Abstract

Several real-life problems as well as problems of theoretical importance within the field of Operations Research are combinatorial in nature. *Combinatorial Optimization* deals with decision-making problems defined on a discrete space. Out of a finite or countably infinite set of feasible solutions, one has to choose the best one according to an objective function. Many of these problems can be modeled on undirected or directed graphs. Some of the most important problems studied in this area include the *Minimum Spanning Tree Problem*, the *Traveling Salesman Problem*, the *Vehicle Routing Problem*, the *Matching Problem*, the *Maximum Flow Problem*. Some combinatorial optimization problems have been modeled on colored (labeled) graphs. The colors can be associated to the vertices as well as to the edges of the graph, depending on the problem. The *Minimum Labeling Spanning Tree Problem* and the *Minimum Labeling Hamiltonian Cycle Problem* are two examples of problems defined on edge-colored graphs.

Combinatorial optimization problems can be divided into two groups, according to their complexity. The problems that are *easy* to solve, i.e. problems polynomially solvable, and those that are *hard*, i.e. for which no polynomial time algorithm exists. Many of the well-known combinatorial optimization problems defined on graphs are hard problems in general. However, if we know more about the structure of the graph, the problems can become more tractable. In some cases, they can even be shown to be polynomial-time solvable. This particularly holds for trees.

In the last 80 years combinatorial optimization problems have been addressed through various modeling and algorithmic approaches, and

many papers proposing algorithms which solve them efficiently have been published. These algorithmic approaches to combinatorial optimization problems can be classified as either *exact* or *heuristic*. Exact techniques are applied in order to find optimal solutions to the problems. Because of the complexity of the problems, exact approaches are not able to prove the optimality for *large instances*. The concept of large instance is related to the problem under consideration. When an exact approach fails, heuristic techniques are usually applied. The aim of a heuristic is to produce good feasible solutions within a reasonable time. Heuristics can also be embedded within exact approaches, e.g. they may be used to generate good initial feasible solutions.

This dissertation is devoted to the study of three different problems of combinatorial optimization defined on graphs and belonging to the class of Spanning Tree and Cycle Cover Problems: the *Rainbow Cycle Cover Problem* (RCCP), the *Rainbow Spanning Forest Problem* (RSFP) and the *Minimum Branch Vertices Spanning Tree Problem* (MBVP).

Given a connected and undirected graph $G = (V, E, L)$ and a coloring function ℓ that assigns a color to each edge of G from the finite color set L, a cycle whose edges have all different colors is called a rainbow cycle. The *Rainbow Cycle Cover Problem* (RCCP) consists of finding the minimum number of disjoint rainbow cycles covering G. The RCCP on general graphs is known to be $\mathcal{NP}$-hard. In this thesis we model and solve the problem. We present an integer linear mathematical formulation and we describe some properties that a rainbow cycle cover must satisfy. Moreover we derive valid inequalities for the RCCP and we solve it by branch-and-cut. Computational results are reported on randomly generated instances.

Given a graph $G = (V, E, L)$ and a coloring function $\ell : E \rightarrow L$ that assigns a color to each edge of G from a finite color set L, the *Rainbow Spanning Forest Problem* (RSFP) consists of finding a spanning forest of G such that the number of rainbow components is minimum. A component of the forest whose edges have all different colors is called rainbow component. The RSFP on general graphs is known to be $\mathcal{NP}$-hard. In this work we prove that the problem is $\mathcal{NP}$-hard on trees and we provide a polynomial case. Moreover we propose two new integer mathematical formulations, (ILP1) and (ILP2), and for the second one we propose some valid inequalities. To solve large instances we present a greedy algorithm and a multi-start scheme applied to the greedy algorithm to improve its results. We show the computational results obtained by solving the ILP1, the greedy algorithm and the multi-start scheme.

Given a connected undirected graph $G = (V, E)$, the *Minimum Branch Vertices Spanning Tree Problem* (MBVP) asks for a spanning tree of G with the minimum number of vertices having degree greater than two in the tree. These are called *branch vertices*. This problem is known to be $\mathcal{NP}$-hard. We model the problem as an integer linear program, with undirected variables and we also investigate some properties. Moreover, we derive the dimension of the polyhedron of integer solutions as well as some valid inequalities and prove than some these are facet defining. We then develop a hybrid formulation containing undirected and directed variables. Both model are solved by branch-and-cut. Computational results show that the hybrid formulation is superior to the undirected formulation and that our branch-and-cut algorithm applied to it solves all benchmark instances to optimality.