\left(1-\frac{1}{V}\right)\right)\right]-3m^2\varphi^2V+$$

$$+2\left[\left(\varphi''+\frac{2}{r}\varphi'+\frac{U'}{2U}\varphi'-\frac{V'}{2V}\varphi'\right)-\left(\varphi''-\frac{V'}{2V}\varphi'\right)\right]=0\,, \tag{3.158}$$

and the effective Klein-Gordon equation is

$$\frac{1}{V}\left[\varphi''+\left(\frac{2}{r}+\frac{U'}{2U}-\frac{V'}{2V}\right)\varphi'\right]-m^2\varphi=-\frac{4\pi G}{3c^2}\rho. \tag{3.159}$$

Finally, the field equations for a spherically symmetric metric can be simplified as

$$(1+2\varphi+r\varphi')\frac{V'}{rV^2}-3m^2\varphi^2-\frac{2}{V}\left(\varphi''+\frac{2}{r}\varphi'\right)=\frac{1+2\varphi}{r^2}\left(\frac{1}{V}-1\right)+\frac{8\pi G}{c^2}\rho, \tag{3.160}$$

$$(1+2\varphi+r\varphi')\frac{U'}{rUV}+3m^2\varphi^2+\frac{4}{rV}\varphi'=\frac{1+2\varphi}{r^2}\left(1-\frac{1}{V}\right), \tag{3.161}$$

$$\frac{1}{V}\left[\varphi''+\left(\frac{2}{r}+\frac{U'}{2U}-\frac{V'}{2V}\right)\varphi'\right]-m^2\varphi=-\frac{4\pi G}{3c^2}\rho. \tag{3.162}$$

It is convenient to recast Eq. (3.160) by removing the second derivative of the scalaron field φ [121]. We multiply Eq. (3.161) by r and Eq. (3.162) by a 2 factor, then in Eq. (3.161) multiplied by r, we isolate the term U'/UV and we get

$$\frac{U'}{UV}=\frac{1+2\varphi}{r^2}\left(1-\frac{1}{V}\right)\frac{r}{(1+2\varphi+r\varphi')}+\frac{3m^2\varphi^2r}{(1+2\varphi'+r\varphi')}-\frac{4r\varphi'}{V(1+2\varphi'+r\varphi')}\,. \tag{3.163}$$

Substituting it in Eq. (3.162), we have

$$\frac{2}{V}\varphi''+\frac{4}{rV}\varphi'+\frac{1+2\varphi}{r}\left(1-\frac{1}{V}\right)\frac{r\varphi'}{(1+2\varphi+r\varphi')}+$$

$$+\frac{(3m^2\varphi^2r)\varphi'}{(1+2\varphi'+r\varphi')}-\frac{4(\varphi')^2}{V(1+2\varphi'+r\varphi')}-\frac{V'}{V^2}\varphi'-2m^2\varphi=-\frac{8\pi G}{3c^2}\rho. \tag{3.164}$$

Then, by summing this last equation to Eq. (3.160), it follows that

$$(1+\varphi)\frac{V'}{rV^2}=\frac{1+2\varphi}{r^2}\left(\frac{1}{V}-1\right)-3m^2\varphi^2+\frac{1+2\varphi}{r^2}\left(\frac{1}{V}-1\right)\frac{r\varphi'}{1+2\varphi+r\varphi'}+$$

$$-\frac{(3m^2\varphi^2r)\varphi'}{1+2\varphi+r\varphi'}+\frac{4(\varphi')^2}{1+2\varphi+r\varphi'}+2m^2\varphi+\frac{16\pi G}{3c^4}\rho \tag{3.165}$$

and, by factorizing the term $\frac{1+2\varphi}{r^2}\left(\frac{1}{V}-1\right)-3m^2\varphi^2$, one has

$$(1+\varphi)\frac{V'}{rV^2}=\left(1+\frac{r\varphi'}{1+2\varphi+r\varphi'}\right)\left[\frac{1+2\varphi}{r^2}\left(\frac{1}{V}-1\right)-3m^2\varphi^2\right]+\frac{4(\varphi')^2}{1+2\varphi+r\varphi'}+2m^2\varphi+\frac{16\pi G}{3c^4}\rho. \tag{3.166}$$

This equation is used in our computations.

3.6.1 Solution of the Stationary Inhomogeneous Klein-Gordon Equation for the Scalar Field φ

In our computations, we consider

$$U(r) = 1 + 2\Phi(r), \qquad V(r) = 1 - 2\Psi(r),$$

we apply the weak field limit conditions

$$\Phi(r) \ll 1, \qquad \Psi(r) \ll 1, \tag{3.167}$$

and consider the conditions

$$r|\Phi'(r)| \ll 1, \qquad r|\Psi'(r)| \ll 1. \tag{3.168}$$

We derive the time-independent inhomogeneous Klein-Gordon equation (3.162). By applying (3.167,3.168) and expanding in Taylor series up to the first order $(1 - 2\Psi(r))^{-1} \simeq (1 + 2\Psi(r))$, we get

$$\varphi'' + \frac{2}{r}\varphi' - m^2\varphi = -\frac{4\pi G}{3c^2}\rho \tag{3.169}$$

which can be written as

$$\nabla^2\varphi - m^2\varphi = -\frac{4\pi G}{3c^2}M\delta(\boldsymbol{x}). \tag{3.170}$$

Here, $r = |\boldsymbol{x}|$. By inserting

$$\varphi(\boldsymbol{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \hat{\varphi}(\boldsymbol{k}) \tag{3.171}$$

in Eq. (3.170), we get the relation, $\hat{\varphi}(\boldsymbol{k}) = 4\pi GM/[3c^2(\boldsymbol{k}^2 + m^2)]$ which replaced in Eq. (3.171) gives

$$\varphi(\boldsymbol{x}) = \frac{4\pi GM}{c^2} \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\boldsymbol{k}\cdot\boldsymbol{x}}}{\boldsymbol{k}^2 + m^2}. \tag{3.172}$$

In spherical coordinates, one obtains

$$\begin{aligned} \varphi(r) = {} & \frac{4\pi GM}{3c^2(2\pi)^3} \int_0^\infty \frac{k^2}{k^2 + m^2} dk \int_0^\pi \sin\theta d\theta \int_{-\pi}^{\pi} e^{ikr\cos\theta} d\phi = \\ & \frac{GM}{3c^2\pi r} \int_0^\infty \frac{k\sin kr}{(k^2 + m^2)} dk = \frac{GM}{3c^2\pi r} \int_{-\infty}^\infty \frac{k\sin kr}{(k^2 + m^2)} dk. \end{aligned} \tag{3.173}$$

Notice that the last integral in Eq. (3.173) is the imaginary part of

$$\int_{-\infty}^\infty \frac{ke^{ikr}}{(k^2 + m^2)} dk. \tag{3.174}$$

We use the residue theorem to solve it. We close the integration path in the upper half of the complex plane, whereby there is a pole at $k = +im$, thus

$$\int_{-\infty}^\infty \frac{ke^{ikr}}{(k^2 + m^2)} dk = 2\pi i \,\mathrm{Res} \frac{ke^{ikr}}{(k^2 + m^2)}\bigg|_{k=+im} = \pi i e^{ikr}\bigg|_{k=+im} = \pi i e^{-mr}. \tag{3.175}$$

Since we have to take the imaginary part, the calculation of the final integral in (3.173) leads to the expression

$$\varphi(r) = \frac{GM}{3c^2}\frac{e^{-mr}}{r} \tag{3.176}$$

which is the required solution of the time-independent Klein-Gordon equation.

3.6.2 Solutions for the Fields Φ and Ψ

Let us consider Eqs. (3.166)-(3.161) and (3.144). Since $\varphi \ll 1$, also the conditions $r|\varphi'| \ll 1$, $r^2m^2\varphi^2 \ll r|\varphi'| \ll 1$ are valid and taken into account. Moreover the term containing φ'^2 is negligible. Then, after straightforward calculations, considering the approximation: $(1-2\Psi(r))^{-1} \simeq (1+2\Psi(r))$, one obtains the following coupled differential equations

$$r\Psi' + \Psi = -m^2r^2\varphi - \frac{8\pi G}{3c^2}\rho r^2, \tag{3.177}$$

$$\Phi' = -2\varphi' - \frac{\Psi}{r}. \tag{3.178}$$

The first equation is easily solved by noting that, $r\Psi' + \Psi = (r\Psi)'$, hence, substituting (3.176), an integration gives

$$\Psi(r) = -\frac{GM}{c^2r} + \frac{GM}{3c^2r}e^{-mr}(1+mr), \tag{3.179}$$

where M is the mass of the point-like source. Using (3.179) and (3.176) in Eq. (3.178), a simple integration provides

$$\Phi(r) = -\frac{GM}{c^2r} - \frac{GM}{3c^2r}e^{-mr}. \tag{3.180}$$

from which follows the final metric (4.17). An equivalent result can be obtained by assuming

$$U(r) = 1 + 2\Phi(r), \qquad V(r) = \frac{1}{1+2\Psi(r)}, \tag{3.181}$$

as done in Ref. [121]. In that paper are derived the same linearized field equations (3.177) and (3.178), with the difference that metric element g_{rr} is $g_{rr} = 1/(1+2\Psi(r))$. In this way, the Taylor expansion of $(1-2\Psi(r))^{-1}$ is not needed during the computation process. Hence, the solution can be written as

$$ds^2 = \left[1 - \frac{r_g}{r}\left(1 + \frac{e^{-mr}}{3}\right)\right]c^2dt^2 - \left[1 + \frac{r_g}{r}\left(1 - \frac{e^{-mr}}{3}(1+mr)\right)\right]dr^2 - r^2(d\theta^2 + \sin\theta^2 d\phi^2). \tag{3.182}$$

with $r_g = 2GM/c^2$. This is the static metric for the $f(R)$-theory. The gravitational potentials contains the Yukawa-like term, which is added to the usual potential of General Relativity. The metric (3.182) is coherent with the one presented in [65]. Moreover, the scalar field deduced by the time-independent inhomogeneous Klein-Gordon equation, necessary to determine the solutions providing the potential $\Phi(r)$ and $\psi(r)$, reads

$$\varphi(r) = \frac{GM}{3c^2}\frac{e^{-mr}}{r}. \tag{3.183}$$

It represents the extra-degree of freedom originated by the point mass M. In our case, the solution of the field equations represented by the metric (3.182) and the scalar degree of freedom (3.183) are simply those of a point-like source.

3.6.3 From Isotropic to Spherically Symmetric Space-times

It is important to emphasize there are a lot of physical scenarios and corresponding models for which spherically symmetry is convenient or simply required, at least as a starting point for subsequent mathematical developments. Furthermore the gravitational potentials as those discussed in the present work, seem to have only a dependence on the mutual spatial distances between the positions of the bodies belonging to a given system or distribution of matter. In particular, radially symmetric potentials lead to central force fields. In general, spherical symmetry in models is an essential point, often one of the most direct ways to deal with problems. Hence in nature the description of a system based on the radial, polar, and spherical symmetry is particularly effective for a great number of physical phenomena; in this case, it leads us to research field solutions and integrals of the equations of motion by adopting such a coordinate system.

In the previous subsection, we performed a possible derivation of the space-time metric for a quadratic $f(R)$-theory by fixing the set of *spherical coordinates* $x^\alpha = (t, r, \theta, \phi)$ from the very beginning, with space-time of the form

$$ds^2 = g_{tt}(t,r)c^2dt^2 - g_{rr}(t,r)dr^2 - r^2 d\Omega, \tag{3.184}$$

with $d\Omega = d\theta^2 + \sin^2\theta d\phi^2$ solid angle. On the other hand, they are connectable by means of a suitable transformation, thus allowing us to switch from one coordinate set to another. Whether the field equations have been solved in *isotropic coordinates* $x^\alpha = (x_0, x_1, x_2, x_3)$, with space-time generally written as

$$ds^2 = g_{00}(x_0, \mathbf{x})c^2dt^2 - g_{ij}(x_0, \mathbf{x})\delta_{ij}dx^i dx^j, \tag{3.185}$$

it is possible to pass to spherically symmetric (and vice versa). In particular the objective can be readily fulfilled by introducing a radial coordinate of the type

$$r^2 = [1 - 2\Psi(|\mathbf{x}|)]|\mathbf{x}|^2, \tag{3.186}$$

with $|\mathbf{x}| = x_i x^i$. Thereby we effect the transformation on the relativistic invariant in isotropic coordinates and working out the computations at first order with respect to the quantity $r_g/|\mathbf{x}|$, we are able to find the desired space-times in spherical coordinates. Examples of spherically symmetric space-times are reported here and read:

- *$f(R)$-gravity*

$$ds^2 = \left[1 - \frac{r_g}{r}\left(1 + \frac{1}{3}\,e^{-m_R r}\right)\right]c^2 dt^2 - \left[1 + \frac{r_g}{r}\left(1 - \frac{1 + m_R r}{3}\,e^{-m_R r}\right)\right]dr^2 - r^2 d\Omega, \tag{3.187}$$

  Coherently with the solution (3.182), obtained by setting the spherical symmetry at the beginning of the calculation process in weak field limit of the $f(R)$-gravity field equation.

- *$f(R, R_{\alpha\beta}R^{\alpha\beta})$-gravity*

$$\begin{aligned} ds^2 \quad &= \left[1 - \frac{r_g}{r}\left(1 + \frac{1}{3}\,e^{-m_R\, r} - \frac{4}{3}\,e^{-m_Y\, r}\right)\right]c^2 dt^2 - \\ &-\left[1 + \frac{r_g}{r}\left(1 - \frac{1 + m_R\, r}{3}\,e^{-m_R\, r} - \frac{2(1 + m_Y\, r)}{3}\,e^{-m_Y r}\right)\right]dr^2 - r^2 d\Omega, \end{aligned} \tag{3.188}$$

- *NonCommutative Spectral Gravity*

$$ds^2 = \left[1 - \frac{r_g}{r}\left(1 - \frac{4}{3}\,e^{-\beta\, r}\right)\right]c^2 dt^2 - \left[1 + \frac{r_g}{r}\left(1 - \frac{5(1 + \beta\, r)}{9}\,e^{-\beta\, r}\right)\right]dr^2 - r^2 d\Omega, \tag{3.189}$$

where $r_g = 2GM/c^2$ is the Schwarzschild radius. We reported the results for each theory here discussed as we are interested in taking advantage of the centrally symmetric metric for the aims of the next chapters, namely the analysis of the periastron advance in planetary and stellar motions as well as the derivation of the theoretical galaxy rotation curves, which will be fitted with the observed data curves for an unexplored sample of spiral galaxies coming from the THINGS catalogue.

3.7 Relativistic STFOG Equations of Orbital Motion for the N-Body System

After we found the solutions for the linearized (weak field) field equations in Standard Post-Newtonian gauge, in this last section we finally determine the relativistic equations of motion for the N-body system of the STFOG, as a general class of ETG. For the metric (3.115), the non-vanishing Christoffel symbols (3.37) in our gauge condition are now writable as

$$\begin{aligned}
&\overset{3}{\Gamma}{}^{0}_{00} = \partial_0\Phi && \overset{2}{\Gamma}{}^{0}_{0i} = \partial_i\Phi \\
&\overset{2}{\Gamma}{}^{i}_{00} = \partial_i\Phi && \overset{2}{\Gamma}{}^{i}_{jk} = \delta_{jk}\partial_i\Psi - \delta_{ij}\partial_k\Psi - \delta_{ik}\partial_j\Psi \\
&\overset{3}{\Gamma}{}^{i}_{0j} = -\delta_{ij}\partial_0\Psi + \frac{1}{2}\Big(\partial_i A_j - \partial_j A_i\Big) && \overset{4}{\Gamma}{}^{i}_{00} = \partial_i(2\Phi^2) - \partial_0 A_i + \partial_0^2(\partial_i X)
\end{aligned} \tag{3.190}$$

Now it becomes more convenient give an explicit form to the vector potential $Z_i = A_i - X_{,0i}$ by executing the partial derivative with respect to time and spatial coordinate on the second term associated to the superpotential X. For a given a-th body, it descends

$$\begin{aligned}
Z_i \quad &= \frac{1}{2}\frac{GM}{|\mathbf{x}-\mathbf{x}_a|}\left[7v_a^i + (\mathbf{v}_a\,\cdot\,\mathbf{n}_a)\,n_a^i\right] - \frac{4\,GMe^{-m_Y|\mathbf{x}-\mathbf{x}_a|}}{|\mathbf{x}-\mathbf{x}_a|}\,v_a^i + \\
&-g(\xi,\eta)\,GM\Bigg\{\left(\frac{\mathbf{v}_a}{m_+^2|\mathbf{x}-\mathbf{x}_a|^3} - \frac{(\mathbf{v}_a\cdot\mathbf{n}_a)n_a^i}{m_+^2|\mathbf{x}-\mathbf{x}_a|^3}\right)\left[1-(1+m_+|\mathbf{x}-\mathbf{x}_a|)e^{-m_+|\mathbf{x}-\mathbf{x}_a|}\right] + \\
&+\left[\left(\frac{1}{|\mathbf{x}-\mathbf{x}_a|} + +2(1+m_+|\mathbf{x}-\mathbf{x}_a|)\right)e^{-m_+|\mathbf{x}-\mathbf{x}_a|} - \frac{2}{m_+^2|\mathbf{x}-\mathbf{x}_a|^3}\right]|\mathbf{n}_a|^2 v_a^i\Bigg\} + \\
&-\Big[(1/3 - g(\xi,\eta))\,GM\Big]\Bigg\{\left(\frac{\mathbf{v}_a}{m_-^2|\mathbf{x}-\mathbf{x}_a|^3} - \frac{(\mathbf{v}_a\cdot\mathbf{n}_a)n_a^i}{m_-^2|\mathbf{x}-\mathbf{x}_a|^3}\right)\left[1-(1+m_-|\mathbf{x}-\mathbf{x}_a|)e^{-m_-|\mathbf{x}-\mathbf{x}_a|}\right] + \\
&+\left[\left(\frac{1}{|\mathbf{x}-\mathbf{x}_a|} + 2(1+m_-|\mathbf{x}-\mathbf{x}_a|)\right)e^{-m_-|\mathbf{x}-\mathbf{x}_a|} - \frac{2}{m_-^2|\mathbf{x}-\mathbf{x}_a|^3}\right]|\mathbf{n}_a|^2 v_a^i\Bigg\} + \\
&+\frac{4GM}{3}\Bigg\{GM\left(\frac{\mathbf{v}_a}{m_Y^2|\mathbf{x}-\mathbf{x}_a|^3} - \frac{(\mathbf{v}_a\cdot\mathbf{n}_a)n_a^i}{m_Y^2|\mathbf{x}-\mathbf{x}_a|^3}\right)\left[1-(1+m_Y|\mathbf{x}-\mathbf{x}_a|)e^{-m_Y|\mathbf{x}-\mathbf{x}_a|}\right] + \\
&+\left[\left(\frac{1}{|\mathbf{x}-\mathbf{x}_a|} + 2(1+m_Y|\mathbf{x}-\mathbf{x}_a|)\right)e^{-m_Y|\mathbf{x}-\mathbf{x}_a|} - \frac{2}{m_Y^2|\mathbf{x}-\mathbf{x}_a|^3}\right]|\mathbf{n}_a|^2 v_a^i\Bigg\}\,,
\end{aligned} \tag{3.191}$$

$$\tag{3.192}$$

whereas $\mathbf{x}_a$ is the position of the particle a, $\mathbf{n}_a = (\mathbf{x}-\mathbf{x}_a)/|\mathbf{x}-\mathbf{x}_a|$ is the unit vector alond the direction $(\mathbf{x}-\mathbf{x}_a)$, $v_a^i = \frac{dx_a^i}{dt}$ is the i-th component of the particle's velocity and

$n_a^i = (x - x_a)_i / |\mathbf{x} - \mathbf{x}_a|$. The first collected term at first raw is the contribution of General Relativity, coinciding with Eq. (3.85), while the other terms are the corrections to GR coming from the Scalar-Tensor-Fourth-Order Gravity. For the NonCommutative Spectral Gravity one gets

$$\begin{aligned} Z_i &= \frac{1}{2}\frac{GM}{|\mathbf{x}-\mathbf{x}_a|}\left[7v_a^i + (\mathbf{v}_a^i \cdot \mathbf{n}_a)\, n_a^i\right] - \frac{4\,GMe^{-\beta|\mathbf{x}-\mathbf{x}_a|}}{|\mathbf{x}-\mathbf{x}_a|}v_a^i + \\ &+\frac{4GM}{3}\left\{GM\left(\frac{\mathbf{v}_a}{\beta^2|\mathbf{x}-\mathbf{x}_a|^3} - \frac{(\mathbf{v}_a\cdot\mathbf{n}_a)n_a^i}{\beta^2|\mathbf{x}-\mathbf{x}_a|^3}\right)\left[1-(1+\beta|\mathbf{x}-\mathbf{x}_a|)e^{-\beta|\mathbf{x}-\mathbf{x}_a|}\right] + \right. \\ &\left. +\left[\left(\frac{1}{|\mathbf{x}-\mathbf{x}_a|} + 2(1+\beta|\mathbf{x}-\mathbf{x}_a|)\right)e^{-\beta|\mathbf{x}-\mathbf{x}_a|} - \frac{2}{\beta^2|\mathbf{x}-\mathbf{x}_a|^3}\right]|\mathbf{n}_a|^2 v_a^i\right\}. \end{aligned} \tag{3.193}$$

Before going on, let us also notice that through the following gauge transformation

$$x_0 = \tilde{x}_0 + X_{,0}\,, \qquad x^i = \tilde{x}^i \tag{3.194}$$

where the tilde is referred to the new coordinates, it is possible to pass to the harmonic coordinates characterizing the de Donder gauge. Indeed, by making this choice, from Eq. (3.42), the metric transforms as

$$g_{00} \to g_{00} + 2X_{,00}\,, \qquad g_{0i} \to g_{0i} + X_{,0i}\,,, \qquad g_{ij} \to g_{ij}\,. \tag{3.195}$$

So in this new gauge, at the required order, the 00-component and the $0i$-component of the metric $g_{\mu\nu}$ results

$$g_{00} = 1 + 2\Phi + 2X_{,00}\,, \qquad g_{0i} = A_i\,. \tag{3.196}$$

As we can see, the cross term g_{0i} associated with the vector potential Z_i has been reduced to the unique vector field A_i, as it occurs in harmonic coordinates. At this point, we point out that with the Christoffel symbols (3.190) in the hands, by virtue of the geodesic equation (3.21) we may directly deduce for a test particle the equations of orbital motion as

$$\begin{aligned} \frac{d^2\mathbf{x}}{dt^2} &= -\nabla(\Phi + 2\Phi^2) - \frac{\partial \mathbf{A}}{\partial t} - \frac{\partial^2}{\partial t^2}\nabla X + \\ &+\mathbf{v}\times(\nabla\times\mathbf{A}) + 3\frac{\partial\Phi}{\partial t}\mathbf{v} + 4\mathbf{v}\,(\mathbf{v}\cdot\nabla)\Phi - \mathbf{v}^2\nabla\Phi \end{aligned} \tag{3.197}$$

where the vector notation has been utilized and the field Φ, $\mathbf{A}$, X are given by Eqs. $(3.116)_{1,2}$, $(3.116)_6$, $(3.116)_{7,8}$. Note that here we make use of assumption (3.32) with respect to the consistency that STFOG must have with GR. Now let us consider the metric we found in the weak field limit for the STFOG and the relativistic Lagrangian of a single a-th particle (3.26) with $a = 1, 2, ..., N$. With respect to the fields Φ, A_i, X, it now becomes

$$L_a = \frac{1}{2}m_a v_a^2 + \frac{1}{8}\frac{m_a v_a^4}{c^2} - m_a c^2\left(\frac{\Phi}{c^2} + \left[A_i - \frac{\partial^2 X}{\partial t \partial x^i}\right]\frac{v_a^i}{c^4} + \frac{3\Phi}{2}\frac{v_a^2}{c^4} - \frac{\Phi^2}{2c^4}\right) \tag{3.198}$$

We can finally obtain its explicit analytical expression. To do this, it must be first noticed that the total Lagrangian of the system is not simply the sum of the N Lagrangians of the single particles with masses m_a. Therefore, to reach the final Lagrangian providing the correct values of the forces that act on every single body for a given motion of the others, we can take the partial derivative of L_a with respect to $\mathbf{x} = \mathbf{x}_a$ and the forces are readily obtained [56; 54]. Thereby we determine the relativistic Lagrangian of the N-body system in Scalar-Tensor-Fourth-Order Gravity, which has the final expression

$$\begin{aligned} L = & \sum_{a=1}^{N}\frac{m_a\mathbf{v}_a^2}{2}\left(1 + 3\sum_{b\neq a}^{N}\frac{Gm_b}{c^2 r_{ab}}\left[1+\zeta(r_{ab})\right]\right) + \sum_{a=1}^{N}\frac{m_a\mathbf{v}_a^4}{8c^2} + \sum_{a=1}^{N}\sum_{b\neq a}^{N}\frac{Gm_a m_b}{2r_{ab}}\left[1+\zeta(r_{ab})\right] + \\ & -\sum_{a=1}^{N}\sum_{b\neq a}^{N}\frac{Gm_a m_b}{4c^2 r_{ab}}\left[7\mathbf{v}_a\cdot\mathbf{v}_b + (\mathbf{v}_a\cdot\mathbf{n}_{ab})(\mathbf{v}_b\cdot\mathbf{n}_{ab})\right] - \sum_{a=1}^{N}\sum_{b\neq a}^{N}\frac{Gm_a m_b}{4c^2}\mathcal{W}(r_{ab}) \\ & -\sum_{a=1}^{N}\sum_{b\neq a}^{N}\sum_{c\neq a}^{N}\frac{G^2 m_a m_b m_c}{2c^2 r_{ab} r_{ac}}\Xi(r_{ab}, r_{ac}), \end{aligned} \tag{3.199}$$

where $\mathbf{x}_a$ indicates the running vector position of the particle a, $\mathbf{v}_a = \dfrac{d\mathbf{x}_a}{dt}$ its velocity, $r_{ab} = |\mathbf{x}_a - \mathbf{x}_b|$ is separation distance between two material points and $\mathbf{n}_{ab} = (\mathbf{x}_a - \mathbf{x}_b)/|\mathbf{x}_a - \mathbf{r}_b|$ the unit vector along the direction $(\mathbf{x}_a - \mathbf{x}_b)$.

Concerning the term $\mathcal{W}(r_{ab})$, one has

$$
\begin{aligned}
\mathcal{W}(r_{ab}) \quad &= -\frac{4e^{-m_Y|\mathbf{x}_a-\mathbf{x}_b|}}{|\mathbf{x}_a-\mathbf{x}_b|}(\mathbf{v}_a\cdot\mathbf{v}_b)+\\
&-g(\xi,\eta)\left\{\left(\frac{(\mathbf{v}_a\cdot\mathbf{v}_b)}{m_+^2|\mathbf{x}_a-\mathbf{x}_b|^3}-\frac{(\mathbf{v}_a\cdot\mathbf{n}_{ab})(\mathbf{v}_b\cdot\mathbf{n}_{ab})}{m_+^2|\mathbf{x}_a-\mathbf{x}_b|^3}\right)\right.\\
&\times\left[1-(1+m_+|\mathbf{x}_a-\mathbf{x}_b|)e^{-m_+|\mathbf{x}_a-\mathbf{x}_b|}\right]+\\
&+\left[\left(\frac{1}{|\mathbf{x}_a-\mathbf{x}_b|}++2(1+m_+|\mathbf{x}_a-\mathbf{x}_b|)\right)e^{-m_+|\mathbf{x}_a-\mathbf{x}_b|}\right.\\
&\left.\left.-\frac{2}{m_+^2|\mathbf{x}_a-\mathbf{x}_b|^3}\right]|(\mathbf{n}_a\cdot\mathbf{n}_b)|^2(\mathbf{v}_a\cdot\mathbf{v}_b)\right\}+\\
&-\Big[(1/3-g(\xi,\eta))\Big]\left\{\left(\frac{(\mathbf{v}_a\cdot\mathbf{v}_b)}{m_-^2|\mathbf{x}_a-\mathbf{x}_b|^3}-\frac{(\mathbf{v}_a\cdot\mathbf{n}_{ab})(\mathbf{v}_b\cdot\mathbf{n}_{ab})}{m_-^2|\mathbf{x}-\mathbf{x}_a|^3}\right)\right.\\
&\times\left[1-(1+m_-|\mathbf{x}_a-\mathbf{x}_b|)e^{-m_-|\mathbf{x}_a-\mathbf{x}_b|}\right]+\\
&\left.+\left[\left(\frac{1}{|\mathbf{x}_a-\mathbf{x}_b|}+2(1+m_-|\mathbf{x}_a-\mathbf{x}_b|)\right)e^{-m_-|\mathbf{x}_a-\mathbf{x}_b|}-\frac{2}{m_-^2|\mathbf{x}_a-\mathbf{x}_b|^3}\right]|(\mathbf{n}_a\cdot\mathbf{n}_b)|^2(\mathbf{v}_a\cdot\mathbf{v}_b)\right\}+\\
&+\frac{4GM}{3}\left\{\left(\frac{(\mathbf{v}_a\cdot\mathbf{v}_b)}{m_Y^2|\mathbf{x}_a-\mathbf{x}_b|^3}-\frac{(\mathbf{v}_a\cdot\mathbf{n}_{ab})(\mathbf{v}_b\cdot\mathbf{n}_{ab})}{m_Y^2|\mathbf{x}_a-\mathbf{x}_b|^3}\right)\right.\\
&\times\left[1-(1+m_Y|\mathbf{x}_a-\mathbf{x}_b|)e^{-m_Y|\mathbf{x}_a-\mathbf{x}_b|}\right]+\\
&+\left[\left(\frac{1}{|\mathbf{x}_a-\mathbf{x}_b|}+2(1+m_Y|\mathbf{x}_a-\mathbf{x}_b|)\right)e^{-m_Y|\mathbf{x}_a-\mathbf{x}_b|}\right.\\
&\left.\left.-\frac{2}{m_Y^2|\mathbf{x}_a-\mathbf{x}_b|^3}\right]|(\mathbf{n}_a\cdot\mathbf{n}_b)|^2(\mathbf{v}_a\cdot\mathbf{v}_b)\right\}.
\end{aligned}
\tag{3.200}
$$

while from Eq. (3.116)$_2$, we have

$$
\zeta(r_{ab})=g(\xi,\eta)\,e^{-m_+r_{ab}}+\left[\frac{1}{3}-g(\xi,\eta)\right]e^{-m_-r_{ab}}-\frac{4}{3}\,e^{-m_Yr_{ab}}
\tag{3.201}
$$

and, therefore, the term $\Xi(r_{ab}, r_{ac})$ reads

$$\begin{aligned}
\Xi(r_{ab}, r_{ac}) \quad &= \Big[1+\zeta(r_{ab})\Big]\Big[1+\zeta(r_{ac})\Big] = \\
& 1 + g(\xi,\eta)\, e^{-m_+ r_{ab}} + \left[\frac{1}{3} - g(\xi,\eta)\right] e^{-m_- r_{ab}} - \frac{4}{3}\, e^{-m_Y r_{ab}} + \\
& + g(\xi,\eta)\, e^{-m_+ r_{ac}} + \left[\frac{1}{3} - g(\xi,\eta)\right] e^{-m_- r_{ac}} - \frac{4}{3}\, e^{-m_Y r_{ac}} + \\
& + g(\xi,\eta)\left[\frac{1}{3} - g(\xi,\eta)\right]\left(e^{-m_+ r_{ab} - m_- r_{ac}} + e^{-m_- r_{ab} - m_+ r_{ac}}\right) + \\
& - \frac{4}{3}\, g(\xi,\eta)\left(e^{-m_+ r_{ab} - m_Y r_{ac}} + e^{-m_Y r_{ab} - m_+ r_{ac}}\right) - \frac{4}{3}\, e^{-m_Y r_{ab}} + \\
& - \frac{4}{3}\left[\frac{1}{3} - g(\xi,\eta)\right]\left(e^{-m_- r_{ab} - m_Y r_{ac}} + e^{-m_Y r_{ab} - m_- r_{ac}}\right) + \\
& + g^2(\xi,\eta)\, e^{-m_+ (r_{ab} - r_{ac})} + \left[\frac{1}{3} - g(\xi,\eta)\right]^2 e^{-m_- (r_{ab} - r_{ac})} + \\
& + \frac{16}{9} e^{-m_Y (r_{ab} - r_{ac})}
\end{aligned} \tag{3.202}$$

These two potential terms both include the Yukawa-like corrections. In the end, the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \mathbf{v}_a} - \frac{\partial L}{\partial \mathbf{x}_a} = 0 \tag{3.203}$$

together with Eq. (3.199) give rise to the most general system of differential equations for which we were searching. In general, by inserting Eq. (3.199) in Eq. (3.203), we have found the relativistic STFOG equations of orbital motions (in the weak field limit) which govern the evolution of the isolated system of N bodies, where each of all the bodies moves along the geodesics of the curved space-time generated by the others. We highlight that by posing $g(\xi\,\eta) = 1/3$, one obtains the relativistic N-body equations of motions for the $f(R, R_{\alpha\beta}R^{\alpha\beta})$-theory. Moreover if in the field's action the $R_{\alpha\beta}R^{\alpha\beta}$ invariant is not present, we have got the equations for case of the $f(R)$-theory, whilst including a non-minimally coupled scalar field ϕ in the field's action, we get the ones for the $f(R,\phi)$-theory. As a special case, now the equations of motion for a system of N particles for NonCommutative Spectral Gravity are finally obtained by simply taking the overall vector potential Z_i in Eq. (3.193) and the potential $\Phi(\mathbf{x})$ in Eq. $(3.136)_1$. On the basis of this dynamical system of differential equations, it is possible to conduct analysis about specific models in several scenarios requiring high accuracy, e.g. as the Solar System, by taking into account also the Post-Newtonian terms and their contribution to the dynamics. We repeat that by Post-Newtonian we mean the relativistic extra-potentials arising from the theory beyond the Newtonian limit. Thanks to Eqs. (3.199) and Eq. (3.203) with respect to the a given Cauchy problem, a straightforward analysis of models like the *relativistic 3-body problem* in the context of a general class of Extended Theory of Gravity, the *relativistic 2-body problem with comparable masses*[17], as well as many other configurations where even the small relativistic perturbations induced by all the objects of the system, are included in order to carry out high precision analysis of the planetary (or stellar) dynamics affected by the presence of the sources.

In the next Chapter, we will see a direct application to the *reduced* 2-body problem, i.e. one of the two masses is negligible and the problem is then restricted to the one-body problem, i.e. the dynamics of a test-particle subject to a central force field produced by the massive dominant source. Despite being a very basic problem, this type of modelling is one of the most common and historically widespread because it is efficient for investigating fundamental problems of mathematical physics and reconstructing many binary astrophysical systems. However, in case of a 2-body system with comparable masses, the space-time generated by the two bodies is diverse (such as the system J0737-3039 constituted by two neutron stars of similar masses) and the simplification is not feasible. However, one can refer to our system (3.199)-(3.203).

[17]Here the mass of one of the two bodies cannot be neglected.

Chapter 4

Periastron Advance: Methods and Applications to the Solar System and the S2 Star

4.1 Epistemological importance and introductory motivations

In a 2-body system like that represented by a *planet* - *Sun* system, the periastron (or perihelion) is the point on the major axis of the ellipse described by a planet around the Sun at which the planet is at its minimal distance from the Sun. On the contrary, the apoastron indicates the point on the major axis at the maximal distance. The Sun is located at one of the two ellipse's focus. Planetary orbits are affected by the perturbing interaction with other objects of the Solar System and usually undergo a shift of the periastron occurring along the direction of orbital motion itself. The shift per revolution is usually very small, but as a cumulative effect it induces a secular variation of the anomalistic angle and rotation of the apsidal line (the line connecting the periastron and the apoastron) which becomes relevant and detectable with ongoing collection of astrometric data. However, the (secular) perturbation induced by other bodies on the analysed object is not the unique way to reproduce anomalistic apsidal precessions of the orbits. The other main manner is due to a modification of the Newtonian law of attraction, e.g. adding potential terms or adding a small perturbation to the exponent of the inverse-squared law of attraction. From a historical point of view, the problem of the absidal precession of the orbits described by celestial objects has always been a landmark topic and characterised some of the most fundamental discoveries, which have been revealed to be connected to the presence of other objects or to the law of gravity.

The issue was connected to the property that a closed elliptical orbit described by a test particle in a central radial force of a bounded *reduced* 2-body system, can be obtained if and only if the test particle is subject to a gravitational force or an elastic force coming from a Newtonian potential or elastic potentials (i.e. $\Phi = -k/r$ or $\Phi = (1/2)\,k\,r^2$). This important result was demonstrated by J. Bertrand in his famous theorem (1873) [133] (see also Arnold [136]) or Landau-Lifshitz [135]). This fact was already noticed by I. Newton.

In fact, I. Newton was actually the first to suppose a generalisation of the inverse squared law of attraction [137] and a modification of it to deal with the apsidal lunar precession, which persisted as a huge problem for a long time until the right combination of perturbing forces was found, induced by the Sun and the other planets. Later on, after having even considered the possibility of a modification of the law of gravity in order to account for Uranus' orbital anomalies U. Le Verrier discovered Neptune by supposing that the anomalies were due to the presence of an unknown (invisible) planet in 1846 [139; 140; 141]. Le Verrier also discovered the observed Mercury perihelion precession, but the analogue attempt to fully explain it with the presence of a new planet (Volcano) was not successful, and it was finally understood only in the context of Einstein theory [60; 73; 74; 54; 59].

In 1915, A. Einstein arrived to a first complete formulation of the theory of General Relativity following the geometrical approach of thinking gravity as manifestation of the curvature of space-time manifold, then not as a real fundamental interaction, based on the introduction of the concept of curvature field for elaborating a theory of gravity. After some erroneous attempts at achieving the right form of the field equations, the explanation of Mercury's perihelion precession was the fundamental step that convinced Einstein to be finally on the right way and, most importantly, also suggested that the conceptual structure of the theory was correct [59] despite the fact that other confirmations were needed[1], as the later measurement of light bending carried out in 1919 by A.S. Eddington [69]. He calculated the precession by starting from the solution of the field equation *in vacuo*. The solution was obtained through a physical approximation, which in practise was the first Post-Newtonian approximation ever made. No further matter was needed but, this time, just

[1]Newtonian gravity could actually be able to account for that discrepancy and in principle still be consistent with the measured anomalistic precession. In fact the Mercury perihelion shift could have been caused by a small flattening at the poles of the Sun, and therefore it was necessary to seek solutions of the equations of motion incorporating the presence of a perturbative potential term given by the quadrupole moment

an additional potential emerging from a new law of gravity provided by the GR. Afterwards, it was inevitably considered a fixed point in the PPN formalism, for the estimations of the Post-Newtonian parameters and the comparison with other theories (see section (3.2.2)). This sequence of events seems to have imposed a sort of epistemological issue between the wider *invisible matter paradigm* and the *new law of gravity paradigm*[2] or dark matter as a superfluid condensate [166].

Since then, predictions regarding the periastron advance further became one of the most important and widespread testing ground for gravitational physics to the point that nowadays it is an inescapable test for new theoretical proposals, the identification of physical constraints that a theory has to respect, or equivalently are an advantageous tool to infer physical bounds to theories beyond General Relativity. It is currently one of the three main dynamical tests in weak field limit together with the deflection of light and the Shapiro time delay. Therefore this analysis plays an important role for the discovery of new physics and the most disparate binary systems are studied to detect phenomenological anomalies.

Besides Einstein, alternative but most commonly known techniques for the calculation of the periastron shift in General Relativity were given by K. Schwarzschild and A. Eddington (we also mention T. Levi-Civita), as well as interesting known resolutions, were proposed by E. T. Whittaker, Robertson [71](using Hamiltonian formalism), S. Chandraskhar (resorting to elliptical integrals) [130] and S. Weinberg [54]. Commonly used techniques are presented in [60] and [53]. Furthermore, Adkins & MacDonell [151] established a method to treat the periastron shift for a wider number of potentials and as a result a class of integrals with respect to the examined potential.

In this chapter, we consider the weak field approximation of Scalar-Tensor Fourth Order Gravity (STFOG), which includes several models of modified gravity and constrain the sizes of the hypothetical new weak forces of STFOG and NonCommutative Spectral Gravity (NCSG) by taking advantage of the Adkins & MacDonell integrals and making use of the data coming from the precession of Planets. The form of the corrections to the Newtonian potential is of the form of a Yukawa-like potential (5th force), i.e. $V(r) = \alpha \frac{e^{-\beta r}}{r}$, where α is the parameter related to the strength of the potential and β to the range of the force. The present data on periastron advance allow to infer a constraint on the free parameter of the gravitational models. Moreover, the Non-Commutative Spectral Gravity (NCSG) is also studied, being a particular case STFOG. Here we show that the precession

[2]If we refer to the Dark Matter problem, the future response could be even more subtle than it appears. Just to mention only one example, we could think about the possibilities of Bose-Einstein condensates that generate corrections to the gravitational field of a galaxy [163; 164; 165]

shift of Planet allows one to improve the bounds on parameter β by several orders of magnitude. Finally, such an analysis is studied to the case of power-like potential, referring in particular to deformation of the Schwarzschild geometry induced by a quintessence field, responsible of the present accelerated phase of the Universe. Afterwards, by relying on the epicyclic perturbation, we elaborate a new resolution method for the determination of the periastron advance. The epicyclic perturbation was already successfully utilized in GR for the perihelion advance computation (see R. Wald [58] or [59]), it is widely employed in the study of galactic physics [122] and it is especially the technique on which Bertrand's theorem was demonstrated [133]. Putting together these concepts, here we demonstrate how this technique can be the starting base to synthesize a generalized resolution method conducting to an *exact solution* for the periastron shift's determination. Such a resolution method is especially applicable for any gravity theory beyond GR, or model within a certain theory, and it incorporates each of all Post-Newtonian contributions with no need to involve numerical methods. By this analytical resolution, we obtain the final result and then deduce the expressions for the examined theories through which a computation of the bounds is performed again, thus improving the previous outcomes.

4.2 The Adkins & MacDonnel's method

In this section, we constrain the sizes of new gravitational forces (inferred in ETG and other scenarios) by making use of the data coming from the precession of planets. For this purpose, we follow the paper by Adkins and MacDonnell [151] (see also [154; 155]), where the precession of Keplerian orbits is calculated under the influence of arbitrary central-force perturbations. In the limit of nearly circular orbits, the perturbed orbit equation takes the form ($u = 1/r$)

$$\frac{d^2u}{d\phi^2} + u = \frac{GM}{h^2} - \frac{g(u)}{h^2} \tag{4.1}$$

where $g(u) = r^2 \frac{F(r)}{m}|_{r=1/u}$ ($\frac{F}{m} = -\nabla V$) and $h^2 = GMa$. $g(u) = 0$ corresponds to the unperturbed solution. We refer to the corrections to the Planets precession induced by the Yukawa-like potential $V_Y(r) = \alpha \frac{e^{-\beta r}}{r}$, and power law (PL) potentials $V_{PL}(r) = \alpha_n r^n$. In GR, the first post-Newtonian correction is a perturbing potential given by $V(r)\Big|_{\rm GR} = -\frac{GMh^2}{c^2 r^3}$, which corresponds to the precession $\Delta\theta_p\Big|_{\rm GR} = \frac{6\pi GM}{c^2 a}$. This gives the well-known 43 arcsec per century when applied to the orbit of Mercury. The correction to the Planet precessions induced by a generic perturbing force $F(z)$ and perturbing potential $V(z)$ is [151]

$$\Delta\theta_p = -\frac{2a^2}{GM\epsilon}\int_{-1}^{1}\frac{dz\, z}{\sqrt{1-z^2}}\frac{F(z)}{(1+\epsilon z)^2} \tag{4.2}$$

$$= -\frac{2a}{GM\epsilon^2}\int_{-1}^{1}\frac{dz\, z}{\sqrt{1-z^2}}\frac{dV(z)}{dz}\,, \tag{4.3}$$

where, for the sake of convenience, the correction $\Delta\theta_p$ is written in terms of the dimensionless integration variable z with a fixed range, while ϵ is the eccentricity ($\epsilon < 1$). The perturbing force $F(z)$ and $V(z)$ is evaluated at radius $r = a/(1+\epsilon z)$. In the following we refer to the Yukawa-like and power-law potentials following from different gravitational theories of gravity.

- **The Yukawa force** - The Yukawa potential (as a correction to the Newtonian potential $V_N = GM/r$) is of the form [152; 153]

$$V_Y(r) = \alpha\frac{e^{-r/\lambda}}{r} \equiv \alpha\frac{e^{-\beta r}}{r} \tag{4.4}$$

  where α and $\lambda \equiv 1/\beta$ are the strength and the range of the interaction, respectively. As we shall see, such a potential occurs in several modified theories of gravity. The precession due to a Yukawa perturbation depends on two parameters: a range parameter $\kappa = a/\lambda = \beta a$ and the eccentricity ϵ, that is, $\Delta\theta_p(\kappa, \epsilon)$, where a is the semi-major axis. According to [151], the correction to the precession is of the integral form

$$\Delta\theta_p(\kappa,\epsilon) = -\frac{2\alpha}{GM\epsilon}\, I_{\epsilon,\beta}\,, \tag{4.5}$$

  where

$$I_{\epsilon,\beta} \equiv \int_{-1}^{1}\frac{dz\, z}{\sqrt{1-z^2}}\left(1+\frac{\kappa}{1+\epsilon z}\right)e^{-\frac{\kappa}{1+\epsilon z}}\,. \tag{4.6}$$

  The behavior of the integral (4.6) is represented in Fig. 4.1 for several Planets.

- **Power Law potential** - The power law potential is of the form

$$V_{PL}(r) = \alpha_q\, r^q\,, \tag{4.7}$$

  where the parameter q assume arbitrary values. The precession (4.2) can be exactly integrated, and leads to [151]

$$\Delta\theta_p(q) = \frac{-\pi\alpha_q}{GM}a^{q+1}\sqrt{1-\epsilon^2}\chi_q(\epsilon)\,, \tag{4.8}$$

  where $\chi_q(\epsilon)$ is written in terms of the Hypergeometric function

$$\chi_q(\epsilon) = q(q+1)\,{}_2F_1\left(\frac{1}{2}-\frac{q}{2}, 1-\frac{q}{2}\,; 2\,; \epsilon^2\right)\,. \tag{4.9}$$

These potentials occur in ETG and in Non-Commutative Spectral Geometry (the Yukawa-like potential), and Quintessence field surrounding a massive gravitational source (the Power Law potential). We shall infer the corrections to periastron advance for Solar System Planets, referring in particular to Mercury, Mars, Jupiter, and Saturn, as well as to S2 star orbiting around Sagittarius A^*.

4.3 Results and Constraints in the Solar System

4.3.1 Scalar-Tensor-Fourth-Order Gravity

As shown before, the STFOG field equations lead to a gravitational potential of the Yukawa-like form ($r = |\mathbf{x}|$)

$$V(r) = \frac{GM}{r}\left(1 + \sum_{i=\pm,Y} F_i e^{-\beta_i r}\right) , \tag{4.10}$$

where F_i and β are the strength and range of the interaction corresponding to each mode $i = +, -, Y$. Referring to the ball-like solution for a non-rotating source[3] (3.135) and to the Eqs. (3.116)-(3.117), and comparing (4.10) with (4.4), it follows the correspondence

$$\alpha \to GMF_i\,, \quad \beta \to \beta_i\,, \quad i = \pm, Y\,. \tag{4.11}$$

with

$$F_+ = g(\xi,\eta)\, F(m_+\mathcal{R})\,, \quad F_- = \left[\frac{1}{3} - g(\xi,\eta)\right] F(m_-\mathcal{R})\,, \quad F_Y = -\frac{4}{3}\, F(m_Y\mathcal{R})\,, \tag{4.12}$$

$$\beta_\pm = m_R\sqrt{\omega_\pm}\,, \qquad \beta_Y = m_Y\,. \tag{4.13}$$

We impose that the periastron shift $\Delta\theta_p(\kappa,\epsilon) = -\frac{2\alpha}{GM\epsilon}\, I_{\epsilon,\beta}$ given by (4.5), where $I_{\epsilon,\beta}$ is defined in (4.6), is lesser than the error η. Fixing $I_{\epsilon,\beta}$ to the maximum values, one gets the bounds on the parameters F_i:

$$|\Delta\theta_p(\kappa,\epsilon)| \lesssim \eta \quad \to \quad |F_i| \lesssim \frac{\eta\epsilon}{2I_{\epsilon,\beta_i}}\,,\ i = \pm, Y\,. \tag{4.14}$$

In Fig. 4.1 are plotted the function $I_{\epsilon,\beta}$ for the Mercury, Mars, Jupiter and Saturn planets. In Table 4.2 are reported the corresponding bounds on F_i. As an illustrative example, we

[3] It means that the g_{0i} mixed term of the metric is set to 0. For example, in certain models like the one we are treating, it is a good assumption when the rotation of the source is so small that its influence can be neglected.

plot $|F_{\pm}(\xi,\eta)|$ in Fig. 4.2, for $m_R = \mathcal{R}^{-1}$. The available values of the parameters $\{\xi,\eta\}$ allow to fix the masses, via Eqs. (3.127), (3.133), (3.128), (3.131), of extra modes arising in Scalar Tensor Fourth Order Gravity. The analysis of Yukawa gravitational potential for $f(R)$ has been carried out in [115]. In the following tables, we report the theoretical and observed values in the Solar System of the periastron shift with respect to the General Relativity and the results on the theoretical constraints inferred on the STFOG.

Table 4.1: Values of periastron advance for the first six planets of the Solar System. In the table we present the values of the eccentricity ϵ, semi-major axis a in meters, the orbital period P in years, the periastron advance predicted in General Relativity (GR).

Planet	ϵ	$a\,(10^{11}\,m)$	$P\,(yrs)$	$\Delta\phi_{GR}\,(''/century)$	$\Delta\phi_{obs}$
Mercury	0.205	0.578	0.24	43.125	42.989 ± 0.500
Venus	0.007	1.077	0.62	8.62	8.000 ± 5.000
Earth	0.017	1.496	1.00	3.87	5.000 ± 1.000
Mars	0.093	2.273	1.88	1.36	1.362 ± 0.0005
Jupiter	0.048	7.779	11.86	0.0628	0.070 ± 0.004
Saturn	0.056	14.272	29.46	0.0138	0.014 ± 0.002

Table 4.2: Bounds on F_i, $i = \pm, Y$ obtained from (4.14) using the values of periastron advance for planets of the Solar System.

Planet	$\|\eta\|$	$I^{\max}_{\epsilon,\beta}$	$\beta^{\max}_i \simeq$	$\|F_i\| \lesssim$
Mercury	0.5	0.18	$4 \times 10^{-11} m^{-1}$	0.28
Mars	5×10^{-4}	0.08	$1.1 \times 10^{-11} m^{-1}$	2.9×10^{-4}
Jupiter	4×10^{-3}	0.04	$2.5 \times 10^{-12} m^{-1}$	2.4×10^{-3}
Saturn	2×10^{-3}	0.05	$2 \times 10^{-13} m^{-1}$	1.1×10^{-3}

NonCommutative Spectral Gravity

The modifications induced by the NCSG action to the Newtonian potentials Φ (and Ψ), Eq. (3.136), are similar to those induced by a Yukawa-like potential (4.4) (fifth-force [168]), with

$$\alpha = \frac{4}{3}GM\,, \quad \beta_{NCSG} = \beta \tag{4.15}$$

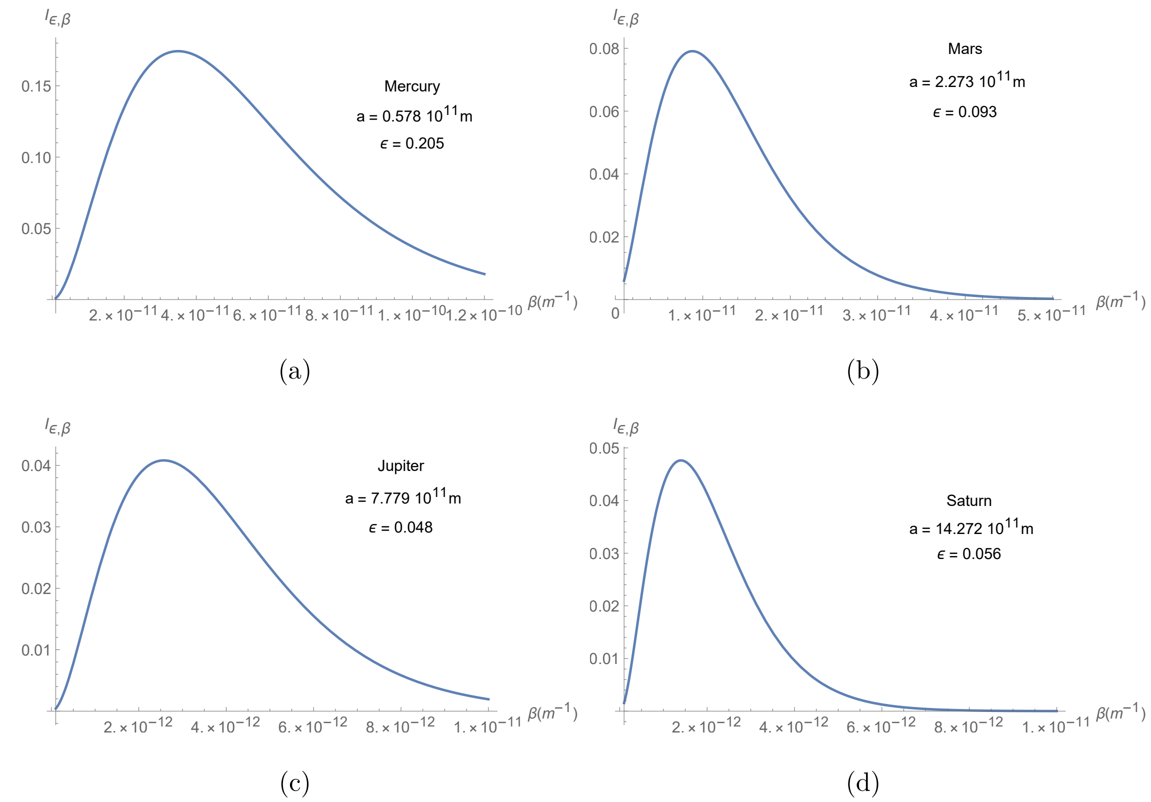
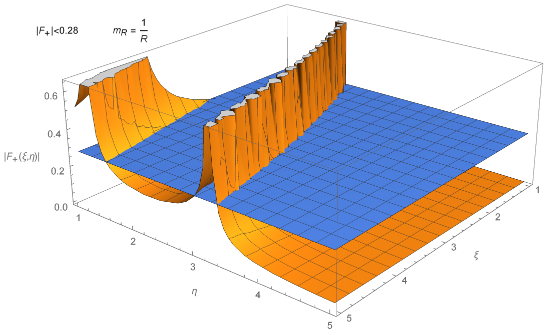


Figure 4.1: (a) $I_{\epsilon,\beta}$ vs β for Mercury. (b) $I_{\epsilon,\beta}$ vs β for Mars. (c) $I_{\epsilon,\beta}$ vs β for Jupiter. (d) $I_{\epsilon,\beta}$ vs β for Saturn.

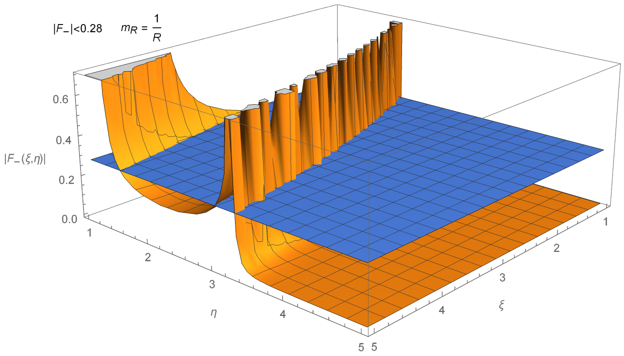
Following the previous section, Eqs. (4.5) (4.14), the periastron advance in NCSG for planets is given by

$$|\Delta\theta_p(\beta,\epsilon)| \lesssim \eta \quad \rightarrow \quad |I_{\epsilon,\beta}| \lesssim I_0\,, \quad I_0 \equiv \frac{3\eta\epsilon}{8}\,, \tag{4.16}$$

where $I_{\epsilon,\beta}$ is defined in (4.6). From Eq, (4.16) one infers the bounds on β, or equivalently an upper bound on λ. Results are reported in Table 4.3 (see also Fig. 4.3). These results show that the bounds on β improve several orders of magnitude compared to those obtained using recent observations of pulsar timing, $\beta \geq 7.55 \times 10^{-13}\mathrm{m}^{-1}$ [169; 170]. The bounds on the parameter β have been obtained in different frameworks. From the Gravity Probe B experiment, one gets $\beta > 10^{-6}\mathrm{m}^{-1}$ [160]. A more stringent constraint on β can be obtained from laboratory experiments designed to test the fifth force, that is, by constraining λ through torsion balance experiments which implies obtaining a stronger lower bound on β (or equivalently an upper bound to the momentum f_0 in NCSG theory). The test masses have a typical size of $\sim$ 10mm and their separation is smaller than their size. As we have already mentioned above, in NCSG one has $|\alpha| \sim \mathcal{O}(1)$, so that the tightest constraint on $\lambda = \beta^{-1}$ provided by Eöt-Wash [171] and Irvine [172] experiments is [180] $\lambda \lesssim 10^{-4}$m, or equivalently $\beta \gtrsim 10^4\mathrm{m}^{-1}$.



(a)



(b)

Figure 4.2: (a) F_+ vs $\{\xi, \eta\}$ for Mercury ($|F_+| \lesssim 0.28$), with $m_R = \frac{1}{\mathcal{R}}$. (b) F_- vs $\{\xi, \eta\}$ for Mercury ($|F_-| \lesssim 0.28$) with $m_R = \frac{1}{\mathcal{R}}$.

Table 4.3: Lower bounds on β obtained from (4.16) using the values of periastron advance for planets of the Solar System.

Planet	η	$I_0 \equiv \frac{3\eta\epsilon}{8}$	$\beta(m^{-1}) >$
Mercury	0.5	0.038	1.0×10^{-10}
Mars	5×10^{-4}	1.36	7.8×10^{-11}
Jupiter	4×10^{-3}	0.0628	2.1×10^{-11}
Saturn	2×10^{-3}	0.0138	8.5×10^{-12}

Quintessence - Dark energy

Here we remind that the solution of Einstein's field equations for a static spherically symmetric quintessence surrounding a black hole in 4 dimension is given by [182; 186]

$$g_{\mu\nu} = \text{diag}\left(-f(r), f^{-1}(r), r^2, r^2 \sin^2\theta\right), \tag{4.17}$$

with

$$f(r) = 1 - \frac{2M}{r} - \frac{c}{r^{3\omega_Q+1}}, \tag{4.18}$$

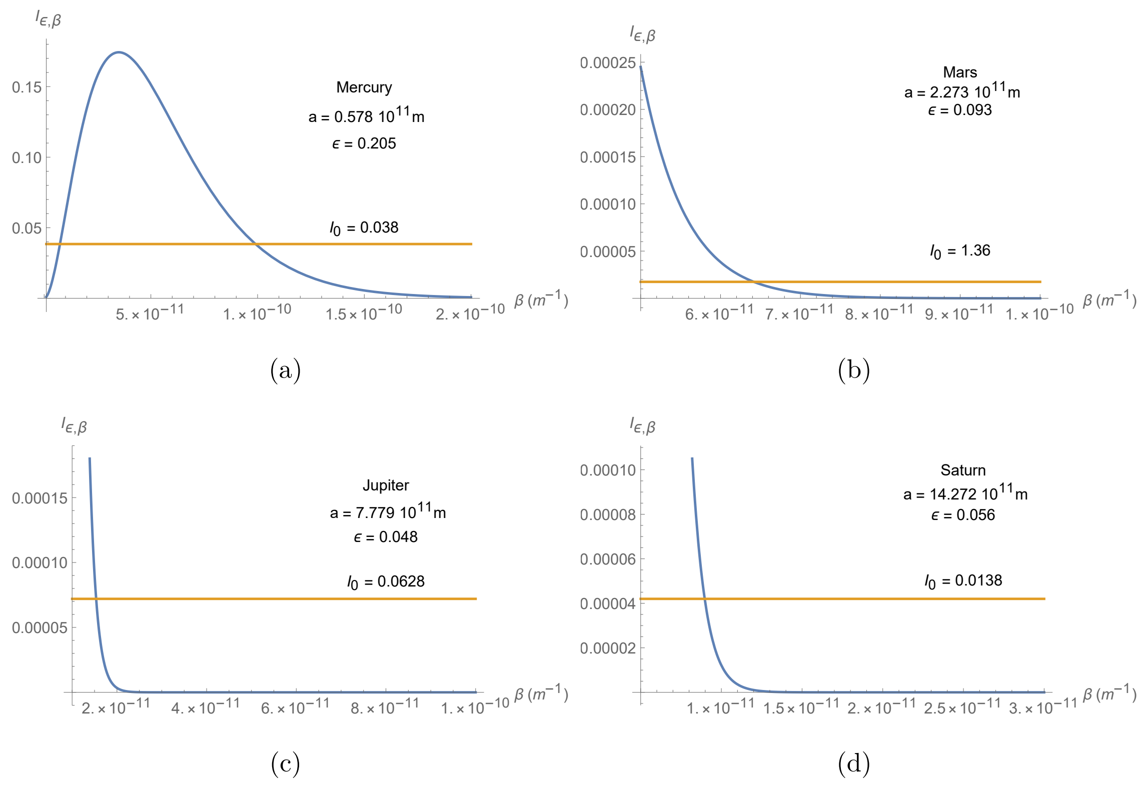


Figure 4.3: (a) $I_{\epsilon,\beta}$ vs β for Mercury. (b) $I_{\epsilon,\beta}$ vs β for Mars. (c) $I_{\epsilon,\beta}$ vs β for Jupiter. (d) $I_{\epsilon,\beta}$ vs β for Saturn.

where ω_Q is the adiabtic index (the parameter of equation of state), $-1 \leqslant \omega_Q \leqslant -\frac{1}{3}$, and c the quintessence parameter. The cosmological constant (ΛCMD model) follows from (4.17) and (2.44) with $\omega_Q = -1$ and $c = \Lambda/3$,

$$f(r) = 1 - \frac{2M}{r} - \frac{\Lambda r^2}{3} \quad . \tag{4.19}$$

The Quintessential potential reads $V_Q = -\frac{c}{r^{3\omega_Q+1}}$, so that comparing with (4.7) one gets

$$q \to -(3\omega_Q + 1) \qquad \alpha_q \to c\,.$$

The precession (4.8) leads to

$$|\Delta\theta_p(\omega_Q, \epsilon)| = \frac{\pi c}{GM} a^{-3\omega_Q} \sqrt{1-\epsilon^2}\chi_{\omega_Q}(\epsilon)\,, \tag{4.20}$$

with

$$\chi_{\omega_Q}(\epsilon) = 3\omega_Q(1+3\omega_Q)\ {}_2F_1\left(\frac{2+3\omega_Q}{2}, \frac{3+3\omega_Q}{2}\,; 2\,; \epsilon^2\right)\,. \tag{4.21}$$

By requiring $|\Delta\theta_p(\omega_Q, \epsilon)| \lesssim \eta$ one gets the bounds on the parameters $\{\omega_Q, c\}$. Results are reported in Table 4.4 and Fig. 4.4 for fixed values of c.

Table 4.4: Values of the parameter ω_Q obtained from (4.20) using the values of periastron advance for planets of the Solar System.

Planet	η	$c(m^{3\omega_Q+1}) \sim$	$\omega_Q \gtrsim$
Mercury	0.5	10^{-25}	-0.86
Mars	5×10^{-4}	10^{-30}	-0.88
Jupiter	4×10^{-3}	10^{-30}	-0.84
Saturn	2×10^{-3}	10^{-30}	-0.82

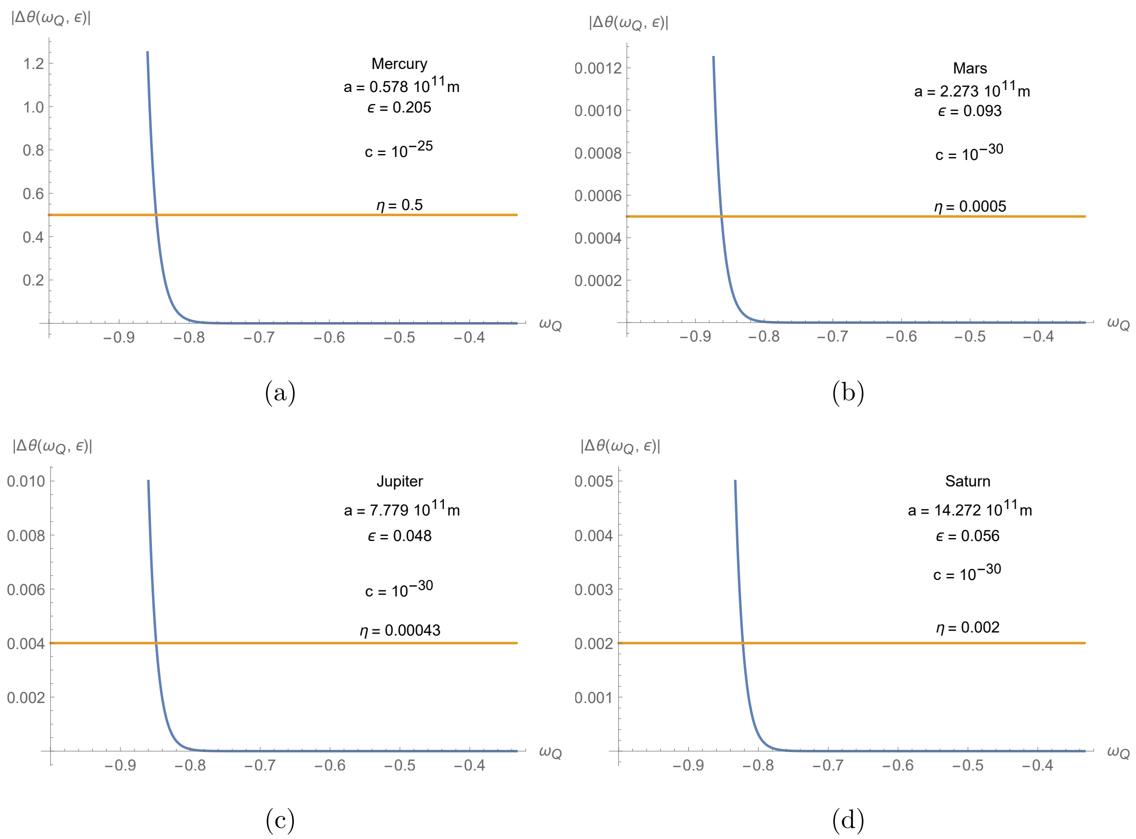


Figure 4.4: (a) $|\Delta\theta(\omega_Q, \epsilon)|$ vs ω_Q for Mercury. (b) $|\Delta\theta(\omega_Q, \epsilon)|$ vs ω_Q for Mars. (c) $|\Delta\theta(\omega_Q, \epsilon)|$ vs ω_Q for Jupiter. (d) $|\Delta\theta(\omega_Q, \epsilon)|$ vs ω_Q for Saturn.

4.3.2 Test on S2 Star

Finally, we briefly conclude our analysis testing the modified gravity predictions for the S2 star orbiting around Sagittarius A*, the Supermassive Black Hole at the center of the Milky Way, which has a mass equal to $M = (4.5 \pm 0.6) \times 10^6 M_\odot$ and a Schwarzschild radius $R_S = 2GM = 1.27 \times 10^{10}\, m$. The S2 Star orbit has an eccentrity $\epsilon = 0.88$ and a semi-major axis $a = 1.52917 \times 10^{14}$m. According to Ref. [162], the periastron advance is (0.2 ± 0.57)deg, hence $\eta = 0.57$ (it is expected that the GRAVITY interferometer can improve such a level of accuracy). We discuss the periastron advance for the gravitational models discussed above:

- **STFOG** - Referring to Scalar-Tensor Fourth Order Gravity, from Eq. (4.14) one gets

$$|\Delta\theta_p(\kappa,\epsilon)| \lesssim \eta \quad \to \quad |F_i| \lesssim \frac{\eta\epsilon}{2I_{\epsilon,\beta_i}} \sim 0.36\,,\ i = \pm, Y\,, \tag{4.22}$$

where in Fig. 4.5(a) is plotted the function $I_{\epsilon,\beta}$ for the S2 star. We have taken the maximum value of $I_{\epsilon,\beta}$ corresponding to $\beta \sim 2 \times 10^{-14} m^{-1}$ (see Fig. 4.5(a)). The analysis of S2 star orbit around the Galactic Centre in $f(\phi, R)$ and $f(R, \Box R)$ has been investigated in [173].

- **NCSG** - The S2 star values $\{\epsilon, \eta, a\}$ imply that, from (4.16),

$$|\Delta\theta_p(\beta,\epsilon)| \lesssim \eta \quad \to \quad |I_{\epsilon,\beta}| \lesssim I_0\,, \quad I_0 \equiv \frac{3\eta\epsilon}{8} \simeq 0.19\,. \tag{4.23}$$

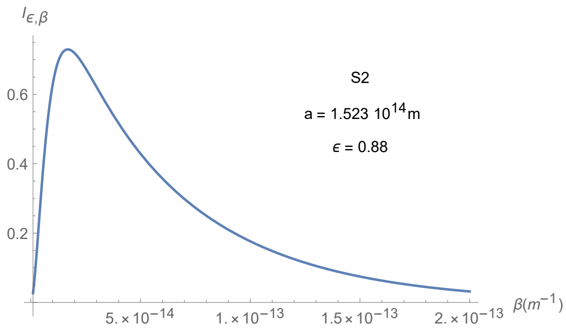
Results are reported in Fig. 4.5(b). We can see that the lower bound on β is $\beta \gtrsim 1.1\times 10^{-13} m^{-1}$. These bounds are compatible with the astrophysical bounds [169; 170].

- **Quintessence** - In the case of Quintessence field deforming the Schwarzschild geometry, Eq. (4.20) implies

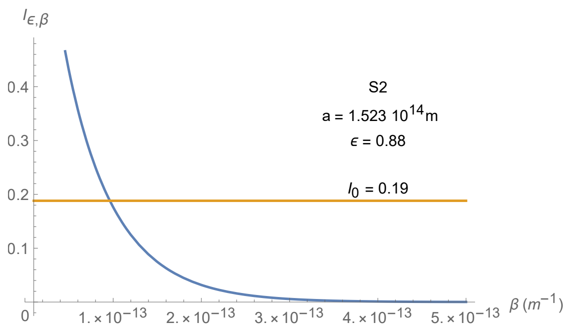
$$|\Delta\theta_p(\omega_Q,\epsilon)| = \frac{\pi c}{GM} a^{-3\omega_Q}\sqrt{1-\epsilon^2}\chi_{\omega_Q}(\epsilon) \lesssim 0.57\,, \tag{4.24}$$

$$\chi_{\omega_Q}(\epsilon) = 3\omega_Q(1+3\omega_Q)\ {}_2F_1\left(\frac{2+3\omega_Q}{2}, \frac{3+3\omega_Q}{2}\,; 2\,; \epsilon^2\right)\,. \tag{4.25}$$

Results are reported in Fig. 4.5 (c), from which it follows that for Quintessence $|\Delta\theta_p(\omega_Q,\epsilon)| \lesssim 0.57$ provided $\omega_Q \gtrsim -0.9$. Therefore, the exact value $\omega_Q = -1$ corresponding to the cosmological constant is excluded from this range of values.



(a)



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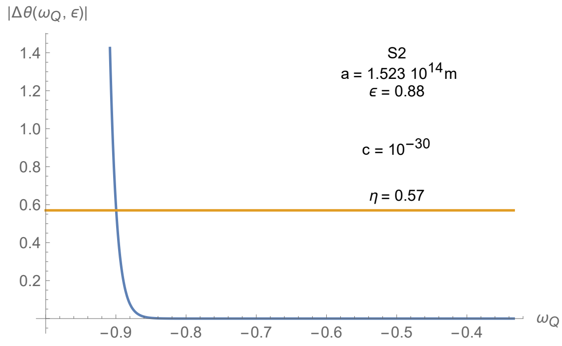


(c)

Figure 4.5: (a) $I_{\epsilon,\beta}$ vs β for S2 star in FOG theories. (b) $I_{\epsilon,\beta}$ vs β for S2 star. (c) $|\Delta\theta\omega_Q,\epsilon|$ vs ω_Q for S2 star. (c is in $m^{3\omega_Q+1}$ units).

4.4 A General Method for the Determination of the Periastron Advance

In this section, we elaborate a generalized method to determine an analytical expression for the periastron advance in the 2-body problem, valid and applicable to *any* theory and model, e.g. ETG, Quintessence fields, but also Non-Local Gravity, GR plus Dark Matter, AdS solution, Reissner-Nordstrom solution, etc., independently of the fact that the solution of the field equation is exact or inferred in the Weak Field limit. The resolution method is based on the mathematical idea of the epicyclic perturbation. Epicycles were first introduced by ellenistic mathematicians and astronomers to reproduce the retrograde observed motion of the planet on the celestial sphere [122]. The final dynamics results as the composition of oscillations around a point moving along the trajectory effected by the body. J. Bertrand used this approach to prove the Theorem stating that the only potentials yielding closed elliptical orbits are the Newtonian and elastic ones. By following an analogue approach, we demonstrate how it is possible to use the epicyclic approximation to synthesise a generalised resolution for the generic ETG (and also adequate for any other model).

From a physical point of view, the epicyclic approximation consists of the fact that an elliptic orbit can be exactly produced by a small perturbation of a stable circular orbit. Since the stable circular trajectory of radius r_0 corresponds to the orbital solution relative to the point of minimum r_0 of the effective potential of a test-particle which moves subject to a central force field as the one given a Schwarzschild Post-Newtonian field (e.g. motion of a satellite around the Earth, or planet around the Sun), this technique involves that we need a Taylor series expansion around the minimum point of the gravitational effective potential. Especially it allows us to incorporate all the Post-Newtonian potentials descending from the entire theory, not only those related to General Relativity. In agreement with Bertrand's theorem, here we decided to deal with the problem by reaching the equation of orbital motion in *second form*, also known as the *Binet equation*. Since we are considering the *restricted* 2-body problem, i.e. test-particle mass moving on the geodesics of Schwarzschild PN space-time, we can precisely reduce to the model of a material point of mass m around a dominant non-rotating spherical source of mass $M \gg m$. Such a method is based just on the assumption on the spherical symmetry of the model. Let us consider a generic spherically symmetric space-time

$$ds^2 = \left[1 + \frac{2}{c^2}\Phi(r)\right]c^2dt^2 - \left[1 - \frac{2}{c^2}\Psi(r)\right]dr^2 - r^2d\Omega, \tag{4.26}$$

where $d\Omega = d\phi^2 + \sin^2\phi\, d\theta^2$, the gravitational potentials $\Phi(r)$ and $\Psi(r)$ are given by Eq. (3.134) with $r = |\mathbf{x}|$ (see also Eqs. (4.10) and (4.11)), and the Lagrangian of the system

$$L = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu. \tag{4.27}$$

We impose the initial conditions $\dot{\phi} = 0$ and $\phi = \pi/2$ in the metric (4.26), so that the motion is planar with respect to the coordinates r and θ. So, L is given by

$$2L = [1 + \frac{2}{c^2}\Phi(r)]c^2\dot{t}^2 - [1 - \frac{2}{c^2}\Psi(r)]\dot{r}^2 - r^2\dot{\theta}^2. \tag{4.28}$$

where the dot indicates the derivative with respect to the proper time. The Euler-Lagrange equations

$$\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{x}^\alpha} - \frac{\partial L}{\partial x^\alpha} = 0\,, \tag{4.29}$$

with respect to coordinate time t and the angle θ, implies the conservation of the quantities

$$E \;=\; \frac{\partial L}{\partial \dot{t}} \;=\; \left[1 + \frac{2}{c^2}\Phi(r)\right]\dot{t}\,, \tag{4.30}$$

$$h \;=\; \frac{\partial L}{\partial \dot{\theta}} \;=\; r^2\dot{\theta}\,, \tag{4.31}$$

which corresponds the conservation of energy (measured by a static observer) and azimuthal angular momentum per unit mass of the test-particle. If we now insert these two relations into the first integral,

$$2L\,=\,[1+\frac{2}{c^2}\Phi(r)]c^2\dot{t}^2-[1-\frac{2}{c^2}\Psi(r)]\dot{r}^2-r^2\dot{\theta}^2\,=\,c^2\,, \tag{4.32}$$

one gets

$$\frac{E^2}{\left[1+\frac{2}{c^2}\Phi(r)\right]}c^2-[1-\frac{2}{c^2}\Psi(r)]\dot{r}^2-\frac{h^2}{r^2}\,=\,c^2\,, \tag{4.33}$$

from which, after some computations, supposing $\Phi\sim\Psi$ and neglecting the higher order terms $\sim O(\varepsilon^4)$ of the type $\sim\bar{v}^2\Phi$ or $\sim\Phi^2$ because irrelevant, we have

$$\frac{1}{2}\dot{r}^2+\Phi(r)+\frac{h^2}{2r^2}\left[1+\frac{2}{c^2}\Phi(r)\right]+\frac{1-E^2}{2}c^2\,=\,0. \tag{4.34}$$

It is now possible to deduce the equation of motion in the suitable *second form* by operating the substitution of the variable $u(\theta)\,\equiv\,1/r$, from which follows $\dot{r}=-h\,\dfrac{du(\theta)}{d\theta}$, where the prime denotes the derivative with respect to angle ϕ. Thus,

$$\left(\frac{du(\theta)}{d\theta}\right)^2+u^2\left[1+\frac{2}{c^2}\Phi(u)\right]-\frac{2}{h^2}\Phi(u)+\frac{1-E^2}{h^2}c^2=0. \tag{4.35}$$

Now we appropriately split the potential Φ into the sum of two separated contributions

$$\Phi\,=\,\Phi_N+\Phi_p \tag{4.36}$$

that is, into the the usual Newtonian potential and the perturbing Yukawa-like potential. Then, differentiating with respect to θ this equation, we finally obtain the *second form* differential equation of the orbit

$$\frac{d^2u(\theta)}{d\theta^2}+u=\frac{GM}{h^2}+\frac{3GM}{c^2}u^2-\frac{1}{h^2}\,\Phi_p'(u)-2\frac{u}{c^2}\,\Phi_p(u)-\frac{u^2}{c^2}\,\Phi_p'(u) \tag{4.37}$$

where the prime denotes the derivative with respect to u. Since the second member can be identified with the function

$$J(u)\,=\,\frac{GM}{h^2}+\frac{3GM}{c^2}u^2-\frac{1}{h^2}\,\Phi_p'(u)-2\frac{u}{c^2}\,\Phi_p(u)-\frac{u^2}{c^2}\,\Phi_p'(u) \tag{4.38}$$

the differential equation equation reads

$$\frac{d^2u(\theta)}{d\theta^2}+u=J(u) \tag{4.39}$$

where we recognize

$$J(u)\,=\,-\frac{1}{h^2}\,V_e'(u) \tag{4.40}$$

as the function associated to the derivative of effective gravitational potential multiplied by $-h^{-2}$, expressed by the second member of Eq. (4.37). We rapidly notice that the first term at the second member leads to the classical elliptic orbit of Newtonian gravity, while the second term is the Post-Newtonian contribution of General Relativity to the central force leading to the Rosette orbit arising from the rotation of the apsidal line. The fourth, fifth, and sixth term represent the Post-Newtonian Yukawa contributions of the ETG to the dynamics. Now we apply the epicyclic perturbation: as the circular motion of radius $u_0 = 1/r_0$ occurs at the point of minimum of the effective potential, namely the potential is such that the motion is stable and the solution u results to be bounded also after a small variation from u_0, in order to describe the elliptic orbit we add a slight perturbation so that

$$u = u_0 + u_\epsilon \tag{4.41}$$

with

$$u_0 = \frac{GM}{h^2} = \frac{1}{a(1-e^2)}, \tag{4.42}$$

obtained from the equation

$$u_0 = \frac{GM}{h^2} + \frac{3GM}{c^2} u_0\,. \tag{4.43}$$

Inserting the relationships (4.41) and (4.42) in the differential equation and expanding in Taylor series around $u_0 = GM/h^2$ the function

$$J(u) \simeq J(u_0) + J'(u_0)u_\epsilon\,, \tag{4.44}$$

where $J'(u_0)$ is the derivative evaluated at the point value u_0, we get

$$u''_\epsilon + n^2 u_\epsilon = 0 \tag{4.45}$$

where

$$n^2 = \Big(1 - J'(u_0)\Big) \tag{4.46}$$

which is the second order harmonic oscillator's differential equation. By an integration of it, we obtain

$$u_\epsilon = u^o_\epsilon \cos(n\theta + f_0). \tag{4.47}$$

with arbitrary constant f_0 set equal to $f_0 = 0$. The periastron occurs when the test-particle arrives at the point of minimum distance on the orbit properly given by the radius r_0 in Eq.

(4.42), and corresponding to a point of maximum of the variable u. Such a maximum point is reached when $\cos(n\theta) = \cos(2\pi) = 1$, that is

$$\cos\left(\sqrt{1 - J'(u_0)}\ \theta\right) = \cos(2\pi) = 1\,, \tag{4.48}$$

from which

$$\theta = 2\pi\left(\sqrt{1 - J'(u_0)}\right)^{-1}. \tag{4.49}$$

By expanding in Taylor series $(1-x)^{-1/2}$, it follows

$$\theta \simeq 2\pi\left(1 + \frac{J'(u_0)}{2}\right) \tag{4.50}$$

and this quickly leads to the final quantity expressing the angular anomalistic precession of the total angle $\theta \simeq 2\pi + 2\pi\,\delta\theta = 2\pi + \Delta\theta$ wiped out by the test-particle, that must be identified with the second term of the last relation as follows

$$\Delta\theta_{ETG} \;=\; \pi J'(u_0) \;=\; -\frac{\pi}{h^2} V_e''(u_0). \tag{4.51}$$

Therefore, by performing a straightforward computation, finally we obtain

$$\Delta\theta_{ETG} \;=\; \Delta\theta_{GR} + \Delta\theta_p \tag{4.52}$$

where

$$\Delta\theta_{GR} = \frac{6\pi GM}{ac^2(1-\epsilon^2)} \tag{4.53}$$

is the General Relativity's contribution to the periastron advance stemming from the first two terms at the second member of Eq. (4.37) and

$$\Delta\theta_p = -\frac{2\pi}{c^2}\,\Phi_p(u_0) - \frac{4\pi u_0}{c^2}\,\Phi_p'(u_0) - \frac{\pi u_0^2}{c^2}\,\Phi_p''(u_0) - \frac{\pi}{h^2}\,\Phi_p''(u_0)\,, \tag{4.54}$$

represents the additional shift containing all the Post-Newtonian corrections to the advance related to the Yukawa-like potentials coming from the theory (see Eqs. (3.116), (3.134), (4.10), (4.11), (4.12)). The derivative of the potentials are evaluated in $u_0 = [a(1-\epsilon^2)]^{-1}$. Putting all together, we find out

$$\Delta\theta_{ETG} \;=\; \frac{6\pi GM}{ac^2(1-\epsilon^2)} - \frac{2\pi}{c^2}\,\Phi_p(u_0) - \frac{4\pi u_0}{c^2}\,\Phi_p'(u_0) - \frac{\pi u_0^2}{c^2}\,\Phi_p''(u_0) - \frac{\pi}{h^2}\,\Phi_p''(u_0). \tag{4.55}$$

The solution is entirely analytical and the periastron advance determination is now simply traced to this final formula, it has a general validity for any theory in the respect of the radial symmetry of the model and its assumptions, without the necessity of choosing a specific method to be used only for a certain theory. Such a resolution method is then an effective product of a generalization of the epiciclyc perturbation technique and it conducts the evaluation of the periastron advance to a simple application of the analytic formula (4.55), independently of the analytic form of the perturbing potential (Yukawa, Power-Law or logarithmic) and their nature (entailed by the theory or induced by other matter). Furthermore the result turns out to be more economical and direct, enabling a fast and simple calculation because the derivatives are much easier than an integration process. It does not require numerical integration techniques whether the analytic form of the potential is too much laborious or even impossible to treat when a given method is employed, for example, as it happens with the Yukawa potentials if one uses the Adkins & MacDonnell integrals.

It comprises *all* Post-Newtonian terms at the required level of accuracy, thus enabling improving several orders of magnitude the previous bounds on the theories. Alternatively, it can be used for testing gravitational effects if the physical parameters of a theory/model have already been estimated. In the end, it also provides an exact mathematical framework for constructing further orbital simulations without making use of numerical techniques or codes.

4.5 Results with Improved Constraints in the Solar System

4.5.1 Scalar-Tensor-Fourth-Order Gravity

It is now possible to repeat the previous analysis by computing the bounds on the theory, which now will include the further Post-Newtonian contributions to the periastron shift. Hence, we pass to determine it for the Scalar-Tensor-Fourth-Order Gravity, NonCommutative Spectral Gravity. On the basis of the potential presented in Eqs. (3.134), (4.10), (4.11), (3.136), through Eq. (4.55) we find

$$\begin{aligned}\Delta\theta_{STFOG} = \quad & \frac{6\pi GM}{ac^2(1-\epsilon^2)} + \sum_{i=\pm Y} \frac{6\pi GMF_i}{ac^2(1-\epsilon^2)}\, e^{-\beta_i a(1-\epsilon^2)} + \\ & + \sum_{i=\pm Y} \frac{4\pi GMF_i}{c^2}\beta_i\, e^{-\beta_i a(1-\epsilon^2)} + \sum_{i=\pm Y} \frac{\pi GMF_i}{c^2}\beta_i^2 a(1-\epsilon^2)\, e^{-\beta_i a(1-\epsilon^2)} + \\ & + \sum_{i=\pm Y} \pi F_i \beta_i^2 a^2 (1-\epsilon^2)^2 e^{-\beta_i a(1-\epsilon^2)}\,, \end{aligned} \tag{4.56}$$

where we recall again that F_i and β_i are the strength and range of the interaction corresponding to each mode $i = +, -, Y$ respectively, and their expressions are given in Eqs. (4.12) and (4.13). As before, it must be imposed again that the periastron shift $\Delta\theta_p(\kappa, \epsilon)$ of STFOG and NCSG given by (4.56) and (4.58) respectively, with $\kappa = \beta a$, is less than the error η. Fixing $\Delta\theta_p(\kappa, \epsilon)$ to the maximum values, one obtains the bounds of the parameters F_i as

$$|\Delta\theta_p(\kappa, \epsilon)| \lesssim \eta \quad \rightarrow \quad |F_i| \lesssim \Theta(\eta, \epsilon, \beta) \qquad i = \pm, Y\,. \tag{4.57}$$

whereas $\Theta(\eta, \epsilon, \beta)$ represents an expression whose value in $\beta_i = \beta_i^{max}$ yields the bound $|F_i|$ with respect to a given known astrometric error η and eccentricity ϵ. In Fig. 4.6 we plot the function $|\Delta\theta_p(\kappa, \epsilon)|$ relative to Mercury, Mars, Jupiter and Saturn. In Table 4.5 are reported the corresponding bounds on F_i and, as we can see, the Post-Newtonian contributions of relativistic origin allow one to achieve a further improvement on the bound of the theory.

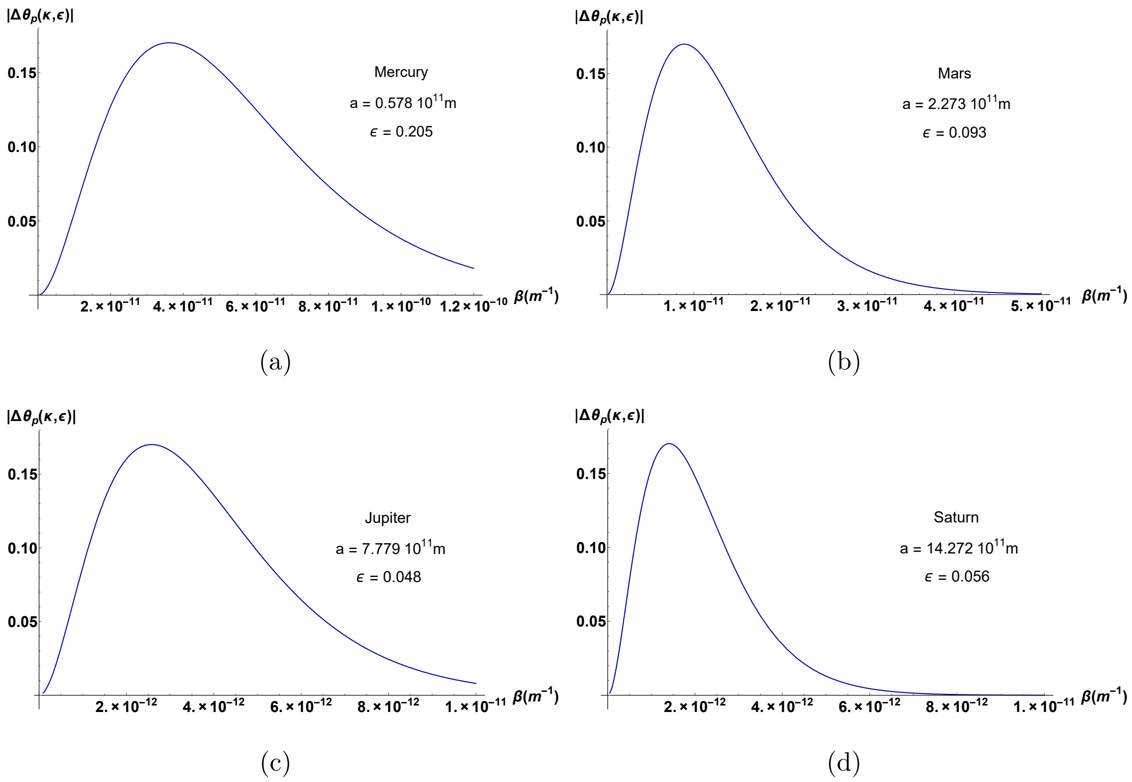


Figure 4.6: (a) $|\Delta\theta_p(\kappa, \epsilon)|$ vs β for Mercury. (b) $|\Delta\theta_p(\kappa, \epsilon)|$ vs β for Mars. (c) $|\Delta\theta_p(\kappa, \epsilon)|$ vs β for Jupiter. (d) $|\Delta\theta_p(\kappa, \epsilon)|$ vs β for Saturn.

Table 4.5: Improved bounds on F_i, $i = \pm, Y$ obtained from (4.57) using the values of periastron advance for planets of the Solar System.

Planet	$\|\eta\|$	$\beta_i^{\max} \simeq$	$\|F_i\| \lesssim$
Mercury	0.5	$3.61 \times 10^{-11} m^{-1}$	0.29
Mars	5×10^{-4}	$8.87 \times 10^{-12} m^{-1}$	2.94×10^{-4}
Jupiter	4×10^{-3}	$2.58 \times 10^{-12} m^{-1}$	2.35×10^{-3}
Saturn	2×10^{-3}	$1.41 \times 10^{-12} m^{-1}$	1.1×10^{-3}

4.5.2 NonCommutative Spectral Gravity

Considering the potential in Eq. (3.136), through Eq. (4.55) we obtain

$$\begin{aligned}\Delta\theta_{NCSG} = \quad & \frac{6\pi GM}{ac^2(1-\epsilon^2)} - \frac{8\pi GM}{ac^2(1-\epsilon^2)}\, e^{-\beta a(1-\epsilon^2)} - \frac{16\pi GM}{3c^2}\beta\, e^{-\beta a(1-\epsilon^2)} + \\ & - \frac{4\pi GM}{3c^2}\beta^2 a(1-\epsilon^2)\, e^{-\beta a(1-\epsilon^2)} - \frac{4\pi}{3}\beta^2 a^2(1-\epsilon^2)^2 e^{-\beta a(1-\epsilon^2)} \,. \end{aligned} \tag{4.58}$$

We recall that the coupling constant of the induced Yukawa-like potential of NCSG is $\alpha = \dfrac{4}{3}GM$ and $\beta = \beta_{NCSG}$ is the range of interaction. Reasoning once again as in the previous section, the periastron advance in NCSG for planets is given by

$$|\Delta\theta_p(\beta,\epsilon)| \lesssim \eta \quad \to \quad |\beta| \lesssim \widetilde{\Theta}(\eta,\epsilon) \tag{4.59}$$

where $\widetilde{\Theta}(\eta,\epsilon)$ is defined as the expression from which we infer the new bounds on β with respect to a certain known value of the astrometric error $|\eta|$ and the eccentricity, or equivalently an upper bound on its characteristic length λ. Results are reported in Table 4.6 (see also Fig. 4.7). These results show that the bounds on β reach a further improvement on their precision $\beta \geq 7.55 \times 10^{-13}\,\mathrm{m}^{-1}$ [169; 170].

4.5.3 Improvements on the Test of S2 Star

- **STFOG** - Referring to Scalar-Tensor Fourth Order Gravity, from Eq. (4.57) we obtain the improved bound

$$|\Delta\theta_p(\kappa,\epsilon)| \lesssim \eta \quad \to \quad |F_i| \lesssim \Theta(\eta,\epsilon,\beta) \sim 0.33 \qquad i = \pm, Y\,. \tag{4.60}$$

In Fig. 4.8(a), we have plotted the function $\Delta\theta_p(\kappa,\epsilon)$ for the S2 star. The maximum value of $\Delta\theta_p(\kappa,\epsilon)$ corresponding to $\beta \simeq 6.04 \times 10^{-14}\,\mathrm{m}^{-1}$ (see Fig. 4.8 (a)) has been considered.

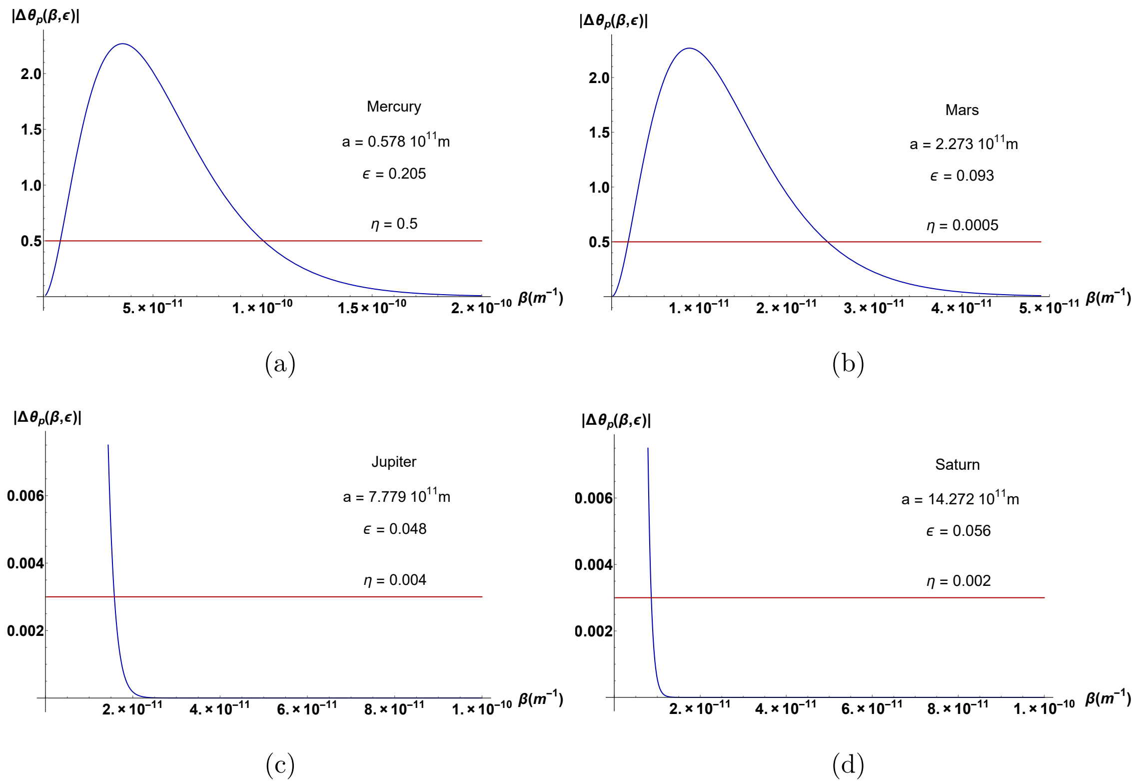


Figure 4.7: (a) $|\Delta\theta_p(\beta,\epsilon)|$ vs β for Mercury. (b) $|\Delta\theta_p(\beta,\epsilon)|$ vs β for Mars. (c) $|\Delta\theta_p(\beta,\epsilon)|$ vs β for Jupiter. (d) $|\Delta\theta_p(\beta,\epsilon)|$ vs β for Saturn.

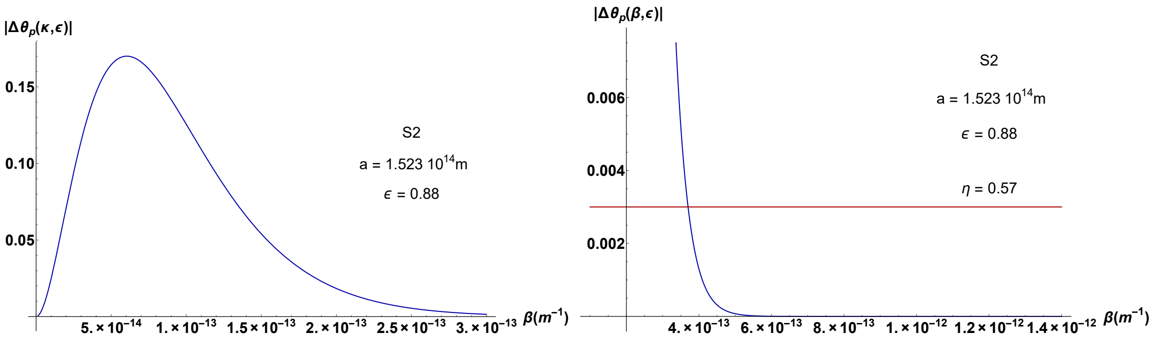
Table 4.6: Improved lower bounds on β obtained from (4.16) using the values of periastron advance for the planets of the Solar System.

Planet	η	$\beta(m^{-1}) >$
Mercury	0.5	1.0×10^{-10}
Mars	5×10^{-4}	6.38×10^{-11}
Jupiter	4×10^{-3}	1.53×10^{-11}
Saturn	2×10^{-3}	8.95×10^{-12}

- **NCSG** - The S2 star values $\{\epsilon, \eta, a\}$, from (4.59), imply that

$$|\Delta\theta_p(\beta,\epsilon)| \lesssim \eta \quad \rightarrow \quad |\beta| \lesssim \tilde{\Theta}(\eta,\epsilon)\,. \tag{4.61}$$

Results are reported in Fig. 4.8(b). We notice that the improved lower bound on β is $\beta \gtrsim 1.62 \times 10^{-13} m^{-1}$. These further bounds are also compatible with the astrophysical bounds [169; 170].



(a) (b)

Figure 4.8: (a) $|\Delta\theta_p(\kappa, \epsilon)|$ vs β for S2 star in Scalar-Tensor-Fourth-Order Gravity (STFOG). (b) $|\Delta\theta_p(\beta, \epsilon)|$ vs β for S2 star in NonCommutative Spectral Gravity (NCSG).

Chapter 5

Galaxy Rotation Curves in Extended Theories of Gravity

In this last chapter, we infer and analyze the galaxy rotation curve in the context of the $f(R)$-theory, the more general Scalar-Tensor-Fourth Gravity and the NonCommutative Spectral Gravity, of which a first ever analysis is also accomplished. In particular, we consider Yukawa-like corrections to the gravitational potentials. By using parametric fits of the velocity curve formulas with observed data of an unexplored sample of galaxies, we derive the numerical values of fundamental physical properties of spiral galaxies (total mass, core radius, and mass-to-light ratio). We consider data coming from the HI Nearby Galaxy Survey catalogue and the THINGS catalogue. Good reproductions of the galactic rotation curves are derived, and for what concerns the metric $f(R)$-theory the numerical predictions are compared with those emerging in the framework of Palatini formalism. Finally, we compare the numerical outcomes for the examined theories with the observed astronomical estimations of the galactic properties.

5.1 Model for the Stellar Motion, Theoretical Curves and Matter Distribution

The solutions presented in Eqs. (3.116), (3.135), (3.134), (3.136), (3.182), and in Eqs. (3.187), (3.188), (3.189), (2.44) referring to the spherically symmetric ones, allow us to determine the rotation curves of galaxies. In fact, as we saw in Chapter 3, for the geodesic principle the material point moves following the geodesics of the space-time manifold locally warped by the presence of a distribution of matter and, thus, the geodesic equation

$$\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0, \tag{5.1}$$

governs the dynamics of a test-particle. This is case when we study the motion of a star around the galactic centre, which can be adequately modelled as the dynamics of a test-particle in a central force field and whose motion occurs around the galactic centre. The galaxy is a self-gravitating system for which the stellar velocities are much lower than the speed of light, i.e. $v \ll c$ and the weak field limit corresponds to the physical regime of the environment. Since at galactic scales the distances involved are very large compared to those of the solar system (or binary systems), the field $\overset{2}{g}_{ij}$ in Eq. (3.13) and its relativistic effects on the dynamical behaviour of a body can be safely neglected, so the Newtonian limit (3.15) is sufficient for our objectives. For the model we adopt the usual radial symmetry and the stars' orbits are assumed to be approximately circular, because this is in good agreement with the astrophysical observations. More complex models with slightly eccentric orbits might be considered, but the approximate circularity assumption turns out to be effective. As showed in section (3.2), in Newtonian limit the geodesics equation becomes

$$\frac{d^2x^i}{dt^2} \simeq -c^2\,\Gamma^i_{00}\,. \tag{5.2}$$

Then we consider the radial component of the geodesics equation

$$\frac{d^2r}{dt^2} = -c^2\Gamma^r_{tt}. \tag{5.3}$$

Since the particle moves non-relativistically by equating the centripetal acceleration $\frac{d^2r}{dt^2} = -\frac{v^2}{r}$ to the second member of Eq. (5.3), one obtains $v^2 = rc^2\Gamma^r_{tt}$, that is

$$v^2(r) = r\frac{\partial\Phi(r)}{\partial r}, \tag{5.4}$$

that leads to the searched theoretical rotation curve with respect to a given theory. This is the galaxies rotation curve in the metric formalism in the framework of the Starobinsky model. In addition, since we want to pass from a simple central point-like source to a suitable model for describing a galaxy, we adopt a profile for the distribution of matter of the HSB (High Surface Brightness) and LSB (Low Surface Brightness) galaxies, whose first form was introduced in Ref. [217] and then presented in Ref. [125] in a slightly modified version (assumed by us) in order to utilize a matter profile closer to that arising from the observed photometric profile. The profile[1] is given by

$$M(r) = M_0 \left(\sqrt{\frac{R_0}{r_c}} \frac{r}{r + r_c} \right)^{3b}, \tag{5.5}$$

where b is a parameter depending on the typology of brightness, M_0 is the total mass, R_0 and r_c are the scale length and the core radius of the galaxy, respectively. By computing the derivatives of the several potentials $\Phi(r)$ from Eqs. (3.182), (3.116), (3.136), (3.189) (2.44) in Eq. (5.4) and inserting the matter profile (5.5) in it, for each theory one finds the final analytical expressions for the theoretical rotation curves of galaxies as follows

- *f(R)-gravity*

$$v(r) = \sqrt{\frac{GM_0}{r}} \left(\sqrt{\frac{R_0}{r_c}} \frac{r}{r + r_c} \right)^{3b/2} \left\{ 1 + \frac{1}{3} e^{-r/\lambda_R} \left(1 + \frac{r}{\lambda_R} \right) \right\}^{1/2}; \tag{5.6}$$

- *Scalar-Tensor-Fourth-Order Gravity*

$$\begin{aligned} v(r) = & \sqrt{\frac{GM_0}{r}} \left(\sqrt{\frac{R_0}{r_c}} \frac{r}{r + r_c} \right)^{3b/2} \left\{ 1 + g(\xi,\eta)\, e^{-r/\lambda_+} \left(1 + \frac{r}{\lambda_+} \right) + \right. \\ & \left. + \left[\frac{1}{3} - g(\xi,\eta) \right] e^{-r/\lambda_-} \left(1 + \frac{r}{\lambda_-} \right) - \frac{4}{3} e^{-r/\lambda_Y} \left(1 + \frac{r}{\lambda_Y} \right) \right\}^{1/2}; \end{aligned} \tag{5.7}$$

- *NonCommutative Spectral Gravity*

$$v(r) = \sqrt{\frac{GM_0}{r}} \left(\sqrt{\frac{R_0}{r_c}} \frac{r}{r + r_c} \right)^{3b/2} \left\{ 1 - \frac{4}{3} e^{-r/\lambda} \left(1 + \frac{r}{\lambda} \right) \right\}^{1/2}. \tag{5.8}$$

These are the theoretical rotation curve formulas that we employ in our numerical analysis. In rotation curves (5.6), (5.10), and (5.8), the characteristic lengths of the gravitational system $\lambda_R = h/m_R c$, $\lambda_i = h/\beta_i c$ with $i = \pm Y$ in Eq. (4.13), and $\lambda = h/\beta_{NCSG}\, c$ (here c is the speed of light and h the Planck constant) are identified with the Compton wavelengths of the massive gravitons associated to the Yukawa-like interaction terms. For the functions $g(\xi, \eta)$ and the other related physical parameters, see Eqs. (3.127), (3.128) and (3.130).

[1] As pointed out in [125], this is a slightly different version than that presented and utilized by Brownstein and Moffat [217] which was improved to deduce a mass profile closer to the one arising from the observed photometric profile

5.2 Curve-fitting with Observed Curves and Prediction on Physical Properties of Galaxies

We consider a sample of spiral galaxies contained in the THINGS catalogue. For HBS spiral galaxies, we set $b = 1$, while for LSB and dwarf galaxies $b = 2$ is a more suitable choice, because for small r the matter profile grows considerably slower [217].

5.2.1 Results for $f(R)$-gravity

Results with fixed b-parameter of luminosity

In order to reproduce the galactic rotational velocity curves, one finds $\lambda_R \simeq 50$ kpc as the length of interaction parameter's value that gives rise the best non-linear fits, corresponding to $m_R \simeq 2 \cdot 10^{-2}$ kpc^{-1}, i.e. to a particle mass of the order $m_R \sim 10^{-28}$ eV. This value is very close to the J. Moffat's ones $\sim 4 \cdot 10^{-2}$ [43; 42; 40; 41; 39]. In Table 5.1, we report the numerical results for the parameters M_0 and r_c of the matter profile model (5.5) obtained by the fittings of the galaxy rotation curve in Eq. (5.6) with the data coming from *The HI Nearby Galaxy Survey* (THINGS) [219; 220; 221]. The list of the sample of HSB galaxies makes possible to compare the outcomes of the $f(R)$-theory in metric formalism with those in Palatini formalism reported in Ref. [125].

In Fig. 5.1, we plot the galaxy rotation curves obtained by the fitting the data coming from the for HSB spiral galaxies of the sample (these refer to the metric formalism). We note a good agreement with the astronomical data, where each single point represents the measured circular velocity at a given distance from the galactic centre. On the contrary, in general we find values for the total amount of mass M_0 significantly larger than the observed values for these galaxies, while for NGC 4736 smaller [230; 232; 233; 234; 235]. Consequently mass-to-light ratios $\Upsilon_\star^B$ are much higher than the values expected by the stellar synthesis population models [232], and lower concerning NGC 4736 [230]. We highlight that these results can improve by considering more complex models, for instance, that can be elaborated with other matter distribution in addition to the Moffat-like profile in Eq. (5.5), which was assumed just as a good starting model.

Now, in order to display a quantitative comparison with the Palatini formalism, it is very useful to report the rotation curve formula[2] in Ref. [125] obtained by the Palatini

[2] In this formula, the energy density $\rho(r)$ can be derived from the equality $M'(r) = \mathcal{M}'(r)$. Here, the modified mass distribution $\mathcal{M}(r)$ must be computed from a definite integral (see [125]) including the correction induced by the conformal factor $\Omega = f'(\hat{R}) = 1 + 2\gamma\hat{R}$ of the metric $\hat{g}_{\alpha\beta} = f'(\hat{R})g_{\alpha\beta}$, with

Galaxy	M_0	$r_c\,(kpc)$	M_{gas}	$R_0\,(kpc)$	L_B	$\Upsilon_\star^B$	χ_r^2
NGC 3031	14.07	2.34	0.48	2.6	3.049	4.45	4.63
NGC 3521	35.08	4.00	1.07	3.3	3.698	9.19	1.85
NGC 3627	4.30	1.87	0.11	3.1	3.076	1.36	0.33
NGC 4736	0.44	0.62	0.05	2.1	1.294	0.30	2.92
NGC 6946	53.25	4.92	0.55	2.9	2.729	19.31	2.97
NGC 7793	13.50	3.35	0.12	1.7	0.511	26.18	5.44

Table 5.1: Numerical values inferred from the fits of the galaxy rotation curve's final formula Eq. (5.6) with the data points relative to the sample of the considered HSB spiral galaxies. Masses are measured in $10^{10}\,M_\odot$ and luminosities in $10^{10}\,L_\odot$. M_0 and r_c are the predicted values of mass and core radius respectively, i.e. the best-fit parameters of the relation which reproduces the data behaviour. $M_{gas} = 4/3 M_{HI}$ and L_B are the total amount of the gaseous thin disk's mass and the luminosity in the B-band reported in Ref. [125] which were computed by means of the data presented in [219], and are needed to compute the mass-to-light ratio. R_0 is the scale length of the galaxy. $\Upsilon_\star^B = M_\star/L_B$ is the mass-to-light ratio measured in $M_\odot/L_\odot$, with $M_\star$ the total stellar mass obtained from the numerical value M_0 of the fit. χ_r^2 is the reduced chi-squared. All numerical computations have been performed through a code developed with Mathematica.

Starobinsky model, which is

$$v^2(r) \approx \frac{GM(r)}{r}\left(1 + \frac{GM(r)}{c^2 r} - \frac{2\pi\kappa\gamma c^2 r^3 \rho^2}{M(r)(1+2\kappa\gamma c^2\rho)^2}\right), \tag{5.9}$$

with the same model of the matter profile in Eq. (5.5), γ is of ther order 10^{-11} and $\kappa = (8\pi G)/c^4$. In Ref. [125], from Eq. (5.9), the authors obtain the numerical results reported in Table 5.2.

Therefore, we find out a situation similar to that derived in the Palatini formalism because the numerical results are of the same order of magnitude. However, the metric formalism provides numerical outcomes lower than those provided by the Palatini formalism. Independently of the comparison between metric and Palatini Starobinsky model, we should expect a good correspondence between the results one gets in metric $f(R)$-theories and their corresponding version in metric-affine formalism, where the connection is also coupled to the matter fields. In this case, because in metric-affine approach a new scalar degree of

$\hat{R} = -\kappa T = c^2\kappa\rho$. Thus, the final conformal factor is $f'(\hat{R}) = 1 + 2\kappa c^2\gamma\rho$.

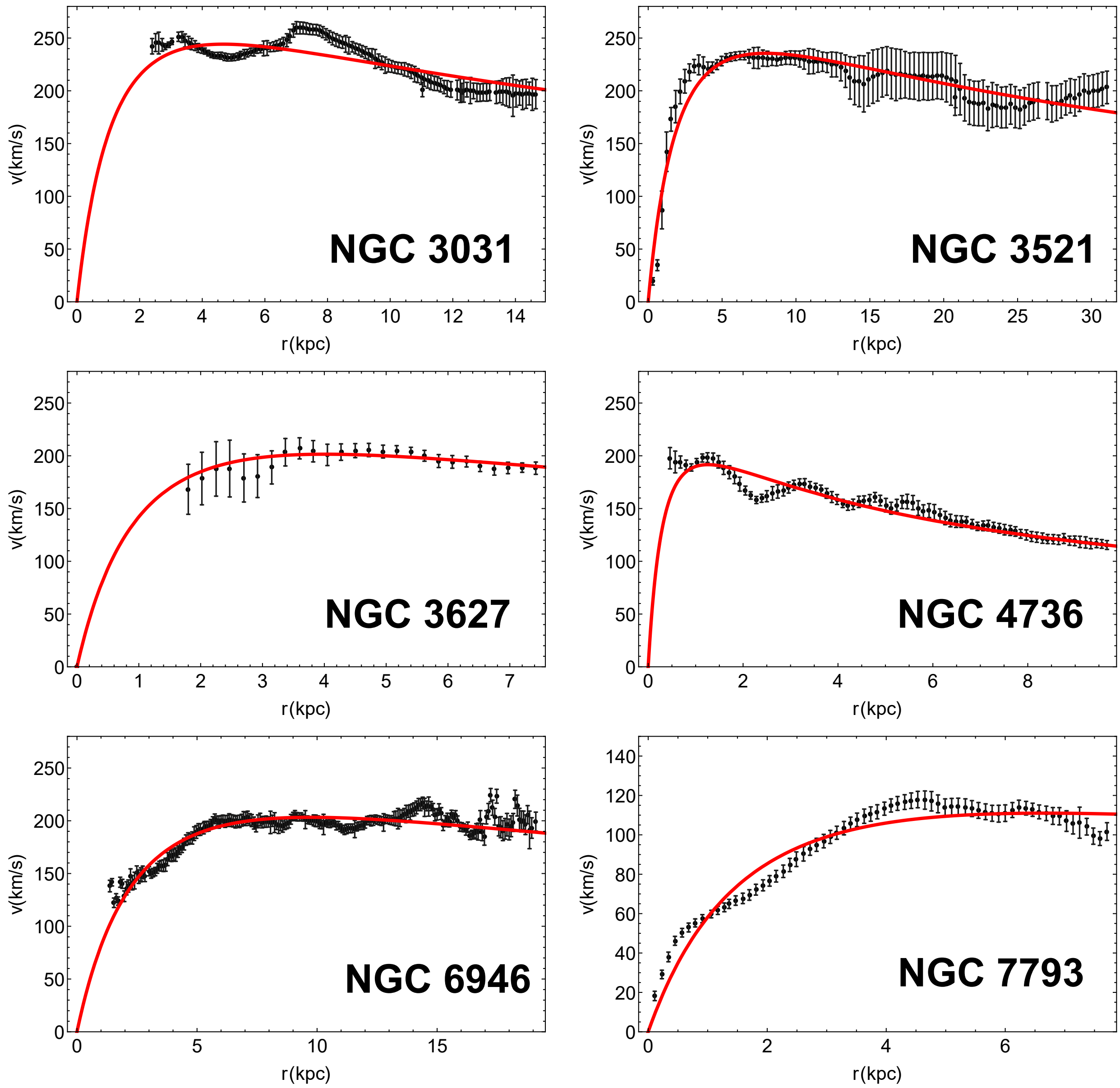


Figure 5.1: Plots of the galaxy rotation curves relative obtained by the fits with the data coming from the for each HSB spiral galaxy of the sample. The velocities are expressed in km/s as a function of the distance from the galactic centre r in kpc. Data points and are coloured in black and error bars in grey, while the continuous red line describes the behaviour of the galaxy rotation curves.

freedom appears (introduced by the connection) and the field equations can be recast in a form similar to the metric $f(R)$-theory, by virtue of a dynamical equivalence.

Galaxy	M_0	$r_c(kpc)$	M_{gas}	$R_0(kpc)$	L_B	$\Upsilon_\star^B$	χ_r^2
NGC 3031	14.86	2.10	0.48	2.6	3.049	4.71	4.88
NGC 3521	38.45	3.69	1.07	3.3	3.698	10.10	1.84
NGC 3627	8.68	2.25	0.11	3.1	3.076	2.78	0.45
NGC 4736	0.53	0.59	0.05	2.1	1.294	0.37	2.41
NGC 6946	78.19	5.09	0.55	2.9	2.729	28.44	2.18
NGC 7793	18.24	3.36	0.12	1.7	0.511	35.45	4.82

Table 5.2: Numerical values presented in Ref. [125] from the fits of the galaxy rotation curve formula in Palatini Starobinsky model with data points of the same sample of the HSB spiral galaxies in Table 1. Masses are measured in $10^{10}\,M_\odot$, luminosities in $10^{10}\,L_\odot$ and the mass-to-light ratios in $M_\odot/L_\odot$. The numerical values on M_0, r_c and $\Upsilon_\star^B$ can be compared with those in Table 1 inferred in metric Starobinsky model.

Results with free b-parameter of luminosity

In this section, we discuss the numerical outcomes of fits by relaxing the hypothesis on the values of the parameter b entering the mass profile (5.5) (it is associated to the brightness emitted by the galaxy). This is due to the fact that, in general, each galaxy has a diverse surface luminosity from another, thus motivating the enlargement of the spectrum of values. For HSB galaxies, the interval of the b-parameter in a neighbourhood of $b = 1$ can be larger or smaller depending on the type of theoretical curve that fits the observed data of a sample of galaxies. For instance, in the literature, an analysis of a different rotation curve is performed in [126]. Here, the authors found better results in the range $0.75 < b < 1.25$. In the case of our analysis referred to the Starobinsky $f(R)$-theory, better fits results emerge if the b-parameter can assume any value in the interval $0.50 < b < 1.50$. As a consequence, the fits of the theoretical curve (5.6) with the data points remarkably improve the predictions of the galactic properties. The results are reported in Table 5.3.

As we can see by a direct comparison with Table 5.1 (outcomes inferred with $b = 1$ in the mass model profile (5.5)), the new fits predict results that are much lower, and generally closer to the estimations [232; 235; 234; 233; 230] on the total mass and the mass-to-light ratios $\Upsilon_\star^B$ for this sample of spiral galaxies. In particular, for NGC 3031, NGC 3521, NGC 3627 and NGC 4736, we notice the results are in agreement with observational values [232; 233; 234; 230]. These results, determined in the framework of the Starobinsky model in metric formalism, enhance the validity of the Extended Theories of Gravity's approach.

Galaxy	M_0	$r_c\,(kpc)$	b	M_{gas}	$R_0\,(kpc)$	L_B	$\Upsilon_\star^B$	χ_r^2
NGC 3031	8.61	1.73	1.24	0.48	2.6	3.049	2.66	4.83
NGC 3521	8.98	1.98	1.39	1.07	3.3	3.698	2.14	1.42
NGC 3627	6.56	2.43	0.84	0.11	3.1	3.076	2.10	0.39
NGC 4736	2.47	1.82	0.51	0.05	2.1	1.294	1.87	2.10
NGC 6946	13.30	2.31	1.49	0.55	2.9	2.729	4.67	5.28
NGC 7793	6.49	2.33	1.18	0.12	1.7	0.511	12.47	7.66

Table 5.3: Numerical values inferred from the fits of the galaxy rotation curves formula Eq. (5.6) with b free to vary in the range $0.50 < b < 1.50$. M_0 and r_c are the predicted values of mass and core radius respectively. Masses are measured in $10^{10}\,M_\odot$ and luminosities in $10^{10}\,L_\odot$. $M_{gas} = 4/3 M_{HI}$ is the total gaseous thin disk's mass, R_0 the scale length of the galaxy, L_B the luminosity in the B-band reported in Ref. [125], computed by means of the [219] catalogue data. $\Upsilon_\star^B = M_\star/L_B$ is the mass-to-light ratio measured in $M_\odot/L_\odot$ and χ_r^2 the reduced chi-squared.

5.2.2 Results for STFOG

Results with fixed b-parameter of luminosity

Regarding the Scalar-Tensor-Fourth-Order Gravity, the numerical code has been found that the theoretical curves match the observed one only for $g(\xi,\eta) \simeq 1/3$ in Eq. (5.10), that is, predictions on the physical galactic parameters are possible only for $\xi \simeq 0$ and $\eta \simeq 1/2$ referring to the physical parameters in Eq. (3.127) linked to the massive modes of the Yukawa-like interaction. It corresponds to the minimum point for the function $g(\xi,\eta)$ respecting the condition $(\eta-1)^2 > \xi$ (see section (3.5.5)). This implies that a good agreement is obtainable only for a minimally coupled scalar field, i.e. $f_{R\phi} \simeq 0$, and imposes to reduce the analysis of the STFOG to the case of $f(R, R_{\alpha\beta}R^{\alpha\beta})$-gravity. Hence, the theoretical curve becomes

$$v(r) \approx \sqrt{\frac{GM_0}{r}}\left(\sqrt{\frac{R_0}{r_c}}\frac{r}{r+r_c}\right)^{3b/2}\left\{1+\frac{1}{3}e^{-r/\lambda_R}\left(1+\frac{r}{\lambda_R}\right)-\frac{4e^{-r/\lambda_Y}}{3}\left(1+\frac{r}{\lambda_Y}\right)\right\}^{1/2} . \quad (5.10)$$

and this one is used to fit the data points for our sample of spiral galaxies. We get a good reproduction of the observed curve, having $\lambda_R \simeq 50$ kpc and $\lambda_Y \simeq 0.05$ kpc as values that match the data. They correspond to masses $m_R \simeq 2\cdot10^{-2}$ and $m_Y \simeq 20$ kpc^{-1}.

In Fig. 5.2, plots the galaxy rotation curves are reported. They are obtained by the fitting the data points coming from the for HSB spiral galaxies of the sample. A good

Galaxy	M_0	$r_c\,(kpc)$	M_{gas}	$R_0\,(kpc)$	L_B	$\Upsilon_\star^B$	χ_r^2
NGC 3031	14.08	2.34	0.48	2.6	3.049	4.46	4.76
NGC 3521	35.02	4.00	1.07	3.3	3.698	9.18	1.84
NGC 3627	4.30	1.88	0.11	3.1	3.076	1.36	0.34
NGC 4736	0.45	0.63	0.05	2.1	1.294	0.31	2.93
NGC 6946	53.26	4.93	0.55	2.9	2.729	19.31	2.98
NGC 7793	13.50	3.35	0.12	1.7	0.511	26.18	5.59

Table 5.4: Numerical values inferred in Scalar-Tensor-Fourth-Order Gravity from the fits of the galaxy rotation curve's final formula Eq. (5.10) with the data points relative to the sample of the considered HSB spiral galaxies. Masses are measured in $10^{10}\,M_\odot$ and luminosities in $10^{10}\,L_\odot$. M_0 and r_c are the predicted values of mass and core radius, respectively, i.e. the best-fit parameters of the relation which reproduces the data behaviour. $M_{gas} = 4/3 M_{HI}$ and L_B are the total amount of the gaseous thin disk mass and the luminosity in the B-band reported in Ref. [125] which were computed by means of the data presented in [219], and are needed to compute the mass-to-light ratio. R_0 is the scale length of the galaxy. $\Upsilon_\star^B = M_\star/L_B$ is the mass-to-light ratio measured in $M_\odot/L_\odot$, with $M_\star$ the total stellar mass obtained from the numerical value M_0 of the fit. χ_r^2 is the reduced chi-squared. All numerical computations have been performed using a code developed with Mathematica.

agreement with the astronomical data is present. However, we find values for the total amount of mass M_0 significantly larger than the observed values for these galaxies, while for NGC 4736 smaller [230; 232; 233; 234; 235], which is a very similar situation to the $f(R)$-gravity (discussed in the preceding section). Also here, the mass-to-light ratios $\Upsilon_\star^B$ results to be much higher than the values expected by the stellar synthesis population models [232] and lower for NGC 4736 [230]. Nevertheless, as previously, we should remark that the model is just a good starting point for more complex models including other matter profiles in addition to the only one we assumed in (5.5).

Results with free b-parameter of luminosity

As in the preceding section, let us now relax again the hypothesis on the values of the parameter b of the mass profile (5.5) associated to the brightness emitted by the galaxy. This enables a first improvement on the galactic parameters. The fits predict results much

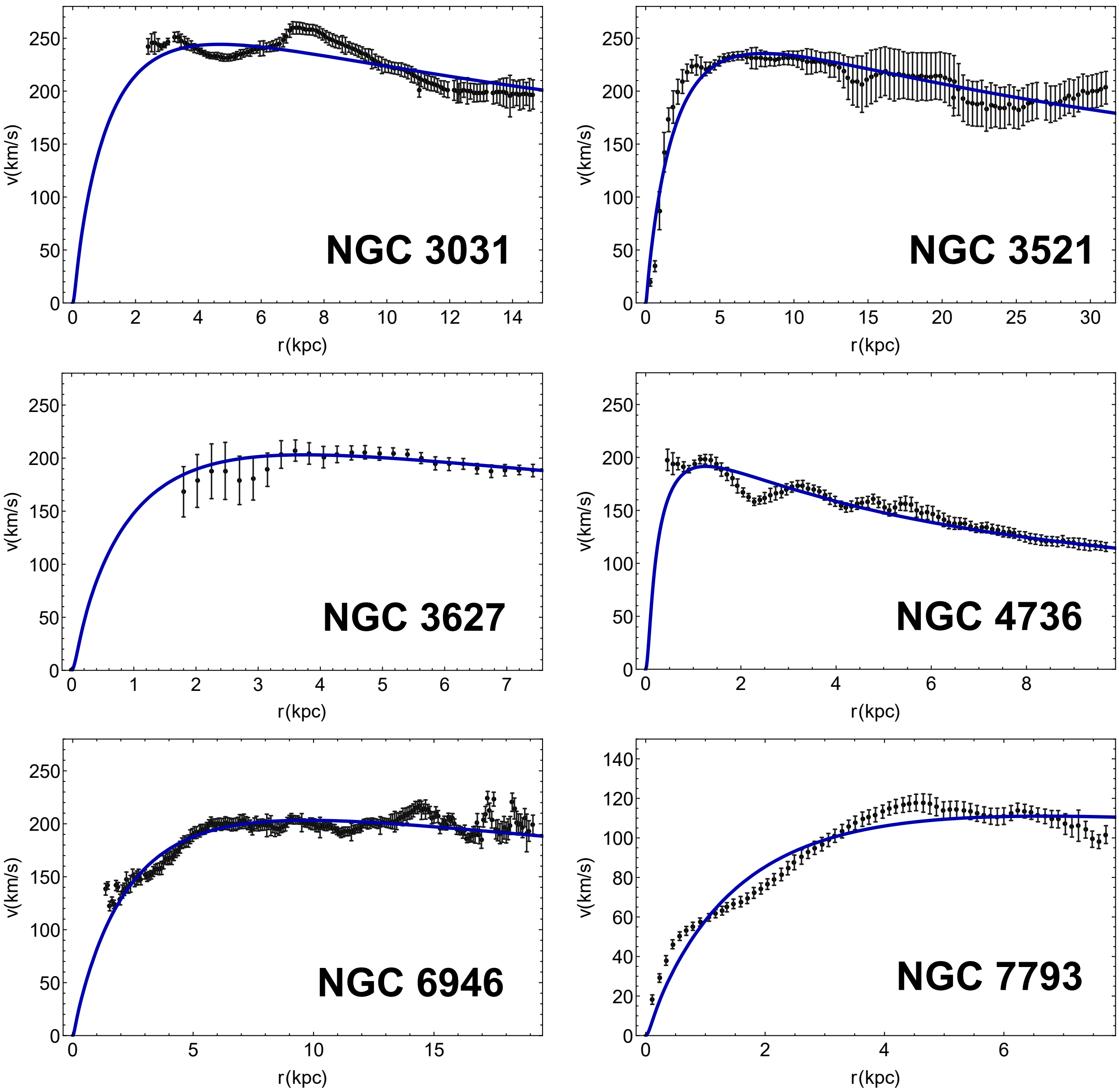


Figure 5.2: Plots of the galaxy rotation curves relative obtained by the fits with the data coming from the for each HSB spiral galaxy of the sample. The velocities are expressed in km/s as a function of the distance from the galactic centre r in kpc. Data points and are coloured in black and error bars in grey, while the continuous blue line describes the behaviour of the galaxy rotation curves.

lower and closer to the estimations [232; 235; 234; 233; 230] on the total mass and the mass-to-light ratios $\Upsilon_\star^B$. However, by a comparison with those in $f(R)$-gravity, we note a bit higher outcomes for NGC 3031 and NGC 3521 with respect to the observed estimations, and the same occurs for NGC 6946 and NGC 7793, while NGC 3627 and NGC 4736 are closer to the agreement with observational values [232; 233; 234; 230].

Galaxy	M_0	$r_c\,(kpc)$	b	M_{gas}	$R_0\,(kpc)$	L_B	$\Upsilon_\star^B$	χ_r^2
NGC 3031	13.93	2.33	1.00	0.48	2.6	3.049	4.41	4.76
NGC 3521	10.24	2.02	1.49	1.07	3.3	3.698	2.48	1.05
NGC 3627	4.30	2.43	1.00	0.11	3.1	3.076	1.36	0.33
NGC 4736	2.47	1.82	0.51	0.05	2.1	1.294	1.87	2.07
NGC 6946	18.73	2.74	1.41	0.55	2.9	2.729	6.66	4.50
NGC 7793	12.56	3.32	0.98	0.12	1.7	0.511	24.34	5.51

Table 5.5: Numerical values inferred in Scalar-Tensor-Fourth-Order Gravity from the fits of the galaxy rotation curves formula Eq. (5.10) with b free to vary in the range $0.50 < b < 1.50$. M_0 and r_c are the predicted values of mass and core radius respectively. Masses are measured in $10^{10}\,M_\odot$ and luminosities in $10^{10}\,L_\odot$. $M_{gas} = 4/3 M_{HI}$ is the total gaseous thin disk's mass, R_0 the scale length of the galaxy, L_B the luminosity in the B-band reported in Ref. [125], computed by means of the [219] catalogue data. $\Upsilon_\star^B = M_\star/L_B$ is the mass-to-light ratio measured in $M_\odot/L_\odot$ and χ_r^2 the reduced chi-squared.

5.2.3 Results for NCSG

Results with fixed b-parameter of luminosity

For what concerns the NonCommutative Spectral Gravity, we perform the analysis once more and find $\lambda_{NCSG} \simeq 1 \cdot 10^{-3}$ kpc as the length of interaction parameter's value that yields the best non-linear fits, corresponding to $\beta_{NCSG} \simeq 1 \cdot 10^3$ kpc^{-1}. In Table 5.6 are reported the numerical results for the parameters M_0 and r_c of the matter profile model (5.5) inferred by the fittings of the galaxy rotation curve in Eq. (5.8) with the data of the THINGS catalogue [219; 220; 221].

As we can see from Fig. 5.3, the galaxy rotation curves reproduced through the fitting process of the data, exhibit a good agreement with the astronomical data of the THINGS catalogue. Also for NonCommutative Spectral Gravity, we obtain values for the total amount of mass M_0 larger than the observed values for these galaxies, while for NGC 4736 smaller [230; 232; 233; 234; 235], analogously to the $f(R)$-gravity. The mass-to-light ratios $\Upsilon_\star^B$ results to be much higher than the values expected by the stellar synthesis population models [232] and lower for NGC 4736 [230]. This ultimately confirms that the idea of more complex models of matter profile added to the starting one (5.5) can be taken into consideration in order to improve the outcomes and therefore to shape a better description

Galaxy	M_0	$r_c\,(kpc)$	M_{gas}	$R_0\,(kpc)$	L_B	$\Upsilon_\star^B$	χ_r^2
NGC 3031	18.45	2.32	0.48	2.6	3.049	5.89	4.64
NGC 3521	45.13	4.00	1.07	3.3	3.698	11, 91	1.84
NGC 3627	5.71	1.87	0.11	3.1	3.076	1.82	0.32
NGC 4736	0.62	0.62	0.05	2.1	1.294	0.30	2.92
NGC 6946	69.13	4.88	0.55	2.9	2.729	25.13	2.93
NGC 7793	17.90	3.35	0.12	1.7	0.511	34.79	5.45

Table 5.6: Numerical values for NonCommutative Spectral Gravity inferred from the fits of the galaxy rotation curve's final formula Eq. (5.8) with the data points relative to the sample of the considered HSB spiral galaxies. Masses are measured in $10^{10}\,M_\odot$ and luminosities in $10^{10}\,L_\odot$. M_0 and r_c are the predicted values of mass and core radius respectively, i.e. the best-fit parameters of the relation which reproduces the data behaviour. $M_{gas} = 4/3 M_{HI}$ and L_B are the total amount of the gaseous thin disk's mass and the luminosity in the B-band reported in Ref. [125] which were computed by means of the data presented in [219], and they are needed to compute the mass-to-light ratio. R_0 is the scale length of the galaxy. $\Upsilon_\star^B = M_\star/L_B$ is the mass-to-light ratio measured in $M_\odot/L_\odot$, with $M_\star$ the total stellar mass obtained from the numerical value M_0 of the fit. χ_r^2 is the reduced chi-squared. All numerical computations have been performed through a code developed with Mathematica.

of the examined spiral galaxies, as well as an asyxsymmetric models.

Results with free b-parameter of luminosity

In this last step, we relax for the last time the hypothesis on the parameter b of the mass profile (5.5) associated to the surface brightness emitted by the galaxy. This enables a first improvement on the galactic parameters. Also here, the fits provide results much lower and closer to the estimations on the total mass and the mass-to-light ratios $\Upsilon_\star^B$ and we notice NGC 3031, NGC 3521, NGC 3627 and NGC 4736 are in good agreement or closer to the expected values. In particular, NGC 4736 is very close to the exact expected value. For NGC 6946, we obtain a mass higher than the observed one (but less distant compared to Table 5.6) and the same result goes for NGC 7793 as well [232; 235; 233; 234; 233; 230].

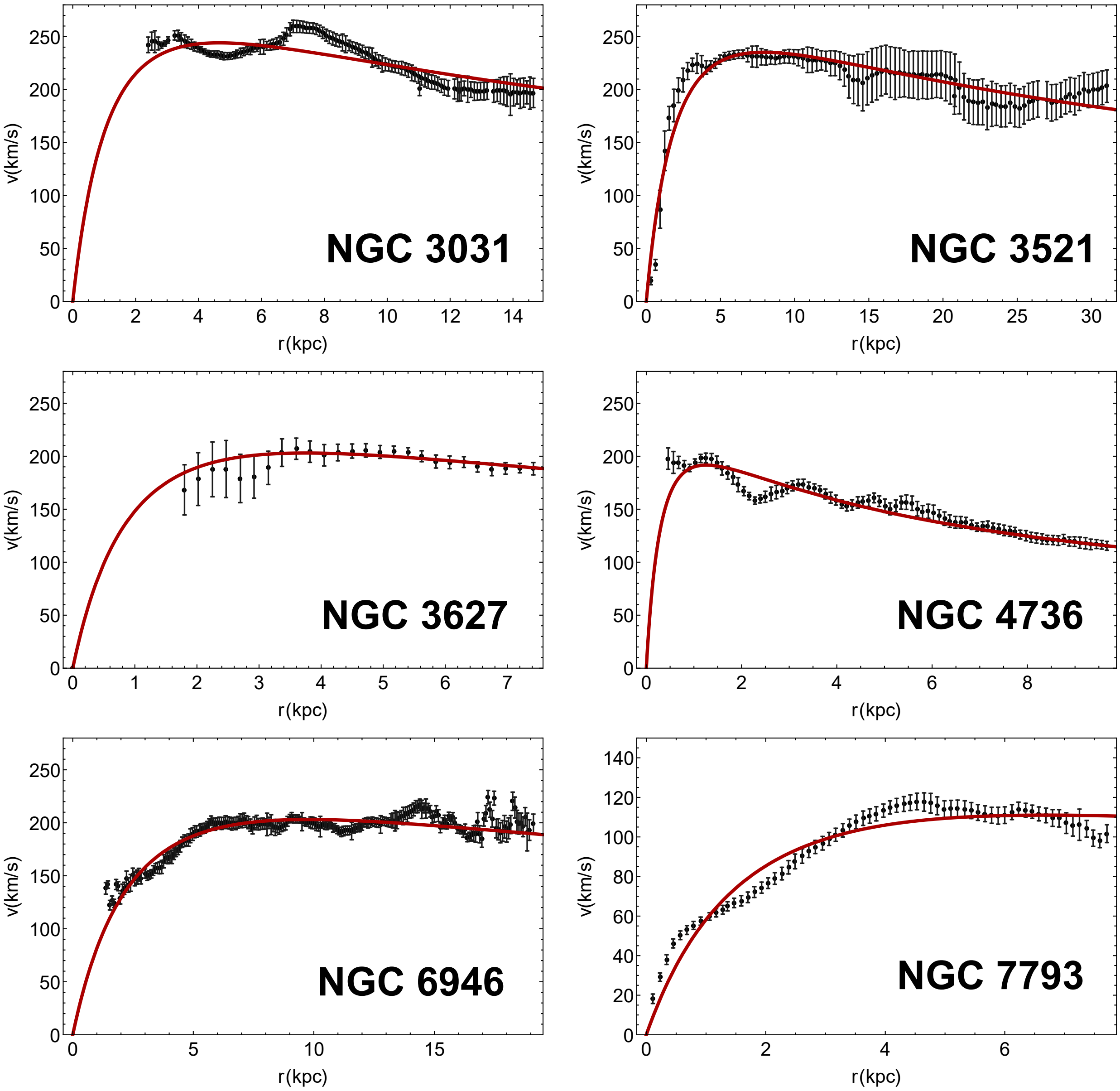


Figure 5.3: Plots of the galaxy rotation curves relative obtained by the fits with the data coming from the for each HSB spiral galaxy of the sample. The velocities are expressed in km/s as a function of the distance from the galactic centre r in kpc. Data points and are coloured in black and error bars in grey, while the continuous crimson line describes the behaviour of the galaxy rotation curves.

Galaxy	M_0	$r_c\,(kpc)$	b	M_{gas}	$R_0\,(kpc)$	L_B	$\Upsilon_\star^B$	χ_r^2
NGC 3031	8.22	1.42	0.94	0.48	2.6	3.049	2.54	4.91
NGC 3521	7.85	1.62	1.49	1.07	3.3	3.698	1.83	4.89
NGC 3627	4.59	1.64	1.10	0.11	3.1	3.076	1.46	0.33
NGC 4736	3.26	1.80	0.51	0.05	2.1	1.294	2.48	2.05
NGC 6946	17.80	2.26	1.49	0.55	2.9	2.729	6.32	4.96
NGC 7793	2.18	3.76	0.94	0.12	1.7	0.511	4.03	4.91

Table 5.7: Numerical values inferred in NonCommutative Spectral Gravity from the fits of the galaxy rotation curves formula Eq. (5.8) with b free to vary in the range $0.50 < b < 1.50$. M_0 and r_c are the predicted values of mass and core radius respectively. Masses are measured in $10^{10}\,M_\odot$ and luminosities in $10^{10}\,L_\odot$. $M_{gas} = 4/3 M_{HI}$ is the total gaseous thin disk's mass, R_0 the scale length of the galaxy, L_B the luminosity in the B-band reported in Ref. [125], computed by means of the [219] catalogue data. $\Upsilon_\star^B = M_\star/L_B$ is the mass-to-light ratio measured in $M_\odot/L_\odot$ and χ_r^2 the reduced chi-squared.

Chapter 6

Discussions and Conclusions

Dark matter and dark energy are the dominant components of the Universe. Dark matter is what produces the observed galactic rotation curves as a manifest anomaly of the galactic dynamics and also affects the dynamics of galaxy clusters. The role of dark energy becomes important on a cosmological scale and, on the isotropic and homogeneity assumptions of the Universe, it is considered the cause of the accelerated expansion of the universe. For what concerns the missing matter problem, several evidences seem to suggest the idea that dark matter is composed of invisible matter. The theoretical attempts following this approach have been conducted to suppose that the missing matter could be constituted by non-baryonic particles like the WIMPs (Weakly Interacting Massive Particles) and a good number of models of possible exotic particles have been taken into account. Unfortunately, until now the experimental projects have not been able to detect it directly. Besides this, tensions due to new discoveries and recent problems emerging at extra- and sub-galactic scales now afflict the ΛCDM-model as initial point of reference of the current Standard Model of Cosmology. In addition, the issues related to the research of a Quantum theory of Gravity as well as of a unifying theory of all interactions definitely conduct us in the direction of the necessity of deeper investigations, further tests and, in particular, to explore the most promising theoretical proposals as well as to work out new theories.

Among these possibilities, referring to the so-called sector of alternative theories of gravity, the geometrical paradigm of the Extended Theories of Gravity has become one of the most widespread over the years by virtue of several successes, both from a theoretical and phenomenological point of view. The reason consists in the fact that the class of ETG makes possible predictions, both at galactic and cosmological scales, in agreement with the observational surveys without implicating invisible matter. From a physical point of view, the effects associated to the dark ingredients of the Universe should be regarded just as a physical manifestation of extra-curvature terms of the geometry of the Universe. Despite this, the debate is still open because there are diverse viable models able to reproduce the majority of the observational evidence and there is no definitive answer. Furthermore, increasingly higher-precision theoretical and experimental tests as well as observational data are needed for decisive steps forward.

The starting point is always General Relativity and its well-tested results. As seen, the main interesting aspect of the ETG is that the theories are conceived as a curvature-based extension of GR, referring to the introduction of higher order scalar curvature invariants in the Einstein-Hilbert Lagrangian density, also motivated as quantum corrections, and leading to Higher Order field equations. Hereby it has been considered the case of Fourth Order Gravity in metric formalism, which relies on the Einstein Equivalence Principle, as a viable reference for the extension of GR. If one also considers the possible presence of a minimally or non-minimally coupled scalar field, as in the approach of the Scalar-Tensor theories, it becomes possible to give rise to a more general ETG, i.e. the Scalar-Tensor-Fourth-Order Gravity (STFOG).

In this thesis, we have presented and investigated the STFOG as a general class of ETG and its main sub-classes like the $f(R)$-, $f(R, \phi)$-, $f(R, R_{\alpha\beta}R^{\alpha\beta})$ theories, as well as its special case represented by the NonCommutative Spectral Gravity. This last one turned out to be the gravitational sector of the NonCommutative Spectral Geometry, the mathematical theory attempting to unify all interactions and proposing that the Standard Model's fields and gravity are packaged into geometry and matter on a Non-commutative space and geometry. Quintessence fields, motivated from M-theory/superstring-inspired models and related to the Dark Energy, have also been considered for an even wider treatment. Since the ETG belongs to the set of theories aiming to enlarge or introduce corrections to Einstein's theory, in accordance with diffeomorphism invariance and general covariance, it should address these physical requirements in order to be consistent with GR and reproduce all the well-established GR's outcomes at the scales of the Solar System and, generally, of binary/stellar systems. This is crucial for every (modified) theory. The ETG should respect the constraints dictated from GR and this leads to the further problem of the determination of the theoretical/experimental bounds of the theory, because one of the consequences is that the law of gravity is not scale invariant and, at the level of weak field and low particle velocities, the Yukawa-like interaction term appears besides the classical Newtonian one and its contributions become increasingly relevant as the scales of the self-gravitating system is larger. Furthermore, mathematical predictions and estimations involving a growing number of astrophysical scenarios and celestial objects of the Universe are an essential point to establish future ways of research and attempt to achieve new physics. One of the main physical scenarios for these analyses is represented by the weak field limit. All this is intertwined with the intriguing issues regarding the motion of celestial bodies constituting gravitating systems like the Solar System, the galaxies, or the dynamics of stellar cluster surrounding Sagittarius A* (the Black Hole at the centre of the Milky Way), and to the aims of this kind of analysis we have developed the arguments of the present work, and then summarise the main results.

Before this, in the first chapter we presented the main features on how it is possible to construct theories beyond General Relativity and then presented the Scalar-Tensor-Fourth-Order Gravity.

In the second chapter, starting from its Lagrangian density $\mathcal{L} = f(R, R_{\alpha\beta}R^{\alpha\beta}, \phi)$ and the relative action, it has been shown how to deduce of the field equations by applying the variational principle. By performing the variation of the action in metric formalism, the deduction was presented in a way that the field equations of the other theories represented by the $f(R)$-gravity, $f(R, \phi)$-gravity and $f(R, R_{\alpha\beta}R^{\alpha\beta})$-gravity, were automatically yielded as particular sub-classes of the general STFOG. Then, the special case of NonCommutative Spectral Gravity was briefly introduced by discussing some of the most substantial aspects, and so it has also been for the Quintessence Field.

In the third chapter, we have finally passed to the fundamental physical regime corresponding to a great number of real (self-)gravitating systems of the Universe: the weak field limit. By starting from the discrimination between strong and weak gravitational field, along with the physical hypothesis of weak field and low velocity compared to the speed of light, we presented an in-depth discussion concerning a distinction between the Weak Field limit, the Newtonian and Post-Newtonian limits, and how it naturally leads in a mathematical method following from these kind of expansions. This is an essential point because the determination of the relativistic equations of motion for a N-body system as the Solar System, involving the Post-Newtonian corrective potentials of the theory, needs the resolution of the field equations in the appropriate physical regime and therefore the space-time metric in which the dynamics of the particles occurs. This is necessary both for the geodesic and for the variational principle. By relying on the consistency of STFOG with General Relativity at Solar System scales, guaranteed by Chameleon's field screening mechanisms underlying the theory, we showed how the Weak Field limit could be sufficient for a complete description of the system. Since it allows the linearization of the field equations, this gives also the possibility to find analytical solutions for the fields $g_{\mu\nu}$. Then we proceeded with the calculations and an important simplification of the linearized equations by means of a suitable gauge transformation. For the physical configuration of a (isolated) system of N bodies, the best choice when we want to investigate the dynamics of interacting celestial objects in a relativistic framework is represented by the *Standard Post-Newtonian gauge*. Because it automatically provides the solution for the vector potential Z_i (in the cross-term metric g_{0i}) comprising all the Post-Newtonian potentials related to the temporal and spatial partial derivatives of the Superpotential X, relevant for a complete description of the dynamics of the interacting particles. Therefore, we have found the solutions of the field equation in such a gauge condition. i.e. the space-metric in isotropic coordinates.

From the resolution of the fourth order partial differential equations, we get that the g_{00} and g_{ij} components coincide with those in harmonic gauge present in the literature [109; 65; 110; 112] and the Yukawa-like corrections of the form $V(r) = GM/r(1 + \alpha e^{-\beta r})$ to the Newtonian gravitational interaction arise, but it differs in the g_{0i} term where the vector potential Z_i also contains the Post-Newtonian contributions connected to the temporal and spatial partial derivatives of the superpotential X (3.116),(3.113). Analogue case for the NonCommutative Spectral Gravity (3.136),(3.137). It has also been shown how to restore the harmonic gauge from Standard-Post-Newtonian gauge. At the end of this chapter, in the context of the STFOG as a representative class of ETG, we have finally determined the relativistic Lagrangian (3.199) for a system of N bodies moving of the geodesics of space-time curved by the presence of the other $N-1$ material points. Such a Lagrangian, inserted in the Euler-Lagrange equations, provides the equations of orbital motion for each body of the system and contains the Post-Newtonian contributions of the theory to the dynamics. A single body moves following the geodesics of the curved space-time generated by the distribution of the others.

This analytical result opens the possibilities for (high-precision) analysis in the context of the Relativistic Celestial Mechanics beyond General Relativity, as it allows one to study and reproduce diverse useful astrophysical models corresponding to real scenarios like the relativistic 3-body problem or the 2-body problem *in case of comparable masses*, as it happens for the binary system system J0737-3039 constituted by two neutron stars of similar mass. In this way, the problem of dealing with specific astrophysical configurations of the gravitating system is solved through the more general differential equations provided by Eqs. (3.199)-(3.203), without having to find the right set of equations for each theory. The achievement straightforwardly leads to the equations of motion for an N-body system for the other sub-classes as well as for the NonCommutative Spectral Gravity. In particular, they reduce to the case of the Einstein-Infeld-Hoffmann equations when GR is restored. In conclusion, it would also be feasible on these bases to build simulations of planetary and stellar systems able to reproduce the dynamics in ETG with the final aim of making a comparison of the final outcomes with other simulations and observational data.

In the fourth chapter, after an epistemological introduction to the importance of the anomalistic precessions in binary systems for the comprehension of new physics, we have studied the periastron advance of Solar System planets in the case in which the gravitational interactions between massive bodies are described by modified theories of gravity. In these models, the corrections to the Newtonian gravitational interaction are of the Yukawa-like form, $V(r) = GM/r(1+\alpha e^{-\beta r})$ (where GM/r is the Newtonian potential), or the power-law form, $V(r) = V_N + \alpha_q r^q$. To compute the corrections to the periastron advance, we have used the results of Ref. [151] in which the general formulas are provided in terms of the central body mass M, and the orbital parameters a and ϵ, the semi-major axis and eccentricity of the orbit, respectively. This two-body system constitutes a good model for many astrophysical scenarios, such as those at the scale of Solar System, constituted by the Sun and a planet, as well as binary system composed by a Super Massive Black Hole and an orbiting star, which are both the most suitable candidates to test a gravitational theory.

In the case of Scalar Tensor Fourth Order Gravity, we find that the parameters of the model are given by (see Eqs. (4.13, 4.12, 4.11) $\alpha \sim F_i$, $\beta \sim \beta_i$, with $i = \pm, Y$, $F_+ = g(\xi,\eta)\, F(m_+\mathcal{R})$, $F_- = \left[\frac{1}{3} - g(\xi,\eta)\right] F(m_-\mathcal{R})$, $F_Y = -\frac{4}{3}\, F(m_Y\mathcal{R})$, $\beta_\pm = m_R\sqrt{\omega_\pm}$, $\beta_Y = m_Y$. The highest value of β_i is $\beta_i \sim 5 \times 10^{-11} m^{-1}$, which leads to the constraint on F_i being $F_i < 0.28$. This allows us to obtain a bound on the massive modes m_i, $i = \pm, Y$, corresponding to the extra modes present in ETG.

In the case of Non-Commutative Spectral Gravity, we have found that the perihelion's shift of planets allows to constrain the parameter β at $\beta > (10^{-11} - 10^{-10}) m^{-1}$ (in this theory the parameter α is given and is of the order $\alpha \sim \mathcal{O}(1)$). This constraint on the parameter β improves several orders of magnitude derived by using the pulsar timing $\beta \geq 7.55 \times 10^{-13} m^{-1}$ [169; 170]. However, these constraints are weaker compared to those obtained from terrestrial experimental data, Eöt-Wash [171] and Irvine [172] experiments is [180], which gives $\beta \gtrsim 10^4 m^{-1}$ (a bound on β has been derived from the Gravity Probe B experiment, giving $\beta > 10^{-6} m^{-1}$ [160]).

We have also studied the Quintessence field surrounding a massive gravitational source. In this case, the parameters characterising the gravitational field are the adiabatic index ω_Q and the quintessence parameter c. The analysis shows that c assumes tiny values, as expected, essentially related to the cosmological constant, while $\omega_Q \gtrsim -(0.9 - 0.8)$, that is, it never assumes the value -1 corresponding to the pure cosmological constant.

The case of the S2 Star around Sagittarius A*, the Super Massive Black Hole at the centre of the Milky Way, has also been studied. In such a case we have found that for STFOG and NCSG $\beta > 10^{-13} m^{-1}$, a bound compatible with astrophysical constraints, while for the quintessence field we have inferred $\omega_Q \gtrsim -0.9$.

Then a resolution method for an effective analytical determination of the periastron advance has been developed, and it was founded on the generalization of the epicyclic perturbation technique, which was already working for the case of General Relativity and widely utilized in the study of stellar motions in galaxies. At the end of the process, we find an analytical formula through which it is possible to obtain the searched advance. It includes all the Post-Newtonian potentials to analyse the dynamics of test-particles subject to a central force field, or equivalently, orbiting in Post-Newtonian Schwarzschild space-time around a central source. By simply starting from very generic assumption of spherical symmetric metric, such a resolution method is valid and can be applied to analyze the restricted 2-body motion in theories beyond General Relativity and models within. Furthermore, it enables simple direct computations for the periastron advance. The results are analytical, and there is no need of numerical integration as it might happen in other approaches.

Therefore, the result has been applied to the Solar System and the S2 star and an analogue analysis for STFOG and NCSG has been performed again because terms of relativistic origin can affect the final result. Thus, we find improvements in the bounds as follows: The highest value of β_i is $\beta_i \simeq 3.61 \times 10^{-11} m^{-1}$, which leads to the constraint in F_i being $F_i < 0.29$. In the case of Non-Commutative Spectral Gravity, we have found that the perihelion's shift of planets allows us to constrain the parameter β at $\beta > (10^{-12}–10^{-10}) m^{-1}$. For the S2 Star around Sagittarius A*, we have found that for STFOG $\beta > 6.04 \times 10^{-14} m^{-1}$, and for NCSG we obtain $\beta > 1.62 \times 10^{-13} m^{-1}$, compatible with astrophysical constraints.

We point out once again that screening mechanism effects operating on Earth and Solar System scales could exist, but could not be effective on larger scales, such as the galactic and extra-galactic scales. Further observations over larger distances could provide limits on both screening mechanisms and higher derivative corrections, in particular on the effective gravitational model here discussed.

In the fifth chapter, we have studied the galaxy rotation curves in the framework of the Extended Theories on Gravity (in metric formalism). In particular, a first ever analysis in NonCommutative Spectral Gravity has been conducted. We reproduced the curves coming from astronomical data and inferred numerical predictions on the physical properties of an unexplored sample of six spiral galaxies such as the total baryonic mass M_0, the core radius r_c and mass-to-light ratios $\Upsilon_\star^B$. We also compare the outcomes with expected values on the basis of current stellar population synthesis models. We used the Starobinsky quadratic model as a working reference example of $f(R)$-theory. This work carried out for the $f(R)$-theory with metric formalism also enables a direct comparison with the outcomes obtained in Palatini's approach as well. We used the observed curves of the HI Nearby Galaxy Survey (THINGS) catalogue.

We determined the theoretical galaxy rotation curve function for the $f(R)$-theory, Scalar-Tensor-Fourth-Order Gravity, then for NonCommutative Spectral Gravity as a general class of ETG, then for NonCommutative Spectral Gravity. To this aim we considered a suitable model for the matter profile and provided the galaxy rotation curve formulas for each theory (5.6), (5.10), (5.8) and then, by means of numerical code (developed in Mathematica), we performed the nonlinear fitting process of the theoretical galaxy rotation curves with the observed ones of the sample. For what concerns the STFOG, in order to achieve a good match with the observed data points of the curves, we have found that the numerical code and the fitting process impose us to reduce to the $f(R, R_{\alpha\beta}R^{\alpha\beta})$-gravity ($g(\xi, \eta) \simeq 1/3$, assumed in $\xi \simeq 0$ and $\eta \simeq 1/2$), i.e. the case of minimally coupled scalar field. It is obtained $\lambda_R \simeq 50$ kpc ($m_R \simeq 2 \cdot 10^{-2}$ kpc^{-1}) as the length of interaction parameter's value that gives rise to the best non-linear fits, while for the STFOG one has $\lambda_R \simeq 50$ kpc and $\lambda_Y \simeq 0.05$ (masses $m_R \simeq 2 \cdot 10^{-2}$ and $m_Y \simeq 20$ kpc^{-1}), while for the NCSG it is $\lambda_{NCSG} \simeq 1 \cdot 10^{-3}$ kpc ($\beta_{NCSG} \simeq 1 \cdot 10^{3}$ kpc^{-1}). The theoretical velocity curves formula provides a satisfying reproduction of the observed curves of the THINGS catalogue (Fig. 5.1) even if, in all three cases, the numerical results of the total baryonic mass and mass-to-light ratios seem to be too large if compared with the estimated values of the stellar population synthesis model and with direct observational measures (Tables 5.1, 5.4, 5.6). However, this model is just a starting point and can be improved, for instance, by means of other models for the matter profile describing spiral galaxies or by considering an axis-symmetric coordinate system. It must be added that the corresponding Newtonian curve does not reproduce well the data.

The results, obtained in the context of the metric formalism and relative to the physical parameters M_0, r_c and $\Upsilon_\star^B$ are different but not far (for NGC 3031, NGC 3521, NGC 4736 close outcomes) from those developed in the context of the Palatini formalism [125]. The metric $f(R)$-gravity gives generally lower numerical values. Especially, if we consider fits where the b-parameter in the matter profile (5.5) can vary in the interval $0.50 < b < 1.50$, we obtain numerical results compatible or much closer to the expectations founded on stellar population synthesis models (see tables 5.3, 5.5, 5.7 and [232; 233; 234; 230]). In $f(R)$-gravity and NonCommutative Spectral Gravity, for NGC 3031, NGC 3521, NGC 3627 and NGC 4736 the results are in agreement with estimated values or close. For NGC 6946 we have got closer values with respect to the previous case, while for NGC 7793 the results on the total ordinary matter and its stellar dominant component are one order of magnitude higher than the expectations. In the case of Scalar-Tensor-Fourth-Order Gravity, we had to reduce to the $f(R, R_{\alpha\beta}R^{\alpha\beta})$-gravity as emerged by the fits. The numerical results are slightly higher than expected for NGC3031 and NGC3521, with values close to the observations for NGC3627 and NGC4736, while roughly double than expected for what regards NGC6946 and two orders of magnitude higher for NGC7793. Once again it must be remarked that more complex models including additional matter profiles can lead to improved results. In the end, in our analysis, the results in more agreement with the datasets and observational surveys are obtained for the $f(R)$-theory and NonCommutative Spectral Gravity.

We remark that, in the context of the ETG, we have not only reproduced the observed galaxy rotation curves but also drawn numerical predictions on physical properties (total baryonic mass, core radius and mass-to-light ratio) on the basis of astronomical data, and these values can be directly compared with those experimentally expected.

List of Papers

1. *Precession shift in curvature based Extended Theories of Gravity and Quintessence fields* - A. Capolupo, G. Lambiase, A. Tedesco, Eur. Phys. J. C - Theoretical Phyisics, 82:286, (2022);

2. *The periastron advance in curvature-based ETG and dark energy, Journal of Physics: Conference Series (Proceedings DICE 2022 Conference)*;

3. *Relativistic Equations of Orbital Motion for the N-body system in weak field limit of Extended Theories of Gravity* - A. Tedesco, A. Capolupo, G. Lambiase (submitted);

4. *A General Method for the Periastron Advance Determination beyond General Relativity and applications to ETG* - A. Tedesco, A. Capolupo, G. Lambiase (submitted);

5. *Galaxy rotation curves in general classes of Extended Gravity* - A. Tedesco, A. Capolupo, G. Lambiase (submitted);

6. *Einstein, Planck and Vera Rubin: relevant encounters between the Cosmological and the Quantum Worlds* - Frontiers in Physics (2021), Sec. Cosmology - Paolo Salucci, Giampiero Esposito, Gaetano Lambiase, Emmanuele Battista, Micol Benetti, Donato Bini, Lumen Boco, Gauri Sharma, Valerio Bozza, Luca Buoninfante, Antonio Capolupo, Salvatore Capozziello, Giovanni Covone, Rocco D'Agostino, Mariafelicia De Laurentis, Ivan De Martino, Giulia De Somma, Elisabetta Di Grezia, Chiara Di Paolo, Lorenzo Fatibene, Viviana Gammaldi, Andrea Geralico, Lorenzo Ingoglia, Andrea Lapi, Giuseppe G. Luciano, Leonardo Mastrototaro, Adele Naddeo, Lara Pantoni, Luciano Petruzziello, Ester Piedipalumbo, Silvia Pietroni, Aniello Quaranta, Paolo Rota, Giuseppe Sarracino, Francesco Sorge, Antonio Stabile, Cosimo Stornaiolo, Antonio Tedesco, Riccardo Valdarnini, Stefano Viaggiu, Andy A. V. Yunge.

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