SOME GROUP PROPERTIES ASSOCIATED WITH TWO-VARIABLE WORDS

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Abstract

Let $w(x, y)$ be a word in two variables and $\mathscr{W}$ the variety determined by w. In this thesis, which includes a work made in collaboration with C. Nicotera [5], we raise the following question: if for every pair of elements a, b in a group G there exists $g \in G$ such that $w(a^g, b) = 1$, under what conditions does the group G belong to $\mathscr{W}$?

We introduce for every $g \in G$ the sets

$$W_L^w(g) = \{a \in G \mid w(g, a) = 1\}$$

and

$$W_R^w(g) = \{a \in G \mid w(a, g) = 1\} \,,$$

where the letters L and R stand for left and right. In [2], M. Herzog, P. Longobardi and M. Maj observed that if a group G belongs to the class $\mathscr{Y}$ of all groups which cannot be covered by conjugates of any proper subgroup, then G is abelian if for every $a, b \in G$ there exists $g \in G$ for which $[a^g, b] = 1$. Hence when G is a $\mathscr{Y}$-group and w is the commutator word $[x, y]$, the set $W_L^w(g) = W_R^w(g)$ is the centralizer of g in G, and the answer to the problem is affirmative. If G belongs to the class $\mathscr{Y}$, we show that, more generally, the problem has a positive answer whenever each subset $W_L^w(g)$ is a subgroup of G, or equivalently, if each subset $W_R^w(g)$ is a subgroup of G. The sets $W_L^w(g)$ and $W_R^w(g)$ can be called the *centralizer-like subsets* associated with the word w. They need not be subgroups in general: we examine some sufficient conditions on the group G ensuring that the sets $W_L^w(g)$ and $W_R^w(g)$ are subgroups of G for all g in G. We denote by $\mathscr{W}_L^w$ and $\mathscr{W}_R^w$ respectively the class of all groups G for which the set $W_L^w(g)$ is a subgroup of G for every $g \in G$ and the class of all groups G for which each subset $W_R^w(g)$ is a subgroup.

In particular, we consider the n-Engel word

$$w(x,y) = [x,_n y],$$

with $n \geq 2$. We say that a group G is in the class $\mathscr{C}_n$ if for every pair of elements $a, b \in G$ there exists $g \in G$ such that $[a^g,_n b] = 1$. L.-C. Kappe and P.M. Ratchford proved [3] that if G is a metabelian group, then the centralizer-like subsets associated with the second variable of w are all subgroups of G. From this property it follows that every metabelian $\mathscr{C}_n$-group is n-Engel. If $n = 2$, then we extend this result to the class of all solvable groups, while if $n > 2$ we prove that any finitely generated solvable $\mathscr{C}_n$-group is m-Engel, for some non-negative integer m.

In Chapter 3 of the thesis, we consider the centralizer-like subsets associated with some commutator words in two variables. First we focus on two-variable words of the form

$$w(x,y) = C_n[y,x],$$

where C_n is a left-normed commutator of weight $n \geq 3$ with entries from the set $\{x, y, x^{-1}, y^{-1}\}$. N.D. Gupta [1] considered a number of group laws of the form

$$C_n = [x,y],$$

observing that any finite or solvable group satisfying such a law is abelian. The question arises whether each group satisfying a law of the form $C_n = [x,y]$ is abelian. L.-C. Kappe and M.J. Tomkinson [4] solved the problem in the case $n = 3$, by showing that the variety of groups satisfying one of the laws of the form $C_3 = [x,y]$ is the variety of the abelian groups. In [6], P. Moravec extended the result to the case $n = 4$.

We show that every locally nilpotent group belongs to the classes $\mathscr{W}_L^w$ and $\mathscr{W}_R^w$ associated with the word w. Moreover, if $w(x,y)$ is one of the 2^{n-1} words of the form

$$[y, x^{\alpha_1}, x^{\alpha_2}, \ldots, x^{\alpha_{n-1}}][y,x]$$

or one of the 2^{n-1} words of the form

$$[x^{\alpha_1}, y, x^{\alpha_2}, \ldots, x^{\alpha_{n-1}}][y,x],$$

where $\alpha_i \in \{-1, 1\}$ for every $i = 1, \ldots, n-1$, then any metabelian group belongs to the class $\mathscr{W}_L^w$. In metabelian groups a symmetry of the centralizer-like subsets associated with the words of the form $w(x, y) = C_n[y, x]$ holds: if

$$w(x, y) = [r_1, r_2, r_3, \ldots, r_n][y, x],$$

with $r_i \in \{x, y, x^{-1}, y^{-1}\}$ for every $i = 1, \ldots, n$, then for every element g in a metabelian group G we have

$$W_R^w(g) = W_L^{\overline{w}}(g)$$

and

$$W_L^w(g) = W_R^{\overline{w}}(g),$$

where $\overline{w}(y, x) = [r_2, r_1, r_3, \ldots, r_n][x, y]$.

In Section 3.2.1 we more specifically investigate the word w when $n = 3$. If G is a metabelian group, for the case $n = 3$ we observe that G belongs to the class $\mathscr{W}_L^w$ for exactly eleven of the words w, by exhibiting counterexamples for the remaining words.

We conclude Chapter 3 investigating the words of the form

$$w(x, y) = (xy)^n y^{-n} x^{-n},$$

for some integer n. They are also called the *n-commutator words*. We prove that $\mathscr{W}_L^w = \mathscr{W}_R^w$, and if the centralizer-like subsets $W_L^w(g)$ and $W_R^w(g)$ are both subgroups of G, then we also have $W_L^w(g) = W_R^w(g)$.

R. Baer introduced the *n-center* $Z(G, n)$ of a group G: it is defined as the set of all elements $g \in G$ which *n-commute* with every element $h \in G$, i.e.

$$(gh)^n = g^n h^n \quad \text{and} \quad (hg)^n = h^n g^n.$$

For every element g in a group G we define the *n-centralizer* $C_G(g, n)$ of g in G as the set of all elements of G which n-commute with g, namely, with our notation,

$$C_G(g, n) = W_L^w(g) \cap W_R^w(g)$$

when w is the n-commutator word. The n-centralizer $C_G(g, n)$ is not necessarily a subgroup, even if the group is metabelian. However, we prove that if a group G is 2-Engel, then $C_G(g, n) = W_L^w(g) = W_R^w(g)$ is a subgroup of G for every $g \in G$.

References

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