ON FRACTIONAL PROBABILISTIC MEAN VALUE THEOREMS, FRACTIONAL COUNTING PROCESSES AND RELATED RESULTS

ABSTRACT

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The thesis collects the outcomes of the author's research carried out in the research group *Probability Theory and Mathematical Statistics* at the Department of Mathematics, University of Salerno, during the doctoral programme "Mathematics, Physics and Applications". The results are at the interface between Fractional Calculus and Probability Theory. While research in probability and applied fields is now well established and enthusiastically supported, the subject of fractional calculus, i.e. the study of an extension of derivatives and integrals to any arbitrary real or complex order, has achieved widespread popularity only during the past four decades or so, because of its applications in several fields of science, engineering and finance. Moreover, the application of the fractional paradigm to probability theory has been carefully but partially explored over the years, especially from the point of view of stochastic processes. The aim of the thesis is to prove some new theorems at the interface between Mathematical Analysis and Probability Theory, and to study rigorously certain new stochastic processes and statistical models constructed on top of some well-known classical results and then generalized by means of fractional calculus.

The dissertation is organized as follows.

In Chapter 1 we give an overview about the main ideas that inspire fractional calculus and about the mathematical techniques for dealing with fractional operators and the related special functions and probability distributions.

In order to develop certain fractional probabilistic analogues of Taylor's theorem

and mean value theorem, in Chapter 2 we introduce the nth-order fractional equilibrium distribution in terms of the Weyl fractional integral and investigate its main properties. Specifically, we show a characterization result by which the nth-order fractional equilibrium distribution is identical to the starting distribution if and only if it is exponential. The nth-order fractional equilibrium density is then used to prove a fractional probabilistic Taylor's theorem based on derivatives of Riemann-Liouville type. A fractional analogue of the probabilistic mean value theorem is thus developed for pairs of nonnegative random variables ordered according to the survival bounded stochastic order. We also provide some related results, both involving the normalized moments and a fractional extension of the variance, and a formula of interest to actuarial science. In conclusion, we discuss the probabilistic Taylor's theorem based on fractional Caputo derivatives.

In Chapter 3 we consider a fractional counting process with jumps of integer amplitude $1, 2, \ldots, k$, whose probabilities satisfy a suitable system of fractional difference-differential equations. We obtain the moment generating function and the probability law of the resulting process in terms of generalized Mittag-Leffler functions. We also discuss two equivalent representations both in terms of a compound fractional Poisson process and of a subordinator governed by a suitable fractional Cauchy problem. The first occurrence time of a jump of fixed amplitude is proved to have the same distribution as the waiting time of the first event of a classical fractional Poisson process, this extending a well-known property of the Poisson process. When $k = 2$ we also express the distribution of the first-passage time of the fractional counting process in an integral form. We then show that the ratios given by the powers of the fractional Poisson process and of the counting process over their means tend to 1 in probability.

In Chapter 4 we propose a generalization of the alternating Poisson process from the point of view of fractional calculus. We consider the system of differential equations governing the state probabilities of the alternating Poisson process and replace the ordinary derivative with a fractional one (in the Caputo sense). This produces a fractional 2-state point process, whose probability mass is expressed in terms of the (two-parameter) Mittag-Leffler function. We then show that it can be recovered also by means of renewal theory arguments. We study the limit state probability, and certain proportions involving the fractional moments of the sub-renewal periods of the process. In order to derive new Mittag-Leffler-like distributions related to the considered process, we then exploit a transformation acting on pairs of stochastically ordered random variables, which is an extension of the equilibrium operator and deserves interest in the analysis of alternating stochastic processes.

In Chapter 5 we analyse a jump-telegraph process by replacing the classical exponential distribution of the interarrival times which separate consecutive velocity changes (and jumps) with a generalized Mittag-Leffler distribution. Such interarrival times constitute the random times of a fractional alternating Poisson process. By means of renewal theory-based arguments, we obtain the forward and backward transition densities of the motion in series form, and prove their uniform convergence. Specific attention is then given to the case of jumps with constant size, for which we also obtain the mean of the process. We conclude the chapter by investigating the first-passage time of the process through a constant positive boundary, providing its formal distribution and suitable lower bounds.

Chapter 6 is dedicated to a stochastic model for competing risks involving the Mittag-Leffler distribution, inspired by fractional random growth phenomena. We prove the independence between the time to failure and the cause of failure, and investigate some properties of the related hazard rates and ageing notions. We also face the general problem of identifying the underlying distribution of latent failure times when their joint distribution is expressed in terms of copulas and the time transformed exponential model. The special case concerning the Mittag-Leffler distribution is approached by means of numerical treatment. We finally adapt the proposed model to the case of a random number of independent competing risks. This leads to certain mixtures of Mittag-Leffler distributions, whose parameters are estimated through the method of moments for fractional moments.