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Fractional derivative of the Riemann zeta function

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To my parents,
Anna and Nicola

and to Camilla Ferlito

Declaration

I hereby declare that except where specific reference is made to the work of others, the contents of this dissertation are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university. This dissertation is my own work and contains nothing which is the outcome of work done in collaboration with others (except my supervisor).

Emanuel Guariglia
April 2017

Abstract

This thesis presents a non-conventional approach in analytic number theory. In particular, fractional calculus is used in order to compute the fractional derivative of the Riemann ζ function, which represents the starting point of this thesis. In particular, the convergence of the real and imaginary parts reveals that the half-plane of convergence depends on the fractional order of the derivative. In order to obtain the aforementioned computation the Ortigueira generalization of the Caputo derivative is used. It emphasizes that these results are a natural generalization of the integer order derivative of ζ. Some interesting properties of $\zeta^{(\alpha)}$ are also presented in order to show the chaotic decay to zero and a promising expression as a complex power series is provided.

The functional equation is given in Chapter 2 together with its simplified forms, in accordance with Apostol (1985) and Spira (1965). Since the Caputo-Ortigueira fractional derivative does not satisfy the generalized Leibniz rule, the Grünwald-Letnikov fractional derivative is introduced. By applying the previous derivative to the asymmetric functional equation of ζ, the functional equation is easily derived. Further properties of this equation are proposed and discussed.

Generalizations of the previous results are given using a Dirichlet series, the Hurwitz ζ function and the Lerch zeta function. Their fractional derivatives are computed together with the associated functional equations. In particular, the Lerch zeta function provides several new results in fractional functional analysis. By introducing Bernoulli numbers an integral representation of $\zeta^{(\alpha)}$ is provided. All the aforementioned results are in accordance with the classical theory of the Riemann ζ function.

In order to investigate the link between $\zeta^{(\alpha)}$ and the distribution of prime numbers, Euler products were recalled and the logarithmic fractional derivative of the Riemann ζ function was computed. The behavior of $\zeta^{(\alpha)}$ on the critical strip was studied by computing the α-order fractional derivative of the classical Dirichlet η function. The convergence half-plane of $\eta^{(\alpha)}$ is given by $\mathrm{Re}\, s > \alpha$, hence $\zeta^{(\alpha)}$ and $\eta^{(\alpha)}$ suggest the strip $\alpha < \mathrm{Re}\, s < 1+\alpha$ as a fractional counterpart of the classical critical strip. Moreover, two signal processing networks associated with $\eta^{(\alpha)}$ and its Fourier transform, respectively, are briefly described.

The symmetry revealed by the one-sided Fourier transform of $\eta^{(\alpha)}$ might be used in order to find a suitable application of $\zeta^{(\alpha)}$ in quantum mechanics.

The fractional derivative of the Riemann ζ function seems to have plenty of promising applications in pure and applied mathematics. In fact, by satisfying the Leibniz rule, $\zeta^{(\alpha)}$ might be generalized in a differential algebra. Several complex functions can be studied in a suitable function space in order to solve a given problem. One of the most famous examples is represented by the Hilbert spaces of entire functions. In particular, de Branges (1986, 1994) linked the Riemann hypothesis with a positivity condition on some particular Hilbert spaces. Despite all controversies around the papers of de Branges (Sabbagh, 2004), it appears to be clear from his work that the Riemann ζ function is strongly related to the Hilbert space theory. By taking into account the widespread interest that the fractional calculus has had in recent years, $\zeta^{(\alpha)}$ can bring interesting results in fractional Hilbert spaces and open new frontiers in research.

Outline

The thesis begins with some remarks on analytic number theory, the theory of zeta functions with their fundamental properties and fractional calculus in Chapter 1. In Chapter 2, the computation of ζ^{α} is given, together with its generalizations to the Hurwitz ζ function and to the Dirichlet series. The convergence of ζ^{α} is studied and its half-plane of convergence is determined. The functional equation of ζ^{α} is described in Chapter 3 and its simplified versions are reported. These results are generalized by using the Hurwitz ζ function and the Lerch zeta function. In Chapter 4, the link between ζ^{α} and the distribution of prime numbers is discussed. By introducing the Dirichlet η function, the fractional counterpart of the critical strip is given. Signal processing networks associated to η^{α} are presented in Chapter 5.

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Emanuel Guariglia
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List of Symbols

General notations

$\mathbb{N}$	set of natural numbers
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$
$\mathbb{P}$	set of prime numbers
$\mathbb{Z}$	set of integer numbers
$\mathbb{Z}_{>0}$	set of positive integers
$\mathbb{R}$	set of real numbers
$\mathbb{C}$	set of complex numbers
$\#A$	cardinality of the set A
(m,n)	greatest common divisor (gcd) of m and n
Li	Eulerian logarithmic integral
$f * g$	convolution product of f and g
$\lfloor x \rfloor$	integer part of the real number x
$\{x\}$	fractional part of the real number x
$\mathscr{C}^k(\Omega)$	functions of class $\mathscr{C}^k$ on Ω
$\mathcal{O}$	big O (Landau notation)
$\gtrsim$	approximately greater than
$f(x) \xrightarrow{x \to x_0} l$	alternative notation for $\lim\limits_{x \to x_0} f(x) = l$
$f \sim g$	asymptotic equivalence between two functions f and g
$\alpha^{\underline{n}}$	falling factorial

Analytic number theory

$d \mid n$, $d \nmid n$	divides, does not divide
μ	Möbius function
ϕ	Euler totient
$f * g$	Dirichlet convolution of f and g
id	identity function
d	divisor function
I	unit function
λ	Liouville function
Λ	von Mangoldt function
π	counting function
χ	Dirichlet character
L	Lerch zeta function (Dirichlet L-series)
x_a	abscissa of absolute convergence
x_c	abscissa of convergence
Γ	gamma function
ζ	Riemann (Hurwitz) ζ function
ξ	Riemann ξ function
$B_n(s)$	Bernoulli polynomials
B_n	Bernoulli numbers

Complex analysis

i	imaginary unit
$\operatorname{Re} z$, $\operatorname{Im} z$	real and imaginary parts of z
$\bar{z}$	complex conjugate of z
$\lvert z\rvert$	modulus of z
$\operatorname{Arg} z$	principal argument of z

$\widehat{f}$	Fourier transform
$\widehat{f}^{+}$	one-sided Fourier transform
$\mathcal{L}$	Laplace transform

Fractional calculus

Y_α	convolution kernel of order α
${}_a\mathrm{D}_t^{-\alpha}$	α-order fractional integral
${}_{\mathrm{RL}}\mathrm{D}^{\alpha}$	α-order Riemann-Liouville fractional derivative
${}_{\mathrm{C}}\mathrm{D}^{\alpha}$	α-order Caputo(-Ortigueira) fractional derivative
${}_{\mathrm{O}}\mathrm{D}^{\alpha}$	Ortigueira fractional derivative

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Chapter 1

Preliminary remarks

1.1 Introduction

The importance of the prime numbers is well known in the international mathematical community. Despite their central role in pure mathematics, in recent years the prime numbers have been widely applied in science as well as engineering. In particular, they have several applications in cryptography, quantum mechanics, biology, etc.

Prime numbers can be used in cryptography due to the extreme difficulty of certain computations like factoring and the discrete logarithm problem. Indeed, the Rivest-Shamir-Adleman cryptosystem and the Diffie-Hellman key exchange are based on them. In particular, the security information depends on the difficulty of finding higher prime numbers (Koblitz, 1994, Chap. 4). Quantum computation represents one of the prevalent computational paradigms for the 21^{st} century (Bennet and Brossard, 2014). In the quantum model, the encrypted data are transmitted via a public channel by using a secret key, a sequence of random bits carried by an invulnerable channel (the so-called quantum channel). Currently, this topic is receiving a lot of attention, since quantum computers are able to find the prime factors of large numbers (Jennewein et al., 2000), despite the presence of different problems associated with any quantum system with links to the classical world (often connected to the accuracy of the results of computation). In recent years, the distribution of prime numbers was widely used in biology. Hibbs (2010) has shown that prime numbers might have a fractal behavior. In his paper, for the first 500 prime numbers, the growth rate is related to the effects of a double-threaded physical model (with a 2-thread and a 4-thread). By taking into account the overlap of these impacts (that is, their corresponding cause and effect, which increment by multiples of 6), a double-helix structure is obtained. Removing the multiple of 6 growth gaps, the 2-4 growth thread happens in a linear form as an alternating sequence of elements from two sets (namely the operator Modulo 6 provides the common structure).

Furthermore, fractal dimension and lacunarity are used for the evaluation of skin lesions, in case of psoriasis. In particular, in molecular biology, some amino acid sequences in genetic matter show patterns of binary representations of prime numbers (Yan et al., 1991). Moreover, since Cattani (2010a,b) has shown that DNA sequences follow some fractal behavior (by observing both correlation matrix and DNA walks) and that the distribution of prime numbers presents hidden fractal shapes and symmetries, the link between DNA and prime numbers appears to be more than a simple hypothesis.

Chapter 1 contains six sections. In Section 1.2, the role of the prime numbers in physics is discussed. In Section 1.3 and 1.4, some general remarks on analytic number theory and the Riemann ζ function, respectively, are provided. Section 1.5 presents two main generalizations of ζ, that is the Dirichlet series and the Hurwitz ζ function. Fractional calculus is presented in Section 1.6 with particular emphasis on its generalizations to the complex plane.

1.2 Prime numbers in physics

Prime numbers have found several legitimate applications in physics. Some applications are industrial (applied mathematics, quantum cryptography, etc.) while others are used only in theoretical physics. In the following, just a few examples of these applications are given, however sufficient to show the many links the prime numbers present in theoretical and experimental physics.

A free Riemann gas (sometimes called a primon gas) is composed of non-interacting particles (primons) and it represents a model used to describe quantum phenomena. The partition function Z of the gas is described by the Riemann ζ function, since

$$Z(T) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \mathrm{e}^{-\beta E_n} = \sum_{n=1}^{\infty} \mathrm{e}^{-\frac{E_n}{K_B T}} = \sum_{n=1}^{\infty} \mathrm{e}^{-\frac{E_0 \log n}{K_B T}} = \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s)\,, \qquad \big(s = E_0/(k_B T)\big)$$

where the energy of each particles is given by $E_n = E_0 \log n$, k_B is the Boltzmann's constant, T is the absolute temperature of the system while $\beta = \dfrac{1}{K_B T}$ (called the thermodynamic beta). Since in $s = 1$, the Riemann ζ function has a simple pole with residue 1 (see section 1.4), there is a critical temperature $T_H = E_0/k_B$ (called Hagedorn temperature), above which the gas cannot be warmed up. A recent result claims that the mean energy density of a bosonic Riemann gas with randomness depends on the distribution of the Riemann zeros (Dueñas and Svaiter, 2015), that is, on the distribution of prime numbers. More specifically, the energy density of the system is expressed in terms of $\frac{\zeta'}{\zeta}$ that is strictly related to the non-trivial Riemann zeros (see section 1.4). In the computation of the average energy density the divergent contributions are avoided by applying an analytic regularization procedure.

Hilbert and Pólya suggested that the behavior of the Riemann ζ function on the critical line $\mathrm{Re}\, s = 1/2$ might depend on an (Hermitian) operator, where the Riemann zeros should be its eigenvalues. Nowadays this is called the Hilbert-Pólya conjecture and it is still unproven. The importance of this conjecture lies in the link established between the distribution of eigenvalues of a random matrix and the critical line. These eigenvalues were studied with several spectral techniques. The physical interpretation of the results of Odlyzko (1987) is that the critical zeros present a "long-range correlation", which has applications in chaos theory (Berry, 1987). Schumayer et al. (2008) have constructed potentials with energy eigenvalues equal to the prime numbers and to the Riemann zeros. Moreover, the multifractal nature of these potentials has been showed using the Rényi dimension. Borwein et al. (2000) have found a connection between the Riemann ζ function and quantum oscillators. Additionally, by using an experimental procedure, they provided the first seven critical zeros with a good approximation. Following this path, critical zeros can be approximated as the eigenvalues of an Hessenberg matrix, which again presents applications in chaos theory. In the 1940s, van der Pol showed his interest in a Fourier decomposition that was a special case of the following relation (Borwein et al., 2000):

$$\zeta(s) = s \int_{-\infty}^{\infty} \mathrm{e}^{-\sigma\omega} \left(\lfloor \mathrm{e}^{\omega} \rfloor - \mathrm{e}^{\omega} \right) e^{-i\omega t} \mathrm{d}t \, , \qquad (s = \sigma + it)$$

which holds for $0 < \sigma < 1$. He designed an electronic circuit to compute the transform above for $\sigma = 1/2$. It is still an open problem whether this circuit can represent a sort of Fourier fast transform to approximate the previous integral. Hence, the work of van der Pol suggests that a device that might reveal the prime numbers could be devised.

In recent years, several authors have begun to investigate the fractal structure of prime numbers. For instance, Ares and Castro (2004) tried to explain the hidden structure of prime numbers through both spin systems and Sierpinski gasket, while Selvam (2014) has discovered that the frequency of occurrence of prime numbers is related to the fractal fluctuations concomitant with inverse power law form for a power spectrum generic to dynamical systems. This study was done with the prime numbers in the first 10 million numbers. It is well known that Cantorian fractal space-time fluctuations are associated with quantum systems (Ord, 1983). The most important result of the work is that the prime number distribution shows quantum-like chaos, because the apparently chaotic fractal fluctuations of a dynamical system exhibit self-similar behavior. Van Zyl and Hutchinson (2003) have provided an interesting result by characterizing a quantum potential whose eigenvalues are prime numbers. Thereafter, they have computed that the fractal dimension of this potential is 1.8, which shows its irregular nature. Wolf (1997) has considered a signal where each component is the count of a prime number on some interval. By using the discrete Fourier transform, the power spectrum shows a $1/f^{\alpha}$ behavior with $\alpha \simeq 1.64$. The invariance of α from the length of the sampled intervals represents a clue of the self-similarity for the distribution of prime numbers. Recently, Cattani and Ciancio (2016) carried out an empiric procedure in order to show that the prime distribution is a quasi self-similar fractal (Batchko, 2014). In particular, by using the correlation matrix to evaluate the invariance of binary images, the authors have shown the fractal nature of the distribution of prime-indexed primes by two parameters, fractal dimension and lacunarity.

1.3 Preliminary remarks on analytic number theory

In this section, the main results and basic properties of analytic number theory are shortly summarized (Apostol, 1998). In particular, the main arithmetical functions will be recalled together with basic concepts of analytic number theory (prime number theorem, Riemann ζ function, etc.

1.3.1 Prime numbers and arithmetical functions

Gauss used to say that "mathematics is the queen of the sciences and number theory is the queen of mathematics". Thus 200 years ago, when number theory did not yet have applications for the real world, mathematicians had already understood its importance. The centrality of number theory in (pure and applied) mathematics is thus clear (Hardy, 1940, p. 33). The most important class of numbers is that of the primes, due to the following important result.

Theorem 1.3.1 (fundamental theorem of arithmetic)**.** *Any $n \in \mathbb{N}_{>1}$ can be uniquely written as a product of primes (up to ordering and unit factors).*

Proof: See Apostol (1998, pp. 17-18).

□

Hence, prime numbers in the set $\mathbb{N}$ play the role of atoms of the world of numbers. This type of numbers can be defined by the concept of divisibility (Apostol, 1998, Chap. 2).

Definition 1.3.2 (divisibility)**.** *Let a, b and c elements of $\mathbb{Z}$. The integer a is said to divide c, and denoted with the notation $a \mid c$, whenever $c = ab$. In the opposite case, in order to* *indicate that a does not divides c, the notation $a \nmid b$ is used. When $a \mid b$, a is also called divisor of b.*

Definition 1.3.3 (prime and composite number)**.** *Let $n \in \mathbb{N}_{>1}$ and let*

$$\mathrm{Div}(n) \stackrel{\text{def}}{=} \{k \in \mathbb{N} : k \mid n\} ,$$

be the set of all the divisor of n and $\#\mathrm{Div}(n)$ its cardinality, respectively. Any natural number $n > 1$ is called prime if $\#\mathrm{Div}(p) = 2$ (i.e. its only divisors are 1 and p). Each $n \in \mathbb{N}_{>1}$ is called composite if is not prime.

By using the above definition, the set of prime numbers, indicated with $\mathbb{P}$, is defined as usual by

$$\mathbb{P} \stackrel{def}{=} \{p \in \mathbb{N} : \#\mathrm{Div}(p) = 2\} .$$

Euclid showed that that $\#\mathbb{P} = \infty$, i.e. the set of the prime numbers is infinite, although their distribution within $\mathbb{N}$ is still an unsolved problem. This represents a significant mathematical challenge and an decided advantage in terms of information security (Koblitz, 1994, Chap. 4). Another important class of integer is that of squarefree numbers, that is, all integers which are not divisible by the square of any prime (Apostol, 1998, Chap. 1). Recalling that the counting function $\pi : \mathbb{R}^+ \to \mathbb{N}$ is defined by

$$\pi(x) = \#\mathbb{P}_x ,$$

where $\mathbb{P}_x \stackrel{\text{def}}{=} \{p \in \mathbb{P} : p \leq x\}$ and $\mathbb{P}_x \subseteq \mathbb{P}$, it appears to be clear that $\pi(x)$ provides the number of primes $\leq x$. By using the counting function, the following important result can be shown (Mollin, 2010).

Theorem 1.3.4 (prime number theorem)**.** *For $x \in \mathbb{R}^+$ it is*

$$\pi(x) \sim \frac{x}{\log x} . \tag{1.1}$$

Proof: See Mollin (2010, p. 221).

□

Eq. (1.1) is an asymptotic estimation, i.e.

$$\pi(x) \xrightarrow{x \to \infty} \frac{x}{\log x} ,$$

that is

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1 \ .$$

The Eulerian logarithmic integral is given (Abramowitz and Stegun, 2014) by

$$\mathrm{Li}(x) \stackrel{\text{def}}{=} \int_2^x \frac{\mathrm{d}t}{\log t} \ , \qquad (x \geq 2)$$

hence the prime number theorem becomes (Mollin, 2010)

$$\mathrm{Li}(x) \sim \pi(x) \ . \tag{1.2}$$

Eq. (1.2) represents the final version of the prime number theorem given by Gauss (1863). Below are some basic definitions which play a fundamental role in number theory.

Definition 1.3.5 (arithmetical function)**.** *A function $f : \mathbb{N} \to \mathbb{C}$ is called an arithmetical function (sometimes number-theoretic function). An arithmetical function f is called multiplicative if*

$$\begin{cases} f(1) \neq 0 \ , \\ f(mn) = f(m)f(n) \ \text{ whenever } \ (m,n) = 1 \ , \end{cases} \tag{1.3}$$

where (m,n) indicates the greatest common divisor of m and n (Apostol, 1998, p. 15)*. If condition* $(1.3)_2$ *holds $\forall m,n \in \mathbb{N}$, the multiplicative function f is called completely multiplicative.*

Some examples of arithmetic functions with an important role in analytic number theory are provided below (Apostol, 1998, Chap. 2).

Definition 1.3.6 (Möbius function)**.** *Let $n \in \mathbb{N}$. The Möbius function, denoted with μ, is an arithmetical function defined as follows*

$$\mu(n) \stackrel{\text{def}}{=} \begin{cases} 1 \ , & n = 1 \ , \\ (-1)^k \ , & n \text{ is squarefree} \ , \\ 0 \ , & \text{otherwise} \ . \end{cases} \tag{1.4}$$

In the previous definition k represents the number of distinct factors in its prime factorization.

Example 1.3.1. *According to* (1.4)*, $\mu(10) = (-1)^2 = 1$, $\mu(30) = (-1)^3 = -1$ and $\mu(20) = 0$ since $\mu(n) = 0$ taking into account the definition of squarefree numbers.*

The Möbius function appears in many different places in number theory. One of its fundamental properties is given below.

Proposition 1.3.7. *Let I be the unit function defined by*

$$I(n) \stackrel{\text{def}}{=} \left\lfloor \frac{1}{n} \right\rfloor = \begin{cases} 1\,, & n = 1\,, \\ 0\,, & n \in \mathbb{N}_{>1}\,. \end{cases}$$

For every $n \in \mathbb{N}$ it is

$$\sum_{d|n} \mu(d) = I(n)\,. \qquad \text{(for all } n\text{)}$$

Proof: See Apostol (1998, p. 25).

□

Definition 1.3.8 (Euler totient)**.** *Let $n \in \mathbb{N}$.The Euler totient, denoted with ϕ, is an arithmetical function defined as the number of positive integers not greater than n and coprime with n.*

Example 1.3.2. *The previous definition can be applied to easily compute the values of $\phi(n)$ for no large values of n. In fact, $\phi(5) = \#\big(\{1,2,3,4\}\big) = 4$, $\phi(6) = \#\big(\{1,5\}\big) = 2$ and $\phi(10) = \#\big(\{1,3,7,9\}\big) = 4$.*

The Euler totient is widely used in number theory, algebra and cryptography. Its main properties are reported below.

Theorem 1.3.9. *Let $n \in \mathbb{N}$ and $p \in \mathbb{P}$. The following assertions hold:*

1) $\phi(p) = p - 1\,,$

2) $\displaystyle\sum_{d|n} \phi(d) = n\,,$

3) $\displaystyle\phi(n) = \sum_{d|n} \mu(d)\frac{n}{d}\,,$

4) $\displaystyle\phi(n) = n\prod_{p|n}\left(1 - \frac{1}{p}\right)\,.$

Proof: See Apostol (1998, Chap. 2).

□

Definition 1.3.10 (Liouville function)**.** *The Liouville function, denoted with λ, is an arithmetical function defined by*

$$\lambda(n) \stackrel{\text{def}}{=} \begin{cases} 1\ , & n = 1\ , \\ (-1)^{a_1+\ldots+a_k}\ , & n = p_1^{a_1} \cdots p_k^{a_k}\ , \end{cases} \tag{1.5}$$

where $p_i \in \mathbb{P}$ and $a_i \in \mathbb{N}$ for $i = 1, \ldots, k$.

According to definition (1.5), the Liouville function is completely multiplicative. The divisor sum of λ is given in the proposition below.

Proposition 1.3.11. *For every $n \in \mathbb{N}$ it is*

$$\sum_{d|n} \lambda(d) = \begin{cases} 1\ , & n \text{ is a square}\ , \\ 0\ , & \text{otherwise}\ . \end{cases}$$

Proof : See Apostol (1998, p. 38).

□

The Liouville function is linked with the Riemann ζ function (see Example 1.5.10) by

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)}\ . \qquad (\operatorname{Re} s > 1) \tag{1.6}$$

Definition 1.3.12 (von Mangoldt function)**.** *Let $n \in \mathbb{N}$. The von Mangoldt function, defined with Λ, is an arithmetical function defined by*

$$\Lambda(n) = \begin{cases} \log p\ , & \text{if } n \text{ is a prime power}\ , \\ 0\ , & \text{otherwise}\ . \end{cases}$$

The von Mangoldt function plays a central role in the distribution of primes since it is strongly linked to the Riemann ζ function (Apostol, 1998, Chap. 12). Its main properties are given below.

Proposition 1.3.13. *For every $n \in \mathbb{N}$ it is*

1) $\log n = \sum_{d|n} \Lambda(d)\ ,$

2) $\Lambda(n) = \sum_{d|n} \mu(d) \log \frac{n}{d} = -\sum_{d|n} \mu(d) \log d\ .$

Proof: See Apostol (1998, p. 32–33).

□

Finally, consider another arithmetical function, the Dirichlet character, which plays a key role in the theory of modular forms (Apostol, 1997, Chap. 6).

Definition 1.3.14 (Dirichlet character)**.** *Let $k \in \mathbb{N}$. The Dirichlet character modulo k, denoted with χ, is a completely multiplicative function χ such that*

1) *χ is periodic of period k, that is $\chi(n+q) = \chi(n), \forall n \in \mathbb{N}$ for every $n \in \mathbb{N}$;*

2) *χ vanishes whenever $(n,k) > 1$, that is $\chi(n) \neq 0$ if and only if $(n,q) = 1$.*

The principal character (Apostol, 1998, p. 138), denoted with χ_1, is such that

$$\chi_1(n) = \begin{cases} 1\,, & (n,k) = 1\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

Example 1.3.3 (character modulo 1)**.** *The constant 1 clearly satisfies the conditions above with $k = 1$. Furthermore, since character modulo 1 must be periodic modulo 1 and equal to 1 in $n = 1$, it follows that $\chi = 1$ is the only Dirichlet character modulo 1. It represents also the principal character modulo 1.*

Example 1.3.4 (character modulo 2)**.** *The coprimality condition implies that a character modulo 2 is 0 for even integers, while the periodicity condition together with the requirement $\chi(1) = 1$ entails that $\chi(n)$ have to be equal to 1 for odd integers. Hence, there is only one character modulo 2, that is the principal character is given by*

$$\chi_1(n) = \begin{cases} 1\,, & (n,2) = 1\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

In the following section one of the most important mathematical functions (especially in analytic number theory), the Riemann ζ function, is introduced.

1.4 Riemann ζ function

Let s be a complex variable such that $s = x + iy$ with $x, y \in \mathbb{R}$. The Riemann ζ function is defined by

$$\zeta(s) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \frac{1}{n^s}\,, \tag{1.7}$$

and it converges for all complex numbers s such that $x > 1$ (Riemann, 1859). The Riemann ζ function has several integral representations, among which the most famous one is given by

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx\,, \qquad (x > 1) \tag{1.8}$$

where

$$\Gamma(t) \stackrel{\text{def}}{=} \int_0^{\infty} x^{t-1} e^{-x} dx\,, \qquad (t \in \mathbb{C}) \tag{1.9}$$

is the gamma function (Apostol, 1998) that converges if $\mathrm{Re}\, t > 0$ (Abramowitz and Stegun, 2014). The Riemann ζ function can be defined only for $x > 1$, otherwise the series in (1.7) would diverge. According to Riemann (1859) this function ζ owns a unique analytical continuation to the entire complex plane, excluding the points $s = 1$, where it presents a simple pole with residue 1. Riemann (1859) discovered the functional equation of ζ, given by

$$\zeta(s) = 2(2\pi)^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\, \zeta(1-s)\,, \qquad (\text{for all } s \in \mathbb{C}) \tag{1.10}$$

that is

$$\zeta(1-s) = 2(2\pi)^{-s} \cos\left(\frac{\pi s}{2}\right) \Gamma(s)\, \zeta(s)\,, \qquad (\text{for all } s \in \mathbb{C}) \tag{1.11}$$

by replacing s with $1-s$. In the literature, equations (1.10) and (1.11) are known as asymmetric forms of the functional equation for ζ since it can also be written in a symmetrical form (Apostol, 1998, Chap. 12) as follows

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right) \zeta(1-s)\,. \qquad (\text{for all } s \in \mathbb{C})\,. \tag{1.12}$$

Several proofs of (1.10) and (1.12) can be found in current literature (see for instance Apostol, 1998, Chap. 12). Introducing a variant of the Riemann ζ function, called Riemann ξ function and defined by

$$\xi(s) \stackrel{\text{def}}{=} \frac{1}{2} s\,(1-s)\, \pi^{-s/2}\Gamma\left(\frac{s}{2}\right) \zeta(s)\,,$$

eq. (1.12) becomes

$$\xi(s) = \xi(1-s) \ , \qquad \text{(for all } s \in \mathbb{C})$$

where ξ is an entire function of s since $\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$ has simple poles at $s=0$ and $s=1$.

Concerning its zeros, it is easy to show (Edwards, 1974) that $\zeta(s)=0$ at the negative even integers, that is $-2,-4-,6,\ldots$ (called trivial zeros). Since ζ is a multiplicative Dirichlet series, the classical theorem about the Euler product (see corollary 4.2.2) holds

$$\zeta(s) = \prod_{p\in\mathbb{P}} \frac{1}{1-p^{-s}} \ , \tag{1.13}$$

The previous classical result shows that there are no zeros in the half-plane $\mathrm{Re}\, s > 1$, so that the only non-trivial zeros must belong to the so-called critical strip $0 < \mathrm{Re}\, s < 1$ (Apostol, 1998). Being $\zeta(\bar{s}) = \overline{\zeta(s)}$, all the zeros of ζ are symmetrically distributed with respect to the real axis $\mathrm{Im}\, s = 0$. According to the Riemann conjecture, all its non-trivial zeros should be distributed along the (critical) line $\mathrm{Re}\, s = 1/2$. The first significant result about this conjecture is due to Hardy (1914), who showed that ζ has infinitely many non-trivial zeros on the critical line (Hardy and Littlewood, 1921).

1.4.1 Uniform convergence

In this subsection, two main results that widely applied in this thesis, are briefly discussed, namely the Weierstrass M-test and the uniform convergence of the Riemann ζ function.

Theorem 1.4.1 (Weieirstrass M-test)**.** *Let $\left(f_j\right)_{j\in\mathbb{N}}$ be a sequence of complex functions defined on a common complex domain D. Assume that there exists a sequence $\left(M_i\right)_{i\in\mathbb{N}}$ of non-negative constants such that the two following conditions*

$$\begin{cases} \left|f_j(z)\right| \leq M_j \ , & \text{(for all } z \in D \text{ and for all } i \geq 1) \\ \displaystyle\sum_{j=1}^{\infty} M_j < \infty \ , & \end{cases} \tag{1.14}$$

hold. Then the series $\sum_{j=1}^{\infty} f_j(z)$ converges uniformly on D.

Proof : See Mathews and Howell (2006, pp. 251–252).

□

Example 1.4.1. *The complex series of general term* $c_n = \dfrac{\sin(z/n)}{n^2+1}$ *converges uniformly on the unit disc* $D(0,1)$. *In fact,* for all $z \in D(0,1)$, *we get*

$$\left| \frac{\sin(z/n)}{n^2+1} \right| < \frac{1}{n^2} \, ,$$

hence from the Weierstrass M-test, taking $M_n = \dfrac{1}{n^2}$ *and being (proposition 3.2.2)*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2) = \frac{\pi^2}{6} \, ,$$

the uniform convergence of $\sum_{n=1}^{\infty} c_n$ *in* $D(0,1)$ *follows immediately.*

In particular, uniform convergence of the Riemann ζ function is a property that is widely used in the thesis since it represents the main hypothesis for interchanging limit, series, derivative and integral signs (see Walnut, 2002, Chap. 1).

Theorem 1.4.2 (uniform convergence of ζ)**.** *The Riemann* ζ *function converges uniformly in the half-plane* $\mathrm{Re}\, s > 1$.

Proof : It is sufficient to show that the series (1.7) converges uniformly in $\mathrm{Re}\, s \geq 1+\delta$, with $\delta > 0$. By using the Weierstrass M-test with $M_n = \dfrac{1}{n^{1+\delta}}$, since

$$\begin{cases} \left| \dfrac{1}{n^s} \right| \leq \dfrac{1}{n^{1+\delta}} \, , \\ \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1+\delta}} < \infty \, , \end{cases} \tag{1.15}$$

it follows that the uniform convergence of the Riemann ζ function in $\mathrm{Re}\, s > 1$, where the domain of uniform convergence follows directly by taking into account $(1.15)_1$.

$\square$

1.5 Generalizations of the Riemann ζ function

Let s be a complex variable such that $s = x + iy$ with $x, y \in \mathbb{R}$. Two generalizations of the Riemann ζ function are provided by both the Hurwitz ζ function and the Dirichlet series.

1.5.1 Hurwitz ζ function

The Hurwitz ζ function is defined by

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} , \qquad (s \in \mathbb{C}) \tag{1.16}$$

with Re$s > 1$ and $a \in \mathbb{R} : 0 < a \leq 1$ (Apostol, 1998, Chap. 12). Thus, $\zeta(s,1) = \zeta(s)$. Like the Riemann ζ function, the Hurwitz ζ function can also be extended analytically for all complex numbers $s \neq 1$; moreover in $s = 1$ it owns a simple pole with residue 1 (Apostol, 1998, Chap. 12). This function admits an integral representation (Abramowitz and Stegun, 2014), which generalizes (1.8), given by

$$\zeta(s,a) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{e^{-at}\, x^{s-1}}{1-e^{-x}} dx , \qquad (x > 1)$$

hence it can be easily expressed in terms of the Mellin transform. In the literature, it was first introduced in the problem of the analytic continuation of the Dirichlet L-function, i.e. a function defined by a Dirichlet L-series (see Sections 1.5.2). In fact, if χ is a Dirichlet character modulo k, it can be shown that (Apostol, 1998, p. 249)

$$L(s,\chi) = k^{-s} \sum_{r=1}^{k} \chi(r)\, \zeta\left(s, \frac{r}{k}\right) , \tag{1.17}$$

that provides a representation of L-functions as a linear combination of Hurwitz ζ functions. Hence, the properties of every L-function depends on those of its associated Hurwitz ζ function by using eq. (1.17).

The functional equation of the Hurwitz ζ function (sometimes called Rademacher's formula) states that given p and q such that $1 \leq p \leq q$ then for all $s \in \mathbb{C}$ (Apostol, 1998, Chap. 12)

$$\zeta\,(s,a) = 2\,(2\pi q)^{s-1}\, \Gamma(1-s) \sum_{m=1}^{q} \sin\left(\frac{\pi s}{2} + \frac{2\pi m p}{q}\right) \zeta\left(1-s, \frac{m}{q}\right) . \tag{1.18}$$

It follows that when $p = q = 1$ the sum in (1.18) reduces to only one term, obtaining eq. (1.11).

1.5.2 Dirichlet series

The series (1.7) is an example of the so-called Dirichlet series.

Definition 1.5.1 (Dirichlet series)**.** *A Dirichlet series is a series of the form* (Apostol, 1998, Chap. 11)

$$F(s) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \ , \qquad (s \in \mathbb{C}) \tag{1.19}$$

where f is an arithmetical function.

Thus, the positive term series associated with (1.19) is given by

$$\sum_{n=1}^{\infty} \left| \frac{f(n)}{n^s} \right| \ . \qquad (s \in \mathbb{C}) \tag{1.20}$$

Now some remarks on the convergence of complex series are briefly discussed (Mathews and Howell, 2006, Chap. 4).

Convergence of complex series

Definition 1.5.2 (pointwise convergence of complex series)**.** *Let $(z_n)_{n\in\mathbb{N}}$ be a complex sequence. A complex infinite series*

$$\sum_{n=1}^{\infty} z_n \ , \tag{1.21}$$

converges to S if it exists $S \in \mathbb{C}$ (called sum of series) for which

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} z_k \ , \tag{1.22}$$

where $(S_n)_{n\in\mathbb{N}}$ is called the sequence of partial sums and defined by

$$\begin{aligned}
&S_1 = z_1 \ , \\
&S_2 = z_1 + z_2 \ , \\
&\vdots \\
&S_n = z_1 + z_2 + \ldots + z_n \ , \\
&\vdots
\end{aligned}$$

The series (1.21) *is said to be absolutely convergent if the series of moduli $\sum_{n=1}^{\infty} |z_n|$ converges, while* (1.21) *is said conditionally convergent if it converges without absolutely converges.*

If the limit (1.22) *is either infinite or do not converge, the series* (1.21) *si said to be either divergent or oscillating, respectively.*

Many of the results concerning the real series theory carry over to the complex case, as the following statement shows.

Proposition 1.5.3. *Let* $(z_n)_{n\in\mathbb{N}}$ *be a complex sequence such that* $z_n = x_n + iy_n$ *and let* $S = \xi + i\eta$ *be a complex number. The complex series* (1.21) *converges to S if and only if*

$$\xi = \lim_{n\to\infty} x_n \quad \text{and} \quad \eta = \lim_{n\to\infty} y_n \ .$$

Proof : See Mathews and Howell (2006, p. 125–126).

□

For a function f defined on a complex domain D, the sequence of functions $(f_n)_{n\in\mathbb{N}}$ at the point $z_0 \in D$ if $\lim_{n\to\infty} f_n(z_0) = f(z_0)$. Hence, for a particular point z_0, it is well know that there exists a positive integer n_{ε,z_0} (that is, depending on both ε and z_0) such that

$$n \geq n_{\varepsilon,z_0} \Rightarrow \left|f_n(z_0) - f(z_0)\right| < \varepsilon \quad (\text{for all } z \in D) \ . \tag{1.23}$$

Thus, for a given value of ε, the integer n_{ε,z_0} has to satisfy (1.23). This is not a case of the uniform convergence. In fact, for a uniformly convergent series, the integer n_{ε,z_0} in (1.23) is substituted by n_ε, that is, it depends only on ε. Hence, the pointwise convergence provides a weaker notion so that the uniform convergence (Mathews and Howell, 2006, Chap. 7) plays a fundamental role in the complex analysis.

Definition 1.5.4 (uniform convergence)**.** *Let* $(f_n)_{n\in\mathbb{N}}$ *be a sequence of complex functions defined on a common complex domain D. The previous sequence converges uniformly to f on D if*

$$\forall\varepsilon > 0 \ \exists n_\varepsilon \in \mathbb{N} : n \geq n_\varepsilon \Rightarrow \left|f_n(z) - f(z)\right| < \varepsilon \ . \qquad (\text{for all } z \in D) \tag{1.24}$$

Moreover, it appears to be clear that if $f_n = S_n$ is the nth partial sum of the series

$$\sum_{k=0}^{\infty} c_k (z-a)^k \ , \qquad (a \in \mathbb{C}) \tag{1.25}$$

the previous definition can be used here and the series (1.25) is said to be uniformly convergent to f on D.

Example 1.5.1. *Let* $f_n(z) = \mathrm{e}^z + \frac{1}{n}$ *be the general term of a complex sequence chosen. It converges uniformly to the complex exponential* $f(z) = \mathrm{e}^z$ *on the entire complex plane,* *since for any $\varepsilon > 0$, the condition* (1.24) *is true for all $z \in \mathbb{C}$ and for all $n \geq n_\varepsilon$, where $n_\varepsilon \in \mathbb{N} : n_\varepsilon > \frac{1}{\varepsilon}$ (Mathews and Howell, 2006, p. 250).*

Convergence of Dirichlet series

Dirichlet series enjoy a large number of useful properties given by the following theorems (see Apostol, 1998, Chap. 11). First, consider the absolute convergence of a Dirichlet series.

Theorem 1.5.5 (absolute convergence)**.** *Suppose the series* (1.20) *does not converge everywhere or diverge everywhere. Under these hypotheses, a real number exists, called abscissa of absolute convergence and denoted by x_a, such that the series* (1.19) *converges absolutely in the half-plane $x > x_a$ and does not converge absolutely in $x < x_a$.*

Proof : See Apostol (1998, p. 225).

□

Hence, its domain of absolute convergence can given by the empty set, $\mathbb{R}$, a half-infinite interval of the form $[x_a, +\infty[$ and an half-infinite interval of the form $]x_a, +\infty[$. In all these cases (Apostol, 1998, Chap. 11), there is a unique $x_a \in [-\infty, +\infty]$ such that

1) series (1.19) converges absolutely $\forall x > x_a$,

2) series (1.19) does not converge absolutely $\forall x < x_a$.

The (unique) number x_a is called abscissa of absolute convergence associated with (1.19). Clearly, x_a presents an infinite value in two cases, i.e. if (1.19) converges absolutely everywhere (assume $x_a = -\infty$) and if it converges absolutely nowhere (assume $x_a = +\infty$) (Apostol, 1998, Chap. 11). The following important result holds, which represents the specular counterpart of theorem 1.5.5 for the usual definition of convergence.

Theorem 1.5.6 (convergence)**.** *Let* (1.19) *be a Dirichlet series not converge everywhere or diverge everywhere. Then there exists a real number, called abscissa of convergence and denoted by x_c, such that it converges in the half-plane $x > x_c$ and does not converge in $x < x_c$.*

Proof : See Apostol (1998, p. 233).

□

In accordance with theorem 1.5.5, if series (1.19) converges everywhere, $x_c = -\infty$, while if it converges nowhere, $x_c = +\infty$ (Apostol, 1998, Chap. 11). Since absolute convergence implies convergence, it is always $x_a \geq x_c$. When $x_a > x_c$, there is an infinite strip $x_c < x < x_a$ where the series (1.19) converges conditionally. It can be proved (see Apostol, 1998, p. 233–234) that $0 \leq x_a - x_c \leq 1$, that is the width of the strip $x_c < x < x_a$ does not exceed 1. The next definition provides a fundamental concept in analytic number theory that gives the possibility to multiply arithmetical functions (Apostol, 1998, p. 29–30).

Definition 1.5.7 (Dirichlet convolution)**.** *Given two arithmetical functions f and g, their Dirichlet convolution (sometimes called Dirichlet product), indicated with $f * g$, is defined as the arithmetical function h such that*

$$(f * g)(n) \stackrel{\text{def}}{=} \sum_{d|n} f(n) g\left(\frac{n}{d}\right) , \tag{1.26}$$

where the sum is extended over all positive divisors d of n.

Example 1.5.2. *Let $f(n) = 1$ and $g(n) = n^\alpha$. Their Dirichlet convolution is given by*

$$(f * g)(n) = 1 * n^\alpha = \sum_{d|n} d^\alpha = \sigma_\alpha(n) ,$$

where σ_α is the divisor function,that is, the sum of the αth powers of the divisors of n (Apostol, 1998, p. 38).

Example 1.5.3. *The Dirichlet series associated with the unit function I is given by*

$$\sum_{n=1}^{\infty} \frac{I(n)}{n^s} = 1 + \sum_{n=2}^{\infty} \frac{0}{n^s} = 1 .$$

Since

$$f * I = I * f = f , \qquad \text{(for all arithmetic functions } f)$$

I represents the unit element for the Dirichlet convolution.

It can easily be shown that the arithmetic functions are a commutative ring, the Dirichlet ring, where sum and product are defined by $(f+g)(n) = f(n) + g(n)$ and (1.26), respectively (Apostol, 1998, Chap. 2). Moreover, for any couple of arithmetical functions, their product can be expressed by the Dirichlet product as claimed in the following statement.

Theorem 1.5.8. *Let f and g be two arithmetical functions that satisfy* (1.19)*,*

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \quad \text{and} \quad G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s} ,$$

and let F and G be absolutely convergent for $x > x_a$ and $x > x_b$, respectively. Under these hypotheses, for all $x > \max\{x_a, x_b\}$ *(that is, in the half-plane where both series converge absolutely), it is*

$$F(s)G(s) = \sum_{n=1}^{\infty} \frac{(f*g)(n)}{n^s} \ .$$

Proof: See Apostol (1998, p. 228).

□

The next statement links the Dirichlet series and uniform convergence.

Proposition 1.5.9. *A Dirichlet series* (1.19) *converges uniformly on compact subsets strictly contained in the half-plane $x > x_c$.*

Proof: See Apostol (1998, p. 235).

□

The convergence properties of Dirichlet series can easily be compared with those of power series. In fact, for a (general) power series

$$\sum_{n=0}^{\infty} a_n s^n \ , \qquad (s \in \mathbb{C})$$

a radius of convergence R can be defined such that the series converges in $|s| < R$ and diverges in $|s| > R$. It follows that each power series has a its own disk of convergence, whereas each Dirichlet series owns a half-plane of convergence. Moreover, a power series represents an analytical function inside its disk of convergence. A similar result for the Dirichlet series, derived as a consequence of theorem 1.5.9, is reported below (Apostol, 1998, p. 236).

Corollary 1.5.10. *The function $F(s)$, defined by* (1.19)*, is a analytic in its half-plane of convergence $x > x_c$. Moreover, in this half-plane of convergence, by differentiating term by term, its derivative is given by*

$$F'(s) = -\sum_{n=1}^{\infty} \frac{f(n)\log n}{n^s} \ . \qquad (x > x_c)$$

Proof: The second part of the theorem will be shown in Example 1.5.6. In order to show the first part, let $F_N(s) = \sum_{n=1}^{N} \frac{f(n)}{n^s}$ denote the partial sums of $F(s)$. Since each term of $f(n)n^{-s} = f(n)e^{-s\log n}$ is the image of an entire function, the functions F_N are also entire. From theorem 1.5.9, it follows that

$$F_N(s) \xrightarrow{N\to+\infty} F(s) \ ,$$

uniformly on compact subsets of the half-plane $x > x_c$. By using the Weierstrass theorem on uniformly convergent sequences of analytic functions (Apostol, 1998, p. 234–235), it becomes clear that F is analytic in every compact subset of $x > x_c$ and thus in the entire half-plane of convergence.

$\square$

The corollary 1.5.10 represents one of the most important properties in the general theory of Dirichlet series. Another consequence of theorem 1.5.9 is that, due to the uniform convergence on compact subsets of the half-plane of convergence, a Dirichlet series can differentiate and integrated term by term. Dirichlet series can be used to obtain approximations of the values of the Riemann ζ function inside the critical strip. Furthermore, the coefficients of these series present several number-theoretical properties (Beliakov and Matiyasevich, 2014).

1.5.3 Examples of Dirichlet series

Previous theorems can be used to evaluate the Dirichlet series of many familiar arithmetic functions, as is shown by the following examples (Apostol, 1998, Chap. 11).

Example 1.5.4 (Riemann ζ function)**.** *It is surely the famous Dirichlet series. From* (1.7) *it is immediate to see that $x_a = x_c = 1$.*

Example 1.5.5. (Dirichlet L series) *It is defined by*

$$L(s,\chi) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \frac{\chi^n}{n^s} , \qquad (s \in \mathbb{C}) , \tag{1.27}$$

in which χ represents the Dirichlet character. If f is bounded, namely $\left|f(n)\right| \leq M$ for all *$n \geq 1$, it follows that $\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ converges absolutely for $x > 1$, thus $x_a = 1$. Thus, if χ is a Dirichlet character the L-series* (1.27) *has $x_a = 1$* (Apostol, 1998, p. 225).

Example 1.5.6 (logarithm)**.** *The first derivative of* (1.19) *is given by*

$$F'(s) = -\sum_{n=1}^{\infty} \log n \, \frac{f(n)}{n^s} .$$

In fact, the partial sums $F_N(s) = \sum_{n=1}^{N} \frac{f(n)}{n^s}$ can be differentiated term by term with derivative $F_N'(s) = -\sum_{n=1}^{N} \log n \, \frac{f(n)}{n^s}$. By using the Weierstrass M-test, we get

$$F_N'(s) \xrightarrow{N\to+\infty} F'(s) ,$$

in the half-plane $x > x_c$ (see theorem 1.5.6). The main consequence of this result is that, for $f(n) = 1$, the function log *has Dirichlet series equal to $-\zeta'$.*

Example 1.5.7. (Möbius function) *Here This is a Dirichlet series with $f = \mu$ that converges absolutely in $x > 1$ like the Riemann ζ function. Using theorem 1.5.8, it follows that*

$$\zeta(s) \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = 1 \,, \qquad (x > 1) \tag{1.28}$$

being their Dirichlet convolution given by $\left\lfloor \frac{1}{n} \right\rfloor$. *From* (1.28) *it can be derived that $\zeta(s) \neq 0$ for $x > 1$ and*

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)} \,. \qquad (x > 1)$$

Example 1.5.8. (identity function) *The Dirichlet series associated is given by*

$$\sum_{n=1}^{\infty} \frac{\mathrm{id}(n)}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^{s-1}} = \zeta(s-1) \,.$$

Hence, it converges absolutely in $x > 2$.

Example 1.5.9. (Euler totient) *Since $\phi = id * \mu$, where $*$ is the Dirichlet convolution, from Examples 1.5.7 and 1.5.8 it follows that*

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)} \,,$$

by using theorem 1.5.8. Hence, it converges absolutely for $x > 2$.

Example 1.5.10. (Liouville function) *Let $f = 1$ and $g = \lambda$. It is*

$$(f * g)(n) = \sum_{d|n} \lambda(n) = \begin{cases} 1, & n = k^2 \text{ for some } k \,, \\ 0, & \text{otherwise} \,. \end{cases} \tag{1.29}$$

hence from theorem 1.5.8, it follows that

$$\zeta(s) \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \sum_{k=1}^{\infty} \frac{1}{k^{2s}} = \zeta(2s) \,.$$

Therefore

$$\frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)} \,. \qquad (x > 1)$$

Example 1.5.11 (divisor function)**.** *Since $d = 1 * 1$ it is*

$$\sum_{n=1}^{\infty} \frac{d(n)}{n^s} = \zeta^2(s) ,$$

by using theorem 1.5.8. Therefore, it converges absolutely for $x > 1$.

Example 1.5.12. (von Mangoldt function) $(1.3.13)_1$ *shows that* $\Lambda * 1 = log$. *Since the Dirichlet series of the function log is given by* $-\zeta'$ *(see Example 1.5.6), it follows that*

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \zeta(s) = -\zeta'(s) ,$$

so that

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = -\frac{\zeta'(s)}{\zeta(s)} , \tag{1.30}$$

where all the series involved converge absolutely for $x > 1$. *The relation* (1.30) *has a crucial role in the (analytic) proof of the prime number theorem. Moreover, since each zero of the Riemann* ζ *function represents a singularity of* (1.30)*, the link between the location of* ζ*-zeros and the distribution of prime numbers appears evident. Moreover, it shows the influence of the location of zeta zeros on the distribution of prime numbers* (see Apostol, 1998, Chaps. 12-13).

1.6 Preliminary remarks on fractional calculus

Fractional calculus has been developed and several applications have emerged in many areas of scientific knowledge since 1974, after the first congress at the University of New Haven (de Oliveira et al., 2014). In the past, Leibniz and Newton had been the first to study problems of fractional calculus, so it is at least as old as the traditional differential calculus. The main problem lies in developing a theory similar to differential calculus, in which the exponent for all its operators (as, for instance, derivative operator, integral operator, etc.) is not integer. Many mathematicians, like Euler or Fourier, have helped to develop this theory, but only Riemann and Liouville have given a significant contribution, introducing the fractional derivative that bears their name. Recall the following

Definition 1.6.1 (convolution product)**.** *The convolution product (or simply convolution) of two functions f and g, denoted with* $(f * g)$*, is defined* (Beerends et al., 2003, chap. 6) *by*

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau) \mathrm{d}\tau , \tag{1.31}$$

whenever the integral 1.31 makes sense.

In the previous definition, f and g are real (or complex) functions. For instance, the previous definition is satisfied whenever f and g both belong to $L^1(\mathbb{R})$. In fact, under this hypothesis. $f * g$ is also in $L^1(\mathbb{R})$ (see Walnut, 2002, pp. 68–72). The symbol $*$ is used to indicate both the convolution product and the Dirichlet convolution. It will cause no confusion, since the Dirichlet convolution is defined only on arithmetic functions. Generally, the fractional calculus is introduced by recalling the Cauchy formula for repeated integration, given by

$$f^{-n}(x) = \frac{1}{(n-1)!} \int_a^x (x-t)^{(n-1)} f(t) \mathrm{d}t \ , \tag{1.32}$$

where $f \in \mathscr{C}^0\left([a,b]\right)$. Relation (1.32) suggests a way to generalize an integral of any real degree α. Being $\Gamma(n-1) = n!$ with $n \in \mathbb{N}$, the idea is to replace in the integral (1.32) the natural number n with some $\alpha \in \left(\mathbb{R}_{>0} \setminus \mathbb{N}\right)$ (Li et al., 2009). The fractional integrals by convolution kernel are introduced first.

Definition 1.6.2 (convolution kernel)**.** *Let $\alpha \in \mathbb{R}_{>0}$. The convolution kernel of order α for fractional integrals, denoted by Y_α, is defined as follows* (Li et al., 2009)*:*

$$Y_\alpha(t) \stackrel{\mathrm{def}}{=} \frac{t_+^{\alpha-1}}{\Gamma(\alpha)} \in L^1_{loc}\left(\mathbb{R}_{>0}\right) \ ,$$

where $L^1_{loc}(\mathbb{R}_{>0})$ is the space of locally integrable functions over $\mathbb{R}_{>0}$ and

$$t_+^{\alpha-1} \stackrel{\mathrm{def}}{=} \begin{cases} t^{\alpha-1} \ , & t > 0 \ , \\ 0 \ , & t \leq 0 \ . \end{cases}$$

Definition 1.6.3 (fractional integral)**.** *Let $f \in \mathscr{C}^0\left([a,b]\right)$ and $\alpha \in \left(\mathbb{R}_{>0} \setminus \mathbb{N}\right)$. The α-order fractional integral (called the Riemann-Liouville integral) of f is denoted with $\mathrm{D}_{t,a}^{-\alpha} f$ and defined by*

$$\mathrm{D}_{t,a}^{-\alpha} f(t) \stackrel{\mathrm{def}}{=} \left(Y_\alpha * f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-x)^{\alpha-1} f(x) \mathrm{d}x \ . \tag{1.33}$$

Since the convolution property holds for Y_α, that is, $Y_\alpha * Y_\beta = Y_{\alpha+\beta}$ for $\alpha, \beta \in \left(\mathbb{R}_{>0} \setminus \mathbb{N}\right)$, it follows (Li et al., 2009) that

$$\mathrm{D}_{t,a}^{-\alpha} \mathrm{D}_{t,a}^{-\beta} = \mathrm{D}_{t,a}^{-\alpha-\beta} \ .$$

Now the fractional differentiation defined by the fractional integral (1.33) can be introduced. There are several definition of fractional derivatives that represent one of its weak points.

Here, two of the commonly used definitions in fractional calculus are reported (Ortigueira, 2011; Pudlubny, 1999).

Definition 1.6.4 (Riemann-Liouville fractional derivative)**.** *Let $f \in \mathscr{C}^0\left([0,t]\right)$ and $m-1 < \alpha < m \in \mathbb{Z}_{>0}$. The α-order Riemann-Liouville fractional derivative of f is defined by*

$$ {}_{\mathrm{RL}}\mathrm{D}^{\alpha} f(t) \stackrel{\text{def}}{=} \frac{\mathrm{d}^m}{\mathrm{d}t^m} \mathrm{D}_{t,0}^{-(m-\alpha)} f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{\mathrm{d}^m}{\mathrm{d}t^m} \int_0^t (t-x)^{m-\alpha-1} f(x)\mathrm{d}x \,. $$

Definition 1.6.5 (Caputo fractional derivative)**.** *Let $f \in \mathscr{C}^0\left([0,t]\right)$ and $m-1 < \alpha < m \in \mathbb{Z}_{>0}$. The α-order Caputo fractional derivative of f is defined by*

$$ {}_{\mathrm{C}}\mathrm{D}^{\alpha} f(t) \stackrel{\text{def}}{=} \mathrm{D}_{t,0}^{-(m-\alpha)} \frac{\mathrm{d}^m}{\mathrm{d}t^m} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-x)^{m-\alpha-1} f^{(m)}(x)\mathrm{d}x \,. $$

In the classical Grünwald-Letnikov fractional derivative $f \in \mathscr{C}^m\left([0,t]\right)$ (see Chapter 2), hence it provides a narrow function space. In order to weaken the conditions on f, definitions 1.6.4 and 1.6.5 are introduced (Li et al., 2009). The links between Riemann-Liouville and Caputo fractional derivatives are discussed in several books. The main difference is represented by the position of the differentiation that is applied before (respectively after) the fractional integral in definition 1.6.4 (respectively 1.6.5). Both definitions can be easily generalized to all real powers (Boyadjiev et al., 2005). Thus, integration and differentiation are unified in one operator, called differintegral (Ortigueira and Coito, 2004). A remarkable result is that the difference between these two definitions is only a singular term that contains the initial value of the function f (Bagley, 2007).

In recent years, fractional calculus has become a gold mine for researchers. In fact, everyday, new properties and new physical applications are discovered. The geometrical interpretation of fractional operators is an especially interesting topic. Since the ordinary derivative is the linear approximation of a smooth function, the fractional derivative might provide a non-linear approximation of the local behavior of non-differentiable functions. The main disadvantage of definitions 1.6.4 and 1.6.5 is that both use an integral. On the other hand, the applications of fractional calculus in physics, continuum mechanics, signal processing and electromagnetism continue to grow (Dalir and Bashour, 2010; Tarasov, 2008).

1.6.1 The fractional derivative in the complex plane

Recently, Ortigueira (2006) has proposed the following generalization of the fractional derivative in the complex plane

$$
{}_{\mathrm{O}}\mathrm{D}^{\alpha} f(s) = \frac{e^{i(\pi-\theta)\alpha}}{\Gamma(-\alpha)} \int_0^{\infty} \frac{f(xe^{i\theta}+s) - \sum_{k=0}^{m} \frac{f^{(k)}(s)}{k!} e^{ik\theta} x^k}{x^{\alpha+1}} \mathrm{d}x \;, \tag{1.34}
$$

in which $m = \lfloor \alpha \rfloor$, $s \in \mathbb{C}$ and $\theta \in [0, 2\pi)$. Starting from the generalized Cauchy integral

$$
\mathrm{D}^{\alpha} f(s) = \frac{\Gamma(\alpha+1)}{2\pi i} \int_C \frac{f(w)}{(w-s)^{\alpha+1}} \mathrm{d}w \;, \qquad (s \in \mathbb{C})
$$

where C is a U-shaped contour that encircles the branch-cut line of $w^{\alpha-1}$ (Li et al., 2009), he has defined the fractional operator (1.34) that provides the fractional derivative in θ-direction of the complex plane. This fractional derivative, called the Ortigueira fractional derivative and denoted with ${}_{\mathrm{O}}\mathrm{D}^{\alpha}$, has several properties. Its most important ones are reported below (Li et al., 2009; Ortigueira and Tenreiro Machado, 2014).

1. *Consistency with integer-order derivative*
   Let $m-1 < \alpha < m \in \mathbb{Z}_{>0}$ and let f be analytic in a region that contains the Hankel contour C. Without loss of generality it can be supposed in (1.34) that $x \in \mathbb{R}_{>0}$. Under these hypotheses, it is easy to show (Li et al., 2009) that

   $$
   \lim_{\alpha \to m^-} {}_{\mathrm{O}}\mathrm{D}^{\alpha} f(s) = f_{\theta}^{(m)}(s) \;,
   $$

   $$
   \lim_{\alpha \to (m-1)^+} {}_{\mathrm{O}}\mathrm{D}^{\alpha} f(s) = f_{\theta}^{(m-1)}(s) \;,
   $$

   where $f_{\theta}^{(m)}$ is the mth directional derivative in θ-direction.

2. *Composition with integer-order derivatives*
   Let A_k be the set of complex functions f of the complex variable s such that

   $$
   \frac{\mathrm{d}^k}{\mathrm{d}s^k} \int_0^{\infty} f(xe^{i\theta}+s)\mathrm{d}x = \int_0^{\infty} \frac{\mathrm{d}^k}{\mathrm{d}s^k} f(xe^{i\theta}+s)\mathrm{d}x \;,
   $$

   in which $\theta \in [0, 2\pi)$ and $k \in \mathbb{N}$. If $f \in A_k$ and $m < \alpha < (m+1) \in \mathbb{Z}_{>0}$, it is easy to show (Li et al., 2009) with a direct computation that

   $$
   \frac{\mathrm{d}^k}{\mathrm{d}s^k} \left({}_{\mathrm{O}}\mathrm{D}^{\alpha} f(s) \right) = {}_{\mathrm{O}}\mathrm{D}^{\alpha} \left(\frac{\mathrm{d}^k}{\mathrm{d}s^k} f(s) \right) \;.
   $$

3. *Composition with itself*
   Let us give $m < \alpha < (m+1) \in \mathbb{Z}_{>0}$, $k < \beta < (k+1) \in \mathbb{Z}_{>0}$ and $f \in A_m$. Li et al. (2009) have shown that

   (a) ${}_{\mathrm{O}}\mathrm{D}^{-\alpha}\left({}_{\mathrm{O}}\mathrm{D}^{\alpha} f(s)\right) = {}_{\mathrm{O}}\mathrm{D}^{\alpha}\left({}_{\mathrm{O}}\mathrm{D}^{-\alpha} f(s)\right)$,

   (b) ${}_{\mathrm{O}}\mathrm{D}^{\alpha}\left({}_{\mathrm{O}}\mathrm{D}^{-\beta} f(s)\right) = {}_{\mathrm{O}}\mathrm{D}^{-\beta}\left({}_{\mathrm{O}}\mathrm{D}^{\alpha} f(s)\right)$,

   (c) ${}_{\mathrm{O}}\mathrm{D}^{\alpha}\left({}_{\mathrm{O}}\mathrm{D}^{\beta} f(s)\right) = {}_{\mathrm{O}}\mathrm{D}^{\beta}\left({}_{\mathrm{O}}\mathrm{D}^{\alpha} f(s)\right) \neq {}_{\mathrm{O}}\mathrm{D}^{\alpha+\beta} f(s)$,

   where (c) holds for $f \in A_{\max[m,k]}$.

For the purpose of this thesis, the most relevant property of the Ortigueira fractional derivative is represented by the possibility to generalize the classical Caputo derivative (Caputo, 1967; de Oliveira et al., 2014) in the complex plane. In fact, by using the Ortigueira derivative, the α-order Caputo derivative along with θ-direction in the complex plane.

Definition 1.6.6 (Caputo-Ortigueira fractional derivative)**.** *Let f be a complex function of the complex variable s and let $m-1 < \alpha < m \in \mathbb{Z}_{>0}$. The α-order Caputo-Ortigueira fractional derivative of f is defined* (Li et al., 2009) *by*

$$
{}_{\mathrm{C}}\mathrm{D}^{\alpha} f(s) \stackrel{\text{def}}{=} {}_{\mathrm{O}}\mathrm{D}^{\alpha-m}\left(f^{(m)}(s)\right) = \frac{e^{i(\pi-\theta)(\alpha-m)}}{\Gamma(m-\alpha)} \int_0^{\infty} \frac{f^{(m)}(xe^{i\theta}+s)}{x^{\alpha-m+1}} \mathrm{d}x \,. \tag{1.35}
$$

In the literature, (1.35) is called the Ortigueira generalization of the Caputo fractional derivative, or simply Caputo-Ortigueira fractional derivative.

Chapter 2

$\zeta^{(\alpha)}$ and generalizations

2.1 Introduction

The Riemann ζ function plays an important role both in number theory and in several applications of quantum mechanics (see Chapter 1). In particular, the Riemann hypothesis can be formulated by using quantum terminology (Pozdnyakov, 2012) and its zeros are associated with the Hamiltonian of a quantum mechanical system (Sierra, 2010).

In recent years, fractional calculus have proven to be a powerful tool in several areas of research, both in theory and in applications, spreading over almost all fields of science and technology (Dalir and Bashour, 2010). Several authors have defined a generalization of the fractional derivative to $\mathbb{C}$. In particular, it has been shown (Owa, 1987) how to extend the fractional derivative to the class of analytical functions f on the unit circle $\mathrm{U} = \{z \in \mathbb{C} : |z| < 1\}$ such that

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \, . \qquad (z \in \mathbb{C}) \tag{2.1}$$

It appears to be clear that the Riemann ζ function does not belong to the previous class since it can not be expressed as (2.1). By using the definition 1.6.6, the fractional derivative of the Riemann ζ function is easily computed. Its convergence domain, which depends on the fractional order of the derivative, is widely studied and plotted. Moreover, the fractional derivates of both the Hurwitz ζ function and the Dirichlet series are computed by using the fractional derivative (1.35), which are in accordance with the classical theory (Apostol, 1998, Chaps. 11–12; Srivastava and Choi, 2011, Chap.2). The previous fractional derivatives have some interesting properties. In particular, the chaotic decay of (2.2) to zero, suggests that the fractional derivative of the Riemann ζ function can be a non-differentiable function around zero. It might open new perspectives in the applications to the theory of dynamical systems.

This chapter consists of four sections. The α-order fractional derivative of the Riemann ζ function is computed in Section 2.2, together with the half-plane of convergence. The fractional derivatives of both the Dirichlet series and the Hurwitz ζ function are given in Section 2.3. Finally, the first properties of $\zeta^{(\alpha)}$ and decay to zero are presented in Section 2.4.

2.2 The fractional derivative of ζ and convergence

In Chapter 1, all the mathematical tools needed to compute the α-order fractional derivative of the Riemann ζ function have been presented.

Theorem 2.2.1 (Guariglia 2015)**.** *Let s be a complex variable such that $s = x+iy$ with $x, y \in \mathbb{R}$ and let $m-1 < \alpha < m \in \mathbb{Z}_{>0}$. The α-order fractional derivative of the Riemann ζ function is given by*

$$\zeta^{(\alpha)}(s) = \mathrm{e}^{i\pi\alpha} \sum_{n=2}^{\infty} \frac{\log^{\alpha} n}{n^s} \,. \tag{2.2}$$

Moreover, the real and imaginary parts of (2.2) *are*

$$\boxed{\begin{aligned} \operatorname{Re}\left(\zeta^{(\alpha)}(s)\right) &= \sum_{n=2}^{\infty} \frac{\log^{\alpha} n}{n^x} \cos(\pi\alpha - y\log n) \,, \\ \operatorname{Im}\left(\zeta^{(\alpha)}(s)\right) &= \sum_{n=2}^{\infty} \frac{\log^{\alpha} n}{n^x} \sin(\pi\alpha - y\log n) \,. \end{aligned}} \tag{2.3}$$

Proof: By using (1.35), it follows that

$$\begin{aligned} \zeta^{(\alpha)}(s) &= \frac{\mathrm{e}^{i(\pi-\theta)(\alpha-m)}}{\Gamma(m-\alpha)} \int_0^{\infty} \frac{\mathrm{d}^m}{\mathrm{d}s^m} \left(\zeta\left(x\mathrm{e}^{i\theta}+s\right)\right) \frac{1}{x^{\alpha-m+1}} \mathrm{d}x \\ &= \frac{\mathrm{e}^{i(\pi-\theta)(\alpha-m)}}{\Gamma(m-\alpha)} \int_0^{\infty} \frac{\mathrm{d}^m}{\mathrm{d}s^m} \left(\sum_{n=1}^{\infty} n^{-s} n^{-x\mathrm{e}^{i\theta}} x^{m-\alpha-1}\right) \mathrm{d}x \,. \end{aligned}$$

Bringing the integral sign under both derivative and summation, gives

$$\zeta^{(\alpha)}(s) = \frac{\mathrm{e}^{i(\pi-\theta)(\alpha-m)}}{\Gamma(m-\alpha)} \frac{\mathrm{d}^m}{\mathrm{d}s^m} \left(\sum_{n=1}^{\infty} n^{-s} \int_0^{\infty} n^{-x\mathrm{e}^{i\theta}} x^{m-\alpha-1} \mathrm{d}x\right) \,. \tag{2.4}$$

Begin by computing the integral on the right hand side (RHS) of (2.4). By a change of variables $x\mathrm{e}^{i\theta} = z$, it follows that

$$
\begin{aligned}
\int_0^\infty n^{-xe^{i\theta}} x^{m-\alpha-1} \mathrm{d}x &= \int_0^\infty n^{-z} \, \mathrm{e}^{-i\theta(m-\alpha)} z^{m-\alpha-1} \mathrm{d}z \\
&= \mathrm{e}^{-i\theta(m-\alpha)} \int_0^\infty \mathrm{e}^{-z\log n} z^{m-\alpha-1} \mathrm{d}z \\
&= \mathrm{e}^{-i\theta(m-\alpha)} \int_0^\infty \mathrm{e}^{-x} x^{m-\alpha-1} \log^{\alpha-m} n \; \mathrm{d}x \\
&= \mathrm{e}^{-i\theta(m-\alpha)} \log^{\alpha-m} n \; \Gamma(m-\alpha) \; ,
\end{aligned} \tag{2.5}
$$

where another change of variable has been performed, i.e. $z \log n = x$ and the definition of gamma function (section 1.4) has been used. Since $m - \alpha > 0$, the last RHS of the continued equality above makes sense. By substituting (2.5) into (2.4), it is

$$
\begin{aligned}
\zeta^{(\alpha)}(s) &= \mathrm{e}^{i\pi(\alpha-m)} \frac{\mathrm{d}^m}{\mathrm{d}s^m} \sum_{n=1}^{\infty} n^{-s} \log^{\alpha-m} n \\
&= \mathrm{e}^{i\pi(\alpha-m)} \sum_{n=1}^{\infty} \log^{\alpha-m} n \, \frac{\mathrm{d}^m}{\mathrm{d}s^m} \left(n^{-s} \right) \; ,
\end{aligned}
$$

where the derivative on the RHS is given by

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s} \left(n^{-s} \right) = -n^{-s} \log n \; , \\
&\frac{\mathrm{d}^2}{\mathrm{d}s^2} \left(n^{-s} \right) = (-1)^2 n^{-s} \log^2 n \; , \\
&\vdots \\
&\frac{\mathrm{d}^m}{\mathrm{d}s^m} \left(n^{-s} \right) = (-1)^m n^{-s} \log^m n \; ,
\end{aligned}
$$

so that

$$
\begin{aligned}
\zeta^{(\alpha)}(s) &= (-1)^m \mathrm{e}^{i\pi(\alpha-m)} \sum_{n=1}^{\infty} \frac{\log^\alpha n}{n^s} \\
&= \mathrm{e}^{i\pi\alpha} \sum_{n=2}^{\infty} \frac{\log^\alpha n}{n^s} \; ,
\end{aligned} \tag{2.6}
$$

In order to show the second part of the theorem, (2.6) has to be written in rectangular form (see Mathews and Howell, 2006, Chap. 1). Since $s = x + iy$, it is

$$n^{-s} = n^{-x}n^{-iy} = n^{-x}\mathrm{e}^{-iy\log n} = n^{-x}\left(\cos(y\log n) - i\sin(y\log n)\right) ,$$

hence

$$\begin{aligned}
\zeta^{(\alpha)}(s) &= \left(\cos(\pi\alpha) + i\sin(\pi\alpha)\right)\sum_{n=2}^{\infty}\frac{\log^{\alpha} n}{n^x}\left(\cos(y\log n) - i\sin(y\log n)\right) \\
&= \sum_{n=2}^{\infty}\frac{\log^{\alpha} n}{n^x}\Big(\cos(\pi\alpha)\cos(y\log n) + \sin(\pi\alpha)\sin(y\log n) \\
&\quad + i\Big(\sin(\pi\alpha)\cos(y\log n) - \cos(\pi\alpha)\sin(y\log n)\Big)\Big) .
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathrm{Re}\left(\zeta^{(\alpha)}(s)\right) &= \sum_{n=2}^{\infty}\frac{\log^{\alpha} n}{n^x}\left(\cos(\pi\alpha)\cos(y\log n) + \sin(\pi\alpha)\sin(y\log n)\right) \\
&= \sum_{n=2}^{\infty}\frac{\log^{\alpha} n}{n^x}\cos(\pi\alpha - y\log n) ,
\end{aligned}$$

and similarly

$$\begin{aligned}
\mathrm{Im}\left(\zeta^{(\alpha)}(s)\right) &= \sum_{n=2}^{\infty}\frac{\log^{\alpha} n}{n^x}\left(\sin(\pi\alpha)\cos(y\log n) - \cos(\pi\alpha)\sin(y\log n)\right) \\
&= \sum_{n=2}^{\infty}\frac{\log^{\alpha} n}{n^x}\sin(\pi\alpha - y\log n) ,
\end{aligned}$$

which completes the proof.

□

2.2.1 Convergence of $\zeta^{(\alpha)}$

By using the classical comparison test and a generalization of the well-known harmonic series, the convergence of $\zeta^{(\alpha)}$ is easily derived.

Theorem 2.2.2 (Comparison test)**.** *Suppose that* $\sum_{n=1}^{\infty} a_n$ *converges and* $0 \leq b_n \leq a_n$*. In these hypotheses, the series* $\sum_{n=1}^{\infty} b_n$ *converges.*

Proof: See Stirling (2009, pp. 99–100).

□

Corollary 2.2.3. *Let $p \in \mathbb{R}$. The series* $\sum_{n=1}^{\infty} \frac{1}{n^p}$ *converges if $p > 1$ and diverges if $p \leq 1$.*

Proof: It directly follows from the convergence domain of the harmonic series.

□

The previous series is called the p-series and it is equal to $\zeta(p)$ for $p > 1$. These two properties are essential for showing the following statement.

Proposition 2.2.4. *Under the same hypotheses of theorem* 2.2.1*,* $(2.3)_1$ *and* $(2.3)_2$ *converge in the half-plane*

$$x > 1 + \alpha \ . \tag{2.7}$$

Proof: Since the sine is a bounded function in $[-1, 1]$, it is

$$\sin x \leq |\sin x| \leq 1 \ ,$$

hence

$$\frac{\log^{\alpha} n}{n^x} \sin(\pi\alpha - y \log n) \leq \frac{\log^{\alpha} n}{n^x} < \frac{n^{\alpha}}{n^x} = \frac{1}{n^{x-\alpha}} \ . \tag{2.8}$$

The last RHS in (2.8) is that of a p-series, hence it converges for

$$x - \alpha > 1 \ .$$

By using the comparison test, it appears to be clear that $\mathrm{Im}\left(\zeta^{(\alpha)}\right)$ also converges in the half-plane (2.7). Similarly, it can be shown that the same holds true for $\mathrm{Re}\left(\zeta^{(\alpha)}\right)$.

□

A direct consequence of the previous theorem is given by the following

Corollary 2.2.5. *Let $\alpha \in \left(\mathbb{R}_{>0} \setminus \mathbb{N}\right)$. $\zeta^{(\alpha)}$ converges in the half-plane* (2.7).

Proof: Since a complex series converges if and only if both the real and imaginary parts converge (Mathews and Howell, 2006, Chap. 4), the proof follows directly from theorem 2.2.4.

□

Figure 2.1 shows that ζ converges for $x > 1$ and its α-order fractional derivative converges for $x > 1 + \alpha$. In Figures 2.2 and 2.3, $\mathrm{Re}\left(\zeta^{(\alpha)}\right)$ and $\mathrm{Im}\left(\zeta^{(\alpha)}\right)$ are shown, respectively, with $\alpha = 0.4$ and the upper limit of the series $n = 60$. In Figure 2.4, $\zeta^{(\alpha)}$ is plotted as a 3D surface of its real part (orange surface) and imaginary part (blue surface).

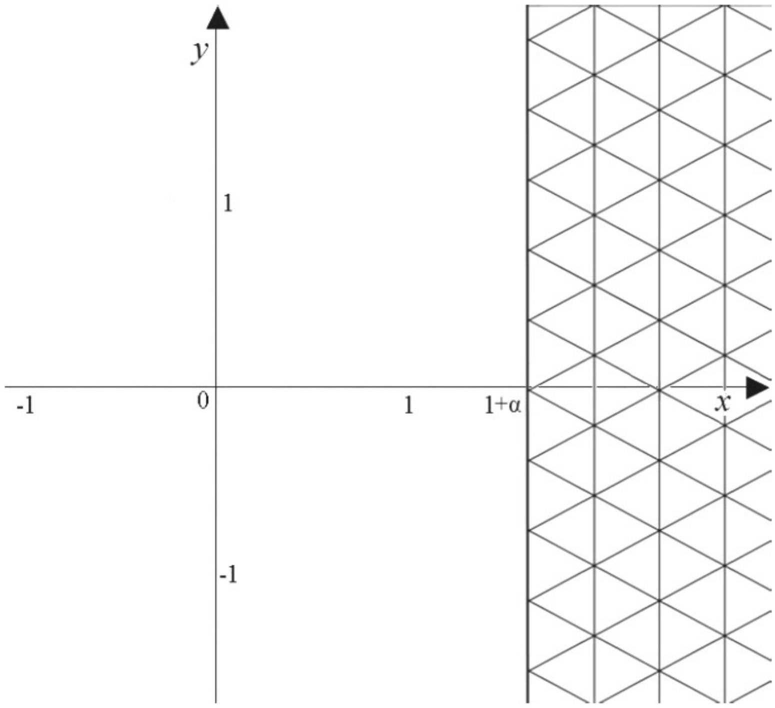


Figure 2.1: Convergence half-plane of $\zeta^{(\alpha)}$.

2.3 Fractional derivatives of the Dirichlet Series and of the Hurwitz ζ function

In this section, the fractional derivative of a function defined by the Dirichlet series (Apostol, 1998, Chap. 11) and by the Hurwitz ζ function are given. They represents a consistent generalization of the computation obtained in the previous section for the Riemann ζ function.

Theorem 2.3.1 (Cattani and Guariglia 2016)**.** *Let s be a complex variable such that $s = x + iy$ with $x, y \in \mathbb{R}$ and let $m-1 < \alpha < m \in \mathbb{Z}_{>0}$. The α-order fractional derivatives of the Dirichlet series and of the Hurwitz ζ function are given by*

$$
\begin{aligned}
F^{(\alpha)}(s) &= \mathrm{e}^{i\pi\alpha} \sum_{n=1}^{\infty} f(n) \frac{\log^{\alpha} n}{n^s} \,, \\
\zeta^{(\alpha)}(s,a) &= \mathrm{e}^{i\pi\alpha} \sum_{n=0}^{\infty} \frac{\log^{\alpha}(n+a)}{(n+a)^s} \,. \qquad (0 < a \leq 1)
\end{aligned}
\tag{2.9}
$$

Proof : By substituting (1.19) into (1.35), it is

$$
\begin{aligned}
F^{(\alpha)}(s) &= \frac{\mathrm{e}^{i(\pi-\theta)(\alpha-m)}}{\Gamma(m-\alpha)} \int_0^{\infty} \frac{\mathrm{d}^m}{\mathrm{d}s^m} \sum_{n=1}^{\infty} \left(f(n)\, n^{-x\mathrm{e}^{i\theta}-s} x^{m-\alpha-1} \right) \mathrm{d}x \\
&= \frac{\mathrm{e}^{i(\pi-\theta)(\alpha-m)}}{\Gamma(m-\alpha)} \frac{\mathrm{d}^m}{\mathrm{d}s^m} \sum_{n=1}^{\infty} f(n)\, n^{-s} \int_0^{\infty} \left(n^{-x\mathrm{e}^{i\theta}} x^{m-\alpha-1} \right) \mathrm{d}x \; .
\end{aligned}
$$

From (2.5), it follows that

$$
\begin{aligned}
F^{(\alpha)}(s) &= \mathrm{e}^{i\pi(\alpha-m)} \frac{\mathrm{d}^m}{\mathrm{d}s^m} \sum_{n=1}^{\infty} f(n)\, n^{-s} \log^{\alpha-m} n \\
&= \mathrm{e}^{i\pi(\alpha-m)} \sum_{n=1}^{\infty} f(n) \log^{\alpha-m} n \; \frac{\mathrm{d}^m}{\mathrm{d}s^m} \left(n^{-s} \right) \; .
\end{aligned}
$$

Since

$$
\frac{\mathrm{d}^m}{\mathrm{d}s^m} \left(n^{-s} \right) = (-1)^m n^{-s} \log^m n \; ,
$$

it is

$$
\begin{aligned}
F^{(\alpha)}(s) &= (-1)^m \mathrm{e}^{i\pi(\alpha-m)} \sum_{n=1}^{\infty} f(n) \frac{\log^{\alpha} n}{n^s} \\
&= \mathrm{e}^{i\pi\alpha} \sum_{n=1}^{\infty} f(n) \frac{\log^{\alpha} n}{n^s} \; .
\end{aligned}
$$

The second part of the theorem can be proven in the same way. In fact, (1.35) for $f(s) = \zeta(s,a)$ becomes

$$
\begin{aligned}
\zeta^{(\alpha)}(s,a) &= \frac{\mathrm{e}^{i(\pi-\theta)(\alpha-m)}}{\Gamma(m-\alpha)} \int_0^{\infty} \frac{\mathrm{d}^m}{\mathrm{d}s^m} \sum_{n=0}^{\infty} \left((n+a)^{-x\mathrm{e}^{i\theta}-s} x^{m-\alpha-1} \right) \mathrm{d}x \\
&= \frac{\mathrm{e}^{i(\pi-\theta)(\alpha-m)}}{\Gamma(m-\alpha)} \frac{\mathrm{d}^m}{\mathrm{d}s^m} \sum_{n=0}^{\infty} (n+a)^{-s} \int_0^{\infty} (n+a)^{-x\mathrm{e}^{i\theta}} x^{m-\alpha-1} \mathrm{d}x \; .
\end{aligned}
$$

The integral in the last RHS is given by

$$
\begin{aligned}
&\int_0^\infty (n+a)^{-z}\, \mathrm{e}^{-i\theta(m-\alpha)}\, z^{m-\alpha-1}\mathrm{d}z \\
&\quad = \mathrm{e}^{-i\theta(m-\alpha)} \int_0^\infty \mathrm{e}^{-z\log(n+a)} z^{m-\alpha-1}\mathrm{d}z \\
&\quad = \mathrm{e}^{-i\theta(m-\alpha)} \int_0^\infty \mathrm{e}^{-x} x^{m-\alpha-1}\ \log^{\alpha-m}(n+a)\,\mathrm{d}x \\
&\quad = \mathrm{e}^{-i\theta(m-\alpha)} \log^{\alpha-m}(n+a)\ \Gamma(m-\alpha)\ ,
\end{aligned}
$$

which was obtained by changing variables twice ($x^{i\theta} = z$ and $z\log(n+a) = x$, respectively). Hence

$$
\begin{aligned}
\zeta^{(\alpha)}(s,a) &= \mathrm{e}^{i\pi(\alpha-m)} \frac{\mathrm{d}^m}{\mathrm{d}s^m} \sum_{n=0}^{\infty} (n+a)^{-s} \log^{\alpha-m}(n+a) \\
&= \mathrm{e}^{i\pi(\alpha-m)} \sum_{n=0}^{\infty} \log^{\alpha-m}(n+a)\ \frac{d^m}{ds^m}\left((n+a)^{-s}\right)\ .
\end{aligned}
$$

Being

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}\left((n+a)^{-s}\right) = -(n+a)^{-s}\log(n+a)\ , \\
&\frac{\mathrm{d}^2}{\mathrm{d}s^2}\left((n+a)^{-s}\right) = (-1)^2\,(n+a)^{-s}\log^2(n+a)\ , \\
&\vdots \\
&\frac{\mathrm{d}^m}{\mathrm{d}s^m}\left((n+a)^{-s}\right) = (-1)^m\,(n+a)^{-s}\log^m(n+a)\ ,
\end{aligned}
$$

we finally have

$$
\begin{aligned}
\zeta^{(\alpha)}(s,a) &= (-1)^m \mathrm{e}^{i\pi(\alpha-m)} \sum_{n=0}^{\infty} \log^{\alpha-m}(n+a)\ (n+a)^{-s}\log^m(n+a) \\
&= \mathrm{e}^{i\pi\alpha} \sum_{n=0}^{\infty} \frac{\log^{\alpha}(n+a)}{(n+a)^s}\ .
\end{aligned}
$$

□

The results (2.9) consistently generalize (2.2), so that the fractional derivative of the Riemann ζ function can be seen in a more general scheme. In fact, if in $F^{\alpha}(s)$ we put $f(n) = 1$, we have $\zeta^{\alpha}(s)$, according to $\zeta(s) = F(s; f(n) = 1)$. Analogously to that, in the Hurwitz ζ function, $(2.9)_2$ becomes (2.2) for $a = 1$. Moreover, theorem 2.2.1 underlines that the Caputo-Ortigueira fractional derivative gives a natural generalization of the integer order derivative of the Riemann ζ function. In fact, the integer derivative of (1.7) is given (Apostol, 1998, Chap. 11) by

$$\zeta^{(k)}(s) = \mathrm{e}^{i\pi k} \sum_{n=1}^{\infty} \frac{\log^k n}{n^s}\,, \qquad (k \in \mathbb{N}_0)$$

which represents the integer counterpart of (2.2).

2.4 Properties of $\zeta^{(\alpha)}$

In this section the main properties of the fractional derivative (2.2) are given. If the variable s is a pure real number $s = -x$, the partial sum (for a fixed n) in (2.2) is a function with rapid decay to zero. For a pure complex number, i.e. $s = -iy$ the real part of $\zeta^{(\alpha)}$ is a slow decay even function (see Figure 2.5) while the imaginary part is a slow decay odd function (Figure 2.6). If the real and imaginary parts of the function (2.2) are plotted in the same plane, then we have the parametric plot of $\zeta^{(\alpha)}$, with $x = 0$ and upper bound $y = 30$ (Figure 2.7). Hence, the parametric plot of (2.2) is a spiral with an asymptotic attractor as origin. However, a zoom around the origin shows that the spiral becomes a self-intersecting, non-differentiable function.

Proposition 2.4.1. *In the half-plane of convergence* (2.7)*, it is*

$$\zeta^{(\alpha)}(s) = \sum_{n=1}^{\infty}\sum_{h=0}^{\infty} \left(-\sum_{k=1}^{\infty} \left(\frac{n-1}{n+1}\right)^{2k-1} \frac{2}{2k-1} \right)^{h+\alpha} \frac{s^h}{h!}\,. \tag{2.10}$$

Proof: Since

$$n^{-s}\log^{\alpha} n = \mathrm{e}^{-s\log n}\log^{\alpha} n = \sum_{h=0}^{\infty} \frac{(-s\log n)^h}{h!}\log^{\alpha} n = \sum_{h=0}^{\infty} \mathrm{e}^{i\pi h}\frac{\log^{h+\alpha} n}{h!} s^h\,,$$

and (Abramowitz and Stegun, 2014, p. 68)

$$\log n = \sum_{k=1}^{\infty} \left(\frac{n-1}{n+1}\right)^{2k-1} \frac{2}{2k-1}\,, \qquad (n > 0)$$

it follows that

$$\begin{aligned}
\zeta^{(\alpha)}(s) &= \mathrm{e}^{i\pi\alpha} \sum_{n=1}^{\infty} \sum_{h=0}^{\infty} \mathrm{e}^{i\pi h} \frac{s^h}{h!} \left(\sum_{k=1}^{\infty} \left(\frac{n-1}{n+1} \right)^{2k-1} \frac{2}{2k-1} \right)^{h+\alpha} \\
&= \sum_{n=1}^{\infty} \sum_{h=0}^{\infty} \mathrm{e}^{i\pi(h+\alpha)} \left(\sum_{k=1}^{\infty} \left(\frac{n-1}{n+1} \right)^{2k-1} \frac{2}{2k-1} \right)^{h+\alpha} \frac{s^h}{h!} \,.
\end{aligned}$$

□

The proposition 2.4.1 shows that the fractional derivative of the Riemann ζ function can also be expressed as a complex power series.

2.4.1 Further properties of $\zeta^{(\alpha)}$

In the following, two important properties of the fractional derivative (2.2) are presented. The first property concerns the composition of $\zeta^{(\alpha)}$ with integer-order derivatives. Since

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\zeta^{(\alpha)}(s)\right) = \mathrm{e}^{i\pi\alpha} \frac{\mathrm{d}}{\mathrm{d}s}\left(\sum_{n=1}^{\infty} \frac{\log^{\alpha} n}{n^s} \right) = \mathrm{e}^{i\pi\alpha} \sum_{n=1}^{\infty} \log^{\alpha} n \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{n^s}\right) = \mathrm{e}^{i\pi(\alpha+1)} \sum_{n=1}^{\infty} \frac{\log^{\alpha+1} n}{n^s} \,,$$

and iterating k times, with $k \in \mathbb{N}$, we get

$$\boxed{\frac{\mathrm{d}^k}{\mathrm{d}s^k}\left(\zeta^{(\alpha)}(s)\right) = \mathrm{e}^{i\pi(\alpha+k)} \sum_{n=1}^{\infty} \frac{\log^{\alpha+k} n}{n^s} \,.} \tag{2.11}$$

Eq. (2.11) shows that the derivatives of ζ dump their order (integer or fractional) on the complex exponent and on the logarithm (inside the summation). The second property provides an interesting characterization of the product of the Riemann ζ function and its α-order fractional derivative and is reported below.

Proposition 2.4.2. *Let s be a complex variable such that $s = x + iy$ with $x, y \in \mathbb{R}$ and let $\alpha \in \left(\mathbb{R}_{>0} \setminus \mathbb{N}\right)$. The product of the Riemann ζ function and its α-order fractional derivative is given by*

$$\zeta^{(\alpha)}(s)\,\zeta(s) = \mathrm{e}^{i\pi\alpha} \sum_{n=1}^{\infty} \sum_{d|n} \frac{\log^{\alpha} d}{n^s} \,, \tag{2.12}$$

in the half-plane $x > 1 + \alpha$.

Proof: From theorem 1.5.8 it is

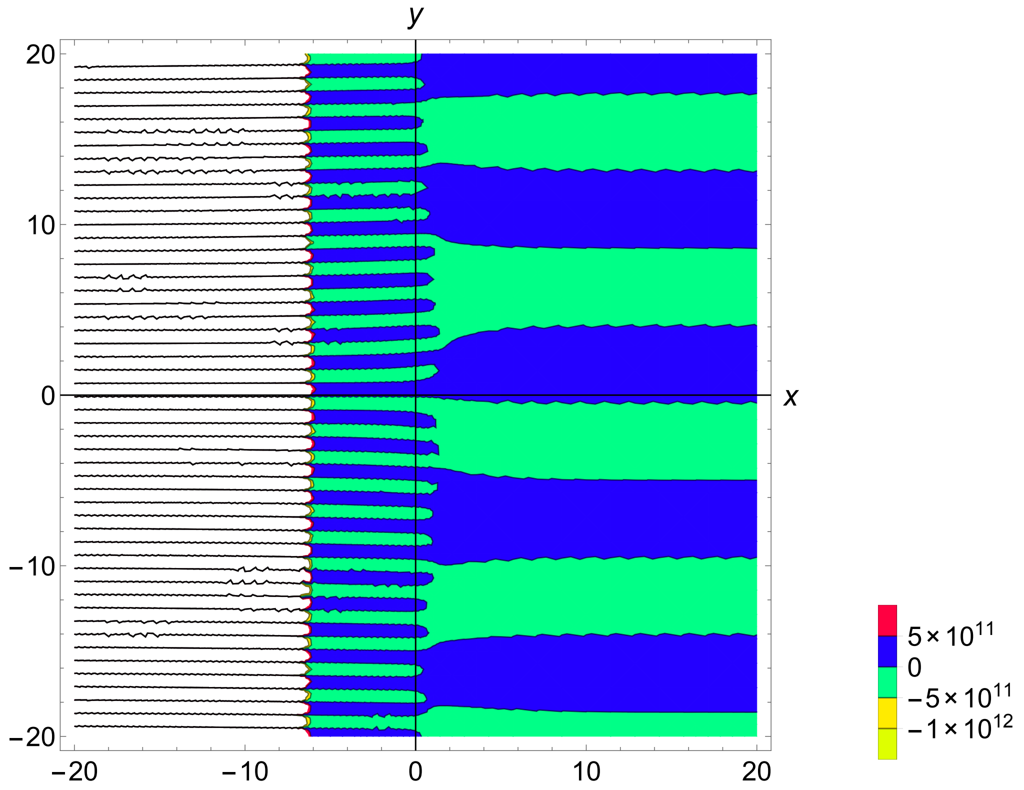
$$\zeta^{(\alpha)}(s)\,\zeta(s) = \mathrm{e}^{i\pi\alpha}\sum_{n=1}^{\infty}\frac{\log^{\alpha} n}{n^s}\sum_{n=1}^{\infty}\frac{1}{n^s} = \mathrm{e}^{i\pi\alpha}\sum_{n=1}^{\infty}\frac{\log^{\alpha} n * 1}{n^s}\ .$$

Since

$$\log^{\alpha} n * 1 = \sum_{d|n}\log^{\alpha} d\ ,$$

and the fractional derivative (2.2) converges in $x > 1+\alpha$, the proof follows.

$\square$



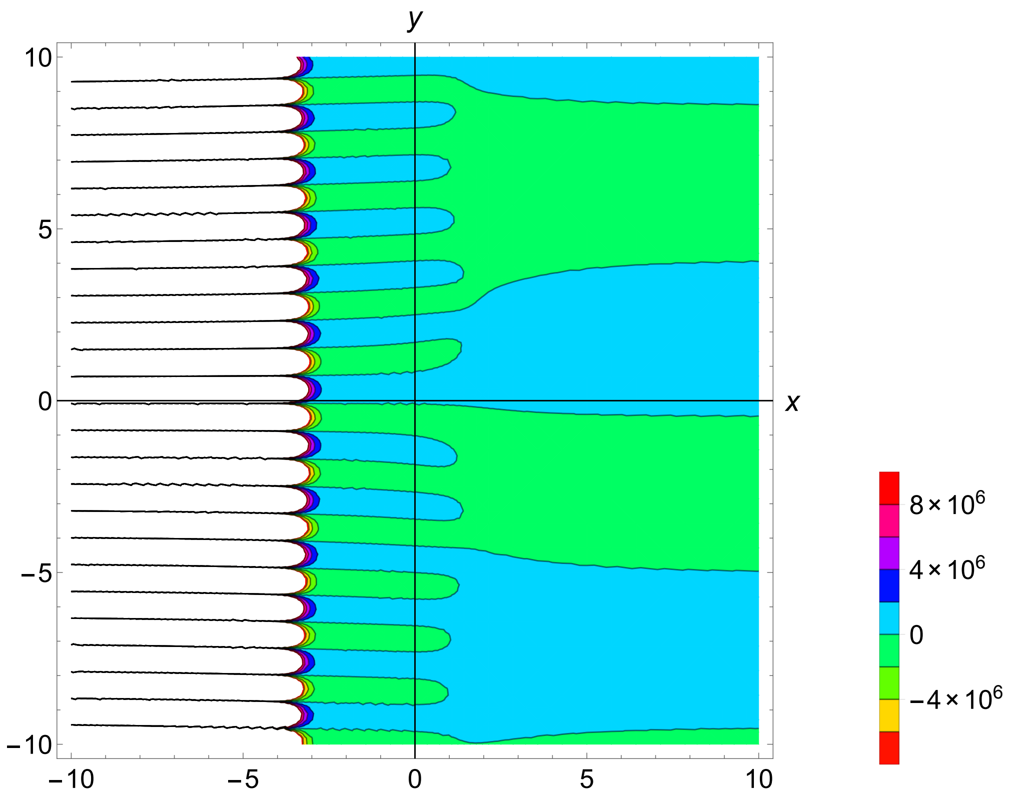
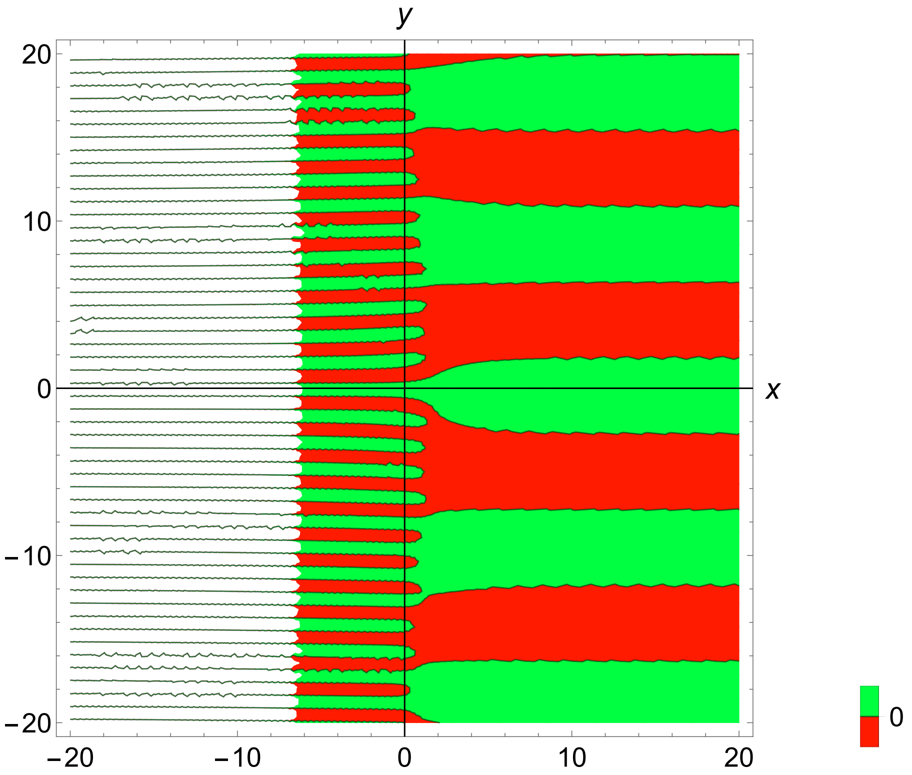


Figure 2.2: Real part of $\zeta^{(\alpha)}$ with $\alpha = 0.4$ and upper limit of the series $n = 60$.



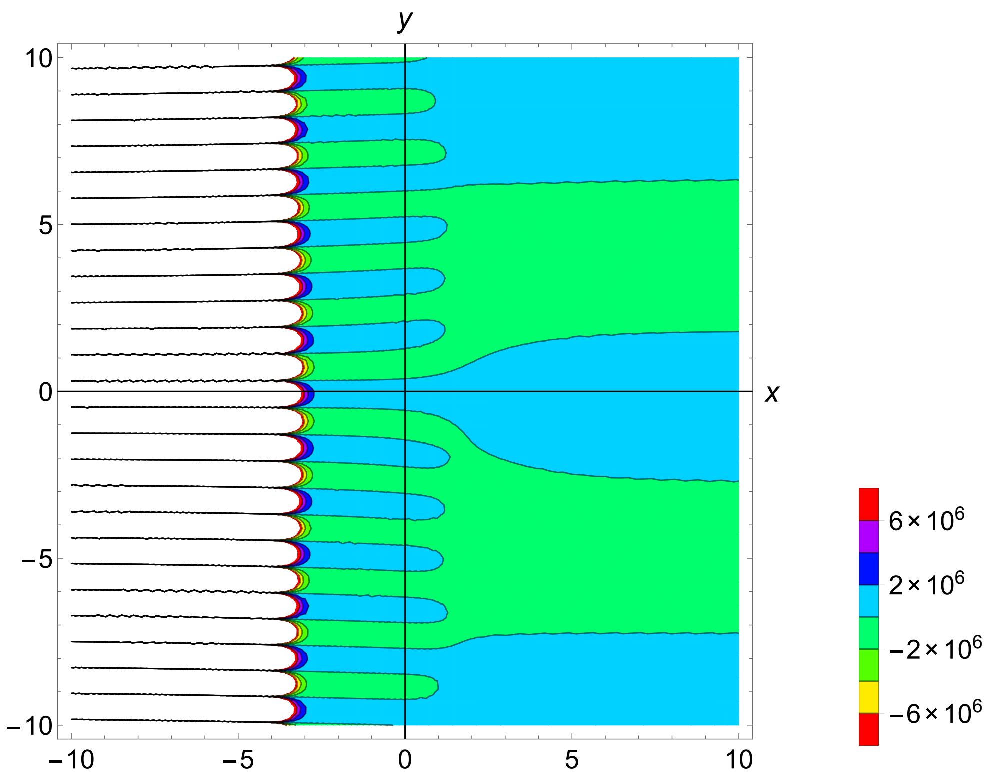


Figure 2.3: Imaginary part of $\zeta^{(\alpha)}$ with $\alpha = 0.4$ and upper limit of the series $n = 60$.

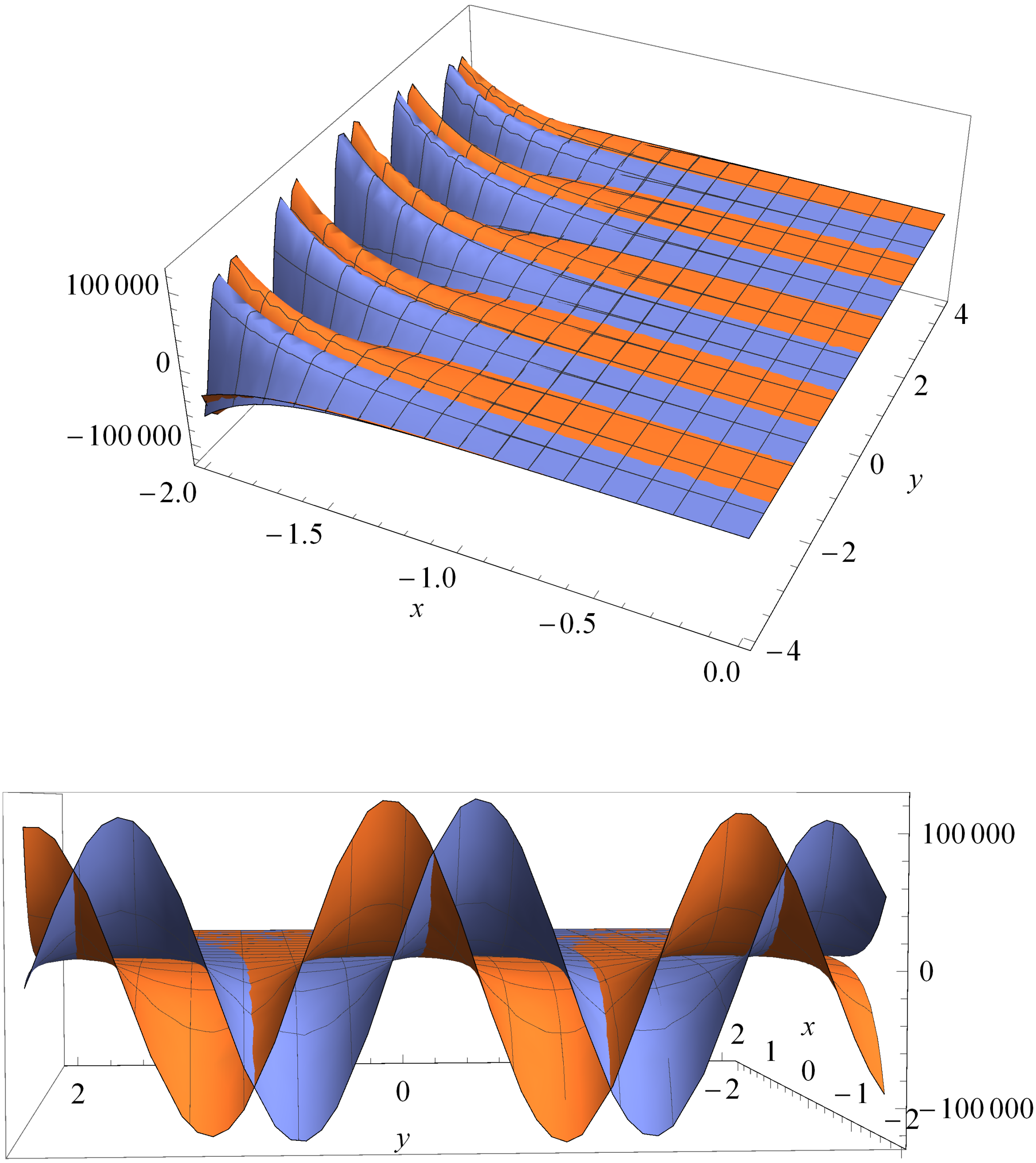


Figure 2.4: $\zeta^{(\alpha)}$ with $\alpha = 0.4$ and upper limit of the series $n = 60$.

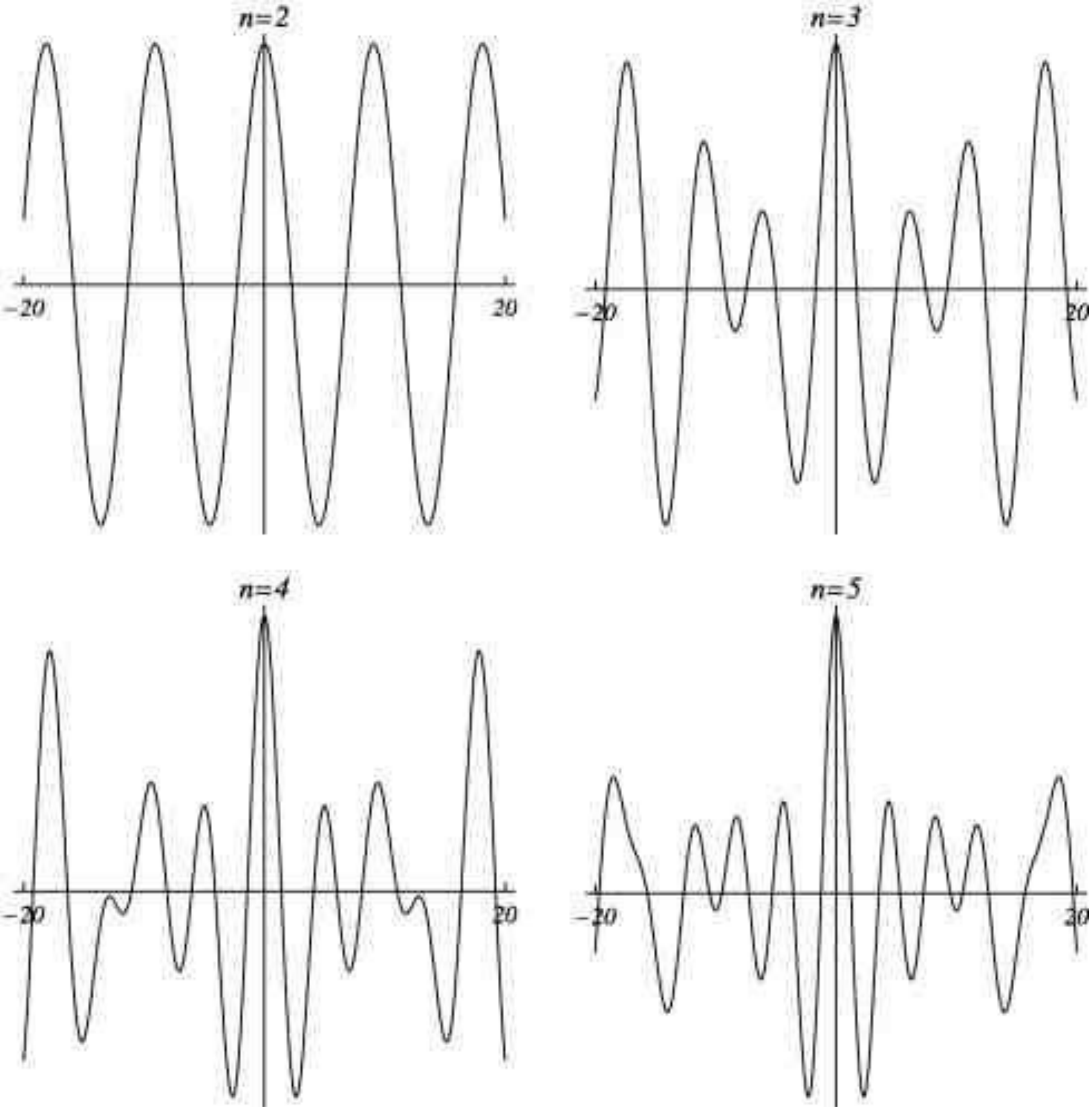


Figure 2.5: Real part of $\zeta^{(\alpha)}$ with $x = 0$, $\alpha = 0.6$ and upper limit of the series $n = 60$..

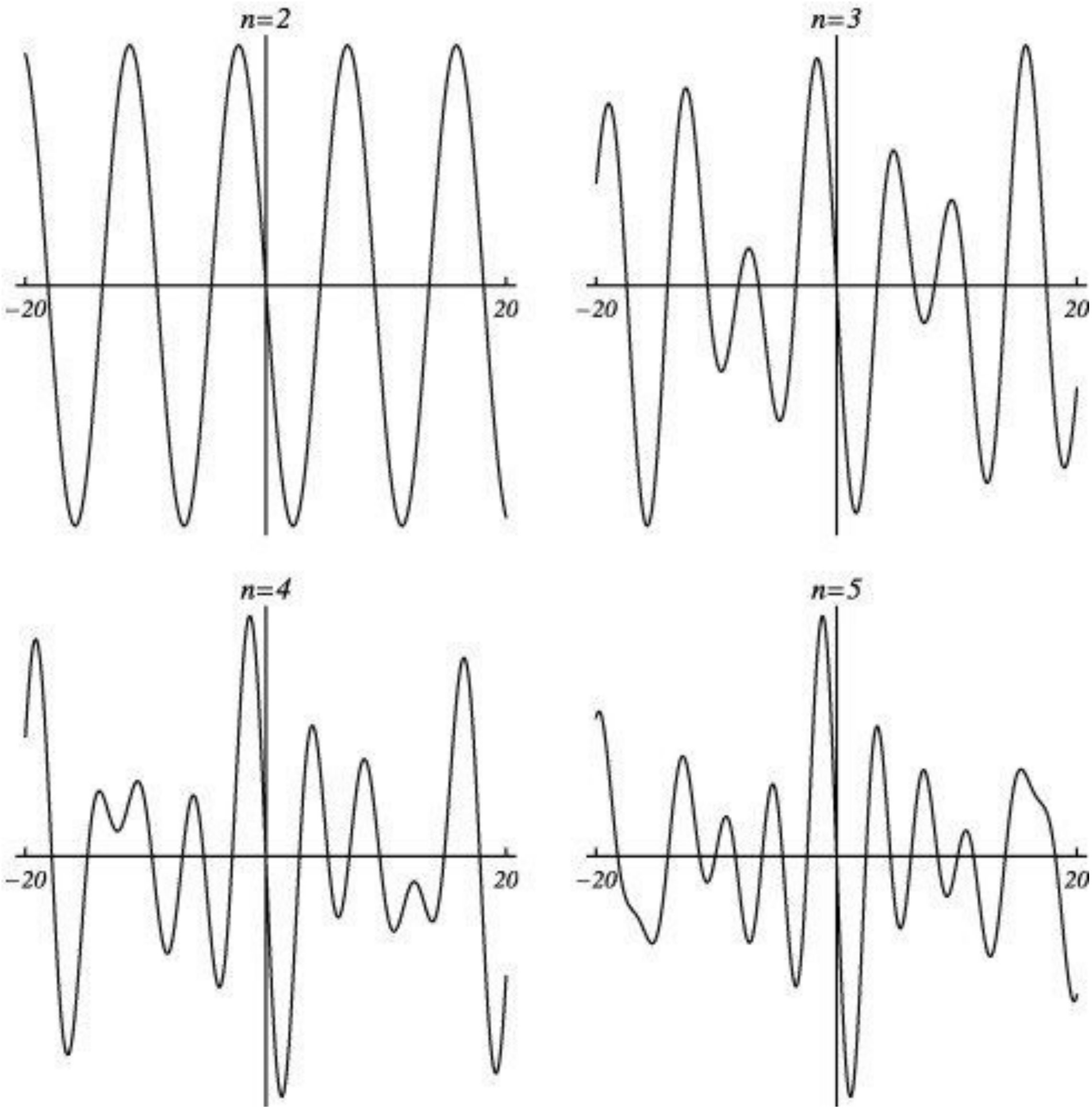


Figure 2.6: Imaginary part of $\zeta^{(\alpha)}$ with $x = 0$, $\alpha = 0.6$ and upper limit of the series $n = 60$..

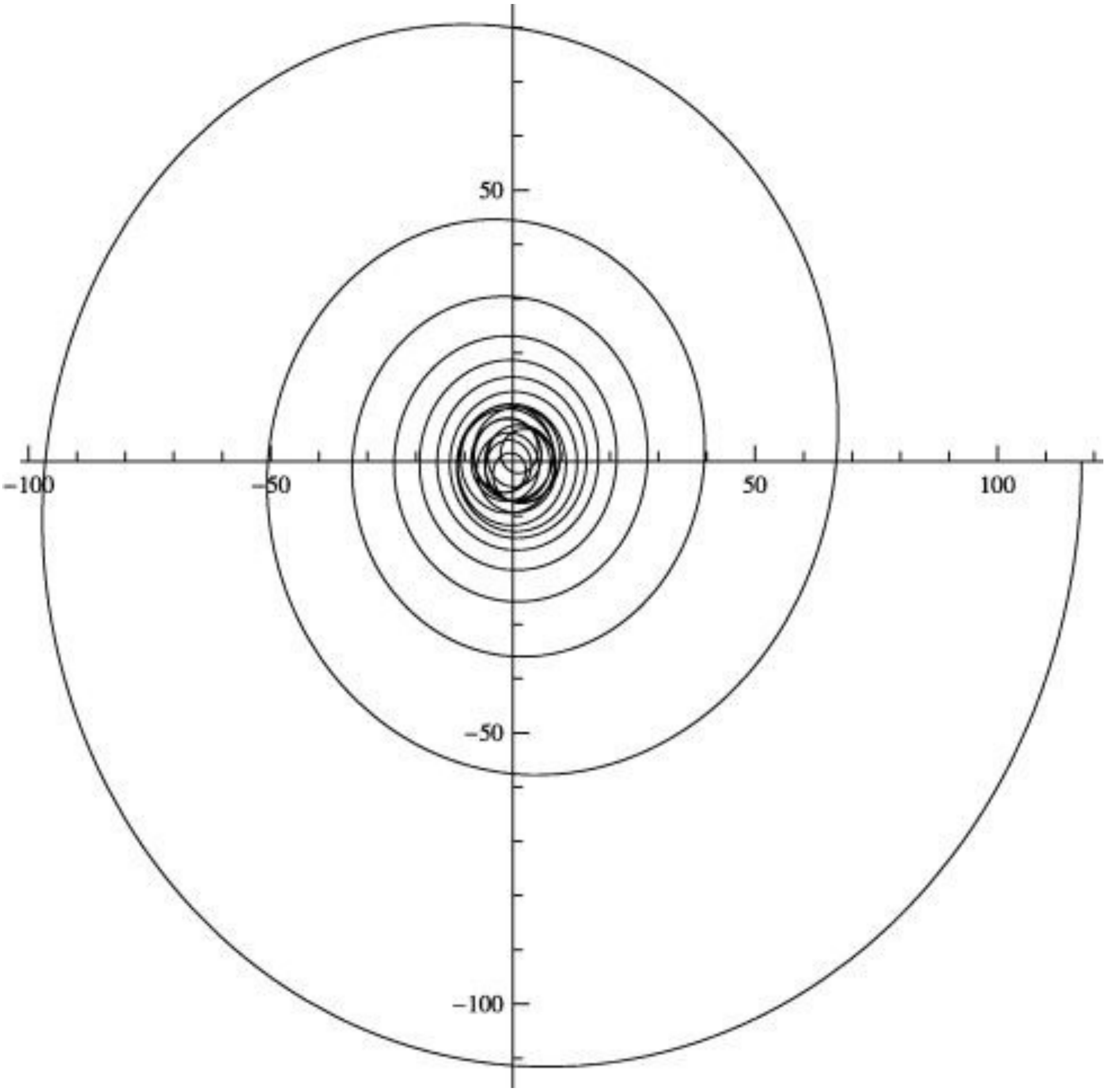


Figure 2.7: Parametric plot of $\zeta^{(\alpha)}$ with $x = 0$, $\alpha = 0.6$ and upper limit of the series $n = 60$.

Chapter 3

Functional equations

3.1 Introduction

In this chapter, the functional equation of $\zeta^{(\alpha)}$ is computed and extensively discussed, together with some of its generalizations. In order to realize these tasks, the α-order fractional derivative of ζ is recomputed by a generalization of the Grünwald-Letnikov fractional derivative (Ortigueira, 2011, Chap. 2). Since it satisfies the generalized Leibniz rule, by applying the previous fractional operator to the asymmetric functional equation of ζ, the result sought is easily derived. By using Bernoulli numbers, an integral representation for $\zeta^{(\alpha)}$ is obtained. According to the classical results of Apostol (1985) and Spira (1965), the functional equation of $\zeta^{(\alpha)}$ is written as a sum of sines and cosines and its simplified version is presented in what follows. By introducing the Hurwitz ζ function and the Lerch zeta function, two main generalizations of the aforementioned functional equation are given. Starting from the so-called Euler summation formula, an integral representation of $(2.9)_2$ is derived. By taking into account that the Grünwald-Letnikov fractional derivative defines a differential C-algebra (see Kolchin, 2012, Chap. 1), the Lerch zeta function provides a further generalization of these results.

The Chapter is organized as follows. Some details on the link between Bernoulli numbers and the Riemann ζ function are given in Section 3.2. The computation of $\zeta^{(\alpha)}$ by the Grünwald-Letnikov fractional derivative and some of its properties are expounded upon in Section 3.3, and an integral representation of $\zeta^{(\alpha)}$ via Bernoulli numbers is reported in Section 3.4. In Section 3.5, results concerning the functional equation of $\zeta^{(\alpha)}$ are given. Section 3.6 and Section 3.7 present a generalization of the functional equation associated with $\zeta^{(\alpha)}$ with respect to the Hurwitz ζ function and with the Lerch zeta function, respectively.

3.2 Bernoulli numbers and the Riemann ζ function

Bernoulli numbers show their importance in several fields of pure and applied mathematics. In particular, in analytic number theory, some values of the Riemann ζ function can be written using them. Bernoulli polynomials $B_n(s)$ are an interesting class of functions of the complex variable s. The functions $B_n = B_n(s)$ are defined by

$$\frac{z\mathrm{e}^{sz}}{\mathrm{e}^z - 1} = \sum_{n=0}^{\infty} \frac{B_n(s)}{n!} z^n \,, \qquad (z \in \mathbb{C} : |z| < 2\pi) \tag{3.1}$$

where the numbers $B_n(0)$ are called Bernoulli numbers and denoted by B_n (Apostol, 1998, Chap. 12). Hence,

$$\frac{z}{\mathrm{e}^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n \,. \qquad (z \in \mathbb{C} : |z| < 2\pi)$$

The main properties of Bernoulli numbers are summarized in the following statement.

Proposition 3.2.1. *Let $n \in \mathbb{N}_0$. The functions* (3.1) *are polynomials of the complex variable s given by*

$$B_n(s) = \sum_{k=0}^{n} \binom{n}{k} B_k \, s^{n-k} \,, \qquad (n \geq 2) \tag{3.2}$$

and satisfy the difference equation

$$B_n(s+1) - B_n(s) = ns^{n-1} \,. \qquad (n \geq 1) \tag{3.3}$$

Proof: See Apostol (1998, pp. 264–265).

□

By evaluating eq. (3.3) for $s = 0$, it is

$$B_n(1) = B_n(0) \,. \qquad (n \geq 2) \tag{3.4}$$

Eqs. (3.3), (3.4) do not provide any formula for computing Bernoulli numbers. On the contrary, they can be recursively obtained via (3.2). In fact, it is easy to compute that $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, etc. Some values of the Riemann ζ function can be expressed in terms of Bernoulli numbers. In particular, the following statement holds.

Proposition 3.2.2. *Let $k \in \mathbb{N}_0$ and let $n \in \mathbb{N}$. The values of ζ for non-positive integers and positive even numbers are given, respectively, by*

$$\zeta(-k) = -\frac{B_{k+1}(1)}{k+1} = \begin{cases} -\dfrac{1}{2}\,, & n = 0\,, \\ -\dfrac{B_{k+1}}{k+1}\,, & n \geq 1\,, \end{cases}$$

and

$$\zeta(2n) = (-1)^{n+1}\,\frac{(2\pi)^{2n} B_{2n}}{2\,(2n)!}\,.$$

Proof: See Apostol (1998, p. 266).

□

Functional equation (1.11) provides no information about $\zeta(2n+1)$ (since both its members vanish). No simple formula for positive odd values of ζ is known. Except for $\zeta(3)$ (whose irrationality was shown by Apéry in 1979), the rationality of $\zeta(2n+1)$ is still an open problem (Apostol, 1998, Chap. 12). Moreover, the derivative of the Riemann ζ function is linked with Bernoulli numbers by the following

Proposition 3.2.3. *Let $n \in \mathbb{N}_0$. The derivative of ζ at the negative even integers is given by*

$$\zeta'(-2n) = (-1)^n\,\frac{(2n)!}{2\,(2\pi)^{2n}}\,\zeta(2n+1)\,. \tag{3.5}$$

In particular, it is

$$\zeta'(0) = -\frac{\log(2\pi)}{2}\,. \tag{3.6}$$

Proof: By differentiating eq. (1.10) term by term, we have

$$\zeta'(s) = \Big(2(2\pi)^{s-1}\,\Gamma(1-s)\,\zeta(1-s)\Big)'\,\sin\left(\frac{\pi s}{2}\right) + 2(2\pi)^{s-1}\,\Gamma(1-s)\,\zeta(1-s)\,\frac{\pi}{2}\,\cos\left(\frac{\pi s}{2}\right)\,.$$

For $s = -2n$, since $\sin(-\pi n) = 0$ and $\cos(-\pi n) = (-1)^n$, it is

$$\zeta'(-2n) = \pi(2\pi)^{-(2n+1)}\,\Gamma(2n+1)\,\zeta(2n+1)\,\frac{\pi}{2}\,(-1)^n\,.$$

Furthermore, eq. (3.5) for $s = 0$ becomes

$$\zeta'(0) = \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n} = -\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} = -\frac{\log(2\pi)}{2}\,,$$

by taking into account that the sum of the alternating harmonic series is $\log(2\pi)$ (Hudelson, 2010).

□

At the end of this section, the classical definition of generalized binomial coefficient is given.

Definition 3.2.4 (generalized binomial coefficient)**.** *Let* $\alpha \in \mathbb{R}$ *and let* $n \in \mathbb{N}_0$*. The binomial coefficient* $\binom{\alpha}{n}$ *is defined* (Graham et al., 1994, pp. 153–154) *as usual by*

$$\binom{\alpha}{n} = \frac{\alpha^{\underline{n}}}{n!} = \frac{\alpha \cdot (\alpha-1) \cdots (\alpha-n+1)}{n!}\,, \tag{3.7}$$

where $\alpha^{\underline{n}}$ *represents the so-called falling factorial* (Graham et al., 1994, pp. 47–48).

3.3 Grünwald-Letnikov fractional derivative and recomputation of $\zeta^{(\alpha)}$

The Caputo-Ortigueira fractional derivative is not particularly suitable for deriving a functional equation for (2.2). Hence, it is been necessary to change the fractional operator. In particular, the problem can be reformulated in terms of the Grünwald-Letnikov fractional derivative (Pudlubny, 1999, Chap. 2), which together with the generalized Leibniz rule, easily provides the functional equation sought. First, the fractional derivative of the Riemann ζ function has to be recomputed by this new fractional derivative in order to show that it is in accordance with (2.2).

Definition 3.3.1 (generalized Grünwald-Letnikov fractional derivative)**.** *Let f be a complex-valued function of the complex variable s and let* $\alpha \in \left(\mathbb{R}_{>0} \setminus \mathbb{N}\right)$*. The α-order Grünwald-Letnikov fractional derivative of f is defined* (Ortigueira, 2011, Chap. 2) *by*

$$\mathrm{D}^{\alpha}_{\theta} f(s) \overset{\text{def}}{=} \mathrm{e}^{-i\theta\alpha} \lim_{|h|\to 0^+} \frac{\sum\limits_{k=0}^{\infty} \binom{\alpha}{k} (-1)^k f(s-kh)}{|h|^{\alpha}}\,, \tag{3.8}$$

where $h \in \mathbb{C}$ *and* $\theta = \operatorname{Arg} h$*.*

The previous definition clearly generalizes the Grünwald-Letnikov fractional derivative based on the incremental ratio (Pudlubny, 1999, Chap. 2) to the whole complex plane. The problem of proving the weakest conditions with regards to the existence of (3.8) appears to be complicated enough, even if some necessary conditions exist. In fact, since binomial coefficients satisfy

$$\left|\binom{\alpha}{k}\right| \le \frac{C}{k^{\alpha+1}}\,, \qquad \text{(for some constant } C>0)$$

the product $f(s)\,\frac{C}{k^{\alpha+1}}$ has to decrease at least as $\frac{C}{k^{\alpha+1}}$ for $k \to \infty$ (Ortigueira, 2011, Chap. 2). In order to get a physical interpretation of (3.8), assume that s is a time variable and that $h \in \mathbb{R}$, i.e. $\theta \in \{0, \pi\}$. In the first case ($\theta = 0$), definition 3.3.1 provides the following fractional derivative

$$\mathrm{D}_f^{\alpha} f(s) = \lim_{h\to 0^+} \frac{\sum_{k=0}^{\infty} \binom{\alpha}{k} (-1)^k f(s-kh)}{h^{\alpha}}\,, \tag{3.9}$$

since $|h| = h \ge 0$. In the literature, this is called the α-order forward Grünwald-Letnikov derivative since the future values cannot be used. By using the terminology of signal theory, (3.9) provides a linear and casual system. In the other case ($\theta = \pi$), the past values cannot be used, hence the associated operator is known as the backward Grünwald-Letnikov derivative (Ortigueira, 2011, Chap. 2). The fractional derivative of the Riemann ζ function can be recomputed by using (3.9).

Theorem 3.3.2. *Let s be a complex variable and let* $\alpha \in \left(\mathbb{R}_{>0} \setminus \mathbb{N}\right)$. *The* α*-order fractional derivative of the Riemann* ζ *function, computed by* (3.9)*, is given by*

$$\zeta^{(\alpha)}(s) = \mathrm{e}^{i\pi\alpha} \sum_{n=2}^{\infty} \frac{\log^{\alpha} n}{n^s}\,. \tag{3.10}$$

Proof: From (3.9), by writing $f(s) = \zeta(s)$, it is

$$\begin{aligned}\zeta^{(\alpha)}(s) &= \lim_{h\to 0^+} \frac{\sum_{k=0}^{\infty} \binom{\alpha}{k} (-1)^k \zeta(s-kh)}{h^{\alpha}} = \lim_{h\to 0^+} \frac{1}{h^{\alpha}} \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} \binom{\alpha}{k} (-1)^k \frac{1}{n^{s-kh}} \\ &= \sum_{n=1}^{\infty} \frac{1}{n^s} \lim_{h\to 0^+} \frac{1}{h^{\alpha}} \sum_{k=0}^{\infty} \binom{\alpha}{k} (-1)^k n^{kh}\,.\end{aligned} \tag{3.11}$$

Taking into account the well know binomial series expansion, it follows that

$$\sum_{k=0}^{\infty} \binom{\alpha}{k} (-1)^k n^{kh} = \sum_{k=0}^{\infty} \binom{\alpha}{k} (-n^h)^k = \left(1 - n^h\right)^{\alpha}\,, \tag{3.12}$$

so that

$$\zeta^{(\alpha)}(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \lim_{h\to 0^+} \frac{\left(1-n^h\right)^\alpha}{h^\alpha} . \tag{3.13}$$

The RHS of (3.11) can converge even if the binomial series (3.12) diverges. In fact,

$$\lim_{h\to 0^+} \left(\frac{1-n^h}{h}\right)^\alpha , \tag{3.14}$$

provides an indeterminate form $\frac{0}{0}$. By L'Hôpital's rule, it is

$$\begin{aligned} \lim_{h\to 0^+} \left(\frac{1-n^h}{h}\right)^\alpha &= \left(\lim_{h\to 0^+} \frac{1-n^h}{h}\right)^\alpha \\ &= \left(-\lim_{h\to 0^+} n^h \log n\right)^\alpha = \mathrm{e}^{i\pi\alpha} \log^\alpha n . \end{aligned} \tag{3.15}$$

By substituting (3.15) into (3.13), the proof follows.

□

Theorem 3.3.2 shows that two different fractional derivatives provide the same result with regards to the computation of $\zeta^{(\alpha)}$. In fact, up till now there has not been a unique definition of the fractional derivative. It represents a weak point of the fractional calculus. A direct consequence of theorem 3.3.2 is reported below.

Corollary 3.3.3. *Under the same hypotheses of theorem* 3.3.2*, it is*

$$\mathrm{D}_f^\alpha (2\pi)^s = (2\pi)^s \,\mathrm{e}^{i\pi\alpha} \log^\alpha (2\pi) . \tag{3.16}$$

Proof : The thesis follows from the theorem 3.3.2 by replacing n with 2π and s with $-s$.

□

Corollary 3.3.3 takes on great importance in the proof of theorem 3.5.1, while the fundamental property that provides the functional equation of $\zeta^{(\alpha)}$ is derived (Ortigueira, 2011, pp. 18–19) from the following

Theorem 3.3.4 (generalized Leibniz rule for D_f^α)**.** *Let f and g be two complex-valued functions of the complex variable s. If f is analytic in a region $D \subseteq \mathbb{C}$, it is*

$$\mathrm{D}_f^\alpha \left(f(s)\, g(s)\right) = \sum_{n=0}^{\infty} \binom{\alpha}{n} f^{(n)}(s)\, g^{(\alpha-n)}(s) . \tag{3.17}$$

Proof: Let ψ be a complex-valued function of s such that $\psi(s) = f(s)\,g(s)$ for every $s \in \mathbb{C}$. The direct difference operator Δ_h is defined (Ortigueira, 2011, pp. 12–13) by

$$\Delta_h \psi(s) = \psi(s) - \psi(s-h) \,, \tag{3.18}$$

where $h \in \mathbb{C}$. Iterating N times Δ_h, it follows that

$$\Delta_h^N \psi(s) = \sum_{k=0}^{N} (-1)^k \binom{N}{k} \psi(s-kh) \,, \tag{3.19}$$

so that

$$\sum_{n=0}^{k} (-1)^n \binom{k}{n} \Delta_h^n \psi(s) = \sum_{n=0}^{k} (-1)^n \binom{k}{n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \psi(s-kh) = \psi(s-kh) \,,$$

that is

$$\psi(s-kh) = \sum_{n=0}^{k} (-1)^n \binom{k}{n} \Delta_h^n \psi(s) \,. \tag{3.20}$$

Eq. (3.19) can easily be extended to the fractional case (Diaz and Osler, 1974) by

$$\Delta_h^\alpha \psi(s) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} \psi(s-kh) \,, \tag{3.21}$$

hence (3.9) becomes

$$\mathrm{D}_f^\alpha \psi(s) = \lim_{h\to 0^+} \frac{\Delta_h^\alpha \psi(s)}{h^\alpha} \,, \tag{3.22}$$

where

$$\Delta_h^\alpha \psi(s) \overset{(3.21)}{=} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(s-kh)\,g(s-kh) \,.$$

Since $g(s-kh)$ can be written by using (3.20), it follows that

$$
\begin{aligned}
\Delta_h^\alpha \psi(s) &= \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(s-kh) \sum_{n=0}^{k} (-1)^n \binom{k}{n} \Delta_h^n g(s) \\
&= \sum_{n=0}^{\infty} (-1)^n \Delta_h^n g(s) \sum_{k=n}^{\infty} (-1)^k \binom{k}{n} \binom{\alpha}{k} f(s-kh) \\
&= \sum_{n=0}^{\infty} \Delta_h^n g(s) \sum_{k=0}^{\infty} (-1)^k \binom{k+n}{n} \binom{\alpha}{k+n} f(s-kh-nh) \\
&= \sum_{n=0}^{\infty} \binom{\alpha}{n} \Delta_h^n g(s) \sum_{k=0}^{\infty} (-1)^k \binom{\alpha-n}{k} f(s-kh-nh) \ ,
\end{aligned}
$$

since

$$
\binom{k+n}{n}\binom{\alpha}{k+n} = \frac{\alpha^{\underline{k+n}}}{n!k!} = \frac{\alpha\cdot(\alpha-1)\cdots(\alpha-(k+n)+1)}{n!k!} = \binom{\alpha}{n}\binom{\alpha-n}{k} \ .
$$

Therefore

$$
\begin{aligned}
\frac{\Delta_h^\alpha \psi(s)}{h^\alpha} &= \frac{\sum_{n=0}^{\infty} \binom{\alpha}{k} \Delta_h^n g(s) \sum_{k=0}^{\infty} (-1)^k \binom{\alpha-n}{k} f(s-kh-nh)}{h^\alpha} \\
&= \sum_{n=0}^{\infty} \binom{\alpha}{n} \frac{\Delta_h^n g(s)}{h^n} \frac{\sum_{k=0}^{\infty} (-1)^k \binom{\alpha-n}{k} f(s-kh-nh)}{h^{\alpha-n}} \xrightarrow{h\to 0^+} \sum_{n=0}^{\infty} \binom{\alpha}{n} f^{(n)}(s) \\
&\quad \cdot g^{(\alpha-n)}(s) \ .
\end{aligned}
$$

By taking into account (3.22), the proof follows.

□

Eq. (3.17) holds in the analytic region $D \subseteq \mathbb{C}$ except over an eventual branch cut line. The hypotheses of theorem 3.3.4 ensure that the RHS of (3.17) is non-commutative, while the commutativity follows if both f and g are analytic in the complex region D. Moreover, a direct consequence of definition (3.22), that will be used in the following, is reported below.

Proposition 3.3.5 (consistency with integer-order derivative)**.** *Let $m-1 < \alpha < m \in \mathbb{Z}_{>0}$ and let f be a complex-valued function of the complex variable s analytic in a region $D \subseteq \mathbb{C}$ such that $\frac{\Delta_h^\alpha f(s)}{h^\alpha}$ (called fractional incremental ratio) is uniformly convergent in D. Under these hypotheses, the α-order forward Grünwald-Letnikov derivative of f is given by*

$$\begin{cases} \mathrm{D}^{\alpha}_{f} f(s) \xrightarrow{\alpha \to m^-} f^{(m)}(s) \ , \\ \\ \mathrm{D}^{\alpha}_{f} f(s) \xrightarrow{\alpha \to (m-1)^+} f^{(m-1)}(s) \ . \end{cases} \tag{3.23}$$

Proof: Taking into account (3.22), it is

$$\lim_{\alpha \to m^-} \mathrm{D}^{\alpha}_{f} f(s) = \lim_{\alpha \to m^-} \lim_{h \to 0^+} \frac{\Delta^{\alpha}_{h} f(s)}{h^{\alpha}} = \lim_{h \to 0^+} \lim_{\alpha \to m^-} \frac{\Delta^{\alpha}_{h} f(s)}{h^{\alpha}} = \lim_{h \to 0^+} \frac{\Delta^{m}_{h} f(s)}{h^{m}} = f^{(m)}(s) \ ,$$

i.e. $(3.23)_1$. Analogously $(3.23)_2$ can be derived, which completes the proof.

□

In order to show the importance of the previous proposition, let us consider the case of the Riemann ζ function. From (3.11), (3.12) and (3.21), it follows that

$$\frac{\Delta^{\alpha}_{h} \zeta(s)}{h^{\alpha}} = \sum_{n=1}^{\infty} \frac{1}{n^s} \left(\frac{1 - n^h}{h} \right)^{\alpha} \ . \tag{3.24}$$

Therefore, by applying the Weierstrass M-test, the Riemann ζ function satisfies the hypotheses of proposition 3.3.5.

3.4 Integral representation of $\zeta^{(\alpha)}$ via Bernoulli numbers

In this section, an integral representation of $\zeta^{(\alpha)}$ in terms of Bernoulli numbers is provided. It is based on the following result (Abramowitz and Stegun, 2014, p. 807)

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + \sum_{r=1}^{n} \frac{B_{2r}}{2r} \binom{s+2r-2}{2r-1} - \binom{s+2n}{2n+1} \int_1^{\infty} \frac{P_{2n+1(x)}}{x^{s+2n+1}} \mathrm{d}x \ , \tag{3.25}$$

where B_{2r} are Bernoulli numbers and

$$P_{2n+1} = (-1)^{n+1} \frac{2(2n+1)!}{(2\pi)^{2n+1}} \sum_{k=1}^{\infty} \frac{\sin 2k\pi x}{k^{2n+1}} \ ,$$

is nothing more than the periodic Bernoulli function (Apostol, 1998, p. 267). Eq. (3.25) gives an integral representation of ζ through Bernoulli numbers and holds in the half-plane $\mathrm{Re}\, s > -2n$ (with $n \in \mathbb{N}$). For simplicity of notation, by introducing

$$
\begin{cases}
Q_m(s) \stackrel{\text{def}}{=} \binom{s+m-1}{m}, & (m \in \mathbb{N}_0) , \\
I_m(s) \stackrel{\text{def}}{=} \int_1^\infty \frac{P_m(x)}{x^{s+m}} \mathrm{d}x , & (m \in \mathbb{N}_0) ,
\end{cases}
\tag{3.26}
$$

eq. (3.25) becomes

$$
\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + \sum_{r=1}^{n} \frac{B_{2r}}{2r} Q_{2r-1}(s) - Q_{2n+1}(s)\, I_{2n+1}(s) . \tag{3.27}
$$

Apostol (1985) has shown that the integer derivative $\zeta^{(k)}$ admits the following integral representation

$$
\zeta^{(k)}(s) = \frac{(-1)^k k!}{(s-1)^{k+1}} + \sum_{r=1}^{n} \frac{B_{2r}}{2r} Q_{2r-1}^{(k)}(s) - \sum_{j=0}^{k} \binom{k}{j} Q_{2n+1}^{(j)}(s)\, I_{2n+1}^{(k-j)}(s) , \tag{3.28}
$$

which holds in the half-plane $\mathrm{Re}\, s > -2n$ (with $n \in \mathbb{N}$). A fractional version of eq. (3.28), that is an integral representation of $\zeta^{(\alpha)}$ via Bernoulli numbers is given in the following

Theorem 3.4.1. *Let s be a complex variable such that* $\mathrm{Re}\, s > 1$ *and let* $\alpha \in \left(\mathbb{R}_{>0} \setminus \mathbb{N}\right)$. *Under these hypotheses, it is*

$$
\boxed{\zeta^{(\alpha)}(s) = \frac{s^\alpha}{s-1} + \sum_{r=1}^{n} \frac{B_{2r}}{2r} Q_{2r-1}^{(\alpha)}(s) - \sum_{j=0}^{\infty} \binom{\alpha}{j} Q_{2n+1}^{(j)}(s)\, I_{2n+1}^{(\alpha-j)}(s) ,} \tag{3.29}
$$

where Q_m and I_m are given by (3.26).

Proof: By applying the fractional operator D_f^α to both members of (3.27), it is

$$
\zeta^{(\alpha)}(s) = \mathrm{D}_f^\alpha\left(\frac{1}{s-1}\right) + \mathrm{D}_f^\alpha\left(\frac{1}{2}\right) + \sum_{r=1}^{n} \frac{B_{2r}}{2r} Q_{2r-1}^{(\alpha)}(s) - \sum_{k=0}^{\infty} \binom{\alpha}{k} Q_{2r-1}^{(k)} I_{2n+1}^{(\alpha-k)} , \tag{3.30}
$$

by taking into account theorem 3.3.4. Being $\alpha > 0$, it follows (Ortigueira, 2011, p. 23) that

$$
\mathrm{D}_f^\alpha\left(\frac{1}{2}\right) = 0 .
$$

In order to compute $\mathrm{D}_f^\alpha\left(\frac{1}{s-1}\right)$, let us recall (Ortigueira, 2011, p. 59) that

$$
\mathcal{L}\left(\mathrm{D}_f^\alpha\left(f(t)\right)\right) = s^\alpha\, \mathcal{L}\left(f(t)\right) , \qquad (\mathrm{Re}\, s > 0)
$$

where $\mathcal{L}$ represents the (two-sided) Laplace transform. From the Laplace transform theory (Beerends et al., 2003, Chap. 12), it is well know that

$$\mathcal{L}\left(\mathrm{e}^{t}\right)=\frac{1}{s-1}\,. \qquad (\mathrm{Re}\, s>1)$$

Since the exponential function is uniformly convergent in its domain, $\mathcal{L}$ and D_f^{α} can be exchanged by using the classical theorems on passing to the limit under the integral sign and on term by term integration (Walnut, 2002, Chap. 1). It follows that

$$\mathrm{D}_f^{\alpha}\left(\frac{1}{s-1}\right)=\mathrm{D}_f^{\alpha}\left(\mathcal{L}\left(\mathrm{e}^{t}\right)\right)=\mathcal{L}\left(\mathrm{D}_f^{\alpha}\left(\mathrm{e}^{t}\right)\right)=\frac{s^{\alpha}}{s-1}\,. \qquad (\mathrm{Re}\, s>1) \tag{3.31}$$

By substituting (3.31) into (3.30) and given that eq. (3.27) holds in $\mathrm{Re}\, s>-2n$, the proof follows directly.

□

The importance of theorem 3.4.1 lies in the link between $\zeta^{(\alpha)}$ and Bernoulli numbers. In order to show the consistency of (3.29), it is sufficient to show that it reduces to (3.25). This property cannot be easily derived by starting from eq. (3.29). Nevertheless, reading the previous proof backwards, eq. (3.30) converges to (3.25) as α approaches 0^{+},.

3.5 Functional equation of $\zeta^{(\alpha)}$

In this section, the functional equation for $\zeta^{(\alpha)}$ is presented. In particular, the α-order fractional derivatives of ζ fulfills the following

Theorem 3.5.1 (Guariglia and Silvestrov 2017)**.** *Let s be a complex variable and let $\alpha\in\left(\mathbb{R}_{>0}\setminus\mathbb{N}\right)$. For all $s\in\mathbb{C}$, it is*

$$\boxed{\begin{aligned}\zeta^{(\alpha)}(s)=2\,(2\pi)^{s-1}\,\mathrm{e}^{i\pi\alpha}\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}A_{n,j,k}^{\alpha}\,\zeta^{(n)}(1-s)\left(-\frac{\pi}{2}\right)^{j}\\ \cdot\sin\left(\frac{\pi}{2}(s+j)\right)\,\frac{\Gamma^{(k)}(1-s)}{\log^{n+j+k-\alpha}(2\pi)}\,,\end{aligned}} \tag{3.32}$$

where $A_{n,j,k}^{\alpha}=\dfrac{\overline{\alpha}^{\,n+j+k}}{n!\,j!\,k!}\,.$

Proof: By applying (3.9) to both members of (1.10), we have

$$\zeta^{(\alpha)}(s) = \frac{1}{\pi}\,\mathrm{D}_f^{\alpha}\left(\zeta(1-s)\,\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\,(2\pi)^s\right)\;. \tag{3.33}$$

For reasons of simplicity, the fractional derivative in the RHS of (3.33) will be indicated with I. The iteration of the generalized Leibniz rule gives

$$\begin{aligned}
\mathrm{I} &= \sum_{n=0}^{\infty}\binom{\alpha}{n}\frac{\mathrm{d}^n}{\mathrm{d}s^n}\Big(\zeta(1-s)\Big)\,\mathrm{D}_f^{\alpha-n}\left(\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\,(2\pi)^s\right)\\
&= \sum_{n=0}^{\infty}\binom{\alpha}{n}\frac{\mathrm{d}^n}{\mathrm{d}s^n}\Big(\zeta(1-s)\Big)\sum_{j=0}^{\infty}\binom{\alpha-n}{j}\frac{\mathrm{d}^j}{\mathrm{d}s^j}\left(\sin\left(\frac{\pi s}{2}\right)\right)\mathrm{D}_f^{\alpha-n-j}\Big(\Gamma(1-s)(2\pi)^s\Big)\\
&= \sum_{n=0}^{\infty}\binom{\alpha}{n}\frac{\mathrm{d}^n}{\mathrm{d}s^n}\Big(\zeta(1-s)\Big)\sum_{j=0}^{\infty}\binom{\alpha-n}{j}\frac{\mathrm{d}^j}{\mathrm{d}s^j}\left(\sin\left(\frac{\pi s}{2}\right)\right)\sum_{k=0}^{\infty}\binom{\alpha-n-j}{k}\\
&\quad\cdot\frac{\mathrm{d}^k}{\mathrm{d}s^k}\Big(\Gamma(1-s)\Big)\,\mathrm{D}_f^{\alpha-n-j-k}(2\pi)^s\;.
\end{aligned}$$

Being

$$\begin{cases}
\dfrac{\mathrm{d}^n}{\mathrm{d}s^n}\Big(\zeta(1-s)\Big) = \mathrm{e}^{i\pi n}\,\zeta^{(n)}(1-s)\;,\\[2ex]
\dfrac{\mathrm{d}^j}{\mathrm{d}s^j}\left(\sin\left(\dfrac{\pi s}{2}\right)\right) = \left(\dfrac{\pi}{2}\right)^j\sin\left(\dfrac{\pi}{2}(s+j)\right)\;,\\[2ex]
\dfrac{\mathrm{d}^k}{\mathrm{d}s^k}\Big(\Gamma(1-s)\Big) = \mathrm{e}^{i\pi k}\,\Gamma^{(k)}(1-s)\;,
\end{cases}$$

and since

$$\begin{cases}
\mathrm{D}_f^{\alpha-n-j-k}(2\pi)^s = (2\pi)^s\,\mathrm{e}^{i\pi(\alpha-n-j-k)}\,\log^{\alpha-n-j-k}(2\pi)\;,\\[2ex]
A^{\alpha}_{n,j,k} \stackrel{\text{def}}{=} \dbinom{\alpha}{n}\dbinom{\alpha-n}{j}\dbinom{\alpha-n-j}{k} = \dfrac{\alpha^{\underline{n+j+k}}}{n!\,j!\,k!}\;,
\end{cases}$$

it follows that

$$\begin{aligned}
\mathrm{I} &= \sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}A^{\alpha}_{n,j,k}\,\zeta^{(n)}(1-s)\left(\frac{\pi}{2}\right)^j\sin\left(\frac{\pi}{2}(s+j)\right)\Gamma^{(k)}(1-s)\\
&\quad\cdot(2\pi)^s\,\mathrm{e}^{i\pi(\alpha-j)}\log^{\alpha-n-j-k}(2\pi) = (2\pi)^s\,\mathrm{e}^{i\pi\alpha}\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}A^{\alpha}_{n,j,k}\\
&\quad\cdot\zeta^{(n)}(1-s)\left(-\frac{\pi}{2}\right)^j\sin\left(\frac{\pi}{2}(s+j)\right)\frac{\Gamma^{(k)}(1-s)}{\log^{n+j+k-\alpha}(2\pi)}\;.
\end{aligned} \tag{3.34}$$

By substituting (3.34) into (3.33), it is

$$\zeta^{(\alpha)}(s) = \frac{1}{\pi}\,\mathrm{e}^{i\pi\alpha}\,(2\pi)^s \sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} A^{\alpha}_{n,j,k}\,\zeta^{(n)}(1-s)\left(-\frac{\pi}{2}\right)^j \sin\left(\frac{\pi}{2}(s+j)\right)$$
$$\cdot \frac{\Gamma^{(k)}(1-s)}{\log^{n+j+k-\alpha}(2\pi)} = 2\,(2\pi)^{s-1}\,\mathrm{e}^{i\pi\alpha}\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} A^{\alpha}_{n,j,k}\,\zeta^{(n)}(1-s)\left(-\frac{\pi}{2}\right)^j$$
$$\cdot \sin\left(\frac{\pi}{2}(s+j)\right)\frac{\Gamma^{(k)}(1-s)}{\log^{n+j+k-\alpha}(2\pi)}\;.$$

Eq. (3.32) holds for all $s \in \mathbb{C}$ since (1.10) is well-defined for every complex number s (Apostol, 1998, Chap. 12).

□

Theorem 3.5.1 gives a functional equation for $\zeta^{(\alpha)}$. Unfortunately, starting from eq. (1.12) and repeating the previous proof, a symmetrical version of (3.32) cannot be determined since the invariance transformation $s \to 1-s$ is no longer preserved by using (3.9) for both members of (1.12). In order to show the consistency of eq. (3.32), it is sufficient to show that it reduces to (1.10) as $\alpha \to 0^+$. Since proposition 3.3.5 holds for the Riemann ζ function, it is

$$\zeta^{(\alpha)}(s) \xrightarrow{\alpha\to 0^+} \zeta(s)\;.$$

The same holds for their RHSs. In fact, even if the RHS of (3.32) appears to be sufficiently complicated, the proof (of the theorem) 3.5.1 can be read backwards in order to obtain (3.33). From $(3.23)_1$, it follows that

$$\mathrm{D}^{\alpha}_{f} f(s) \xrightarrow{\alpha\to 0^+} f^{(0)}(s) = f(s)\;, \tag{3.35}$$

hence the RHS of (3.32) converges to the RHS of (1.10) as α goes to 0^+. The simulations showed that the RHS of (3.32) is far more complicated than that of (1.10), since each of its approximations with finite upper limits has produced a buffer overflow in several numerical tools for mathematical computation.

3.5.1 Simplified version

In order to minimize the computational cost of eq. (3.32), the approach proposed by Apostol (1985) and Spira (1965) for the integer order derivative of ζ was followed. The computational cost of the above-mentioned RHS was considerably reduced to only one infinite series (see theorem 3.5.4). In the literature, Apostol and Spira were the first researchers to investigate the properties of the integer derivative $\zeta^{(k)}$. In particular, Spira discovered the following functional equation

$$\begin{aligned}(-1)^k \zeta^{(k)}(1-s) = 2\,(2\pi)^{-s} \sum_{h=0}^{k}\sum_{n=0}^{k} \left(a_{hkn} \cos\left(\frac{\pi s}{2}\right) + b_{hkn} \sin\left(\frac{\pi s}{2}\right)\right) \\ \cdot \Gamma^{(h)}(s)\zeta^{(n)}(s)\ ,\end{aligned} \tag{3.36}$$

where a_{hkn} and b_{hkn} are constants. By replacing s with $1-s$, eq. (3.36) becomes

$$\begin{aligned}\zeta^{(k)}(s) = 2\,(2\pi)^{s-1}\,\mathrm{e}^{i\pi k} \sum_{h=0}^{k}\sum_{n=0}^{k} \left(a_{hkn} \cos\left(\frac{\pi}{2}(1-s)\right) + b_{hkn} \sin\left(\frac{\pi}{2}(1-s)\right)\right) \\ \cdot \Gamma^{(h)}(1-s)\zeta^{(n)}(1-s)\ .\end{aligned} \tag{3.37}$$

The importance of eqs. (3.36), (3.37) lies in the fact that they provide the functional equation of $\zeta^{(k)}$ in terms of sines and cosines. Furthermore, Apostol (1985) showed that they can be easily written in terms of complex exponentials. A modified version of this result is reported below.

Proposition 3.5.2. *Let $k \in \mathbb{N}$. For all $s \in \mathbb{C}$, it is*

$$\begin{aligned}\zeta^{(k)}(s) = \sum_{h=0}^{k} \binom{k}{h} \mathrm{e}^{i\pi(k-h)} \left(\mathrm{e}^{(1-s)w} w^{k-h} - \mathrm{e}^{(1-s)\overline{w}+i\pi}\,(\overline{w})^{k-h}\right) \\ \cdot \left(\Gamma(1-s)\,\zeta(1-s)\right)^{(h)}\ ,\end{aligned} \tag{3.38}$$

where $w = -\log(2\pi) - i\pi/2$.

Proof: Eq. (1.10) can be written as follows

$$\begin{aligned}\zeta(s) &= \Gamma(1-s)\zeta(1-s)\,(2\pi)^{s-1}\, 2\sin\left(\frac{\pi s}{2}\right) \\ &= \Gamma(1-s)\zeta(1-s)\left(\mathrm{e}^{(1-s)[-\log(2\pi)-i\pi/2]} - \mathrm{e}^{(1-s)[-\log(2\pi)+i\pi/2]-i\pi}\right)\ ,\end{aligned}$$

being

$$\begin{cases} (2\pi)^{s-1} = \mathrm{e}^{(s-1)\log(2\pi)} \,, \\ 2\sin\left(\dfrac{\pi s}{2}\right) = \dfrac{\mathrm{e}^{i\frac{\pi s}{2}} - \mathrm{e}^{-i\frac{\pi s}{2}}}{i} = \mathrm{e}^{i\frac{\pi}{2}(s-1)} - \mathrm{e}^{-i\frac{\pi}{2}(s+1)} \,, \\ \mathrm{e}^{-i\frac{\pi}{2}(s+1)} = \mathrm{e}^{-i\frac{\pi}{2}s+i\frac{\pi}{2}-i\pi} = \mathrm{e}^{i\frac{\pi}{2}(1-s)-i\pi} \,. \end{cases}$$

By introducing the complex function ψ given by

$$\psi(s,w,z) \stackrel{\mathrm{def}}{=} \Gamma(s)\zeta(s)\,\mathrm{e}^{sw+z} \,, \qquad (z \in \mathbb{C})$$

it is

$$\zeta(s) = \psi(1-s,w) - \psi(1-s,\overline{w},-i\pi) \,, \tag{3.39}$$

with $\psi(1-s,w) = \psi(1-s,w,0)$. Differentiating (3.39) k times, we have

$$\zeta^{(k)}(s) = \frac{\mathrm{d}^k}{\mathrm{d}s^k}\big(\psi(1-s,w)\big) - \frac{\mathrm{d}^k}{\mathrm{d}s^k}\big(\psi(1-s,\overline{w},-i\pi)\big) \,.$$

Since

$$\begin{aligned} \frac{\mathrm{d}^k}{\mathrm{d}s^k}\big(\psi(1-s,w,z)\big) &= \frac{\mathrm{d}^k}{\mathrm{d}s^k}\Big(\Gamma(1-s)\zeta(1-s)\,\mathrm{e}^{(1-s)w+z}\Big) \\ &= \sum_{h=0}^{k}\binom{k}{h}\big(\Gamma(1-s)\,\zeta(1-s)\big)^{(h)}\Big(\mathrm{e}^{(1-s)w+z}\Big)^{(k-h)} \,, \end{aligned}$$

and

$$\Big(\mathrm{e}^{(1-s)w+z}\Big)^{(k-h)} = \mathrm{e}^{i\pi(k-h)}\,w^{(k-h)}\,\mathrm{e}^{(1-s)w+z} \,,$$

it follows that

$$\frac{\mathrm{d}^k}{\mathrm{d}s^k}\big(\psi(1-s,w,z)\big) = \sum_{h=0}^{k}\binom{k}{h}\mathrm{e}^{i\pi(k-h)}\,\mathrm{e}^{(1-s)w+z}\,w^{(k-h)}\big(\Gamma(1-s)\,\zeta(1-s)\big)^{(h)} \,.$$

Therefore

$$
\begin{aligned}
\zeta^{(k)}(s) = \sum_{h=0}^{k} \binom{k}{h} \mathrm{e}^{i\pi(k-h)} \left(\mathrm{e}^{(1-s)w}\, w^{(k-h)} - \mathrm{e}^{(1-s)\overline{w}+i\pi}\, (\overline{w})^{(k-h)} \right) \\
\cdot \left(\Gamma(1-s)\,\zeta(1-s) \right)^{(h)} ,
\end{aligned}
$$

given that the complex exponential is a $2\pi i$ – periodic function.

□

Eq. (3.38) differs from the classical results of Apostol (1985), since it is derived from (1.10) instead of (1.11). In order to find a fractional counterpart of (3.38), the following result is given.

Lemma 3.5.3. *Let s be a complex variable, let $\alpha \in \left(\mathbb{R}_{>0} \setminus \mathbb{N}\right)$ and let $h \in \mathbb{N}_0$. For all $s \in \mathbb{C}$ and for all $w \in \mathbb{C}$ such that* $\mathrm{Re}\, w < 0$*, it is*

$$
\mathrm{D}_f^{\alpha-h} \left(\mathrm{e}^{(1-s)w} \right) = \mathrm{e}^{i\pi(\alpha-h)} w^{\alpha-h} \mathrm{e}^{(1-s)w} ,
$$

where $h \in \mathbb{N}_0$.

Proof : The fractional operator (3.9) for $f(s) = \mathrm{e}^{(1-s)w}$ becomes

$$
\mathrm{D}_f^{\alpha-h} \left(\mathrm{e}^{(1-s)w} \right) = \mathrm{e}^{(1-s)w} \lim_{l \to 0^+} \frac{\sum_{k=0}^{\infty} (-1)^k \binom{\alpha-h}{k} \mathrm{e}^{klw}}{l^{\alpha-h}} . \tag{3.40}
$$

The series in the RHS of (3.40) converges to $g(w) = \left(1 - \mathrm{e}^{lw}\right)^{\alpha-h}$ if $\left|\mathrm{e}^{lw}\right| < 1$, that is $\mathrm{Re}\, w < 0$. By using L'Hôpital's rule, we finally get

$$
\begin{aligned}
\mathrm{D}_f^{\alpha-h} \left(\mathrm{e}^{(1-s)w} \right) &= \mathrm{e}^{(1-s)w} \lim_{l \to 0^+} \frac{\left(1 - \mathrm{e}^{wl}\right)^{\alpha-h}}{l^{\alpha-h}} = \mathrm{e}^{(1-s)w} \left(\lim_{l \to 0^+} \frac{1 - \mathrm{e}^{wl}}{l} \right)^{\alpha-h} \\
&= \mathrm{e}^{i\pi(\alpha-h)}\, w^{\alpha-h}\, \mathrm{e}^{(1-s)w} .
\end{aligned}
$$

□

A generalization of (3.38) to the fractional case is provided by the next statement.

Theorem 3.5.4. *Let s be a complex variable and let $\alpha \in \left(\mathbb{R}_{>0} \setminus \mathbb{N}\right)$. For all $s \in \mathbb{C}$, eq.* (3.32) *can be rewritten as*

$$\zeta^{(\alpha)}(s) = \sum_{h=0}^{\infty} \binom{\alpha}{h} \mathrm{e}^{i\pi(\alpha-h)} \left(\mathrm{e}^{(1-s)w} w^{\alpha-h} - \mathrm{e}^{(1-s)\overline{w}+i\pi} \left(\overline{w}\right)^{\alpha-h}\right) \cdot \left(\Gamma(1-s)\,\zeta(1-s)\right)^{(h)} , \tag{3.41}$$

where $w = -\log(2\pi) - i\pi/2$ *.*

Proof : By applying the fractional operator (3.9) to both members of (3.39), it is

$$\zeta^{(\alpha)}(s) = \mathrm{D}_f^{\alpha} \psi(1-s, w) - \mathrm{D}_f^{\alpha} \psi(1-s, \overline{w}, -i\pi) , \tag{3.42}$$

so that

$$\begin{aligned} \mathrm{D}_f^{\alpha} \psi(1-s, w, z) &= \sum_{h=0}^{\infty} \binom{\alpha}{h} \left(\Gamma(1-s)\,\zeta(1-s)\right)^{(h)} \left(\mathrm{e}^{(1-s)w+z}\right)^{(\alpha-h)} \\ &= \sum_{h=0}^{\infty} \binom{\alpha}{h} \left(\Gamma(1-s)\,\zeta(1-s)\right)^{(h)} \mathrm{e}^{z} \left(\mathrm{e}^{(1-s)w}\right)^{(\alpha-h)} , \end{aligned}$$

being the constants transparent in D_f^{α}. Therefore, taking into account lemma 3.5.3, it follows that

$$\mathrm{D}_f^{\alpha-h} \psi(1-s, w, z) = \sum_{h=0}^{\infty} \binom{\alpha}{h} \mathrm{e}^{i\pi(\alpha-h)} w^{\alpha-h} \mathrm{e}^{(1-s)w+z} \left(\Gamma(1-s)\,\zeta(1-s)\right)^{(h)} . \tag{3.43}$$

Since eq. (1.10) is well defined in the whole complex plane $\mathbb{C}$ (Apostol, 1998, Chap. 12) and by substituting (3.43) into (3.42), the proof follows.

□

The RHS of (3.41) has less computational cost than that of (3.32). Moreover, the fractional counterpart of eq. (3.37) is given by the following

Theorem 3.5.5. *Under the same hypothesis of theorem* 3.5.4*, it is*

$$\boxed{\begin{aligned} \zeta^{(\alpha)}(s) = 2\,(2\pi)^{s-1}\,\mathrm{e}^{i\pi\alpha} \sum_{h=0}^{\infty}\sum_{n=0}^{\infty} \left(a_{h\alpha n} \sin\left(\frac{\pi s}{2}\right) + b_{h\alpha n} \cos\left(\frac{\pi s}{2}\right)\right) \\ \cdot \Gamma^{(h)}(1-s)\zeta^{(n)}(1-s) , \qquad (\text{for all } s \in \mathbb{C}) \end{aligned}} \tag{3.44}$$

where the coefficients $a_{h\alpha n}$ *and* $b_{h\alpha n}$ *are given by*

$$\begin{cases} a_{h\alpha n} \stackrel{\text{def}}{=} \displaystyle\sum_{j=0}^{\infty} \frac{A_{h,j,n}^{\alpha}}{\log^{h+j+n-\alpha}(2\pi)} \left(-\frac{\pi}{2}\right)^{j} \cos\left(\frac{\pi j}{2}\right) \,, \\ b_{h\alpha n} \stackrel{\text{def}}{=} \displaystyle\sum_{j=0}^{\infty} \frac{A_{h,j,n}^{\alpha}}{\log^{h+j+n-\alpha}(2\pi)} \left(-\frac{\pi}{2}\right)^{j} \sin\left(\frac{\pi j}{2}\right) \,. \end{cases}$$

Proof: Eq. (3.44) follows by substituting the identity

$$\sin\left(\frac{\pi}{2}(s+j)\right) = \sin\left(\frac{\pi s}{2}\right)\cos\left(\frac{\pi j}{2}\right) + \cos\left(\frac{\pi s}{2}\right)\sin\left(\frac{\pi j}{2}\right) \,,$$

into (3.32).

□

The previous theorem takes on a particular relevance. In fact, eq. (3.44) can be suitable for several applications in harmonic analysis since it expresses the functional equation of $\zeta^{(\alpha)}$ as a sum of sines and cosines. Therefore, eqs. (3.41), (3.44) represent two different and interesting forms of (3.32).

3.6 Generalizations to the Hurwitz ζ function

In the literature, Apostol (1985) showed that the analysis developed in Section 3.5.1 can be extended to the Hurwitz ζ function. In this section, the results given in the previous section are generalized with respect to the Hurwitz ζ function.

Theorem 3.6.1. *Let s be a complex variable, let $\alpha \in \left(\mathbb{R}_{>0} \setminus \mathbb{N}\right)$ and let p and q be two integers such that $1 \le p \le q$. For all $s \in \mathbb{C}$, it is*

$$\boxed{\begin{aligned} \zeta^{(\alpha)}\left(s, \frac{p}{q}\right) &= 2\,(2\pi q)^{s-1}\,\mathrm{e}^{i\pi\alpha} \sum_{h=0}^{\infty}\sum_{j=0}^{\infty}\sum_{n=0}^{\infty} A_{h,j,n}^{\alpha} \frac{\Gamma^{(h)}(1-s)}{\log^{h+j+n-\alpha}(2\pi q)} \left(-\frac{\pi}{2}\right)^{j} \\ &\cdot \sum_{m=1}^{q} \sin\left(\frac{\pi}{2}(s+j) + \frac{2\pi m p}{q}\right) \zeta^{(n)}\left(1-s, \frac{m}{q}\right) \,. \end{aligned}} \tag{3.45}$$

Proof: The generalized Leibniz rule can also be applied here. From eq. (1.18) it follows that

$$
\begin{aligned}
\zeta^{(\alpha)}\left(s,\frac{p}{q}\right) &= 2\,\mathrm{D}_f^{\alpha}\left((2\pi q)^{s-1}\,\Gamma(1-s)\sum_{m=1}^{q}\sin\left(\frac{\pi s}{2}+\frac{2\pi mp}{q}\right)\zeta\left(1-s,\frac{m}{q}\right)\right)\\
&= 2\sum_{h=0}^{\infty}\binom{\alpha}{h}\frac{\mathrm{d}^h}{\mathrm{d}s^h}\Big(\Gamma(1-s)\Big)\,\mathrm{D}_f^{\alpha-h}\left((2\pi q)^{s-1}\sum_{m=1}^{q}\sin\left(\frac{\pi s}{2}+\frac{2\pi mp}{q}\right)\right.\\
&\quad\left.\cdot\,\zeta\left(1-s,\frac{m}{q}\right)\right) = 2\sum_{h=0}^{\infty}\binom{\alpha}{h}\frac{\mathrm{d}^h}{\mathrm{d}s^h}\Big(\Gamma(1-s)\Big)\sum_{j=0}^{\infty}\sum_{m=1}^{q}\binom{\alpha-h}{j}\\
&\quad\cdot\frac{\mathrm{d}^j}{\mathrm{d}s^j}\left(\sin\left(\frac{\pi s}{2}+\frac{2\pi mp}{q}\right)\right)\mathrm{D}_f^{\alpha-h-j}\left((2\pi q)^{s-1}\,\zeta\left(1-s,\frac{m}{q}\right)\right)\\
&= 2\sum_{h=0}^{\infty}\binom{\alpha}{h}\frac{\mathrm{d}^h}{\mathrm{d}s^h}\Big(\Gamma(1-s)\Big)\sum_{j=0}^{\infty}\sum_{m=1}^{q}\binom{\alpha-h}{j}\frac{\mathrm{d}^j}{\mathrm{d}s^j}\left(\sin\left(\frac{\pi s}{2}+\frac{2\pi mp}{q}\right)\right)\\
&\quad\cdot\sum_{n=0}^{\infty}\binom{\alpha-h-j}{n}\frac{\mathrm{d}^n}{\mathrm{d}s^n}\left(\zeta\left(1-s,\frac{m}{q}\right)\right)\mathrm{D}_f^{\alpha-h-j-n}\Big((2\pi q)^{s-1}\Big)\ ,
\end{aligned}
\tag{3.46}
$$

Since

$$
\begin{cases}
\dfrac{\mathrm{d}^h}{\mathrm{d}s^h}\Big(\Gamma(1-s)\Big) = \mathrm{e}^{i\pi h}\,\Gamma^{(h)}(1-s)\ ,\\[2ex]
\dfrac{\mathrm{d}^j}{\mathrm{d}s^j}\left(\sin\left(\dfrac{\pi s}{2}+\dfrac{2\pi mp}{q}\right)\right) = \left(\dfrac{\pi}{2}\right)^j\sin\left(\dfrac{\pi}{2}(s+j)+\dfrac{2\pi mp}{q}\right)\ ,\\[2ex]
\dfrac{\mathrm{d}^n}{\mathrm{d}s^n}\left(\zeta\left(1-s,\dfrac{m}{q}\right)\right) = \mathrm{e}^{i\pi n}\,\zeta\left(1-s,\dfrac{m}{q}\right)\ ,
\end{cases}
$$

and

$$
\begin{cases}
\mathrm{D}_f^{\alpha-h-j-n}\Big((2\pi q)^{s-1}\Big) = (2\pi q)^{s-1}\,\mathrm{e}^{i\pi(\alpha-h-j-n)}\,\log^{\alpha-h-j-n}(2\pi q)\ ,\\[2ex]
\dbinom{\alpha}{h}\dbinom{\alpha-h}{j}\dbinom{\alpha-h-j}{n} = \dfrac{\alpha^{\overline{h+j+n}}}{h!\,j!\,n!} = A^{\alpha}_{h,j,n}\ ,
\end{cases}
$$

hence

$$
\begin{aligned}
\zeta^{(\alpha)}\left(s,\frac{p}{q}\right) = 2\sum_{h=0}^{\infty}\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}A^{\alpha}_{h,j,n}\frac{\Gamma^{(h)}(1-s)}{\log^{h+j+n-\alpha}(2\pi q)}(2\pi q)^{s-1}\,\mathrm{e}^{i\pi(\alpha-j)}\left(\frac{\pi}{2}\right)^j\\
\cdot\sum_{m=1}^{q}\sin\left(\frac{\pi}{2}(s+j)+\frac{2\pi mp}{q}\right)\,\zeta\left(1-s,\frac{m}{q}\right)\ .
\end{aligned}
$$

Eq. (3.45) holds for all $s \in \mathbb{C}$ since (1.10) is well defined in the whole complex plane $\mathbb{C}$ (Apostol, 1998, Chap. 12), therefore the proof follows.

$\square$

In order to show the consistency of (3.45), it is sufficient to show that it reduces to (1.18). The Hurwitz ζ function fulfills hypotheses of proposition 3.3.5, hence

$$\zeta^{(\alpha)}\left(s, \frac{p}{q}\right) \xrightarrow{\alpha \to 0^+} \zeta\left(s, \frac{p}{q}\right) .$$

Even if the RHS of (3.45) appears to be sufficiently complicated, the proof (of the theorem) 3.6.1 can be read backwards until eq. (3.46). From $(3.23)_1$, we have

$$\mathrm{D}_f^{\alpha} f(s) \xrightarrow{\alpha \to 0^+} f^{(0)}(s) = f(s) ,$$

therefore the RHS of (3.46) converges to the RHS of (1.18) as α goes to 0^+. Equivalent forms of theorem 3.5.4 and theorem 3.5.5 hold also for the Hurwitz ζ function and are reported below, respectively.

Theorem 3.6.2. *Under the same hypotheses of theorem* 3.6.1*, eq.* (3.45) *can be rewritten by*

$$\begin{aligned} \zeta^{(\alpha)}\left(s, \frac{p}{q}\right) = \sum_{m=1}^{q} \sum_{h=0}^{\infty} \binom{\alpha}{h} & \left(\mathrm{e}^{(1-s)w_q + i\frac{2\pi mp}{q}} \, w^{\alpha-h} - \mathrm{e}^{(1-s)\overline{w}_q + i\left(\pi - \frac{2\pi mp}{q}\right)} \, (\overline{w})^{\alpha-h}\right) \\ & \cdot \left(\Gamma(1-s)\, \zeta\left(1-s, \frac{m}{q}\right)\right)^{(h)} , \qquad \text{(for all } s \in \mathbb{C}) \end{aligned} \tag{3.47}$$

where $w_q = -\log(2\pi q) - i\pi/2$.

Proof: Since

$$\begin{aligned} 2\sin\left(\frac{\pi s}{2} + \frac{2\pi mp}{q}\right) = \frac{\mathrm{e}^{i\left(\frac{\pi s}{2} + \frac{2\pi mp}{q}\right)} - \mathrm{e}^{-i\left(\frac{\pi s}{2} + \frac{2\pi mp}{q}\right)}}{i} & = \mathrm{e}^{i\left(\frac{\pi}{2}(s-1) + \frac{2\pi mp}{q}\right)} \\ & - \mathrm{e}^{-i\left(\frac{\pi}{2}(s+1) + \frac{2\pi mp}{q}\right)} , \end{aligned}$$

it is

$$2\,(2\pi q)^{s-1}\sin\left(\frac{\pi s}{2}+\frac{2\pi mp}{q}\right)=\mathrm{e}^{(1-s)\left[-\log(2\pi q)-i\frac{\pi}{2}\right]+i\frac{2\pi mp}{q}}$$
$$-\,\mathrm{e}^{(1-s)\left[-\log(2\pi q)+i\frac{\pi}{2}\right]-i\left(\pi+\frac{2\pi mp}{q}\right)}\,,$$

so that

$$\begin{aligned}\zeta\left(s,\frac{p}{q}\right)&=\sum_{m=1}^{q}\Gamma(1-s)\,\zeta\left(1-s,\frac{m}{q}\right)2\,(2\pi q)^{s-1}\sin\left(\frac{\pi s}{2}+\frac{2\pi mp}{q}\right)=\sum_{m=1}^{q}\Gamma(1-s)\\&\cdot\zeta\left(1-s,\frac{m}{q}\right)\left(\mathrm{e}^{(1-s)\left[-\log(2\pi q)-i\frac{\pi}{2}\right]+i\frac{2\pi mp}{q}}-\mathrm{e}^{(1-s)\left[-\log(2\pi q)+i\frac{\pi}{2}\right]-i\left(\pi+\frac{2\pi mp}{q}\right)}\right)\\&=\sum_{m=1}^{q}\Gamma(1-s)\zeta\left(1-s,\frac{m}{q}\right)\left(\mathrm{e}^{(1-s)\left[-\log(2\pi q)-i\frac{\pi}{2}\right]+i\frac{2\pi mp}{q}}\right.\\&\left.-\,\mathrm{e}^{(1-s)\left[-\log(2\pi q)+i\frac{\pi}{2}\right]+i\left(\pi-\frac{2\pi mp}{q}\right)}\right)\,,\end{aligned}\tag{3.48}$$

since the complex exponential is a $2\pi i-$periodic function. The natural extension of ψ (introduced in the proof of proposition 3.5.2) for the Hurwitz ζ function is given by

$$\psi_q\left(s,\frac{m}{q},w_q,z\right)\stackrel{\text{def}}{=}\Gamma(s)\,\zeta\left(s,\frac{m}{q}\right)\mathrm{e}^{sw_q+z}\,,\qquad(z\in\mathbb{C})$$

where $w_q=-\log(2\pi q)-i\pi/2$. Hence, eq. (3.48) becomes

$$\zeta\left(s,\frac{p}{q}\right)=\sum_{m=1}^{q}\left(\psi_q\left(1-s,\frac{m}{q},w_q,i\frac{2\pi mp}{q}\right)-\psi_q\left(1-s,\frac{m}{q},\overline{w}_q,i\left(\pi-\frac{2\pi mp}{q}\right)\right)\right)\,.$$

By proceeding as in the proof of theorem 3.5.4, we have

$$\begin{aligned}\zeta^{(\alpha)}\left(s,\frac{p}{q}\right)=\sum_{m=1}^{q}&\left(\mathrm{D}_f^{\alpha}\psi_q\left(1-s,\frac{m}{q},w_q,i\frac{2\pi mp}{q}\right)\right.\\&\left.-\,\mathrm{D}_f^{\alpha}\psi_q\left(1-s,\frac{m}{q},\overline{w}_q,i\left(\pi-\frac{2\pi mp}{q}\right)\right)\right)\,.\end{aligned}\tag{3.49}$$

Taking into account that (3.43) holds here by replacing $\psi(1-s,w,z)$ and $\zeta(1-s)$ with $\psi_q\left(1-s,\frac{m}{q},w_q,z\right)$ and $\zeta\left(1-s,\frac{m}{q}\right)$, respectively, it follows that

$$\begin{aligned}\mathrm{D}_f^{\alpha-h}\psi\left(1-s,\frac{m}{q},w,z\right) &= \sum_{h=0}^{\infty}\binom{\alpha}{h}\mathrm{e}^{i\pi(\alpha-h)}\,w_q^{\alpha-h}\,\mathrm{e}^{(1-s)w_q+z} \\ &\quad\cdot\left(\Gamma(1-s)\,\zeta\left(1-s,\frac{m}{q}\right)\right)^{(h)} .\end{aligned} \tag{3.50}$$

Given that eq. (1.10) is well defined in the whole complex plane, by substituting (3.50) into (3.49) the proof follows.

□

Theorem 3.6.3. *Under the same hypotheses of theorem* 3.6.1*, it is*

$$\begin{aligned}\zeta^{(\alpha)}\left(s,\frac{p}{q}\right) &= 2\,(2\pi q)^{s-1}\,\mathrm{e}^{i\pi\alpha}\sum_{h=0}^{\infty}\sum_{n=0}^{\infty}\Gamma^{(h)}(1-s)\sum_{m=1}^{q}\left(a_{h\alpha mn}^{p,q}\sin\left(\frac{\pi}{2}s\right)\right. \\ &\quad\left.+\,b_{h\alpha mn}^{p,q}\cos\left(\frac{\pi}{2}s\right)\right)\zeta^{(n)}\left(s,\frac{m}{q}\right) , \qquad (\text{for all } s\in\mathbb{C})\end{aligned} \tag{3.51}$$

where the coefficients $a_{h\alpha mn}^{p,q}$ and $b_{h\alpha mn}^{p,q}$ are given by

$$\begin{cases} a_{h\alpha mn}^{p,q} \overset{\text{def}}{=} \displaystyle\sum_{j=0}^{\infty}\frac{A_{h,j,n}^{\alpha}}{\log^{h+j+n-\alpha}(2\pi q)}\left(-\frac{\pi}{2}\right)^{j}\cos\left(\frac{\pi}{2}j+\frac{2\pi mp}{q}\right) , \\ b_{h\alpha mn}^{p,q} \overset{\text{def}}{=} \displaystyle\sum_{j=0}^{\infty}\frac{A_{h,j,n}^{\alpha}}{\log^{h+j+n-\alpha}(2\pi q)}\left(-\frac{\pi}{2}\right)^{j}\sin\left(\frac{\pi}{2}j+\frac{2\pi mp}{q}\right) . \end{cases} \tag{3.52}$$

Proof : Eq. (3.51) follows by substituting the identity

$$\begin{aligned}\sin\left(\frac{\pi}{2}(s+j)+\frac{2\pi mp}{q}\right) &= \sin\left(\frac{\pi}{2}s+\left(\frac{\pi}{2}j+\frac{2\pi mp}{q}\right)\right) \\ &= \sin\left(\frac{\pi}{2}s\right)\cos\left(\frac{\pi}{2}j+\frac{2\pi mp}{q}\right)+\cos\left(\frac{\pi}{2}s\right)\sin\left(\frac{\pi}{2}j+\frac{2\pi mp}{q}\right) ,\end{aligned}$$

into (3.45).

□

The coefficients (3.52) are constant as in theorem 3.5.5. Theorem 3.6.3 represents a consistent generalization of theorem 3.5.5 since eq. (3.51) reduces to (3.44) for $p = q = 1$. In particular, the functional equations of both $\zeta^{(\alpha)}$ and $(2.9)_2$ can be written in terms of sines and cosines.

3.6.1 Integral representation via Euler summation formula

The integral representation of $(2.9)_2$ is presented here. It is based on the Euler summation formula, which approximates a finite sum by an integral.

Theorem 3.6.4 (Euler summation formula)**.** *Let* $f \in \mathscr{C}^1\left([y,x]\right)$ *such that* $0 < y < x$*. It holds that*

$$\sum_{y<n\leq x} f(n) = \int_y^x f(t)\mathrm{d}t + \int_y^x \left(t - \lfloor t\rfloor\right) f'(t)\mathrm{d}t + f(x)\left(\lfloor x\rfloor - x\right) - f(y)\left(\lfloor y\rfloor - y\right) \ .$$

Proof : See Apostol (1998, pp. 54–55).

□

Apostol (1985) provided an integral representation for the k-order integer derivative of (1.16). Its fractional generalization is given by the following

Theorem 3.6.5. *Let s be a complex variable such that* $\mathrm{Re}\, s > -1$*, let* $\alpha \in \left(\mathbb{R}_{>0} \setminus \mathbb{N}\right)$ *and let* $a \in \mathbb{R} : 0 < a \leq 1$*. The integral representation of* $(2.9)_2$ *is given by*

$$\boxed{\begin{aligned}\zeta^{(\alpha)}(s,a) = \mathrm{e}^{i\pi\alpha}\Bigg(&\frac{\log^\alpha a}{2a^s} + a^{1-s}\sum_{j=0}^{\infty}\alpha^{\underline{j}}\,\frac{\log^{\alpha-j} a}{(s-1)^{j+1}} - s(s+1)\\ &\cdot\int_0^\infty \frac{\varphi_2(x)\log^\alpha(x+a)}{(x+a)^{s+2}}\,\mathrm{d}x + \alpha(2s+1)\int_0^\infty \frac{\varphi_2(x)\log^{\alpha-1}(x+a)}{(x+a)^{s+2}}\,\mathrm{d}x\\ &-\alpha(\alpha-1)\int_0^\infty \frac{\varphi_2(x)\log^{\alpha-2}(x+a)}{(x+a)^{s+2}}\,\mathrm{d}x\Bigg)\ ,\end{aligned}} \tag{3.53}$$

where

$$\varphi_2(x) \overset{\text{def}}{=} \int_0^x \left(t - \lfloor t\rfloor - 1\right)\mathrm{d}t\ ,$$

is a $1-$*periodic function* (Apostol, 1985) *satisfying the following condition*

$$\varphi_2(x) = \frac{1}{2}x(x-1)\ . \qquad (0 \leq x \leq 1)$$

Proof : The Euler summation formula gives the following representation (Apostol, 1985) for the Hurwitz ζ function

$$\zeta\,(s,a) = a^{-s}\left(\frac{1}{2}+\frac{a}{s-1}\right) - s(s+1)\int_0^\infty \frac{\varphi_2(x)}{(x+a)^{s+2}}\mathrm{d}x\,, \qquad (\mathrm{Re}\, s > -1) \tag{3.54}$$

and since D_f^α is a linear operator, it is

$$\zeta^{(\alpha)}\,(s,a) = \mathrm{D}_f^\alpha\left(\frac{1}{2a^s}\right) + \mathrm{D}_f^\alpha\left(\frac{a^{1-s}}{s-1}\right) - \mathrm{D}_f^\alpha\left(s(s-1)\int_0^\infty \frac{\varphi_2(x)}{(x+a)^{s+2}}\mathrm{d}x\right)\,. \tag{3.55}$$

From (3.10) it follows that

$$\mathrm{D}_f^\alpha\left(\frac{1}{2a^s}\right) = \frac{\mathrm{e}^{i\pi\alpha}}{2}\frac{\log^\alpha a}{a^s}\,. \tag{3.56}$$

By taking into account the generalized Leibniz rule (theorem 3.3.4), we have

$$\mathrm{D}_f^\alpha\left(\frac{a^{1-s}}{s-1}\right) = \sum_{j=0}^\infty \binom{\alpha}{j}\left(\frac{1}{s-1}\right)^{(j)}\left(\frac{1}{a^{s-1}}\right)^{(\alpha-j)}\,.$$

Since

$$\left(\frac{1}{s-1}\right)^{(j)} = \mathrm{e}^{i\pi j}\,\frac{j!}{(s-1)^{j+1}}\,,$$

and

$$\left(\frac{1}{a^{s-1}}\right)^{(\alpha-j)} = \mathrm{e}^{i\pi(\alpha-j)}\frac{\log^{\alpha-j} a}{a^{s-1}}\,,$$

it is

$$\begin{aligned}\mathrm{D}_f^\alpha\left(\frac{a^{1-s}}{s-1}\right) &= \sum_{j=0}^\infty \binom{\alpha}{j} j!\frac{\mathrm{e}^{i\pi\alpha}}{(s-1)^{j+1}}\frac{\log^{\alpha-j} a}{a^{s-1}}\\ &= \mathrm{e}^{i\pi\alpha}\,a^{1-s}\sum_{j=0}^\infty \alpha^{\underline{j}}\,\frac{\log^{\alpha-j} a}{(s-1)^{j+1}}\,.\end{aligned} \tag{3.57}$$

Analogously, we have

$$\mathrm{D}_f^\alpha\left(s(s+1)\int_0^\infty \frac{\varphi_2(x)}{(x+a)^{s+2}}\mathrm{d}x\right) = \sum_{m=0}^\infty \binom{\alpha}{m}\big(s(s+1)\big)^{(m)}\int_0^\infty \varphi_2(x)\left(\frac{1}{(x+a)^{s+2}}\right)^{(\alpha-m)}\mathrm{d}x\,. \tag{3.58}$$

Being

$$\big(s(s+1)\big)^{(m)} = 0\,, \qquad (\forall m > 2)$$

the series in (3.58) reduces to only three terms, hence

$$\begin{aligned}
\mathrm{D}_f^{\alpha}\left(s(s-1)\int_0^{\infty}\frac{\varphi_2(x)}{(x+a)^{s+2}}\mathrm{d}x\right) &= \binom{\alpha}{0}s(s+1)\,\mathrm{e}^{i\pi\alpha}\int_0^{\infty}\frac{\varphi_2(x)\log^{\alpha}(x+a)}{(x+a)^{s+2}}\mathrm{d}x \\
&+\binom{\alpha}{1}(2s+1)\,\mathrm{e}^{i\pi(\alpha-1)}\int_0^{\infty}\frac{\varphi_2(x)\log^{\alpha-1}(x+a)}{(x+a)^{s+2}}\mathrm{d}x \\
&+\binom{\alpha}{2}\,2\,\mathrm{e}^{i\pi(\alpha-2)}\int_0^{\infty}\frac{\varphi_2(x)\log^{\alpha-2}(x+a)}{(x+a)^{s+2}}\mathrm{d}x \\
&= s(s+1)\,\mathrm{e}^{i\pi\alpha}\int_0^{\infty}\frac{\varphi_2(x)\log^{\alpha}(x+a)}{(x+a)^{s+2}}\mathrm{d}x \\
&-\alpha(2s+1)\mathrm{e}^{i\pi\alpha}\int_0^{\infty}\frac{\varphi_2(x)\log^{\alpha-1}(x+a)}{(x+a)^{s+2}}\mathrm{d}x \\
&+\alpha(\alpha-1)\,\mathrm{e}^{i\pi\alpha}\int_0^{\infty}\frac{\varphi_2(x)\log^{\alpha-2}(x+a)}{(x+a)^{s+2}}\mathrm{d}x\,.
\end{aligned} \tag{3.59}$$

By substituting (3.56), (3.57), and (3.59) into (3.55), the proof follows.

□

According to (3.53), we have

$$\begin{aligned}
\zeta^{(\alpha)}(0,a) = \mathrm{e}^{i\pi\alpha}\Bigg(&\frac{\log^{\alpha}a}{2} + a\sum_{j=0}^{\infty}\alpha^{\underline{j}}\frac{\log^{\alpha-j}a}{(-1)^{j+1}} + \alpha\int_0^{\infty}\frac{\varphi_2(x)\log^{\alpha-1}(x+a)}{(x+a)^2}\mathrm{d}x \\
&-\alpha(\alpha-1)\int_0^{\infty}\frac{\varphi_2(x)\log^{\alpha-2}(x+a)}{(x+a)^2}\mathrm{d}x\Bigg)\,,
\end{aligned}$$

and since $\zeta^{(\alpha)}(s,1) = \zeta^{(\alpha)}(s)$, it follows that

$$\begin{aligned}
\zeta^{(\alpha)}(0) &= \mathrm{e}^{i\pi\alpha}\left(\alpha\int_0^{\infty}\frac{\varphi_2(x)\log^{\alpha-1}(x+1)}{(x+1)^2}\mathrm{d}x - \alpha(\alpha-1)\int_0^{\infty}\frac{\varphi_2(x)\log^{\alpha-2}(x+1)}{(x+1)^2}\mathrm{d}x\right) \\
&= \alpha\,\mathrm{e}^{i\pi\alpha}\int_0^{\infty}\frac{\varphi_2(x)\log^{\alpha-1}(x+1)}{(x+1)^2}\Big(1-(\alpha-1)\log^{-1}(x+1)\Big)\,\mathrm{d}x\,.
\end{aligned}$$

Hence

$$\zeta^{(\alpha)}(0) \xrightarrow{\alpha\to 1} \zeta'(0) = -1 - \int_0^\infty \frac{\varphi_2(x)}{(x+1)^2}\mathrm{d}x = -1 - \int_1^\infty \frac{\varphi_2(x)}{x^2}\mathrm{d}x = -\frac{\log(2\pi)}{2} \;,$$

by taking into account (Apostol, 1969, p. 616) that

$$1 + \int_1^\infty \frac{\varphi_2(x)}{x^2}\mathrm{d}x = \frac{\log(2\pi)}{2} \;.$$

Therefore, eq. (3.53) is in accordance with the classical theory of the Riemann ζ function since it reduced to (3.6) as $\alpha \to 1$.

3.7 Fractional derivative of the Lerch zeta function and functional equation

In this section, the Lerch zeta function is introduced in order to compute its fractional derivative and to discuss the associated functional equation.

3.7.1 Remarks

First, some preliminaries on the Lerch zeta function are given.

Definition 3.7.1 (Lerch zeta function)**.** *Let s be a complex variable, let* $\lambda \in \mathbb{R}$ *and let* $a \in \mathbb{R} : 0 < a \leq 1$. *The Lerch zeta function, denoted by L, is given* (Laurinčikas and Garunkštis, 2002, Chap. 2) *by*

$$L(\lambda, s, a) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{\mathrm{e}^{2\pi i \lambda n}}{(n+a)^s} \;, \tag{3.60}$$

with

$$\operatorname{Re} s > \begin{cases} 1\;, & \lambda \in \mathbb{Z}\;, \\ 0\;, & \lambda \notin \mathbb{Z}\;. \end{cases} \tag{3.61}$$

Thus, the Lerch zeta function reduces to the Hurwitz ζ function for $\lambda \in \mathbb{Z}$ and $L(\lambda \in \mathbb{Z}, s, 1) = \zeta(s,1) = \zeta(s)$. Moreover, it is

$$L\left(\frac{1}{2}, s, 1\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = \eta(s) \;,$$

hence L also generalizes the Dirichlet η function (see Chapter 4). The Lerch zeta function is a particular case of a general Dirichlet series (Apostol, 1997, Chap. 8), that is

$$\sum_{n=0}^{\infty} a_n \mathrm{e}^{-\lambda_n s} , \tag{3.62}$$

where $(\lambda_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence of real numbers such that $\lambda_n \xrightarrow{n\to\infty} \infty$. It appears to be clear that for the Lerch zeta function $a_n = \mathrm{e}^{2\pi i\lambda n}$ and $\lambda_n = \log(n+a)$. According to the fundamental convergence theorem on (3.62), the Lerch zeta function converges uniformly on compact subsets of $\mathrm{Re}\, s > 1$ (Apostol, 1997, Chap. 8; Laurinčikas and Garunkštis, 2002, Chap. 2). The Lerch zeta function (like the Riemann ζ function and the Hurwitz ζ function) is analytically continuable to an entire function (Laurinčikas and Garunkštis, 2002, Chap. 2). A generalization of L is given by the so-called Lerch transcendent function, denoted with Φ and defined (Srivastava and Choi, 2011, Chap. 2) as follows

$$\Phi(z,s,a) \overset{\mathrm{def}}{=} \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} . \qquad (s\in\mathbb{C})$$

with $a \notin \mathbb{Z}_{\leq 0}$ and $z\in\mathbb{C} : |z| \leq 1$. In particular, the previous definition makes sense whenever

$$\begin{cases} s\in\mathbb{C} , & |z|<1 , \\ \mathrm{Re}\, s > 1 , & |z| = 1 . \end{cases}$$

Obviously, $L(\lambda,s,a) = \Phi\left(\mathrm{e}^{2\pi i\lambda}, s, a\right)$. Fermi-Dirac and Bose-Einstein distributions can be written by using the function Φ. Under the assumption that $0<\lambda<1$, Lerch (1883) derived the following three-term functional equation (Laurinčikas and Garunkštis, 2002, pp. 22–23)

$$\begin{aligned} L(\lambda,1-s,a) = \frac{\Gamma(s)}{(2\pi)^s} \bigg(& \mathrm{e}^{\frac{i\pi s}{2} - 2\pi i a\lambda} L(-a,s,\lambda) \\ & + \mathrm{e}^{\frac{-i\pi s}{2} + 2\pi i a(1-\lambda)} L(a,s,1-\lambda) \bigg) , \qquad (\text{for all } s\in\mathbb{C}) \end{aligned} \tag{3.63}$$

that is

$$\begin{aligned} L(\lambda,s,a) = \frac{\Gamma(1-s)}{(2\pi)^{1-s}} \bigg(& \mathrm{e}^{\frac{i\pi(1-s)}{2} - 2\pi i a\lambda} L(-a,1-s,\lambda) + \mathrm{e}^{\frac{-i\pi(1-s)}{2} + 2\pi i a(1-\lambda)} \\ & \cdot L(a,1-s,1-\lambda) \bigg) . \qquad (\text{for all } s\in\mathbb{C}) \end{aligned} \tag{3.64}$$

Eq. (3.64) reduces to (1.10) for $\lambda \in \mathbb{Z}$ and $a = 1$ since $L(\lambda \in \mathbb{Z}, s, 1) = \zeta(s)$, so that the previous functional equation is in accordance with the classical theory of the Riemann ζ function. Moreover, an approximation of the Lerch zeta function by finite sum is given (Garunkštis, 2004; Laurinčikas and Garunkštis, 2002, pp. 32–34) by

$$L(\lambda, s, a) = \sum_{n=0}^{x} \frac{\mathrm{e}^{2\pi i \lambda n}}{(n+a)^s} + \mathcal{O}\left(x^{-\sigma}\right) , \qquad (s = \sigma + it) \tag{3.65}$$

where the RHS is written by Landau notation (Graham et al., 1994, Chap. 9) and

$$\begin{cases} 0 < \lambda < 1 \, , \\ \sigma > 0 \, , \\ |t| \leq \pi \lambda x \, . \end{cases} \tag{3.66}$$

However, the RHS of (3.65) often appears to be too long for applications. An approximate functional equation for (3.60), which gives more precise results, is reported below.

Theorem 3.7.2. *Let* $0 < \lambda \leq 1$, $0 < a \leq 1$ *and let* $s = \sigma + it$ *be a complex variable with* $\sigma, t \in \mathbb{R} : 0 < \sigma \leq 1$ *and* $t \geq 1$. *Under the following conditions*

$$\begin{cases} y = \left(t/2\pi\right)^{1/2} \, , \\ q = \lfloor y \rfloor \, , \\ k = \lfloor y - a \rfloor \, , \\ \beta = q - k \, . \end{cases}$$

it is

$$\begin{aligned} L(\lambda, s, a) = & \sum_{n=0}^{k} \frac{\mathrm{e}^{2\pi i \lambda n}}{(n+a)^s} + \left(\frac{2\pi}{t}\right)^{\sigma - \frac{1}{2} + it} \mathrm{e}^{it + \frac{\pi i}{4} - 2\pi i \{\lambda\} a} \sum_{n=0}^{q} \frac{\mathrm{e}^{-2\pi i a n}}{(n+\lambda)^{1-s}} \\ & + \left(\frac{2\pi}{t}\right)^{\frac{\sigma}{2}} \mathrm{e}^{\pi i f(\lambda, t, a)} \psi\left(2y - q - k - \{\lambda\} - a\right) + \mathcal{O}\left(t^{\frac{\sigma-2}{2}}\right) , \end{aligned} \tag{3.67}$$

where the functions f *and* ψ *are given, respectively, by*

$$\begin{aligned} f(\lambda, t, a) = & -\frac{t}{2\pi} \log \frac{t}{2\pi \mathrm{e}} - \frac{7}{8} + \frac{1}{2}\left(a^2 - \{\lambda\}^2\right) - a\beta \\ & + 2y\left(\beta + \{\lambda\} - a\right) - \frac{1}{2}(q+k) - \{\lambda\}\left(\beta + a\right) , \end{aligned}$$

and

$$\psi(b) = \frac{\cos\left(\pi\left(\frac{b^2}{2} - b - \frac{1}{8}\right)\right)}{\cos(\pi b)} .$$

Proof: See Laurinčikas and Garunkštis (2002, pp. 53–59).

□

It is not hard to show that eq. (3.67) holds uniformly in λ and a (Garunkštis, 2004).

3.7.2 Functional equation of $L^{(\alpha)}$

First and foremost, the fractional derivative of the Lerch zeta function is computed.

Theorem 3.7.3. *Let $\lambda \in \mathbb{R}$ and let $\alpha \in \left(\mathbb{R}_{>0} \setminus \mathbb{N}\right)$. Moreover, let s be a complex variable such that the conditions* (3.61) *hold. The α-order fractional derivative of L is given by*

$$L^{(\alpha)}(\lambda, s, a) = \mathrm{e}^{i\pi\alpha} \sum_{n=0}^{\infty} \frac{\log^{\alpha}(n+a)}{(n+a)^s} \mathrm{e}^{2\pi i\lambda n} . \qquad (0 < a \leq 1) \tag{3.68}$$

Proof: As in the proof of theorem 3.3.2, it is

$$\begin{aligned} \mathrm{D}_f^{\alpha} L(\lambda, a, s) &= \lim_{h\to 0^+} \frac{\sum_{k=0}^{\infty} \binom{\alpha}{k} (-1)^k L(s-kh)}{h^{\alpha}} = \lim_{h\to 0^+} \frac{1}{h^{\alpha}} \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \binom{\alpha}{k} (-1)^k \frac{\mathrm{e}^{2\pi i\lambda n}}{(n+a)^{s-kh}} \\ &= \sum_{n=0}^{\infty} \frac{\mathrm{e}^{2\pi i\lambda n}}{(n+a)^s} \lim_{h\to 0^+} \frac{1}{h^{\alpha}} \sum_{k=0}^{\infty} \binom{\alpha}{k} (-1)^k (n+a)^{kh} , \end{aligned}$$

and

$$\sum_{k=0}^{\infty} \binom{\alpha}{k} (-1)^k (n+a)^{kh} = \sum_{k=0}^{\infty} \binom{\alpha}{k} (-(n+a)^h)^k = \left(1 - (n+a)^h\right)^{\alpha} ,$$

It follows that

$$\mathrm{D}_f^{\alpha} L(\lambda, a, s) = \sum_{n=0}^{\infty} \frac{\mathrm{e}^{2\pi i\lambda n}}{(n+a)^s} \lim_{h\to 0^+} \frac{\left(1 - (n+a)^h\right)^{\alpha}}{h^{\alpha}} . \tag{3.69}$$

By L'Hôpital's rule, we have

$$\begin{aligned}\lim_{h\to 0^+}\left(\frac{1-(n+a)^h}{h}\right)^\alpha &= \left(\lim_{h\to 0^+}\frac{1-(n+a)^h}{h}\right)^\alpha = \left(-\lim_{h\to 0^+}(n+a)^h\log(n+a)\right)^\alpha \\ &= \mathrm{e}^{i\pi\alpha}\log^\alpha(n+a)\ .\end{aligned} \tag{3.70}$$

By substituting (3.70) into (3.69), the proof follows directly.

□

Eq. (3.68) consistently generalizes (3.10) since $L^{(\alpha)}(\lambda\in\mathbb{Z},s,a)=\zeta^{(\alpha)}(s,a)$. By using the generalized Leibniz rule, a three-term functional equation for $L^{(\alpha)}$ can easily be derived and is reported below.

Theorem 3.7.4. *Let $\lambda\in\mathbb{R}$, $0<a\le 1$ and let $\alpha\in\left(\mathbb{R}_{>0}\setminus\mathbb{N}\right)$. Moreover, let $s=\sigma+it$ be a complex variable with with $\sigma,t\in\mathbb{R}$. The following assertions hold.*

(i) *If $0<\lambda<1$, for all $s\in\mathbb{C}$ we have*

$$\begin{aligned}L^{(\alpha)}(\lambda,s,a) &= \frac{\mathrm{e}^{i\pi\alpha}}{(2\pi)^{1-s}}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{m}C^{\alpha}_{m,j,k}\Bigg(\mathrm{e}^{\frac{i\pi(1-s)}{2}-2\pi ia\lambda}\left(\frac{i\pi}{2}\right)^k L^{(m-k)}(-a,1-s,\lambda) \\ &\quad+\mathrm{e}^{\frac{-i\pi(1-s)}{2}-2\pi ia(1-\lambda)}\left(-\frac{i\pi}{2}\right)^k L^{(m-k)}(a,1-s,1-\lambda)\Bigg)\frac{\Gamma^{(j)}(1-s)}{\log^{m+j-\alpha}(2\pi)}\ ,\end{aligned} \tag{3.71}$$

where $C^{\alpha}_{m,j,k}=\dfrac{\alpha^{\underline{m+j}}}{j!k!(m-k)!}$.

(ii) *Under the conditions* (3.66)*, it is*

$$L^{(\alpha)}(\lambda,s,a)=\mathrm{e}^{i\pi\alpha}\sum_{0\le n\le x}\frac{\log^{(\alpha)}(n+a)}{(n+a)^s}\,\mathrm{e}^{2\pi i\lambda n}+\mathcal{O}\left(\frac{\log^\alpha x}{x^\sigma}\right)\ . \tag{3.72}$$

Proof: (i) By applying D^α_f to both members of (3.64), it follows that

$$\begin{aligned}L^{(\alpha)}(\lambda,s,a) &= \frac{1}{2\pi}\mathrm{D}^\alpha_f\Bigg(\Bigg(\mathrm{e}^{\frac{i\pi(1-s)}{2}-2\pi ia\lambda}L(-a,1-s,\lambda)+\mathrm{e}^{\frac{-i\pi(1-s)}{2}+2\pi ia(1-\lambda)} \\ &\quad\cdot L(a,1-s,1-\lambda)\Bigg)\Gamma(1-s)\,(2\pi)^s\Bigg)\ .\end{aligned} \tag{3.73}$$

The RHS of (3.73) is given by

$$\begin{aligned}
&\mathrm{D}_f^{\alpha}\Bigg(\Big(\mathrm{e}^{\frac{i\pi(1-s)}{2}-2\pi ia\lambda}L(-a,1-s,\lambda)+\mathrm{e}^{\frac{-i\pi(1-s)}{2}+2\pi ia(1-\lambda)}L(a,1-s,1-\lambda)\Big)\\
&\quad\cdot\Gamma(1-s)\,(2\pi)^s\Bigg)\overset{(3.17)}{=}\sum_{m=0}^{\infty}\binom{\alpha}{m}\frac{\mathrm{d}^m}{\mathrm{d}s^m}\Bigg(\mathrm{e}^{\frac{i\pi(1-s)}{2}-2\pi ia\lambda}L(-a,1-s,\lambda)+\mathrm{e}^{\frac{-i\pi(1-s)}{2}+2\pi ia(1-\lambda)}\\
&\quad\cdot L(a,1-s,1-\lambda)\Bigg)\sum_{j=0}^{\infty}\binom{\alpha-m}{j}\frac{\mathrm{d}^j}{\mathrm{d}s^j}\big(\Gamma(1-s)\big)\,\mathrm{D}_f^{\alpha-m-j}(2\pi)^s\,.
\end{aligned}$$

Since

$$\begin{aligned}
&\frac{\mathrm{d}^m}{\mathrm{d}s^m}\Big(\mathrm{e}^{\frac{i\pi(1-s)}{2}-2\pi ia\lambda}L(-a,1-s,\lambda)+\mathrm{e}^{\frac{-i\pi(1-s)}{2}+2\pi ia(1-\lambda)}L(a,1-s,1-\lambda)\Big)\\
&=\frac{\mathrm{d}^m}{\mathrm{d}s^m}\Big(\mathrm{e}^{\frac{i\pi(1-s)}{2}-2\pi ia\lambda}L(-a,1-s,\lambda)\Big)+\frac{\mathrm{d}^m}{\mathrm{d}s^m}\Big(\mathrm{e}^{\frac{-i\pi(1-s)}{2}+2\pi ia(1-\lambda)}L(a,1-s,1-\lambda)\Big)\\
&=\sum_{k=0}^{m}\binom{m}{k}\Bigg(\mathrm{e}^{\frac{i\pi(1-s)}{2}-2\pi ia\lambda}\left(\frac{-i\pi}{2}\right)^k L^{(m-k)}(-a,1-s,\lambda)+\mathrm{e}^{\frac{-i\pi(1-s)}{2}-2\pi ia(1-\lambda)}\left(\frac{i\pi}{2}\right)^k\\
&\quad\cdot L^{(m-k)}(a,1-s,1-\lambda)\Bigg)\mathrm{e}^{i\pi(m-k)}\,,
\end{aligned}$$

and

$$\begin{cases}
\mathrm{D}_f^{\alpha-m-j}(2\pi)^s=(2\pi)^s\,\mathrm{e}^{i\pi(\alpha-m-j)}\log^{\alpha-m-j}(2\pi)\,,\\
\dfrac{\mathrm{d}^j}{\mathrm{d}s^j}\big(\Gamma(1-s)\big)=\mathrm{e}^{i\pi j}\Gamma^{(j)}(1-s)\,,\\
C^{\alpha}_{m,j,k}\overset{\text{def}}{=}\dbinom{\alpha}{m}\dbinom{\alpha-m}{j}\dbinom{m}{k}=\dfrac{\alpha^{\underline{m+j}}}{j!k!(m-k)!}\,,\\
\left(\pm\dfrac{i\pi}{2}\right)^k=\mathrm{e}^{i\pi k}\left(\mp\dfrac{i\pi}{2}\right)^k\Rightarrow\left(\pm\dfrac{i\pi}{2}\right)^k\mathrm{e}^{i\pi(m-k)}=\left(\mp\dfrac{i\pi}{2}\right)^k\mathrm{e}^{i\pi m}\,,
\end{cases}$$

it is

$$\begin{aligned}
L^{(\alpha)}(\lambda,a,s)=\frac{\mathrm{e}^{i\pi\alpha}}{2\pi}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{m}C^{\alpha}_{m,j,k}\Bigg(&\mathrm{e}^{\frac{i\pi(1-s)}{2}-2\pi ia\lambda}\left(\frac{i\pi}{2}\right)^k L^{(m-k)}(-a,1-s,\lambda)\\
&+\mathrm{e}^{\frac{-i\pi(1-s)}{2}-2\pi ia(1-\lambda)}\left(-\frac{i\pi}{2}\right)^k L^{(m-k)}(a,1-s,1-\lambda)\Bigg)(2\pi)^s\frac{\Gamma^{(j)}(1-s)}{\log^{m+j-\alpha}(2\pi)}\,,
\end{aligned}$$

therefore eq. (3.71) follows directly. (ii) From eq. (3.65), it follows that

$$L^{(\alpha)}(\lambda,a,s) = \mathrm{D}_f^{\alpha}\left(\sum_{0\leq n\leq x}\frac{\mathrm{e}^{2\pi i\lambda n}}{(n+a)^s}\right) + \mathrm{D}_f^{\alpha}\left(\mathcal{O}\left(x^{-\sigma}\right)\right) . \tag{3.74}$$

From $(2.9)_2$ we have

$$\mathrm{D}_f^{\alpha}\left(\sum_{0\leq n\leq x}\frac{\mathrm{e}^{2\pi i\lambda n}}{(n+a)^s}\right) = \mathrm{e}^{i\pi\alpha}\sum_{0\leq n\leq x}\frac{\log^{(\alpha)}(n+a)}{(n+a)^s}\,\mathrm{e}^{2\pi i\lambda n} , \tag{3.75}$$

and being $\sigma = \mathrm{Re}\, s$, by using the classical definition of the Grünwald-Letnikov fractional derivative (Pudlubny, 1999, Chap. 2), it is

$$\mathrm{D}_f^{\alpha}\left(\mathcal{O}\left(x^{-\sigma}\right)\right) = \mathcal{O}\left(\frac{\log^{\alpha} x}{x^{\sigma}}\right) . \tag{3.76}$$

Consequently, by substituting (3.75) and (3.76) into (3.74), the approximation (3.72) follows. □

Eqs. (3.71) and (3.72) clearly represent the fractional counterpart of (3.64) and (3.65), respectively. In particular, eq. (3.72) provides a finite-sum approximation of $L^{(\alpha)}$. The results given in theorem 3.7.4 are in accordance with the theory of the Lerch zeta function (Laurinčikas and Garunkštis, 2002, Chaps. 2–4).

Chapter 4

On the critical strip of $\zeta^{(\alpha)}$

4.1 Introduction

In this chapter, the link between $\zeta^{(\alpha)}$ and the distribution of prime numbers is studied and discussed extensively. In the first part of the chapter, Euler products are briefly described and the logarithmic fractional derivative of the Riemann ζ function is computed. Furthermore, the α-order fractional derivative of the Dirichlet η function is explicitly computed and discussed in order to better investigate the behavior of $\zeta^{(\alpha)}$ on the critical strip and its main properties. In fact, η has the same zeros on the critical line (Borwein et al., 2008, Chap. 5), so that this function can provide better knowledge of prime numbers through the Riemann ζ. The convergence of $\eta^{(\alpha)}$ is carefully studied by showing that it converges for $\mathrm{Re}\, s > \alpha$, hence the two functions $\eta^{(\alpha)}$ and $\zeta^{(\alpha)}$ characterize the complex strip $\alpha < \mathrm{Re}\, s < 1+\alpha$ as η and ζ do for the classical critical strip.

Chapter 4 is outlined as follows. In the next section, Euler products are introduced in order to discuss the link between $\zeta^{(\alpha)}$ and prime numbers. In Section 4.3, the α-order fractional derivative of the Dirichlet η function is explicitly computed and its half-plane of convergence is studied. The investigation into the link between $\zeta^{(\alpha)}$ and $\eta^{(\alpha)}$ is introduced in this section and is developed in Section 4.4 together with the fractional counterpart of the critical strip.

4.2 Prime numbers and $\zeta^{(\alpha)}$

The problem of the link between $\zeta^{(\alpha)}$ and the prime numbers is here presented and discussed. In particular, in the first part some remarks on the representation of the Dirichlet series as an infinite product over the set of all prime numbers are given, while in the second part, the link between $\zeta^{(\alpha)}$ and the distribution of prime numbers is proposed and discussed.

4.2.1 Remarks on Euler products

The Dirichlet series can be introduced in order to generalize the Euler formula (1.13). Euler (1737) provided the following statement (Apostol, 1998, pp. 230–231), which represents a fundamental step for our purpose.

Theorem 4.2.1. *Let s be a complex variable and suppose f is a multiplicative function. Under the hypothesis that $\sum_{n=1}^{\infty} f(n)$ is absolutely convergent, it is*

$$\sum_{n=1}^{\infty} f(n) = \prod_{p\in\mathbb{P}} \left\{1 + f(p) + f(p^2) + \cdots \right\} . \tag{4.1}$$

Moreover, if f is completely multiplicative, we have

$$\sum_{n=1}^{\infty} f(n) = \prod_{p\in\mathbb{P}} \frac{1}{1 - f(p)} . \tag{4.2}$$

Proof: Let

$$P(x) = \prod_{p\leq x} \left\{1 + f(p) + f(p^2) + \cdots \right\} , \tag{4.3}$$

be the finite product extended on all prime numbers $p \leq x$. Since the RHS of (4.3) is the product of a finite number of absolutely convergent series, they can be multiplied and the terms can be rewritten without modifying the sum. It is evident that its typical term is given by

$$f\left(p_1^{k_1}\right) f\left(p_2^{k_2}\right) \cdots f\left(p_t^{k_t}\right) = f\left(p_1^{k_1} p_2^{k_2} \cdots p_t^{k_t}\right) .$$

Hence, from theorem 1.3.1, it is

$$P(x) = \sum_{n\in C} f(n) ,$$

where $C = \{n \in \mathbb{N} : p \mid n \Rightarrow p \leq x\}$ being $p \in \mathbb{P}$. Consequently, we get

$$\sum_{n=1}^{\infty} f(n) - P(x) = \sum_{n\notin C} f(n) .$$

It follows that

$$\left|\sum_{n=1}^{\infty} f(n) - P(x)\right| \le \sum_{n \notin C} |f(n)| \le \sum_{n>x} |f(n)| \xrightarrow{x\to\infty} 0\,,$$

given that $\sum\limits_{n=1}^{\infty} |f(n)|$ is convergent. Hence

$$P(x) \xrightarrow{x\to\infty} \sum_{n=1}^{\infty} f(n)\,.$$

Recall that an infinite product of the form $\prod(1+c_n)$ converges absolutely whenever the associated series $\sum c_n$ converges absolutely (Knopp, 1990, Chap. 7). Since

$$\sum_{p\le x} \left| f(p) + f(p^2) + \cdots \right| \le \sum_{p\le x} \left(|f(p)| + \left|f(p^2)\right| + \cdots \right) \le \sum_{n=2}^{\infty} |f(n)| \xrightarrow{x\to\infty} 0\,,$$

and since all the partial sums are bounded, the following series

$$\sum_{p\le x} \left| f(p) + f(p^2) + \cdots \right|\,,$$

converges therefore the product in (4.1) converges absolutely. Whenever f is completely multiplicative, it follows that $f(p^k) = f(p)^k$ so that each series on the RHS of (4.1) is nothing more than a geometric series with sum $\dfrac{1}{1-f(p)}$.

□

In eqs. (4.1), (4.2), the infinite product is usually called the Euler product of the associated series. The previous theorem performs an important role in analytic number theory owing to the following

Corollary 4.2.2. *Let s be a complex variable such that $s = x + iy$ with $x, y \in \mathbb{R}$ and let the Dirichlet series* (1.19) *be absolutely convergent for $x > x_a$. It follows that*

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_{p\in\mathbb{P}} \left\{ 1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \cdots \right\}\,. \qquad (x > x_a) \tag{4.4}$$

Furthermore, if f is completely multiplicative, it is

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_{p\in\mathbb{P}} \frac{1}{1-f(p)p^{-s}}\,. \qquad (x > x_a) \tag{4.5}$$

Proof: It follows directly from applying theorem 4.2.1 to absolutely convergent Dirichlet series.

□

The previous corollary can be applied by using many of the arithmetical functions presented in Chapter 1. In fact, from Section 1.5, it is

$$f = I \Rightarrow \zeta(s) = \prod_{p\in\mathbb{P}} \frac{1}{1-p^{-s}}\,, \qquad (\mathrm{Re}\,s > 1)$$

$$f = \mu \Rightarrow \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \prod_{p\in\mathbb{P}} (1-p^{-s})\,, \qquad (\mathrm{Re}\,s > 1)$$

$$f = \phi \Rightarrow \sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)} = \prod_{p\in\mathbb{P}} \frac{1-p^{-s}}{1-p^{1-s}}\,, \qquad (\mathrm{Re}\,s > 2)$$

$$f = \lambda \Rightarrow \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)} = \prod_{p\in\mathbb{P}} \frac{1}{1+p^{-s}}\,. \qquad (\mathrm{Re}\,s > 1)$$

The scientific literature contains no one further generalization of (1.13), that is, a non-multiplicative version of (4.4). Hence, this represents an interesting open problem in the theory of Dirichlet series. Nevertheless, in the next subsection the link between $\zeta^{(\alpha)}$ and the distribution of prime numbers is discussed is some detail.

4.2.2 The role of $\zeta^{(\alpha)}$ in the distribution of prime numbers

Theorem 4.2.1 cannot utilized to obtain a link between $\zeta^{(\alpha)}$ and the distribution of prime numbers, since $\log^{\alpha}$ is a non-multiplicative function. A partial result towards the representation sought can be obtained by the logarithmic fractional derivative of the Riemann ζ function, which is given by the following

Theorem 4.2.3. *Let s be a complex variable such that $s = x+iy$ with $x, y \in \mathbb{R}$ and let $\alpha \in \left(\mathbb{R}_{>0} \setminus \mathbb{N}\right)$. In the half-plane of convergence $x > 1$, it is*

$$ {}_{\mathrm{C}}\mathrm{D}^{\alpha} \log \zeta(s) = \mathrm{e}^{i\pi\alpha} \sum_{p\in\mathbb{P}} \sum_{k=1}^{\infty} p^{-ks} \log^{\alpha} p \;\, k^{\alpha-1}\,. \tag{4.6}$$

where the fractional derivative ${}_{\mathrm{C}}\mathrm{D}^{\alpha}$ is computed on all the complex numbers $x\mathrm{e}^{i\theta}$ such that $\mathrm{Re}(x\mathrm{e}^{i\theta}) > 0$.

Proof : From (1.13), it follows that

$$\log\zeta(s) = \log\left(\prod_{p\in\mathbb{P}}\frac{1}{1-p^{-s}}\right) = \sum_{p\in\mathbb{P}}\log\frac{1}{1-p^{-s}} = -\sum_{p\in\mathbb{P}}\log\left(1-p^{-s}\right),$$

hence

$$\begin{aligned}
{}_{\mathrm{C}}\mathrm{D}^{\alpha}\log\zeta(s) &= \frac{\mathrm{e}^{i(\pi-\theta)(\alpha-m)}}{\Gamma(m-\alpha)}\int_0^{\infty}\frac{\mathrm{d}^m}{\mathrm{d}s^m}\left(\log\zeta\left(s+x\mathrm{e}^{i\theta}\right)\right)\frac{\mathrm{d}x}{x^{\alpha-m+1}} \\
&= -\frac{\mathrm{e}^{i(\pi-\theta)(\alpha-m)}}{\Gamma(m-\alpha)}\sum_{p\in\mathbb{P}}\frac{\mathrm{d}^{m-1}}{\mathrm{d}s^{m-1}}\int_0^{\infty}\frac{\mathrm{d}}{\mathrm{d}s}\left(\log\left(1-p^{-s-x\mathrm{e}^{i\theta}}\right)\right)\frac{\mathrm{d}x}{x^{\alpha-m+1}},
\end{aligned}$$

by taking into account the uniform convergence of the Riemann ζ function. Being

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\log\left(1-p^{-s-x\mathrm{e}^{i\theta}}\right)\right) = \frac{p^{-s-x\mathrm{e}^{i\theta}}}{1-p^{-s-x\mathrm{e}^{i\theta}}}\log p,$$

and by using the change of variables $x\mathrm{e}^{i\theta} = z$, it follows that

$$\begin{aligned}
{}_{\mathrm{C}}\mathrm{D}^{\alpha}\log\zeta(s) &= -\frac{\mathrm{e}^{i(\pi-\theta)(\alpha-m)}}{\Gamma(m-\alpha)}\sum_{p\in\mathbb{P}}\log p\,\frac{\mathrm{d}^{m-1}}{\mathrm{d}s^{m-1}}\left(p^{-s}\int_0^{\infty}\frac{p^{-x\mathrm{e}^{i\theta}}x^{m-\alpha-1}}{1-p^{-s-x\mathrm{e}^{i\theta}}}\mathrm{d}x\right) \\
&= -\frac{\mathrm{e}^{i(\pi-\theta)(\alpha-m)}}{\Gamma(m-\alpha)}\sum_{p\in\mathbb{P}}\log p\,\frac{\mathrm{d}^{m-1}}{\mathrm{d}s^{m-1}}\left(p^{-s}\int_0^{\infty}\frac{p^{-z}z^{m-\alpha-1}}{1-p^{-s-z}}\mathrm{e}^{-i\theta(m-\alpha)}\mathrm{d}z\right) \qquad (4.7) \\
&= -\frac{\mathrm{e}^{i\pi(\alpha-m)}}{\Gamma(m-\alpha)}\sum_{p\in\mathbb{P}}\log p\,\frac{\mathrm{d}^{m-1}}{\mathrm{d}s^{m-1}}\left(p^{-s}\int_0^{\infty}\frac{p^{-z}z^{m-\alpha-1}}{1-p^{-s-z}}\mathrm{d}z\right).
\end{aligned}$$

Since (Mathews and Howell, 2006, p. 141)

$$\frac{1}{1-p^{-s-z}} = \sum_{k=0}^{\infty}p^{-k(s+z)}, \qquad \left(\left|p^{-s-z}\right|<1\right), \tag{4.8}$$

it is

$$\begin{aligned}
\int_0^{\infty}\frac{p^{-z}z^{m-\alpha-1}}{1-p^{-s-z}}\mathrm{d}z &= \sum_{k=0}^{\infty}p^{-ks}\int_0^{\infty}p^{-(k+1)z}z^{m-\alpha-1}\mathrm{d}z \\
&= \sum_{k=0}^{\infty}p^{-ks}\int_0^{\infty}\mathrm{e}^{-x}\frac{x^{m-\alpha-1}}{(k+1)^{m-\alpha}\log^{m-\alpha}p}\mathrm{d}x \qquad (4.9) \\
&= \Gamma(m-\alpha)\log^{\alpha-m}p\sum_{k=0}^{\infty}p^{-ks}(k+1)^{\alpha-m},
\end{aligned}$$

where being $m-\alpha>0$, the last RHS of the continued equality above makes sense. By substituting (4.9) into (4.7), it follows that

$$ {}_{\mathrm{C}}\mathrm{D}^{\alpha}\log\zeta(s) = -\mathrm{e}^{i\pi(\alpha-m)}\sum_{p\in\mathbb{P}}\log^{\alpha-m+1}p\,\frac{\mathrm{d}^{m-1}}{\mathrm{d}s^{m-1}}\left(\sum_{k=0}^{\infty}p^{-s(k+1)}(k+1)^{\alpha-m}\right) . $$

Thus,

$$ \sum_{k=0}^{\infty}p^{-s(k+1)}(k+1)^{\alpha-m} = \sum_{k=1}^{\infty}p^{-sk}\,k^{\alpha-m} , $$

and since

$$ \begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}s}\left(p^{-sk}\right) = p^{-sk}\log p\,(-k) , \\ &\frac{\mathrm{d}^2}{\mathrm{d}s^2}\left(p^{-sk}\right) = p^{-sk}\log^2 p\,(-k)^2 , \\ &\vdots \\ &\frac{\mathrm{d}^{m-1}}{\mathrm{d}s^{m-1}}\left(p^{-sk}\right) = p^{-sk}\log^{m-1}p\,\mathrm{e}^{i\pi(m-1)}\,k^{m-1} , \end{aligned} $$

consequently

$$ \begin{aligned} {}_{\mathrm{C}}\mathrm{D}^{\alpha}\log\zeta(s) &= \mathrm{e}^{i\pi\alpha}\sum_{p\in\mathbb{P}}\log^{\alpha}p\sum_{k=1}^{\infty}p^{-sk}k^{\alpha}k^{-1} \\ &= \mathrm{e}^{i\pi\alpha}\sum_{p\in\mathbb{P}}\sum_{k=1}^{\infty}p^{-sk}\log^{\alpha}p\;k^{\alpha-1} . \end{aligned} $$

Let $\sigma+it$ be the rectangular form of the complex number $z=x\mathrm{e}^{i\theta}$. The condition (4.8) is satisfied being

$$ \begin{cases} x>1 , \\ \sigma>1 , \\ \left|p^{-s-z}\right| = \left|\dfrac{1}{p^s}\right|\left|\dfrac{1}{p^z}\right| = \dfrac{1}{p^x}\dfrac{1}{p^{\sigma}} = p^{-x-\sigma} , \end{cases} $$

which completes the proof.

$\square$

Eq. 4.6 represents a fractional counterpart of the integer case. In fact, it is easy to show that

$$\frac{\mathrm{d}^m}{\mathrm{d}s^m}\log\zeta(s) = \mathrm{e}^{i\pi m}\sum_{p\in\mathbb{P}}\sum_{k=1}^{\infty} p^{-ks}\,\log^m p\;k^{m-1}\,, \qquad (m\in\mathbb{Z}_{>0}) \tag{4.10}$$

therefore eq. (4.10) can be simply derived by (4.6) by replacing m with α. Theorem 4.2.3 does not clearly solve the problem of the link between $\zeta^{(\alpha)}$ and the distribution of prime numbers. In fact, eq. 4.6 gives no information on it. Eulerian-type proofs have shown no appreciable results due to the presence of $\log^{\alpha} n$ in (3.10). A direct computation by Caputo-Ortigueira fractional derivative does not appear possible, since infinite product (1.13) and integral symbol cannot be interchanged. Therefore, the link between $\zeta^{(\alpha)}$ and the distribution of prime numbers is currently an open problem. By taking into account the introduction of the fractional critical strip (see Section 4.4), the proposed problem can provide interesting results in the near future.

4.3 The fractional derivative of the Dirichlet η function

In this section, the fractional derivative of the Dirichlet η function is computed and its half-plane of convergence is determined.

Definition 4.3.1 (Dirichlet η function)**.** *Let s be a complex variable. The Dirichlet η function (called also alternating Riemann ζ function) is given* (Srivastava and Choi, 2011, pp. 384–385) *by*

$$\eta(s) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}\,. \qquad (\mathrm{Re}\,s>0) \tag{4.11}$$

By a direct computation it is

$$\begin{aligned}\eta(s) &= 1-2^{-s}+3^{-s}-4^{-s}+\ldots = 1+\left(-2\cdot 2^{-s}+2^{-s}\right)+3^{-s}+\\ &\quad+\left(-2\cdot 4^{-s}+4^{-s}\right)+\ldots = \left(1-2^{1-s}\right)\zeta(s)\,,\end{aligned} \tag{4.12}$$

that is

$$\zeta(s) = \frac{\eta(s)}{1-2^{1-s}}\,. \qquad (\mathrm{Re}\,s>1) \tag{4.13}$$

Since $\eta(1)$ coincides with the harmonic series, that is $\eta(1)=\log 2$, the pole of ζ at $s=1$ is canceled by the vanishing of the factor $1-2^{1-s}$. It follows that the half-plane of convergence associated with (4.11) is $\operatorname{Re} s > 0$ (Adams, 2005), hence the introduction of an alternating factor $(-1)^{n-1}$ within the series provides an extension of the convergence domain. It can also be shown that the Riemann hypothesis is true if and only if the zeros of η, which belong to the strip $0 < \operatorname{Re} s < 1$, are also distributed along the critical line (Borwein et al., 2008, Chap. 5). Considering the correspondent domains of convergence, it appears to be clear that η, as with ζ, can be used to investigate the zeros belonging to the critical line. In fact, both ζ and η have the same zeros on the critical line $\operatorname{Re} s = 1/2$. The α-order fractional derivative of η can easily be computed by both (1.35) and (3.9). From (3.10), it follows directly that

$$\eta^{(\alpha)}(s) = \mathrm{e}^{i\pi\alpha} \sum_{n=2}^{\infty} (-1)^{n-1} \frac{\log^{\alpha} n}{n^s} \; . \tag{4.14}$$

The convergence of the series in (4.14) is given by the following

Theorem 4.3.2. *Let s be a complex variable such that $s = x + iy$ with $x, y \in \mathbb{R}$ and let $\alpha \in \left(\mathbb{R}_{>0} \setminus \mathbb{N}\right)$. The α-order fractional derivative of* (4.11) *is a complex function of s which converges absolutely in the half-plane*

$$x > \alpha \; . \tag{4.15}$$

Proof : Since $\eta^{(\alpha)}$ and $\zeta^{(\alpha)}$ differ only in the alternating factor $(-1)^{n-1}$, from (2.3) it follows that

$$\begin{cases} \operatorname{Re}\left(\eta^{(\alpha)}(s)\right) = \displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\log^{\alpha} n}{n^x} \cos(\pi\alpha - y \log n) \; , \\ \operatorname{Im}\left(\eta^{(\alpha)}(s)\right) = \displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\log^{\alpha} n}{n^x} \sin(\pi\alpha - y \log n) \; . \end{cases}$$

Being $\sin x \leq |\sin x| \leq 1$ for all $x \in \mathbb{R}$, it is

$$\begin{aligned} (-1)^{n-1} \frac{\log^{\alpha} n}{n^x} \sin(\pi\alpha - y\log n) &\leq (-1)^{n-1} \frac{\log^{\alpha} n}{n^x} \left|\sin(\pi\alpha - y\log n)\right| \\ &\leq (-1)^{n-1} \frac{\log^{\alpha} n}{n^x} < (-1)^{n-1} \frac{n^{\alpha}}{n^x} = \frac{(-1)^{n-1}}{n^{x-\alpha}} \; . \end{aligned}$$

Clearly, $\dfrac{(-1)^{n-1}}{n^{x-\alpha}}$ represents the general term of $\eta(x-\alpha)$ which converges for $x - \alpha > 0$ (see Figure 4.1), therefore $\operatorname{Im}\left(\eta^{(\alpha)}\right)$ converges absolutely in the half-plane (4.15) by using the comparison test (theorem 2.2.2). Analogously, the same holds for $\operatorname{Re}\left(\eta^{(\alpha)}\right)$.

□

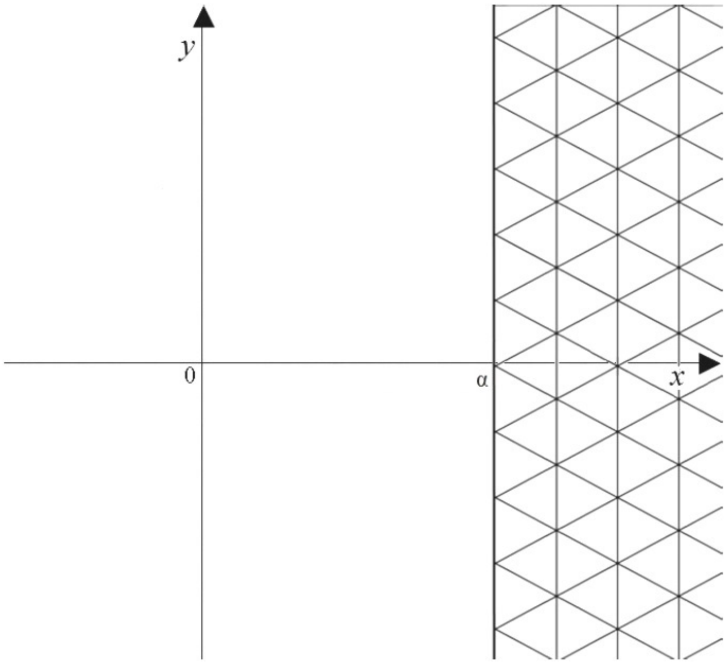


Figure 4.1: Convergence half-plane of $\eta^{(\alpha)}$.

Interestingly, η and its α-order fractional derivative converge for $x > 0$ and $x > \alpha$, respectively. Moreover, by taking into account (4.14) and the link between η and ζ functions (4.13), it can be shown that

Theorem 4.3.3. *Let s be a complex variable and let $\alpha \in \left(\mathbb{R}_{>0} \setminus \mathbb{N}\right)$. The relation between $\eta^{(\alpha)}$ and $\zeta^{(\alpha)}$ is given by*

$$\eta^{(\alpha)}(s) = \zeta^{(\alpha)}(s) - \mathrm{e}^{i\pi\alpha} \cdot 2^{1-s} \sum_{n=1}^{\infty} \frac{\log^{\alpha}(2n)}{n^s} \ . \tag{4.16}$$

Proof: From (4.14), it is

$$\begin{aligned}
\eta^{(\alpha)}(s) &= \mathrm{e}^{i\pi\alpha}\left(-\frac{\log^{\alpha} 2}{2^s} + \frac{\log^{\alpha} 3}{3^s} - \frac{\log^{\alpha} 4}{4^s} + \dots\right) \\
&= \mathrm{e}^{i\pi\alpha}\left(\left(-2\frac{\log^{\alpha} 2}{2^s} + \frac{\log^{\alpha} 2}{2^s}\right) + \frac{\log^{\alpha} 3}{3^s} + \left(-2\frac{\log^{\alpha} 4}{4^s} + \frac{\log^{\alpha} 4}{4^s}\right) + \dots\right) \\
&= \mathrm{e}^{i\pi\alpha} \sum_{n=1}^{\infty} \frac{\log^{\alpha} n}{n^s} - \mathrm{e}^{i\pi\alpha} \cdot 2 \sum_{n=1}^{\infty} \frac{\log^{\alpha}(2n)}{(2n)^s} = \zeta^{(\alpha)}(s) - \mathrm{e}^{i\pi\alpha} \cdot 2^{1-s} \sum_{n=1}^{\infty} \frac{\log^{\alpha}(2n)}{n^s} \ .
\end{aligned}$$

$\square$

4.4 Fractional counterpart of the critical strip

In Section 4.3, the α-order fractional derivative of the Dirichlet η function has been computed in order to single out the connection with $\zeta^{(\alpha)}$, on the half-plane of convergence, and on the critical strip. In the previous section, it was also recalled that η owns as many zeros as ζ on the same critical line and converges in the half-plane $\mathrm{Re}\, s > 0$, with a well-defined behavior in the critical strip $0 < \mathrm{Re}\, s < 1$.

On the other hand, $\zeta^{(\alpha)}$ converges for $\mathrm{Re}\, s > 1+\alpha$ while the convergence half-plane of $\eta^{(\alpha)}$ is $\mathrm{Re}\, s > \alpha$. Hence, $\zeta^{(\alpha)}$ and $\eta^{(\alpha)}$ suggest the strip $\alpha < \mathrm{Re}\, s < 1+\alpha$ as a fractional counterpart of the critical strip (Cattani et al., 2017). By comparing these two complex strips, the fractional operator ${}_{\mathrm{C}}\mathrm{D}^{\alpha}$ implies a positive shift in the half-plane of convergence by an amount equal to α. Therefore, every α-order fractional derivative is associated with a unique complex strip $\alpha < \mathrm{Re}\, s < 1+\alpha$, to be considered as the fractional strip corresponding to the classical critical strip (Figure 4.2). Consequently

$$\left(\alpha, 1+\alpha\right) \xrightarrow{\alpha\to 0} \left(0, 1\right) .$$

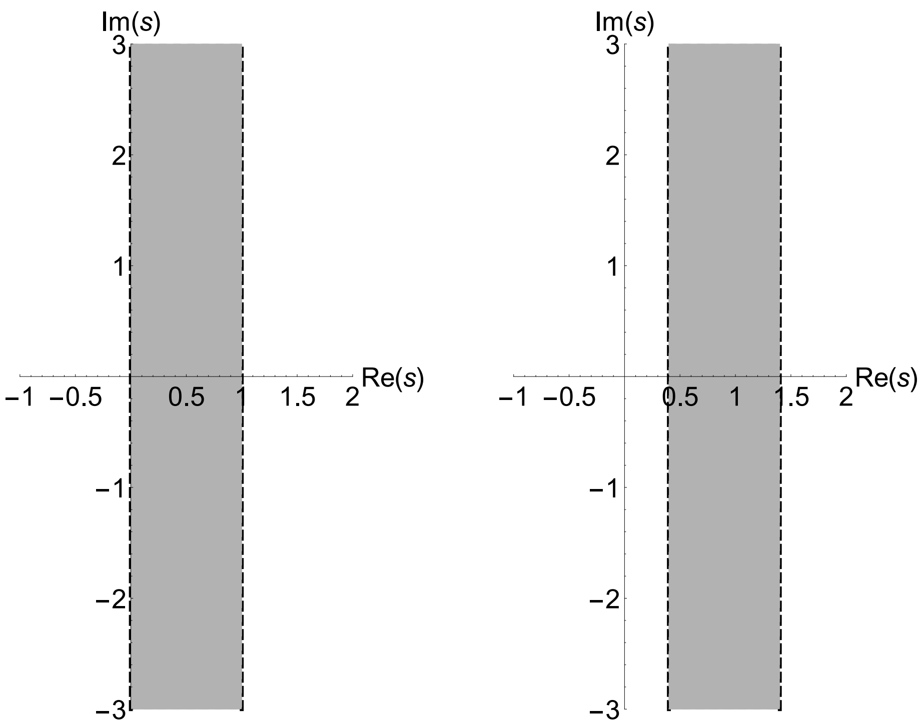


Figure 4.2: Critical strip (on the left side) and fractional critical strip with $\alpha = 0.4$ (on the right side).

The aforementioned fractional counterpart opens up new scenarios in research. In fact, some properties of the critical strip can be invariant by passing to $\alpha < \operatorname{Re} s < 1 + \alpha$ (distribution of the zeros, zero-free regions, etc.). They make $\zeta^{(\alpha)}$ extremely interesting for applications in analytic number theory and applied science. In fact, some applications of $\eta^{(\alpha)}$ in signal processing with interesting perspectives in the theory of quantum circuits, are discussed in Chapter 5.

4.4.1 Series characterising the fractional critical strip

The aim of this section is to obtain a fractional counterpart of (4.13), i.e. to express series (4.16) in terms of ζ^{α}. Since the Caputo-Ortigueira fractional derivative does not satisfy the generalized Leibniz rule, the investigation on the infinite series (4.16) appears to be complicated enough. On the other hand, eq. (4.13) can also be derived by using the forward Grünwald-Letnikov fractional derivative, which satisfies the generalized Leibniz rule. Hence, by applying (3.9) to both sides of (4.13), it is

$$\eta^{(\alpha)}(s) = \zeta^{(\alpha)}(s) - \mathrm{D}_f^{\alpha}\left(2^{1-s}\zeta(s)\right) . \tag{4.17}$$

The following statement holds.

Theorem 4.4.1. *Let s be a complex variable and let $\alpha \in \left(\mathbb{R}_{>0} \setminus \mathbb{N}\right)$. The functions $\eta^{(\alpha)}$ and $\zeta^{(\alpha)}$ are linked by*

$$\boxed{\eta^{(\alpha)}(s) = \zeta^{(\alpha)}(s) - 2^{1-s} \sum_{k=0}^{\infty} \binom{\alpha}{k} \log^k 2 \, \mathrm{e}^{i\pi k} \zeta^{(\alpha-k)}(s) \, .} \tag{4.18}$$

Moreover, the series

$$\sum_{n=1}^{\infty} \frac{\log^{\alpha}(2n)}{n^s} \, ,$$

can be expressed in terms of $\zeta^{(\alpha)}$ via

$$\sum_{n=1}^{\infty} \frac{\log^{\alpha}(2n)}{n^s} = \sum_{k=0}^{\infty} \binom{\alpha}{k} \log^k 2 \, \mathrm{e}^{i\pi k} \zeta^{(\alpha-k)}(s) \, . \tag{4.19}$$

Proof : Since D_f^{α} satisfies the generalized Leibniz rule, it is

$$
\begin{aligned}
\mathrm{D}_f^\alpha\left(2^{1-s}\zeta(s)\right) &= \sum_{k=0}^{\infty}\binom{\alpha}{k}\left(2^{1-s}\right)^{(k)}\zeta^{(\alpha-k)}(s) \\
&= \sum_{k=0}^{\infty}\binom{\alpha}{k} 2^{1-s}\log^k 2\, \mathrm{e}^{i\pi k}\,\zeta^{(\alpha-k)}(s) \\
&= 2^{1-s}\sum_{k=0}^{\infty}\binom{\alpha}{k}\log^k 2\, \mathrm{e}^{i\pi k}\,\zeta^{(\alpha-k)}(s)\ ,
\end{aligned}
$$

Therefore, from eq. (4.17), we have

$$
\eta^{(\alpha)}(s) = \zeta^{(\alpha)}(s) - 2^{1-s}\sum_{k=0}^{\infty}\binom{\alpha}{k}\log^k 2\, \mathrm{e}^{i\pi k}\zeta^{(\alpha-k)}(s)\ . \tag{4.20}
$$

Comparing (4.16) with (4.20), the proof follows.

□

Eq. (4.18) is nothing other than the fractional counterpart of (4.12). In fact, we get

$$
\begin{aligned}
&\zeta^{(\alpha)}(s) - 2^{1-s}\sum_{k=0}^{\infty}\binom{\alpha}{k}\log^k 2\, \mathrm{e}^{i\pi k}\,\zeta^{(\alpha-k)}(s) \xrightarrow{\alpha\to 0^+} \zeta(s) - 2^{1-s}\,\zeta(s) \\
&= \zeta(s)\left(1-2^{1-s}\right)\ .
\end{aligned}
$$

Furthermore, (4.19) also represents a fractional generalization since

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\frac{\log(2n)}{n^s} = \log 2\cdot\zeta(s) - \zeta'(s) = \sum_{k=0}^{1}\binom{1}{k}\log^k 2\,(-1)^{1-k}\,\zeta^{(1-k)}(s)\ , \\
&\sum_{n=1}^{\infty}\frac{\log^2(2n)}{n^s} = \log^2 2\cdot\zeta(s) - 2\log(2)\cdot\zeta'(s) + \zeta''(s) = \sum_{k=0}^{2}\binom{2}{k}\log^k 2\,(-1)^{2-k}\,\zeta^{(2-k)}(s)\ , \\
&\vdots \\
&\sum_{n=1}^{\infty}\frac{\log^m(2n)}{n^s} = \sum_{k=0}^{m}\binom{m}{k}\log^k 2\,(-1)^{m-k}\,\zeta^{(m-k)}(s) = \sum_{k=0}^{m}\binom{m}{k}\log^k 2\,\mathrm{e}^{i\pi(m-k)}\,\zeta^{(m-k)}(s)\ ,
\end{aligned}
\tag{4.21}
$$

where $m \in \mathbb{N}$. Hence, the proposition below unifies both cases.

Corollary 4.4.2. *Let s be a complex variable and let $\alpha \in \mathbb{R}_{>0}$. The series*

$$\sum_{n=1}^{\infty} \frac{\log^{\alpha}(2n)}{n^s} \, ,$$

can be expressed in terms of $\zeta^{(\alpha)}$ *by*

$$\boxed{\sum_{n=1}^{\infty} \frac{\log^{\alpha}(2n)}{n^s} = \sum_{k=0}^{\tilde{k}} \binom{\alpha}{k} \log^k 2 \; e^{i\pi k} \zeta^{(\alpha-k)}(s) \, ,} \tag{4.22}$$

in which $\tilde{k}$ *is an element of* $\widetilde{\mathbb{R}}_{>0} = \mathbb{R}_{>0} \cup \{\infty\}$ *and given by*

$$\tilde{k} \stackrel{\text{def}}{=} \begin{cases} \alpha \, , & \alpha \in \mathbb{N} \, , \\ \infty \, , & \alpha \in \left(\mathbb{R}_{>0} \setminus \mathbb{N}\right) \, . \end{cases}$$

Proof : (4.19) for $\alpha \in \left(\mathbb{R}_{>0} \setminus \mathbb{N}\right)$ and $(4.21)_3$ for $\alpha \in \mathbb{N}$ show that the proposition holds for all $\alpha \in \mathbb{R}_{>0}$.

□

Chapter 5

Application to signal processing

5.1 Introduction

In this chapter, suitable signal processing networks associated with $\eta^{(\alpha)}$ and $\zeta^{(\alpha)}$ are presented. The Riemann hypothesis can be recast in terms of signal theory by designing signal processing networks with respect to the Riemann ζ function and to the Dirichlet η function. The signal processing paradigm associated with the Riemann ζ function is called a discrete log-time system, since it only presents weighted ideal delay units with delays that fall on an logarithmic time grid (see Adams, 2005). In particular, a variant of the discrete log-time systems is presented . The main difference between this signal processing paradigm and the discrete log-time system is that the proposed model is based on a non-multiplicative operator. Consequently, the network associated with $\eta^{(\alpha)}$ and $\zeta^{(\alpha)}$ cannot be characterized by using the Euler product formula (1.13). Moreover, the Fourier transform of $\eta^{(\alpha)}$ is also computed and its associated signal processing network is briefly described. The symmetry showed by the one-sided Fourier transform of $\eta^{(\alpha)}$ can be used in the analysis of the Riemann quantum circuits (see Ramos and Mendes, 2014). Their analysis provides an interesting intersection between signal processing theory and analytic number theory.

Chapter 5 is organized as follows. In section 5.2 a signal processing network based on $\eta^{(\alpha)}$ is presented. Section 5.3 presents the Fourier transform of $\eta^{(\alpha)}$ as filter bank. Finally, the the one-sided Fourier transform of $\eta^{(\alpha)}$ and its symmetry are presented in Section 5.4.

5.2 Signal processing model associated to $\eta^{(\alpha)}$

In this section, a signal processing interpretation of $\eta^{(\alpha)}$ by using some analogies with classical systems theory (Beerends et al., 2003, parts 3–4) is given. In order to compute an integral transform of (4.14), let us shift the complex plane by $1/2$ with the change of variables

$$\tilde{s} = s - \frac{1}{2} \, ,$$

where $\tilde{s}$ represents the Laplace transform variable. Since

$$n^s = n^{1/2} n^{\tilde{s}} = n^{1/2} \mathrm{e}^{\tilde{s} \log n} \, ,$$

it is

$$\begin{aligned} \eta^{(\alpha)}(s) &= \mathrm{e}^{i\pi\alpha} \sum_{n=2}^{\infty} (-1)^{n-1} \frac{\log^\alpha n}{n^s} = \mathrm{e}^{i\pi\alpha} \left(-\frac{\log^\alpha 2}{2^s} + \frac{\log^\alpha 3}{3^s} - \frac{\log^\alpha 4}{4^s} + \dots \right) \\ &= \mathrm{e}^{i\pi\alpha} \left(-\frac{\log^\alpha 2}{2^{1/2}} \mathrm{e}^{-\tilde{s}\log 2} + \frac{\log^\alpha 3}{3^{1/2}} \mathrm{e}^{-\tilde{s}\log 3} - \frac{\log^\alpha 4}{4^{1/2}} \mathrm{e}^{-\tilde{s}\log 4} + \dots \right) \, . \end{aligned} \tag{5.1}$$

From the systems theory (Beerends et al., 2003, part 4), it is well know that

$$\mathcal{L}\left(\delta(t-a)\right) = \mathrm{e}^{-as} \, , \qquad (a > 0)$$

where $\mathcal{L}$ and δ are the Laplace transform and the Dirac delta, respectively. Therefore, (5.1) can be viewed as the summation of weighted ideal delay units, where each term has a Laplace transfer function given by $\mathrm{e}^{i\pi\alpha} \dfrac{(\log n)^\alpha}{n^{1/2}} \mathrm{e}^{-\tilde{s}\log n}$, hence it represents the transfer function of some network. Figure 5.1 shows how (5.1) can be interpreted as a network of linear weighted delays where a FIR implementation (that is, a filter with a finite impulse response) is also possible by using a tapped delay line (Adams, 2005; Horner, 1987, Chap. 7) instead of the delay blocks. Figure 5.1 shows a linear network with weighted ideal delay units having delays that can be modeled as discrete log-time (DLT) systems. In fact, (5.1) shows that the delay falls on $\log n$ time grid. The operator wld defined (Adams, 2005) by

$$\mathrm{wld}(n) = \frac{1}{n^{1/2}} \mathrm{e}^{-\tilde{s}\log n} \, , \qquad (n \in \mathbb{N})$$

can be introduced in order to represent η. Analogously, $\eta^{(\alpha)}$ can be rewritten by introducing the α-order fractional counterpart of wld, that is

$$\mathrm{wld}_\alpha(n) = \mathrm{e}^{i\pi\alpha} \frac{\log^\alpha n}{n^{1/2}} \mathrm{e}^{-\tilde{s}\log n} \, . \qquad (n \in \mathbb{N})$$

Hence, (5.1) can be rewritten as follows

$$\eta^{(\alpha)}(s) = -\mathrm{wld}_\alpha(2) + \mathrm{wld}_\alpha(3) - \mathrm{wld}_\alpha(4) + \dots \ ,$$

and despite the fact that wld is a multiplicative function, its fractional counterpart wld_α is no longer multiplicative.

DLT systems have different properties such as series connection, time-shift and convolution (Adams, 2005). In particular, the impulse response of two DLT systems (which is the convolution of their impulse responses), plays a fundamental role. Further, at a fixed time $t = \log k$, the output is obtained by adding all paths through the network having a delay equal to $\log k$. Since wld is multiplicative, the output at time $t = \log p$, with $p \in \mathbb{P}$, can present only a single element of delay (from theorem 1.3.1). Clearly, the same cannot be true for wld_α (due to the lack of multiplicativity). Therefore, the network associated with $\eta^{(\alpha)}$ is more complicated than that associated with η. In particular, it cannot be characterized by using the Euler product (1.13) unlike the network associated with η (Adams, 2005).

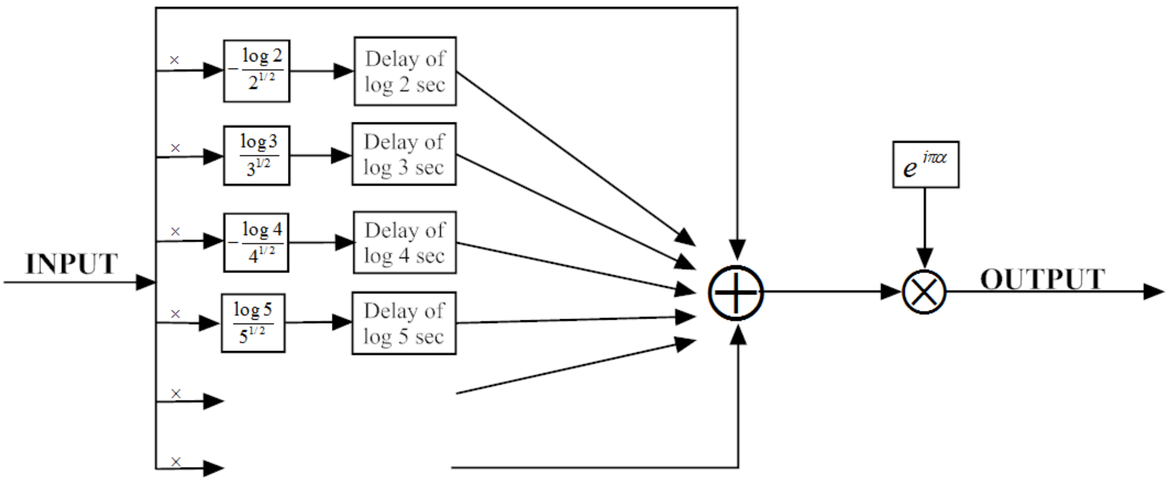


Figure 5.1: Signal processing network for $\eta^{(\alpha)}$.

5.3 Fourier transform of $\eta^{(\alpha)}$ as filter bank

In order to investigate the relation between $\zeta^{(\alpha)}$ and its Fourier transform, the Fourier transform of $\eta^{(\alpha)}$ is computed (Cattani et al., 2017). In particular, analogously to the integer order derivative, it can be shown that

Theorem 5.3.1. *Let s be a complex variable such that $s = x + iy$ with $x, y \in \mathbb{R}$ and let $\alpha \in \left(\mathbb{R}_{>0} \setminus \mathbb{N}\right)$. Under these hypotheses, it is*

$$\widehat{f^{(\alpha)}}(\omega) = (i\omega)^{\alpha} \widehat{f}(\omega) , \tag{5.2}$$

where $\widehat{f}$ is the Fourier transform defined as follows

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} f(s)\, e^{-i\omega s}\, \mathrm{d}s . \tag{5.3}$$

Proof: See Li et al. (2009).

$\square$

However, in what follows we will consider the integral transform of complex functions that are defined only in the half-plane $\mathrm{Re}\, s > 0$, so that we have to limit ourselves to integral transforms defined in the half-plane. In particular, the Laplace transform $\mathcal{L}$ is given (Beerends et al., 2003, part 4) by

$$\mathcal{L}\left(f(t)\right) \overset{\text{def}}{=} \int_{0}^{\infty} f(t)\, e^{-st}\, \mathrm{d}t \ ,$$

while the one-sided Fourier transform is defined by

$$\widehat{f}^{+}(\omega) = \int_{0}^{\infty} f(s)\, e^{-i\omega s}\, \mathrm{d}s \ . \tag{5.4}$$

Alternatively, (5.3) and (5.4) can also be indicated with $\mathcal{F}$ and $\mathcal{F}^{+}$, respectively. Hence, we can show that

Theorem 5.3.2. *Under the same hypotheses of theorem* 5.3.1*, the Fourier transform of $\zeta^{(\alpha)}$ is given by*

$$\widehat{\zeta^{(\alpha)}}(y) = (iy)^{\alpha} \sum_{n=1}^{\infty} \mathcal{L}\left(\mathrm{e}^{-iy\log n}\right) \ . \tag{5.5}$$

Proof: From (5.2) by taking $f(s) = \zeta(s)$, we get

$$\widehat{\zeta^{(\alpha)}}(y) = (iy)^{\alpha}\, \widehat{\zeta}(y) , \tag{5.6}$$

and since

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^{s}} = \sum_{n=1}^{\infty} e^{-s\log n} = \sum_{n=1}^{\infty} \mathcal{L}\left(\delta(t - \log n)\right) \ ,$$

we have

$$\widehat{\zeta}(y) = \sum_{n=1}^{\infty} \mathcal{F}\left(\mathcal{L}\left(\delta(t-\log n)\right)\right) = \sum_{n=1}^{\infty} \mathcal{L}\left(\mathcal{F}\left(\delta(t-\log n)\right)\right) = \sum_{n=1}^{\infty} \mathcal{L}\left(\mathrm{e}^{-iy\log n}\right) , \quad (5.7)$$

by taking into account that $s = x+iy$. By substituting (5.7) into (5.6), the proof follows.

□

Analogously to (5.7), it is

$$\widehat{\eta}(y) = \sum_{n=1}^{\infty} (-1)^{n-1} \mathcal{L}\left(\mathrm{e}^{-iy\log n}\right) , \quad (5.8)$$

hence, by using (5.5), we get

$$\widehat{\eta^{(\alpha)}}(y) = (iy)^{\alpha} \sum_{n=1}^{\infty} (-1)^{n-1} \mathcal{L}\left(\mathrm{e}^{-iy\log n}\right) . \quad (5.9)$$

It follows that the Fourier transform of $\eta^{(\alpha)}$ provides another signal processing network. In Figure 5.2 the RHS of (5.9) is represented as a Laplace filter bank where the main block can be simplified by using a fast numerical method for the Laplace transform (see Rokhlin, 1988).

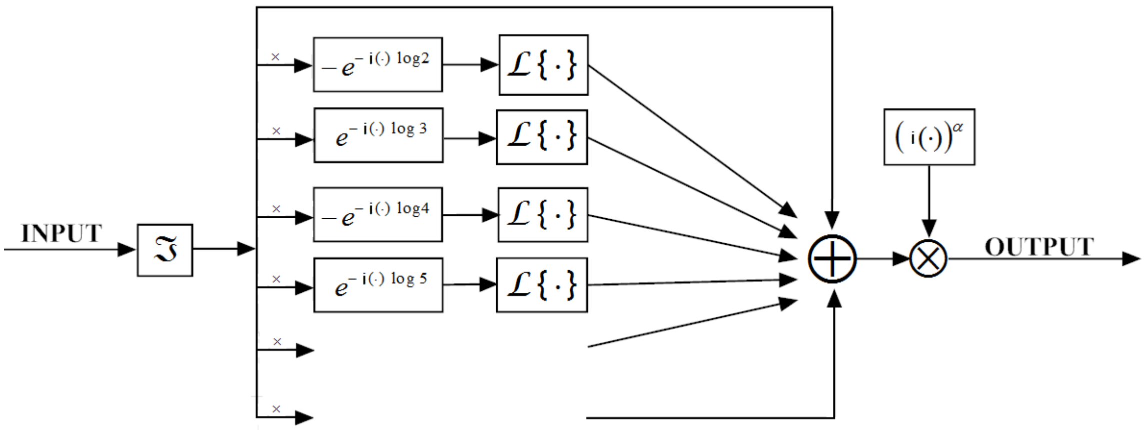


Figure 5.2: $\widehat{\eta^{(\alpha)}}$ as Laplace bank filter.

5.4 Symmetry of $\widehat{\eta^{(\alpha)}}^{+}$

The one-sided Fourier transform provides information only for a causal signal (Beerends et al., 2003, part 4). However, it does not represent a restriction in Physics and Engineering applications (such as radio-frequency circuits or general telecommunications systems) since the input is almost always a causal signal. Consequently, the explicit computation of $\widehat{\eta^{(\alpha)}}^{+}$ is given and discussed here, together with its symmetry.

Theorem 5.4.1. *Under the same hypotheses of theorem* 5.3.1*, the double one-sided Fourier transform of* $\eta^{(\alpha)}$ *is given by*

$$\boxed{\widehat{\eta^{(\alpha)}}^{+}(\omega,\xi) = \mathrm{e}^{i\pi\alpha}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\log^{\alpha} n}{(\log n+\xi)(i\log n-\omega)}\ .} \tag{5.10}$$

Proof : From the theory of n-dimensional Fourier transforms (see e.g. Beerends et al., 2003, Part 3), for all $f \in L^1(\mathbb{R}^2)$ it is

$$f(x_1,x_2) = f(x_1)f(x_2) \Longrightarrow \widehat{f}^{\,+}(\omega,\xi) = \widehat{f}^{\,+}(\omega)\,\widehat{f}^{\,+}(\xi)\ ,$$

and being $n^{-s} = n^{-x}n^{-iy}$, we get

$$\begin{aligned}
\widehat{n^{-s}}^{+} &= \widehat{n^{-x}}^{+}\ \widehat{n^{-iy}}^{+} = \int_0^{\infty} n^{-x}e^{-i\omega x}\,\mathrm{d}x \int_0^{\infty} n^{-iy}e^{-i\xi y}\,\mathrm{d}y \\
&= \int_0^{\infty}\mathrm{e}^{-x\log n}e^{-i\omega x}\,\mathrm{d}x\int_0^{\infty}\mathrm{e}^{-iy\log n}e^{-i\xi y}\,\mathrm{d}y = \int_0^{\infty}\mathrm{e}^{-x(\log n+i\omega)}\,\mathrm{d}x \\
&\cdot\int_0^{\infty}\mathrm{e}^{-iy(\log n+\xi)}\,\mathrm{d}y = \frac{1}{-\log n - i\omega}\quad e^{-x\,(\log n+i\omega)}\Big|_0^{\infty}\frac{i}{\log n+\xi} \\
&\cdot e^{-iy\,(\log n+\xi)}\Big|_0^{\infty} = \frac{1}{\log n+i\omega}\,\frac{-i}{\log n+\xi} = \frac{1}{(\log n+\xi)(i\log n-\omega)}\ .
\end{aligned}$$

Since

$$\widehat{\zeta^{(\alpha)}}^{+}(\omega,\xi) = \sum_{n=1}^{\infty}\mathcal{F}^{+}\left({}_{\mathrm{C}}\mathrm{D}^{\alpha}\left(n^{-s}\right)\right)\ ,$$

and

$${}_{\mathrm{C}}\mathrm{D}^{\alpha}\left(n^{-s}\right) = \mathrm{e}^{i\pi\alpha}\,\frac{\log^{\alpha} n}{n^{s}}\ ,$$

it follows that the double one-sided Fourier transform of $\zeta^{(\alpha)}$ is given by

$$
\begin{aligned}
\widehat{\zeta^{(\alpha)}}^{+}\big(\omega,\xi\big) &= \mathrm{e}^{i\pi\alpha}\sum_{n=1}^{\infty}\log^{\alpha} n\ \widehat{n^{-s}}^{+} \\
&= \mathrm{e}^{i\pi\alpha}\sum_{n=1}^{\infty}\frac{\log^{\alpha} n}{(\log n+\xi)(i\log n-\omega)}\ .
\end{aligned}
\tag{5.11}
$$

Taking into account (4.14), the proof follows.

□

Figures 5.3 and 5.4 show the real part and imaginary part of $\zeta^{(\alpha)}$, respectively. Likewise, in Figures 5.5 and 5.6 the real part and imaginary part of $\eta^{(\alpha)}$ are represented, respectively. In each of these figures, areas where the function becomes nonreal are excluded (white color). Furthermore, the surfaces are split in the presence of discontinuities. Both surfaces show a symmetry induced by the Fourier transform, given by the following

Proposition 5.4.2. *The real (respectively imaginary) part of both* (5.10) *and* (5.11) *is an odd (respectively even) function with regard to* ω.

Proof: Since

$$
\begin{aligned}
\frac{1}{(\log n+\xi)(i\log n-\omega)} &= \frac{1}{-\omega(\log n+\xi)+i\log n\,(\log n+\xi)} \\
&= \frac{-\omega(\log n+\xi)-i\log n\,(\log n+\xi)}{\omega^2(\log n+\xi)^2+\log^2 n\,(\log n+\xi)^2}\ ,
\end{aligned}
$$

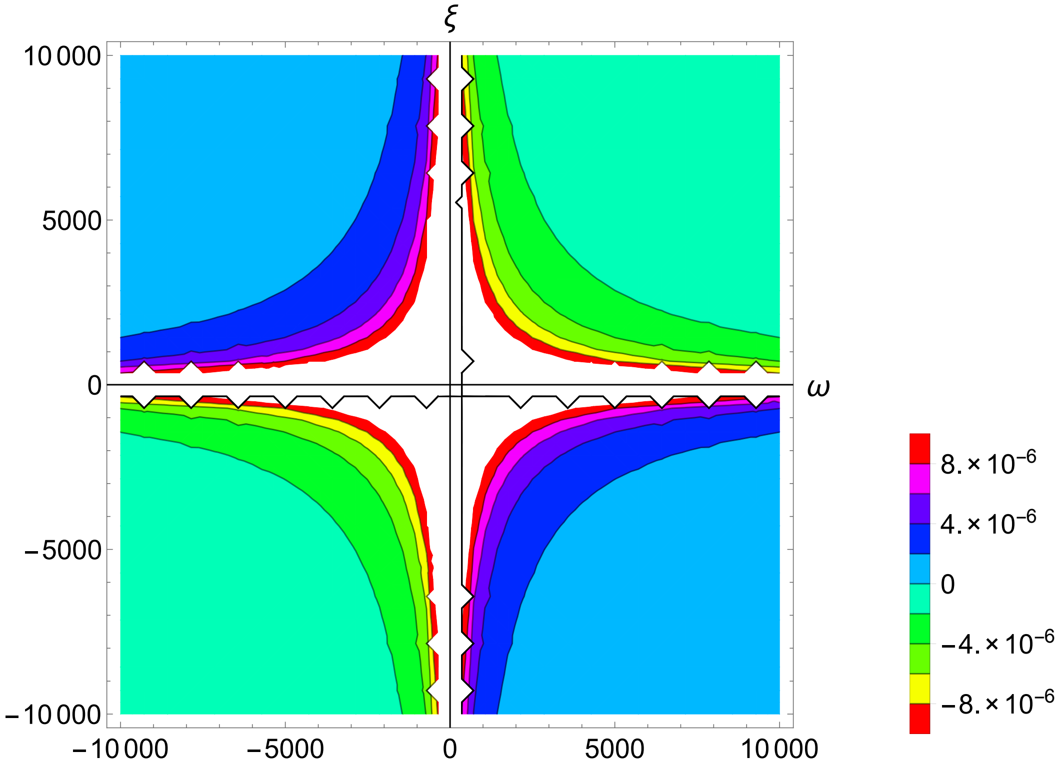
by taking into account (5.10), it follows that

$$
\begin{cases}
\mathrm{Re}\left(\widehat{\eta^{(\alpha)}}^{+}(-\omega,\xi)\right) = -\,\mathrm{Re}\left(\widehat{\eta^{(\alpha)}}^{+}(\omega,\xi)\right)\ , \\
\mathrm{Im}\left(\widehat{\eta^{(\alpha)}}^{+}(-\omega,\xi)\right) = \mathrm{Im}\left(\widehat{\eta^{(\alpha)}}^{+}(\omega,\xi)\right)\ ,
\end{cases}
$$

The same result holds for (5.11) since (5.10) and (5.11) differ only in the alternating factor $(-1)^n$. Hence the assertion of the proposition follows.

□

Let z be the axes orthogonal to the $\omega\xi$-plane. Geometrically, proposition 5.4.2 shows that the real (respectively imaginary) parts of both $\widehat{\eta^{(\alpha)}}^{+}$ and $\widehat{\zeta^{(\alpha)}}^{+}$ are symmetrical with respect to the ξ-axis (respectively ξz-plane). More details on the aforementioned complex functions are shown in Figures 5.7 and 5.8. In both figures, the real part (orange surface) and imaginary part (blue surface) overlap in the half-plane $\xi \gtrsim 0$ while they present an extremely irregular behavior in $\xi < 0$. Integral transforms (5.10) and (5.11) can be applied in the analysis of the so-called Riemann quantum circuits. In fact, the (double) one-sided Fourier transform is already used in circuits theory and Riemann quantum circuits have become very popular in recent years (see Ramos and Mendes, 2014) by opening new frontiers in research.



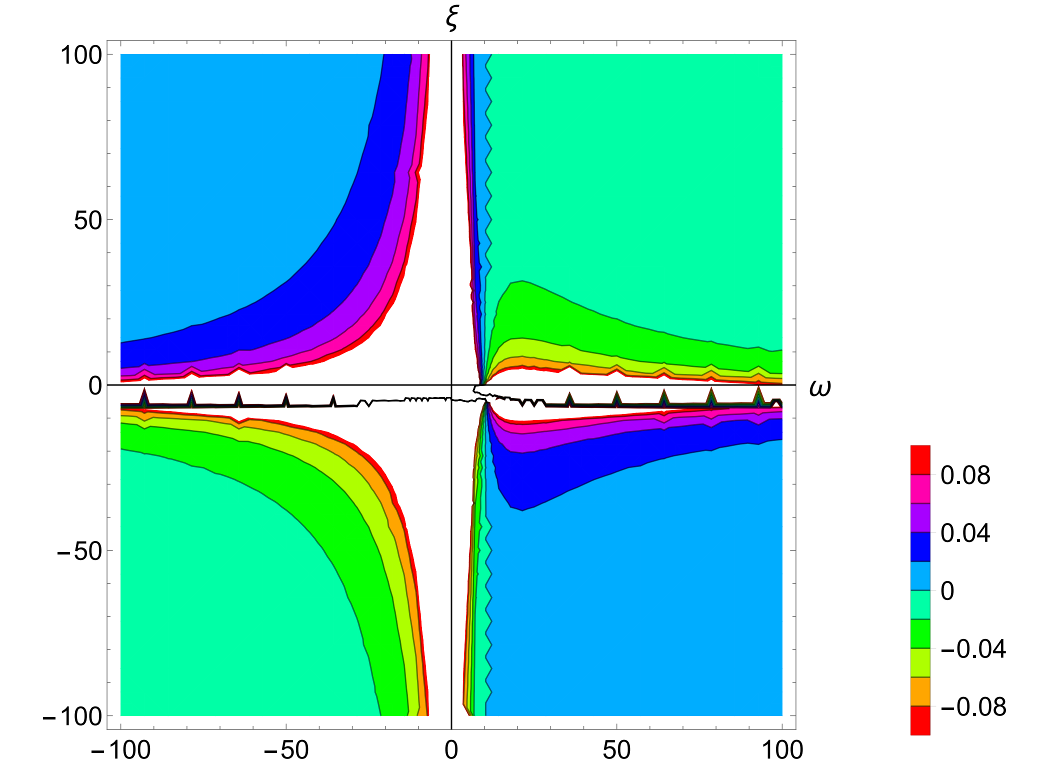


Figure 5.3: Real part of $\widehat{\zeta^{(\alpha)}}^{+}$ with $\alpha = 0.4$ and upper limit of the series $n = 60$.

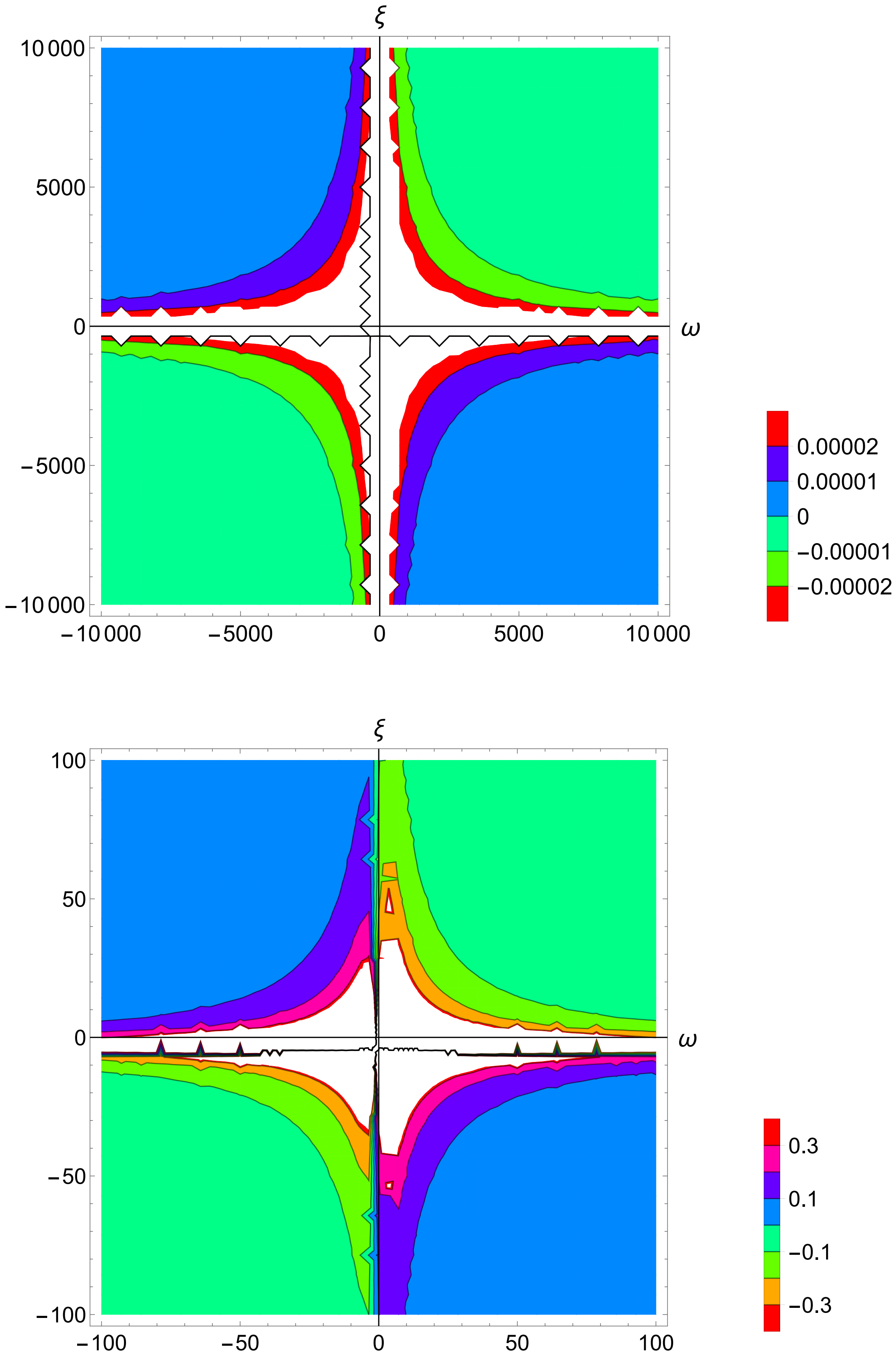
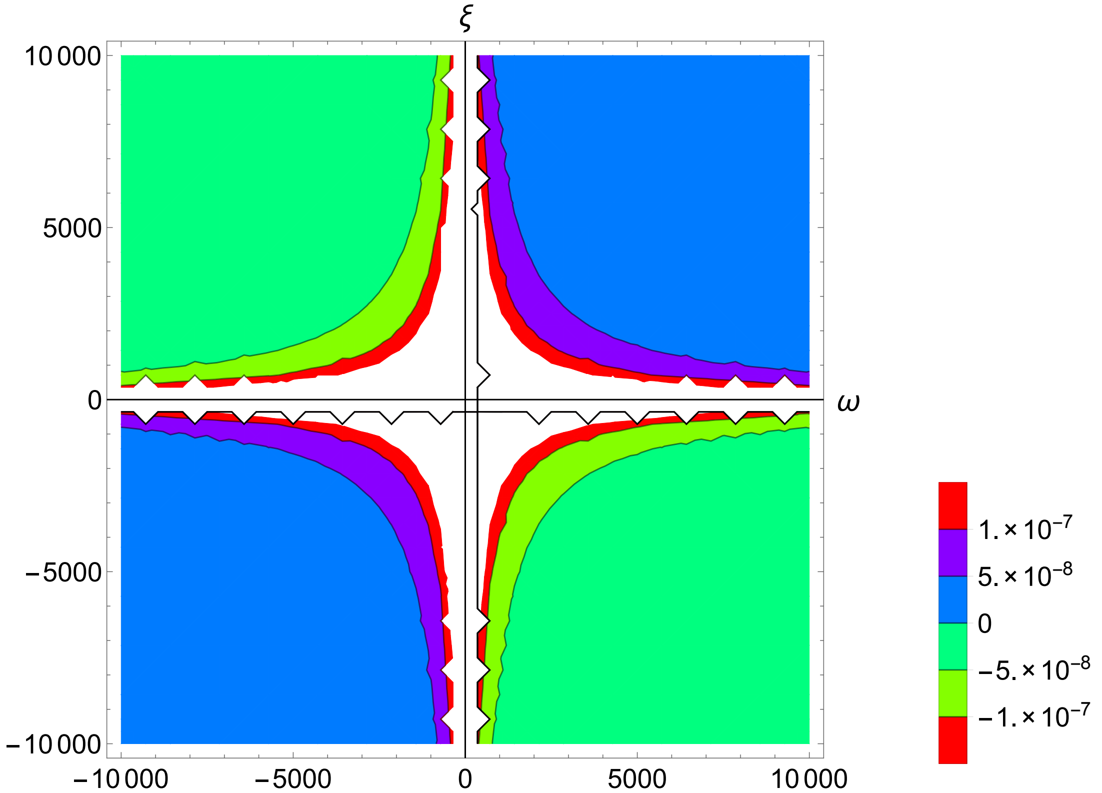


Figure 5.4: Imaginary part of $\widehat{\zeta^{(\alpha)}}^{+}$ with $\alpha = 0.4$ and upper limit of the series $n = 60$.



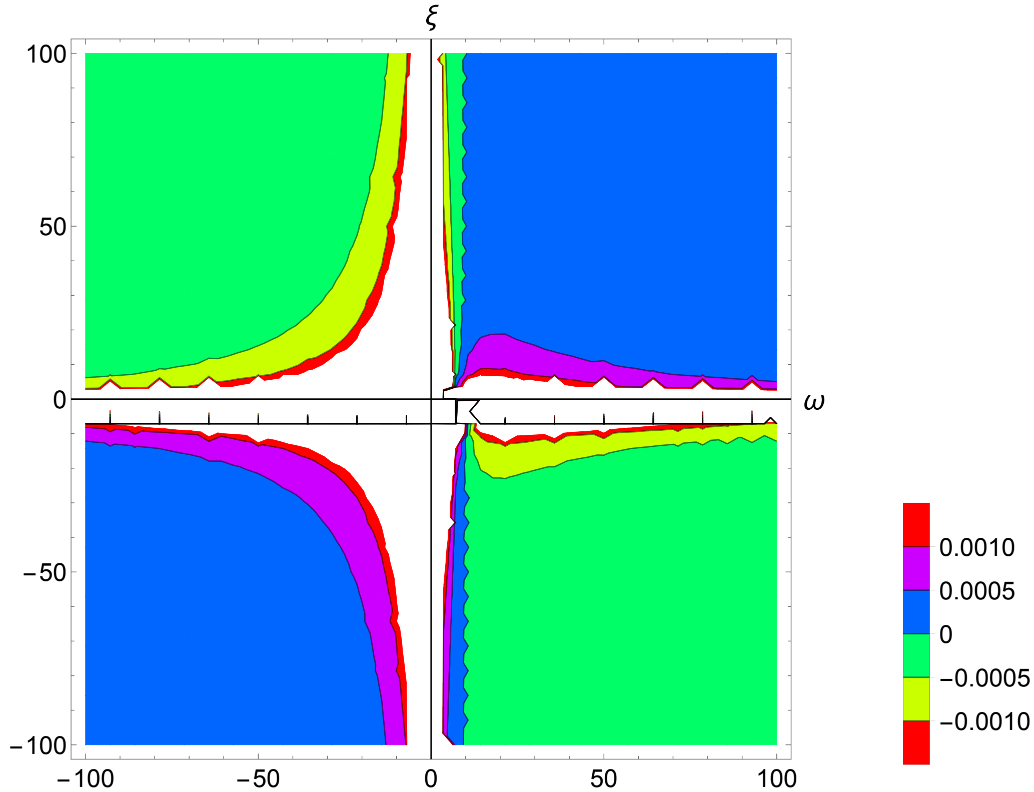
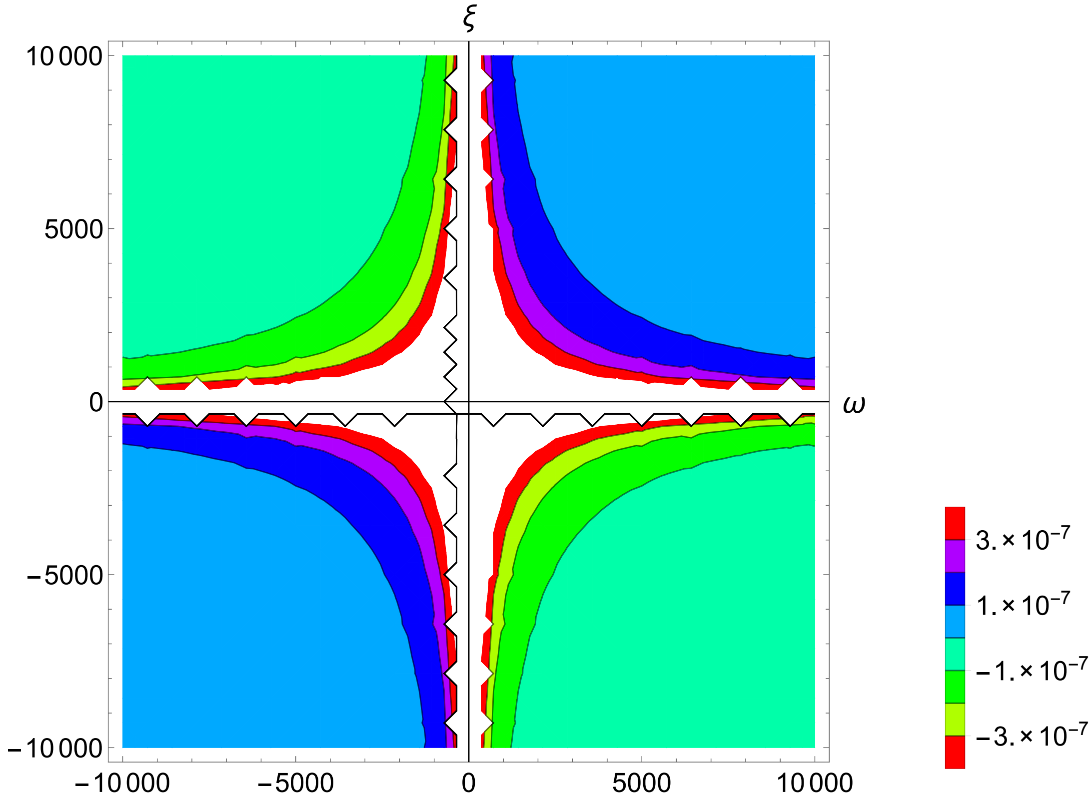


Figure 5.5: Real part of $\widehat{\eta^{(\alpha)}}^{+}$ with $\alpha = 0.4$ and upper limit of the series $n = 60$.



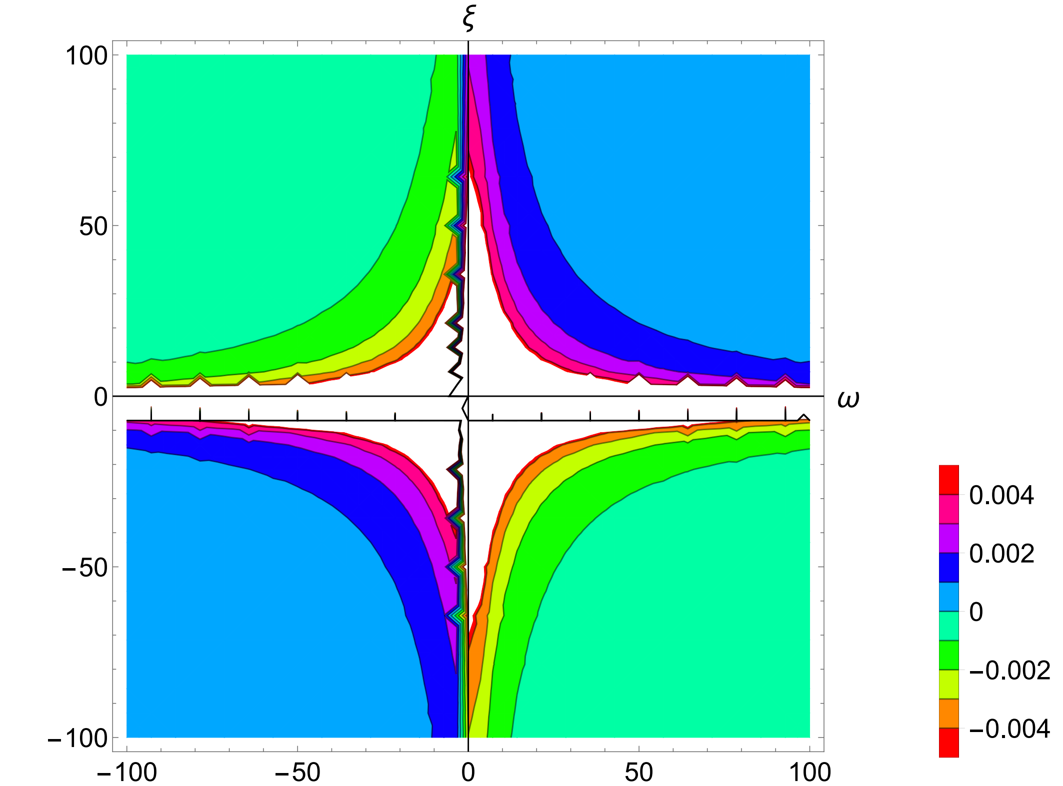


Figure 5.6: Imaginary part of $\widehat{\eta^{(\alpha)}}^{+}$ with $\alpha = 0.4$ and upper limit of the series $n = 60$.

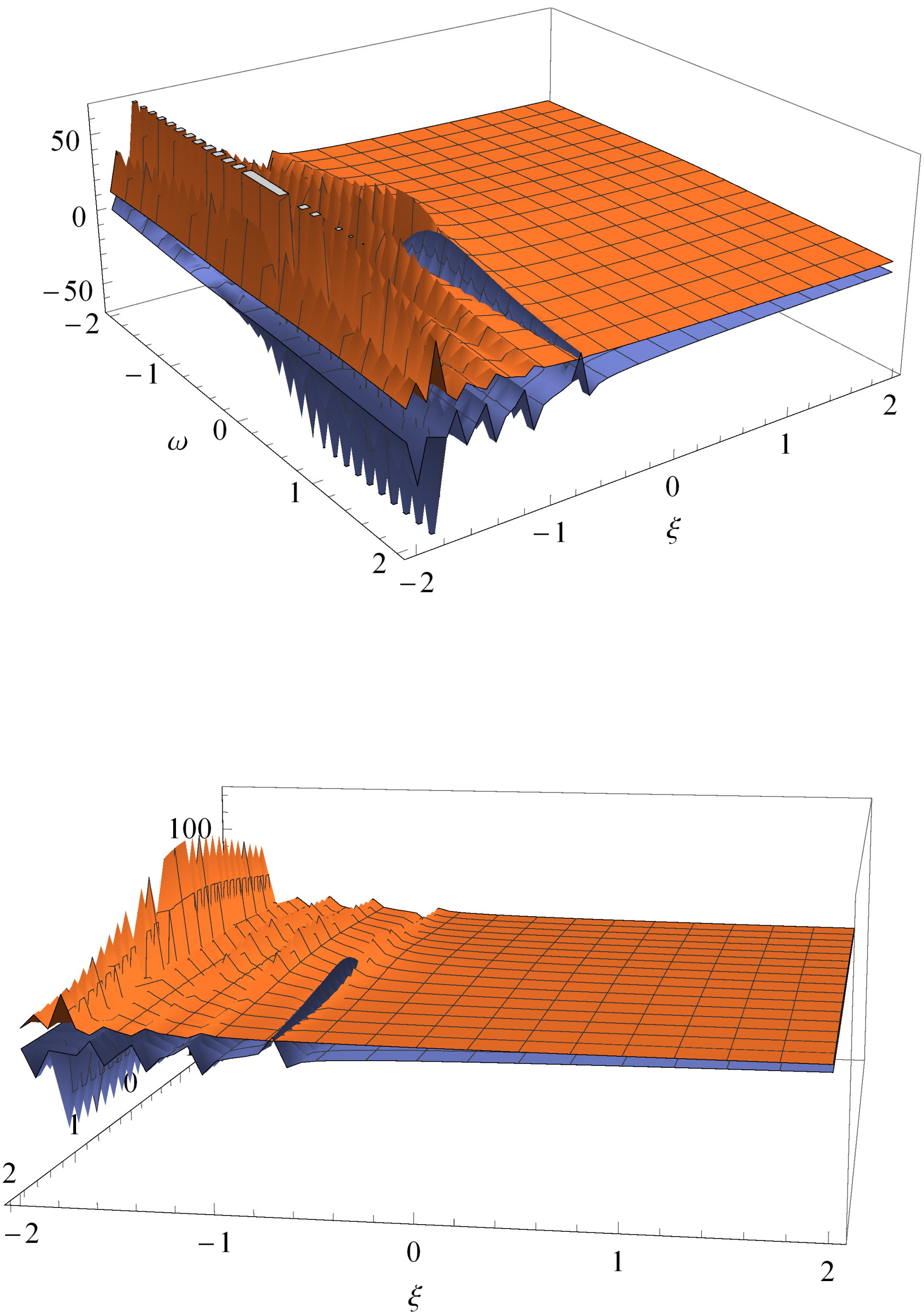


Figure 5.7: $\widehat{\zeta^{(\alpha)}}^{+}$ with $\alpha = 0.4$ and upper limit of the series $n = 60$.

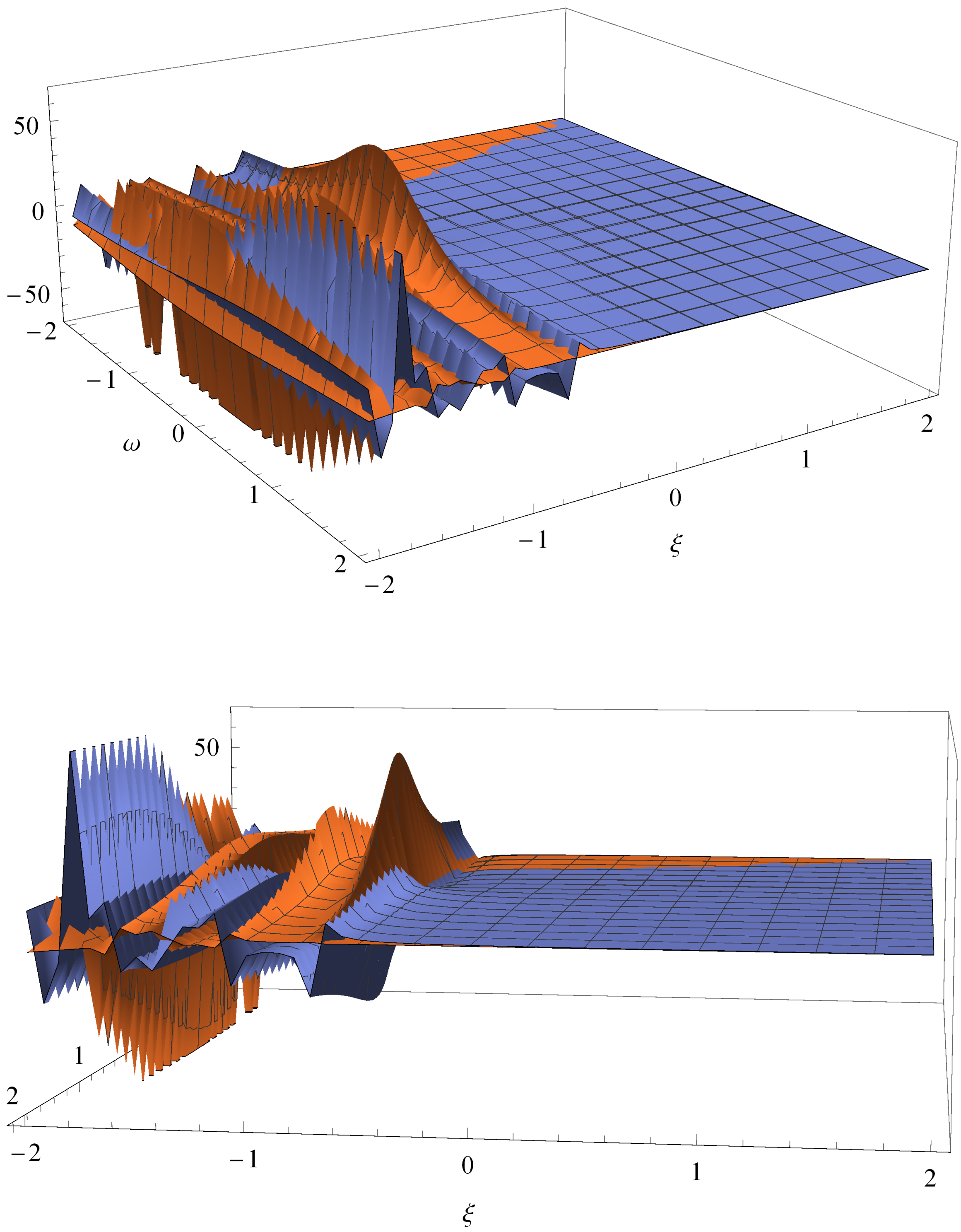


Figure 5.8: $\widehat{\eta^{(\alpha)}}^{+}$ with $\alpha = 0.4$ and upper limit of the series $n = 60$.

Conclusion

In this thesis, the α-order fractional derivative of the Riemann ζ function was computed, using the Ortigueira generalization of the Caputo derivative to the complex plane. The computation of its real and imaginary parts shows that the half-plane of convergence depends on the fractional order α. In order to generalize this result, the fractional derivatives of the Hurwitz ζ function and of the Dirichlet series were computed. A chaotic decay to zero was shown that might suggest that the fractional derivative of the Riemann ζ function is a non-differentiable function around zero (Chapter 2).

The fractional derivative $\zeta^{(\alpha)}$ has been recomputed by the Grünwald-Letnikov fractional derivative in order to determine its functional equation. Moreover, an integral representation of $\zeta^{(\alpha)}$ via Bernoulli numbers was given, which represents the fractional version based on a result of Apostol (1985). Thereafter, following the same approach of Apostol (1985) and Spira (1965), a simplified version of this functional equation was found, as well as its generalization to the Hurwitz ζ function and Lerch zeta function (Chapter 3).

The problem of the link between $\zeta^{(\alpha)}$ and the distribution of prime numbers was discussed in computing the logarithmic fractional derivative of the Riemann ζ function. By introducing the Dirichlet η function, it was shown that the complex strip $\alpha < \mathrm{Re}\, s < 1 + \alpha$ represents the fractional counterpart of the critical strip. It was emphasized how the infinite series associated with the strip $\alpha < \mathrm{Re}\, s < 1 + \alpha$ can be written in terms of $\zeta^{(\alpha)}$ (Chapter 4). Two signal processing networks associated with $\eta^{(\alpha)}$ and its Fourier transform were presented in order to provide an application in quantum signal processing. Moreover, the symmetry of both $\widehat{\zeta^{(\alpha)}}^{+}$ and $\widehat{\eta^{(\alpha)}}^{+}$ was shown (Chapter 5).

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