Particle mixing,
two level systems and gauge theory

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Abstract

In this thesis I will discuss the theory of two level systems and the theory of the oscillating particles in quantum field theory.
In the first chapter I will consider the time evolution of a two level system, a qubit, to show that it has inside a local in time gauge invariant evolution equation. I construct the covariant derivative operator and show that it is related to the free energy. The gauge invariance of the time evolution of the two level system is analogous to the phenomenon of birefringence.I also show that the two level systems present a Berry-like and an Anandan-Aharonov phase. Finally, I discuss entropy environment effects and the distance in projective Hilbert space between two level states to show that the last one is properly related to the Aharonov - Anandan phase.
In the second chapter I review the result obtained in QFT for particle mixing, analyzing the theoretical construction and the oscillation formula in the fermion case.I will emphasize the differences between the quantum mechanics formulas and the QFT formulas.The unitary inequivalence between the flavor and the mass eigenstates is also shown and the structure of the current for charged fields is finally discussed. I found a non - perturbative vacuum structure for the mixing particles that, among the other things, will lead to a non zero contribution to the value of the cosmological constant (chapter 3).
Several links between first and second chapter will arise from this thesis and will shed the light on the fact that it is possible to construct a generic two level quantum field theory, that is an extension of the quantum mechanics bit theory in a quantum field theory framework.

Contents

Introduction

A two level system, namely a qubit, is a system that oscillate between two different configurations i.e an ammonia molecule[1], an electron, a neutrino[2]. In the first chapter the gauge theory paradigm is applied to the time evolution of a qubit and after this we study the geometric phase properties of these systems.
As a matter of fact, it is already known [1] that a Berry-like phase and the related gauge field connection [2] play an important role in quantum computing.Here it is recognized the role of a gauge field showing, by the explicit construction of the covariant derivative, what are the physical links through this gauge structure and thermodynamical operators such as the free energy operator, and showing how it provides an analogy of the qubit system with the birefringence phenomenon. It is found that the time evolution for the qubit shows that such a system can be seen as embedded in a gauge field background that preserve the invariance under local in time gauge transformations.Such a gauge structure also simulates the propagation through a birefringent medium.
I finally compute the static entropy, dynamic entropy and the distance between the qubit states in the Hilbert space using the the Fubini-Study metric.
In the second chapter, particle mixing and oscillation, that is an important topic in modern Particle Physics, is discussed.
The experimental evidences that comes from neutrino experiments [3] demonstrate that each neutrino can be seen effectively as a two or a three level system and that flavor oscillation occurs.
Mixing and oscillation phenomenon happen even in the boson case but here I focus my attention on the fermion case.This phenomenon seems to be the most interesting one beyond the physics of standard model.Neutrino mixing theory is crucial in explaining the puzzle of solar neutrinos [4].
I emphasize that neutrino oscillations ore possible only if neutrinos are massive particle [5].
I also observe, however, that many problems are still not solved, see for example the nature of neutrinos masses, the value of these masses compared to the masses of other leptons, the same origin and generation of the particle mixing and oscillations.
Studying the puzzle of mixing and oscillation from a QFT point of view we come to a unitarily inequivalent representation problem, where, as we will see, the problem appears of the choice of a proper Hilbert space that is dif-

[1]The ammonia molecule is an intresting system that oscillate between a configuration that have the nitrogen atom up and a configuration that have the nitrogen atom down.

[2]This is the system which has inspired the ideas on which this thesis is based.

ferent for mixed and unmixed fields.
Clearly this picture is in contrast to the one proposed by the Quantum Mechanics where the Von Neumann theorem holds and by which only one Hilbert space is admitted when the considered system has a finite number of degrees of freedom. Thus in quantum field theory we define two different Hilbert spaces, one for the flavor fields and one for the mass fields.
I emphasize that, as it will be shown in this thesis, the difference between the mass and flavor fields consists in the fact that the mass part of the hamiltonian of the system is diagonal if written using the first ones and is not diagonal if written using the second ones. It is properly the non diagonal term that causes the mixing.
Many studies have been done for fermions and for bosons. Here we propose the QFT analysis conducted for fermions [6] where one of the most interesting results, that comes out from the unitarily in-equivalence of the Hilbert spaces, is the orthogonality between the flavor and mass vacuum. We calculate the exact oscillation formulae for fermion mixing [7] using the framework of Quantum Field Theory. As we will see, a new additional term and energy dependence of amplitude will come out naturally.
In the third chapter of this thesis the vacuum structure obtained considering the mixing is shown to lead to a non zero contribution to the dark energy [8], [9]. In particular I will show how was the situation in the early universe, and in the universe at present epoch. In the second case I point out separately the contribution due to neutrinos and due to quarks.

Chapter 1

Analysis of two level quantum systems.

1.1 Gauge structure for time evolution of two level systems.

1.1.1 Oscillation of a two level system.

I apply a gauge theory paradigm to the time evolution of a generic qubit, computing the covariant derivative and the gauge potential.
I start the analysis considering the familiar example of a two level system [11] i.e. any system which might be described by the orthonormal basis of two unit (pure state) vectors $|0\rangle$ and $|1\rangle$, $\langle i|j\rangle = \delta_{ij}$, $i, j = 0, 1$. These states are obviously eigenstates of the hamiltonian:

$$H = \omega_1|0\rangle\langle 0| + \omega_2|1\rangle\langle 1|; \tag{1.1}$$

so that:

$$\begin{aligned} H|0\rangle &= \omega_1|0\rangle, \\ H|1\rangle &= \omega_2|1\rangle\,. \end{aligned} \tag{1.2}$$

ω_1 and ω_2 denote the two different values of the quantum number (energy, or charge, or spin, etc.) that characterize respectively the states $|0\rangle$ and $|1\rangle$. We might consider them to be the energie[1] eigenvalues.
Rotating in the plane $\{|0\rangle, |1\rangle\}$, one may then prepare, at some initial time $t_0 = 0$, the superposition of states:

$$|\phi\rangle \;=\; \alpha\,|0\rangle \;+\; \beta\,|1\rangle\,, \tag{1.3}$$

$$|\psi\rangle \;=\; -\beta\,|0\rangle \;+\; \alpha\,|1\rangle\,, \tag{1.4}$$

[1] i.e. the frequencies in natural units $h = 1 = c$

α and β satisfying the relations

$$|\alpha|^2 + |\beta|^2 = 1 \tag{1.5}$$

and

$$\alpha^* \beta - \alpha \beta^* = 0\,. \tag{1.6}$$

Thus in full generality we can set:

$$\begin{aligned} \alpha &= e^{i\gamma_1} \cos\theta \\ \beta &= e^{i\gamma_2} \sin\theta \end{aligned} \tag{1.7}$$

with $\gamma_1 = \gamma_2 + n\pi$, $n = 0, 1, 2...$ [2].
In the preparation process we may have a limited control on the system and sometimes no control [13, 14, 15].
Sometimes, however, is possible to have a good precision in the above mentioned initialization problem; this are the case of nuclear magnetic resonance and electron spin resonant systems [16].
Without loss of generality we start considering

$$\alpha = \beta \;= 1\,, \tag{1.8}$$

or equivalently

$$\gamma_1 = \gamma_2 = 0\,, \tag{1.9}$$

to have the two initial superpositions of states:

$$|\phi\rangle \;=\; \cos\theta|0\rangle + \sin\theta|1\rangle\,, \tag{1.10}$$

$$|\psi\rangle \;=\; -\sin\theta|0\rangle + \cos\theta|1\rangle\,. \tag{1.11}$$

At time t we have:

$$|\phi(t)\rangle \;=\; e^{-iHt}|\phi(0)\rangle = e^{-i\omega_1 t}(\cos\theta|0\rangle + e^{-i(\omega_2-\omega_1)t}\sin\theta|1\rangle)\;, \tag{1.12}$$

$$|\psi(t)\rangle \;=\; e^{-iHt}|\psi(0)\rangle = e^{-i\omega_1 t}(-\sin\theta|0\rangle \;+\; e^{-i(\omega_2-\omega_1)t}\cos\theta|1\rangle)\,, \tag{1.13}$$

Obviously, time evolution preserves the orthonormality of the states $|\phi\rangle$ and $|\psi\rangle$ at any time t.
Inverting the equations (1.12) and (1.13), and using equation (1.1) we obtain the expression for H at any t in function of the state $|\phi(t)\rangle$ and $|\psi(t)\rangle$.

$$H = \omega_{\phi\phi}|\phi(t)\rangle\langle\phi(t)| + \omega_{\psi\psi}|\psi(t)\rangle\langle\psi(t)| + \omega_{\phi\psi}(|\phi(t)\rangle\langle\psi(t)| + |\psi(t)\rangle\langle\phi(t)|)\;, \tag{1.14}$$

[2] We note that $|\phi\rangle$ and $|\psi\rangle$ are not eigenstates of H and that is because $\omega_1 \neq \omega_2$.

$\omega_{\phi\phi}$, $\omega_{\psi\psi}$ and $\omega_{\phi\psi}$ are given by the following time-independent expectation values [3].

$$\omega_{\phi\phi} = \omega_1 \cos^2\theta + \omega_2 \sin^2\theta = \langle\phi(t)|\, i\partial_t\, |\phi(t)\rangle \;, \tag{1.15}$$

$$\omega_{\psi\psi} = \omega_1 \sin^2\theta + \omega_2 \cos^2\theta = \langle\psi(t)|\, i\partial_t\, |\psi(t)\rangle \;, \tag{1.16}$$

$$\omega_{\phi\psi} = \frac{1}{2}(\omega_2 - \omega_1)\, \sin 2\theta = \langle\psi(t)|\, i\partial_t\, |\phi(t)\rangle \;, \tag{1.17}$$

$$\omega_{\phi\psi} = \omega_{\psi\phi} \tag{1.18}$$

We stress that the $\omega_{\phi\psi}$ "mixed term", that is responsible of the "oscillations" between the states $|\phi(t)\rangle$ and $|\psi(t)\rangle$, appears in H because $\omega_2 - \omega_1 \neq 0$ and the "angle of mixing" θ is not equal to zero[4].
In fact we have:

$$H|\phi(t)\rangle = \omega_{\phi\phi}\, |\phi(t)\rangle \,+\, \omega_{\phi\psi}\, |\psi(t)\rangle\,; \tag{1.19}$$

$$H|\psi(t)\rangle = \omega_{\psi\psi}\, |\psi(t)\rangle \,+\, \omega_{\phi\psi}\, |\phi(t)\rangle\,. \tag{1.20}$$

1.1.2 Gauge interpretation of the time evolution of a two level system.

The time evolution of a two level system introduced above can be interpreted as due to a gauge background. I will explain how to achieve to this interpretation by considering the evolution of the doublet:

$$|\zeta(t)\rangle = \begin{pmatrix} |\phi(t)\rangle \\ |\psi(t)\rangle \end{pmatrix}. \tag{1.21}$$

Since

$$H|\phi(t)\rangle = i\,\partial_t\, |\phi(t)\rangle\,, \tag{1.22}$$

$$H|\psi(t)\rangle = i\,\partial_t\, |\psi(t)\rangle\,, \tag{1.23}$$

remembering equations (1.1), (1.12) and (1.13)), we get the evolution equations:

$$i\,\partial_t\, |\zeta(t)\rangle \,=\, \omega_d\, |\zeta(t)\rangle \,+\, \omega_{\phi\psi}\, \sigma_1\, |\zeta(t)\rangle\,; \tag{1.24}$$

[3]Remember that $H|\chi(t)\rangle = i\partial_t|\chi(t)\rangle$ where $|\chi(t)\rangle = |\psi(t)\rangle$, $|\phi(t)\rangle$

[4]This situation appear even in the mixing of fermions, of bosons and in general of particles with different masses [17].

where

$$\omega_d = \begin{pmatrix} \omega_{\phi\phi} & 0 \\ 0 & \omega_{\psi\psi} \end{pmatrix}, \tag{1.25}$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{1.26}$$

If now we consider that

$$\begin{aligned} \delta\omega = \omega_{\psi\psi} - \omega_{\phi\phi} = \\ = (\omega_1 - \omega_2)(\sin^2\theta - \cos^2\theta) = \\ = (\omega_2 - \omega_1)\cos 2\theta\,, \end{aligned} \tag{1.27}$$

we have:

$$\tan 2\theta = \frac{2\omega_{\phi\psi}}{\delta\omega}. \tag{1.28}$$

If we put

$$g = \tan 2\theta, \tag{1.29}$$

we have:

$$\omega_{\phi\psi} = \frac{1}{2} g\delta\omega. \tag{1.30}$$

Substituting in (1.28) we obtain:

$$i\,\partial_t\,|\zeta(t)\rangle \;=\; \omega_d\,|\zeta(t)\rangle \;+\; \frac{1}{2} g A_1\,|\zeta(t)\rangle\,; \tag{1.31}$$

where $A_1 = \frac{1}{2}\delta\omega\sigma_1$.
From this formula we can write the covariant derivative[5]:

$$D_t \;=\; \partial_t \;+\; i\,\omega_{\phi\psi}\,\sigma_1 \;= \partial_t \;+\; i\,g\,A_0^{(1)}\,\sigma_1\,, \tag{1.32}$$

where g plays the role of a coupling constant and

$$A_0^{(1)} = \frac{1}{2}\delta\omega \tag{1.33}$$

plays the role of a non abelian gauge field [6].
Now we can write the equation of motion (1.31) in the form:

$$iD_t\,|\zeta(t)\rangle \;=\; \omega_d\,|\zeta(t)\rangle\,. \tag{1.34}$$

[5]We will also use the notation : $A_0 = A_0^{(1)}\,\sigma_1 \;=\; \frac{1}{2}\,\delta\omega\,\sigma_1$

[6]a similar situation occurs in the different context of neutrino mixing, see ref. [18]

If now we set:

$$|\zeta'(t)\rangle = e^{-ig\,\lambda(t)\,\sigma_1}|\zeta(t)\rangle\,, \tag{1.35}$$

and

$$D'_t \,=\, \partial_t \,+\, i\,g\,(A_0^{(1)}\,\sigma_1 \,+ \partial_t\,\lambda(t)\,\sigma_1\,,) \tag{1.36}$$

we can easily see that:

$$iD'_t\,|\zeta'(t)\rangle \,=\, \omega_d\,|\zeta'(t)\rangle\,, \tag{1.37}$$

By this we can define the operator:

$$U(t) \,\equiv\, e^{-ig\,\lambda(t)\,\sigma_1}. \tag{1.38}$$

So we can write:

$$U(t)\,(iD_t\,|\zeta(t)\rangle) \,=\, iD'_t\,U(t)\,|\zeta(t)\rangle \tag{1.39}$$

and also

$$g\,A_0^{(1)'}\,\sigma_1 \,=\, U(t)\,g\,A_0^{(1)}\,\sigma_1\,U^{-1}(t) \,+\, i\,(\partial_t\,U(t))\,U^{-1}(t)\,. \tag{1.40}$$

The same situation (see Eq. (1.36)) happens for a gauge field transformation. The above result can be expressed by saying that the evolution in time of the vector doublet $|\zeta(t)\rangle$, that is our two level system, can be described using a non abelian gauge field that couple the two single component of the qubit among themselves.This situation occurs because we have to preserve the invariance of the dynamics against local in time gauge transformations (phase fluctuations). Finally we note that since the only non-vanishing component of A_μ is A_0 (which is a constant $(A_0 \,\equiv\, \frac{1}{2}\,\delta\omega\,\sigma_1)$), the field strength $F_{\mu\nu}$ is identically zero.
This is a feature which, for example, occurs in the case where the gauge potential is a pure gauge and the gauge function is not singular.

1.2 Gauge evolution in time of a two level system as a birefringence phenomenon.

1.2.1 Time evolution of the qubit seen as a birefringence phenomenon

The purpose of this paragraph is to show that the time evolution above described can be recognised as a birefringence phenomenon. To do this we observe that the time evolution of the eigenstates $|0\rangle$ and $|1\rangle$ of the hamiltonian is described by:

$$\begin{pmatrix} |0(t)\rangle \\ |1(t)\rangle \end{pmatrix} = \begin{pmatrix} e^{-i\omega t} & 0 \\ 0 & e^{-i\omega t} \end{pmatrix} \begin{pmatrix} |0(0)\rangle \\ |1(0)\rangle \end{pmatrix}, \tag{1.41}$$

in"the vacuum"[7] and:

$$\omega = 2\pi\nu \tag{1.42}$$

So the "propagation speed" of the the wave packet that corresponds to each eigenstate $|0\rangle$ or $|1\rangle$ is given by:

$$v_0 = \lambda\,\nu \tag{1.43}$$

We will show now what happens if, instead of considering the vacuum medium we consider a medium that has a different refringent behavior for each of the states $|0\rangle$ and $|1\rangle$. Suppose now that instead of "the vacuum" the propagation happens in a medium that presents different refraction indexes, for example n_1 for $|0\rangle$ and n_2 for $|1\rangle$. To be more precise, this means that given a length ℓ it will be crossed in a time t_1 for $|0\rangle$ and in a time t_2 for $|1\rangle$ and this times are given by:

$$t_1 = \frac{\ell}{v_1} = \frac{\ell\, n_1}{v_0} = t\, n_1\ ; \tag{1.44}$$

and

$$t_2 = \frac{\ell}{v_2} = \frac{\ell\, n_2}{v_0} = t\, n_2\ . \tag{1.45}$$

Here we have considered the same time t of propagation in the birefringence medium but we have different speeds of propagation considering v_1 for $|0\rangle$ and v_2 for $|1\rangle$.

Now time evolution for $|0\rangle$ is described by the phase factor

$$e^{-i\omega t_1} = e^{-i\omega_1 t} \tag{1.46}$$

[7]"The vacuum" means in absence of any "external field.", even in absence of the gauge field.

and by

$$e^{-i\omega t_2} = e^{-i\omega_2 t} \tag{1.47}$$

for $|1\rangle$, where we have applied the sequent substitutions:

$$\begin{aligned} \omega\, t_1 \;&=\; \omega \frac{\ell}{v_0} n_1 = \\ = 2\,\pi\,\nu\,t\,n_1 = 2\,\pi\,\nu_1\,t \;&=\; \omega_1\,t \end{aligned} \tag{1.48}$$

for $|0\rangle$ and

$$\begin{aligned} \omega\, t_2 \;&=\; \omega \frac{\ell}{v_0} n_2 = \\ = 2\,\pi\,\nu\,t\,n_2 = 2\,\pi\,\nu_2\,t \;&=\; \omega_2\,t \end{aligned} \tag{1.49}$$

for $|1(t)\rangle$.

In the above written formulas we have considered:

$$\lambda_1\,\nu \;=\; v_1; \tag{1.50}$$

$$n_1 \;=\; \frac{v_0}{v_1} \;=\; \frac{\nu_1}{\nu} \tag{1.51}$$

for $|0(t)\rangle$ and

$$\lambda_2\,\nu \;=\; v_2; \tag{1.52}$$

$$n_2 \;=\; \frac{v_0}{v_2} \;=\; \frac{\nu_2}{\nu} \tag{1.53}$$

for $|1(t)\rangle$

We now can see that considering the dynamical evolution of the two states in a birefringent medium is a manner to consider two
different phase instead of one.
So now we are again in the same puzzle exposed in the first chapter, because the time evolution of the doublet $\begin{pmatrix} |0(t)\rangle \\ |1(t)\rangle \end{pmatrix}$ can be now written as:

$$\begin{pmatrix} |0(t)\rangle \\ |1(t)\rangle \end{pmatrix} = \begin{pmatrix} e^{-i\omega_1 t} & 0 \\ 0 & e^{-i\omega_2 t} \end{pmatrix} \begin{pmatrix} |0(0)\rangle \\ |1(0)\rangle \end{pmatrix} \tag{1.54}$$

If now we operate in the same way that we did in chapter 1 and construct again at time $t_0 = 0$ the pure states:

$$|\phi\rangle \;=\; |0\rangle + |1\rangle\,, \tag{1.55}$$

$$|\psi\rangle \;=\; -|0\rangle + |1\rangle\,. \tag{1.56}$$

and let them evolve in time we have:

$$\begin{pmatrix} |\phi(t)\rangle \\ |\psi(t)\rangle \end{pmatrix} = e^{-i\omega_1 t} \begin{pmatrix} \cos\theta & e^{-i(\omega_2-\omega_1)t}\sin\theta \\ -\sin\theta & e^{-i(\omega_2-\omega_1)t}\cos\theta \end{pmatrix} \begin{pmatrix} |0\rangle \\ |1\rangle \end{pmatrix}, \tag{1.57}$$

that is exactly the matrix form that unify the equations (1.12) and (1.13). So starting from there we can evolve all the puzzle that we presented in the above chapter in the same way or, in other words, we have shown that, provided that $\omega_1 \neq \omega_2$, for $\theta \neq \frac{\pi}{4} + \frac{n\pi}{2}$, the effect of time evolution through the refractive medium is equivalent to the effect of the background gauge field

$$\begin{aligned} A_0^{(1)} &= \frac{1}{2}(\omega_2 - \omega_1)\ \cos 2\theta = \\ &= \frac{1}{2}\omega(n_2 - n_1)\ \cos 2\theta. \end{aligned} \tag{1.58}$$

This situation obviously disappear if the propagation happens in the vacuum where

$$n_1 = n_2 = n_0 = 1 \Rightarrow \tag{1.59}$$

$$\Rightarrow \omega_1 = \omega = \omega_2. \tag{1.60}$$

1.3 Free energy interpretation of the gauge background.

In this section we will show how we can interpret the results of previous sections in a "thermodynamical" way. To be more precise we will relate the **free energy** of our two level system to the hamiltonian and our gauge potential.
Let us first re write the equation (1.24)

$$
\begin{aligned}
i\,\partial_t\,|\zeta(t)\rangle \;=\; \omega_d\,|\zeta(t)\rangle \quad &+ \quad \omega_{\phi\psi}\,\sigma_1\,|\zeta(t)\rangle\,; && (1.61)\\
\Rightarrow i\,\partial_t\,|\zeta(t)\rangle \,-\, \omega_{\phi\psi}\,\sigma_1\,|\zeta(t)\rangle \quad &= \quad \omega_d\,|\zeta(t)\rangle\,; && (1.62)\\
(H \;-\; \omega_{\phi\psi}\,\sigma_1)\,|\zeta(t)\rangle \quad &= \quad \omega_d\,|\zeta(t)\rangle && (1.63)
\end{aligned}
$$

As we can see we have written the covariant derivative as $H \,-\, \omega_{\phi\psi}\,\sigma_1$ where again σ_1 is the first Pauli Matrix.

It is possible now to write the operator F:

$$F \;=\; (H \,-\, \omega_{\phi\psi}\,\sigma_1), \tag{1.64}$$

Which can be identified with the free energy if we provided that we define:

$$\omega_{\phi\psi}\,\sigma_1 = gA_0 = TS\,; \tag{1.65}$$

where $S \,=\, A_0$ is the entropy of the system and $T \,=\, g$ is its temperature.
Therefore we see that the free energy (1.64) controls the time evolution of the two level system, namely a qubit, and the gauge field plays the role of the entropy.
It is interesting to see that TS is written in terms of the states $|\phi\rangle$ and $|\psi\rangle$ as.

$$TS = \omega_{\phi\psi}(|\phi(t)\rangle\langle\psi(t)| + |\psi(t)\rangle\langle\phi(t)|)\ . \tag{1.66}$$

1.4 Geometrical invariants.

I now write a geometrical phase that is effectively a Berry like phase and I will compute it.
As done in Appendix A, I consider the time evolution of each one of the states $|\phi\rangle$ and $|\psi\rangle$[8].

$$\begin{aligned} H|\phi(t)\rangle &= i\frac{\partial}{\partial t}|\phi(t)\rangle &(1.67)\\ H|\psi(t)\rangle &= i\frac{\partial}{\partial t}|\psi(t)\rangle &(1.68)\end{aligned}$$

We recall that the dynamic part of the time evolution of is governed by the "energy" that pertains to each one, respectively $\omega_{\phi\phi}$ for $|\phi\rangle$ and $\omega_{\psi\psi}$ for $|\psi\rangle$; considering that the dynamical phase of the time evolution of $|\phi\rangle$ is given by $e^{-}i\int_0^t \omega_{\phi\phi}dt'$ and equally for $|\psi\rangle$ is given by $e^{-i\int_0^t \omega_{\psi\psi}dt'}$.
If we now consider the amount of time that goes from $t = 0$ to $t = T = \frac{2\pi}{\omega_2-\omega_1}$ referring to the equations (1.12), (1.13) we will have:

$$\begin{aligned} |\phi(T)\rangle &= e^{i\varphi}|\phi(0)\rangle\,; &(1.69)\\ |\psi(T)\rangle &= e^{i\varphi}|\psi(0)\rangle, &(1.70)\end{aligned}$$

where

$$\varphi \equiv -\frac{2\pi\omega_1}{\omega_2-\omega_1}\,, \quad (1.71)$$

is the total phase factor that we have in this amount of time.
As expected we have:

$$\begin{aligned} \varphi &\neq -i\int_0^T \omega_{\phi\phi}dt' &(1.72)\\ \varphi &\neq -i\int_0^T \omega_{\psi\psi}dt' &(1.73)\end{aligned}$$

So we can write:

$$\begin{aligned} \varphi &= \beta_{\phi\phi} - i\int_0^T \omega_{\phi\phi}dt' &(1.74)\\ \varphi &= \beta_{\psi\psi} - i\int_0^T \omega_{\psi\psi}dt' &(1.75)\end{aligned}$$

[8]remember that we are in natural unit, $\hbar = c = 1$

Here $\beta_{\phi\phi}$ and $\beta_{\psi\psi}$ are pure geometrical phases, i.e. Berry like phases, and are originated exquisitely by the mixing term $\omega_{\phi\psi}(|\phi(t)\rangle\langle\psi(t)| + |\psi(t)\rangle\langle\phi(t)|)$ of the hamiltonian.
We can obtain the two geometrical phases inverting the equations (1.74) and (1.75) obtaining:

$$\beta_{\phi\phi} \;=\; \varphi + i\int_0^T \omega_{\phi\phi} dt'\,; \tag{1.76}$$

$$\beta_{\psi\psi} \;=\; \varphi + i\int_0^T \omega_{\psi\psi} dt'\,. \tag{1.77}$$

Recalling the equations (1.12) and (1.13) we have:

$$\beta_{\phi\phi} \;=\; \varphi + i\int_0^T \langle\phi|\partial_t|\phi\rangle dt'\,; \tag{1.78}$$

$$\beta_{\psi\psi} \;=\; \varphi + i\int_0^T \langle\psi|\partial_t|\psi\rangle dt'\,. \tag{1.79}$$

The above formulas for $\beta_{\phi\phi}$ and $\beta_{\psi\psi}$ are formally identical to (A.11) for the canonical Berry phase, as shown in Appendix A. Finally, by performing

integrals in equations (1.78) and (1.79), we have:

$$\beta_{\phi\phi} \;=\; 2\pi \sin^2\theta\,; \tag{1.80}$$

$$\beta_{\psi\psi} \;=\; 2\pi \cos^2\theta\,; \tag{1.81}$$

i.e.

$$\beta_{\phi\phi} + \beta_{\psi\psi} \;=\; 1. \tag{1.82}$$

These geometrical phases play an important role in the oscillation phenomenon of the two level systems that we analyzed. We can adopt each one of them as a counter of the number of oscillation.
In fact if we evaluate them for a time $t = nT$ instead of $t = T$ we obtain:

$$\beta_{\phi\phi} \;=\; 2n\pi \sin^2\theta\,; \tag{1.83}$$

$$\beta_{\psi\psi} \;=\; 2n\pi \cos^2\theta\,. \tag{1.84}$$

So, by measuring one of them, for example $\beta_{\phi\phi}$ we can get the number n of oscillation occurred.

Another important geometrical quantity is the Anandan - Aharonov invariant.
For a more detailed explanation of this invariant we refer to appendix A and

to reference [12], here we limit our attention to the calculation of it for our two level system, namely a qubit, case.
Referring to the equation (A.14) our $\Delta E(t)^2$ is given by:

$$\Delta E(t)^2 = \Delta\,\omega^2 = \Delta\,\omega^2_{\phi\phi} = \Delta\,\omega^2_{\psi\psi}\,;\Rightarrow \quad (1.85)$$

$$\Rightarrow \Delta E(t)^2 = \langle\xi(t)|H^2|\xi(t)\rangle - \langle\xi(t)|H|\xi(t)\rangle^2 = \omega^2_{\phi\psi}\,. \quad (1.86)$$

Where $\xi = \phi,\, \psi$.
So our geometrical invariant is given by:

$$s = 2\int \omega_{\phi\psi} dt. \quad (1.87)$$

Considering [39], for $t = nT$ one obtains:

$$s = 2n\pi \sin(2\theta) \quad (1.88)$$

Immediately we can see that:

$$\int \langle\zeta(t)|TS\sigma_1|\zeta(t)\rangle\; dt = \int \langle\zeta(t)|g\,A_0^{(1)}|\zeta(t)\rangle\; dt = 2\int \omega_{\phi\psi}\, dt = s, \quad (1.89)$$

That give us the correlation between s, TS, and $g\,A_0^{(1)}$.
It is interesting to note that the relation between TS and the variance of the energy $\Delta\,E = \omega_{\phi\psi}$ is through the non-diagonal elements of H, namely it is proportional to the energy gap, $\omega_2 - \omega_1$, between the two levels (cf. Eq. (1.88) and (1.17)).
Finally, it is instructive to analyze the invariant s in terms of the distance between states in the Hilbert space.
We generically denote by $|\xi(t)\rangle$ either $|\phi(t)\rangle$ or $|\psi(t)\rangle$.
Their evolution is governed by the Schrödinger equation

$$i\;\partial_t|\xi(t)\rangle = \; H|\xi(t)\rangle\,, \quad (1.90)$$

Referring to appendix A, equation (A.19)[9], to references([11]) and ([12]) we have:

$$\begin{aligned} ds^2 &= 2g_{\mu\nu}dZ^\mu d\overline{Z}^\nu = \\ &= 4(1 - |\langle\xi(t)|\xi(t+dt)\rangle|^2), \end{aligned} \quad (1.91)$$

where $g_{\mu\nu}$ is the Fubini - Study metric.
Doing the same calculation that we did to obtain (A.18) we have:

$$|\langle\xi(t)|\xi(t+dt)\rangle|^2 = 1 - dt^2\Delta\omega^2_{\xi\xi}\; + \; O(dt^3)\,, \quad (1.92)$$

[9] Here we adopt $|\xi(t)'\rangle = |\xi(t+dt)\rangle = e^{-iE_\xi(t+dt)}|\xi(t)\rangle$ or in other words we are considering $\Pi = e^{-iH(t+dt)}$ as our projection operator.

so we finally:

$$\begin{aligned} ds^2 = 2g_{\mu\nu}dZ^\mu d\overline{Z}^\nu = \\ = 4(dt^2\Delta\omega_{\xi\xi}^2 + O(dt^3)) \\ = 4(dt^2\omega_{\phi\psi}^2 + O(dt^3)), \end{aligned} \tag{1.93}$$

Thus we showed the connection between $\omega_{\phi\psi}$ that is responsible of the oscillation and of the mixing and the Fubini - Study metric using the Aharonov - Anandan invariant.

Chapter 2

Particle fermion mixing in QFT.

In this second part of the thesis I summarize the results obtained by the analysis of the puzzle of fermion mixing. Among the results that we will encounter, of particular importance is the formal difference between the mixing formula in QFT and the ones obtained by the quantum mechanics theory.
Another important result is the orthogonality between the "flavor" vacuum and the "mass" vacuum.
We will encounter other important features of the fermion mixing as a contribution to the dark energy that can be obtained from QFT fermion mixing.
For simplicity we will refer to neutrinos for which we have a large amount of experimental data, see for example ref. [3].

2.1 The vacuum structure

I focus the attention on the theoretical structure of fermion mixing treated in quantum field theory; in particular I am going to treat neutrinos as fields and we will surprisingly find that the oscillation formulas of neutrinos here have in addition a Bogoliubov term.
I first start checking that the Fock space of the flavor fields is unitary inequivalent to the Fock space of the mass field in the infinite volume limit.
The flavor states exhibit the structure of SU(2) coherent states, as we will see below.
We will get the new formulas for neutrino oscillation [25, 26] in QFT and then we will study the current structure for mixed fields [27].

The mixing relations written for the fields([28]):

$$\nu_e(x) = \nu_1(x)\cos\theta + \nu_2(x)\sin\theta\,; \tag{2.1}$$
$$\nu_\mu(x) = -\nu_1(x)\sin\theta + \nu_2(x)\cos\theta\,. \tag{2.2}$$

Where $\nu_e(x)\,,\nu_\mu(x)$ are the flavor neutrino fields and $\nu_1(x)\,,\nu_2(x)$ are the neutrino fields with definite masses $m_1\,,m_2$.
In terms of creator and of annihilator operators, as free Dirac fields, $\nu_1(x)\,,\nu_2(x)$ can be written as:

$$\nu_1(x) = \frac{1}{\sqrt{V}}\sum_{\mathbf{k},r}\left[u^r_{\mathbf{k},1}\alpha^r_{\mathbf{k},1}(t) + v^r_{\text{-}\mathbf{k},1}\beta^{r\dagger}_{\text{-}\mathbf{k},1}(t)\right]\,; \tag{2.3}$$
$$\nu_2(x) = \frac{1}{\sqrt{V}}\sum_{\mathbf{k},r}\left[u^r_{\mathbf{k},2}\alpha^r_{\mathbf{k},2}(t) + v^r_{\text{-}\mathbf{k},2}\beta^{r\dagger}_{\text{-}\mathbf{k},2}(t)\right]\,; \tag{2.4}$$

where we have:

$$u^1_{\mathbf{k},i} = \left(\frac{\omega_{k,i}+m_i}{2\omega_{k,i}}\right)^{\frac{1}{2}}\begin{pmatrix} 1 \\ 0 \\ \frac{k_3}{\omega_{k,i}+m_i} \\ \frac{k_1+ik_2}{\omega_{k,i}+m_i}\end{pmatrix}\,; \tag{2.5}$$

$$u^2_{\mathbf{k},i} = \left(\frac{\omega_{k,i}+m_i}{2\omega_{k,i}}\right)^{\frac{1}{2}}\begin{pmatrix} 0 \\ 1 \\ \frac{k_1-ik_2}{\omega_{k,i}+m_i} \\ \frac{-k_3}{\omega_{k,i}+m_i}\end{pmatrix}\,; \tag{2.6}$$

$$v^1_{-\mathbf{k},i} = \left(\frac{\omega_{k,i}+m_i}{2\omega_{k,i}}\right)^{\frac{1}{2}}\begin{pmatrix} \frac{-k_3}{\omega_{k,i}+m_i} \\ \frac{-k_1-ik_2}{\omega_{k,i}+m_i} \\ 1 \\ 0\end{pmatrix}\,; \tag{2.7}$$

$$v^2_{-\mathbf{k},i} = \left(\frac{\omega_{k,i}+m_i}{2\omega_{k,i}}\right)^{\frac{1}{2}}\begin{pmatrix} \frac{-k_1+ik_2}{\omega_{k,i}+m_i} \\ \frac{k_3}{\omega_{k,i}+m_i} \\ 0 \\ 1\end{pmatrix}\,. \tag{2.8}$$

Moreover we have:

$$\alpha^r_{\mathbf{k},i}(t) = \alpha^r_{\mathbf{k},i}\,e^{-i\omega_{k,i}t}\,, \tag{2.9}$$
$$\beta^{r\dagger}_{\mathbf{k},i}(t) = \beta^{r\dagger}_{\mathbf{k},i}\,e^{i\omega_{k,i}t}\,, \tag{2.10}$$
$$\omega_{k,i} = \sqrt{\mathbf{k}^2+m_i^2}. \tag{2.11}$$

with $i = 1, 2$.

As usual, the operator $\alpha^r_{\mathbf{k},i}$ and $\beta^r_{\mathbf{k},i}$ are the mass annihilation operators that applied to the mass vacuum give:

$$\alpha^r_{\mathbf{k},i}|0\rangle_{12} = \beta^r_{\mathbf{k},i}|0\rangle_{12} = 0\,; \tag{2.12}$$

where

$$|0\rangle_{1,2} \equiv |0\rangle_1 \otimes |0\rangle_2. \tag{2.13}$$

We write now the mass fields anti-commutation relation:

$$\left\{\nu_i^\alpha(x), \nu_j^{\beta\dagger}(y)\right\}_{t=t'} = \delta^3(\mathbf{x}-\mathbf{y})\delta_{\alpha\beta}\delta_{ij}, \qquad \alpha,\beta = 1,...4, \tag{2.14}$$

And also the annihilator/creator anti-commutation relation:

$$\left\{\alpha^r_{\mathbf{k},i}, \alpha^{s\dagger}_{\mathbf{q},j}\right\} \;=\; \delta_{\mathbf{kq}}\delta_{rs}\delta_{ij}\,; \tag{2.15}$$

$$\left\{\beta^r_{\mathbf{k},i}, \beta^{s\dagger}_{\mathbf{q},j}\right\} \;=\; \delta_{\mathbf{kq}}\delta_{rs}\delta_{ij}, \qquad i,j = 1,2. \tag{2.16}$$

All other anti- commutation relation are equal to zero.
We also write the completeness and orthonormality relations:

$$u^{r\dagger}_{\mathbf{k},i}u^s_{\mathbf{k},i} = v^{r\dagger}_{\mathbf{k},i}v^s_{\mathbf{k},i} \;=\; \delta_{rs}, \tag{2.17}$$

$$u^{r\dagger}_{\mathbf{k},i}v^s_{-\mathbf{k},i} = v^{r\dagger}_{-\mathbf{k},i}u^s_{\mathbf{k},i} \;=\; 0, \tag{2.18}$$

$$\sum_r (u^r_{\mathbf{k},i}u^{r\dagger}_{\mathbf{k},i} + v^r_{-\mathbf{k},i}v^{r\dagger}_{-\mathbf{k},i}) \;=\; 1. \tag{2.19}$$

The mass fields $\nu_1(x)$ and $\nu_2(x)$ are related to the free hamiltonian whose mass term [27] is:

$$H \;=\; m_1\,\nu_1^\dagger\nu_1 + m_2\,\nu_2^\dagger\nu_2 \tag{2.20}$$

Instead, the flavor fields are related to the hamiltonian [1]

$$H \;=\; m_{ee}\,\nu_e^\dagger\nu_e + m_{\mu\mu}\,\nu_\mu^\dagger\nu_\mu + m_{e\mu}\left(\nu_e^\dagger\nu_\mu + \nu_\mu^\dagger\nu_e\right) \tag{2.21}$$

where [2]

$$m_{ee} \;=\; m_1\cos^2\theta + m_2\sin^2\theta, \tag{2.22}$$

$$m_{\mu\mu} \;=\; m_1\sin^2\theta + m_2\cos^2\theta, \tag{2.23}$$

$$m_{e\mu} \;=\; (m_2 - m_1)\sin\theta\cos\theta. \tag{2.24}$$

$$m_{e\mu} \;=\; m_{\mu e} \tag{2.25}$$

[1] We refer only to the mass term here also.
[2] Note the analogy with the eqs. (1.15),(1.16),(1.17),(1.18)

In QFT the basic dynamics, i.e. the Lagrangian and the resulting field equations, is given in terms of Heisenberg (or interacting) fields but the physical observables are expressed in terms of asymptotic free (in or out) fields.
The free fields, say for definitiveness the infields, are obtained by the weak limit of the Heisenberg fields for time $t \to -\infty$.
In QFT, there exist infinitely unitary non-equivalent representations of the anti-commutation relations of the fields[3]; considering that for any one of this representation the asymptotic free fields are different, we have a different set of observables for any representation.
Thus before has a physical important meaning: if to any unitary non equivalent representation corresponds the same dynamics [4] but a different set of observables, that means that to any representation corresponds a different physical phase of the system.
We have written above we stress that is a very rude and naive approximation to assume that interacting fields and free fields share the same vacuum state and the same Fock space representation. Taking into account what we said

above, we will investigate the structure of the Fock spaces $\mathcal{H}_{1,2}$ and $\mathcal{H}_{e,\mu}$ relative to $\nu_1(x), \nu_2(x)$ and $\nu_e(x), \nu_\mu(x)$, respectively.
We show that $\mathcal{H}_{1,2}$ and $\mathcal{H}_{e,\mu}$ become orthogonal in the infinite volume limit, in particular the free field vacuum and the interacting field vacuum will became orthogonal.
We will achieve this result starting first by constructing the Fock spaces $\mathcal{H}_{1,2}$ and $\mathcal{H}_{e,\mu}$ in the finite volume limit, then we will extend our volume V to the infinite limit.
The relations (2.1) and (2.2) can be seen as originated by a generator $G_\theta(t)$ as we can see below:

$$\nu_e^\alpha(x) = G_\theta^{-1}(t)\nu_1^\alpha(t)G_\theta(t), \tag{2.26}$$
$$\nu_\mu^\alpha(x) = G_\theta^{-1}(t)\nu_2^\alpha(t)G_\theta(t). \tag{2.27}$$

Where α is the helicity of the field. To get the generator $G_\theta(t)$ we start from (2.1) and (2.2) and consider that:

[3] That means that we can represent the same Lie group of fields by different non equivalent representations.

[4] Remember that Lagrangian and field equation depends only to Heisenberg fields that are not representation dependent.

$$\begin{aligned}
\nu_e^\alpha(x)|_{\theta=0} &= \nu_1^\alpha(x) && (2.28)\\
G_\theta^{-1}(t)\nu_1(t)G_\theta(t)|_{\theta=0} &= \nu_1^\alpha(x) && (2.29)\\
G_\theta^{-1}(t)|_{\theta=0} &= I; && (2.30)\\
G_\theta(t)|_{\theta=0} &= I && (2.31)
\end{aligned}$$

From equations (2.30),(2.31) we can easily see that $G_\theta(t)$ must be such that:

$$G_\theta(t) = e^{\theta S^\alpha(x)} \Rightarrow G_\theta^{-1}(t) = e^{-\theta S^\alpha(x)}; \tag{2.32}$$

where $\delta\theta \rightarrow 0$ when $\theta \rightarrow 0$. Still following the equations (2.30),(2.31) we have:

$$d\nu_e^\alpha(x)/d\theta = \nu_2^\alpha(x)\,, \tag{2.33}$$

and

$$\left(\frac{dG_\theta^{-1}(t)}{d\theta}\nu_1^\alpha(x)G_\theta(t) + G_\theta^{-1}(t)\nu_1^\alpha(x)\frac{dG_\theta(t)}{d\theta}\right)|_{\theta=0} = \nu_2^\alpha(x) \tag{2.34}$$

$$-S^\alpha(x)e^{-\theta S^\alpha(x)}\nu_1^\alpha(x)e^{\theta S^\alpha(x)} + e^{-\theta S^\alpha(x)}\nu_1^\alpha(x)S^\alpha(x)e^{\theta S^\alpha(x)}|_{\theta=0} = \nu_2^\alpha(x) \tag{2.35}$$

$$-S^\alpha(x)\nu_1^\alpha(x) + \nu_1^\alpha(x)S^\alpha(x) = \nu_2^\alpha(x) \tag{2.36}$$

$$[\nu_1^\alpha(x)\,,\, S^\alpha(x)] = \nu_2^\alpha(x). \tag{2.37}$$

By this we have found that

$$S^\alpha(x) = \hat{a}\nu_2^\alpha(x) + terms\,that\,commute\,with\;\nu_1^\alpha(x). \tag{2.38}$$

To calculate the value of $\hat{a}$ we evaluate the commutator$[\nu_1(x), S^\alpha(x)]$.

$$\begin{aligned}
[\nu_1(x), S^\alpha(x)] &= [\nu_1(x)\,,\, \hat{a}\nu_2^\alpha(x)] && (2.39)\\
[\nu_1^\alpha(x)\,,\, \hat{a}\nu_2^\alpha(x)] &= \nu_2^\alpha(x) && (2.40)\\
\nu_1^\alpha(x)\hat{a}\nu_2^\alpha(x) - \hat{a}\nu_1^\alpha(x)\nu_2^\alpha(x) - \hat{a}\{\nu_2^\alpha(x)\,,\,\nu_1^\alpha(x)\} &= \nu_2^\alpha(x) && (2.41)
\end{aligned}$$

Remembering that $\{\nu_2^\alpha(x)\,,\,\nu_1^\alpha(x)\} = 0$ we obtain:

$$\begin{aligned}
(\nu_1^\alpha(x)\hat{a} - \hat{a}\nu_1^\alpha(x))\nu_2^\alpha(x) = \nu_2^\alpha(x) &\Rightarrow && (2.42)\\
[\nu_1^\alpha(x)\,,\,\hat{a}] = I &\Rightarrow && (2.43)\\
\hat{a} = \nu_1^{\alpha\dagger}(x). & && (2.44)
\end{aligned}$$

We thus result we obtain:

$$S^\alpha(x) = \nu_1^{\alpha\dagger}\nu_2^\alpha + terms\, that\, commute\, with\; \nu_1^\alpha(x). \tag{2.45}$$

We now consider the relation

$$\frac{d\nu_\mu^\alpha(x)}{d\theta}|_{\theta=0} = \nu_1^\alpha(x)\cdots \tag{2.46}$$
$$\cdots - S^\alpha(x)\nu_2^\alpha(x) + \nu_2^\alpha(x)S^\alpha(x) = -\nu_1^\alpha(x) \Rightarrow \tag{2.47}$$
$$[\nu_2^\alpha(x)\,,\, S^\alpha(x)] = -\nu_1^\alpha(x) \tag{2.48}$$

Equation (2.48) says to us that:

$$S^\alpha(x) = \nu_1^{\alpha\dagger}(x)\nu_2^\alpha(x) + \hat{b}\nu_1^\alpha(x) + pieces\, that\, commute\, with\; \nu_1^\alpha(x)\, and\, with\, \nu_2^\alpha(x). \tag{2.49}$$

Substituting now $S^\alpha(x)$ into (2.48) and doing the same calculations that we did to obtain $\hat{a}$ we obtain:

$$\hat{b} = -\nu_2^{\alpha\dagger}(x) \tag{2.50}$$
$$\begin{aligned} S^\alpha(x) &= \nu_1^{\alpha\dagger}(x)\nu_2^\alpha(x) - \nu_2^{\alpha\dagger}(x)\nu_1^\alpha(x) \\ &+ pieces\, that\, commute\, with\; \nu_1^\alpha(x)\, and\, \nu_2^\alpha(x). \end{aligned} \tag{2.51}$$

Naturally we can exclude the terms that commute with ν_1^α and ν_2^α to attain finally[5]:

$$S^\alpha(x) = \nu_1^{\alpha\dagger}(x)\nu_2^\alpha(x) - \nu_2^{\alpha\dagger}(x)\nu_1^\alpha(x) \tag{2.52}$$

Summing on α and integrating into all the volume V we have:

$$S(t) = \int_V d^3x\, \nu_1^\dagger(x)\nu_2(x) - \nu_2^\dagger(x)\nu_1(x). \tag{2.53}$$

From (2.53) we have:

$$G_\theta(t) = e^{\theta \int_V d^3x\,(\nu_1^\dagger(x)\nu_2(x) - \nu_2^\dagger(x)\nu_1(x)}). \tag{2.54}$$

Note that $G_\theta(t)$ is a unitary operator at finite volume V.

[5] We recall that for our interaction the pieces that commute with ν_1^α and ν_2^α do not contribute to it.

By introducing the operators:

$$S_+(t) \;=\; \int_V d^3x\, \nu_1^\dagger(x)\nu_2(x); \tag{2.55}$$

$$S_-(t) \;=\; \int_V d^3x\, \nu_2^\dagger(x)\nu_1(x) = (S_+(t))^\dagger. \tag{2.56}$$

we can rewrite (2.54) as follows:

$$G_\theta(t) = e^{\theta(S_+(t)-S_-(t))},. \tag{2.57}$$

The operator of Casimir [6], that here we call S_0 is given by:

$$S_0 = \int_V d^3x\,(\nu_1^\dagger(x)\nu_1(x) + \nu_2^\dagger(x)\nu_2(x)), \tag{2.58}$$

and together with:

$$S_3 = \int_V d^3x\,(\nu_1^\dagger(x)\nu_1(x) - \nu_2^\dagger(x)\nu_2(x)), \tag{2.59}$$

close the $su(2)$ algebra, in fact we have:

$$[S_+(t)\,,\,S_-(t)] \;=\; 2S_3\,; \tag{2.60}$$

$$[S_3\,,\,S_\pm(t)] \;=\; \pm S_\pm(t)\,; \tag{2.61}$$

$$[S_0\,,\,S_3] \;=\; [S_0\,,\,S_\pm(t)] = 0\,. \tag{2.62}$$

Thus we can see that $S_\theta(t) \in SU(2)$ or equivalently $G_\theta(t) \in su(2)$.

Expanding the above mentioned operators respect to the momentum we have:

$$\begin{aligned} S_+(t) \;\equiv\; & \sum_{\mathbf{k}} S_+^{\mathbf{k}}(t) = \sum_{\mathbf{k}}\sum_{r,s}\Big(u^{r\dagger}_{\mathbf{k},1}\, u^s_{\mathbf{k},2}\, \alpha^{r\dagger}_{\mathbf{k},1}(t)\, \alpha^s_{\mathbf{k},2}(t) \;+\; v^{r\dagger}_{-\mathbf{k},1}\, u^s_{\mathbf{k},2}\, \beta^r_{-\mathbf{k},1}(t)\, \alpha^s_{\mathbf{k},2}(t) + \\ & + u^{r\dagger}_{\mathbf{k},1}\, v^s_{-\mathbf{k},2}\, \alpha^{r\dagger}_{\mathbf{k},1}(t)\, \beta^{s\dagger}_{-\mathbf{k},2}(t) \;+\; v^{r\dagger}_{-\mathbf{k},1}\, v^s_{-\mathbf{k},2}\, \beta^r_{-\mathbf{k},1}(t)\, \beta^{s\dagger}_{-\mathbf{k},2}(t)\Big)\,, \end{aligned} \tag{2.63}$$

$$\begin{aligned} S_-(t) \;\equiv\; & \sum_{\mathbf{k}} S_-^{\mathbf{k}}(t) = \sum_{\mathbf{k}}\sum_{r,s}\Big(u^{r\dagger}_{\mathbf{k},2}\, u^s_{\mathbf{k},1}\, \alpha^{r\dagger}_{\mathbf{k},2}(t)\, \alpha^s_{\mathbf{k},1}(t) \;+\; v^{r\dagger}_{-\mathbf{k},2}\, u^s_{\mathbf{k},1}\, \beta^r_{-\mathbf{k},2}(t)\, \alpha^s_{\mathbf{k},1}(t) + \\ & + u^{r\dagger}_{\mathbf{k},2}\, v^s_{-\mathbf{k},1}\, \alpha^{r\dagger}_{\mathbf{k},2}(t)\, \beta^{s\dagger}_{-\mathbf{k},1}(t) \;+ v^{r\dagger}_{-\mathbf{k},2}\, v^s_{-\mathbf{k},1}\, \beta^r_{-\mathbf{k},2}(t)\, \beta^{s\dagger}_{-\mathbf{k},1}(t)\Big)\,, \end{aligned} \tag{2.64}$$

[6] As we will see the Casimir is proportional to the total charge.

$$S_3 \equiv \sum_{\mathbf{k}} S_3^{\mathbf{k}} = \frac{1}{2}\sum_{\mathbf{k},r} \left(\alpha_{\mathbf{k},1}^{r\dagger}\alpha_{\mathbf{k},1}^{r} - \beta_{-\mathbf{k},1}^{r\dagger}\beta_{-\mathbf{k},1}^{r} - \alpha_{\mathbf{k},2}^{r\dagger}\alpha_{\mathbf{k},2}^{r} + \beta_{-\mathbf{k},2}^{r\dagger}\beta_{-\mathbf{k},2}^{r} \right), \quad (2.65)$$

$$S_0 \equiv \sum_{\mathbf{k}} S_0^{\mathbf{k}} = \frac{1}{2}\sum_{\mathbf{k},r} \left(\alpha_{\mathbf{k},1}^{r\dagger}\alpha_{\mathbf{k},1}^{r} - \beta_{-\mathbf{k},1}^{r\dagger}\beta_{-\mathbf{k},1}^{r} + \alpha_{\mathbf{k},2}^{r\dagger}\alpha_{\mathbf{k},2}^{r} - \beta_{-\mathbf{k},2}^{r\dagger}\beta_{-\mathbf{k},2}^{r} \right). \quad (2.66)$$

We observe that the operatorial structure of Eqs.(2.63) and (2.64) is the one of the rotation generator and of the Bogoliubov generator. We can easily see that:

$$\begin{aligned}
\left[S_+^{\mathbf{k}}(t), S_-^{\mathbf{k}}(t)\right] &= 2S_3^{\mathbf{k}}, && (2.67)\\
\left[S_3^{\mathbf{k}}, S_\pm^{\mathbf{k}}(t)\right] &= \pm S_\pm^{\mathbf{k}}(t), && (2.68)\\
\left[S_0^{\mathbf{k}}, S_3^{\mathbf{k}}\right] &= \left[S_0^{\mathbf{k}}, S_\pm^{\mathbf{k}}\right] = 0, && (2.69)\\
\left[S_\pm^{\mathbf{k}}(t), S_\pm^{\mathbf{p}}(t)\right] = \left[S_3^{\mathbf{k}}, S_\pm^{\mathbf{p}}(t)\right] &= \left[S_3^{\mathbf{k}}, S_3^{\mathbf{p}}\right] = 0, \qquad \mathbf{k} \neq \mathbf{p}. && (2.70)
\end{aligned}$$

So the $su(2)$ algebra structure holds for any $\mathbf{k}$.

$$SU(2) = \bigotimes_{\mathbf{k}} SU(2)_{\mathbf{k}}\,. \quad (2.71)$$

Now we will establish the relation between the flavor Hilbert space $\mathcal{H}_{e,\mu}$ and the mass Hilbert space $\mathcal{H}_{1,2}$ evaluating the generic element ${}_{1,2}\langle a|\nu_1^\alpha(x)|b\rangle_{1,2}$[7]. Inverting the equation (2.26) we can write:

$${}_{1,2}\langle a|\nu_1^\alpha(x)|b\rangle_{1,2} = {}_{1,2}\langle a|G_\theta(t)\nu_e^\alpha(x)G_\theta^{-1}(t)|b\rangle_{1,2} \quad (2.72)$$

Recalling that $\nu_e^\alpha(x)$ is an operator of the Hilbert space $\mathcal{H}_{e,\mu}$, the equation (2.72) says to us that $G_\theta^{-1}(t)|b\rangle_{1,2} \in \mathcal{H}_{e,\mu}$, i.e. $G_\theta(t)$ maps $\mathcal{H}_{1,2}$ into $\mathcal{H}_{e,\mu}$. In finite volume V we have:

$$|0(t)\rangle_{e,\mu} = G_\theta^{-1}(t)\; |0\rangle_{1,2}\,. \quad (2.73)$$

We can write the state of one particle in the Hilbert space $\mathcal{H}_{e\mu}$:

$$\begin{aligned}
\alpha_{\mathbf{k},\nu_e}^r(t)\; |0(t)\rangle_{e,\mu} &= G_\theta^{-1}(t)\; \alpha_{\mathbf{k},1}^r(t)\; |0\rangle_{1,2} = 0,\\
\alpha_{\mathbf{k},\nu_\mu}^r(t)\; |0(t)\rangle_{e,\mu} &= G_\theta^{-1}(t)\; \alpha_{\mathbf{k},2}^r(t)\; |0\rangle_{1,2} = 0,\\
\beta_{\mathbf{k},\nu_e}^r(t)\; |0(t)\rangle_{e,\mu} &= G_\theta^{-1}(t)\; \beta_{\mathbf{k},1}^r(t)\; |0\rangle_{1,2} = 0, \qquad (2.74)\\
\beta_{\mathbf{k},\nu_\mu}^r(t)\; |0(t)\rangle_{e,\mu} &= G_\theta^{-1}(t)\; \beta_{\mathbf{k},2}^r(t)\; |0\rangle_{1,2} = 0,
\end{aligned}$$

[7] we can do the same thing if we calculate ${}_{1,2}\langle a|\nu_2^\alpha(x)|b\rangle_{1,2}$

then we have:

$$\begin{aligned}
\alpha^r_{\mathbf{k},\nu_e}(t) &\equiv G^{-1}_\theta(t)\, \alpha^r_{\mathbf{k},1}(t)\, G_\theta(t), \\
\alpha^r_{\mathbf{k},\nu_\mu}(t) &\equiv G^{-1}_\theta(t)\, \alpha^r_{\mathbf{k},2}(t)\, G_\theta(t), \\
\beta^r_{\mathbf{k},\nu_e}(t) &\equiv G^{-1}_\theta(t)\, \beta^r_{\mathbf{k},1}(t)\, G_\theta(t), \\
\beta^r_{\mathbf{k},\nu_\mu}(t) &\equiv G^{-1}_\theta(t)\, \beta^r_{\mathbf{k},2}(t)\, G_\theta(t).
\end{aligned} \tag{2.75}$$

Taylor expanding $G_\theta(t)$ and $G^{-1}_\theta(t)$ and substituting into (2.75) we obtain the flavour annihilation operators:

$$\begin{aligned}
\alpha^r_{\mathbf{k},\nu_e}(t) &= \cos\theta\, \alpha^r_{\mathbf{k},1}(t) + \sin\theta \sum_s \left[u^{r\dagger}_{\mathbf{k},1} u^s_{\mathbf{k},2}\, \alpha^s_{\mathbf{k},2}(t) + u^{r\dagger}_{\mathbf{k},1} v^s_{-\mathbf{k},2}\, \beta^{s\dagger}_{-\mathbf{k},2}(t) \right], \\
\alpha^r_{\mathbf{k},\nu_\mu}(t) &= \cos\theta\, \alpha^r_{\mathbf{k},2}(t) - \sin\theta \sum_s \left[u^{r\dagger}_{\mathbf{k},2} u^s_{\mathbf{k},1}\, \alpha^s_{\mathbf{k},1}(t) + u^{r\dagger}_{\mathbf{k},2} v^s_{-\mathbf{k},1}\, \beta^{s\dagger}_{-\mathbf{k},1}(t) \right], \\
\beta^r_{-\mathbf{k},\nu_e}(t) &= \cos\theta\, \beta^r_{-\mathbf{k},1}(t) + \sin\theta \sum_s \left[v^{s\dagger}_{-\mathbf{k},2} v^r_{-\mathbf{k},1}\, \beta^s_{-\mathbf{k},2}(t) + u^{s\dagger}_{\mathbf{k},2} v^r_{-\mathbf{k},1}\, \alpha^{s\dagger}_{\mathbf{k},2}(t) \right], \\
\beta^r_{-\mathbf{k},\nu_\mu}(t) &= \cos\theta\, \beta^r_{-\mathbf{k},2}(t) - \sin\theta \sum_s \left[v^{s\dagger}_{-\mathbf{k},1} v^r_{-\mathbf{k},2}\, \beta^s_{-\mathbf{k},1}(t) + u^{s\dagger}_{\mathbf{k},1} v^r_{-\mathbf{k},2}\, \alpha^{s\dagger}_{\mathbf{k},1}(t) \right].
\end{aligned} \tag{2.76}$$

Without loosing generality, we can choose the rest frame where:

$$\mathbf{k} = (0, 0, |\mathbf{k}|); \tag{2.77}$$

obtaining:

$$\begin{aligned}
\alpha^r_{\mathbf{k},\nu_e}(t) &= \cos\theta\, \alpha^r_{\mathbf{k},1}(t) + \sin\theta \left(|U_\mathbf{k}|\, \alpha^r_{\mathbf{k},2}(t) + \epsilon^r\, |V_\mathbf{k}|\, \beta^{r\dagger}_{-\mathbf{k},2}(t) \right) \\
\alpha^r_{\mathbf{k},\nu_\mu}(t) &= \cos\theta\, \alpha^r_{\mathbf{k},2}(t) - \sin\theta \left(|U_\mathbf{k}|\, \alpha^r_{\mathbf{k},1}(t) - \epsilon^r\, |V_\mathbf{k}|\, \beta^{r\dagger}_{-\mathbf{k},1}(t) \right)
\end{aligned} \tag{2.78}$$

$$\begin{aligned}
\beta^r_{-\mathbf{k},\nu_e}(t) &= \cos\theta\, \beta^r_{-\mathbf{k},1}(t) + \sin\theta \left(|U_\mathbf{k}|\, \beta^r_{-\mathbf{k},2}(t) - \epsilon^r\, |V_\mathbf{k}|\, \alpha^{r\dagger}_{\mathbf{k},2}(t) \right) \\
\beta^r_{-\mathbf{k},\nu_\mu}(t) &= \cos\theta\, \beta^r_{-\mathbf{k},2}(t) - \sin\theta \left(|U_\mathbf{k}|\, \beta^r_{-\mathbf{k},1}(t) + \epsilon^r\, |V_\mathbf{k}|\, \alpha^{r\dagger}_{\mathbf{k},1}(t) \right),
\end{aligned} \tag{2.79}$$

with $\epsilon^r = (-1)^r$ and

$$|U_\mathbf{k}| \equiv u^{r\dagger}_{\mathbf{k},i} u^r_{\mathbf{k},j} = v^{r\dagger}_{-\mathbf{k},i} v^r_{-\mathbf{k},j}\ , \qquad |V_\mathbf{k}| \equiv \epsilon^r\ u^{r\dagger}_{\mathbf{k},1} v^r_{-\mathbf{k},2} = -\epsilon^r\ u^{r\dagger}_{\mathbf{k},2} v^r_{-\mathbf{k},1}$$

where $i, j = 1, 2$ and $i \neq j$ and we have:

$$|U_{\mathbf{k}}| = \left(\frac{\omega_{k,1}+m_1}{2\omega_{k,1}}\right)^{\frac{1}{2}} \left(\frac{\omega_{k,2}+m_2}{2\omega_{k,2}}\right)^{\frac{1}{2}} \left(1+\frac{\mathbf{k}^2}{(\omega_{k,1}+m_1)(\omega_{k,2}+m_2)}\right) \tag{2.80}$$

$$|V_{\mathbf{k}}| = \left(\frac{\omega_{k,1}+m_1}{2\omega_{k,1}}\right)^{\frac{1}{2}} \left(\frac{\omega_{k,2}+m_2}{2\omega_{k,2}}\right)^{\frac{1}{2}} \left(\frac{k}{(\omega_{k,2}+m_2)}-\frac{k}{(\omega_{k,1}+m_1)}\right) \tag{2.81}$$

$$|U_{\mathbf{k}}|^2 + |V_{\mathbf{k}}|^2 = 1. \tag{2.82}$$

The flavor fields can be expanded in the same operational bases of the mass fields, i.e.:

$$\nu_e(\mathbf{x}, t) = \frac{1}{\sqrt{V}}\sum_{\mathbf{k},r} e^{i\mathbf{k}.\mathbf{x}} \left[u^r_{\mathbf{k},1}\alpha^r_{\mathbf{k},\nu_e}(t) + v^r_{-\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},\nu_e}(t)\right], \tag{2.83}$$

$$\nu_\mu(\mathbf{x}, t) = \frac{1}{\sqrt{V}}\sum_{\mathbf{k},r} e^{i\mathbf{k}.\mathbf{x}} \left[u^r_{\mathbf{k},2}\alpha^r_{\mathbf{k},\nu_\mu}(t) + v^r_{-\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},\nu_\mu}(t)\right], \tag{2.84}$$

At level of annihilator operator we can see that the structure of mixing is made by a Bogoliubov transformation nested into a rotation.
We will see below that this mixing structure will be confirmed even at fields level.
As we can see this two, transformation could be not disentangled, so (2.78, 2.79) are different from a simple rotation and from a simple Bogoliubov. We

will now obtain the explicit expression of $|0\rangle_{e,\mu}$ at finite volume, then we will do the infinite volume limit.
Using the gaussian decomposition [29] we have :

$$G^{-1}_\theta(t) = exp[\theta(S_- - S_+)] = exp(-tan\theta\ S_+)\ exp(-2ln\ cos\theta\ S_3)\ exp(tan\theta\ S_-) \tag{2.85}$$

where $0 \leq \theta < \frac{\pi}{2}$.
Then Eq.(2.73) becomes:

$$|0\rangle_{e,\mu} = \prod_{\mathbf{k}} |0\rangle^{\mathbf{k}}_{e,\mu} = \prod_{\mathbf{k}} exp(-tan\theta\ S^{\mathbf{k}}_+) exp(-2ln\ cos\theta\ S^{\mathbf{k}}_3)\ exp(tan\theta\ S^{\mathbf{k}}_-)|0\rangle_{1,2}\ . \tag{2.86}$$

-+- The right hand of the above equation can be calculate considering:

$$S_3^{\mathbf{k}}|0\rangle_{1,2} = 0 \ , \tag{2.87}$$
$$S_\pm^{\mathbf{k}}|0\rangle_{1,2} \neq 0 \ , \tag{2.88}$$
$$(S_\pm^{\mathbf{k}})^2|0\rangle_{1,2} \neq 0 \ , \tag{2.89}$$
$$(S_\pm^{\mathbf{k}})^3|0\rangle_{1,2} = 0 \tag{2.90}$$

and other relations that are calculated in appendix C, so the expression for $|0\rangle_{e,\mu}$ becomes:

$$|0\rangle_{e,\mu} = \prod_{\mathbf{k}} |0\rangle^{\mathbf{k}}_{e,\mu} = \prod_{\mathbf{k}} \Big[1 + \sin\theta\cos\theta \left(S_-^{\mathbf{k}} - S_+^{\mathbf{k}}\right) + \frac{1}{2}\sin^2\theta\cos^2\theta \left((S_-^{\mathbf{k}})^2 + (S_+^{\mathbf{k}})^2\right) + \\ - \sin^2\theta S_+^{\mathbf{k}} S_-^{\mathbf{k}} + \frac{1}{2}\sin^3\theta\cos\theta \left(S_-^{\mathbf{k}}(S_+^{\mathbf{k}})^2 - S_+^{\mathbf{k}}(S_-^{\mathbf{k}})^2\right) + \frac{1}{4}\sin^4\theta (S_+^{\mathbf{k}})^2(S_-^{\mathbf{k}})^2 \Big] |0\rangle_{1,2} \ . \tag{2.91}$$

Obviously, see eq. (2.73), $|0\rangle_{e,\mu}$ is normalized to one.

Now we will show the orthogonality between $|0\rangle_{e,\mu}$ and $|0\rangle_{1,2}$ at infinite volume. We have

$${}_{1,2}\langle 0|0\rangle_{e,\mu} = \prod_{\mathbf{k}} \left(1 - \sin^2\theta \ {}_{1,2}\langle 0|S_+^{\mathbf{k}} S_-^{\mathbf{k}}|0\rangle_{1,2} + \frac{1}{4}\sin^4\theta \ {}_{1,2}\langle 0|(S_+^{\mathbf{k}})^2(S_-^{\mathbf{k}})^2|0\rangle_{1,2} \right) \tag{2.92}$$

where (see Appendix C)

$$\begin{aligned} & {}_{1,2}\langle 0|S_+^{\mathbf{k}} S_-^{\mathbf{k}}|0\rangle_{1,2} = \\ = \ & {}_{1,2}\langle 0| \left(\sum_{\sigma,\tau}\sum_{r,s} \left[v^{\sigma\dagger}_{-\mathbf{k},1}\, u^{\tau}_{\mathbf{k},2}\right] \left[u^{s\dagger}_{\mathbf{k},2}\, v^{r}_{-\mathbf{k},1}\right] \beta^{\sigma}_{-\mathbf{k},1}(t)\, \alpha^{\tau}_{\mathbf{k},2}(t)\, \alpha^{s\dagger}_{\mathbf{k},2}(t)\, \beta^{r\dagger}_{-\mathbf{k},1}(t) \right) |0\rangle_{1,2} = \\ = \ & \sum_{r,s} |\ v^{r\dagger}_{-\mathbf{k},1}\, u^{s}_{\mathbf{k},2}\ |^2 \equiv 2|V_{\mathbf{k}}|^2 \end{aligned} \tag{2.93}$$

and proceeding in a similar way we find:

$${}_{1,2}\langle 0|(S_+^{\mathbf{k}})^2(S_-^{\mathbf{k}})^2|0\rangle_{1,2} = 2|V_{\mathbf{k}}|^4 \ . \tag{2.94}$$

$|V_{\mathbf{k}}|^2$ depends on $\mathbf{k}$ only through its modulus and it has values always in the interval $[0, \frac{1}{2}[$. This function has a maximum at $|\mathbf{k}| = \sqrt{m_1 m_2}$ [8] and we

[8] $\sqrt{m_1 m_2}$ is the scale of the condensation density

have $|V_{\mathbf{k}}|^2 = 0$ when $m_1 = m_2$.Also, $|V_{\mathbf{k}}|^2 \to 0$ when $k \to \infty$.
Finally we have

$$
{}_{1,2}\langle 0|0\rangle_{e,\mu} = \prod_{\mathbf{k}} \left(1 - \sin^2\theta\, |V_{\mathbf{k}}|^2\right)^2 \equiv \prod_{k} \Gamma(\mathbf{k}) = \prod_{\mathbf{k}} e^{ln\,\Gamma(\mathbf{k})} = e^{\sum_{\mathbf{k}} ln\,\Gamma(\mathbf{k})}. \tag{2.95}
$$

Given the properties ok $|V_{\mathbf{k}}|^2$ we have that $\Gamma(\mathbf{k}) < 1$ for any value of $\mathbf{k}$ and for any value of the parameters m_1 and m_2.
Using the continuous limit relation $\sum_{\mathbf{k}} \to \frac{V}{(2\pi)^3}\int d^3\mathbf{k}$, for the infinite volume limit we obtain:

$$
\lim_{V\to\infty} {}_{1,2}\langle 0|0\rangle_{e,\mu} = \lim_{V\to\infty} e^{\frac{V}{(2\pi)^3}\int d^3\mathbf{k}\, ln\,\Gamma(\mathbf{k})} = 0 \tag{2.96}
$$

The relation (2.96) goes to zero because the infinite volume limit takes into account the contribution due to values $\mathbf{k} \to \infty$ [9] that eliminate the UV divergence of $\int d^3\mathbf{k}\, ln\,\Gamma(\mathbf{k})$.
In conclusion we showed that the flavor states are generalised $SU(2)$ coherent stases, this leads us to two fundamental results: first we have that the flavor vacuum state is different than the mass vacuum state [10] at finite volume, second we have that at infinite volume limit the flavor vacuum is orthogonal to the mass vacuum.
Thus we see that the usual identification of the flavor vacuum with the mass vacuum is only an approximation and we showed the limit of applicability of this approximation.
What we said above shows the absolutely non trivial nature of the mixing transformations (2.1, 2.2).

If we exhibit the explicit expression of$|0\rangle^{\mathbf{k}}_{e,\mu}$, at time $t = 0$, in the reference frame for which $\mathbf{k} = (0, 0, |\mathbf{k}|)$ (see Appendix D) we obtain:

$$
\begin{aligned}
|0\rangle^{\mathbf{k}}_{e,\mu} &= \prod_{r} \Big[(1 - \sin^2\theta\, |V_{\mathbf{k}}|^2) - \epsilon^r \sin\theta\, \cos\theta\, |V_{\mathbf{k}}| (\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2} + \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}) + \\
&+ \epsilon^r \sin^2\theta\, |V_{\mathbf{k}}||U_{\mathbf{k}}| (\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},1} - \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},2}) + \sin^2\theta\, |V_{\mathbf{k}}|^2 \alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2}\alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}\Big] |0\rangle_{1,2}
\end{aligned} \tag{2.97}
$$

As we can see from (2.97) the expression of the flavor vacuum $|0\rangle^{\mathbf{k}}_{e,\mu}$ involves four pairs of massive particles.
The number of condensed particles is given by:

[9] In other words it takes into account the infrared limit.

[10] To be more precise each flavor state is different to the respective mass state

$$ {}_{e,\mu}\langle 0|\alpha^{r\dagger}_{\mathbf{k},i}\alpha^{r}_{\mathbf{k},i}|0\rangle_{e,\mu} \;=\; {}_{e,\mu}\langle 0|\beta^{r\dagger}_{\mathbf{k},i}\beta^{r}_{\mathbf{k},i}|0\rangle_{e,\mu} \;=\; \sin^2\theta\,|V_{\mathbf{k}}|^2\,, \qquad i=1,2\,. \tag{2.98} $$

2.2 Generalized transformation of the mixing.

Let us now start by introducing a new notation to describe the flavor fields:

$$ \nu_\sigma(x) = G^{-1}_\theta(t)\,\nu_j(x)\,G_\theta(t) = \frac{1}{\sqrt{V}}\sum_{\mathbf{k},r}\left[u^r_{\mathbf{k},j}\alpha^r_{\mathbf{k},\nu_\sigma}(t) + v^r_{-\mathbf{k},j}\beta^{r\dagger}_{-\mathbf{k},\nu_\sigma}(t)\right]e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{2.99} $$

Obviously we have also:

$$ \left(\begin{array}{c}\alpha^r_{\mathbf{k},\nu_\sigma}(t)\\ \beta^{r\dagger}_{-\mathbf{k},\nu_\sigma}(t)\end{array}\right) = G^{-1}_\theta(t)\left(\begin{array}{c}\alpha^r_{\mathbf{k},j}(t)\\ \beta^{r\dagger}_{-\mathbf{k},j}(t)\end{array}\right)G_\theta(t) \tag{2.100} $$

If we explicit the generator $G_\theta(t)$ in the frame where $\mathbf{k}=(0,0,|\mathbf{k}|)$ we have:

$$ \left(\begin{array}{c}\alpha^r_{\mathbf{k},\nu_e}(t)\\ \alpha^r_{\mathbf{k},\nu_\mu}(t)\\ \beta^{r\dagger}_{-\mathbf{k},\nu_e}(t)\\ \beta^{r\dagger}_{-\mathbf{k},\nu_\mu}(t)\end{array}\right) = \left(\begin{array}{cccc} c_\theta & s_\theta\,|U_{\mathbf{k}}| & 0 & s_\theta\,\epsilon^r\,|V_{\mathbf{k}}| \\ -s_\theta\,|U_{\mathbf{k}}| & c_\theta & s_\theta\,\epsilon^r\,|V_{\mathbf{k}}| & 0 \\ 0 & -s_\theta\,\epsilon^r\,|V_{\mathbf{k}}| & c_\theta & s_\theta\,|U_{\mathbf{k}}| \\ -s_\theta\,\epsilon^r\,|V_{\mathbf{k}}| & 0 & -s_\theta\,|U_{\mathbf{k}}| & c_\theta \end{array}\right)\left(\begin{array}{c}\alpha^r_{\mathbf{k},1}(t)\\ \alpha^r_{\mathbf{k},2}(t)\\ \beta^{r\dagger}_{-\mathbf{k},1}(t)\\ \beta^{r\dagger}_{-\mathbf{k},2}(t)\end{array}\right) \tag{2.101} $$

where $c_\theta \equiv \cos\theta$, $s_\theta \equiv \sin\theta$, $\epsilon^r \equiv (-1)^r$.
In Eq.(2.99) it can be used an eigenfunction with arbitrary masses μ_σ, so the transformation (2.100),see [30, 31], becomes:

$$ \nu_\sigma(x) \;=\; \frac{1}{\sqrt{V}}\sum_{\mathbf{k},r}\left[u^r_{\mathbf{k},\sigma}\widetilde{\alpha}^r_{\mathbf{k},\nu_\sigma}(t) + v^r_{-\mathbf{k},\sigma}\widetilde{\beta}^{r\dagger}_{-\mathbf{k},\nu_\sigma}(t)\right]e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{2.102} $$

u_σ and v_σ are the helicity eigenfunction with definite mass μ_σ.
Using this basis we simplify considerably the calculations with respect to the above mentioned choice [31]. The generalized flavor operator will be denoted by a tilde to distinguish the one above defined.

Let us consider again the set of free fermion[11] field operators composed, for simplicity, by only two elements, i.e. assume our annihilators are:

$$\left(\begin{array}{c}\alpha^r_{\mathbf{k},i}(t)\\ \beta^{r\dagger}_{-\mathbf{k},i}(t)\end{array}\right) \ , \quad \text{with} \quad i = 1,2. \, , \tag{2.103}$$

the non zero masses are m_1 and m_2, with $m_1 \neq m_2$.
The wave functions u_i and v_i[12] satisfy the free Dirac equations

$$(i \not{k} + m_i)u_i = 0 \quad , \qquad (i \not{k} - m_i)v_i = 0 \quad , \tag{2.104}$$

respectively.
We denote the with $|0\rangle_{1,2} = |0\rangle_1 \otimes |0\rangle_2$ the vacuum state that is annihilated by $\left(\begin{array}{c}\alpha^r_{\mathbf{k},i}(t)\\ \beta^{r\dagger}_{-\mathbf{k},i}(t)\end{array}\right)$. As we know, in quantum field theory there exist many unitarily inequivalent representations of the anti-commutation (commutation for bosons) relations of the fields operators [33] [34] so we can consider another set of annihilation operators:

$$\left(\begin{array}{c}\widetilde{\alpha}^r_{\mathbf{k},i}(t)\\ \widetilde{\beta}^{r\dagger}_{-\mathbf{k},i}(t)\end{array}\right) \ , \quad i = 1,2, \tag{2.105}$$

Suppose now that the wave functions that corresponds to the operators of Eqs. (2.105) satisfy the free fields Dirac equations:

$$(i \not{k} + \mu_i)\widetilde{u}_i = 0 \quad , \qquad (i \not{k} - \mu_i)\widetilde{v}_i = 0 \quad , \tag{2.106}$$

As we can see, the mass now is considered as a mass parameter[13] i.e. m_i[14] are mass parameters for the fields $\nu_i(x)$ and μ_i are mass parameters for the fields $\widetilde{\nu_i}(x)$.
The two sets of operators are related in the following way:

$$\left(\begin{array}{c}\widetilde{\alpha}^r_{\mathbf{k},i}(t)\\ \widetilde{\beta}^{r\dagger}_{-\mathbf{k},i}(t)\end{array}\right) = I^{-1}_\mu(t)\left(\begin{array}{c}\alpha^r_{\mathbf{k},i}(t)\\ \beta^{r\dagger}_{-\mathbf{k},i}(t)\end{array}\right) I_\mu(t) \quad . \tag{2.107}$$

The operator $I_\mu(t)$ is given by:

$$I_\mu(t) = \prod_{\mathbf{k},r} \exp\left\{ i \sum_i \xi_i^{\mathbf{k}} \left[\alpha^{r\dagger}_{\mathbf{k},i}(t)\, \beta^{r\dagger}_{-\mathbf{k},i}(t) + \beta^{r}_{-\mathbf{k},i}(t)\, \alpha^{r}_{\mathbf{k},i}(t)\right]\right\} \tag{2.108}$$

[11] All the conclusions that we achieve here remain valid even in boson case.
[12] We omit the index **k** whenever no confusion arises
[13] A parameter with the dimension of a mass.
[14] In this particular case coincide with the masses

Obviously if $\mu_i = m_i$ the operator $I_\mu(t)$ transforms into the identity operator and this is simply due to the fact that $\xi_i^{\mathbf{k}} \equiv (\widetilde{\chi}_i - \chi_i)/2$ and $\cot \widetilde{\chi}_i = |\mathbf{k}|/\mu_i$, $\cot \chi_i = |\mathbf{k}|/m_i$.
Substituting (2.108) into (2.107) we obtain the matrix form of (2.107)[15], i.e. :

$$\begin{pmatrix} \widetilde{\alpha}^r_{\mathbf{k},1}(t) \\ \widetilde{\alpha}^r_{\mathbf{k},2}(t) \\ \widetilde{\beta}^{r\dagger}_{-\mathbf{k},1}(t) \\ \widetilde{\beta}^{r\dagger}_{-\mathbf{k},2}(t) \end{pmatrix} = \begin{pmatrix} \rho_1^{\mathbf{k}} & 0 & i\,\lambda_1^{\mathbf{k}} & 0 \\ 0 & \rho_2^{\mathbf{k}} & 0 & i\,\lambda_2^{\mathbf{k}} \\ i\,\lambda_1^{\mathbf{k}} & 0 & \rho_1^{\mathbf{k}} & 0 \\ 0 & i\lambda_2^{\mathbf{k}} & 0 & \rho_2^{\mathbf{k}} \end{pmatrix} \begin{pmatrix} \alpha^r_{\mathbf{k},1}(t) \\ \alpha^r_{\mathbf{k},2}(t) \\ \beta^{r\dagger}_{-\mathbf{k},1}(t) \\ \beta^{r\dagger}_{-\mathbf{k},2}(t) \end{pmatrix} \tag{2.109}$$

The matrix elements of the transformation matrix are given in the sequent way:

$$\begin{aligned} \rho_i^{\mathbf{k}}\delta_{rs} = \widetilde{u}^{r\dagger}_{\mathbf{k},i}\, u^s_{\mathbf{k},i} \;&=\; \widetilde{v}^{r\dagger}_{-\mathbf{k},i}\, v^s_{-\mathbf{k},i} \;\;\equiv\;\; \cos\xi_i^{\mathbf{k}}\,\delta_{rs} \\ i\,\lambda_i^{\mathbf{k}}\delta_{rs} = \widetilde{u}^{r\dagger}_{\mathbf{k},i} v^s_{-\mathbf{k},i} &= \widetilde{v}^{r\dagger}_{-\mathbf{k},i} u^s_{\mathbf{k},i} \;\;\equiv\;\; i\;\sin\xi_i^{\mathbf{k}}\delta_{rs} \end{aligned} \tag{2.110}$$

with $i = 1, 2$ and $\widetilde{\omega}_{k,i} = \sqrt{\mathbf{k}^2 + \mu_i^2}$.
The vacuum that is annihilated by the $(\widetilde{\alpha}^r_{\mathbf{k},i},\, \widetilde{\beta}^r_{-\mathbf{k},i})$ operators is

$$|\widetilde{0}(t)\rangle_{1,2} \equiv I_\mu^{-1}(t)|0\rangle_{1,2} \quad . \tag{2.111}$$

Relation (2.107) is nothing but a Bogoliubov transformation that relate fields operators $(\alpha^r_{\mathbf{k},i},\, \beta^r_{-\mathbf{k},i})$ and $(\widetilde{\alpha}^r_{\mathbf{k},i}$,that have different mass parameters, i.e. μ_i and m_i.
In the infinite volume limit the Hilbert spaces where $\alpha^r_{\mathbf{k},i}$ and $\widetilde{\alpha}^r_{\mathbf{k},i}$ are respectively defined these operators turn out to be unitarily inequivalent.
As we can see, the $\xi_i^{\mathbf{k}}$ acts like a label that specify the unitarily inequivalent Hilbert spaces that have different values of the μ_i mass parameter among themselves.
Obviously $|\widetilde{0}(t)\rangle_{1,2}$ is not annihilated by $\alpha^r_{\mathbf{k},i}$ and $\beta^r_{\mathbf{k},i}$ and it is not an eigenstate of the number operators:

$$N_{\alpha_i} = \sum_r \int d^3\mathbf{k}\; \alpha^{r\dagger}_{\mathbf{k},i}\alpha^r_{\mathbf{k},i}, \qquad N_{\beta_i} = \sum_r \int d^3\mathbf{k}\; \beta^{r\dagger}_{\mathbf{k},i}\beta^r_{\mathbf{k},i} \tag{2.112}$$

Also $|0\rangle_{1,2}$ is not annihilated by $\widetilde{\alpha}^r_{\mathbf{k},i}$ and $\widetilde{\beta}^r_{\mathbf{k},i}$ and it is not an eigenstate of the number operators:

[15]Written for both the index $i = 1\,,2$

$$\widetilde{N}_{\alpha_i}(t) = \sum_r \int d^3\mathbf{k}\, \widetilde{\alpha}^{r\dagger}_{\mathbf{k},i}(t)\widetilde{\alpha}^{r}_{\mathbf{k},i}(t), \qquad \widetilde{N}_{\beta_i}(t) = \sum_r \int d^3\mathbf{k}\, \widetilde{\beta}^{r\dagger}_{\mathbf{k},i}(t)\widetilde{\beta}^{r}_{\mathbf{k},i}(t). \tag{2.113}$$

If we perform the expectation values of N_{α_i} and N_{β_i} on $|\widetilde{0}(t)\rangle_{1,2}$ we obtain:

$$_{1,2}\langle\widetilde{0}(t)|N_{\alpha_i}|\widetilde{0}(t)\rangle_{1,2} = {}_{1,2}\langle\widetilde{0}(t)|N_{\beta_i}|\widetilde{0}(t)\rangle_{1,2} = \sin^2\xi_i^{\mathbf{k}} \ , \tag{2.114}$$

If we also perform the expectation value of $\widetilde{N}_{\alpha_i}(t)$ and $\widetilde{N}_{\beta_i}(t)$ on $|0\rangle_{1,2}$ we obtain:

$$_{1,2}\langle 0|\widetilde{N}_{\alpha_i}(t)|0\rangle_{1,2} = {}_{1,2}\langle 0|\widetilde{N}_{\beta_i}(t)|0\rangle_{1,2} = \sin^2\xi_i^{\mathbf{k}} \ . \tag{2.115}$$

So the number operator, say N_{α_i}, is not an invariant under the Bogoliubov transformation (2.107). This results is not really surprising us because Bogoliubov transformations introduce a *new* unitary inequivalent(i.e. physically inequivalent) Hilbert space in the infinite volume limit. In other words, through the Bogoliubov transformation we introduce a new set of asymptotic fields[16], so there are infinitely many sets of asymptotic fields, each one associated to its specific representation. The choice of the set of fields operators, i.e. the choice of the Hilbert space of states that describe our system, is then dictated by the physical conditions which are actually realized and where the system is.
If we define the state $|\widetilde{\alpha}^r_{\mathbf{k},i}\rangle$ as $|\widetilde{\alpha}^r_{\mathbf{k},i}(0)\rangle = \widetilde{\alpha}^{r\dagger}_{\mathbf{k},i}(0)|\widetilde{0}\rangle$, we have:

$$\begin{aligned}\langle\widetilde{\alpha}^r_{\mathbf{k},i}(0)|\,\widetilde{N}_{\alpha_i}(t)\,|\widetilde{\alpha}^r_{\mathbf{k},i}(0)\rangle &= |\{\widetilde{\alpha}^r_{\mathbf{k},i}(t), \widetilde{\alpha}^{r\dagger}_{\mathbf{k},i}(0)\}|^2 \\ &= ||\rho_i^{\mathbf{k}}|^2 e^{i(\widetilde{\omega}_{k,i}-\omega_{k,i})t} + |\lambda_i^{\mathbf{k}}|^2 e^{i(\widetilde{\omega}_{k,i}+\omega_{k,i})t}|^2,\end{aligned} \tag{2.116}$$

with $i = 1, 2$. Equation(2.116) shows that the expectation value of the time dependent number operator is not preserved by the transformation (2.107) applied both to states and operators. Consider the total charge operator

$$Q_i = N_{\alpha_i} - N_{\beta_i} \tag{2.117}$$

with $i = 1, 2$, instead of considering the single number operators N_{α_i} and N_{β_i} we have that:

$$\widetilde{Q}_i \ = \ Q_i, \tag{2.118}$$

[16] A new set of quasi-particles.

i.e. Q_i, for free fields, is conserved under the Bogoliubov transformation (2.107).
The expectation value of Q_i at time t on the state at $t = 0$ is given by:

$$\langle \widetilde{\alpha}^r_{\mathbf{k},i}(0) \mid \widetilde{Q}_i(t) \mid \widetilde{\alpha}^r_{\mathbf{k},i}(0)\rangle = \langle \alpha^r_{\mathbf{k},i}(0) \mid Q_i(t) \mid \alpha^r_{\mathbf{k},i}(0)\rangle, \tag{2.119}$$

since we have:

$$|\{\widetilde{\alpha}^r_{\mathbf{k},i}(t), \widetilde{\alpha}^{r\dagger}_{\mathbf{k},i}(0)\}|^2 + |\{\widetilde{\beta}^{r\dagger}_{\mathbf{k},i}(t), \widetilde{\alpha}^{r\dagger}_{\mathbf{k},i}(0)\}|^2 = |\{\alpha^r_{\mathbf{k}_i}(t), \alpha^{r\dagger}_{\mathbf{k},i}(0)\}|^2 + |\{\beta^{r\dagger}_{\mathbf{k},i}(t), \alpha^{r\dagger}_{\mathbf{k},i}(0)\}|^2. \tag{2.120}$$

So we have shown that the above mentioned expectation value is independent from mass parameters.
If we consider, instead of the Bogoliubov (2.107) a Bogoliubov nested into a rotation [17] we obtain similar results.
We clarified that quantum fields can have different mass parameters because of it is intrinsic into QFT, so this feature is *independent from the occurrence of the mixing*. Also we have seen that the choose of the mass parameter must be justified on the ground of physical reasons [35, 36, 37].
Obviously in the mixing problem the choice $\mu_e \equiv m_1$, $\mu_\mu \equiv m_2$ is motivated by the fact that m_1, m_2 are the mass-eighenfields, so it is the only one that have a physic relevance.
Going forward, we will use the charge operator instead of the each number operator; this operator describes the relative population densities in a beam. In case of fermion oscillation it is relate to the lepton charge.

2.3 Current structure for field mixing.

Consider a doublet of free fields with masses m_1 and m_2 with $m_1 \neq m_2$.
The lagrangian density that describe the behavior of the above mentioned fields is:

$$\mathcal{L}(x) \,=\, \bar{\Psi}_m(x)\,(i\,\not\partial - \mathsf{M}_d)\,\Psi_m(x)\,, \tag{2.121}$$

where $\Psi_m^T = (\nu_1, \nu_2)$ and $\mathsf{M}_d = diag(m_1, m_2)$.
The label m indicates that we are dealing with mass eighenfields, in the following the flavor eighenfields are labelled by f.
The above lagrangian is invariant under global $U(1)$ phase transformations like:

[17] In this way we reproduce the mixing fields situation that we have seen before.

$$\Psi'_m(x) = e^{i\alpha}\Psi_m(x), \tag{2.122}$$

so we have the conservation of the Noether charge:

$$Q_= \int I^0(x)d^3\mathbf{x} \tag{2.123}$$

with

$$I^\mu(x) = \bar{\Psi}_m(x)\gamma^\mu\Psi_m(x) \tag{2.124}$$

that is the current obtained applying Noether's theorem. Q is the total charge of the system, i.e. the total lepton number.
Let us now consider a global $SU(2)$ transformation:

$$\Psi'_m(x) = e^{i\alpha_j\cdot\tau_j}\Psi_m(x) \qquad\qquad j = 1, 2, 3. \tag{2.125}$$

with α_j real constants, and τ_j being the Pauli matrices. The lagrangian (2.121) is not invariant under the above mentioned transformation because $m_1 \neq m_2$. Referring to (2.121) if we consider the gauge transformation:

$$\psi_m(x) \to \psi'_m(x) = e^{i\alpha_j\tau_j}\psi_m(x), \tag{2.126}$$

with:

$$\tau_j \equiv \left(I, \frac{\sigma_1}{2}, \frac{\sigma_2}{2}, \frac{\sigma_3}{2}\right), \tag{2.127}$$

where I is the identity matrix, $\sigma_j,\ with\, j = 1,\ 2,\ 3.$ are the Pauli matrices and $\alpha_j,\ with\, j = 1,\ 2,\ 3.$ that are constants. Obviously (2.126) is an $SU(2)\otimes U(1)$ transformation.
We can power expand $e^{i\alpha_j\tau_j}$ obtaining:

$$e^{i\alpha_j\tau_j} = \frac{\sum_n (i\alpha_j\tau_j)^n}{n!}. \tag{2.128}$$

For not too big values α_j we can approximate (2.128) in the following way:

$$e^{i\alpha_j\tau_j} = I + i\alpha_j\tau j. \tag{2.129}$$

So we can re write the equation (2.126) as follows:

$$\psi'(x) = \frac{\sum_n^\infty (i\alpha_j\tau_j)^n}{n!}\psi(x). \tag{2.130}$$

If we now calculate:

$$\Delta\mathcal{L}(x) = \mathcal{L}'(x) - \mathcal{L}(x), \tag{2.131}$$

we have:

$$\Delta L(x) = \overline{\psi'_m(x)}\,(i\gamma^\mu\partial_\mu - M_d)\,\psi'_m(x) - \overline{\psi_m(x)}\,(i\gamma^\mu\partial_\mu - M_d)\,\psi_m(x) \quad (2.132)$$

Remembering that $\overline{\psi'_m(x)} = \psi'^\dagger_m(x)\gamma_0$ we obtain:

$$\overline{\psi'_m(x)} = \left(\frac{\sum_n^\infty (i\alpha_j\tau_j)^n}{n!}\psi(x)\right)^\dagger \gamma_0 = \psi(x)^\dagger(x)\left(\frac{\sum_n^\infty (i\alpha_j\tau_j)^n}{n!}\right)^\dagger \gamma_0 \Rightarrow \quad (2.133)$$

$$\mathcal{L}'(x) = \psi(x)^\dagger(x)\left(\frac{\sum_n^\infty (i\alpha_j\tau_j)^n}{n!}\right)^\dagger \gamma_0\,(i\gamma^\mu\partial_\mu - M_d)\left(\frac{\sum_n^\infty (i\alpha_j\tau_j)^n}{n!}\right)\psi_m(x). \quad (2.134)$$

The first term of the equation (2.134) becomes[18]:

$$\begin{aligned}
\psi(x)^\dagger\left(\frac{\sum_n^\infty (i\alpha_j\tau_j)^n}{n!}\right)^\dagger \gamma_0\,(i\gamma^\mu\partial_\mu)\left(\frac{\sum_n^\infty (i\alpha_j\tau_j)^n}{n!}\right)\psi_m(x) &= \\
\psi(x)^\dagger\left(\frac{\sum_n^\infty (i\alpha_j\tau_j)^n}{n!}\right)^\dagger \left(\frac{\sum_n^\infty (i\alpha_j\tau_j)^n}{n!}\right)\gamma_0\,(i\gamma^\mu\partial_\mu)\,\psi_m(x) &= \\
\psi(x)^\dagger\,(i\gamma^\mu\partial_\mu)\,\psi_m(x) \Rightarrow &\qquad (2.135)
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}'(x) = \psi(x)^\dagger\,(i\gamma^\mu\partial_\mu)\,\psi_m(x) - \\
\psi(x)^\dagger(x)\left(\frac{\sum_n^\infty (i\alpha_j\tau_j)^n}{n!}\right)^\dagger \gamma_0 M_d\left(\frac{\sum_n^\infty (i\alpha_j\tau_j)^n}{n!}\right)\psi_m(x) \Rightarrow \\
\Delta\mathcal{L}(x) = \overline{\psi_m(x)}\left(M_d - \left(\frac{\sum_n^\infty (i\alpha_j\tau_j)^n}{n!}\right)^\dagger M_d\left(\frac{\sum_n^\infty (i\alpha_j\tau_j)^n}{n!}\right)\right)\psi_m(x).
\end{aligned} \quad (2.136)$$

Separating in the two sums the even and the odd terms we obtain:

$$\begin{aligned}
\Delta\mathcal{L}(x) &= \overline{\psi_m(x)}\,(M_d - (-i\tau_j\sin(\alpha_j) + \cos(\alpha_j))\,M_d\,(i\tau_j\sin(\alpha_j) + \cos(\alpha_j)))\,\psi_m(x) = \\
&\overline{\psi_m(x)}\,\big(M_d + i\sin^2(\alpha_j)\tau_j M_d\tau_j + i\sin(\alpha_j)\cos(\alpha_j)[\tau_j, M_d] - \cos^2(\alpha_j)M_d\big)\,\psi_m(x).
\end{aligned} \quad (2.137)$$

---

[18]Note that: $\tau_0 = I_{4\times4}\,;\tau_1 = \frac{1}{2}\begin{pmatrix} 0_{2\times2} & I_{2\times2} \\ I_{2\times2} & 0_{2\times2}\end{pmatrix}$; $\tau_2 = \frac{1}{2}\begin{pmatrix} 0_{2\times2} & -\imath I_{2\times2} \\ \imath I_{2\times2} & 0_{2\times2}\end{pmatrix}$; $\tau_1 = \frac{1}{2}\begin{pmatrix} I_{2\times2} & 0_{2\times2} \\ 0_{2\times2} & I_{2\times2}\end{pmatrix}$; where 2×2 indicate 2×2 matrices. So the τ matrices operate on $\begin{pmatrix}\nu_1(x) \\ \nu_2(x)\end{pmatrix}$. and note that $\left(\frac{\sum_n^\infty (i\alpha_j\tau_j)^n}{n!}\right)^\dagger \left(\frac{\sum_n^\infty (i\alpha_j\tau_j)^n}{n!}\right) = I_{4\times4}$

In the limit of $\alpha_j \to 0$[19]we obtain:

$$\delta\mathcal{L}(x) = \lim_{\alpha_j \to 0} \Delta\mathcal{L}(x) = \overline{\psi_m(x)} i\alpha_j [\tau_j, M_d]\psi_m(x). \tag{2.138}$$

By Noether theorem the currents of a complex field are given by:

$$J^\mu(x) = i\left\{\Psi^{\dagger\alpha}(x)\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\Psi^{\dagger\alpha}(x)\right)} - \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\Psi^{\alpha}(x)\right)}\Psi^\alpha(x)\right\}, \tag{2.139}$$

so, in our case:

$$J^\mu_{m,j}(x) = \bar{\Psi}_m(x)\ \gamma^\mu\ \tau_j\ \Psi_m(x), \qquad\qquad j = 1, 2, 3. \tag{2.140}$$

Explicitly we have:

$$J^\mu_{m,1}(x) = \frac{1}{2}\left[\bar{\nu}_1(x)\ \gamma^\mu\ \nu_2(x) + \bar{\nu}_2(x)\ \gamma^\mu\ \nu_1(x)\right], \tag{2.141}$$

$$J^\mu_{m,2}(x) = \frac{i}{2}\left[\bar{\nu}_1(x)\ \gamma^\mu\ \nu_2(x) - \bar{\nu}_2(x)\ \gamma^\mu\ \nu_1(x)\right], \tag{2.142}$$

$$J^\mu_{m,3}(x) = \frac{1}{2}\left[\bar{\nu}_1(x)\ \gamma^\mu\ \nu_1(x) - \bar{\nu}_2(x)\ \gamma^\mu\ \nu_2(x)\right]. \tag{2.143}$$

and,

$$J^\mu_{m,0}(x) = \frac{1}{2}\left[\bar{\nu}_1(x)\ \gamma^\mu\ \nu_1(x) + \bar{\nu}_2(x)\ \gamma^\mu\ \nu_2(x)\right] \tag{2.144}$$

Obviously the related charges are given by:

$$Q_{m,j}(t) = \int J^0_{m,j}(x)d^3\mathbf{x}, \qquad\qquad j = 1, 2, 3, \tag{2.145}$$

and satisfy $su(2)$ algebra:

$$[Q_{m,i}(t), Q_{m,j}(t)] = i\varepsilon_{ijk}Q_{m,k}(t). \tag{2.146}$$

The Casimir operator is proportional to the total conserved charge, as we can see:

$$Q_{m,0} = \frac{1}{2}Q. \tag{2.147}$$

By the fact that M_d is diagonal we have that even $Q_{m,3}$ is conserved, this also implies that Q_1 and Q_2, witch are the charges of the two separated neutrinos ($\nu_1(x)$ and $\nu_2(x)$), are conserved:

$$Q_1 = \frac{1}{2}Q + Q_{m,3}, \tag{2.148}$$

$$Q_2 = \frac{1}{2}Q - Q_{m,3}\,. \tag{2.149}$$

[19]Up to first order in α_j

If we do the normal ordering, we obtain:

$$: Q_i :\equiv \int d^3\mathbf{x} \, : \nu_i^\dagger(x) \, \nu_i(x) := \sum_r \int d^3\mathbf{k} \, \left(\alpha^{r\dagger}_{\mathbf{k},i} \alpha^r_{\mathbf{k},i} \, - \, \beta^{r\dagger}_{-\mathbf{k},i} \beta^r_{-\mathbf{k},i} \right), \quad (2.150)$$

where $i = 1, 2$ and the $: .. :$ denotes normal ordering with respect to the vacuum $|0\rangle_{1,2}$.
As we can easily verify, neutrino states with definite masses defined as

$$|\nu^r_{\mathbf{k},i}\rangle = \alpha^{r\dagger}_{\mathbf{k},i} |0\rangle_{1,2}, \qquad i = 1, 2, \quad (2.151)$$

are then eigenstates of $: Q_1 :$ and $: Q_2 :$, which can be identified with the lepton charges in the absence of mixing. The above mentioned charges are nothing but the Noether charges of the free fields $\nu_1(x)$ and $\nu_2(x)$.
The transformations induced by the generators τ_1, τ_2, τ_3 are

$$\Psi_m' = \left(\begin{array}{cc} \cos\theta_1 & i \sin\theta_1 \\ i \sin\theta_1 & \cos\theta_1 \end{array} \right) \Psi_m, \quad (2.152)$$

$$\Psi_m' = \left(\begin{array}{cc} \cos\theta_2 & \sin\theta_2 \\ -\sin\theta_2 & \cos\theta_2 \end{array} \right) \Psi_m, \quad (2.153)$$

$$\Psi_m' = \left(\begin{array}{cc} e^{i\theta_3} & 0 \\ 0 & e^{-i\theta_3} \end{array} \right) \Psi_m \quad (2.154)$$

and the transformation induced by $Q_{m,2}(t)$:

$$\Psi_f(x) = e^{-2i\theta Q_{m,2}(t)} \, \Psi_m(x) \, e^{2i\theta Q_{m,2}(t)} \quad (2.155)$$

corresponds exactly to the matrix form of the mixing transformations (2.1), (2.2).

I will consider now the lagrangian written in the flavor basis:

$$\mathcal{L}(x) \, = \, \bar{\Psi}_f(x) \, (i \, \partial\!\!\!/ - \mathsf{M}) \, \Psi_f(x) \, , \quad (2.156)$$

where $\Psi_f^T = (\nu_e, \, \nu_\mu)$, $\mathsf{M} = \left(\begin{array}{cc} m_e & m_{e\mu} \\ m_{e\mu} & m_\mu \end{array} \right)$ and f indicate the flavor basis.
Considering the transformation:

$$\Psi_f'(x) = e^{i\alpha_j \cdot \tau_j} \Psi_f(x) \qquad j = 0, 1, 2, 3 \quad (2.157)$$

and repeating the same steps that we have done to obtain the formula (2.138) we have the analogous formula:

$$\delta\mathcal{L}(x) = i\alpha_j \bar{\Psi}_f(x) \, [\tau_j, \mathsf{M}] \, \Psi_f(x) = -\alpha_j \partial_\mu J^\mu_{f,j}(x), \quad (2.158)$$

Again by Noether theorem we obtain the currents:

$$J^{\mu}_{f,j}(x) = \bar{\Psi}_f(x)\, \gamma^{\mu}\, \tau_j\, \Psi_f(x)\,, \tag{2.159}$$

or explicitly:

$$J^{\mu}_{f,1}(x) = \frac{1}{2}\left[\bar{\nu}_e(x)\, \gamma^{\mu}\, \nu_{\mu}(x) + \bar{\nu}_{\mu}(x)\, \gamma^{\mu}\, \nu_e(x)\right], \tag{2.160}$$

$$J^{\mu}_{f,2}(x) = \frac{i}{2}\left[\bar{\nu}_e(x)\, \gamma^{\mu}\, \nu_{\mu}(x) - \bar{\nu}_{\mu}(x)\, \gamma^{\mu}\, \nu_e(x)\right], \tag{2.161}$$

$$J^{\mu}_{f,3}(x) = \frac{1}{2}\left[\bar{\nu}_e(x)\, \gamma^{\mu}\, \nu_e(x)\; - \bar{\nu}_{\mu}(x)\gamma^{\mu}\, \nu_{\mu}(x)\right], \tag{2.162}$$

$$J^{\mu}_{f,0}(x) = \frac{1}{2}\left[\bar{\nu}_e(x)\, \gamma^{\mu}\, \nu_e(x) + \bar{\nu}_{\mu}(x)\, \gamma^{\mu}\, \nu_{\mu}(x)\right]. \tag{2.163}$$

From these formulas we obtain the flavor charges:

$$Q_{f,j}(t) = \int J^{0}_{f,j}(x) d^3\mathbf{x} \qquad j = 0, 1, 2, 3, \tag{2.164}$$

that satisfy again the $su(2) \bigotimes u(1)$ algebra:

$$[Q_{f,i}(t), Q_{f,j}(t)] = i\varepsilon_{i,jk} Q_{f,k}(t)\,. \tag{2.165}$$

Defining the electronic and the muonic charges as follows:

$$Q_{\nu_e}(t) \equiv \frac{1}{2}Q + Q_{f,3}(t), \tag{2.166}$$

$$Q_{\nu_\mu}(t) \equiv \frac{1}{2}Q - Q_{f,3}(t), \tag{2.167}$$

where $Q = Q_{\nu_\mu}(t) + Q_{\nu_\mu}(t)$ is the total conserved charge, we have that those charges are not conserved because $Q_{f,3}(t)$ is not conserved by the fact that the off diagonal terms of M are not null. Any way the Casimir operator is again proportional to the total conserved charge:

$$Q_{f,0} = Q_0 = \frac{Q}{2}. \tag{2.168}$$

If we do the normal ordering we obtain:

$$\begin{aligned} :: Q_{\nu_\sigma}(t) :: \quad &\equiv \quad \int d^3\mathbf{x} \; :: \nu^{\dagger}_{\sigma}(x)\, \nu_{\sigma}(x) :: \\ &= \quad \sum_r \int d^3\mathbf{k} \left(\alpha^{r\dagger}_{\mathbf{k},\nu_\sigma}(t) \alpha^{r}_{\mathbf{k},\nu_\sigma}(t) \; - \; \beta^{r\dagger}_{-\mathbf{k},\nu_\sigma}(t) \beta^{r}_{-\mathbf{k},\nu_\sigma}(t) \right) \end{aligned} \tag{2.169}$$

where $\sigma = e, \mu$, and $::\ ...\ ::$ denotes normal ordering with respect to $|0\rangle_{e,\mu}$.

Note that

$$:: Q_{\nu_\sigma}(t) ::=\ G_\theta^{-1}(t) : Q_j : G_\theta(t), \tag{2.170}$$

with $\sigma = e,\ \mu,\ j = 1,\ 2$.
The flavor states at any time t are defined as eigenstates of the charge

$$:: Q_{\nu_e}(t) :: |\nu^r_{\mathbf{k},e}(t)\rangle\ =\ |\nu^r_{\mathbf{k},e}(t)\rangle \qquad ; \qquad :: Q_{\nu_\mu}(t) :: |\nu^r_{\mathbf{k},\mu}(t)\rangle\ =\ |\nu^r_{\mathbf{k},\mu}(t)\rangle \tag{2.171}$$

$$:: Q_{\nu_e}(t) ::\ |\nu^r_{\mathbf{k},\mu}(t)\rangle\ =:: Q_{\nu_\mu}(t) ::\ |\nu^r_{\mathbf{k},e}(t)\rangle\ =\ 0, \quad :: Q_{\nu_\sigma}(t) ::\ |0(t)\rangle_{e,\mu}\ =\ 0. \tag{2.172}$$

Where

$$|\nu^r_{\mathbf{k},\sigma}(t)\rangle \equiv \alpha^{r\dagger}_{\mathbf{k},\nu_\sigma}(t)|0(t)\rangle_{e,\mu}, \qquad \sigma = e, \mu \tag{2.173}$$

Same situation happens for antiparticles.
These results are not trivial since the usual Pontecorvo states [28]:

$$\begin{aligned} |\nu^r_{\mathbf{k},e}\rangle_P &= \cos\theta\ |\nu^r_{\mathbf{k},1}\rangle\ +\ \sin\theta\ |\nu^r_{\mathbf{k},2}\rangle & (2.174)\\ |\nu^r_{\mathbf{k},\mu}\rangle_P &= -\sin\theta\ |\nu^r_{\mathbf{k},1}\rangle\ +\ \cos\theta\ |\nu^r_{\mathbf{k},2}\rangle\ , & (2.175)\end{aligned}$$

are not eigenstates of the flavor charges operators [38]. To see better it, consider that the expectation values of the flavor charges on the Pontecorvo states are (we consider for simplicity $t = 0$).

$$\begin{aligned} {}_P\langle\nu^r_{\mathbf{k},e}| :: Q_{\nu_e}(0) :: |\nu^r_{\mathbf{k},e}\rangle_P &= \cos^4\theta + \sin^4\theta \\ &+ 2|U_{\mathbf{k}}|\sin^2\theta\cos^2\theta + \sum_r \int d^3\mathbf{k}, & (2.176)\end{aligned}$$

and

$$_{1,2}\langle 0| :: Q_{\nu_e}(0) :: |0\rangle_{1,2} = \sum_r \int d^3\mathbf{k}, \tag{2.177}$$

which are both infinite.
The above mentioned infinity can be removed by normal ordering respect to the mass vacuum but we also have the problem that the expectation values:

$$_{1,2}\langle 0|(: Q_{\nu_e}(0) :)^2|0\rangle_{1,2} = 4\sin^2\theta\cos^2\theta \int d^3\mathbf{k}|V_{\mathbf{k}}|^2, \tag{2.178}$$

$$\begin{aligned} {}_P\langle\nu^r_{\mathbf{k},e}|(: Q_{\nu_e}(0) :)^2|\nu^r_{\mathbf{k},e}\rangle_P &= \cos^6\theta + \sin^6\theta & (2.179)\\ &+ \sin^2\theta\cos^2\theta\left[2|U_{\mathbf{k}}| + |U_{\mathbf{k}}|^2 + 4\int d^3\mathbf{k}|V_{\mathbf{k}}|^2\right], \end{aligned}$$

are both infinite and they make the corresponding quantum fluctuation divergent.
So the correct flavor states are the ones in (2.173) and the correct normal ordered charge operators are the ones in (2.170).

2.4 Oscillations in neutrino mixing.

Following what we have explained before, using the Pontecorvo's states produces a violation of lepton charge conservation both in the production and in the detection vertices. Of course in the presence of mixing the leptonic charge is violated, but not in the production or in the detection vertex, this violation occurs during time evolution (flavor oscillations), due to the form of the weak interaction.
Let us show it in more specific details; I define the following quantities:

$$A_0 \equiv {}_P\langle \nu^r_{\mathbf{k},e}| : Q_{\nu_e}(0) : |\nu^r_{\mathbf{k},e}\rangle_P = \cos^4\theta + \sin^4\theta + 2|U_{\mathbf{k}}|\sin^2\theta\cos^2\theta < 1, \tag{2.180}$$

$$1 - A_0 \equiv {}_P\langle \nu^r_{\mathbf{k},e}| : Q_{\nu_\mu}(0) : |\nu^r_{\mathbf{k},e}\rangle_P = 2\sin^2\theta\cos^2\theta - 2|U_{\mathbf{k}}|\sin^2\theta\cos^2\theta > 0, \tag{2.181}$$

for any $\theta \neq 0$, $\mathbf{k} \neq 0$ and for $m_1 \neq m_2$.
Obviously we have:

$${}_P\langle \nu^r_{\mathbf{k},e}| : Q_{\nu_e}(0) : |\nu^r_{\mathbf{k},e}\rangle_P + {}_P\langle \nu^r_{\mathbf{k},e}| : Q_{\nu_\mu}(0) : |\nu^r_{\mathbf{k},e}\rangle_P = 1. \tag{2.182}$$

Consider, for example, an ideal experiment in which neutrinos (or other oscillating fermions) are produced and detected by means of some charged weak interaction process. What will be measured in the experiment will be obviously the number of accompanying leptons, say (anti-) electrons in the source and in the detector. I will indicate with N^S_e the number of electrons revealed at the source and with $N^D_e(t)$ the number of electrons revealed at the detector, usually one assumes that $N^S_{\nu_e} = N^S_e$ and $N^D_{\nu_e}(t) = N^D_e(t)$, where $N^S_{\nu_e}$ and $N^D_{\nu_e}(t)$ are the neutrinos produced in the source and those detected, respectively. The usual Pontecorvo oscillating probability formulas are then given by:

$$\frac{N^D_e(t)}{N^S_e} = \frac{N^D_{\nu_e}(t)}{N^S_{\nu_e}} = 1 - \sin^2 2\theta \sin^2\left(\frac{\Delta\omega}{2}t\right) = 1 - P(t) \tag{2.183}$$

$$\frac{N^D_\mu(t)}{N^S_e} = \frac{N^D_{\nu_\mu}(t)}{N^S_{\nu_e}} = \sin^2 2\theta \sin^2\left(\frac{\Delta\omega}{2}t\right) = P(t). \tag{2.184}$$

I take now into account (2.178, 2.179) so we have that the vacuum uctuation for neutrinos has got a divergence, and it means that in the generation vertex there is an instantaneous leptonic charge violations for them. Considering that the total leptonic charge in any vertex must be conserved, we have that only a part of electron neutrinos produced is accompanied by an anti electron23. I denote these quantities as:

$$\tilde{N}_e^S = A_0 N_{\nu_e}^S \tag{2.185}$$

$$\tilde{N}_e^D(t) = A_0 N_{\nu_e}^D(t) + (1 - A_0) N_{\nu_\mu}^D(t). \tag{2.186}$$

$(1 - A_0) N_{\nu_\mu}^D(t)$ is the part of the electronic neutrinos that arrives at the detecting vertex initially as muon neutrinos and that becomes electron neutrinos by a vacuum oscillation in the same vertex.
So the oscillation formulas becomes:

$$\frac{\tilde{N}_e^D(t)}{\tilde{N}_e^S} \;=\; \frac{A_0 N_{\nu_e}^D(t) + (1 - A_0) N_{\nu_\mu}^D(t)}{A_0 N_{\nu_e}^S} = 1 - \frac{2A_0 - 1}{A_0} P(t). \tag{2.187}$$

The above written formula is substantially different from the usual Pontecorvo's formula used to describe the oscillation (2.183) [20].

2.5 The exact formula for neutrino oscillations

The flavor vacuum state at time $t = 0$ is $|0\rangle_{e,\mu}$ and the one electron neutrino state is (for $\mathbf{k} = (0, 0, |\mathbf{k}|)$):

$$|\nu_{\mathbf{k},e}^r\rangle \equiv \alpha_{\mathbf{k},e}^{r\dagger}|0\rangle_{e,\mu} = \Big[\cos\theta\, \alpha_{\mathbf{k},1}^{r\dagger} + |U_\mathbf{k}|\ \sin\theta\ \alpha_{\mathbf{k},2}^{r\dagger} + \epsilon^r\ |V_\mathbf{k}|\ \sin\theta\ \alpha_{\mathbf{k},2}^{r\dagger}\alpha_{\mathbf{k},1}^{r\dagger}\beta_{-\mathbf{k},1}^{r\dagger}\Big]\, |0\rangle_{1,2}\,. \tag{2.188}$$

Last term is a multi - particle component that disappear in relativistic limit $|\mathbf{k}| \gg \sqrt{m_1 m_2}$: in this limit the Pontecorvo state is recovered.
It is not possible to compare directly the neutrino state at time t with the one at time $t = 0$ given in Eq.(2.188).
Recalling (2.72) indeed in the reference frame for which $\mathbf{k} = (0, 0, |\mathbf{k}|)$ we have:

[20]The Pontecorvo's formula is however recovered in the relativistic limit.Indeed, for $|\mathbf{k}| \gg \sqrt{m_1 m_2}$ we have $|U_\mathbf{k}| \longrightarrow 1$ and $A_0 = 1$.

$$
\begin{aligned}
&|\nu^r_{\mathbf{k},e}(t)\rangle \equiv \alpha^{r\dagger}_{\mathbf{k},e}(t)|0(t)\rangle_{e,\mu} = e^{-i:H:t}|\nu^r_{\mathbf{k},e}(0)\rangle \\
= \;& e^{-i\omega_{k,1}t}\Big[\cos\theta\,\alpha^{r\dagger}_{\mathbf{k},1} + |U_{\mathbf{k}}|e^{-i\Omega^k_- t}\,\sin\theta\,\alpha^{r\dagger}_{\mathbf{k},2} - \epsilon^r\,|V_{\mathbf{k}}|e^{-i\Omega^k_+ t}\,\sin\theta\,\alpha^{r\dagger}_{\mathbf{k},1}\alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}\Big] \\
\times\;& G^{-1}_{\mathbf{k},s\neq r}(\theta,t)\prod_{\mathbf{p}\neq\mathbf{k}} G^{-1}_{\mathbf{p}}(\theta,t)|0\rangle_{1,2}\,, \qquad (2.189)
\end{aligned}
$$

where $\Omega^k_+ \equiv \omega_{k,2}+\omega_{k,1}$, $\Omega^k_- \equiv \omega_{k,2}-\omega_{k,1}$, and

$$
:H: = H -_{1,2}\langle 0|H|0\rangle_{1,2} = H + 2\int d^3\mathbf{k}\,\Omega^k_+ = \sum_{i,r}\int d^3\mathbf{k}\,\omega_{k,i}[\alpha^{r\dagger}_{\mathbf{k},i}\alpha^r_{\mathbf{k},i} + \beta^{r\dagger}_{\mathbf{k},i}\beta^r_{\mathbf{k},i}]\,, \qquad (2.190)
$$

is the Hamiltonian normal ordered with respect to the vacuum $|0\rangle_{1,2}$ [21]. From the above written equations, considering Appendix E, we have

$$
\lim_{V\to\infty}\langle\nu_e(t)|\nu_e(0)\rangle \;=\; 0. \qquad (2.191)
$$

We found this because $|0\rangle_{e,\mu}$ is not eigenstate of the free Hamiltonian H so one find:

$$
\lim_{V\to\infty}\;{}_{e,\mu}\langle 0\mid 0(t)\rangle_{e,\mu} = 0 \qquad (2.192)
$$

Thus, in the limit$V\to\infty$, at different times we have unitarily inequivalent flavor vacua. It is direct consequence of the fact that flavor states are not mass eigenstates, moreover, the unitarily inequivalence implies that we cannot directly compare flavor states at different times.
Let us now consider the flavor charge operators,in the Heisemberg representation we have:

$$
{}_{e,\mu}\langle 0|Q_{\nu_e}(t)|0\rangle_{e,\mu} ={}_{e,\mu}\langle 0|Q_{\nu_\mu}(t)|0\rangle_{e,\mu} = 0, \qquad (2.193)
$$

and the oscillation formula for the flavor charges are given by [7]:

$$
\begin{aligned}
\mathcal{Q}^{\mathbf{k}}_{\nu_e\to\nu_e}(t) &= \langle\nu^r_{\mathbf{k},e}|::Q_{\nu_e}(t)::|\nu^r_{\mathbf{k},e}\rangle \\
&= \left|\left\{\alpha^r_{\mathbf{k},\nu_e}(t),\alpha^{r\dagger}_{\mathbf{k},\nu_e}(0)\right\}\right|^2 + \left|\left\{\beta^{r\dagger}_{-\mathbf{k},\nu_e}(t),\alpha^{r\dagger}_{\mathbf{k},\nu_e}(0)\right\}\right|^2, \qquad (2.194)
\end{aligned}
$$

[21]We have also used the notation $G(\theta,\,t) = \prod_{\mathbf{p}}\prod_s G_{\mathbf{p},\,s}(\theta,\,t)$.

$$\begin{aligned}\mathcal{Q}^{\mathbf{k}}_{\nu_e\to\nu_\mu}(t) &= \langle \nu^r_{\mathbf{k},e}| :: Q_{\nu_\mu}(t) :: |\nu^r_{\mathbf{k},e}\rangle \\ &= \left|\left\{\alpha^r_{\mathbf{k},\nu_\mu}(t), \alpha^{r\dagger}_{\mathbf{k},\nu_e}(0)\right\}\right|^2 + \left|\left\{\beta^{r\dagger}_{-\mathbf{k},\nu_\mu}(t), \alpha^{r\dagger}_{\mathbf{k},\nu_e}(0)\right\}\right|^2 .\end{aligned} \tag{2.195}$$

Total charge is obviously conserved:

$$\mathcal{Q}^{\mathbf{k}}_{\nu_e\to\nu_e}(t) + \mathcal{Q}^{\mathbf{k}}_{\nu_e\to\nu_\mu}(t) = 1. \tag{2.196}$$

Explicitly equations (2.193), (2.194) are:

$$\mathcal{Q}^{\mathbf{k}}_{\nu_e\to\nu_e}(t) = 1 - \sin^2(2\theta)\left[|U_{\mathbf{k}}|^2\sin^2\left(\frac{\omega_{k,2}-\omega_{k,1}}{2}t\right) + |V_{\mathbf{k}}|^2\sin^2\left(\frac{\omega_{k,2}+\omega_{k,1}}{2}t\right)\right], \tag{2.197}$$

$$\mathcal{Q}^{\mathbf{k}}_{\nu_e\to\nu_\mu}(t) = \sin^2(2\theta)\left[|U_{\mathbf{k}}|^2\sin^2\left(\frac{\omega_{k,2}-\omega_{k,1}}{2}t\right) + |V_{\mathbf{k}}|^2\sin^2\left(\frac{\omega_{k,2}+\omega_{k,1}}{2}t\right)\right]. \tag{2.198}$$

2.6 Consideration on the oscillating formulas.

We conclude the chapter with some considerations about the oscillation formulas Eqs.(2.197),(2.198). These formulas are exact and have two differences with respect to the Pontecorvo formulas: the amplitudes are energy dependent, and there is an additional oscillating term. They have the sense of statistical averages. This is because thy present the structure of a many-body theory, where it has no sense to talk about single particle states. The approximate Pontecorvo result is recovered in the relativistic limit. Indeed for $|\mathbf{k}| \gg \sqrt{m_1 m_2}$ we have $|U_{\mathbf{k}}|^2 \longrightarrow 1$ and $|V_{\mathbf{k}}|^2 \longrightarrow 0$ and the results of [28] about oscillation are recovered.

2.7 Non-cyclic phases for neutrino oscillations in QFT

We now study the Aharonov-Anandan geometric invariant in the context of QFT. In this manner we will show a connection between two level systems that we have studied before and the neutrino mixing. The structure of the neutrino flavor vacuum has got multi particle components even in two flavor case. This component makes non cyclic time evolution associated to them. As we can see, the flavor state in the reference frame for which $\mathbf{k} = (0, 0, |\mathbf{k}|)$ are:

$$
\begin{aligned}
|\nu^r_{\mathbf{k},e}(t)\rangle &\equiv \alpha^{r\dagger}_{\mathbf{k},e}(t)|0(t)\rangle_{e,\mu} = e^{-i:H:t}|\nu^r_{\mathbf{k},e}(0)\rangle, \qquad (2.199)\\
&= e^{-i\omega_{k,1}t}\Big[\cos\theta\,\alpha^{r\dagger}_{\mathbf{k},1} + |U_{\mathbf{k}}|e^{-i\Omega^k_- t}\ \sin\theta\ \alpha^{r\dagger}_{\mathbf{k},2}\\
&- \epsilon^r\ |V_{\mathbf{k}}|e^{-i\Omega^k_+ t}\ \sin\theta\ \alpha^{r\dagger}_{\mathbf{k},1}\alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}\Big]\ G^{-1}_{\mathbf{k},s\neq r}(\theta,t)\prod_{\mathbf{p}\neq\mathbf{k}} G^{-1}_{\mathbf{p}}(\theta,t)|0\rangle_{1,2}\,,
\end{aligned}
$$

$$
\begin{aligned}
|\nu^r_{\mathbf{k},\mu}(t)\rangle &\equiv \alpha^{r\dagger}_{\mathbf{k},\mu}(t)|0(t)\rangle_{e,\mu} = e^{-i:H:t}|\nu^r_{\mathbf{k},\mu}(0)\rangle, \qquad (2.200)\\
&= e^{-i\omega_{k,2}t}\Big[\cos\theta\,\alpha^{r\dagger}_{\mathbf{k},2} - |U_{\mathbf{k}}|e^{i\Omega^k_- t}\ \sin\theta\ \alpha^{r\dagger}_{\mathbf{k},1}\\
&+ \epsilon^r\ |V_{\mathbf{k}}|e^{-i\Omega^k_+ t}\ \sin\theta\ \alpha^{r\dagger}_{\mathbf{k},1}\alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},2}\Big]\ G^{-1}_{\mathbf{k},s\neq r}(\theta,t)\prod_{\mathbf{p}\neq\mathbf{k}} G^{-1}_{\mathbf{p}}(\theta,t)|0\rangle_{1,2}\,,
\end{aligned}
$$

where $\Omega^k_+ \equiv \omega_{k,2} + \omega_{k,1}$, $\Omega^k_- \equiv \omega_{k,2} - \omega_{k,1}$, and

$$
\begin{aligned}
:H: &= H -_{1,2}\langle 0|H|0\rangle_{1,2} = H + 2\int d^3\mathbf{k}\,\Omega^k_+ \qquad (2.201)\\
&= \sum_i\sum_r\int d^3\mathbf{k}\,\omega_{k,i}[\alpha^{r\dagger}_{\mathbf{k},i}\alpha^r_{\mathbf{k},i} + \beta^{r\dagger}_{\mathbf{k},i}\beta^r_{\mathbf{k},i}]\ ,
\end{aligned}
$$

is the Hamiltonian normal ordered with respect to the vacuum $|0\rangle_{1,2}$. the multi-particle components disappear in the relativistic limit $|\mathbf{k}| \gg \sqrt{m_1 m_2}$, where $|U_{\mathbf{k}}|^2 \to 1$ and $|V_{\mathbf{k}}|^2 \to 0$ and the quantum mechanical Pontecorvo's states are recovered.
By formulas (2.199) and (2.200) we see that the non-cyclic time evolution of mixed neutrino states is due to the presence of two oscillation frequencies, namelyΩ_+ and Ω_+. The Berry like phase studied before is not applicable in the QFT mixing formalism, since quantities like $\langle\nu_\sigma(t)|\nu_\sigma(t')\rangle$ are zero for $t \neq t'$ in the infinite volume limit [38]. On the contrary the Aharonov Anandan invariant defined in [12] is suitable for the present case because it is well defined in the case of non-cyclic transformation in time and do not have products of states at different times.
Let us now consider the quantity:

$$
s_{\sigma,\tau}(t) = 2\int_0^t \Delta E_{\sigma,\tau}\,dt\ , \qquad (2.202)
$$

where $\Delta E \equiv \Delta E^r_{\mathbf{k}}$ and σ, τ are labels used to compute the uncertainties $\Delta E_{\sigma,\tau}$ in the above mentioned integrals. Computing $\Delta E_{\sigma,\sigma}$ with $\sigma = e, \mu$ by using $: H :$ we have:

$$\Delta E^2_{\sigma,\sigma} = \langle \nu^r_{\mathbf{k},\sigma}(t)|\,(: H :)^2\,|\nu^r_{\mathbf{k},\sigma}(t)\rangle - \langle \nu^r_{\mathbf{k},\sigma}(t)|: H :|\nu^r_{\mathbf{k},\sigma}(t)\rangle^2, \quad \sigma = e, \mu. \tag{2.203}$$

Using Eqs.(G.1), (G.3), and (G.2), (G.4), we obtain

$$\Delta E^2_{e,e} = \sin^2\theta\,\cos^2\theta\,\left[(\Omega^k_-)^2 + 4\,\omega_{k,1}\omega_{k,2}|V_{\mathbf{k}}|^2\right] + 4\,\omega^2_{k,1}\,\sin^4\theta\,|U_{\mathbf{k}}|^2|V_{\mathbf{k}}|^2, \tag{2.204}$$

$$\Delta E^2_{\mu,\mu} = \sin^2\theta\,\cos^2\theta\,\left[(\Omega^k_-)^2 + 4\,\omega_{k,1}\omega_{k,2}|V_{\mathbf{k}}|^2\right] + 4\,\omega^2_{k,2}\,\sin^4\theta\,|U_{\mathbf{k}}|^2|V_{\mathbf{k}}|^2. \tag{2.205}$$

$\Delta E_{e,\mu}$ in QFT is given by

$$\Delta E_{e,\mu} = \langle \nu^r_{\mathbf{k},e}(t)|: H :|\nu^r_{\mathbf{k},\mu}(t)\rangle = \langle \nu^r_{\mathbf{k},\mu}(t)|: H :|\nu^r_{\mathbf{k},e}(t)\rangle = \Omega^k_-\,\sin\theta\,\cos\theta\,|U_{\mathbf{k}}|\,. \tag{2.206}$$

Defining, at time t, the multi-particle flavor states (their explicit expressions are given in Appendix F):

$$|\nu^r_{\mathbf{k},e\bar{e}\mu}(t)\rangle \equiv \alpha^{r\dagger}_{\mathbf{k},e}(t)\,\beta^{r\dagger}_{-\mathbf{k},e}(t)\,\alpha^{r\dagger}_{\mathbf{k},\mu}(t)\,|0(t)\rangle_{e,\mu}\,, \tag{2.207}$$

$$|\nu^r_{\mathbf{k},\mu\bar{\mu}e}(t)\rangle \equiv \alpha^{r\dagger}_{\mathbf{k},\mu}(t)\,\beta^{r\dagger}_{-\mathbf{k},\mu}(t)\,\alpha^{r\dagger}_{\mathbf{k},e}(t)\,|0(t)\rangle_{e,\mu}\,, \tag{2.208}$$

we have also the following non-zero expectation values:

$$\Delta E_{\mu\bar{e}e,e} = \langle \nu^r_{\mathbf{k},\mu\bar{e}e}(t)|: H :|\nu^r_{\mathbf{k},e}(t)\rangle, \quad \Delta E_{e\bar{\mu}\mu,e} = \langle \nu^r_{\mathbf{k},e\bar{\mu}\mu}(t)|: H :|\nu^r_{\mathbf{k},e}(t)\rangle, \tag{2.209}$$

$$\Delta E_{\mu\bar{e}e,\mu} = \langle \nu^r_{\mathbf{k},\mu\bar{e}e}(t)|: H :|\nu^r_{\mathbf{k},\mu}(t)\rangle, \quad \Delta E_{e\bar{\mu}\mu,\mu} = \langle \nu^r_{\mathbf{k},e\bar{\mu}\mu}(t)|: H :|\nu^r_{\mathbf{k},\mu}(t)\rangle, \tag{2.210}$$

whose explicit expressions are given in Appendix G.
Now note that $\Delta E^2_{e,e}$ and $\Delta E^2_{\mu,\mu}$ can be also obtained as follows

$$\Delta E^2_{e,e} = \Delta E^2_{e,\mu} + \Delta E^2_{\mu\bar{e}e,e} + \Delta E^2_{e\bar{\mu}\mu,e}\,, \tag{2.211}$$

$$\Delta E^2_{\mu,\mu} = \Delta E^2_{e,\mu} + \Delta E^2_{\mu\bar{e}e,\mu} + \Delta E^2_{e\bar{\mu}\mu,\mu}\,. \tag{2.212}$$

Eqs.(2.211), (2.212) represent a generalization of the relations (1.85), (1.86) to the case of QFT flavor states; they take into account the multiparticle

components due to the condensate structure of the flavor vacuum.
Explicitly the expressions for the various Aharonov Anandan invariant $s_{\sigma,\tau}$, with $\sigma,\tau = e, \mu, e\bar{\mu}\mu, \mu\bar{e}e$ are given by:

$$s_{e,e}(t) \;=\; 2\,t\,\sin\theta\,\sqrt{\cos^2\theta\,\left[(\Omega^k_-)^2 + 4\,\omega_{k,1}\omega_{k,2}|V_{\mathbf{k}}|^2\right] + 4\,\omega^2_{k,1}\,\sin^2\theta\,|U_{\mathbf{k}}|^2|V_{\mathbf{k}}|^2}\,, \tag{2.213}$$

$$s_{\mu,\mu}(t) \;=\; 2\,t\,\sin\theta\,\sqrt{\cos^2\theta\,\left[(\Omega^k_-)^2 + 4\,\omega_{k,1}\omega_{k,2}|V_{\mathbf{k}}|^2\right] + 4\,\omega^2_{k,2}\,\sin^2\theta\,|U_{\mathbf{k}}|^2|V_{\mathbf{k}}|^2}\,, \tag{2.214}$$

$$s_{e,\mu}(t) \;=\; \Omega^k_-\,t\,\sin 2\theta\,|U_{\mathbf{k}}|\,, \tag{2.215}$$
$$s_{\mu\bar{e}e,e}(t) \;=\; s_{e\bar{\mu}\mu,\mu}(t) \;=\; \epsilon^r\,\Omega^k_+\,t\,\sin 2\theta\,|V_{\mathbf{k}}|\,, \tag{2.216}$$
$$s_{e\bar{\mu}\mu,e}(t) = 4\,\epsilon^r\,\omega_{k,1}\,t\,\sin^2\theta\,|U_{\mathbf{k}}|\,|V_{\mathbf{k}}|\,, \tag{2.217}$$
$$s_{\mu\bar{e}e,\mu}(t) \;=\; -4\,\epsilon^r\,\omega_{k,2}\,t\,\sin^2\theta\,|U_{\mathbf{k}}|\,|V_{\mathbf{k}}|\,. \tag{2.218}$$

From Eqs.(2.214)-(2.218) we see that in the relativistic limit, $\mathbf{k} \gg \sqrt{m_1 m_2}$, where $|V_{\mathbf{k}}| \to 0$, $|U_{\mathbf{k}}| \to 1$, we have

$$s_{\mu\bar{e}e,e} = s_{e\bar{\mu}\mu,e} = s_{\mu\bar{e}e,\mu} = s_{e\bar{\mu}\mu,\mu} = 0. \tag{2.219}$$

When we are in that limit, from Appendix C and Eqs. (2.204), (2.205), we have

$$\Delta E_{e,e} = \Delta E_{\mu,\mu} = \Delta E_{e,\mu} = \Omega^k_-\,\sin\theta\,\cos\theta\,. \tag{2.220}$$

In particular, if the time t is set $t = 2n\pi/\Omega^k_-$, the quantum mechanical result is consistently recovered and the geometric invariants

$$s_{e,e} = s_{\mu,\mu} = s_{e,\mu} = 2n\pi\sin 2\theta \tag{2.221}$$

coincide with the one given in Eq.(1.89). Since $|0\rangle_{1,2}$ and $|0\rangle_{e,\mu}$ are unitary inequivalent states in the infinite volume limit, two different normal orderings must be defined, one with respect to the vacuum $|0\rangle_{1,2}$ for fields with definite masses, as usual denoted by : ... :, and one with respect to the vacuum for fields with definite flavor $|0\rangle_{e,\mu}$, denoted by :: ... :: . $\Delta E_{\sigma,\tau}$ can be then computed by using : H : as done above or with :: H ::. The Hamiltonian normal ordered with respect to the vacuum $|0\rangle_{e,\mu}$ is given by

$$::H:: \equiv H \,-\, {}_{e,\mu}\langle 0|H|0\rangle_{e,\mu} \,=\, H \,+\, 2\int d^3\mathbf{k}\,\Omega^k_+\,(1 - 2\,|V_{\mathbf{k}}|^2\sin^2\theta)\,. \tag{2.222}$$

Taking into account the expectation values of :: H :: on the flavor states given in Appendix G, we have:

$$\Delta E_{e,\mu} \;=\; \langle\nu^r_{\mathbf{k},e}(t)|:H:|\nu^r_{\mathbf{k},\mu}(t)\rangle \;=\; \langle\nu^r_{\mathbf{k},e}(t)|::H::|\nu^r_{\mathbf{k},\mu}(t)\rangle\,. \tag{2.223}$$

On the other hand, defining the uncertainties $\Delta\widetilde{E}_{\sigma,\sigma}$ as

$$\Delta\widetilde{E}^2_{\sigma,\sigma} = \langle\nu^r_{\mathbf{k},\sigma}(t)|(:: H ::)^2|\nu^r_{\mathbf{k},\sigma}(t)\rangle - \langle\nu^r_{\mathbf{k},\sigma}(t)| :: H :: |\nu^r_{\mathbf{k},\sigma}(t)\rangle^2, \qquad \sigma = e, \mu\,, \tag{2.224}$$

and by using the relations in Appendix G, we have:

$$\Delta\widetilde{E}^2_{e,e} = \Delta E^2_{e,e}\,, \tag{2.225}$$
$$\Delta\widetilde{E}^2_{\mu,\mu} = \Delta E^2_{\mu,\mu}\,, \tag{2.226}$$

that means that, $\Delta E^2_{\sigma,\sigma}$ are independent of the normal ordering used, $: H :$ or $:: H ::$.
We can also see that by comparing the expectation values of $: H :$ and $:: H ::$ presented in Appendix G, we obtain that: $\Delta E_{e,\mu}$, $\Delta E_{\mu\bar{e}e,e}$, $\Delta E_{e\bar{\mu}\mu,e}$, $\Delta E_{\mu\bar{e}e,\mu}$, $\Delta E_{e\bar{\mu}\mu,\mu}$ are also independent of the particular normal ordering used. This implies that Eqs.(2.213)-(2.214) are normal ordering independent.

2.7.1 Discussion and Conclusions

We now conclude the argument treated above resuming some links obtained by discussing the above paragraph, and giving some further comments.
Eqs.(2.213) - (2.218) are the generalization of Eq.(2.221) that coincide with Eq.(1.89). Indeed quantum mechanics neutrinos can be seen as a particular two level system so what we have seen in general about them in quantum mechanic can be extended to quantum field theory.

Let me now define the operator:

$$\begin{aligned} H'(t) \;\equiv\; & \sum_r \int d^3\mathbf{k} \Big[\omega_{ee} \Big(\alpha^{r\dagger}_{\mathbf{k},e}(t)\alpha^r_{\mathbf{k},e}(t) \,+\, \beta^{r\dagger}_{-\mathbf{k},e}(t)\beta^r_{-\mathbf{k},e}(t)\Big) \\ +\;& \omega_{\mu\mu} \Big(\alpha^{r\dagger}_{\mathbf{k},\mu}(t)\alpha^r_{\mathbf{k},\mu}(t) + \beta^{r\dagger}_{-\mathbf{k},\mu}(t)\beta^r_{-\mathbf{k},\mu}(t)\Big) \\ +\;& \omega_{\mu e}\Big(\alpha^{r\dagger}_{\mathbf{k},e}(t)\alpha^r_{\mathbf{k},\mu}(t) \,+\, \alpha^{r\dagger}_{\mathbf{k},\mu}(t)\alpha^r_{\mathbf{k},e}(t) \\ +\;& \beta^{r\dagger}_{-\mathbf{k},e}(t)\beta^r_{-\mathbf{k},\mu}(t) \,+\, \beta^{r\dagger}_{-\mathbf{k},\mu}(t)\beta^r_{-\mathbf{k},e}(t)\Big)\Big]. \end{aligned} \tag{2.227}$$

$$\omega_{ee} \equiv \omega_{k,1}\, \cos^2\theta \,+ \omega_{k,2}\, \sin^2\theta, \tag{2.228}$$
$$\omega_{\mu\mu} \equiv \omega_{k,1}\, \sin^2\theta \,+ \omega_{k,2}\, \cos^2\theta \tag{2.229}$$

and

$$\omega_{\mu e} \equiv \Omega^k_-\, \sin\theta\cos\theta. \tag{2.230}$$

We also have:

$$\begin{aligned}
\langle\nu^r_{\mathbf{k},e}(t)|\, H'(t)\,|\nu^r_{\mathbf{k},e}(t)\rangle &= \omega_{k,1}\cos^2\theta \,+\, \omega_{k,2}\sin^2\theta\,, &(2.231)\\
\langle\nu^r_{\mathbf{k},\mu}(t)|\, H'(t)\,|\nu^r_{\mathbf{k},\mu}(t)\rangle &= \omega_{k,1}\sin^2\theta \,+\, \omega_{k,2}\cos^2\theta\,, &(2.232)\\
\langle\nu^r_{\mathbf{k},e}(t)|\, H'(t)\,|\nu^r_{\mathbf{k},\mu}(t)\rangle &= \Omega^k_-\sin\theta\cos\theta\,, &(2.233)
\end{aligned}$$

$$\begin{aligned}
&\langle\nu^r_{\mathbf{k},\mu\bar{e}e}(t)|\, H'(t)\,|\nu^r_{\mathbf{k},e}(t)\rangle \,=\, \langle\nu^r_{\mathbf{k},\mu\bar{e}e}(t)|\, H'(t)\,|\nu^r_{\mathbf{k},\mu}(t)\rangle \,=\\
&=\, \langle\nu^r_{\mathbf{k},e\bar{\mu}\mu}(t)|\, H'(t)\,|\nu^r_{\mathbf{k},e}(t)\rangle \,=\, \langle\nu^r_{\mathbf{k},e\bar{\mu}\mu}(t)|\, H'(t)\,|\nu^r_{\mathbf{k},\mu}(t)\rangle = 0\,. \qquad (2.234)
\end{aligned}$$

The above expectation values tell to us that contributions from the flavor vacuum condensate have been eliminated.
Indeed, Eqs.(2.231) and (2.232) coincide with Eqs.(1.15), and (1.16) derived in the two level system case. Moreover the uncertainties in the energy $H'(t)$ of the multi-particle states (2.207), (2.208) are zero, like for QM two level systems.
Finally we can define the invariant:

$$s'_e = s'_\mu = 2\int_0^{nT} \Delta E'\, dt \,=\, 2n\pi\sin 2\theta\,, \qquad (2.235)$$

where

$$T = 2n\pi/\Omega^k_- \qquad (2.236)$$

and

$$\begin{aligned}
\Delta E'^2_{e,e} = \Delta E'^2_{\mu,\mu} &= \Delta E'^2_{e,\mu} = \langle\nu^r_{\mathbf{k},\sigma}(t)|\, H'^2(t)\,|\nu^r_{\mathbf{k},\sigma}(t)\rangle - \langle\nu^r_{\mathbf{k},\sigma}(t)|\, H'(t)\,|\nu^r_{\mathbf{k},\sigma}(t)\rangle^2\\
&= \langle\nu^r_{\mathbf{k},e}(t)|\, H'(t)\,|\nu^r_{\mathbf{k},\mu}(t)\rangle^2 \,=\, (\Omega^k_-)^2\sin^2\theta\,\cos^2\theta\,, \qquad (2.237)
\end{aligned}$$

with $\sigma = e, \mu$.

Chapter 3

Particle mixing and connection with dark energy.

The aim of this chapter is to show that the non-perturbative vacuum structure associated with neutrino and quark mixing leads to a non–zero contribution to the value of the dark energy. At the present epoch, contributions compatible with the evaluated upper bound on the dark energy come from vacuum condensates due to particle mixing with adiabatic index close to -1.
An increasing bulk of data [40]-[41] has been accumulated in the last few years and indicates that the geometry of the universe is spatially flat and that it is in a phase of accelerated expansion. This accelerated expansion is thought to be induced by *dark energy* theorized as a non-clustered fluid with negative pressure. This picture have acquired many strengths from more precise measurements of the CMBR spectrum, due to the WMAP experiment [42] and by the extension of the SNeIa Hubble diagram to redshifts higher than 1 [43].
In this chapter we show that particle mixing might contribute to the dark energy budget of the universe [44, 45, 46, 47] and establish a link between cosmology and particle physics. In particular we will analyze the possible contribution to dark energy due to the vacuum condensation of neutrino and quark mixing in two regimes of the currently adopted scheme in literature [48]:
A) the regime of the matter dominated universe with adiabatic index $w = p/\rho$ ranging between 0 and 1/3; and
B) the present epoch regime, dark energy dominated universe, with $w \simeq -1$.
We find that, at the present epoch, $w \simeq -1$ imposes constraints on the vacuum condensate leading to dark energy values compatible with those inferred from observations.
First we will discuss the particle mixing condensate in the early and in the

present epoch, second we present explicit computations of the mixing contributions to the dark energy at the present epoch.
We outline the QFT formalism for fermion mixing [49, 50, 51, 52, 53, 54, 55] in Appendix F and in Appendix G a useful computation is reported . The

contribute of dark energy component to the total matter-energy density is $\Omega_\Lambda \simeq 0.7$. So, physically motivated cosmological models should undergo, at least, three phases: an early accelerated inflationary phase, an intermediate standard matter dominated (decelerated) phase and a final, today observed, dark energy dominated (accelerated) phase.
In other words we have to take into account some form of *dark energy* which evolves from early epochs inducing the today observed acceleration.
We put now our attention to the the energy density due to the vacuum condensate arising from particle mixing because it gives a contribution to the vacuum energy which evolves **dynamically**. We will also consider the contribution due the quark mixing condensate to complete the analysis. The calculation here presented is performed for Dirac fermion fields in a Minkowski space-time. It can be extended to curved space-times, as it is showed in many papers that will be resumed in a forthcoming review.
The energy-momentum tensor density $\mathcal{T}_{\mu\nu}(x)$ for the fermion fields ψ_i, $i = 1, 2, 3$ [33], is

$$:\mathcal{T}_{\mu\nu}(x) := \frac{i}{2} : \left(\bar{\Psi}_m(x)\gamma_\mu \overleftrightarrow{\partial}_\nu \Psi_m(x)\right) : \tag{3.1}$$

which can be written as [56]

$$\begin{aligned} :\mathcal{T}_{\mu\nu}(x): &= :\Sigma_{\mu\nu}(x): + :V_{\mu\nu}(x): \\ &= :\left\{\frac{i}{2}\left(\bar{\Psi}_m(x)\gamma_\mu \overleftrightarrow{\partial}_\nu \Psi_m(x)\right) - \eta_{\mu\nu}\left[\frac{i}{2}\bar{\Psi}_m(x)\gamma^\alpha \overleftrightarrow{\partial}_\alpha \Psi_m(x)\right]\right. \\ &+ \left.\eta_{\mu\nu}\left[\bar{\Psi}_m(x)\,\mathsf{M}_d\,\Psi_m(x)\right]\right\}:, \end{aligned} \tag{3.2}$$

where

$$:\Sigma_{\mu\nu}(x): = :\left\{\frac{i}{2}\left(\bar{\Psi}_m(x)\gamma_\mu \overleftrightarrow{\partial}_\nu \Psi_m(x)\right) - \eta_{\mu\nu}\left[\frac{i}{2}\bar{\Psi}_m(x)\gamma^\alpha \overleftrightarrow{\partial}_\alpha \Psi_m(x)\right]\right\}:, \tag{3.3}$$

$$:V_{\mu\nu}(x): = \eta_{\mu\nu} : \left[\bar{\Psi}_m(x)\,\mathsf{M}_d\,\Psi_m(x)\right]:, \tag{3.4}$$

$\mathsf{M}_d = diag(m_1, m_2, m_3)$ [1] and $\Psi_m = (\psi_1, \psi_2, \psi_3)^T$.
We recall that $\mathcal{T}_{00}$ and $\mathcal{T}_{jj}$ do not depend on time in the Minkowski metric.

[1] m_1, m_2, m_3 are the quarks's masses.

The energy momentum tensor density of the vacuum condensate is given by

$$\mathcal{T}^{cond}_{\mu\nu}(x) = {}_f\langle 0(t)| : \mathcal{T}_{\mu\nu}(x) : |0(t)\rangle_f \,, \tag{3.5}$$

where $|0(t)\rangle_f$ is the vacuum for the flavor fields and the normal ordering is with respect to the vacuum $|0\rangle_m$ for the massive fields (see Appendix F).

As said above, we consider two cases:

1_{st}) The early universe epochs,where the vacuum is not required to be space-time invariant since Lorentz invariance is broken [57]. Thus the vacuum expectation values of the energy momentum tensor density in such epochs may be space-time dependent, i.e.

$$\mathcal{T}^{cond}_{\mu\nu}(x) = \frac{i}{2}\, {}_f\langle 0(t)| : \Big(\bar{\Psi}_m(x)\gamma_\mu \overleftrightarrow{\partial}_\nu \Psi_m(x)\Big) : |0(t)\rangle_f \,. \tag{3.6}$$

2_{nd}) The present epoch where $\mathcal{T}_{\mu\nu}(x)$ is space - time independent because the breaking of the Lorentz invariance is very small (negligible). Thus, in this case (present epoch), terms in $\mathcal{T}_{\mu\nu}(x)$ carrying space-time derivatives $\partial_\mu \sim k_\mu = (\omega_k, k_j)$, must give vanishing contributions to the vacuum expectation values. Then, in the present epoch, for the kinematical part $\Sigma^{cond}_{\mu\nu}$ of $\mathcal{T}^{cond}_{\mu\nu}$ we have

$$\Sigma^{cond}_{\mu\nu} \;=\; {}_f\langle 0(t)| : \Sigma_{\mu\nu}(x) : |0(t)\rangle_f \;\simeq 0 \tag{3.7}$$

and the energy momentum tensor density of the vacuum condensate is given by:

$$\mathcal{T}^{cond}_{\mu\nu} \simeq \; {}_f\langle 0(t)| : V_{\mu\nu}(x) : |0(t)\rangle_f \;=\; \eta_{\mu\nu}\; {}_f\langle 0(t)| : \bar{\Psi}_m(x)\, \mathsf{M}_d\, \Psi_m(x) : |0(t)\rangle_f \,. \tag{3.8}$$

In the following we compute the contributions ρ_{mix} and p_{mix} to the vacuum energy density and to the vacuum pressure in the 1_{st} and 2_{nd} case.

3.1 Early universe epochs

We now focus our attention on the Early universe epochs; the contribution ρ_{mix} of the particle mixing to the vacuum energy density is given by the $(0, 0)$ component of the energy momentum tensor density of the vacuum condensate given in Eq.(3.6):

$$\rho_{mix} \equiv \frac{1}{V}\,\eta^{00}\int d^3x\;\mathcal{T}^{cond}_{00}(x)\;. \tag{3.9}$$

In terms of the annihilation and creation operators of ψ_1, ψ_2 and ψ_3, we have

$$\rho_{mix} \;=\; \sum_{i,r}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\omega_{k,i}\Big({}_f\langle 0|\alpha^{r\dagger}_{\mathbf{k},i}\alpha^{r}_{\mathbf{k},i}|0\rangle_f + {}_f\langle 0|\beta^{r\dagger}_{\mathbf{k},i}\beta^{r}_{\mathbf{k},i}|0\rangle_f\Big)\,, \qquad i=1,2,3\,. \tag{3.10}$$

Introducing the cut-off K, Eq.(3.10) becomes

$$\rho_{mix} \;=\; \frac{2}{\pi}\sum_i\int_0^K dk\;k^2\;\omega_{k,i}\;\mathcal{N}^{\mathbf{k}}_i\,, \qquad i=1,2,3\,. \tag{3.11}$$

Here $\omega_{k,i}=\sqrt{k^2+m_i^2}$ and $\mathcal{N}^{\mathbf{k}}_i$ are the numbers of particles condensed in the vacuum given in Appendix H.

The contribution p_{mix} of particle mixing to the vacuum pressure is given by the (j,j) component of the energy momentum tensor density of the vacuum condensate given in Eq.(3.6):

$$p_{mix} \;=\; -\frac{1}{V}\,\eta^{jj}\int d^3x\;\mathcal{T}^{cond}_{jj}(x)\;. \tag{3.12}$$

where no summation on the index j is intended. We have

$$p_{mix} \;=\; -\eta^{jj}\sum_{i,r}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{k_jk_j}{\omega_{k,i}}\Big({}_f\langle 0|\alpha^{r\dagger}_{\mathbf{k},i}\alpha^{r}_{\mathbf{k},i}|0\rangle_f + {}_f\langle 0|\beta^{r\dagger}_{\mathbf{k},i}\beta^{r}_{\mathbf{k},i}|0\rangle_f\Big)\,, \tag{3.13}$$

with $i=1,2,3$. If we have the isotropy of the vacuum condensate momenta we can write: $\mathcal{T}^{cond}_{11}=\mathcal{T}^{cond}_{22}=\mathcal{T}^{cond}_{33}$, then:

$$p_{mix} \;=\; \frac{2}{3\,\pi}\sum_i\int_0^K dk\,k^2\,\frac{k^2}{\omega_{k,i}}\,\mathcal{N}^{\mathbf{k}}_i\,. \tag{3.14}$$

Considering Eqs.(3.11) and (3.14) one obtains the adiabatic index w_{mix} of the particle mixing condensate, $w_{mix}\equiv p_{mix}/\rho_{mix}$ in function of the momentum cut-off K.

This means that in the case considered now, the condensate "have got the behavior" of a perfect fluid of dust and radiation at the extreme values of the cut-off. To be more precise,it behaves as radiation in the relativistic regime ($w_{mix} \simeq 1/3$) and as dark matter in the non-relativistic regime ($w_{mix} \simeq 0$). So, in the early Universe and in the regions in which the breaking of Lorentz invariance of the vacuum is not negligible, the condensate could give rise to the dark matter component of the Universe and also the particle mixing condensate does not give contributions to the "standard" dark energy (the adiabatic index w_{mix} assumes in such epoch, as we said, values in the range $0 \leq w_{mix} \leq 1/3$). This gives the possibility to achieve the large scale structure formation as requested in a standard matter-radiation dominated regime and is in complete agreement with the WMAP results [57].

3.2 Universe at present epoch

Estimates from WMAP data show that the current universe consists of 5% of ordinary matter, 23% dark matter, 72% dark energy [57]. Lorentz invariance of the vacuum is now unbroken, then the energy momentum tensor density of the vacuum condensate comes almost completely from space-time independent condensate contributions and from Eq.(3.8), we have:

$$\eta_{\mu\nu} \sum_i m_i \int \frac{d^3x}{(2\pi)^3}\, {}_f\langle 0| : \bar{\psi}_i(x)\psi_i(x) : |0\rangle_f \; = \eta_{\mu\nu}\, \rho_{mix}. \tag{3.15}$$

Since the vacuum condensate is homogeneous and isotropic and its energy-momentum tensor density is given by

$$\mathcal{T}^{cond}_{\mu\nu} = diag(\rho_{mix}\,, p_{mix}\,, p_{mix}\,, p_{mix}\,), \tag{3.16}$$

by comparing this expression with Eq.(3.15) and using $\eta_{\mu\nu} = diag(1, -1, -1, -1)$, we obtain the state equation: $\rho_{mix} \simeq -p_{mix}$, i.e. the adiabatic index at the present epoch is

$$w_{mix} \;=\; p_{mix}/\rho_{mix} \;\simeq\; -1\;. \tag{3.17}$$

We can see that the vacuum condensate, coming from particle mixing, at the present epoch behaves as a cosmological constant [58].

ρ_{mix} computed from Eq.(3.15) thus turns out to be

$$\rho_{mix} \;\simeq\; \frac{2}{\pi} \sum_i \int_0^K dk\, k^2\, \frac{m_i^2}{\omega_{k,i}}\, \mathcal{N}_i^{\mathbf{k}}\,, \tag{3.18}$$

which, by using Eqs.(H.5), (H.6), (H.7) in Appendix H, can be written as

$$\begin{aligned}\rho_{mix} &\simeq \frac{2}{\pi}\int_0^K dk\, k^2 \Big\{ \frac{m_1^2}{\omega_{k,1}} \left(s_{12}^2 c_{13}^2\, |V_{12}^{\mathbf{k}}|^2 + s_{13}^2\, |V_{13}^{\mathbf{k}}|^2\right) \\ &+ \frac{m_2^2}{\omega_{k,2}} \left[\left(s_{12}^2 c_{23}^2 + c_{12}^2 s_{23}^2 s_{13}^2\right) |V_{12}^{\mathbf{k}}|^2 + s_{23}^2 c_{13}^2\, |V_{23}^{\mathbf{k}}|^2\right] \\ &+ \frac{m_3^2}{\omega_{k,3}} \left[\left(c_{12}^2 s_{23}^2 + s_{12}^2 c_{23}^2 s_{13}^2\right) |V_{23}^{\mathbf{k}}|^2 + \left(s_{12}^2 s_{23}^2 + c_{12}^2 c_{23}^2 s_{13}^2\right) |V_{13}^{\mathbf{k}}|^2\right] \Big\} \\ &- \frac{4}{\pi} s_{12} c_{23} c_{12} s_{23} s_{13} c_\delta \int_0^{K_\Lambda} dk\, k^2 \Big\{ \frac{m_2^2}{\omega_{k,2}}\, |V_{12}^{\mathbf{k}}|^2 + \frac{m_3^2}{\omega_{k,3}} \left[|V_{23}^{\mathbf{k}}|^2 - |V_{13}^{\mathbf{k}}|^2\right] \Big\}\,,\end{aligned} \tag{3.19}$$

where $c_\delta = \cos\delta$. Note that ρ_{mix} also depends on the CP violating phase δ. Let us observe that the value of the integral is conditioned by the presence in the integrand of the $|V_{ij}^{\mathbf{k}}|^2$ factors.The integral, and thus ρ_{mix}, would be zero for $|V_{ij}^{\mathbf{k}}|^2 = 0$ for any $|\mathbf{k}|$. The $|V_{ij}^{\mathbf{k}}|^2$'s account for the vacuum condensate (Eqs. (H.5) - (H.7)) and $|V_{ij}^{\mathbf{k}}|^2$ goes to zero only for large momenta, getting its maximum value for $|\mathbf{k}| \approx \sqrt{m_i m_j}$ for any $i, j = 1, 2, 3$ [53]. We note that the integral (3.18) diverges in K as $m_i^4 \log(2K/m_j)$, with $i, j = 1, 2, 3$ [47].

3.3 Contribute of the particle mixing to the condensate at the present epoch.

Let us now show that the very small breaking of the Lorentz invariance of the flavor vacuum at the present epoch constrains the value of the cut-off on the momenta and consequently the value of the dark energy contributions due to the particle mixing.

Eq.(3.10) and the identity $\omega_{k,i} = \frac{k^2}{\omega_{k,i}} + \frac{m_i^2}{\omega_{k,i}}$ show that the energy density induced by the particle mixing condensate can be written as

$$\rho_{mix} = \Sigma_{mix} + V_{mix} \tag{3.20}$$

where the kinematic term Σ_{mix} and the potential term V_{mix} are respectively given by

$$\Sigma_{mix} = \frac{2}{\pi}\sum_i \int_0^K dk\, k^2\, \frac{k^2}{\omega_{k,i}}\, \mathcal{N}_i^{\mathbf{k}}\,, \tag{3.21}$$

and

$$V_{mix} = \frac{2}{\pi}\sum_i \int_0^K dk\, k^2\, \frac{m_i^2}{\omega_{k,i}}\, \mathcal{N}_i^{\mathbf{k}}\,. \tag{3.22}$$

By comparing Eqs.(3.18), (3.20) and (3.22), we observe that, at the present epoch, $\rho_{mix} \simeq V_{mix}$; in other words, $\Sigma_{mix} \ll V_{mix}$. The Lorentz invariance of the flavor vacuum imposes a very small value of the cut-off on the momenta as we can see:

$$K \ll \sqrt[3]{m_1 m_2 m_3}\,. \tag{3.23}$$

We consider the adiabatic expansion of a sphere of volume V. Let p denote the pressure at which the sphere expands. The total energy, $E = \rho\mathsf{V}$, is not conserved since the pressure does work.

We have: $dE = -p\, d\mathsf{V}$. That is $\rho\, d\mathsf{V} + \mathsf{V}\, d\rho = -p\, d\mathsf{V}$, that can be written as

$$d[(\rho + p)\mathsf{V}] = 0\,, \tag{3.24}$$

from which

$$\rho + p = \frac{const}{\mathsf{V}}\,. \tag{3.25}$$

Then if the volume is very large ($\mathsf{V} \to \infty$), that is in the bulk of the Universe, i.e. far from the Universe "boundaries", we have $\rho \simeq -p$ and the adiabatic index is $w = p/\rho \simeq -1$.

From Eq.(3.25) we have

$$\rho = \frac{const}{\mathsf{V}} - p. \tag{3.26}$$

Moreover, taking into account that the energy density can be written as

$$\rho = \Sigma + V \tag{3.27}$$

where Σ and V are the kinetic and the potential terms respectively, we have $\rho = \Sigma + V \simeq -p$ for a volume $\mathsf{V} \to \infty$. If we consider now the condition $\Sigma \ll V$ due to the very small breaking of the Lorentz invariance at the present epoch, we obtain $\rho \simeq V \simeq -p$. Using such a relation and by equating the two expressions of ρ: $\rho = \frac{const}{\mathsf{V}} - p = \Sigma + V$, we find that, for very large volume V, the kinematic term is approximatively given by

$$\Sigma \simeq \frac{const}{\mathsf{V}} \simeq 0\,. \tag{3.28}$$

From Eqs.(3.27) and (3.28), we have, at the present epoch,

$$\rho = \Sigma - p \simeq -p \tag{3.29}$$

. So in the case of the flavor vacuum condensate, the state equation we obtain at the present epoch is

$$w_{mix} = \frac{p_{mix}}{\Sigma_{mix} - p_{mix}} \,. \tag{3.30}$$

Note that

$$\Sigma_{mix} - p_{mix} \neq 0, \tag{3.31}$$

since $\Sigma_{mix} \ll p_{mix}$. Eq.(3.30) shows that, since at the present epoch $\Sigma_{mix} \to 0$, then $w_{mix} \to -1$. Moreover, since Σ_{mix} and p_{mix} are function of the cut-off on the momenta K, then Eq.(3.30) gives an expression of w_{mix} as function of K:

$$w_{mix} = w_{mix}(K). \tag{3.32}$$

We now estimate the contributions given to the dark energy by the particle mixing condensates for different values of w_{mix} close to -1, both for neutrino and for quark mixing condensates.

3.3.1 Neutrino mixing condensate contribution

Let Ψ_f represents the flavor neutrino fields: $\Psi_f^T = (\nu_e, \nu_\mu, \nu_\tau)$ and Ψ_m denotes the neutrino fields with definite masses, m_1, m_2, m_3:

$$\Psi_m^T = (\nu_1, \nu_2, \nu_3). \tag{3.33}$$

The experimental values of squared mass differences and mixing angles are respectively:

$$\Delta m_{12}^2 = 7.9 \times 10^{-5} eV^2, \tag{3.34}$$
$$\Delta m_{23}^2 = 2.3 \times 10^{-3} eV^2, \tag{3.35}$$
$$s_{12}^2 = 0.31, \tag{3.36}$$
$$s_{23}^2 = 0.44\,, \tag{3.37}$$
$$s_{13}^2 = 0.009\,. \tag{3.38}$$

If we consider the normal hierarchy case $|m_3| \gg |m_{1,2}|$, we consider values of the neutrino masses such that the experimental values of squared mass difference are satisfied, as for example:

$$m_1 = 4.6 \times 10^{-3} eV, \tag{3.39}$$
$$m_2 = 1 \times 10^{-2} eV, \tag{3.40}$$
$$m_3 = 5 \times 10^{-2} eV. \tag{3.41}$$

Then the condition Eq.(3.24) for neutrinos reads

$$K \ll 1.2 \times 10^{-2} eV\,. \tag{3.42}$$

We find that contributions to the dark energy compatible with its estimated upper bound, $\rho_{\nu-mix} \sim 10^{-47} GeV^4$, are obtained for values of the adiabatic index $w_{\nu-mix}$ of the neutrino mixing dark energy component:

$$-0.98 \leq w_{\nu-mix} \leq -0.97\,. \tag{3.43}$$

Eq.(3.42) is in agreement with the constraint on the equation of state of the dark energy given by the combination of WMAP and Supernova Legacy Survey (SNLS) data: $w = -0.967^{+0.073}_{-0.072}$ and with the constraint given by combining WMAP, large-scale structure and supernova data: $w = -1.08 \pm 0.12$ [60].
Values of $w_{\nu-mix} < -0.98$ leads to negligible contributions of $\rho_{\nu-mix}$. [2]

3.3.2 Quark mixing condensate contribution

The quark mixing is expressed as:

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}\,, \tag{3.44}$$

where $V = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$ is the CKM matrix [61]. For the values of the quark masses given in Ref.[61], we have:

$$K \ll 120 MeV\,. \tag{3.45}$$

We find that the exact Lorentz invariance of the quark mixing condensate $w_{q-mix} = -1$ (Σ_{q-mix} is 16 orders less than V_{q-mix}), at the present epoch, leads to a contribution to the dark energy that is compatible with its estimated upper bound: $\rho_{q-mix} = 1.5 \times 10^{-47} GeV^4$. We remark that very small deviations from the value $w_{q-mix} = -1$ give rise to contributions of ρ_{q-mix} that are beyond the accepted upper bound of the dark energy.

The computation of ρ_{mix} turns out to be sensible to small variations in the values of the particle masses and of Δm^2. [3]

[2] The results we found are dependent on the neutrino mass values one uses.
[3] Our results are therefore dependent on the mass values one uses.

Chapter 4

Conclusions

We summarize Briefly here the results obtained in the first chapter and then proceed to link them to the second chapter of this thesis. Lastly we comment the results obtained in chapter 3. As we have seen before a fundamental link between two level quantum systems and QFT fermion systems consists into the Aharonov Anandan invariant. As we have seen this geometrical invariant, instead of the Berry phase, can be both defined for quantum two level systems and for quantum fermion mixing particles; this result and the presence of an underlying gauge structure for the first one lead us to suppose that is possible to write a "quantum field two level theory"[1]. We stress although that is possible in QFT of neutrino mixing to define an hamiltonian $H'(t)$ (2.227) that leads us to value of the Aharonov Anandan invariant formally identical to the one obtained in for two level systems:

$$s'_e = s'_\mu = 2\int_0^{nT} \Delta E'\, dt\; = 2n\pi \sin 2\theta\,, \tag{4.1}$$

Concerning the chapter 3 instead, we emphasize that the structure of mixing implies a vacuum condensate that can give reason of the presence of dark energy. We reviewed briefly some results obtained about ω_{mix} both for early and for present epoch, and these seems to be in agreement with the evaluable experimental data that we have, especially those that comes from WMAP experiments. For the present epoch we calculate separately the two contributions due to neutrinos and to quark mixing.

[1]Here we mean a quantum field mathematical formalism that is independent from particular particle family (neutrinos, kaons....etc) as we do in a quantum mechanics framework; we will treat it in a further work.

Appendix A

Brief introduction to geometrical invariants

Consider a quantum system described by an Hamiltonian $\hat{H}$ and being in a stationary state. To be more precise we consider an eighenstate $|E_n\rangle$ of $\hat{H}$ or an evolved state

$$e^{-\frac{i}{\hbar}E_n t}|E_n\rangle. \tag{A.1}$$

The measurement on this state will always give the same result excepted for a variation of the dynamical phase, that is:

$$e^{-\frac{i}{\hbar}\Delta E_n t}. \tag{A.2}$$

where ΔE_n is the difference between two eigenvalues of the hamiltonian $\hat{H}$. Suppose that the hamiltonian has an external dependence by some parameters $\overrightarrow{R}(t) \equiv (\alpha_1(t), ...)$ that depends from the time too.
Suppose also that this parameters remain constant for a while, or in other words that the hamiltonian do not vary for a time interval.In this interval the time evolution of an eigenstate of the hamiltonian due to the Shrödinger equation is[1]:

$$\begin{aligned}
\hat{H}(\overrightarrow{R})|n(t)\rangle &= i\hbar\frac{\partial}{\partial t}|n(t)\rangle \Rightarrow \\
\Rightarrow E_n|n(t)\rangle &= i\hbar\frac{\partial}{\partial t}|n(t)\rangle \Rightarrow \\
\Rightarrow |n(t)\rangle &= e^{-\frac{i}{\hbar}\int_0^t(dt' E_n)}|n(0)\rangle\,.
\end{aligned} \tag{A.3}$$

[1]Remember that in this interval of time $\overrightarrow{R}$ do not vary so we write $|(n)(t)\rangle$ to put in evidence that we are considering the time evolution of the eigenstate due only to the Shrödinger equation

To be more general we have to consider the relative initial phase $e^{i\gamma_n}$ where the initial eigenstate $|n(0)\rangle$ could be to have finally

$$|n(t)\rangle = e^{-\frac{i}{\hbar}\int_0^t(dt' E_n)} e^{i\gamma_n}|n(0)\rangle \,. \tag{A.4}$$

If we now vary the parameters $\overrightarrow{R(t)}$ slowly in time, considering that even γ_n depends on $\overrightarrow{R(t)}$, by the adiabatic theorem we have that the system evolve in time from an eigenstate to another eigenstate of the hamiltonian.
So we have that in an adiabatic transformations of $\hat{H}(\overrightarrow{R}(t))$ in the parameter space the relative phases remain each other independent time after time. Varying $\overrightarrow{R}(t)$ from 0 to time t we have [2]:

$$|n(t)\rangle = e^{-\frac{i}{\hbar}\int_0^t\left(dt' E_{\overrightarrow{R_n}}(t)\right)} e^{i\gamma_n(\overrightarrow{R(t)})}|n(\overrightarrow{R}(t))\rangle \,, \tag{A.5}$$

and we have the following Shrödinger equation:

$$\begin{aligned}
\hat{H}(\overrightarrow{R(t)})|n(t)\rangle = i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle \Rightarrow \\
\Rightarrow \hat{H}(\overrightarrow{R(t)}) e^{-\frac{i}{\hbar}\int_0^t\left(dt' E_{\overrightarrow{R_n}}(t')\right)} e^{i\gamma_n(\overrightarrow{R(t)})}|(n\overrightarrow{R})(t)\rangle = \\
= i\hbar\frac{\partial}{\partial t}\left(e^{-\frac{i}{\hbar}\int_0^t\left(dt' E_{\overrightarrow{R_n}}(t')\right)} e^{i\gamma_n(\overrightarrow{R(t)})}|(n\overrightarrow{R})(t)\rangle.\right) \Rightarrow \\
\Rightarrow \hat{H}(\overrightarrow{R(t)})|(n\overrightarrow{R})(t)\rangle e^{-\frac{i}{\hbar}\int_0^t\left(dt' E_{\overrightarrow{R_n}}(t')\right)} e^{i\gamma_n(\overrightarrow{R(t)})} = \\
= i\hbar\left(-\frac{i}{\hbar}E_n(\overrightarrow{R}(t)) + i\dot{\gamma}_n(\overrightarrow{R}(t))\right)|n(t)\rangle + \\
i\hbar\left(e^{-\frac{i}{\hbar}\int_0^t\left(dt' E_{\overrightarrow{R_n}}(t')\right)} e^{i\gamma_n(\overrightarrow{R(t)})}|\nabla_R(n\overrightarrow{R})(t)\rangle.\right)\cdot\dot{\overrightarrow{R}}_n(t) \,.
\end{aligned} \tag{A.6}$$

Applying $\hat{H}(\overrightarrow{R}(t))$ on $|(n\overrightarrow{R})(t)\rangle$ we obtain:

$$\begin{aligned}
\hat{H}(\overrightarrow{R(t)})|(n\overrightarrow{R})(t)\rangle e^{-\frac{i}{\hbar}\int_0^t\left(dt' E_{\overrightarrow{R_n}}(t')\right)} e^{i\gamma_n(\overrightarrow{R(t)})} = \\
E_n(\overrightarrow{R(t)})|(n\overrightarrow{R})(t)\rangle e^{-\frac{i}{\hbar}\int_0^t\left(dt' E_{\overrightarrow{R_n}}(t')\right)} e^{i\gamma_n(\overrightarrow{R(t)})} = E_n(\overrightarrow{R(t)})|n(t)\rangle \Rightarrow
\end{aligned} \tag{A.7}$$

[2]Note that $|n(0)\rangle$ is an eigenstate of the hamiltonian at time 0, $|n(t)\rangle$ is only the time Shrödinger evolution of $|n(0)\rangle$ (no variation of $\overrightarrow{R}$ happened in time) and $|n(\overrightarrow{R}(t))\rangle$ is the eigenstate $|n(t)\rangle$ after the variation of $\overrightarrow{R}$ in time (no Shrödinger time evolution happened in this time for the eigenstate). Cause of the variation in time of $\overrightarrow{R}$ we also have $\gamma_n \to \gamma_n(\overrightarrow{R(t)})$ and $E_n \to E_{\overrightarrow{R_n}}(t)$, note that γ_n and E_n do not depends strictly on time or in other words do not vary for Srodinger time evolution.

$$0 = -\hbar\dot{\gamma}_n(\overrightarrow{R}(t))|n(t)\rangle + i\hbar\left(e^{-\frac{i}{\hbar}\int_0^t\left(dt' E_{\overrightarrow{R_n}}(t')\right)}e^{i\gamma_n(\overrightarrow{R(t)})}|\nabla_R(n\overrightarrow{R})(t)\rangle.\right)\cdot\dot{\overrightarrow{R}}_n(t) \Rightarrow$$
$$\Rightarrow \dot{\gamma}_n(\overrightarrow{R}(t))|\psi(t)\rangle = i\left(e^{-\frac{i}{\hbar}\int_0^t\left(dt' E_{\overrightarrow{R_n}}(t')\right)}e^{i\gamma_n(\overrightarrow{R(t)})}|\nabla_R(n\overrightarrow{R})(t)\rangle\right)\cdot\dot{\overrightarrow{R}}_n(t)\,. \tag{A.8}$$

Multiplying on the left by $\langle\psi(t)|$:

$$\dot{\gamma}_n(\overrightarrow{R}(t)) = \langle n\overrightarrow{R})(t)|\nabla_R(n\overrightarrow{R})(t)\rangle\cdot\dot{\overrightarrow{R}}_n(t)\,. \tag{A.9}$$

Integrating (A.9) from $t = 0$ to $t = T$, where T is the time the needed to let the parameters $\overrightarrow{R}(t)$ turn in their initial values, we have:

$$\begin{aligned}\gamma_n(T) &= i\int_0^T dt\cdot\dot{\overrightarrow{R}}_n(t)\langle n\overrightarrow{R})(t)|\nabla_R(n\overrightarrow{R})(t)\rangle \Rightarrow\\ &\Rightarrow \gamma_n(C) = i\oint\overrightarrow{dR}\,\langle n\overrightarrow{R})(t)|\nabla_R(n\overrightarrow{R})(t)\rangle\,,\end{aligned} \tag{A.10}$$

where C denote the cycle computed in the parameter space by $\overrightarrow{R}$.
What we found is the so-called Berry phase, that as we can see is a pure parametrical dependent phase and does not depend on the specific cyclic path in the parameter space.
Sometimes is better to compute the Berry phase considering a total phase φ and the Berry phase as the difference between φ and the dynamical phase:

$$\gamma_n(C) = \varphi + \frac{1}{\hbar}\int_0^t dt' E_{\overrightarrow{R_n}}(t) = \varphi + \frac{1}{\hbar}\int_0^t dt'\langle n\overrightarrow{R})(t)|\frac{\partial}{\partial t}|(n\overrightarrow{R})(t)\rangle\,. \tag{A.11}$$

Let us now consider the Aharonov - Anandan geometrical invariant.
This invariant has the advantages that does not depend on the adiabatic transformation and remains valid even for non cyclic transformation.
We start considering again a physical system that evolves according to the Schrödinger equation:

$$\hat{H}(t)|\psi(t)\rangle = i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle\,. \tag{A.12}$$

Note that now we do not make any hypotheses on the time dependence of $\hat{H}(t)$ Let $\mathcal{H}$ be the Hilbert space of the states $|\psi(t)\rangle$ and let Π be the projection map between $\mathcal{H}$ and another Hilbert space $\mathcal{P}$, defined as :

$$\Pi(|\psi(t)\rangle) \equiv \{|\psi(t)'\rangle : |\psi(t)'\rangle = c|\psi(t)\rangle\}\,. \tag{A.13}$$

Where c is a complex number. For an isolated system to move in $\mathcal{P}$ it is necessary and sufficient that it is not a stationary state or in other words that it has a non zero value of the energy uncertainty defined by:

$$\Delta E(t)^2 = \langle\psi(t)|\hat{H}(t)^2|\psi(t)\rangle - \langle\psi(t)|\hat{H}(t)|\psi(t)\rangle^2\,. \tag{A.14}$$

We will now show that the quantity

$$s = 2\int \frac{\Delta E(t)}{\hbar}dt\,. \tag{A.15}$$

is independent on the particular $\hat{H}(t)$ used to transport the state along a curve Γ in $\mathcal{P}$ hence it is a pure geometrical quantity as it is the Berry Phase. To do this we Taylor expand to the second order the state:$|\psi(t+dt)\rangle$. Bearing in mind the formula (A.12) we obtain:

$$\begin{aligned} |\psi(t+dt)\rangle = |\psi(t)\rangle - i\frac{dt}{\hbar}\hat{H}(t) + \\ -\frac{dt^2}{2\hbar}\left(i\frac{d\hat{H}(t)}{dt} + \frac{1}{\hbar}\hat{H}(t)^2|\psi(t)\rangle\right) + \mathcal{O}(dt^3)\,. \end{aligned} \tag{A.16}$$

Remembering that $\hat{H}(t)$ is Hermitian we obtain:

$$\begin{aligned} |\langle\psi(t)|\psi(t+dt)\rangle|^2 = \\ = (\langle\psi(t)|\psi(t+dt)\rangle)^\dagger(\langle\psi(t)|\psi(t+dt)\rangle) = \\ = 1 - \frac{dt^2\Delta E^2}{\hbar^2} + \mathcal{O}(dt^3)\,, \end{aligned} \tag{A.17}$$

therefore

$$ds = 2\frac{\Delta E}{\hbar} \tag{A.18}$$

is independent of the phases of $|\psi(t)\rangle$ and $|\psi(t+dt)\rangle$ but it depends only on the points in $\mathcal{P}$ to witch they project.
There are infinite Hamiltonians that would evolve the state of our system along a given curve Γ in $\mathcal{P}$, they generally produce different phase factors for the state vector in every single instant of time, but they all give the samequantity for s. In other words s is invariant and to be more precise it's a geometrical invariant that give to us the distance along Γ as measured by the Fubini-Study metric as we can see by the relation:

$$\begin{aligned} ds^2 &= 2g_{\mu\nu} dZ^\mu d\overline{Z}^\nu = \\ &= 4(1 - |\langle\psi(t)|\psi'(t)\rangle|^2)\,, \end{aligned} \tag{A.19}$$

where Z^μ are coordinates in $\mathcal{P}$ and where $g_{\mu\nu}$ is the Fubini-Study metric in the projective Hilbert space $\mathcal{P}$, as asserted in reference [12].

Appendix B

Mixed states, entropy and environment effects.

Now I exploit the Schmit decomposition theorem (see e.g. [19, 20]). One can always "double" the system under study; denote it by $\mathcal{A}$.
The "doubled" system, denoted by $\tilde{\mathcal{A}}$, is introduced in such a way to work in the composite Hilbert space $\mathcal{H}_{\mathcal{A},\tilde{\mathcal{A}}} \equiv \mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\tilde{\mathcal{A}}}$ with states $|\Psi_{\mathcal{A},\tilde{\mathcal{A}}}\rangle = \sum_n \sqrt{w_n}\,|a_n\,\tilde{a}_n\rangle \in \mathcal{H}_{\mathcal{A},\tilde{\mathcal{A}}}$, $\sum_n w_n = 1$. The density matrix for mixed states of the system $\mathcal{A}$, $\rho^{\mathcal{A}} = \sum_n w_n |a_n\rangle\langle a_n|$, is obtained by tracing out the system $\tilde{\mathcal{A}}$:

$$\rho^{\mathcal{A}} = \sum_n w_n |a_n\rangle\langle a_n| = \sum_{nm} \sqrt{w_n w_m}\,|a_n\rangle\langle a_m|\,\mathrm{Tr}\left(|\tilde{a}_n\rangle\langle\tilde{a}_m|\right) = \mathrm{Tr}_{\tilde{\mathcal{A}}}\left(\rho^{\mathcal{A}\oplus\tilde{\mathcal{A}}}\right)\,, \tag{B.1}$$

where the relation $\langle\tilde{a}_m|\tilde{a}_n\rangle = \delta_{nm}$ has been used.
One can show that the "tilde" system $\tilde{\mathcal{A}}$ can be interpreted as the thermal bath or reservoir for the original system $\mathcal{A}$ [21, 22] and the free energy and the entropy can be defined[1]. The state $|\Psi_{\mathcal{A},\tilde{\mathcal{A}}}\rangle$ is recognized to be an entangled state of the tilde and non-tilde modes and the entropy provides a measure of the entanglement [18, 23].

Now I will show the computation of the static (linear) entropy for the qubit states $|\phi(t)\rangle$ and $|\psi(t)\rangle$ given by Eqs. (1.8) and (1.9), respectively.
One introduces the states $|\tilde{0}\rangle$ and $|\tilde{1}\rangle$ as

$$|0\rangle \;\rightarrow\; |0\rangle \otimes |\tilde{0}\rangle\,, \tag{B.2}$$

$$|1\rangle \;\rightarrow\; |1\rangle \otimes |\tilde{1}\rangle\,, \tag{B.3}$$

[1] Such a construction is equivalent to the GNS construction in the C^*-algebra formalism and requires the quantum field theory framework [21, 22].

and uses Eqs. (B.2) and (B.3) in Eqs. (1.8) and (1.9). The density matrices for the states in $\mathcal{H}_{\mathcal{S},\tilde{\mathcal{S}}}$, where $\mathcal{S} = \{0,1\}$ and $\tilde{\mathcal{S}} = \{\tilde{0},\tilde{1}\}$, are denoted by $\rho_\xi = |\xi(t),\tilde{\xi}(t)\rangle\langle\xi(t),\tilde{\xi}(t)|$ where $\xi = \phi,\psi$ and $\tilde{\xi} = \tilde{\phi},\tilde{\psi}$. The reduced density matrix $\rho_\phi^{\mathcal{S}}$ (and similarly for $\rho_\psi^{\mathcal{S}}$) is obtained by tracing out the tilde-system $\tilde{\mathcal{S}}$, and vice-versa. Thus one obtains:

$$\rho_\phi^{\mathcal{S}} = Tr_{\tilde{\mathcal{S}}}\,\rho_\phi = \cos^2\theta\,|0\rangle\,\langle 0| + \sin^2\theta\,|1\rangle\,\langle 1|\,, \tag{B.4}$$

$$\rho_\phi^{\tilde{\mathcal{S}}} = Tr_{\mathcal{S}}\,\rho_\phi = \cos^2\theta\,|\tilde{0}\rangle\,\langle\tilde{0}|\, + \sin^2\theta\,|\tilde{1}\rangle\,\langle\tilde{1}|\,. \tag{B.5}$$

The static linear entropies S_L associated to the reduced matrices $\rho_\phi^{\mathcal{S}}$ and $\rho_\phi^{\tilde{\mathcal{S}}}$ are then:

$$S_L[\rho_\phi^{\mathcal{S}}] = 2\,(1 - Tr_{\mathcal{S}}[(\rho_\phi^{\mathcal{S}})^2]) = \sin^2(2\theta), \tag{B.6}$$

$$S_L[\rho_\phi^{\tilde{\mathcal{S}}}] = 2\,(1 - Tr_{\tilde{\mathcal{S}}}[(\rho_\phi^{\tilde{\mathcal{S}}})^2]) = \sin^2(2\theta). \tag{B.7}$$

Recall that

$$\sin^2 2\theta = \frac{4}{\omega_-^2}\,\omega_{\phi\psi}^2 \tag{B.8}$$

(cf. Eq. (1.17)), where

$$\omega_- \equiv \omega_2 - \omega_1 \neq 0. \tag{B.9}$$

. For the dynamic entropy we have to consider the state $|\phi(t)\rangle$ in Eq.(1.8) [2] and express it in terms of the states $|\phi(0)\rangle$ and $|\psi(0)\rangle$:

$$|\phi(t)\rangle = A_{\phi\phi}(t)|\phi(0)\rangle + A_{\phi\psi}(t)|\psi(0)\rangle, \tag{B.10}$$

where $A_{\phi\phi}(t)$ and $A_{\phi\psi}(t)$ are the amplitudes:

$$A_{\phi\phi}(t) = \langle\phi(0)|\phi(t)\rangle = e^{-i\omega_1 t}\cos^2\theta + e^{-i\omega_2 t}\sin^2\theta, \tag{B.11}$$

$$A_{\phi\psi}(t) = \langle\psi(0)|\phi(t)\rangle = e^{-i\omega_1 t}\sin\theta\cos\theta + e^{-i\omega_2 t}\sin\theta\cos\theta\,, \tag{B.12}$$

respectively. The tilde-states $|\tilde{\phi}\rangle$ and $|\tilde{\psi}\rangle$ are introduced, for any t, as :

$$|\phi\rangle \to |\phi\rangle \otimes |\tilde{\phi}\rangle\,, \tag{B.13}$$

$$|\psi\rangle \to |\psi\rangle \otimes |\tilde{\psi}\rangle\,. \tag{B.14}$$

[2] we can proceed in a similar way for $|\psi(t)\rangle$

The reduced density matrices are now, for any t,

$$\rho_\phi^{\mathcal{R}} = Tr_{\tilde{\mathcal{R}}}\rho_\phi = |A_{\phi\phi}(t)|^2 \, |\phi\rangle \, \langle\phi| + \, |A_{\phi\psi}(t)|^2 \, |\psi\rangle \, \langle\psi|, \tag{B.15}$$

$$\rho_\phi^{\tilde{\mathcal{R}}} = Tr_{\mathcal{R}}\rho_\phi = \, |A_{\phi\phi}(t)|^2 \, |\tilde{\phi}\rangle \, \langle\tilde{\phi}| + \, |A_{\phi\psi}(t)|^2 \, |\tilde{\psi}\rangle \, \langle\tilde{\psi}|, \tag{B.16}$$

where $\mathcal{R} = \{\phi, \psi\}$ and $\tilde{\mathcal{R}} = \{\tilde{\phi}, \tilde{\psi}\}$.
The dynamic entropies S_L are

$$S_L(\rho_\phi^{\mathcal{R}}) = 2\,(1 - Tr_{\mathcal{R}}[(\rho_\phi^{\mathcal{R}})^2]) = 4\,|A_{\phi\phi}(t)|^2\,|A_{\phi\psi}(t)|^2 = 4\,P_{\phi\to\phi}(t)\,P_{\phi\to\psi}(t), \tag{B.17}$$

$$S_L(\rho_\phi^{\tilde{\mathcal{R}}}) = 2\,(1 - Tr_{\tilde{\mathcal{R}}}[(\rho_\phi^{\tilde{\mathcal{R}}})^2]) = 4\,|A_{\phi\phi}(t)|^2\,|A_{\phi\psi}(t)|^2 = 4\,P_{\phi\to\phi}(t)\,P_{\phi\to\psi}(t), \tag{B.18}$$

where $P_{\phi\to\phi}(t)$ and $P_{\phi\to\psi}(t)$ are the probabilities of the transitions $\phi \to \phi$ and $\phi \to \psi$:

$$P_{\phi\to\psi}(t) = \sin^2(2\theta)\sin^2\left(\frac{\omega_2 - \omega_1}{2}t\right), \tag{B.19}$$

$$P_{\phi\to\phi}(t) = 1 - \sin^2(2\theta)\sin^2\left(\frac{\omega_2 - \omega_1}{2}t\right), \tag{B.20}$$

respectively.

Appendix C

Formulas for the flavor vacuum

We use the formulas (2.67) - (2.69) and the equations (2.87 - 2.90) to obtain:

$$S_+^{\mathbf{k}} S_-^{\mathbf{k}} |0\rangle_{1,2} = S_-^{\mathbf{k}} S_+^{\mathbf{k}} |0\rangle_{1,2} \qquad (S_+^{\mathbf{k}})^2 (S_-^{\mathbf{k}})^2 |0\rangle_{1,2} = (S_-^{\mathbf{k}})^2 (S_+^{\mathbf{k}})^2 |0\rangle_{1,2}$$

$$(S_+^{\mathbf{k}})^2 S_-^{\mathbf{k}} |0\rangle_{1,2} = S_-^{\mathbf{k}} (S_+^{\mathbf{k}})^2 |0\rangle_{1,2} + 2 S_+^{\mathbf{k}} |0\rangle_{1,2}$$

$$(S_-^{\mathbf{k}})^2 S_+^{\mathbf{k}} |0\rangle_{1,2} = S_+^{\mathbf{k}} (S_-^{\mathbf{k}})^2 |0\rangle_{1,2} + 2 S_-^{\mathbf{k}} |0\rangle_{1,2}$$

$$(S_+^{\mathbf{k}})^3 S_-^{\mathbf{k}} |0\rangle_{1,2} = 6 (S_+^{\mathbf{k}})^2 |0\rangle_{1,2} \qquad (S_+^{\mathbf{k}})^4 S_-^{\mathbf{k}} |0\rangle_{1,2} = 0$$

$$(S_+^{\mathbf{k}})^3 (S_-^{\mathbf{k}})^2 |0\rangle_{1,2} = 6 S_-^{\mathbf{k}} (S_+^{\mathbf{k}})^2 |0\rangle_{1,2} \qquad (S_-^{\mathbf{k}})^3 (S_+^{\mathbf{k}})^2 |0\rangle_{1,2} = 6 S_+^{\mathbf{k}} (S_-^{\mathbf{k}})^2 |0\rangle_{1,2}$$

$$(S_+^{\mathbf{k}})^4 (S_-^{\mathbf{k}})^2 |0\rangle_{1,2} = 24 (S_+^{\mathbf{k}})^2 |0\rangle_{1,2} \qquad (S_+^{\mathbf{k}})^5 (S_-^{\mathbf{k}})^2 |0\rangle_{1,2} = 0$$

$$S_+^{\mathbf{k}} S_-^{\mathbf{k}} (S_+^{\mathbf{k}})^2 |0\rangle_{1,2} = 4 (S_+^{\mathbf{k}})^2 |0\rangle_{1,2} \qquad S_-^{\mathbf{k}} S_+^{\mathbf{k}} (S_-^{\mathbf{k}})^2 |0\rangle_{1,2} = 4 (S_-^{\mathbf{k}})^2 |0\rangle_{1,2}$$

$$S_3^{\mathbf{k}} S_-^{\mathbf{k}} S_+^{\mathbf{k}} |0\rangle_{1,2} = 0 \qquad S_3^{\mathbf{k}} (S_+^{\mathbf{k}})^2 (S_-^{\mathbf{k}})^2 |0\rangle_{1,2} = 0$$

$$(S_3^{\mathbf{k}})^n S_-^{\mathbf{k}} |0\rangle_{1,2} = (-1)^n S_-^{\mathbf{k}} |0\rangle_{1,2} \qquad (S_3^{\mathbf{k}})^n (S_-^{\mathbf{k}})^2 |0\rangle_{1,2} = (-2)^n (S_-^{\mathbf{k}})^2 |0\rangle_{1,2}$$

$$(S_3^{\mathbf{k}})^n S_+^{\mathbf{k}} |0\rangle_{1,2} = S_+^{\mathbf{k}} |0\rangle_{1,2} \qquad (S_3^{\mathbf{k}})^n (S_+^{\mathbf{k}})^2 |0\rangle_{1,2} = 2^n (S_+^{\mathbf{k}})^2 |0\rangle_{1,2}$$

$$S_3^{\mathbf{k}} S_-^{\mathbf{k}} (S_+^{\mathbf{k}})^2 |0\rangle_{1,2} = S_-^{\mathbf{k}} (S_+^{\mathbf{k}})^2 |0\rangle_{1,2} \qquad S_3^{\mathbf{k}} S_+^{\mathbf{k}} (S_-^{\mathbf{k}})^2 |0\rangle_{1,2} = -S_+^{\mathbf{k}} (S_-^{\mathbf{k}})^2 |0\rangle_{1,2}$$

Appendix D

The flavor vacuum

To calculate $|0\rangle^{\mathbf{k}}_{e,\mu}$ it is useful to choose $\mathbf{k} = (0, 0, |\mathbf{k}|)$.
In the above mentioned reference frame the operators $S^{\mathbf{k}}_{+}$, $S^{\mathbf{k}}_{-}$, $S^{\mathbf{k}}_{3}$ are written as follows:

$$S^{\mathbf{k}}_{+} \equiv \sum_{\mathbf{k},r} S^{\mathbf{k},r}_{+} = \sum_{r} \left(U^{*}_{\mathbf{k}}\ \alpha^{r\dagger}_{\mathbf{k},1}\alpha^{r}_{\mathbf{k},2} - \epsilon^{r}\ V^{*}_{\mathbf{k}}\ \beta^{r}_{-\mathbf{k},1}\alpha^{r}_{\mathbf{k},2} + \epsilon^{r}\ V_{\mathbf{k}}\ \alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2} + U_{\mathbf{k}}\ \beta^{r}_{-\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2} \right) \tag{D.1}$$

$$S^{\mathbf{k}}_{-} \equiv \sum_{k,r} S^{k,r}_{-} = \sum_{r} \left(U_{\mathbf{k}}\ \alpha^{r\dagger}_{\mathbf{k},2}\alpha^{r}_{\mathbf{k},1} + \epsilon^{r}\ V^{*}_{\mathbf{k}}\ \beta^{r}_{-\mathbf{k},2}\alpha^{r}_{\mathbf{k},1} - \epsilon^{r}\ V_{\mathbf{k}}\ \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1} + U^{*}_{\mathbf{k}}\ \beta^{r}_{-\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1} \right) \tag{D.2}$$

$$S^{\mathbf{k}}_{3} \equiv \sum_{\mathbf{k},r} S^{\mathbf{k},r}_{3} = \frac{1}{2} \sum_{\mathbf{k},r} \left(\alpha^{r\dagger}_{\mathbf{k},1}\alpha^{r}_{k,1} - \beta^{r\dagger}_{-\mathbf{k},1}\beta^{r}_{-\mathbf{k},1} - \alpha^{r\dagger}_{\mathbf{k},2}\alpha^{r}_{\mathbf{k},2} + \beta^{r\dagger}_{-\mathbf{k},2}\beta^{r}_{-\mathbf{k},2} \right)\ , \tag{D.3}$$

where $U_{\mathbf{k}}$, $V_{\mathbf{k}}$ are defined in Eqs.(2.80)-(2.81) and $\epsilon^{r} = (-1)^{r}$.
Using the Gaussian decomposition, $|0\rangle^{\mathbf{k}}_{e,\mu}$ can be written as

$$|0\rangle^{\mathbf{k}}_{e,\mu} = \prod_{r} exp(-tan\theta\ S^{\mathbf{k},r}_{+}) exp(-2ln\ cos\theta\ S^{\mathbf{k},r}_{3})\ exp(tan\theta\ S^{\mathbf{k},r}_{-})|0\rangle_{1,2} \tag{D.4}$$

where $0 \leq \theta < \frac{\pi}{2}$.
The final expression for $|0\rangle^{\mathbf{k}}_{e,\mu}$ in terms of $S^{\mathbf{k},r}_{\pm}$ and $S^{\mathbf{k},r}_{3}$ is then

$$|0\rangle^{\mathbf{k}}_{e,\mu} = \prod_r \left[1 + \sin\theta\cos\theta\left(S^{\mathbf{k},r}_- - S^{\mathbf{k},r}_+\right) - \sin^2\theta\; S^{\mathbf{k},r}_+ S^{\mathbf{k},r}_-\right] |0\rangle_{1,2}\ . \tag{D.5}$$

Appendix E

Orthogonality of flavor vacuum states and flavor states at different times

The product of two vacuum states at different times $t \neq t'$ (we put for simplicity $t' = 0$) is

$$_{e,\mu}\langle 0|0(t)\rangle_{e,\mu} = \prod_k C_k^2(t) = e^{2\sum_k lnC_k(t)} \tag{E.1}$$

with

$$\begin{aligned} C_k(t) \;\; &\equiv \;\; (1-\sin^2\theta\,|V_k|^2)^2 + \;2\;\sin^2\theta\;\cos^2\theta\,|V_k|^2\;e^{-i(\omega_{k,2}+\omega_{k,1})t} + \\ &+ \;\;\sin^4\theta\,|V_k|^2\;|U_k|^2\left(e^{-2i\omega_{k,1}t}\;+\;e^{-2i\omega_{k,2}t}\;\right)+ \\ &+ \;\;\sin^4\theta\,|V_k|^4\;e^{-2i(\omega_{k,2}+\omega_{k,1})t}\;. \end{aligned} \tag{E.2}$$

In the infinite volume limit we obtain (note that $|C_k(t)| \leq 1$ for any value of k, t, and of the parameters θ, m_1, m_2):

$$\lim_{V\to\infty}\;_{e,\mu}\langle 0|0(t)\rangle_{e,\mu} = \lim_{V\to\infty}\;exp\left[\frac{2V}{(2\pi)^3}\int d^3k\;\;(ln\;|C_k(t)|\;+\;i\alpha_k(t))\right] = 0 \tag{E.3}$$

with $|C_k(t)|^2 = Re[C_k(t)]^2 + Im[C_k(t)]^2$ and $\alpha_k(t) = \tan^{-1}\left(Im[C_k(t)]/Re[C_k(t)]\right)$. Thus we have orthogonality of the vacua at different times.

Now we can show the orthogonality of flavor states at different times.

We define the electron neutrino state at time t with momentum $\mathbf{k}$ as

$$|\nu_{\mathbf{k},e}(t)\rangle = \alpha^{r\dagger}_{\mathbf{k},e}(t)|0(t)\rangle_{e,\mu}. \tag{E.4}$$

The flavor vacuum is explicitly given by[1]

$$|0(t)\rangle_{e,\mu} = \prod_{\mathbf{p}} G^{-1}_{\mathbf{p},\theta}(t)|0\rangle_{1,2}, \tag{E.5}$$

then, we have

$$\begin{aligned}\langle\nu_{\mathbf{k},e}(0)|\nu_{\mathbf{k},e}(t)\rangle &= {}_{e,\mu}\langle 0|\alpha^{r}_{\mathbf{k},e}(0)\alpha^{r\dagger}_{\mathbf{k},e}(t)|0(t)\rangle_{e,\mu} \\ &= \prod_{\mathbf{p}}\prod_{\mathbf{q}} {}_{1,2}\langle 0|G_{\mathbf{p},\theta}(0)\alpha^{r}_{\mathbf{k},e}(0)\alpha^{r\dagger}_{\mathbf{k},e}(t)G^{-1}_{\mathbf{q},\theta}(t)|0\rangle_{1,2}.\end{aligned} \tag{E.6}$$

With $\mathbf{p} \neq \mathbf{q}$ the mixing generators commute, then we put $\mathbf{p} = \mathbf{q}$:

$$\langle\nu_{\mathbf{k},e}(0)|\nu_{\mathbf{k},e}(t)\rangle = \prod_{\mathbf{p}} {}_{1,2}\langle 0|G_{\mathbf{p},\theta}(0)\alpha^{r}_{\mathbf{k},e}(0)\alpha^{r\dagger}_{\mathbf{k},e}(t)G^{-1}_{\mathbf{p},\theta}(t)|0\rangle_{1,2}. \tag{E.7}$$

$\alpha^{r\dagger}_{\mathbf{k},e}$ acts only on vacuum with momentum $\mathbf{k}$, then

$$\begin{aligned}\langle\nu_{\mathbf{k},e}(0)|\nu_{\mathbf{k},e}(t)\rangle &\propto {}_{e,\mu}\langle 0^{\mathbf{k}}|\alpha^{r}_{\mathbf{k},e}(0)\alpha^{r\dagger}_{\mathbf{k},e}(t)|0^{\mathbf{k}}(t)\rangle_{e,\mu}\prod_{\mathbf{p}\neq\mathbf{k}} {}_{1,2}\langle 0|G_{\mathbf{p},\theta}(0)G^{-1}_{\mathbf{p},\theta}(t)|0\rangle_{1,2} \\ &= {}_{e,\mu}\langle 0^{\mathbf{k}}|\alpha^{r}_{\mathbf{k},e}(0)\alpha^{r\dagger}_{\mathbf{k},e}(t)|0^{\mathbf{k}}(t)\rangle_{e,\mu}\ {}_{e,\mu}\langle 0|0(t)\rangle_{e,\mu}.\end{aligned} \tag{E.8}$$

By using the Eq.(E.3), in the infinite volume limit we obtain the orthogonality of flavor states at different times.

[1]To be precise, the mass vacuum is to be understood as $|0\rangle_{1,2} = |0\rangle^{\mathbf{k}_1}_{1,2} \bigotimes |0\rangle^{\mathbf{k}_2}_{1,2} \bigotimes |0\rangle^{\mathbf{k}_3}_{1,2}.....$

Appendix F

QFT flavor states

The explicit expression for $|0\rangle_{e,\mu}$ at time $t = 0$ in the reference frame for which $\mathbf{k} = (0, 0, |\mathbf{k}|)$ is

$$
\begin{aligned}
|0\rangle_{e,\mu} &= \prod_{r,\mathbf{k}} \Big[(1 - \sin^2\theta\, |V_{\mathbf{k}}|^2) - \epsilon^r \sin\theta\, \cos\theta\, |V_{\mathbf{k}}|(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2} + \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}) + \\
&+ \epsilon^r \sin^2\theta\, |V_{\mathbf{k}}||U_{\mathbf{k}}|(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},1} - \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},2}) + \sin^2\theta\, |V_{\mathbf{k}}|^2 \alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2}\alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}\Big] |0\rangle_{1,2}.
\end{aligned}
\tag{F.1}
$$

Eq.(F.1) exhibits the condensate structure of the flavor vacuum $|0\rangle_{e,\mu}$. The important point is that ${}_{1,2}\langle 0|0(t)\rangle_{e,\mu} \to 0$, for any t, in the infinite volume limit [32]. Thus, in such a limit the Hilbert spaces $\mathcal{H}_{1,2}$ and $\mathcal{H}_{e,\mu}$ turn out to be unitarily inequivalent spaces.

The explicit form of the multi-particle states defined in equations:

$$
|\nu^r_{\mathbf{k},e\bar{e}\mu}(t)\rangle \equiv \alpha^{r\dagger}_{\mathbf{k},e}(t)\, \beta^{r\dagger}_{-\mathbf{k},e}(t)\, \alpha^{r\dagger}_{\mathbf{k},\mu}(t)\, |0(t)\rangle_{e,\mu}\,, \tag{F.2}
$$

$$
|\nu^r_{\mathbf{k},\mu\bar{\mu}e}(t)\rangle \equiv \alpha^{r\dagger}_{\mathbf{k},\mu}(t)\, \beta^{r\dagger}_{-\mathbf{k},\mu}(t)\, \alpha^{r\dagger}_{\mathbf{k},e}(t)\, |0(t)\rangle_{e,\mu}\,, \tag{F.3}
$$

is

$$
\begin{aligned}
|\nu^r_{\mathbf{k},e\bar{e}\mu}(t)\rangle \;\; &= \;\; -\Big[\cos\theta\;\alpha^{r\dagger}_{\mathbf{k},1}\alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}\,e^{-i(2\omega_{k,1}+\omega_{k,2})t} \,+\, \epsilon^r\;|V_{\mathbf{k}}|\;\sin\theta\;\alpha^{r\dagger}_{\mathbf{k},1}\,e^{-i\omega_{k,1}t} \\
&+|U_{\mathbf{k}}|\;\sin\theta\;\alpha^{r\dagger}_{\mathbf{k},1}\alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},2}\,e^{-i(\omega_{k,1}+2\omega_{k,2})t}\Big]G^{-1}_{\mathbf{k},s\neq r}(\theta,t)\prod_{\mathbf{p}\neq\mathbf{k}}G^{-1}_{\mathbf{p}}(\theta,t)|0\rangle_{1,2}\,,
\end{aligned}
\tag{F.4}
$$

$$
\begin{aligned}
|\nu^r_{\mathbf{k},\mu\bar{\mu}e}(t)\rangle \;\; &= \;\; \Big[\cos\theta\;\alpha^{r\dagger}_{\mathbf{k},1}\alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},2}\,e^{-i(\omega_{k,1}+2\omega_{k,2})t} \,-\, \epsilon^r\;|V_{\mathbf{k}}|\;\sin\theta\;\alpha^{r\dagger}_{\mathbf{k},2}\,e^{-i\omega_{k,2}t} \\
&-|U_{\mathbf{k}}|\;\sin\theta\;\alpha^{r\dagger}_{\mathbf{k},1}\alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}\,e^{-i(2\omega_{k,1}+\omega_{k,2})t}\Big]G^{-1}_{\mathbf{k},s\neq r}(\theta,t)\prod_{\mathbf{p}\neq\mathbf{k}}G^{-1}_{\mathbf{p}}(\theta,t)|0\rangle_{1,2}\,.
\end{aligned}
\tag{F.5}
$$

Appendix G

Expectation values of $: H :$ and $:: H ::$

The flavor states introduced in the Appendix F are used in computing the following expectation values for the Hamiltonian $: H :$, $:: H ::$. We have:

$$\langle \nu^r_{\mathbf{k},e}(t)| : H : |\nu^r_{\mathbf{k},e}(t)\rangle \; = \; \omega_{k,1}\,(\cos^2\theta \,+\, 2\,\sin^2\theta\,|V_{\mathbf{k}}|^2) \,+\, \omega_{k,2}\,\sin^2\theta\,, \tag{G.1}$$
$$\langle \nu^r_{\mathbf{k},\mu}(t)| : H : |\nu^r_{\mathbf{k},\mu}(t)\rangle \; = \; \omega_{k,2}\,(\cos^2\theta \,+\, 2\,\sin^2\theta\,|V_{\mathbf{k}}|^2) \,+\, \omega_{k,1}\,\sin^2\theta\,, \tag{G.2}$$

$$\begin{aligned}\langle \nu^r_{\mathbf{k},e}(t)|\,(: H :)^2\,|\nu^r_{\mathbf{k},e}(t)\rangle \; &= \; \omega^2_{k,1}\,(\cos^2\theta \,+\, 4\,\sin^2\theta\,|V_{\mathbf{k}}|^2)\\ &+\; \omega^2_{k,2}\,\sin^2\theta \,+\, 4\,\omega_{k,1}\,\omega_{k,2}\sin^2\theta\,|V_{\mathbf{k}}|^2\,,\end{aligned} \tag{G.3}$$

$$\begin{aligned}\langle \nu^r_{\mathbf{k},\mu}(t)|(: H :)^2|\nu^r_{\mathbf{k},\mu}(t)\rangle \; &= \; \omega^2_{k,2}\,(\cos^2\theta \,+\, 4\,\sin^2\theta\,|V_{\mathbf{k}}|^2)\\ &+\; \omega^2_{k,1}\,\sin^2\theta \,+\, 4\,\omega_{k,1}\,\omega_{k,2}\sin^2\theta\,|V_{\mathbf{k}}|^2\,.\end{aligned} \tag{G.4}$$

$$\langle \nu^r_{\mathbf{k},e\bar{\mu}\mu}(t)| : H : |\nu^r_{\mathbf{k},e}(t)\rangle \; = \; 2\,\epsilon^r\,\omega_{k,1}\,\sin^2\theta\,|U_{\mathbf{k}}|\,|V_{\mathbf{k}}|\,, \tag{G.5}$$
$$\langle \nu^r_{\mathbf{k},\mu\bar{e}e}(t)| : H : |\nu^r_{\mathbf{k},\mu}(t)\rangle \; = \; -2\,\epsilon^r\,\omega_{k,2}\,\sin^2\theta\,|U_{\mathbf{k}}|\,|V_{\mathbf{k}}|\,, \tag{G.6}$$
$$\langle \nu^r_{\mathbf{k},e\bar{\mu}\mu}(t)| : H : |\nu^r_{\mathbf{k},\mu}(t)\rangle \; = \; \langle \nu^r_{\mathbf{k},\mu\bar{e}e}(t)| : H : |\nu^r_{\mathbf{k},e}(t)\rangle \; = \; \epsilon^r\,\Omega^k_+\,\sin\theta\,\cos\theta\,|V_{\mathbf{k}}|, \tag{G.7}$$

The Hamiltonian normal ordered with respect to the flavor vacuum $:: H ::$ satisfies the following relations:

$$\langle \nu^r_{\mathbf{k},e}(t)| :: H :: |\nu^r_{\mathbf{k},e}(t)\rangle \; = \; \omega_{k,1}\,\cos^2\theta \,+\, \omega_{k,2}\,\sin^2\theta\,(1 - 2\,|V_{\mathbf{k}}|^2)\,, \tag{G.8}$$
$$\langle \nu^r_{\mathbf{k},\mu}(t)| :: H :: |\nu^r_{\mathbf{k},\mu}(t)\rangle \; = \; \omega_{k,2}\,\cos^2\theta \,+\, \omega_{k,1}\,\sin^2\theta\,(1 - 2\,|V_{\mathbf{k}}|^2)\,, \tag{G.9}$$
$$\langle \nu^r_{\mathbf{k},e}(t)| :: H :: |\nu^r_{\mathbf{k},\mu}(t)\rangle \; = \; \langle \nu^r_{\mathbf{k},\mu}(t)| :: H :: |\nu^r_{\mathbf{k},e}(t)\rangle = \Omega^k_-\,\sin\theta\,\cos\theta\,|U_{\mathbf{k}}|\,, \tag{G.10}$$

$$
\begin{aligned}
\langle \nu^r_{\mathbf{k},e}(t)|\,(:: H ::)^2\,|\nu^r_{\mathbf{k},e}(t)\rangle &= \omega^2_{k,1}\,(\cos^2\theta\,+4\,\sin^4\theta\,|U_{\mathbf{k}}|^2\,|V_{\mathbf{k}}|^2) \\
&+ \omega^2_{k,2}\,\sin^2\theta\,(1-4\,\sin^2\theta\,|U_{\mathbf{k}}|^2\,|V_{\mathbf{k}}|^2) \qquad (G.11)
\end{aligned}
$$

$$
\begin{aligned}
\langle \nu^r_{\mathbf{k},\mu}(t)|\,(:: H ::)^2\,|\nu^r_{\mathbf{k},\mu}(t)\rangle &= \omega^2_{k,2}\,(\cos^2\theta\,+4\,\sin^4\theta\,|U_{\mathbf{k}}|^2\,|V_{\mathbf{k}}|^2) \\
&+ \omega^2_{k,1}\,\sin^2\theta\,(1-4\,\sin^2\theta\,|U_{\mathbf{k}}|^2\,|V_{\mathbf{k}}|^2) \qquad (G.12)
\end{aligned}
$$

Finally we have:

$$
\begin{aligned}
\langle \nu^r_{\mathbf{k},e\bar{\mu}\mu}(t)|\,::H::\,|\nu^r_{\mathbf{k},e}(t)\rangle &= \langle \nu^r_{\mathbf{k},e\bar{\mu}\mu}(t)|\,:H:\,|\nu^r_{\mathbf{k},e}(t)\rangle \qquad (G.13) \\
\langle \nu^r_{\mathbf{k},\mu\bar{e}e}(t)|\,::H::\,|\nu^r_{\mathbf{k},\mu}(t)\rangle &= \langle \nu^r_{\mathbf{k},\mu\bar{e}e}(t)|\,:H:\,|\nu^r_{\mathbf{k},\mu}(t)\rangle \qquad (G.14)
\end{aligned}
$$

$$
\begin{aligned}
&\langle \nu^r_{\mathbf{k},\mu\bar{e}e}(t)|\,::H::\,|\nu^r_{\mathbf{k},e}(t)\rangle = \langle \nu^r_{\mathbf{k},e\bar{\mu}\mu}(t)|\,::H::\,|\nu^r_{\mathbf{k},\mu}(t)\rangle = \\
&= \langle \nu^r_{\mathbf{k},e\bar{\mu}\mu}(t)|\,:H:\,|\nu^r_{\mathbf{k},\mu}(t)\rangle = \langle \nu^r_{\mathbf{k},e\bar{\mu}\mu}(t)|\,:H:\,|\nu^r_{\mathbf{k},\mu}(t)\rangle \qquad (G.15)
\end{aligned}
$$

.

Appendix H

Particle mixing in Quantum Field Theory

We use the CKM matrix as mixing matrix $\mathcal{U}$ for three fields: $\Psi_f(x) = \mathcal{U}\,\Psi_m(x)$ where $\Psi_f(x)$ are fields (neutrinos or quarks) with definite flavor and $\Psi_m(x)$ are fields with definite masses. The mixing transformation can be written as $\psi_\sigma^\alpha(x) \equiv G_\theta^{-1}(t)\,\psi_i^\alpha(x)\,G_\theta(t)$, where $(\sigma, i) = (A, 1), (B, 2), (C, 3)$, and the generator is

$$G_\theta(t) = G_{23}(t)\,G_{13}(t)\,G_{12}(t)\,, \tag{H.1}$$

where

$$G_{12}(t) \equiv \exp\left[\theta_{12}\int d^3x\,\Big(\psi_1^\dagger(x)\psi_2(x) - \psi_2^\dagger(x)\psi_1(x)\Big)\right], \tag{H.2}$$

$$G_{23}(t) \equiv \exp\left[\theta_{23}\int d^3x\,\Big(\psi_2^\dagger(x)\psi_3(x) - \psi_3^\dagger(x)\psi_2(x)\Big)\right], \tag{H.3}$$

$$G_{13}(t) \equiv \exp\left[\theta_{13}\int d^3x\,\Big(\psi_1^\dagger(x)\psi_3(x)e^{-i\delta} - \psi_3^\dagger(x)\psi_1(x)e^{i\delta}\Big)\right]. \tag{H.4}$$

The numbers of particles condensed in the vacuum for any r are:

$$\mathcal{N}_1^{\mathbf{k}} = {}_f\langle 0(t)|N_{\alpha_1}^{\mathbf{k},r}|0(t)\rangle_f = {}_f\langle 0(t)|N_{\beta_1}^{\mathbf{k},r}|0(t)\rangle_f = s_{12}^2 c_{13}^2\,|V_{12}^{\mathbf{k}}|^2 + s_{13}^2\,|V_{13}^{\mathbf{k}}|^2\,, \tag{H.5}$$

$$\mathcal{N}_2^{\mathbf{k}} = {}_f\langle 0(t)|N_{\alpha_2}^{\mathbf{k},r}|0(t)\rangle_f = {}_f\langle 0(t)|N_{\beta_2}^{\mathbf{k},r}|0(t)\rangle_f = \left|-s_{12}c_{23} + e^{i\delta}\,c_{12}s_{23}s_{13}\right|^2\,|V_{12}^{\mathbf{k}}|^2 + s_{23}^2 c_{13}^2\,|V_{23}^{\mathbf{k}}|^2\,, \tag{H.6}$$

$$\begin{gathered}\mathcal{N}_3^{\mathbf{k}} = {}_f\langle 0(t)|N_{\alpha_3}^{\mathbf{k},r}|0(t)\rangle_f = {}_f\langle 0(t)|N_{\beta_3}^{\mathbf{k},r}|0(t)\rangle_f = \\ \left|-c_{12}s_{23} + e^{i\delta}\,s_{12}c_{23}s_{13}\right|^2|V_{23}^{\mathbf{k}}|^2 + \left|s_{12}s_{23} + e^{i\delta}\,c_{12}c_{23}s_{13}\right|^2|V_{13}^{\mathbf{k}}|^2\,.\end{gathered} \tag{H.7}$$

Since the vacuum $|0\rangle_m$ for the massive fields is unitarily inequivalent to the vacuum $|0(t)\rangle_f$ for the mixed (flavored) fields at time t, for any t, two different normal orderings must be defined, respectively with respect to $|0\rangle_m$, as usual denoted by : ... :, and with respect to $|0(t)\rangle_f$, denoted by :: ... :: . The Hamiltonian normal ordered with respect to the vacua $|0\rangle_m$ and $|0(t)\rangle_f$ is given by

$$:H: = H -_m \langle 0|H|0\rangle_m = H + 2\sum_i \int d^3\mathbf{k}\, \omega_{k,i} = \sum_i \sum_r \int d^3\mathbf{k}\, \omega_{k,i}[\alpha^{r\dagger}_{\mathbf{k},i}\alpha^r_{\mathbf{k},i} + \beta^{r\dagger}_{\mathbf{k},i}\beta^r_{\mathbf{k},i}] \,, \tag{H.8}$$

$$:: H :: \;\equiv\; H - {}_f\langle 0(t)|H|0(t)\rangle_f = H + 2\sum_i \int d^3\mathbf{k}\, \omega_{k,i} - 4\sum_i \int d^3\mathbf{k}\, \omega_{k,i}\, \mathcal{N}^{\mathbf{k}}_i \,, \tag{H.9}$$

respectively. Note that $H = \int d^3x\, \mathcal{T}_{00}$. The state $|0(t)\rangle_f$ is a condensate of massive particle-antiparticle pairs. We point out that the difference of energy between $|0(t)\rangle_f$ and $|0\rangle_m$ represents the energy of the condensed neutrinos given in Eqs.(H.5)-(H.7)

$${}_f\langle 0(t)| :H: |0(t)\rangle_f \;=\; {}_f\langle 0(t)|H|0(t)\rangle_f - {}_m\langle 0|H|0\rangle_m = 4\sum_i \int d^3\mathbf{k}\, \omega_{k,i}\, \mathcal{N}^{\mathbf{k}}_i \,. \tag{H.10}$$

Now we give the proof of

$${}_f\langle 0(t)| : \mathcal{T}_{\mu\nu}(x) : |0(t)\rangle_f \;=\; {}_f\langle 0| : \mathcal{T}_{\mu\nu}(x) : |0\rangle_f \,. \tag{H.11}$$

From Eqs.

$$\rho_{mix} \equiv \frac{1}{V}\, \eta^{00} \int d^3x\, \mathcal{T}^{cond}_{00}(x) \,. \tag{H.12}$$

and

$$\rho_{mix} \;=\; \frac{2}{\pi}\sum_i \int_0^K dk\, k^2\, \omega_{k,i}\, \mathcal{N}^{\mathbf{k}}_i \,, \qquad i = 1,2,3\,. \tag{H.13}$$

we see that the (0,0) component of the energy momentum tensor of the vacuum condensate is given by:

$$\int d^3x\, \mathcal{T}^{cond}_{00}(x) \;=\; \int d^3x\, {}_f\langle 0(t)| : \mathcal{T}_{00}(x) : |0(t)\rangle_f \;=\; 4\sum_i \int d^3\mathbf{k}\, \omega_{k,i}\, \mathcal{N}^{\mathbf{k}}_i \,, \tag{H.14}$$

which coincides with (H.10) and shows that ${}_f\langle 0(t)| : \mathcal{T}_{00}(x) : |0(t)\rangle_f$ is time independent since the numbers $\mathcal{N}_i^{\mathbf{k}}$ (cf. (H.5)-(H.7)) are time independent in the Minkowski metric.

Considering now that the (j,j) components of the energy momentum tensor of the vacuum condensate are given by the equations:

$$p_{mix} \;=\; -\frac{1}{V}\,\eta^{jj} \int d^3x\; \mathcal{T}_{jj}^{cond}(x)\;; \tag{H.15}$$

$$p_{mix} \;=\; \frac{2}{3\,\pi}\sum_i \int_0^K dk\, k^2\, \frac{k^2}{\omega_{k,i}}\,\mathcal{N}_i^{\mathbf{k}}\,; \tag{H.16}$$

we have

$$\int d^3x\; {}_f\langle 0(t)| : \mathcal{T}_{jj}(x) : |0(t)\rangle_f \;=\; 4\sum_i \int d^3\mathbf{k}\, \frac{k_j k_j}{\omega_{k,i}}\,\mathcal{N}_i^{\mathbf{k}}\,, \tag{H.17}$$

(no summation on j), we again see that ${}_f\langle 0(t)| : \mathcal{T}_{jj}(x) : |0(t)\rangle_f$ is time independent. This completes the diagonal part of Eq.(H.11).

For the component $(0,j)$, $j \neq 0$, of the energy momentum tensor of the condensate we have

$$\int d^3x\; {}_f\langle 0(t)|\mathcal{T}_{0j}(x)|0(t)\rangle_f \;=$$

$$\sum_i\sum_r \int d^3\mathbf{k}\frac{k_j}{2}\; {}_f\langle 0(t)\Big(\alpha^{r\dagger}_{\mathbf{k},i}\alpha^{r}_{\mathbf{k},i} - \alpha^{r\dagger}_{-\mathbf{k},i}\alpha^{r}_{-\mathbf{k},i} + \beta^{r\dagger}_{\mathbf{k},i}\beta^{r}_{\mathbf{k},i} - \beta^{r\dagger}_{-\mathbf{k},i}\beta^{r}_{-\mathbf{k},i}\Big)|0(t)\rangle_f\,, \tag{H.18}$$

then

$$\int d^3x\; {}_f\langle 0(t)|\mathcal{T}_{0j}(x)|0(t)\rangle_f \;=\; 4\sum_i \int d^3\mathbf{k}\,\frac{k_j}{2}\,\Big(\mathcal{N}_i^{\mathbf{k}} - \mathcal{N}_i^{\mathbf{k}}\Big) \;=\; 0\,. \tag{H.19}$$

In a similar way, the (j,l), $j \neq l$ component can be written as

$$\int d^3x\; {}_f\langle 0(t)|\mathcal{T}_{j\,l}(x)|0(t)\rangle_f \;=$$

$$\sum_i\sum_r \int d^3\mathbf{k}\frac{k_j k_l}{2\,\omega_{k,i}}\; {}_f\langle 0(t)\Big(\alpha^{r\dagger}_{\mathbf{k},i}\alpha^{r}_{\mathbf{k},i} - \alpha^{r\dagger}_{-\mathbf{k},i}\alpha^{r}_{-\mathbf{k},i} + \beta^{r\dagger}_{\mathbf{k},i}\beta^{r}_{\mathbf{k},i} - \beta^{r\dagger}_{-\mathbf{k},i}\beta^{r}_{-\mathbf{k},i}\Big)|0(t)\rangle_f\,, \tag{H.20}$$

then

$$\int d^3x \; {}_f\langle 0(t)|\mathcal{T}_{j\,l}(x)|0(t)\rangle_f \;=\; 4\sum_i \int d^3\mathbf{k}\, \frac{k_j k_l}{2\,\omega_{k,i}} \left(\mathcal{N}_i^{\mathbf{k}} - \mathcal{N}_i^{\mathbf{k}}\right) \;=\; 0\,. \quad \text{(H.21)}$$

Summing up, Eq.(H.11) holds for any μ and ν. Moreover, since from Eqs.(H.19) and (H.21), we have ${}_f\langle 0(t)|\mathcal{T}_{0j}(x)|0(t)\rangle_f \;=\; 0$, $j \neq 0$, and ${}_f\langle 0(t)|\mathcal{T}_{j\,l}(x)|0(t)\rangle_f \;=\; 0$, $j \neq l$ respectively, then the energy-momentum tensor density of the vacuum condensate is given by

$$\mathcal{T}_{\mu\nu}^{cond} = diag(\mathcal{T}_{00}^{cond}\,, \mathcal{T}_{11}^{cond}\,, \mathcal{T}_{22}^{cond}\,, \mathcal{T}_{33}^{cond}\,); \quad \text{(H.22)}$$

i.e. the vacuum condensate is homogeneous and isotropic. By using Eqs.(H.12) and (H.15), we have

$$\mathcal{T}_{\mu\nu}^{cond} = diag(\rho_{mix}\,, p_{mix}\,, p_{mix}\,, p_{mix}\,). \quad \text{(H.23)}$$

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