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# MULTISCALE WAVELET ANALYSIS FOR INTEGRAL AND DIFFERENTIAL PROBLEMS

**Candidato: Aleksey Kudreyko**

**Supervisor**
Prof. Carlo Cattani

**Ph.D. Coordinator**
Prof. Patrizia Longobardi

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UNIVERSITY OF SALERNO
DEPARTMENT OF
MATHEMATICS AND COMPUTER SCIENCE

The undersigned below certify that they have read and recommend to the Dottorato di ricerca for acceptance a thesis entitled "**Multiscale Wavelet Analysis for Integral and Differential Problems**" by **Aleksey Kudreyko** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy in Mathematics**.

Dated: 28 February, 2011

Research Supervisor: _____
Prof. Carlo Cattani

External Committee: _____
Prof. Nail G. Migranov
(Bashkir State University)

_____
Prof. Paolo Maremonti
(Seconda Università degli Studi di Napoli)

_____
Prof. Roberto Musmanno
(Università degli Studi della Calabria)

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Author: **Aleksey Kudreyko**

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*This thesis is dedicated to my parents for their love, endless support and encouragement.*

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Aleksey Kudreyko ©

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# Acronyms

| | |
|---|---|
| DWT | Discrete Wavelet Transform |
| FT | Fourier transform |
| IE(s) | Integral Equation(s) |
| KdV | Korteweg-de Vries (equation) |
| MRA | Multiresolution Analysis |
| ODE | Ordinary Differential Equation |
| PDE(s) | Partial Differential Equation(s) |
| PHW | Periodic Harmonic Wavelet |
| WT | Wavelet Transform |

# Sunto

Oggetto della seguente ricerca è l'analisi di problemi differenziali e integrali, utilizzando wavelet armoniche e wavelet armoniche periodiche. Si dimostra che le wavelet periodiche costituiscono una base completa per le funzioni $L^2[0;1]$ e formano un'analisi multiscala. L'analisi multirisoluzione può essere brevemente considerata come la decomposizione di $L^2[0;1]$ in un insieme completo di sottospazi di wavelet dipendenti da un fattore di scala. Pertanto gli operatori integrali e differenziali e le funzioni $L^2(\mathbb{R})$ vengono studiati come funzioni di scala mediante le corrispondenti proiezioni in questi sottospazi di wavelet. In particolare, vengono sviluppati quattro principali argomenti:

- sono state individuate le condizioni per applicare una data famiglia di wavelets alla soluzione di un data problema differenziale o integrale;

- si è dimostrato che la precisione di questo approccio cresce esponenzialmente quando decresce il numero dei momenti nulli e del parametro di scala;

- soluzioni wavelet di equazioni differenziali a derivate parziali nonlineari di dimensione bassa sono state confrontate con altri metodi di soluzioni;

- l'approccio basato sull'uso delle wavelet è stato applicato anche per ricerca di soluzioni di alcune equazioni integrali di Fredholm e insieme al metodo di Galerkin per risolvere equazioni integrali Fredholm di dimensioni due.

# Abstract

The object of the present research is wavelet analysis of integral and differential problems by means of harmonic and circular wavelets. It is shown that circular wavelets constitute a complete basis for $L^2[0;1]$ functions, and form multiresolution analysis. Multiresolution analysis can be briefly considered as a decomposition of $L^2[0;1]$ into a complete set of scale depending subspaces of wavelets. Thus, integral operators, differential operators, and $L^2(\mathbb{R})$ functions were investigated as scale depending functions through their projection onto these subspaces of wavelets. In particular:

- conditions when a certain wavelet can be applied for solution of integral or differential problem are given;

- it is shown that the accuracy of this approach exponentially grows when increasing the number of vanishing moments and scaling parameter;

- wavelet solutions of low-dimensional nonlinear partial differential equations are compared with other methods;

- wavelet-based approach is applied to low-dimensional Fredholm integral equations and the Galerkin method for two-dimensional Fredholm integral equations.

# Preface

One of the main directions of the development of mathematical knowledge during the past 25 years is the application of wavelets for solution of mathematical problems. This topic has been popular since the middle of the 1980s, which is explained by a large number of publications.

Wavelets were introduced relatively recently, i.e. in the beginning of the 1980s. They attracted a considerable interest from the mathematical community and scientists of many different disciplines, where wavelets had promising applications. As a consequence of this interest, there have appeared many books on this subject and a large volume of research articles.

Wavelets are functions which satisfy certain mathematical requirements and used in representation of data or other functions. The idea of representing a function as a superposition of basis functions is not new. Approximation using superposition of functions has existed since the early 1800's, when Joseph Fourier discovered that it is possible to superpose sines and cosines to represent other functions. However, in the wavelet analysis, the scale that we use to look at data plays a special role. Wavelet algorithms process data at different scales or resolutions. If we look at a signal through a large "window", we would notice great features. Similarly, if we look at a signal through a small "window", we would notice small features. Generally

speaking, the result of the wavelet analysis is to see forest, trees and leafes. This makes wavelets interesting and useful in the data analysis.

Wavelet analysis attracted a great attention of researchers from many subjects, which include acoustics [19], nuclear engineering [26], subband coding [73], signal and image processing, neurophysiology [34], music [46, 59], magnetic resonance imaging [9], speech discrimination, optics, analysis of fractals [14], turbulence, earthquake engineering [33], radar technology [39], and pure mathematical applications such as solution of partial differential equations (PDEs) [10, 11, 43, 53]. It is worth to emphasize that the purpose of the wavelet analysis is rather modest. It helps to describe hidden characterictics of signals. However, it does not claim to explain the nature of phenomena, although it can give valuable suggestions.

The *purpose* of the present work is the research of application of PHWs for solution of nonlinear PDEs. During the selection of practical materials author gave preference to two types of equations:

- Fluid dynamics equations;

- General integral equations, which depend on arbitrary functions. Exact solutions of such equations represent a special interest for testing of new approaches.

The *main results* of this research consist in the following:

1. It was shown that periodic harmonic wavelets satisfy the axioms of the multiresolution analysis (MRA). This property is necessary for solution of PDEs as well as integral equations (IEs) at different levels of multiscale approximation.

2. The obtained numerical results of solution of nonlinear PDEs and IEs were compared with the analytical ones and other numerical methods. This comparison

has shown that the wavelet basis is a good alternative to traditional methods.

3. The error of the projection of solution of PDEs and integral equations on the space of periodic wavelets was analytically estimated.

4. It was shown that the wavelet-Galerkin method for solution of integral equations in two dimensions yields sparse matrices, which reduce the computational costs.

Therefore, our research is *motivated* by the need of creation of highly effective analytical and numerical schemes for solution of PDEs and integral equations.

*Scientific and practical significance of the work.* Mathematical models describe a variety of physical and engineering problems and processes, which can be represented by PDEs and IEs. Therefore, it is getting necessary to have advanced mathematical tools to deal with such kind of equations. The application of wavelets for solution of PDEs and IEs significally extends our opportunities in dealing with some types of integral or/and differential problems.

The advantages of application of wavelets for solution of PDEs consist in the following:

- A variety of wavelet bases enables us to choose such $\psi(x)$ that it produces a maximum number of wavelet coefficients $a_{j,k}$, which are close to zero.

- Approximation by means of analytically defined basis functions is always better than pure numerical computations.

- Quick decay of basis functions makes wavelets useful in approximation of rapidly oscillating functions. Therefore, this approach is more stuitable for problems with localized phenomena.

The advantage of the choice of circular (or periodized) wavelets is that they are defined analytically, and a superposition of these functions is also an analytically defined function. The obtained results can be useful in creation of efficient computer codes for solution of PDEs (as well as IEs) and the further development of the theory of PDEs and IEs.

*Original materials for the defense of the thesis.*

1. Proof of the theorem, which states that PHW form multiscale analysis.
2. Multiscale solution of PDEs on the basis of connection coefficients and the discrete harmonic wavelet transform.
3. Solution of integral equations by means of the collocation method with PHW as basis functions.
4. Error estimation of the projection of a function on the space of periodic wavelets.
5. Wavelet-Galerkin approach for solution of the Fredholm type integral equations in two dimensions.

*Publications and practice of the work.* The main results of the thesis were published in the following articles: [82] – [94], and presented on the International Conference on Computer Science and its Applications (Perugia, 30 June – 2 July, 2008 and Fukuoka, 23 – 26 March, 2010); All-Russian school-conference for students, PhD students and young researchers (Bashkir State University, Ufa 2008, 2009); International Conference "Days on Diffraction" (Saint-Petersburg, 26 – 30 June, 2009); Doctoral Seminar (25 December, 2009) at the Bashkir State Pedagogical University (Ufa) on the Department of Theoretical Physics; Doctoral Seminar (13 October, 2010) at the Katholieke Universiteit Leuven on the Department of Computer Science.

*Structure of the thesis.* PhD thesis consists of the Introduction, four chapters, Conclusion and Appendix. It contains 127 pages, 11 figures and 94 references.

The motivation, scientific novelty of the results, and the defense materials are given in the Introduction.

Chapter 1 is devoted to the overview of wavelets for expansion of functions. The novelty of the wavelet theory consists in the discovery of a new class of special functions, which possess several specific properties.

Chapter 2 shows general theory of harmonic wavelets in the $L^2(\mathbb{R})$ space, and presents periodic wavelets in $L^2[0;1]$. This Chapter also shows that periodic wavelets constitute orthogonal basis in $L^2[0;1]$. The proof of this fact allows us to proceed to solution of several problems.

Chapters 3 and 4 constitute the original part of the dissertation. The literature survey showed that the efforts of previous researchers were not focused on periodic wavelets and its application for integral and differential problems. Chapter 3 discusses the application of PHW for solution of the Burgers equation. The corresponding connection coefficients were derived and discussed. A special attention was given to approximation properties of periodic wavelets. For this reason, different values of the viscosity coefficient were used. The numerical results show that PHW are able to approximate rapidly changing functions.

The solution of the KdV equation shows that there appears a problem of boundary effects because periodic wavelets do not form a complete basis on a finite interval for non-periodic functions. The obtained wavelet solution is compared with the analytical one. The error estimation, which unavoidably appears due to the projection of solution on the space of periodic wavelets is given in Chapter 3.

The main result of Chapter 4 consists in the wavelet-Galerkin approach for solution of two-dimensional integral equations. In particular, it was shown that computational difficulties in solution of such IEs can be overpassed by involving wavelets as basis functions in the Galerkin method. This gives us an opportunity to operate with pseudo-sparse matrices, and thus obtaining additional speedup for no cost.

# Chapter 1

# Introduction to wavelet analysis

**Overview.** The preliminaries on wavelets are provided in this chapter. First, we briefly introduce wavelets in general, then we consider orthogonal in $L^2(\mathbb{R})$ wavelet basis and the related definitions.

## 1.1 Introduction to wavelets

In many cases we deal with functions $f(x)$, which are square integrable, and defined on the $\mathbb{R}$. It is known that such functions form finite-dimensional Hilbert space $L^2(\mathbb{R})$:

$$L^2(\mathbb{R}) = \left\{ f(x) : \mathbb{R} \to \mathbb{C}; \int\limits_{-\infty}^{\infty} |f(x)|^2 dx < \infty \right\},$$

which always has orthonormal bases. Square integrable functions form an inner product space whose scalar product is given by

$$\langle f, g \rangle_{L^2(\mathbb{R})} = \int\limits_{-\infty}^{\infty} f(x) g^*(x) dx \,,$$

where $g^*(x)$ is the complex conjugate of $g(x) \in L^2(\mathbb{R})$ and $f(x) \in L^2(\mathbb{R})$.

Let us introduce the definition of a wavelet function – the main concept discussed in the thesis.

**Definition 1.1.1.** Wavelet is a function $\psi(x) \in L^2(\mathbb{R})$, which depends on two parameters $j, k \in \mathbb{Z}$, such that the set of functions

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$$

forms an orthogonal basis in the Hilbert space $L^2(\mathbb{R})$, and fulfill some additional properties.

Note that, the term *wavelet* does not have a unique definition. For practical use of wavelets, it is important to know criteria (properties) when we can say that this function is a wavelet [1].

1. *Localization.* This property of wavelets follows from the belonging of $\psi_{j,k}(x)$ to $L^2(\mathbb{R})$.

2. *Zero mean*
$$\int_{-\infty}^{\infty} \psi(x)\, dx = 0\,.$$

3. *Boundedness*
$$\int_{-\infty}^{\infty} |\psi(x)|^2\, dx < \infty\,.$$

4. *Self-similarity of basis.* It means that all wavelets of family $\{\psi_{j,k}(x);\, j, k \in \mathbb{Z}\}$ have the same number of oscillations as $\psi(x)$.

In view of Definition 1.1.1, each wavelet has the associated orthogonal system $\{\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)\,|\,j, k \in \mathbb{Z}\}$, which will be called a wavelet basis in $L^2(\mathbb{R})$. The proceeding section is devoted to orthogonal wavelet bases.

## 1.2 Orthogonal wavelet bases

Two functions $f$ and $g$ are called orthogonal if their inner product $\langle f, g \rangle_{L^2(\mathbb{R})}$ is zero for $f \neq g$. Our search for orthogonal wavelets begins with multiresolution approximations. For $f(x) \in L^2(\mathbb{R})$, the partial sum of wavelet coefficients $\sum_{k=-\infty}^{\infty} \langle f(x), \psi_{j,k}(x) \rangle \psi_{j,k}(x)$ can be interpretated as the difference between two approximations of $f(x)$ on the resolutions $2^{-j+1}$ and $2^{-j}$. Multiresolution approach computes the approximation of functions at various resolutions with orthogonal projections on different spaces $\{V_j\}_{j \in \mathbb{Z}}$. Subsection 1.2.1 shows that multiresolution approximations entirely characterized by a particular discrete filter, which governs the loss of information across the resolutions. These discrete filters provide a simple procedure for designing and synthesizing orthogonal wavelet bases.

### 1.2.1 Multiresolution approximations

Multiresolution approximations are based on two fundamental concepts: nested subspaces and orthonormal bases. The first permits decomposition of information into different scales, the second concept allows the construction of fast and stable algorithms. On the top of these concepts, there is a third ingredient: the invariance of basis functions with respect to certain translations. Thus, all the basis functions are nothing else than scaled and translated versions of a *mother function*. This section formalizes multiresolution approximations, and sets the ground for the construction of orthogonal wavelets.

Approximation of a function $f(x)$ at resolution $2^{-j}$ is specified by a discrete grid of samples which provides local averages of $f(x)$ over neighborhoods of size proportional

to $2^j$. Multiresolution approximation is composed of embedded grids of approximation. More formally, the approximation of a function at resolution $2^{-j}$ is defined as orthogonal projection on the space $V_j \subset L^2(\mathbb{R})$. The space $V_j$ regroups all possible approximations at resolution $2^{-j}$. Orthogonal projection of $f$ is the function $f_j \in V_j$ which minimizes the norm: $||f - f_j||_{L^2(\mathbb{R})}$. To avoid the confusion, let us note that the scaling parameter $2^j$ is the inverse of the resolution $2^{-j}$. The following definition of MRA was introduced by S. Mallat [49].

**Definition 1.2.1.** A sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R})$ is a multiresolution approximation if the following 6 properties are satisfied:

i. $\forall (j,k) \in \mathbb{Z}^2,\ f(x) \in V_j \Leftrightarrow f(x - 2^j k) \in V_j\,,$

ii. $\forall j \in \mathbb{Z},\ V_j \subset V_{j+1}\,,$

iii. $\forall j \in \mathbb{Z},\ f(x) \in V_j \Leftrightarrow f\left(\frac{x}{2}\right) \in V_{j+1}\,,$

iv. $\lim\limits_{j\to+\infty} V_j = \bigcap\limits_{j=-\infty}^{\infty} V_j = \{0\}\,,$

v. $\lim\limits_{j\to+\infty} V_j = \text{Closure}\left(\bigcup\limits_{j=-\infty}^{\infty} V_j\right) = L^2(\mathbb{R})\,.$

vi. There exists a function $\varphi$ such that $\{\varphi(x-k)\}_{k\in\mathbb{Z}}$ is a Riesz basis[1] of $V_0$.

Let us give a set of statements constructed to describe these mathematical properties. Property (i) means that $V_j$ is invariant with respect to any translation proportional to the scale $2^j$. As we will see later, this space can be assimilated to a uniform grid with intervals $2^j$, which characterizes approximation at resolution level $2^{-j}$. The inclusion (ii) is a causal property which demonstrates that approximation

[1] In a Hilbert space, an unconditional basis is also called a Riesz basis.

at resolution $2^{-j}$ contains all the necessary information to compute approximation at a coarser resolution $2^{-j-1}$. Dilating functions in space $V_j$ by 2 enlargers all details by factor 2, and (iii) guarantees that it defines an approximation at a coarser resolution $2^{-j-1}$. When the resolution $2^{-j}$ converges to 0, property (iv) implies that we lose all the details of function $f$, and

$$\lim_{j \to -\infty} ||\mathcal{P}_{V_j} f(x)||_{L^2(\mathbb{R})} = 0\,,$$

where $\mathcal{P}_{V_j}$ is the operator of orthogonal projection of $f(x)$ on the space of wavelets. The complete definition of $\mathcal{P}_{V_j}$ will be given later. On the other hand, when the resolution $2^{-j}$ tends to $+\infty$, property (v) imposes that the approximation of a function converges to the original function:

$$\lim_{j \to +\infty} ||f(x) - \mathcal{P}_{V_j} f(x)||_{L^2(\mathbb{R})} = 0\,.$$

When resolution $2^{-j}$ increases, the decay rate of the approximation error

$$||f(x) - \mathcal{P}_{V_j} f(x)||_{L^2(\mathbb{R})}$$

depends on the regularity of $f(x)$.

### 1.2.2 Detailed spaces $W_j$

The given nested subspaces in Definition 1.2.1, we define as $W_j$, which are orthogonal complements of $V_j$ in $V_{j+1}$, i.e. $V_j \perp W_j$, and

$$V_{j+1} = V_j \oplus W_j\,, \tag{1.2.1}$$

where $\oplus$ denotes the direct sum of orthogonal subspaces. Consider two spaces: $V_0$ and $V_N$. The recursive application of formula (1.2.1) gives:

$$V_N = V_0 \oplus \left( \bigoplus_{j=0}^{N-1} W_j \right) .$$

Thus, any function in $V_N$ can be expressed as a linear combination of functions in $V_0$ and $W_j$, $j = 0, 1, \ldots, N-1$; hence it can be analyzed separately at different scales. Multiresolution analysis has received its name from this separation of scales.

### 1.2.3 Basic scaling function and mother wavelet

In view of Definition 1.2.1, the set $\{\varphi(x-k)\}_{k\in\mathbb{Z}}$ is an orthogonal basis in $V_0$, and it follows that

$$\{\varphi(2^j x - k)\}_{j,k\in\mathbb{Z}} \tag{1.2.2}$$

is an orthogonal basis in space $V_j$. Note that (1.2.2) is the function $\varphi(2^j x)$, which is translated by $k/2^j$, i.e. it becomes narrower, and translations get smaller when $j$ grows. Since the squared norm of one of these functions is

$$\int_{-\infty}^{\infty} |\varphi(2^j x - k)|^2 dx = 2^{-j} \int_{-\infty}^{\infty} |\varphi(y)|^2 dy = 2^{-j} ,$$

it follows that $\{2^{j/2}\varphi(2^j x - k)\}_{j,k\in\mathbb{Z}}$ is an orthonormal basis for $V_j$.

In a similar way, as shown in reference [23], there exists a function $\psi(x)$ such that $\{2^{j/2}\psi(2^j x - k)\}_{j,k\in\mathbb{Z}}$ is an orthonormal basis in $W_j$. We call $\varphi(x)$ basic scaling function, and $\psi(x)$ the mother wavelet. It is convenient to introduce the following notations:

$$\varphi_{j,k}(x) = 2^{j/2}\varphi(2^j x - k) ;$$
$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k) ,$$

and

$$\varphi_k(x) = \varphi_{0,k}(x)\,;$$
$$\psi_k(x) = \psi_{0,k}(x)\,.$$

At this point it is useful to present the definition of a scaling function [23].

**Definition 1.2.2.** If $\varphi \in L^2(\mathbb{R})$ satisfies the following conditions:

i. $\hat{\varphi}$ is continuous at the origin, and $\hat{\varphi}(0) = 1$, where $\hat{\varphi}$ is the Fourier transform (A.1.9) of $\varphi$;

ii. $\exists\, M > 0$ such that $\sum_{k\in\mathbb{Z}} |\hat{\varphi}(\omega + 2\pi k)|^2 \leq M$, $\omega \in \mathbb{R}$;

iii. $\hat{\varphi}(2\omega) = H_0(\omega)\hat{\varphi}(\omega)$, where $H_0$ is a $2\pi n$-periodic bounded function, $n \in \mathbb{Z}$.

Then $\varphi$ is called a scaling function.

Since $\psi_{j,k}(x) \in W_j$, it immediately follows that $\psi_{j,k}$ is orthogonal to $\varphi_{j,k}$ because $\varphi_{j,k} \in V_j$ and $V_j \perp W_j$. Also, because all $\{W_j\}$ are mutually orthogonal, it follows that the wavelets are orthogonal across their scales [23]. Therefore, we have the orthogonality relations

$$\int_{-\infty}^{\infty} \varphi_{j,k}(x)\varphi_{j,l}(x)dx = \delta_{k,l}\,;$$
$$\int_{-\infty}^{\infty} \psi_{i,k}(x)\psi_{j,l}(x)dx = \delta_{i,j}\delta_{k,l}\,;$$
$$\int_{-\infty}^{\infty} \varphi_{i,k}(x)\psi_{j,l}(x)dx = 0\,, \quad j \geq i$$

where $i, j, k, l \in \mathbb{Z}$, and $\delta_{k,l}$ is the Kronecher delta.

### 1.2.4 Expansion of a function in $V_N$

A function $f(x) \in V_N$ can be expanded in various ways. For example, there is a pure scaling function expansion

$$f(x) = \sum_{l=-\infty}^{\infty} c_{N,l}\varphi_{N,l}(x)\,, \quad x \in \mathbb{R}$$

where

$$c_{N,l} = \int_{-\infty}^{\infty} f(x)\varphi_{N,l}(x)dx\,.$$

There is also the wavelet expansion

$$f(x) = \sum_{k=-\infty}^{\infty} c_k\varphi_k(x) + \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} d_{j,k}\psi_{j,k}(x)\,, \quad x \in \mathbb{R} \tag{1.2.3}$$

where

$$\begin{aligned} c_k &= \int_{-\infty}^{\infty} f(x)\varphi_k(x)dx\,, \\ d_{j,k} &= \int_{-\infty}^{\infty} f(x)\psi_{j,k}(x)dx\,, \end{aligned} \tag{1.2.4}$$

and $c_k, d_{j,k} \in \mathbb{C}$. The first sum in (1.2.3) with scaling functions contains local averages of $f(x)$ at a certain scale. The second term includes all details of a certain scale. The most visible difference with respect to the Fourier series (A.1.1) is the fact that now we have two types of basis functions, which are indexed by scaling and translation parameters. From the orthonormality of scaling functions and wavelets we find that

$$||f||^2_{L^2(\mathbb{R})} = \sum_{k=-\infty}^{\infty} |c_{N,k}|^2 = \sum_{k=-\infty}^{\infty} |c_k|^2 + \sum_{j=0}^{N-1}\sum_{k=-\infty}^{\infty} |d_{j,k}|^2\,,$$

which represents Parseval's equation for wavelets. Another distinct difference between wavelet and Fourier series is that $\varphi_{j,k}$ can be represented by many functions unified by Definition 1.1.1.

**Definition 1.2.3.** Let $\mathcal{P}_{V_j}$ and $\mathcal{P}_{W_j}$ denote the operators, which project any function $f(x) \in L^2(\mathbb{R})$ orthogonally onto $V_j$ and $W_j$ respectively. Then,

$$\mathcal{P}_{V_j} f(x) = \sum_{k=-\infty}^{\infty} c_{j,k} \varphi_{j,k}(x) \,;$$
$$\mathcal{P}_{W_j} f(x) = \sum_{k=-\infty}^{\infty} d_{j,k} \psi_{j,k}(x) \,,$$

where

$$c_{j,k} = \int_{-\infty}^{\infty} f(x) \varphi_{j,k}(x) dx \,;$$
$$d_{j,k} = \int_{-\infty}^{\infty} f(x) \psi_{j,k}(x) dx \,;$$

and

$$\mathcal{P}_{V_N} f(x) = \mathcal{P}_{V_0} f(x) + \sum_{j=0}^{N-1} \mathcal{P}_{W_j} f(x) \,.$$

The most important property of such decomposition is that $\mathcal{P}_{V_N} f(x)$ has various discrete scales. Let us note once again that the family of functions $\varphi_{j,k}(x)$ forms orthogonal basis in $V_j$. The orthogonal supplement $V_j$ to $V_{j+1}$ is called $W_j$. The subspaces $W_j$ form a mutually orthogonal set.

**Remark.** Space $W_j$ can be chosen such that it is not orthogonal to $V_j$. In this case MRA will lead to the so-called bi-orthogonal wavelets. We will not discuss this point, but only mention that bi-orthogonal wavelets are more flexible [23].

### 1.2.5 Dilation equation and wavelet equation

Since $V_0 \subset V_1$, then any function $f(x) \in V_0$ can be expanded in terms of basis functions of space $V_1$. In particular, $\varphi(x) \equiv \varphi_{0,0}(x) \in V_0$. Therefore,

$$\varphi(x) = \sum_{k=-\infty}^{\infty} a_k \varphi_{1,k}(x) = \sqrt{2} \sum_{k=-\infty}^{\infty} a_k \varphi(2x - k) \,,$$

where

$$a_k = \int_{-\infty}^{\infty} \varphi(x)\varphi_{1,k}(x)dx\,.$$

For a compactly supported scaling function, only a finite number of coefficients $\{a_k\}$ will be nonzeros, then we can write:

$$\varphi(x) = \sqrt{2}\sum_{k=0}^{D-1} a_k\varphi(2x-k)\,. \tag{1.2.5}$$

Equation (1.2.5) is the fundamental in the wavelet theory, and it is known as the dialation equation. A constant $D$ is an even positive integer called the wavelet genus, and numbers $a_0, a_1, \ldots, a_{D-1}$ are called the filter coefficients. The scaling function is uniquely characterized (up to a constant) by these coefficients. One may think that the chosen norm for coefficients $\{a_k\}$ with the factor $\sqrt{2}$ is free. However, it is determined *a posteriori* in fast algorithms [23].

According to equation (1.2.5), we can write a relation for the mother wavelet $\psi(x)$. Since $\psi(x) \in W_0$ and $W_0 \subset V_1$, we can expand $\psi(x)$ as follows:

$$\psi(x) = \sqrt{2}\sum_{k=0}^{D-1} b_k\varphi(2x-k)\,, \tag{1.2.6}$$

where the filter coefficients are

$$b_k = \int_{-\infty}^{\infty} \psi(x)\varphi_{1,k}(x)dx\,,$$

and $\sum_k |b_k|^2 < \infty$. Equation (1.2.6) is the wavelet equation.

# Conclusion of Chapter 1

The first fundamental contributions toward the decomposition of a function with respect to basis functions emerged in works by Brook Taylor in 1715 and Joseph Fourier in 1807. The novelty of the wavelet decomposition consists in the discovery

of a special class of functions, which satify a series of special conditions, and are capable to represent any function or a signal providing highly effective algorithms for data processing.

The concept of the MRA was introduced by Stephane Mallat in 1988, and the paradigm for constructing wavelets was established. Many books exist and a huge number of papers, where many of these formulae are written. Expansion (1.2.3) and formulae (1.2.4) are the most important in the problem of projecting function on the space of wavelets, but there is still no answer to the question: which wavelet is better for a particular problem. The answer exists only on the level of reccomendations.

# Chapter 2

# Harmonic wavelets

**Overview.** Using the definitions presented in Chapter 1, we proceed our discussion to harmonic wavelets defined on the real line $\mathbb{R}$. Then we introduce periodic harmonic wavelets defined on the circle $\mathbb{T}$ (or $[0;1)$). The discrete harmonic wavelet transform is considered in detail.

## 2.1 Choosing a wavelet basis

Mathematical approximation theory suggests to choose a basis that can construct precise approximations with a linear combination of a small number of vectors selected inside the basis. Wavelet bases possess the ability of efficient approximation of particular classes of functions with few nonzero wavelet coefficients. This is true not only for fast calculations but also for data compression and noise reduction. The design of $\psi$ must be optimized to produce a maximum number of wavelet coefficients (1.2.4) which are close to zero, and take less computational costs. A function $f$ has a few non-negligible wavelet coefficients if most of the fine-scale (high-resolution) wavelet coefficients are small. This mostly depends on the regularity of function $f$, the number of vanishing moments in wavelet $\psi$, and the size of its support. Unsuccessful choice of a wavelet can cause a deadlock in the solution of a particular problem.

## 2.2 Introduction into harmonic wavelet analysis

In order to analyze problems of vibration (e.g. [33]), D.E. Newland proposed [57, 58] wavelets whose spectrum is confined exactly to an octave band. It was suggested that the "level" of a signal's multiresolution would be interchangeable with its frequency band and the interpretation of the frequency content, which is inimitable to engineers would be easier. In addition, for the convenience of the further analysis, it would be better to operate with such functions, whose FT was compact and which could, if possible, be constructed from simple functions.

The wavelets considered in this chapter are called harmonic wavelets and periodic harmonic wavelets; they possess all the mentioned properties and constitute a specific, but a representative example of wavelets in general.

## 2.3 Harmonic wavelets

Let us consider a function $\hat{\psi}(\omega)$, whose FT is defined as follows:

$$\hat{\psi}(\omega) = \begin{cases} 1/2\pi \quad \text{for} \quad 2\pi \leq \omega < 4\pi\,, \\ 0 \quad \text{elsewhere}\,. \end{cases} \tag{2.3.1}$$

Then, by calculating the inverse FT (A.1.12) of $\hat{\psi}(\omega)$, we see that the corresponding function is:

$$\psi(x) = \frac{e^{4\pi i x} - e^{2\pi i x}}{2\pi i x}\,, \tag{2.3.2}$$

and it is called a mother function of a harmonic wavelet [58].

Thus, the mother function of Newland's harmonic wavelet represents a complex-valued function, which means that it can return the information about both amplitude and phase; and it is better adapted for capturing oscillatory behavior [72]. The plot of function $\psi(x)$ with real and imaginary parts is shown in Fig. 2.1.

By changing the argument in (2.3.2) from $x$ to $(2^j x - k)$, where $j, k \in \mathbb{Z}$ the shape of the wavelet does not change, but its horizontal scale is compressed by factor $2^j$, and its position is translated by $k$ units at the new scale (which is $k/2^j$ units at the

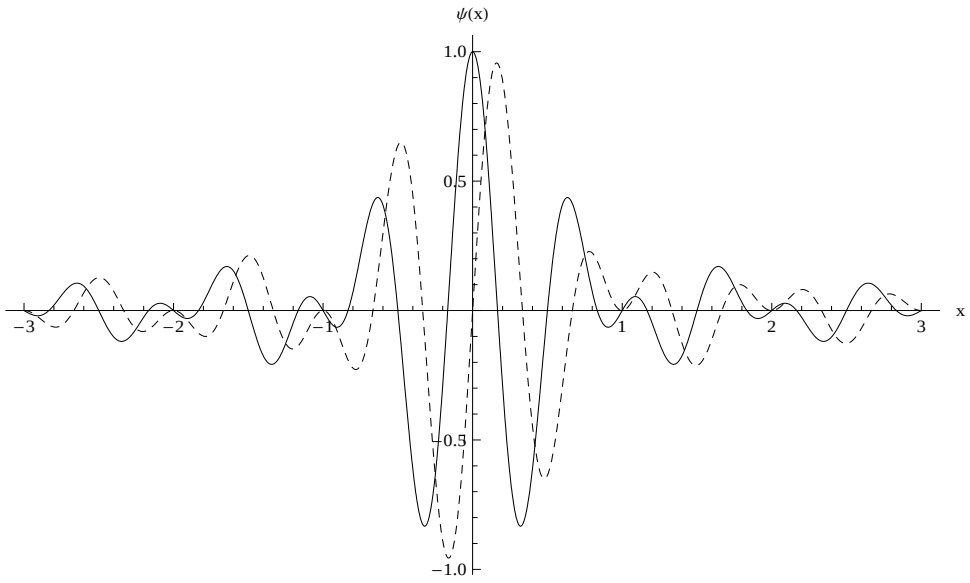


Figure 2.1: *Real (solid) and imaginary (dashed) parts of the mother function of the harmonic wavelet.*

original scale). We call $j$ the scaling parameter because it scales the width of the support, and $k$ is the dilation parameter because it translates the support interval. The value of $j$ determines the "level" of the wavelet. At level $j = 0$ wavelet's FT occupies the bandwidth from $2\pi$ to $4\pi$, and at level $j$ it occupies the bandwidth from $2\pi\,2^j$ to $4\pi\,2^j$ which is $j$ octaves higher up the frequency scale. In other words, the compression of $\psi(x)$ in the $x$-domain by factor $2^j$, spreads out (or dilates) $\hat{\psi}(\omega)$ in the $\omega$ (or frequency) – domain by replacing $\hat{\psi}(\omega)$ into $\hat{\psi}(\omega/2^j)$. The translation of $\psi(x)$ by $k$ units involves rotation of its FT in the complex plane of $\hat{\psi}(\omega)$. Harmonic wavelets possess several specific properties, which are described in the proceeding sections.

### 2.3.1 Orthogonality

Because of the simplicity of the FT of harmonic wavelets, orthogonality can be easily demonstrated in the frequency domain, but first let us consider the FT of $\psi(2^j x - k)$.

**Theorem 2.3.1.** *[57] If $\hat{\psi}(\omega)$ and $\hat{\psi}_{j,k}(\omega)$ are Fourier transforms of $\psi(x)$ and $\hat{\psi}_{j,k}(x) = \psi(2^j x - k)$, then the Fourier transform of $\psi_{j,k}(x)$ is*

$$\hat{\psi}_{j,k}(\omega) = 2^{-j}\, e^{-i\omega k/2^j} \hat{\psi}\left(\frac{\omega}{2^j}\right). \tag{2.3.3}$$

*Proof.* Indeed, if $\hat{\psi}(\omega)$ and $\hat{\psi}_{j,k}(\omega)$ are FTs of $\psi(x)$ and $\psi(2^j x - k)$. Then,

$$\hat{\psi}_{j,k}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi\left(2^j x - k\right) e^{-i\omega x} dx\,.$$

Assuming that $z = 2^j x - k$, we obtain:

$$\hat{\psi}_{j,k}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi(z)\, e^{-i\omega(z+k)2^{-j}} 2^{-j} dz = 2^{-j}\, e^{-i\omega k/2^j}\, \hat{\psi}\left(\frac{\omega}{2^j}\right),$$

which completes the proof. □

It follows that the Fourier transforms of the successive levels of harmonic wavelets decrease during the propagation of their bandwidth, as shown in Fig. 2.2. For $\omega < 0$, they are always zero. Then the orthogonality we can write as follows:

$$\int_{-\infty}^{\infty} \psi(x)\, \psi\left(2^j x - k\right) dx = 0\,, \quad \forall j, k \in \mathbb{Z}\,. \tag{2.3.4}$$

The proof derives from Theorem 2.3.1.

**Theorem 2.3.2.** *[57] If $\psi_{j,k}(x), \psi_{l,h}(x) \in \mathbb{C}$, where $x \in \mathbb{R}$ and their Fourier transforms are $\hat{\psi}_{j,k}(\omega)$ and $\hat{\psi}_{l,h}(\omega)$, then there exists the following property*

$$\int_{-\infty}^{\infty} \psi_{j,k}(x)\, \psi_{l,h}(x)\, dx = 2\pi \int_{-\infty}^{\infty} \hat{\psi}_{j,k}(\omega)\, \hat{\psi}_{l,h}(-\omega)\, d\omega. \tag{2.3.5}$$

*Proof.* If $\langle \psi_{j,k}(x), \psi_{l,h}(x) \rangle_{L^2(\mathbb{R})} = 0$ with the Fourier transforms $\hat{\psi}_{j,k}(\omega)$ and $\hat{\psi}_{l,h}(\omega)$ so that

$$\psi_{j,k}(x) = \int_{-\infty}^{\infty} \hat{\psi}_{j,k}(\omega)\, e^{i\omega x} d\omega\,; \quad \psi_{l,h}(x) = \int_{-\infty}^{\infty} \hat{\psi}_{l,h}(\omega)\, e^{i\omega x} d\omega\,,$$

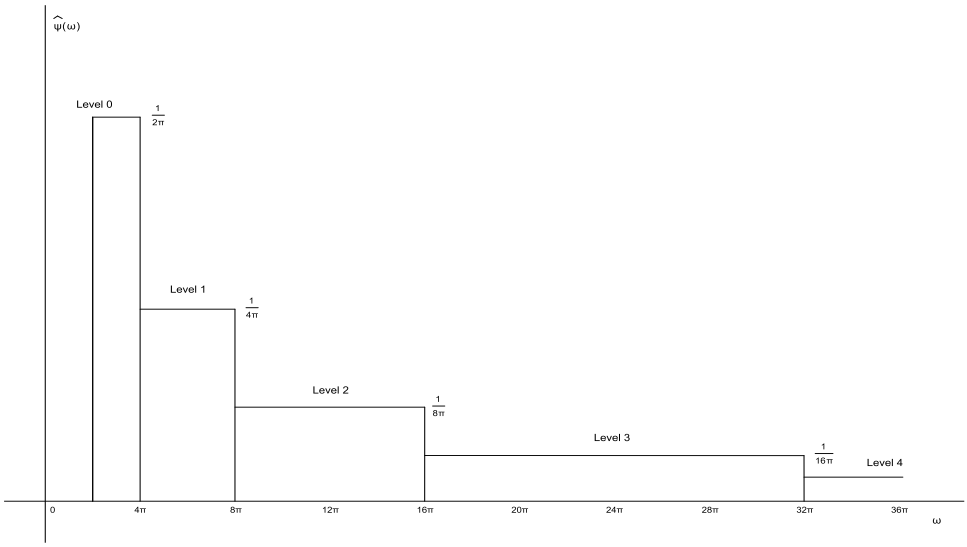


Figure 2.2: *Fourier transforms of complex harmonic wavelets of different levels.*

then its substitution into equality (2.3.5) yields

$$\begin{aligned} &\int_{-\infty}^{\infty} \psi_{j,k}(x)\,\psi_{l,h}(x)\,dx = \\ &\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} d\omega_{j,k} \int_{-\infty}^{\infty} \hat{\psi}_{j,k}(\omega_{j,k})\,\hat{\psi}_{l,h}(\omega_{l,h})\,e^{i\left(\omega_{j,k}+\omega_{l,h}\right)x} d\omega_{l,h}\,. \end{aligned} \tag{2.3.6}$$

From the theory of $\delta$-functions we know that its FT is given by

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(x-c)\,e^{-i\omega x}dx = \frac{1}{2\pi}e^{-i\omega c},$$

therefore, the inverse FT is

$$\int_{-\infty}^{\infty} e^{i\omega(x-c)}d\omega = 2\pi\delta(x-c)\,.$$

Using the result (2.3.6), we obtain

$$\int_{-\infty}^{\infty} e^{i\left(\omega_{j,k}+\omega_{l,h}\right)x} dx = 2\pi\delta\left(\omega_{j,k}+\omega_{l,h}\right). \tag{2.3.7}$$

The substitution of (2.3.7) into (2.3.6), and the integration over $\omega_{l,h}$ yields us the equality:

$$\int_{-\infty}^{\infty} \psi_{j,k}(x)\,\psi_{l,h}(x)\,dx = 2\pi\int_{-\infty}^{\infty} \hat{\psi}_{j,k}(\omega)\,\hat{\psi}_{l,h}(-\omega)\,d\omega,$$

after replacing $\omega_{j,k}$ by $\omega$. Also, by replacing $\psi_{l,h}(x)$ in (2.3.6) by $\psi^*_{l,h}(x)$, the corresponding result is

$$\int_{-\infty}^{\infty} \psi_{j,k}(x)\,\psi^*_{l,h}(x)\,dx = 2\pi\int_{-\infty}^{\infty} \hat{\psi}_{j,k}(\omega)\,\hat{\psi}^*_{l,h}(\omega)\,d\omega. \tag{2.3.8}$$

Thus, the theorem is proved. □

When $\psi_{j,k}(x)$ and $\psi_{l,h}(x)$ are two harmonic wavelets, then they have one-sided FT, as shown in Fig. 2.2, so that the product $\langle\hat{\psi}_{j,k}(\omega)\,\hat{\psi}_{l,h}(-\omega)\rangle$ must always be zero. Therefore, the right-hand side of identity (2.3.5) is always zero, and (2.3.4) must be always true. In addition, we need to know the conditions for which

$$\int_{-\infty}^{\infty} \psi(x)\,\psi^*\left(2^j x-k\right)dx = 0, \tag{2.3.9}$$

where the asterisk over $\psi$ means the complex conjugate of $\psi(2^j x-k)$. In this case, since (the proof see above)

$$\int_{-\infty}^{\infty} \psi_{j,k}(x)\,\psi^*_{l,h}(x)\,dx = 2\pi\int_{-\infty}^{\infty} \hat{\psi}_{j,k}(\omega)\,\hat{\psi}^*_{l,h}(\omega)\,d\omega, \tag{2.3.10}$$

it is obvious that wavelets of different levels are always orthogonal to each other. This is because their Fourier transforms occupy different frequency bands, therefore their scalar product $<\hat{\psi}_{j,k}(\omega),\hat{\psi}_{l,h}(\omega)>$ always equals to zero for $j\neq 0$.

Wavelet translation within a level is the result of progressive rotation of the wavelet's FT in the frequency domain. This may be seen from formula (2.3.3). Assuming $\hat{\psi}_{l,h}(\omega) = e^{-i\omega k}\hat{\psi}_{j,k}(\omega)$, then its substitution into (2.3.10) yields us equality

$$\int_{2\pi}^{4\pi} e^{i\omega k} d\omega = 0\,, \tag{2.3.11}$$

where we have taken into account the support interval of $\hat{\psi}_{j,k}(\omega)$. The integral will be equal to zero if

$$e^{4\pi ik} = e^{2\pi ik}. \tag{2.3.12}$$

Therefore, all wavelets translated by any number of unit intervals are orthogonal to each other.

The result was found for the wavelet of the zeroth level defined by the FT (2.3.1). For the other levels, the same result applies, except that the unit interval is now that for the wavelet level concerned. For example, for level $j$, the unit interval is $1/2^j$, and the translation must be through any multiple of distance $1/2^j$.

The conclusion is: the family of wavelets defined by function

$$\psi_{j,k}(x) = \frac{e^{4\pi i\left(2^j x-k\right)} - e^{2\pi i\left(2^j x-k\right)}}{2\pi i\left(2^j x-k\right)} \tag{2.3.13}$$

forms an orthogonal basis in $L^2(\mathbb{R})$. Wavelets of different levels (different $j$) are always orthogonal; wavelets of the same level are orthogonal if one is translated with respect to the other by a unit interval (different $k$). In orthogonal wavelet analysis, the number of convolutions at each scale is proportional to the width of the wavelet basis at that scale. This produces a wavelet spectrum that contains discrete "blocks" of wavelet power and is useful for signal processing because it gives the most compact representation of the signal.

### 2.3.2 Scaling function of harmonic wavelet

Fig. 2.2 shows the Fourier transforms of levels for $j \geq 0$. For the octave bands defined by $j < 0$, the same sequence as shown in Fig. 2.2 may be maintained, in this case the

resolution of $f(x)$ embraces wavelets of all orders including vanishing small octave bands when $j \to -\infty$. The theory of MRA shows [16, 23, 58] that all negative orders can be rolled together into a small order (which is referred to $-1$), which covers the whole of the residual frequency band from 0 to $2\pi$ as shown in Fig. 2.2.

Functions generated by calculating the inverses of the Fourier transforms in Fig. 2.3 represent scaling functions.

**Definition 2.3.1.** Function $\varphi(x)$, which is generated from a unit box (unit length and height) is called the scaling function [58].

The even Fourier transform

$$\hat{\varphi}_e(\omega) = \begin{cases} 1/4\pi \quad \text{for} \quad -2\pi \leq \omega < 2\pi, \\ 0 \quad \text{elsewhere}, \end{cases} \tag{2.3.14}$$

gives an even scaling function

$$\varphi_e(x) = \frac{\sin 2\pi x}{2\pi x},$$

and the odd Fourier transform

$$\hat{\varphi}_o(\omega) = \begin{cases} i/4\pi \quad \text{for} \quad -2\pi \leq \omega < 0, \\ -i/4\pi \quad \text{for} \quad 0 \leq \omega < 2\pi, \\ 0 \quad \text{elsewhere}, \end{cases} \tag{2.3.15}$$

gives an odd scaling function

$$\varphi_o(x) = -\frac{\cos 2\pi x - 1}{2\pi x}$$

so that, defining the complex scaling function $\varphi(x)$ by

$$\varphi(x) = \varphi_e(x) + i\varphi_o(x),$$

we find that

$$\varphi(x) = \frac{\sin 2\pi x}{2\pi x} - \frac{i(\cos 2\pi x - 1)}{2\pi x} = \frac{e^{2\pi i x} - 1}{2\pi i x}. \tag{2.3.16}$$

Its Fourier transform is

$$\hat{\varphi}(\omega) = \hat{\varphi}_e(\omega) + i\hat{\varphi}_o(\omega)$$

so that, from (2.3.14) and (2.3.15),

$$\hat{\varphi}(\omega) = \begin{cases} 1/2\pi & \text{for} \quad 0 \leq \omega < 2\pi, \\ 0, & \text{elsewhere,} \end{cases} \tag{2.3.17}$$

as shown in Fig. 2.3 c.

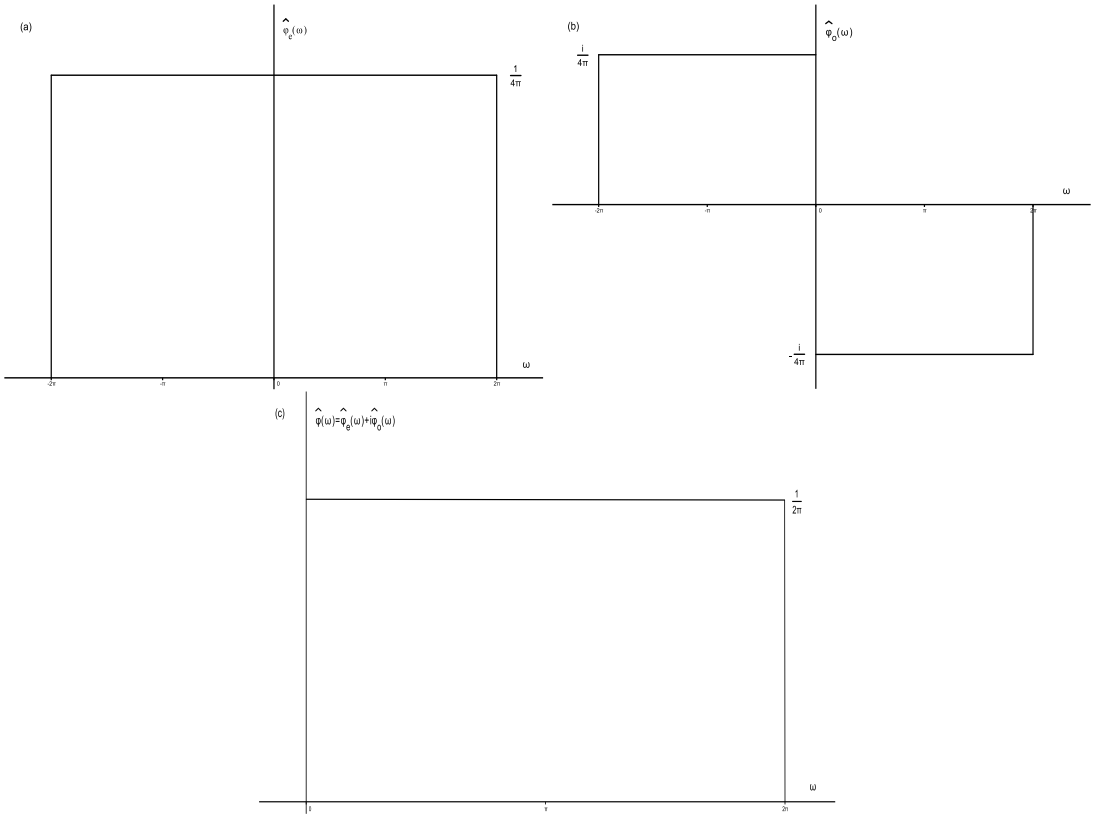


Figure 2.3: *Fourier transforms of the* $(a)$ *even,* $(b)$ *odd and* $(c)$ *complex scaling function.*

Graphs of the real and imaginary parts of function (2.3.16) are illustrated in Fig. 2.4. It is easy to see that $\varphi(x)$ is orthogonal to its own unit translations. In addition, the scaling function is orthogonal to $\psi(x)$ because its Fourier transforms are confined to separate the frequency bands.

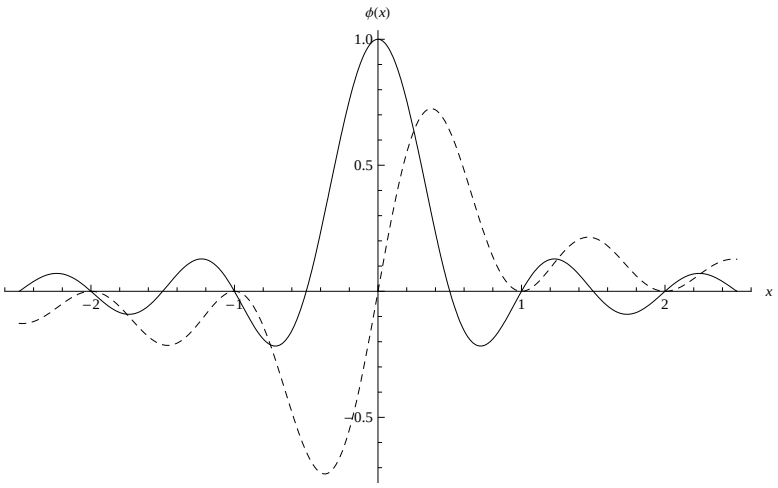


Figure 2.4: *Real part (solid) and imaginary parts (dashed) of the harmonic scaling function.*

**Remark.** In view of the wavelet equation (1.2.6), function $\psi(x)$ comprises a finite linear combination of $\varphi(2x)$, then the algorithms, which include scaling functions can be replaced by wavelets.

### 2.3.3 Normalization

Using the definitions of $\psi(x)$ by (2.3.2) and $\varphi(x)$ by (2.3.16), we can be present the following theorems.

**Theorem 2.3.3.** [*58*] *Harmonic wavelets are normalized functions, i.e.*

$$\int_{-\infty}^{\infty} \left|\psi\left(2^j x - k\right)\right|^2 dx = \frac{1}{2^j}. \tag{2.3.18}$$

*Proof.* Recalling identity (2.3.8) we have,

$$\int_{-\infty}^{\infty} \psi(x)\,\psi^*(x)\,dx = 2\pi \int_{-\infty}^{\infty} \hat{\psi}(\omega)\,\hat{\psi}^*(\omega)\,d\omega\,;$$

therefore,

$$\int_{-\infty}^{\infty} \psi\left(2^j x-k\right) \psi^*\left(2^j x-k\right) dx = 2\pi\, 2^{-2j} \int_{-\infty}^{\infty} \hat{\psi}\left(\frac{\omega}{2^j}\right) \hat{\psi}^*\left(\frac{\omega}{2^j}\right) d\omega.$$

As we know from (2.3.1), $\hat{\psi}\left(\omega/2^j\right) = 1/2\pi$, where $2\pi\, 2^j \leq \omega < 4\pi\, 2^j$; finally, we have

$$\begin{aligned}\int_{-\infty}^{\infty} \left|\psi\left(2^j x-k\right)\right|^2 dx =& 2\pi\, 2^{-2j} \int_{-\infty}^{\infty} \hat{\psi}\left(\frac{\omega}{2}\right) \hat{\psi}^*\left(\frac{\omega}{2}\right) d\omega \\ =& 2\pi\, 2^{-2j} \int_{2\pi 2^j}^{4\pi 2^j} \left(\frac{1}{2\pi}\right)^2 d\omega = \frac{1}{2^j}\,.\end{aligned}$$

Thus, the identity given in (2.3.18) holds. □

Analogously we can show the normalization of scaling functions.

**Theorem 2.3.4.** *Scaling functions of harmonic wavelets are normalized, i.e.*

$$\int_{-\infty}^{\infty} |\varphi\left(x-k\right)|^2 dx = 1\,. \tag{2.3.19}$$

*Proof.* Recalling equation (2.3.16), and making the corresponding substitution into (2.3.19), we obtain:

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{e^{2\pi i(x-k)}-1}{2\pi i\left(x-k\right)} \times \frac{e^{-2\pi i(x-k)}-1}{2\pi\left(-i\right)\left(x-k\right)} dx &= \int_{-\infty}^{\infty} \frac{2-e^{2\pi i(x-k)}-e^{-2\pi i(x-k)}}{4\pi^2\left(x-k\right)^2} dx \\ &= \frac{1}{\pi^2} \int_{-\infty}^{\infty} \frac{\sin^2\pi\left(x-k\right)}{\left(x-k\right)^2} dx = 1.\end{aligned}$$

□

*Proof.* (more general [57]).

$$\int_{-\infty}^{\infty} \varphi\left(x-k\right) \varphi^*\left(x-k\right) dx = 2\pi \int_{-\infty}^{\infty} \hat{\varphi}\left(\omega\right) \hat{\varphi}^*\left(\omega\right) d\omega.$$

Recalling (2.3.17), we come to the following result

$$\int_{-\infty}^{\infty} |\varphi(x-k)|^2 dx = 2\pi \int_{0}^{2\pi} \left(\frac{1}{2\pi}\right)^2 d\omega = 1\,,$$

which completes the proof of the normalization of harmonic wavelets. □

By (2.3.4), we can write

$$\int_{-\infty}^{\infty} \psi^2\left(2^j x - k\right) dx = 0\,.$$

And in the same way we can establish that

$$\int_{-\infty}^{\infty} \varphi^2(x-k)\, dx = 0\,. \tag{2.3.20}$$

It is important to have these relations for expansion of an arbitrary function $f(x)$ (real or complex) in terms of complex harmonic wavelets. Examples of some applied problems of function expansion in terms of harmonic and Shannon wavelets can be found in references [11, 13, 15].

### 2.3.4 Property of vanishing moments

Another important property of scaling functions is their ability to represent polynomials up to a certain order $P-1$ [49]. More precisely, it is required that

$$x^p = \sum_{k=-\infty}^{\infty} M_k^p \varphi(x-k), \quad x \in \mathbb{R}, \quad p = 0, 1, \ldots, P-1 \tag{2.3.21}$$

where

$$M_k^p = \int_{-\infty}^{\infty} x^p \varphi(x-k)\, dx, \quad k \in \mathbb{Z}, \quad p = 0, 1, \ldots, P-1 \tag{2.3.22}$$

and $M_k^p$ denotes the $p$-th moment of $\varphi(x-k)$. Equation (2.3.21) can be translated into the orthogonality condition involving wavelets by taking the inner product with

$\psi(x)$. In particular, we obtain:

$$\int_{-\infty}^{\infty} x^p \psi(x) dx = \sum_{k=-\infty}^{\infty} M_k^p \int_{-\infty}^{\infty} \varphi(x-k)\psi(x) dx = 0\,.$$

Using the orthogonality of $\psi(x)$ and $\varphi(x)$, we can get the property of $P$ vanishing moments [49, 78]:

$$\int_{-\infty}^{\infty} x^p \psi(x) dx = 0, \quad x \in \mathbb{R},\, p = 0, 1, \ldots, P-1. \tag{2.3.23}$$

Thus, a wavelet has $P$ vanishing moments if it is orthogonal to all polynomials of degree $P-1$ or smaller.

### 2.3.5 Multiresolution of complex $f\,(x)$

The goal of the WT is to decompose any arbitrary function $f(x) \in L^2(\mathbb{R})$ into an infinite summation of wavelets at different scales according to the expansion [58]

$$f\,(x) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} a_{j,k} \psi\left(2^j x - k\right) \tag{2.3.24}$$

or

$$f\,(x) = \sum_{k=-\infty}^{\infty} a_{\varphi,k} \varphi\,(x-k) + \sum_{j=0}^{\infty} \sum_{k=-\infty}^{\infty} a_{j,k} \psi\left(2^j x - k\right). \tag{2.3.25}$$

Wavelets of all levels are included in the first formula; in the second expansion, all negative levels are replaced by scaling functions and its translations. The scaling function coefficients $\{a_{\varphi,k}\}_{k=-\infty}^{\infty}$ can be interpreted as a local weighted average of $f(x)$ in the region where $\varphi_k$ is nonzero. On the other hand, wavelet coefficients $\{a_{j,k}\}$ represent the opposite property, i.e. the details of $f(x)$ which are lost in the weighted average.

Equations (2.3.24) and (2.3.25) assume that $f\,(x) \in \mathbb{R}$, and the wavelets are derived from the solution of two scale dilation equations with real coefficients, and there is only one wavelet for each pair of $j$ and $k$. For harmonic wavelets there are, as

we have seen, two wavelets for each $j, k$ pair, namely an even real wavelet $\psi_e\left(2^j x - k\right)$ and an odd complex wavelet $\psi_o\left(2^j x - k\right)$ (See Fig. 2.1). Since these functions are combined together by (2.3.13) into a single complex wavelet $\psi\left(2^j x - k\right)$, the wavelet coefficients $\{a_{j,k}\}$ will be also complex, i.e. $a_{j,k} \in \mathbb{C}$. The expansion coefficients can be computed using the orthogonality relations (1.2.4). We shall define a pair of complex wavelet coefficients by the following integrals:

$$a_{j,k} = 2^j \int_{-\infty}^{\infty} f(x)\, \psi^*\left(2^j x - k\right) dx; \quad \widetilde{a}_{j,k} = 2^j \int_{-\infty}^{\infty} f(x)\, \psi\left(2^j x - k\right) dx\,, \tag{2.3.26}$$

and the corresponding pair of complex coefficients in the terms of scaling functions,

$$a_{\varphi,k} = \int_{-\infty}^{\infty} f(x)\, \varphi^*(x-k)\, dx, \quad \widetilde{a}_{\varphi,k} = \int_{-\infty}^{\infty} f(x)\, \varphi(x-k)\, dx\,. \tag{2.3.27}$$

For a real-valued function $f(x) \in \mathbb{R}$, the following relations exist: $\widetilde{a}_{j,k} = a^*_{j,k}$ and $\widetilde{a}_{\varphi,k} = a^*_{\varphi,k}$. So, one can cut the number of independent wavelet coefficients in half. But to allow $f(x)$ to be a complex function, we will distinguish between $\widetilde{a}_{j,k}$ and $a^*_{j,k}$.

**Theorem 2.3.5.** [57, 58] *In terms of these coefficients, the contribution of a single complex wavelet to the function $f(x)$ is*

$$a_{j,k}\psi\left(2^j x - k\right) + \widetilde{a}_{j,k}\psi^*\left(2^j x - k\right)\,,$$

*and expansion formulae* (2.3.24) *and* (2.3.25) *become*

$$f(x) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \left\{a_{j,k}\psi\left(2^j x - k\right) + \widetilde{a}_{j,k}\psi^*\left(2^j x - k\right)\right\}, \tag{2.3.28}$$

*or alternatively in the basis of wavelets for $j \geq 0$ supplemented by the scaling function $\varphi$:*

$$\begin{aligned} f(x) = &\sum_{k=-\infty}^{\infty} \left\{a_{\varphi,k}\varphi(x-k) + \widetilde{a}_{\varphi,k}\varphi^*(x-k)\right\} + \\ &\sum_{j=0}^{\infty} \sum_{k=-\infty}^{\infty} \left\{a_{j,k}\psi\left(2^j x - k\right) + \widetilde{a}_{j,k}\psi^*\left(2^j x - k\right)\right\}. \end{aligned} \tag{2.3.29}$$

The conjugate counterpart in expansions (2.3.28) and (2.3.29) is explained by the mutual cancellation of the complex terms. Let us proof the identity of (2.3.28) and (2.3.29).

*Proof.* It is obvious from (2.3.28) and (2.3.29) that the following identity of the residual terms

$$\sum_{k=-\infty}^{\infty} a_{\varphi,k}\varphi\left(x-k\right) = \sum_{j=-\infty}^{-1}\sum_{k=-\infty}^{\infty} a_{j,k}\psi\left(2^{j}x-k\right), \tag{2.3.30}$$

exists, and analogously for $\widetilde{a}_{\varphi,k}$, $\varphi^{*}\left(x-k\right)$, $\widetilde{a}_{j,k}$ and $\psi^{*}\left(2^{j}x-k\right)$, where coefficients $a_{\varphi,k}$ are defined by (2.3.27). And substituting

$$f\left(x\right) = \int_{-\infty}^{\infty} \hat{f}\left(\omega\right) e^{i\omega x} d\omega\,;$$

$$\varphi\left(x-k\right) = \int_{-\infty}^{\infty} \hat{\psi}\left(\omega\right) e^{-i\omega k} e^{i\omega k} d\omega\,,$$

we find

$$a_{\varphi,k} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} d\omega_1 \int_{-\infty}^{\infty} \hat{f}\left(\omega_1\right)\hat{\psi}^{*}\left(\omega_2\right) e^{i\omega_2 k} e^{i(\omega_1-\omega_2)x} d\omega_2\,.$$

If we recall the identity

$$\int_{-\infty}^{\infty} e^{i(\omega_1-\omega_2)x} dx = 2\pi\delta\left(\omega_1-\omega_2\right), \tag{2.3.31}$$

we obtain

$$a_{\varphi,k} = 2\pi \int_{-\infty}^{\infty} \hat{f}\left(\omega\right)\hat{\psi}^{*}\left(\omega\right) e^{i\omega k} d\omega.$$

And the left-hand side of (2.3.30) becomes

$$\sum_{k=-\infty}^{\infty} a_{\varphi,k}\varphi\left(x-k\right) = \sum_{k=-\infty}^{\infty} 2\pi \int_{-\infty}^{\infty} d\omega_1 \int_{-\infty}^{\infty} \hat{f}\left(\omega_1\right)\hat{\psi}^{*}\left(\omega_1\right) e^{i(\omega_1-\omega_2)k} e^{i\omega_2 x} d\omega_2\,.$$

The summation in the right-hand side involves only the term $e^{i(\omega_1-\omega_2)k}$. This can be evaluated using the Poisson summation formula[2]

$$\frac{1}{2l}\sum_{k=-\infty}^{\infty} e^{-\frac{ixk\pi}{l}} = 2\pi \sum_{m=-\infty}^{\infty} \delta\left(\omega_2-\omega_1-2\pi m\right).$$

Assuming that $x=\omega_2-\omega_1$, $\pi/l=1$, we get

$$\sum_{k=-\infty}^{\infty} e^{i(\omega_1-\omega_2)k} = 2\pi \sum_{m=-\infty}^{\infty} \delta\left(\omega_2-\omega_1-2\pi m\right).$$

Then,

$$\begin{aligned}&\sum_{k=-\infty}^{\infty} a_{\varphi,k}\varphi\left(x-k\right) = \\ &\sum_{k=-\infty}^{\infty} (2\pi)^2 \int_{-\infty}^{\infty} d\omega_1 \int_{-\infty}^{\infty} \hat{f}\left(\omega_1\right)\hat{\psi}^*\left(\omega_1\right)\hat{\psi}\left(\omega_2\right)e^{i\omega_2 x}\delta\left(\omega_2-\omega_1-2\pi m\right)d\omega_2\,.\end{aligned} \tag{2.3.32}$$

It turns out that only one term in the new summation needs to be included, since the product $\hat{\psi}^*\left(\omega_1\right)\hat{\psi}\left(\omega_2\right)$ equals zero unless its arguments belong to the same frequency band $0 \le \omega < 2\pi$, which corresponds to $m=0$. If $\omega_1=\omega_2$, then $\hat{\psi}^*\left(\omega_1\right)\hat{\psi}\left(\omega_2\right) = 1/(2\pi)^2$. Hence,

$$\sum_{k=-\infty}^{\infty} a_{\varphi,k}\varphi\left(x-k\right) = \int_0^{2\pi} \hat{f}\left(\omega\right)e^{i\omega x}d\omega.$$

The same analysis can be applied to the right-hand side of equality (2.3.30). Beginning from (2.3.27),

$$a_{j,k} = 2^j \int_{-\infty}^{\infty} f\left(x\right)\psi^*\left(2^j x-k\right)dx$$

[2]For appropriate function $f$, the Poisson summation formula may be stated as: $\sum_{k=-\infty}^{\infty} f(x+kT) = \frac{1}{T}\sum_{m=-\infty}^{\infty} \hat{f}\left(\frac{m}{T}\right)\exp\left(2\pi i\frac{mx}{T}\right)$, where $\hat{f}$ is the FT of $f$, and $f$ is a continious and integrable function. For full details see e.g. [64], p.222.

and transferring this to the frequency domain by implementing the Fourier transform, we obtain

$$\psi^*\left(2^j x - k\right) = 2^{-j} \int_{-\infty}^{\infty} \hat{\psi}^*\left(\frac{\omega}{2^j}\right) e^{i\omega k/2^j} e^{-i\omega x} d\omega\,.$$

Then we have

$$a_{j,k} = \int_{-\infty}^{\infty} d\omega_1 \int_{-\infty}^{\infty} \hat{f}\left(\omega_1\right) \hat{\psi}\left(\frac{\omega_2}{2^j}\right) e^{i\omega_2 k} e^{i(\omega_1-\omega_2)x} d\omega_2\,.$$

Taking into account equality (2.3.31), we get

$$a_{j,k} = 2\pi \int_{-\infty}^{\infty} \hat{f}\left(\omega\right) \hat{\psi}^*\left(\frac{\omega}{2^j}\right) e^{i\omega k/2^j} d\omega.$$

Then, substituting $a_{j,k}$ and $\psi\left(2^j x - k\right)$ into the right-hand side of (2.3.30), we get

$$\begin{aligned} &\sum_{j=-\infty}^{-1} \sum_{k=-\infty}^{\infty} a_{j,k} \psi\left(2^j x - k\right) = \sum_{j=-\infty}^{-1} \sum_{k=-\infty}^{\infty} 2\pi\, 2^{-j} \\ &\int_{-\infty}^{\infty} d\omega_1 \int_{-\infty}^{\infty} \hat{f}\left(\omega_1\right) \hat{\psi}^*\left(\frac{\omega_1}{2^j}\right) \hat{\psi}\left(\frac{\omega_2}{2^j}\right) e^{i\omega_2 x} e^{i(\omega_1-\omega_2)k/2^j} d\omega_2\,. \end{aligned} \tag{2.3.33}$$

Taking the summation over $k$ first and the Poisson summation, we have

$$\sum_{k=-\infty}^{\infty} e^{i(\omega_1-\omega_2)k/2^j} = 2\pi\, 2^j \sum_{m=-\infty}^{\infty} \delta\left(\omega_2 - \omega_1 - 2\pi\, 2^j m\right).$$

From (2.3.1), the product $\hat{\psi}^*\left(\omega_1/2^j\right) \hat{\psi}\left(\omega_2/2^j\right)$ is zero unless $m = 0$ in the latter equality when it is $1/(2\pi)^2$. The conclusion is that (2.3.33) gives

$$\begin{aligned} \sum_{j=-\infty}^{-1} \sum_{k=-\infty}^{\infty} a_{j,k} \psi\left(2^j x - k\right) &= \sum_{j=-\infty}^{\infty} \int_{2\pi\, 2^j}^{2\pi\, 2^{j+1}} \hat{f}\left(\omega\right) e^{i\omega x}\, d\omega \\ &= \int_{0}^{2\pi} \hat{f}\left(\omega\right) e^{i\omega x}\, d\omega\,, \end{aligned}$$

which is the same as $\displaystyle\sum_{k=-\infty}^{\infty} a_{\varphi,k} \varphi\left(x - k\right) = \int_{0}^{2\pi} \hat{f}\left(\omega\right) e^{i\omega x} d\omega.$ □

The band-limited character of the FTs of $\varphi(x)$ and $\psi(x)$ provides the basis of a formal proof of the equivalence of expressions (2.3.28) and (2.3.29). Because of the orthogonality of $\psi(2^j x - k)$ and $\varphi(x - k)$ together with their additional properties (2.3.18) – (2.3.20), it is straightforward to check that the coefficients in the expansions (2.3.28) and (2.3.29) can be found by making the convolutions (2.3.26) and (2.3.27). If $f(x) \in \mathbb{R}$, the real part of $a_{j,k}$ is a half of the amplitude of the even wavelet $\psi_e(2^j x - k)$ in the expansion; minus the imaginary part of $a_{j,k}$ is a half of the amplitude of the corresponding odd wavelet $\psi_o(2^j x - k)$ in the expansion.

Using the band-limited structure of the FT of $\psi(2^j x - k)$, it can be demonstrated that

$$\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} 2^{-j} \left\{ |a_{j,k}|^2 + |\widetilde{a}_{j,k}|^2 \right\} = \int_{-\infty}^{\infty} |f(x)|^2 dx \,.$$

Thus, the set of functions $\{\psi(2^j x - k) \,|\, j, k \in \mathbb{Z}\}$ is a tight "frame" in the terminology of the wavelet theory [23]. In other words, the band-limited harmonic wavelet defined by (2.3.13) provides a complete set of basis functions in the $L^2(\mathbb{R})$-space for expanding an arbitrary function $f(x)$, i.e.:

$$\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty \,.$$

### 2.3.6 Decay of wavelet coefficients

In order to get the convergence of series (2.3.29), it is natural to present a question of the decay of wavelet coefficients.

The $P$ vanishing moments (2.3.23) have an important consequence for the wavelet coefficients $\{a_{j,k}\}$ and $\{\widetilde{a}_{j,k}\}$ (2.3.26): they decrease rapidly for a smooth function. Furthermore, if a function has a discontinuity in one of its derivatives, then the wavelet coefficients will decrease slowly only close to that discontinuity and maintain fast decay where the function is smooth. This property makes wavelets particularly suitable for representing piecewise smooth functions. The decay of wavelet coefficients can be expressed in the following theorem.

**Theorem 2.3.6.** *[49] Let $P = D/2$ be the number of vanishing moments for a wavelet $\psi_{j,k}(x)$, and let $f(x) \in \mathcal{C}^P(\mathbb{R})$. Then the wavelet coefficients defined by* (2.3.26) *decay as follows:*

$$|a_{j,k}| \leq C_P 2^{-j\left(P+\frac{1}{2}\right)} \max_{\xi \in I_{j,k}} |f^{(P)}(\xi)|\,,$$

*where $C_P$ is a constant independent of $j, k$, $f(x)$, and $I_{j,k} = supp\{\psi_{j,k}\} = \left[\dfrac{k}{2}; \dfrac{k+D-1}{2^j}\right]$.*

*Proof.* Let us write the Taylor series expansion for $f(x)$, $x \in I_{j,k}$ in the neighborhood of $x = k/2^j$

$$f(x) = \left(\sum_{p=0}^{P-1} f^{(p)}(k/2^j) \frac{\left(x - \frac{k}{2^j}\right)^p}{p!}\right) + f^{(P)}(\xi) \frac{\left(x - \frac{k}{2^j}\right)^P}{P!}\,, \tag{2.3.34}$$

where $\xi \in [k/2^j; x]$ and under the assumption that $\exists\, f^{(P)}(x)$ in the support interval $I_{j,k}$. The substitution of (2.3.34) into (2.3.26), and restricting the integration to the support integral, we get

$$\begin{aligned} a_{j,k} &= \int_{I_{j,k}} f(x)\psi_{j,k}(x)dx \\ &= \left(\sum_{p=0}^{P-1} f^{(p)}(k/2^j) \frac{1}{p!} \int_{I_{j,k}} \left(x - \frac{1}{2^j}\right)^p \psi_{j,k}(x)dx\right) \\ &+ \frac{1}{P!} \int_{I_{j,k}} f^{(P)}(\xi) \left(x - \frac{k}{2^j}\right)^P \psi_{j,k}(x)dx\,. \end{aligned}$$

Recall that $\xi$ depends on $x$, therefore $f^{(P)}(\xi)$ is not a constant, and must remain under the integral.

Let us now consider integrals, where $p = 0, 1, 2, \ldots, P-1$. Taking into account

the support interval $I_{j,k}$ and the substitution $y = 2^j x - k$, we obtain:

$$\begin{aligned}
&\int_{k/2^j}^{(k+D-1)/2^j} \left(x - \frac{k}{2^j}\right)^p 2^{j/2}\psi(2^j x - k)dx \\
&\qquad = 2^{j/2} \int_0^{D-1} \left(\frac{y}{2^j}\right)^p \psi(y) 2^{-j} dy \\
&\qquad = 2^{-j(p+1/2)} \int_0^{D-1} y^p \psi(y) dy \\
&\qquad = 0, \quad p = 0, 1, \ldots, P-1
\end{aligned}$$

because of the $P$ vanishing moments. Therefore, the wavelet coefficient is determined from the reminder term alone. Hence,

$$\begin{aligned}
|a_{j,k}| &= \frac{1}{P!} \left| \int_{I_{j,k}} f^{(P)}(\xi) \left(x - \frac{k}{2^j}\right)^P 2^{j/2}\psi(2^j x - k)dx \right| \\
&\leq \frac{1}{P!} \max_{\xi \in I_{j,k}} |f^{(P)}(\xi)| \int_{I_{j,k}} \left| \left(x - \frac{k}{2^j}\right)^P 2^{j/2}\psi(2^j x - k) \right| dx \\
&= 2^{-j(P+1/2)} \frac{1}{P!} \max_{\xi \in I_{j,k}} |f^{(P)}(\xi)| \int_0^{D-1} |y^p \psi(y)| dy \,.
\end{aligned}$$

Defining

$$C_P = \frac{1}{P!} \int_0^{D-1} |y^p \psi(y)| dy \,,$$

we obtain the desired inequality. □

Theorem 2.3.6 shows that if $f(x)$ behaves like a polynomial of a degree less than $P$ within the interval $I_{j,k}$, then $f^{(P)} \equiv 0$; and the corresponding wavelet coefficients $\{a_{j,k}\}$ are equal to zero. If $f^{(P)}(x)$ is different from zero, then the coefficients will decay exponentially with respect to the scaling parameter $j$. If $f(x)$ has a discontinuity in its derivative of the order less or equal to $P$, then Theorem 2.3.6 does not hold

for wavelet coefficients located at the discontinuity. However, wavelet coefficients away from the discontinuity are not affected. Thus, the coefficients in the wavelet expansion reflect only local properties of $f(x)$, and isolated discontinuities do not affect on the convergence away from discontinuities. This means that functions, which are piecewise and smooth, have many small wavelet coefficients in their expansions. Therefore, they can be well represented by a few wavelet coefficients.

### 2.3.7 Periodic harmonic wavelets

In recent years, so-called periodic wavelets became one of the basic tools for investigation of periodic processes in numerical methods. One way to construct periodic wavelets is the periodization of $L^2(\mathbb{R})$ wavelets.

The aim of this subsection is to introduce a unified approach to periodic wavelets, and construct periodic harmonic wavelets with the related decomposition and reconstruction algorithms of function $f(x)$ on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. The main point in this paragraph is that a wavelet basis in $L^2(\mathbb{R})$ yields complete orthonormal system in $L^2(\mathbb{T})$ (identified quite often as $L^2[0;1]$), and

$$L^2[0;1] \stackrel{def}{=} \left\{ f(x) \,:\, f(x) = f(x+1),\, \mathbb{R} \to \mathbb{C};\, \int_0^1 |f(x)|^2 dx < \infty \right\}.$$

There exist two approaches to restrict wavelets to bounded sets. One involves its modification by requiring those which overlap on the bound to satisfy certain boundary conditions [2]. Another approach on $\mathbb{R}^1$ involves periodization of a scaling function and the corresponding wavelets.

**Definition 2.3.2.** Let $\varphi \in L^2(\mathbb{R})$ and $\psi \in L^2(\mathbb{R})$ be the basic scaling function and the basic wavelet from MRA. For $\forall j,k \in \mathbb{Z}$ we define the 1 - periodic scaling function

$$\varphi_{j,k}^{per}(x) = \sum_{r=-\infty}^{\infty} \varphi_{j,k}(x+r) = 2^{j/2} \sum_{r=-\infty}^{\infty} \varphi(2^j(x+r)-k), \quad x \in \mathbb{R} \tag{2.3.35}$$

and the 1 - periodic wavelet

$$\psi_{j,k}^{per}(x) = \sum_{r=-\infty}^{\infty} \psi_{j,k}(x+r) = 2^{j/2} \sum_{r=-\infty}^{\infty} \psi(2^j(x+r)-k), \quad x \in \mathbb{R}. \tag{2.3.36}$$

The 1-periodicity can be verified as follows:

$$\varphi_{j,k}^{per}(x+1) = \sum_{r=-\infty}^{\infty} \varphi_{j,k}(x+r+1) = \sum_{m=-\infty}^{\infty} \varphi_{j,k}(x+m) = \varphi_{j,k}^{per}\,,$$

and in precisely the same way we can show that $\psi_{j,k}^{per}(x+1) = \psi_{j,k}^{per}(x)$. If the support of a wavelet $\psi_{j,k}$ is out of $[0;1]$ on the left side, the missing part is matched by $\psi_{j,2^j+k}$ on the right side.

The following theorem demonstrates that periodic wavelets $\psi_{j,k}^{per}$ together with periodized scaling functions $\varphi_{j,k}^{per}$ generate orthogonal basis in $L^2[0;1]$.

**Theorem 2.3.7.** *[49] For* $\forall N \leq 0$

$$\left\{\psi_{j,k}^{per} \,|\, -\infty < j \leq N, 0 \leq k < 2^{-j}\,, \quad \varphi_{N,k}^{per} \,|\, 0 \leq n < 2^{-N}\right\} \tag{2.3.37}$$

*is an orthogonal basis of* $L^2[0;1]$.

*Proof.* Let $\alpha(x), \beta(x) \in L^2(\mathbb{R})$. If $\langle \alpha(x), \beta(x+r)\rangle = 0$ for $\forall\, r \in \mathbb{Z}$, then

$$\int_0^1 \alpha^{per}(x)\beta^{per}(x)dx = 0\,. \tag{2.3.38}$$

To verify (2.3.38), we recall Definition 2.3.2, and get

$$\begin{aligned}\int_0^1 \alpha^{per}(x)\beta^{per}dx &= \int_{-\infty}^{\infty} \alpha(x)\beta^{per}(x)dx \\ &= \sum_{-\infty}^{\infty} \int_{r=-\infty}^{\infty} \alpha(x)\beta(x+r)dx = 0\,.\end{aligned}$$

Since $\{\psi_{j,k} \,|\, -\infty < j \leq N,\, k \in \mathbb{Z},\, \varphi_{N,k} \,|\, k \in \mathbb{Z}\}$ is orthogonal in $L^2(\mathbb{R})$, we can verify that any two different wavelets or scaling functions $\alpha^{per}$ and $\beta^{per}$ in (2.3.37) have

necessarily a non-periodized version which satisfies the orthogonality $\langle \alpha(x), \beta(x+r)\rangle = 0$ for $\forall\, r \in \mathbb{Z}$ in $L^2[0;1]$.

To prove that this family generates basis in $L^2[0;1]$, we extend $f \in L^2[0;1]$ with zeros outside $[0;1]$, and decompose it in terms of wavelets for $L^2(\mathbb{R})$:

$$f = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \langle f, \psi_{j,k}\rangle \psi_{j,k} + \sum_{k=-\infty}^{\infty} \langle f, \varphi_{N,k}\rangle \varphi_{N,k}\,. \tag{2.3.39}$$

This zero extension is periodized by Definition 2.3.2. Periodization of (2.3.39) proves that $f$ can be decomposed over the periodized wavelet family (2.3.37) in $L^2[0;1]$. □

Periodic wavelets are constructed for decomposing periodic square integrable functions into different frequency bands for yielding local information within each frequency band [65]. Periodization is a standard technique in the Fourier analysis. Our construction is based on periodization of orthogonal harmonic wavelets. Periodic scaling functions can be constructed by a standard procedure [23], which is defined as follows:

$$\varphi^{per}(x) = \sum_{k=-\infty}^{\infty} \varphi(x-k) \tag{2.3.40}$$

on a unit interval. By substituting $\varphi(x-k)$ with its Fourier transform, we have:

$$\varphi^{per}(x) = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{\varphi}(\omega)\, e^{-i\omega k} e^{-i\omega x} d\omega\,,$$

and from the equality

$$\sum_{k=-\infty}^{\infty} e^{i(\omega_1-\omega_2)k} = 2\pi \sum_{m=-\infty}^{\infty} \delta(\omega_2 - \omega_1 - 2\pi m)$$

it immediately follows that

$$\sum_{k=-\infty}^{\infty} e^{-i\omega k} = 2\pi \sum_{m=-\infty}^{\infty} \delta(\omega - 2\pi m). \tag{2.3.41}$$

Since, from (2.3.17), function $\hat{\varphi}(\omega)$ equals zero except the band $0 \leq \omega < 2\pi$, only the term with $m = 0$ needs to be retained in (2.3.41), therefore,

$$\varphi^{per}(x) = 2\pi \int_{-\infty}^{\infty} \hat{\varphi}(\omega) e^{i\omega x} \delta(\omega)\, d\omega\,,$$

which with (2.3.17) gives

$$\varphi^{per}(x) = 1. \tag{2.3.42}$$

The same argument applies when $\varphi(x-k)$ in (2.3.40) is replaced by its complex conjugate. For harmonic wavelet function $\psi(2^j x - k)$, its periodic (circular) equivalent is

$$\begin{aligned} \psi^{per}\left(2^j x - k\right) &= \sum_{m=-\infty}^{\infty} \psi\left(2^j (x-m) - k\right) = \\ \sum_{m=-\infty}^{\infty} 2^{-j} &\int_{-\infty}^{\infty} e^{-i\omega k/2^j} e^{-i\omega m} \hat{\psi}\left(\frac{\omega}{2^j}\right) e^{i\omega x} d\omega\,, \end{aligned} \tag{2.3.43}$$

where $k = 0, \ldots, 2^j - 1$. In order to compute the summation over $m$, recall the identity (2.3.32). Since $\hat{\psi}(\omega/2^j) = 1/2\pi$ for $2\pi\, 2^j \leq \omega < 2\pi\, 2^{j+1}$ and zero elsewhere. The only values of $m$, which need to be considered in (2.3.41), are $m = 2^j, \ldots, 2^{j+1} - 1$. Substituting equation (2.3.41) into (2.3.43) and completing the integration, we get:

$$\psi^{per}\left(2^j x - k\right) = 2^{-j} \sum_{m=2^j}^{2^{j+1}-1} e^{2\pi i m\left(x - \frac{k}{2^j}\right)} \quad , \tag{2.3.44}$$

where $k = 0, \ldots, 2^j - 1$. In particular, for the zero-level, the circular wavelet is

$$\psi_{0,0}(x) = e^{2\pi i x}.$$

Fig. 2.5 shows real and imaginary parts of periodic harmonic wavelets for some of the first three scales at the selected positions.

It is clear from (2.3.44) that a circular (periodic) wavelet of level $j$ has $2^j$ discrete harmonic frequencies, which are $2\pi\, 2^j$, $2\pi\,(2^j + 1)$, $2\pi\,(2^j + 2)$, $\ldots$, $2\pi\,(2^{j+1} - 1)$. It

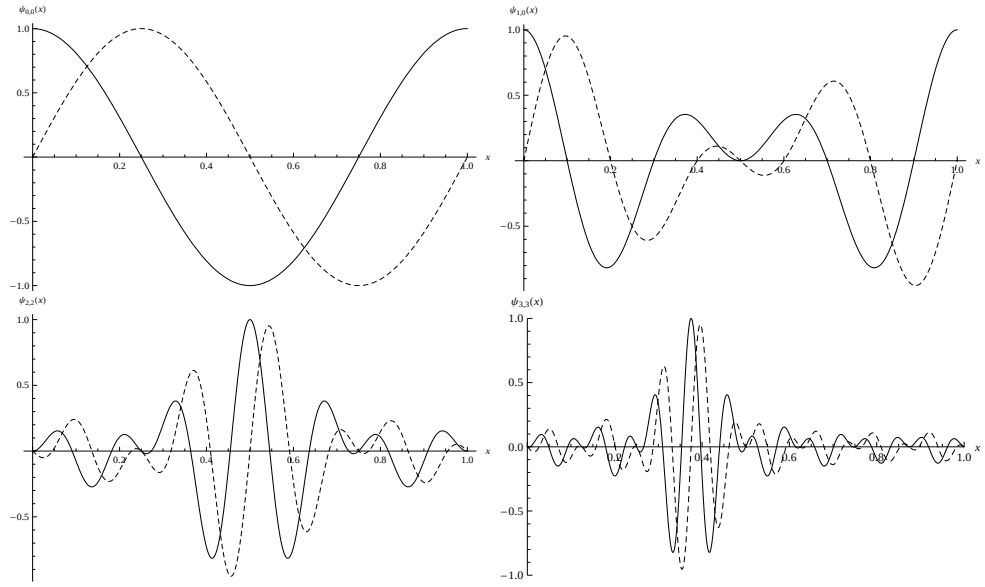

Figure 2.5: *Real (solid) and imaginary (dashed line) parts of the periodic harmonic wavelets* $\psi_{0,0}(x)$, $\psi_{1,0}(x)$, $\psi_{2,2}(x)$, $\psi_{3,3}(x)$.

follows that a train of constant amplitude circular wavelets harmonic wavelets can model a pure tone provided that it had one of these frequencies.

The limits of integration in the orthogonality condition (2.3.9) are given from $-\infty$ to $\infty$, but the integrand in each case is only nonzero for the finite length of the shortest wavelet involved. In the case of PHW, the limits of integration must be from $x = 0$ to $x = 1$. Therefore, formulae (2.3.26) and (2.3.27) can be written as follows:

$$a_{j,k} = 2^j \int_0^1 f(x)\, \psi^*\left(2^j x - k\right) dx\,; \quad \widetilde{a}_{j,k} = 2^j \int_0^1 f(x)\, \psi\left(2^j x - k\right) dx\,, \tag{2.3.45}$$

and the corresponding coefficient for the terms of scaling function is

$$a_{\varphi,k} = \int_0^1 f(x)\, dx\,. \tag{2.3.46}$$

**Corollary.** From the existance of FTs for harmonic wavelets for $j \geq 0$ (see (2.3.13) and Fig.2.2), we can conclude that $\psi^{per}_{j,k}(x) = 0$ for $j \leq -1$, $j, k \in \mathbb{Z}$, $x \in \mathbb{R}$.

### 2.3.8 Discrete harmonic wavelet transform

The discrete wavelet transform (DWT) is an algorithm for computing (2.3.45) and (2.3.46) when $f(x)$ is sampled at equally spaced points over $0 \leq x < 1$ [58]. In the following, we will assume that $f(x)$ represents one period of a periodic function and that the scaling and wavelet functions wrap around the interval as described above. The algothithm of the DWT was introduced by S. Mallat [23], and it is called Mallat's pyramid algorithm.

An algorithm to compute $\varphi(x-k)$ and $\psi(2^j x-k)$ from their defining Fourier transforms and then to make the convolutions of (2.3.26) and (2.3.27) can be set up in a straightforward way, but it is not efficient numerically and the following alternative approach is better.

Let us consider a function $f(x) \in L^2[0;1]$, which is sampled at equally spaced points, and represented by the following sequence:

$$f_r, \quad r = 0, 1, \ldots, N-1$$

where $N = 2^n$. Using the discrete Fourier transform, the corresponding Fourier coefficients are

$$\hat{f}_m = \frac{1}{N}\sum_{r=0}^{N-1} f_r e^{-2\pi i m r/N}, \quad m = 0, 1, \ldots, N-1\,.$$

Let us note, that

$$\hat{f}_{N-m} = \frac{1}{N}\sum_{r=0}^{N-1} f_r e^{-2\pi i (N-m) r/N} = \frac{1}{N}\sum_{r=0}^{N-1} f_r e^{-2\pi i r} e^{\frac{2\pi i m r}{N}} = \hat{f}_m^*\,,$$

where $\hat{f}_m \in \mathbb{C}$ except $\hat{f}_0, \hat{f}_{N/2} \in \mathbb{R}$.

Furthermore, we will consider a coefficient $a_{j,k}$, defined by the first formula in (2.3.26). Firstly, we will substitute $\psi_{j,k}^*(x)$ in terms of its Fourier transform of function (2.3.13)

$$\psi_{j,k}^*(x) = \frac{1}{2^j}\int_{2\pi 2^j}^{4\pi 2^j} \frac{1}{2\pi} e^{i\omega k/2^j} e^{-i\omega x} d\omega$$

into the first formula of (2.3.26), and we obtain the following integral

$$a_{j,k} = \frac{1}{2\pi} \int_{2\pi 2^j}^{4\pi 2^j} e^{i\omega k/2^j} d\omega \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx, \tag{2.3.47}$$

where we have changed the order of integration, assuming that $f(x) < \infty$ everywhere. The second integral over $x$ represents the FT of $f(x)$ multiplied by $2\pi$, and then equation (2.3.47) becomes

$$a_{j,k} = \int_{2\pi 2^j}^{4\pi 2^j} \hat{f}(\omega) e^{-i\omega k/2^j} d\omega. \tag{2.3.48}$$

Next, we replace the integral by a summation. In this case we step through the frequency band increments of $2\pi$ (in such a way that there are $2^j$ steps in level $j$). Then $\hat{f}(\omega)$ is replaced by the discrete coefficients $\hat{f}_{2^j+s}$, where $\hat{f}_{2^j+s} = 2\pi \hat{f}(\omega = 2\pi(2^j + s))$, and integral (2.3.48) becomes a summation

$$a_{2^j+k} = \sum_{s=0}^{2^j-1} \hat{f}_{2^j+s} e^{2\pi i s k/2^j}, \quad k = 0, \ldots, 2^j - 1. \tag{2.3.49}$$

This identity represents the inverse discrete Fourier transform for the sequence of frequency coefficients $\hat{f}_{2^j+s}$. We arranged these coefficients in wavelet levels (which are synonymous with the octave bands) in the table below (Table 2.1).

Consider a level $j$ with $2^j$ coefficients $\{a_{2^j+k}\}$, where $k = 0, \ldots, 2^j - 1$. Each of these coefficients defines a complex amplitude of a wavelet whose Fourier transform is described by the Fourier coefficients in the frequency band $\hat{f}_{2^j+k},\ k = 0, \ldots, 2^j - 1$. The first wavelet in this sequence with the amplitude $a_{2^j}$ has a constant spectral density of relative level $1/2^j$ (see Fig. 2.2). Therefore, it contributes $a_{2^j}/2^j$ to the general coefficient $\hat{f}_m$, $2^j \leq m < 2^{j+1}$. The second wavelet on level $j$ has amplitude $a_{2^j+1}$. It is translated $1/2^j$ with respect to its neighbour, and therefore its Fourier transform is rotated by $e^{-i\omega k/2^j}$ with $k = 1$. Since $\hat{f}_m$ is the Fourier coefficient for frequency $\omega = 2\pi m$, the contribution of $a_{2^j+1}$ to $\hat{f}_m$ is $a_{2^j+1} e^{\left(-2\pi m i/2^j\right)/2^j}$. Combining

| Wavelet level | Coefficient in wavelet transform |
|---|---|
| -1 | $a_0$ |
| 0 | $a_1$ |
| 1 | $a_2, a_3$ |
| 2 | $a_4, a_5, a_6, a_7$ |
| 3 | $a_8$ to $a_{15}$ |
| – | – |
| $j$ | $a_{2^j}$ to $a_{2^{j+1}-1}$ |
| – | – |
| $n-2$ | $a_{N/4}$ to $a_{N/2-1}$ |
| $n-1$ | $a_{N/2}$ |

Table 2.1: Wavelet levels and its corresponding coefficients in the wavelet transform.

all the contributions from $k = 0, \ldots, 2^j - 1$, we obtain:

$$\hat{f}_m = 2^{-j} \sum_{k=0}^{2^j-1} a_{2^j+k} e^{-2\pi i m k/2^j}. \tag{2.3.50}$$

We are left with a remarkable conclusion that the wavelet coefficients can be obtained by computing the inverse discrete FT of the successive blocks of the Fourier coefficients of the function $f$.

# Conclusion of Chapter 2

In this chapter we introduced harmonic wavelets, and derived the corresponding PHWs. Periodic harmonic wavelets represent orthogonal functions with a unit period, and form a basis for the $L^2[0;1]$ functions. Another important feature of PHWs is its ability to represent high-frequency oscillations, which enables to approximate any smooth function with strong gradients. The theory of PHWs concerns not only one wavelet, but all periodic wavelets!

Regarding the harmonic wavelets, then there appears the only disadvantage: their decay rate is relatively low (proportional to $x^{-1}$), and therefore it can not be a powerful tool in applications when a function or a signal is strictly localized in time or

space. However, this is a penalty for the wavelet, whose the Fourier transform is restricted in frequency domain.

The mentioned Mallat's algorithm for computing the DWT involves a sequential filtering operation. At each level of the transform, a high-pass filter separates fine structure to give the wavelet-coefficients at this level. The algorithm described in this Chapter represents a parallel algorithm in the sense that all the wavelet-coefficients $\{a_s \,|\, s = 0, \ldots, N-1\}$ can be computed simultaneously (at the same time). Mallat's algorithm is replaced by a flat sandwich with Fast Fourier Transform operations at only two levels of the sandwich.

# Chapter 3

# Multiscale solution of low-dimensional nonlinear PDEs

**Overview.** This Chapter shows that PHWs build periodic MRA. Strategies for handling nonlinear PDEs are shortly discussed. We consider the problem of solution of nonlinear PDEs in the 1D space by using PHWs as basis functions in $L^2[0;1]$. We assume that boundary conditions are periodic and make use of periodized harmonic wavelets described in the previous chapter. The absolute error of the projection of solution on the space of periodic wavelets is analitically estimated.

## 3.1 Introduction

Even though the field of the wavelet theory has had a great influence on the other fields, such as signal processing and image analysis, it is not yet clear whether it will have such influence on numerical methods for solution of PDEs and IEs.

In the early 1990s researchers were very optimistic because it seemed that special properties of wavelets would automatically lead to efficient methods of solution of PDEs and IEs. The reason for this optimism was because many nonlinear PDEs and IEs have solutions, which contain local phenomena and interaction between several scales (e.g. turbulence, multi-frequency oscillations). Such solutions can often be represented in wavelet basis. Therefore, it was believed that the efficient wavelet

based numerical schemes for solving such problems would follow from the compression properties of wavelets.

However, this early optimism remains to be honored. Wavelets have not had the expected impact on differential equations, partly because the computational work is not necessarily reduced by applying wavelet compression – even though the solution is sparsely represented in a wavelet basis. In the following paragraphs we will discuss the most promising approach and compare our results with other numerical methods.

Schematically, wavelet based methods for solution of PDEs can be separated in the following classes:

**Class 1. Methods based on scaling function expansions.** The unknown solution is expanded in scaling functions at some chosen level $N$ and is solved by the Galerkin or collocation approach. Because of their compact support, the scaling functions can be regarded as alternatives to splines or the piecewise polynomials used in finite element schemes. While this approach is important in its own right, it cannot exploit wavelet compression. However, this approach has many points of interest. Lenard Jameson [38] has shown that one obtains a method, which exhibits super convergence at the grid points, the order of approximation is twice as large as that of the projection of the solution onto the space spanned by scaling functions. J. Waldén has shown [76] that the size of the differentiation filters grows faster than optimal centered finite difference method of the same order. Finally, we mention that Teresa Regińska [68] used scaling functions to regularize the solution of the sideways heat equation. By expanding the solution in scaling functions, high frequency components can be filtered away and continuous dependence of the initial condition is restored.

**Class 2. Methods based on wavelet expansions.** A PDE is solved by taking the Galerkin approach as in the first class. However, in this case the unknown solution is expressed in terms of wavelets instead of scaling functions. So, wavelet expression can be applied either to the solution or to the differential operator. Several different approaches have been considered for exploiting the sparsity of a wavelet representation. Those, who peformed this method are S. Mallat [3] and S. Bertoluzza

[5].

Y. Liu et al. [47] employed the wavelet-collocation method for solution of transient problems with steep gradients. They have shown that the Daubechies orthonormal basis in the collocation method performs an accurate, stable and convergent algorithm to describe evolution processes.

In another approach, linear operators such as differentiation are computed in the wavelet domain and nonlinear operators such as squaring, in physical domain. This is one of the most common approaches and was used by [17], [28] and [40]. This approach involves a number of transformations between the physical domain and the wavelet domain in each time step, and this can involve considerable difficulties. Hence, the wavelet compression potential of the solution must be very large for this approach to be feasible.

An interesting aspect of the wavelet approach is that certain operators represented with respect to a wavelet basis become sparser, when we go to the higher powers [25]. From this property, one can obtain an efficient time-stepping scheme for certain evolution equations. This method was employed in [17] to solve the heat equation.

We will call methods which belong to Class 1 and 2 as projection methods.

**Class 3. Wavelets and finite differences.** In the third approach, wavelets are used to derive adaptive finite-difference methods. Instead of expanding solution in terms of wavelets, the wavelet transform is used to determine where the finite difference grid must be refined or coarsened in order to represent the optimal solution. In this case, the computational costs are low because one works with point values in the physical representation. One approach was developed by Lenard Jameson [38] under the name Wavelet Optimized Finite Difference Method (WOFD). This method finds an approximation to the solution found on the finest scale across the whole domain. The efficiency depends on the rate of data compression, i.e. how fast derivative change along the space domain.

**Class 4. Other methods.** There are a few approaches that can implement wavelets, but do not fit into the previous classes. The examples are operator wavelets

[80], the method of travelling waves [63], and wavelet preconditioning [7].

Operator wavelets are wavelets, which are (bi-)orthogonal with respect to the inner product designed for the particular differential operator. This is not a general method, since it works only for certain operators.

In the method of travelling waves, initial condition is expanded, involving only a few terms of wavelets, which propagate in time. A disadvantage of this method is that these few wavelets can be unable to express the solution after it has been propagated in a long time domain.

It was shown by G. Beylkin in [7] that any finite difference matrix of representation of periodized differential operators can be preconditioned so that the condition number is $\mathcal{O}(1)$. Furthermore, if the finite difference matrix is represented in a wavelet basis, then the preconditioner is a diagonal matrix. Thus, wavelets play an auxiliary role that they provide ways to reduce the condition number of the operator.

This Chapter of the thesis, refers to Class 2 and the Galerkin approach. Also, it is shown that PWH form MRA.

## 3.2 Periodic multiresolution analysis

The idea of the decomposition of a function into a sum of approximate $\varphi(x)$ and detailed $\psi(x)$ terms by using orthogonal and biorthogonal wavelets was realized in the MRA (e.g. see [23]).

In view of Definition 2.3.2 and the assumption that (2.3.35) and (2.3.36)

$$\psi_{j,k}^{per}(x) = \sum_{r\in\mathbb{Z}} \psi_{j,k}(x+r)\,, \quad \varphi_{j,k}^{per}(x) = \sum_{r\in\mathbb{Z}} \varphi_{j,k}(x+r)\,, \tag{3.2.1}$$

are bounded functions, many of the properties of non-periodic scaling functions and wavelets carry over to their periodized versions restricted to the interval $[0;1]$. As we already know, one of such properties is the MRA. The definition of MRA in $L^2(\mathbb{R})$ can be adapted to the $L^2[0;1]$-space as follows:

**Definition 3.2.1.** A sequence of subspaces $\{V_j\}$ in the $L^2[0;1]$-space is called a periodic MRA of $L^2[0;1]$ if the subsequent conditions hold true:

i. *Nested.* $V_j \subset V_{j+1} \ldots \subset L^2[0;1]$ ;

ii. for $j = 0, 1, 2, \ldots$ the system $\varphi_{j,k}^{per}$ with $k = 0, 1, \ldots, 2^j - 1$ is an orthonormal basis in $V_j$ ;

iii. for $j = 0, 1, 2, \ldots$ the system $\{1, \psi_{s,k}^{per}\}$ for $s = 0, 1, \ldots, j-1$ and $k = 0, 1, \ldots, 2^{s-1}$ is an orthonormal basis in $V_j$ ;

iv. $\bigcup_{j=0}^{\infty} V_j$ is the dense in $L^2[0;1]$, so the system $\{\psi_{j,k}^{per}\}$ is a complete orthonormal system in $L^2[0;1]$.

By Theorem 2.3.7, PHWs form basis in $L^2[0;1]$. To get the full benefit from the PHW, we need to show that these functions satisfy the conditions of periodic MRA.

A periodic scaling function of level $j$ generates subspaces $\{V_j^{per}\}$. The *nested* property follows, since the set of multiples of $\{2^{-j} \,|\, j \geq 0\}$ is contained in the set of multiples of $\{2^{-j-1}\}$. Intuitively, approximation of the function in $V_j^{per}$ is capable of capturing details of a signal down to resolution of $2^{-j}$.

The *density* condition (iv) means that approximation of a function by harmonic scaling function captures all details of the function as $j$ gets larger.

To discuss the separation condition, we should note that $j$ can be negative as well as positive in the definition of $V_j^{per}$. If $f(x) \in V_{-j}$ for $j > 0$, then $f(x)$ must be a linear combination of $\{\psi(2^{-j}x - k) \,|\, j, k \in \mathbb{Z}\}$, whose elements are constant within $[0; 2^j]$. When $j$ is increasing, these intervals get larger [8]. Then we can formulate the following theorem.

**Theorem 3.2.1.** *(C.Cattani and A.Kudreyko [92]) Periodic harmonic wavelets* (2.3.44) *fulfill axioms* $(i)$ – $(iv)$ *of the MRA.*

*Proof.* Recall Definitions 1.1.1, 1.2.1 and formulae (2.3.18), (2.3.19), from which follow that scaling function and harmonic wavelet $\varphi, \psi \in L^2(\mathbb{R})$. Assume that for $s < j$,

we have $\varphi_{s,k} = \sum_{r\in\mathbb{Z}} \alpha_r \varphi_{j,r}$ and $\psi_{s,k} = \sum_{r\in\mathbb{Z}} \beta_r \varphi_{j,r}$. From formulae (3.2.1) and the orthogonality of $\varphi$ and $\psi$ in $L^2(\mathbb{R})$, we infer that

$$\begin{aligned}
\sum_{r\in\mathbb{Z}} |\alpha_r| &= \sum_{r\in\mathbb{Z}} | < \varphi_{s,k}, \varphi_{j,r} > | \\
&\leq 2^{s/2+j/2} \sum_{r\in\mathbb{Z}} \int_{-\infty}^{\infty} |\varphi(2^s x - k)| * |\varphi(2^j x - r)| dx \\
&= 2^{s/2+j/2} \int_{-\infty}^{\infty} |\varphi(2^s x - k)| * \varphi^{per}(2^j x) dx \\
&\leq C\, 2^{s/2+j/2} \int_{-\infty}^{\infty} |\varphi(2^s x - k)| dx < \infty\,,
\end{aligned}$$

and in the same way we can get $\sum_{r\in\mathbb{Z}} |\beta_r| < \infty$. It implies that $\varphi^{per}_{s,k} \in V^{per}_j$ and $\psi^{per}_{s,k} \in W^{per}_j$. This gives the proof of issue (i). Thus, we have to check the orthogonality in (ii) – (iv). For $j, j' \geq 0$ we have

$$\begin{aligned}
\int_0^1 \psi^{per}_{j,k}(x) \psi^{per*}_{j',k'}(x) dx &= \int_0^1 \sum_{r=-\infty}^{\infty} \psi_{j,k}(x+r) \psi^{per*}_{j',k'}(x) dx \\
&= \sum_{r=-\infty}^{\infty} \int_r^{r+1} \psi_{j,k}(y) \psi^{per*}_{j',k'}(y-r) dy \\
&= \sum_{r=-\infty}^{\infty} \int_r^{r+1} \psi_{j,k}(y) \psi^{per*}_{j',k'}(y) dy \\
&= \int_{-\infty}^{\infty} \psi_{j,k}(y) \psi^{per*}_{j',k'}(y) dy\,.
\end{aligned} \tag{3.2.2}$$

Using Definition 2.3.2, we can write

$$\psi^{per}_{j,k}(x) = 2^{j/2} \sum_{r=-\infty}^{\infty} \psi(2^j x + 2^j r - k) = \sum_{r=-\infty}^{\infty} \psi_{j,k-2^r}(x) \tag{3.2.3}$$

Because $\psi(x)$ is a compactly supported mother wavelet, the supports of the terms do not overlap if $2^j$ is sufficiently large. Using formula (3.2.3) for the second function in

(3.2.2), and involving the orthogonality relation for harmonic wavelets (2.3.13), we obtain

$$\int_0^1 \psi_{j,k}^{per}(x)\psi_{j',k'}^{per*}(x)dx = \sum_{r=-\infty}^{\infty} \int_{-\infty}^{\infty} \psi_{j,k}(x)\psi_{j,k-2^j r}(x)dx$$
$$= \delta_{j,j'} \sum_{r=-\infty}^{\infty} \delta_{k,k'-2^j r}\,.$$

If $j = j'$, then $\delta_{j,j'} = 1$, and $\delta_{k,k'-2^j r}$ contribute only when $r = 0$ and $k = k'$ because $k, k' \in [0; 2^j - 1]$. Hence,

$$\int_0^1 \psi_{j,k}^{per}(x)\psi_{j',k'}^{per*}(x)dx = \delta_{j,j'}\delta_{k,k'}\,.$$

The use of a similar analysis can establish the relation:

$$\int_0^1 \varphi_{j,k}^{per}(x)\psi_{j',k'}^{per*}(x)dx = 0\,.$$

Therefore, the system, which appears in item (iv) of the properties of the MRA is orthonormal. If we repeat calculations (3.2.2) for $j = j'$ and $\varphi$ instead of $\psi$, we can obtain that item (ii) holds true. Thus, $\dim V_j = 2^j$.

In order to show the proof of (iv), let us consider the orthogonal projection $\mathcal{P}_{V_j^{per}}$ from $L^2[0;1]$ onto $V_j^{per}$. From (ii) we infer that

$$\mathcal{P}_{V_j} f = \sum_{k=0}^{2^j-1} < f, \varphi_{j,k}^{per} > \varphi_{j,k}^{per}\,. \tag{3.2.4}$$

Suppose we fixed an exponential $e^{2\pi irt}$, and let us calculate the $r-$th Fourier coefficient of $\mathcal{P}_{V_j}(e^{2\pi irt})$. From expansion (3.2.4) and the consequences of Parseval's identity (see

e.g. [58] or [64]) it follows that

$$\begin{aligned}\mathcal{P}_{V_j}(e^{2\pi irt}) &= \sum_{k=0}^{2^j-1} < e^{2\pi irt}, \varphi_{j,k}^{per} >< \varphi_{j,k}^{per}, e^{2\pi irt} > \\ &= \sum_{k=0}^{2^j-1} | < \varphi_{j,k}^{per}, e^{2\pi irt} > |^2 \\ &= \sum_{k=0}^{2^j-1} |\hat{\varphi}_{j,k}^{per}(r)|^2 = 2\pi |\hat{\varphi}_{j,k}^{per}(r)|^2 \,.\end{aligned} \tag{3.2.5}$$

We can conclude that $\mathcal{P}_{V_j}(e^{2\pi irt}) \to 1$ as $j \to \infty$. Since $||\mathcal{P}_{V_j}|| = 1$ (because $\mathcal{P}_{V_j}$ is an orthogonal projection) and $\{e^{2\pi is}\}_{s\in\mathbb{Z}}$ is an orthonormal system in $L^2([0;1])$, we infer that $\mathcal{P}_{V_j}(e^{2\pi is})$ tends in $L^2[0;1]$ to $e^{2\pi irt}$ as $j \to \infty$. This implies that for every trigonometric polynomial $f : \mathcal{P}_{V_N}(f) \to f$ in $L^2[0;1]$. Since (the Weierstrass approximation theorem, 1885) trigonometric polynomials are the dense in $L^2[0;1]$, we conclude that $\bigcup_{j=0}^{\infty} V_j^{per}$ is the dense in space $L^2[0;1]$. □

Periodized wavelets and scaling functions generate MRA in space $L^2[0;1]$ on a unit interval.

**Definition 3.2.2.**

$$\begin{aligned}V_j^{per} &= \text{span}\left\{\varphi_{j,k}^{per}(x),\, x \in [0;1]\right\}_{k=0}^{2^j-1}\,, \\ W_j^{per} &= \text{span}\left\{\psi_{j,k}^{per}(x),\, x \in [0;1]\right\}_{k=0}^{2^j-1}\,.\end{aligned}$$

It turns out that spaces $\{V_j^{per}\}$ are nested [23] as in the non-periodic MRA, i.e.

$$V_0^{per} \subset V_1^{per} \subset \ldots \subset L^2[0;1]\,.$$

In addition, the orthogonality relations between scaling functions and wavelets imply that

$$V_j^{per} \bigoplus W_j^{per} = V_{j+1}^{per}\,.$$

Using these argumented spaces, we obtain the following result:

$$L^2[0;1] = V_0^{per} \bigoplus \left(\bigoplus_{j=0}^{\infty} W_j^{per}\right)\,.$$

Combining (2.3.42) and (2.3.44), we get

$$\left\{1, \left\{\left\{\psi_{j,k}^{per}\right\}_{k=0}^{2^j-1}\right\}_{j=0}^{\infty}\right\},$$

which represents an orthonormal basis for $L^2[0;1]$ functions. We note that according to (2.3.42), the scaling function no longer appears explicitly, since we replaced it by the constant 1.

**Remark.** One usually thinks, and it is very natural that $V_j^{per} = \bigoplus_j V_j$. One has to be aware that there are functions $f \in V_j$ for which periodization is not properly defined.

At this point it is important to present the problem between the relation of approximate and analytical solutions. The proceeding paragraph is devoted to the answer to this question.

## 3.3 Approximation properties of multiresolution spaces

Consider the approximation error for a family of generalized periodic wavelets. Let $f(x) \in L^2[0;1]$, and assume that its periodic expansion (2.3.29) is $P$ times differentiable everywhere. Then let us define the approximation error as follows:

$$e_N^{per}(x) = f(x) - \mathcal{P}_{V_N^{per}} f(x), \quad x \in [0;1]$$

where $\mathcal{P}_{V_N^{per}} f(x)$ is the orthogonal projection of $f(x)$ onto the approximation space as stated in Definition 3.2.2 (see also Chapter 1.2.2). Using the wavelet periodic expansion (2.3.29), we find that

$$\mathcal{P}_{V_N^{per}} f(x) = \sum_{k=0}^{\infty} a_{\varphi,k} \varphi^{per}(x-k) + \sum_{j=0}^{N-1} \sum_{k=0}^{2^j-1} a_{j,k} \psi_{j,k}^{per}(x)\,. \tag{3.3.1}$$

On passing to the limit as $N \to \infty$, we get $f(x)$:

$$f(x) = \sum_{k=0}^{\infty} a_{\varphi,k} \varphi^{per}(x-k) + \sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} a_{j,k} \psi_{j,k}^{per}(x)\,. \tag{3.3.2}$$

Then, by subtracting (3.3.2) from (3.3.1), we obtain the expression for the error $e_N^{per}$ in terms of wavelets at scales $j \geq N$:

$$e_N^{per}(x) = \sum_{j=N}^{\infty} \sum_{k=0}^{2^j-1} a_{j,k} \psi_{j,k}^{per}(x) \,. \tag{3.3.3}$$

Define

$$C_{\psi^{per}} = \max_{x \in I_{j,k}} |\psi^{per}(2^j x - k)| = \max_{y \in [0, D-1]} |\psi^{per}(y)| \,.$$

Since $\max\limits_{x \in I_{j,k}} |\psi_{j,k}^{per}(x)| = 2^{j/2} C_\psi$, and in view of Theorem 2.3.6 (Theorem of decay of wavelet coefficients), we find that

$$|a_{j,k} \psi_{j,k}^{per}(x)| \leq C_P 2^{-jP} \max_{\xi \in I_{j,k}} |f^{(P)}(\xi)| C_{\psi^{per}} \,,$$

where

$$\text{supp}(\psi_{j,k}^{per}) = I_{j,k} = \left[ \frac{k}{2^j}, \frac{k + D - 1}{2^j} \right] .$$

We conclude that there are at most $D-1$ intervals $I_{j,k}$ containing a given value of $x$. Thus, for any $x$ only $D-1$ terms in the inner summation in (3.3.3) are nonzero. Let $I_j$ be a union of all these intervals, i.e.

$$I_j(x) = \bigcup_{\{l : x \in I_{j,l}\}} I_{j,l} \,,$$

and assume

$$\mu_j^P(x) = \max_{\xi \in I_j(x)} |f^{(P)}(\xi)| \,.$$

Then we can establish a common bound for all terms in the inner sum:

$$\sum_{k=0}^{2^j-1} |a_{j,k} \psi_{j,k}^{per}| \leq C_{\psi^{per}} C_P 2^{-jP} (D-1) \mu_j^P(x) \,.$$

The outer sum over $j$ can be evaluated using the fact that

$$\mu_N^P(x) \geq \mu_{N+1}^P(x) \geq \mu_{N+2}^P(x) \geq \ldots$$

and we can establish the bound

$$|e_N^{per}(x)| \leq C_{\psi^{per}} C_P (D-1) \mu_N^P(x) \sum_{j=N}^{\infty} 2^{-jP}$$
$$= C_{\psi^{per}} C_P (D-1) \mu_N^P(x) \frac{2^{-NP}}{1-2^{-P}} .$$

Thus, we see that for an arbitrary, but fixed $x$, the approximation error will be bounded as follows:

$$|e_N^{per}(x)| = \mathcal{O}(2^{-NP}) . \tag{3.3.4}$$

This is an exponential decay with respect to the resolution $N$. Furthermore, the greater number of vanishing moments $P$ of a periodic wavelet increases the rate of decay. Formula (3.3.4) involves big $\mathcal{O}$ notation, and allows us to concentrate on the decay rate. It is worth to remind that the description of a function in terms of big $\mathcal{O}$ notation usually only provides upper bound of the decay rate of a function. (See A. Kudreyko and C. Cattani [91, 93]).

To get the full benefit of (3.3.4), we must compare it with the reminder term in the Fourier series. Let $\eta_x(x) = f(x) - f_n(x)$, where $f_n(x)$ is the Fourier expansion (A.1.1) of $f(x)$. Let us suppose that $f^{(m)}(x)$ exists, although its continuity is required only up to $f^{(m)}(x)$. As described in reference [4], the reminder term can be estimated as follows:

$$|\eta_n(x)| < \frac{2}{(n+1/2)^{m-1}} \frac{\ln \pi(n+1/2)}{\pi(n+1/2)} |f^{(m)}(x)|_{\max} .$$

As we see from the previous formula, the decay rate of the reminder term is proportional to a polynomial. We know that exponential decay tends to 0 faster than polynomial one.

## 3.4 Harmonic wavelet solution of the Burgers equation

The approach we take uses the combination of the standard Galerkin techniques and multiscale periodic basis. Numerical solutions of the Burgers equations have been a

subject of discussions in many papers (e.g. [32, 43, 53, 55]). From the conventional methods of solution we would like to mention the following ones:

- Fourier spectral method (Appendix B);
- Finite element and finite difference method (Appendix B);
- Galerkin and collocation methods (Appendix B).

As one can see, the solution of the Burgers equation is well-studied. However, the application of wavelets for solution of PDEs is not yet well-studied. Therefore, this approach for the Burgers equation represents a nontrivial problem.

### 3.4.1 Statement of the problem

Let us consider the Burgers equation of the following view:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = 0, \tag{3.4.1}$$

where $u(x,t)$ is the velocity and $\nu$ is the kinematic viscosity[3], with the initial condition

$$u(x,0) = \sin 2\pi x \tag{3.4.2}$$

and the periodic boundary conditions

$$u(x,t) = u(x+1,t), \quad t \geq 0.$$

Burgers' equation is the simplest example of a nonlinear partial differential equation. Solutions of Burgers' equation consist of stationary or moving shocks, and capturing such behavior is an important test of our approach.

It is generally acknowledged that the Burgers equation can be solved analitically by the Cole-Hopf transformation [62], where it is observed that the solution of (3.4.1)

[3]The ratio of the viscosity of a fluid to its density.

can be expressed as $u(x,t) = -2\nu\dfrac{\varphi'_x}{\varphi}$, where $\varphi = \varphi(x,t)$ is the solution of the heat equation with the initial condition

$$\varphi(x,0) = e^{-\frac{1}{4\pi\nu}\int u(x,t)dx}\,.$$

Our approach assumes the implementation of the Galerkin method for solution of (3.4.1) with PHWs (3.4.3) as basis functions

$$\psi_{j,k}(x) = 2^{-j}\sum_{m=2^j}^{2^{j+1}-1} e^{2\pi i m\left(x-\frac{k}{2^j}\right)}\,. \tag{3.4.3}$$

There always presents a frequent question in the application of wavelets for solution of PDEs: how do we choose the best wavelet? Even for the restricted class of harmonic wavelets described here, the same question needs to be answered. Unfortunately there is still no any clear answer to this question [60]. Any signal (function) can be decomposed into many different families of orthogonal wavelets, and for signal processing in real time, computational efficiency maybe the essential feature. However, for the analysis of vibration records, that is usually not important compared with the need to achieve a time-frequency map with high resolution. The effective deconstruction of such a record requires a narrow bandwidth to achieve the frequency resolution, and a wide bandwidth to achieve the time resolution.

### 3.4.2 Connection coefficients

A natural starting point for the projection methods is the problem of computation of connection coefficients. If we restrict ourselves to quadratic terms, then the generalized definitions of connection coefficients can be written out as follows [11]:

$$L^{(l)jr}_{ks} \overset{def}{=} \left\langle \frac{d^l\psi_{j,k}}{dx^l}, \psi_{r,s} \right\rangle; \tag{3.4.4}$$

$$N^{(n)jpr}_{kqs} \overset{def}{=} \left\langle \psi_{j,k}\frac{d^n\psi_{p,q}}{dx^n}, \psi_{r,s} \right\rangle; \tag{3.4.5}$$

$$\Theta^{(n,m)\,jpr}_{kqs} \overset{def}{=} \left\langle \frac{d^n\psi_{j,k}}{dx^n}\frac{d^m\psi_{p,q}}{dx^m}, \psi_{r,s} \right\rangle, \tag{3.4.6}$$

where $l$, $m$ and $n$ are the orders of the derivatives. We will assume that these derivatives are well-defined. Since in our case, we deal with nonlinear $u\dfrac{\partial u}{\partial x}$ and linear term $\dfrac{\partial^2 u}{\partial x^2}$, and therefore we will consider connection coefficients defined by (3.4.4) and (3.4.5). For the simplicity of further notation we will skip parameters $l$ and $n$, assuming that $l = 2$, and $n = 1$.

PHWs (3.4.3) are orthogonal functions on $[0; 1]$; therefore they fulfill orthogonal constraint, i.e. $\langle \psi_{j,k}, \psi_{m,h} \rangle_{L^2[0;1]} = \delta_{j,m}\delta_{k,h}$. The recursive formulae for the connection coefficients [10] can be easily derived from their definitions (3.4.4) and (3.4.5).

$$
\begin{aligned}
L_{ks}^{jr} =& \left\langle \frac{d^2\psi_{j,k}}{dx^2},\ \psi_{r,s} \right\rangle = \int_0^1 \frac{d^2\psi_{j,k}}{dx^2}\psi_{r,s}^* dx = \\
& - 2^{-j-r+2}\pi^2 \sum_{m_j=2^j}^{2^{j+1}-1} \sum_{m_r=2^r}^{2^{r+1}-1} m_j^2 \int_0^1 e^{2\pi i\left[(m_r-m_j)x-\left(\frac{m_r s}{2^r}-\frac{m_j k}{2^j}\right)\right]} dx = \\
& - 2^{-j-r+2} \sum_{m_j=2^j}^{2^{j+1}-1} \sum_{m_r=2^r}^{2^{r+1}-1} (\pi m_j)^2 e^{2\pi i\left(\frac{m_r s}{2^r}-\frac{m_j k}{2^j}\right)} \delta_{m_j,m_r};
\end{aligned}
\tag{3.4.7}
$$

$$
\begin{aligned}
N_{kqs}^{jpr} =& \left\langle \psi_{j,k}\frac{d}{dx}\psi_{p,q}, \psi_{r,s} \right\rangle = \int_0^1 \psi_{j,k}\frac{d\psi_{p,q}}{dx}\psi_{r,s}^* dx = -\frac{2i\pi}{2^{j+p+r}} * \\
& \sum_{m_j=2^j}^{2^{j+1}-1} \sum_{m_p=2^p}^{2^{p+1}-1} \sum_{m_r=2^r}^{2^{r+1}-1} m_p \cdot e^{2\pi i\left(\frac{m_j k}{2^j}+\frac{m_p q}{2^p}-\frac{m_r s}{2^r}\right)} \delta_{m_j+m_p,m_r},
\end{aligned}
\tag{3.4.8}
$$

where matrix (3.4.8) describes nonlinear interactions caused by the convective derivative.

As can be seen from (3.4.7) and (3.4.8), there is a certain freedom in the choice of the range of variation of index $j$. It is obvious that the maximum value of $j$ determines the spatial resolution of the method. The maximum scale $j_{\max} = n$ can be chosen from the statement of the problem (e.g., initial condition). The range of variation of index $k$ is closely related with the boundary conditions of the problem. In the case of periodic problem, the number of functions at level $j$ is determined from the condition

of the complete basis, i.e. the dilation of functions must cover the whole circle. It is evident that every proceeding scale doubles the number of basis functions (3.4.3).

### 3.4.3 The Galerkin approach for harmonic wavelet solution of the Burgers equation

The Galerkin methods are known as a class of methods for converting a continuous operator problem (such as a differential equation) into a discrete problem. The idea of the Galerkin method lies in the choice of such basis functions, which satisfy the boundary conditions. It is obvious that PHW satisfy the boundary value problem of the Burgers equation with initial condition (3.4.2).

The main advantage of the Galerkin method follows from its idea: once, when we have chosen the suitable scaling functions, we should not mind any more about the boundary conditions. However, there is a disadvantage in this method, it works only for simple boundary conditions as for this problem.

According to the idea of the Galerkin approach, we express $u(x,t)$ in terms of the time-dependent amplitudes with the space-dependent bases, i.e. we look for a projection of the solution of the Burgers equation (3.4.1) in the form [53]

$$\mathcal{P}_{V_n} u(x,t) = \sum_{j=0}^{n-1} \sum_{k=0}^{2^j-1} a_{j,k}(t)\, \psi_{j,k}(x). \tag{3.4.9}$$

This wavelet-based expansion is similar to the Fourier expansion. Due to the vanishing moments of wavelet functions, we know that the wavelet transform of a function automatically places significant coefficients in a neighborhood of large gradients. This is a great difference with the traditional methods (Appendix B). Also, let us introduce the inner product in $L^2[0,1]$

$$\langle f, g \rangle = \int_0^1 f(x)\, g^*(x)\, dx, \tag{3.4.10}$$

where the "asterisk" over $g(x) \in L^2[0;1]$ denotes its complex conjunction. Substituting series (3.4.9) into (3.4.1) and applying (3.4.10), one obtains the following system

of complex amplitude evolution equations which play the role of a finite-dimensional projection of the Burgers equation onto the wavelet space $\mathcal{P}_{V_n}$,

$$\frac{da_{r,s}}{dt} - \nu \sum_{j=0}^{n-1} \sum_{k=0}^{2^j-1} L_{ks}^{jr} a_{j,k} + \sum_{j=0}^{n-1} \sum_{p=0}^{n-1} \sum_{k=0}^{2^j-1} \sum_{q=0}^{2^p-1} N_{kqs}^{jpr} a_{j,k} a_{p,q} = 0\,. \tag{3.4.11}$$

The initial values for $a_{r,s}$ can be obtained by using the discrete Newland transform (see Section 2.3.8). The solution of system (3.4.11) was obtained by using the Adams method for ODEs.

The presence of the Kronecker delta factor in connection coefficients (3.4.7), (3.4.8) provides a formal constraint on the mode selection based on scale criteria. We deduce the selection rule for triadic scale interaction $\{j, p, r\}$ to be $m_j + m_p - m_r = 0$. This relation also suggests that only forward cascading processes are allowed, i.e. for a set of scale triads $\{j, p, r\}$ only $j$ and $p$ that are smaller or equal to the observational scale $r$ give nonzero contributions to the nonlinear terms. Smaller values of $j, p, r$ imply larger scales.

In order to reduce the computational costs in the wavelet Galerkin scheme, especially for high Reynolds number problems ($Re = u_0 L/\nu$, where $u_0$ and $L$ is the amplitude of the velocity and the length of the calculation area), one has to reduce the number of space-scaling degrees of freedom further by exploiting the localization properties of wavelets. This approximation utilizes the finite spectral support of harmonic wavelets to the optimum. The second reduction is for the spatial modes (i.e. $\{k, q, s\}$) carried out based on the localization of the wavelet in physical space. In general, wavelet $\psi_{j,k}(x)$ can be considered as approximately localized function [24] within an interval given by

$$I_{jk} = \left[\frac{k-1}{2^j},\ \frac{k+1}{2^j}\right].$$

Thus, for a triad interaction coefficient $N_{kqs}^{jpr}$ with the reference scale $r$ and position $s$, we retain position indices, satisfying the support overlapping criteria,

$$I_{jk} \cap I_{rs} \neq 0, \quad I_{pq} \cap I_{rs} \neq 0.$$

In order to illustrate the applicability of our scheme, we may vary kinematic viscosity $\nu$, and plot the solution on the map: $x \times t \to [0;1] \times [0;1]$ for $n = 10$. The results for partial solutions of (3.4.11) for different values of the kinematic viscosity are displayed in Figs. 3.1 – 3.2.

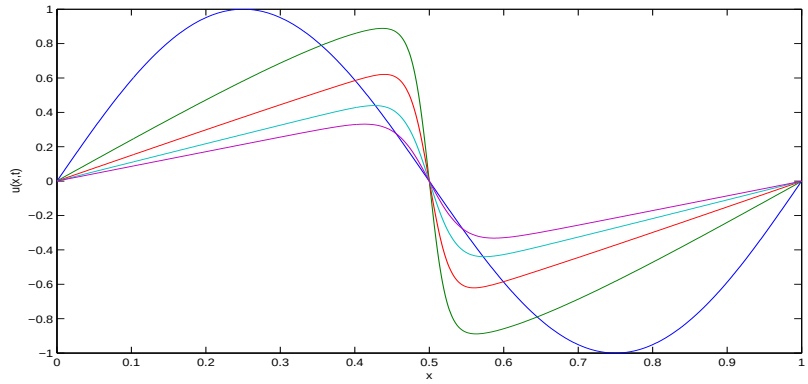


Figure 3.1: *Wavelet solution of the Burgers equation for* $\nu = 0.01$*, from* $t = 0$ *to* $t = 1$ *with the backup time* $\Delta t = 0.25$.

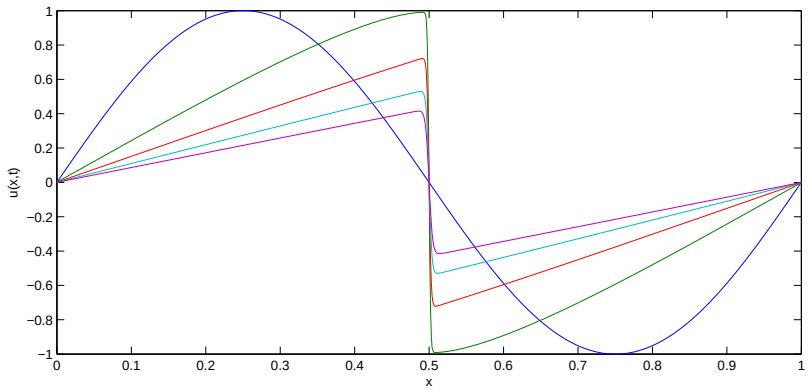


Figure 3.2: *Wavelet solution of the Burgers equation for* $\nu = 0.001$*, from* $t = 0$ *to* $t = 1$ *with the backup time* $\Delta t = 0.25$.

It is observed that when time evolves, the solution develops into a shock wave due to the formation of higher frequency harmonics, and eventually decays. This

is consistent with the qualitative property of the analytical solution of the Burgers equation. The criterion for the relaxation of wave propagation was chosen in the form

$$\varepsilon_{\max} = |\mathcal{P}_{V_{j+1}} u(x,t) - \mathcal{P}_{V_j} u(x,t)| \leq 5 \cdot 10^{-5} .$$

Also, there are other wavelet based studies on solution of nonlinear PDEs. One of such methods was presented by Beylkin and Keiser [40], who employed the method of semigroups to transform the given PDE into an integral equation before its discretization. The connection coefficients in their work are determined using the two-scale difference equation and the moments condition.

The Burgers equation is the simplest model, which describes the competition between nonlinear advection and linear dissipation, and widely studied as the one-dimensional model of three-dimensional Navier-Stokes equations. Our calculation of the nonlinear term highlights the complexity of spatial-scale modes interactions with the scale selection rule explaining generation of smaller scale structures.

## 3.5 Harmonic wavelet solution of the Korteweg-de Vries equation

The Korteweg-de Vries equation is a mathematical model of waves on shallow water surfaces (See e.g. [41, 66]). It is particularly notable as the prototypical example of exactly solvable model (which explains our choice of the problem), that is, a nonlinear PDE whose solutions can be exactly and precisely specified. The solutions include prototypical examples of solitons. The KdV can be solved by means of the inverse scattering transform. The mathematical theory behind the KdV equation is rich and interesting, and in a broad sense, it has always been a topic of active mathematical research.

### 3.5.1 Statement of the problem

It is well-known that a wide class of one-dimensional nonlinear waves in a weakly dispersive medium is governed by the Korteweg-de Vries equation (KdV)

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x_*} + \frac{\partial^3 u}{\partial x_*^3} = 0\,, \tag{3.5.1}$$

where $t$ is time $0 \le t < \infty$, $x_* \in \mathbb{R}$, $u(x_*, t) \in \mathbb{R}$. This equation describes evolution of a nonlinear wave in the moving coordinate system at the speed of long-wave perturbation. The physical value of function $u(x_*, t)$ can be referred to the perturbation of velocity or pressure on shallow water.

Equation (3.5.1) describes media with negative dispersion, when the phase velocity of linear waves decays as the wave-number grows, and media with positive dispersion, when the phase velocity grows with the wave-number. The difference between these two cases consists in the following: in the first case, the coordinate $x_*$ is measured in the direction of propagation of waves; in the second case, it is measured in the reverse direction.

Not to mention that the KdV equation is well-studied, and different methods of its solution exist, the approach we propose also deserves attention because it involves wavelet method, which is getting more involved in the modern mathematical physics and applied mathematics [10, 30, 53, 56]. In this case, the KdV equation is the model of a nonlinear process, which we use in order to approve a new mathematical approach and tricks for solution of nonlinear PDEs.

### 3.5.2 Connection coefficients

Consider again the computation of connection coefficients. The scalar product of two functions on the unit interval $[0; 1]$ is: $\langle f, g\rangle_{L^2[0;1]} \equiv \int_0^1 f(x)g^*(x)dx$, where $g^*(x)$ means the complex conjugate. As we already know from Theorem 3.2.1, PHW are complex and orthonormal functions, i.e. $\langle \psi_{j,k}, \psi_{m,h}\rangle_{L^2[0;1]} = \delta_{j,k}\delta_{m,h}$. Therefore, taking

into account formulae (3.4.4) and (3.4.5), we can define the connection coefficients as follows:

$$L_{ks}^{jr} \stackrel{def}{=} \left\langle \frac{d^3\psi_{j,k}}{dx^3}, \psi_{r,s} \right\rangle,$$

$$N_{kqs}^{jpr} \stackrel{def}{=} \left\langle \psi_{j,k} \frac{d\psi_{p,q}}{dx}, \psi_{r,s} \right\rangle.$$

Let us find the recursive formulae for linear and nonlinear connection coefficients. It follows from the first and third derivatives of (3.4.3) and the definition of connection coefficients $L_{ks}^{(3)jr}$, $N_{kqs}^{jpr}$ that

$$\begin{aligned} L_{ks}^{jr} &= \left\langle \frac{d^3\psi_{j,k}}{dx^3}, \psi_{r,s} \right\rangle = \int_0^1 \frac{d^3\psi_{j,k}}{dx^3} \psi_{r,s}^* dx = \\ &= 2^{-(j+r)+3} \sum_{m_j=2^j}^{2^{j+1}-1} \sum_{m_r=2^r}^{2^{r+1}-1} (\pi m_j)^3 \, i e^{2\pi i \left( \frac{m_r s}{2^r} - \frac{m_j k}{2^j} \right)} \delta_{m_j m_r}. \end{aligned} \tag{3.5.2}$$

As suggested in (3.5.2), we can derive the recursive expression for the nonlinear connection coefficient

$$\begin{aligned} N_{kqs}^{jpr} &= \left\langle \psi_{j,k} \frac{d\psi_{p,q}}{dx}, \psi_{r,s} \right\rangle = \\ &- \frac{2\pi i}{2^{j+p+r}} \sum_{m_j=2^j}^{2^{j+1}-1} \sum_{m_p=2^p}^{2^{p+1}-1} \sum_{m_r=2^r}^{2^{r+1}-1} m_p \cdot e^{2\pi i \left( \frac{m_j k}{2^j} + \frac{m_p q}{2^p} - \frac{m_r s}{2^r} \right)} \delta_{m_j + m_p, m_r}. \end{aligned} \tag{3.5.3}$$

The set of wavelet functions $\{\psi_{j,k}(x)\}$ does not form a complete basis for a finite interval of a non-periodic localized in the space solution. It unavoidably appears the problem of boundary effects. Therefore, the fulfillment of the boundary conditions presents a difficulty. Although, it is possible to construct an orthonormal wavelet basis for a finite interval. Nevertheless, the boundary effects for functions with a compact support will be localized in the neighbourhood of both end points of the interval. These effects can be neglected in the majority of problems.

### 3.5.3 Wavelet-Galerkin method for solution of the KdV equation

By implementing the standard approach in the Galerkin method, we can assume that the unknown function (or weak solution) can be represented as follows:

$$\mathcal{P}_{V_n} u\,(x,t) = \sum_{j=0}^{n-1} \sum_{k=0}^{2^j-1} a_{j,k}(t)\psi_{j,k}(x), \tag{3.5.4}$$

on the interval $x \in [0;1]$ with the given initial and boundary conditions[4] for (3.5.1)

$$\begin{aligned} &u(x_*,0) = -4\mathrm{sech}^2(x_*)\,; \\ &u(x_{*,min},t) = u(x_{*,max},t)\,. \end{aligned} \tag{3.5.5}$$

The solution of the KdV equation (3.5.1) is searched in the map $x_* \times t \to [x_{*,min}; x_{*,max}] \times [0; t_{max}]$.

In order to proceed the solution, we must introduce new variables:

$$x = \frac{1}{L}(x_* - x_{*,min}), \quad L = x_{*,max} - x_{*,min}.$$

Then equation (3.5.1) with boundary and initial conditions (3.5.5) becomes

$$\begin{aligned} &\frac{\partial u}{\partial t} - \frac{6}{L}u(x,t)\frac{\partial u}{\partial x} + \frac{1}{L^3}\frac{\partial^3 u}{\partial x^3} = 0\,, \\ &u(x,0) = -4\mathrm{sech}^2\left(L\left(x + \frac{x_{*,min}}{L}\right)\right)\,; \\ &u(0,t) = u(1,t)\,. \end{aligned} \tag{3.5.6}$$

The substitution of expansion (3.5.4) into rescaled KdV equation (3.5.6) yields us the following expression:

$$\begin{aligned} \sum_{j=0}^{n-1}\sum_{k=0}^{2^j-1}\frac{da_{j,k}}{dt}\psi_{j,k} - \frac{6}{L}\sum_{j=0}^{n-1}\sum_{p=0}^{n-1}\sum_{k=0}^{2^j-1}\sum_{q=0}^{2^p-1} a_{j,k}a_{p,q}\psi_{j,k}\frac{d\psi_{p,q}}{dx} + \\ \frac{1}{L^3}\sum_{j=0}^{n-1}\sum_{k=0}^{2^j-1} a_{j,k}\frac{d^3\psi_{j,k}}{dx^3} = 0\,. \end{aligned} \tag{3.5.7}$$

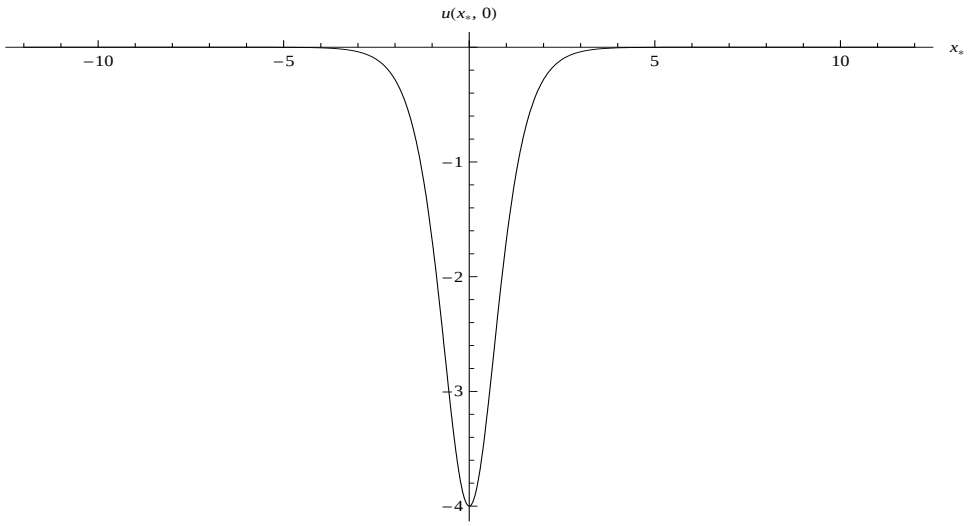


Figure 3.3: *Initial condition for the KdV equation.*

The graph, which represents initial condition (3.5.5) is shown in Fig. 3.3.

The product between equation (3.5.7) and $\psi^*_{r,s}(x)$, taking into account the orthogonality of the basis functions and the definition of the connection coefficients (3.5.2) and (3.5.3) is

$$\frac{da_{r,s}}{dt} - \frac{6}{L}\sum_{j=0}^{n-1}\sum_{p=0}^{n-1}\sum_{k=0}^{2^j-1}\sum_{q=0}^{2^p-1} N^{jpr}_{kqs} a_{j,k} a_{p,q} + \frac{1}{L^3}\sum_{j=0}^{n-1}\sum_{k=0}^{2^j-1} L^{jr}_{ks} a_{j,k} = 0\,. \tag{3.5.8}$$

This means that we have reduced our PDE (3.5.1) to the system of ordinary differential equations. We will skip the detailed analysis of system (3.5.8), which should be solved in order to find the Galerkin solution.

### 3.5.4 Results and analysis of the solution

In order to illustrate the results of the proposed approach, we plot a graph, which displays the solution of the KdV equation (3.5.1) for $n = 8$ on the map $x_* \times t \to [-12; 12] \times [0; 1]$ (See Fig. 3.4).

[4]In fluid dynamics, a nonlinear and exact periodic wave solution of the KdV equation is called a cnoidal wave. A cnoidal wave, characterised by sharper crests and flatter troughs than in a sine wave.

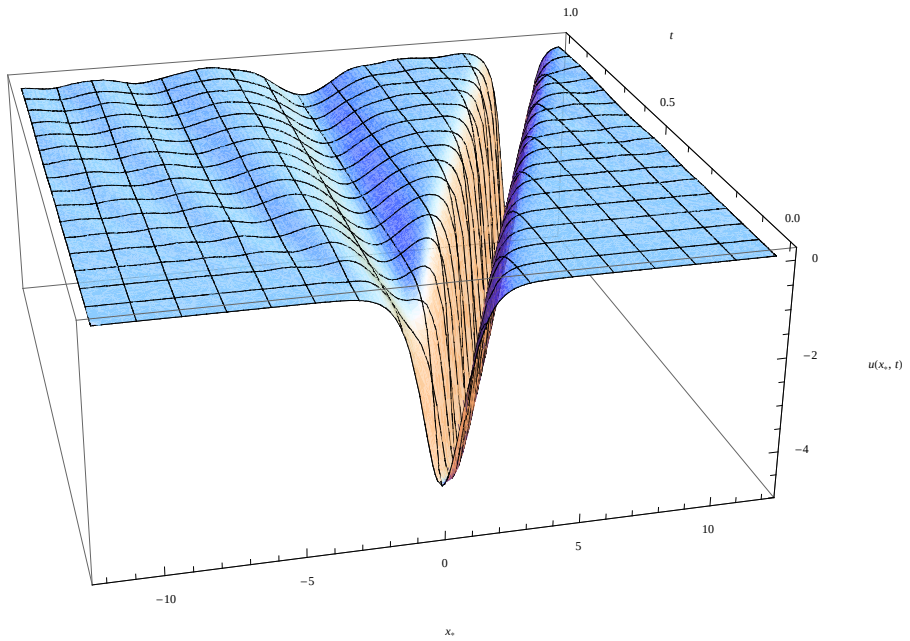


Figure 3.4: *Wavelet solution for evolution of wave propagation.*

Now we are in a position to find the unknown wavelet coefficients, $\{a_{j,k}\}$ at the initial moment $t = 0$ by employing the discrete Newland transform (see Section 2.3.8 and reference [58]):

1. The interval $[0; 1]$ is divided into $2^n$ equal subintervals; the discrete values of the function $u(x, 0)$ are defined.

2. The Fourier transform of the discrete values of the initial function is computed, and the Fourier coefficients are $\{\hat{u}_i\}$, where $i = 0, \ldots, 2^n - 1$.

3. By taking the inverse Fourier transform, we can write out the set of coefficients $\{a_{j,k}\}$, which gives us the initial conditions for solution of ODE (3.5.8).

The solution of the KdV equation (3.5.8) was obtained by using the Adams method, and then making the substitution of the corresponding wavelet coefficients $\{a_{j,k}\}$ into expansion (3.5.4). The result of our computations is shown in Fig. 3.4, and it represents the evolution of wave propagation in dispersive medium.

In order to give a numerical value of the error, recall that (3.5.1) has analytical solution for several initial conditions with the periodic boundary conditions. In particular, one of such functions is:

$$u(x,t) = -2\text{sech}^2(x-4t)\,.$$

The maximum value of absolute error $\varepsilon_{\max} = |\mathcal{P}_{V_n}u(x,t) - u(x,t)|$ was compared with the analytical solution for several values of $t$, and the results are presented in Table 3.1.

| $n$ | $t$ | $\varepsilon_{\max}$ |
|---|---|---|
| 7 | 0.25 | $3.4 \cdot 10^{-3}$ |
| 7 | 0.75 | $7.5 \cdot 10^{-3}$ |
| 8 | 0.25 | $1.4 \cdot 10^{-4}$ |
| 8 | 0.75 | $5.6 \cdot 10^{-4}$ |

Table 3.1: Maximum absolute error for the given values of time.

Let us note that the overall the error is contributed by the discrete wavelet transform and the numerical solution of system (3.5.8). Indeed, the exactness of the projection of the solution of equation (3.5.1) in the finite space with basis functions depends on the level of approximation $n$ (See C. Cattani and A. Kudreyko [89]).

The application of the Galerkin method for solution of nonlinear PDEs depends if the recursive formulae were found. In general, the computation of connection coefficients represents rather difficult problem, and there are two visible reasons. The first reason is that wavelets are not always analytically defined functions by a single formula. The second reason is that a single formula, which defines a family of wavelets does not always yield the corresponding connection coefficients.

There are three important facts to remark about the wavelet approximation.

1. High resolution of a rapidly changing function is the consequence of a large number of wavelet coefficients appearing at fine scales.

2. The fact that the error is restricted to a small neighbourhood of the discontinuity

is the result of the "locality" of wavelets. The behaviour of $f(x)$ at one location affects only the coefficients of wavelets close to that location.

3. Most of the linear part of $f(x)$ is represented exactly.

## Conclusion of Chapter 3

Examples of numerical solutions of PDEs were given. In particular, we developed a wavelet approach for nonlinear Burgers' and the KdV equations. We may conclude that PHW adequately characterize solutions with strong gradients, and allow to reduce the number of connection coefficients in the description of hierarchical processes with localized structures.

It is evident that this approach can be applied for many other types of differential equations. However, there are several restrictions and generalizations for this method:

- It is supposed that $u(x,t)$ is differentiable everywhere.
- A function $u(x,t)$, which is to be projected on the space of wavelets, must belong to the same space of function $\psi(x)$.
- Periodic boundary conditions.
- The solution of a problem should exhibit a (traveling) wave, which can explain the choice of wavelets.

It was observed that the Galerkin method with PHW as basis functions is stable even for small viscosities. However, the main difficulty of this method consists in large sizes of matrices, which describe nonlinear terms of equations (in our case, it is $N$). There exist some potential opportunities in reduction of the computational costs. For example, it is possible to create numerical schemes, which automatically set up the scale and the number of basis functions for a certain space and time intervals (e.g. [38]). Whereas it is easy to obtain the wavelet expansion of a known function,

it can be exceedingly difficult to obtain the wavelet coefficients for PDEs. Similar conclusions were reached in [17, 28, 38].

This methodology can be easily applied for solution of other nonlinear partial differential equations.

We conclude by pointing out that we have a competitive wavelet based method for PDEs and there is always a finer level of approximation inherent to all wavelet approaches. Another interesting problem is that compression errors tend to accumulate when we go to higher levels [86]. This puts severe restriction on the amount of admissible levels. Also, when errors are introduced, the solution may become smooth, which means that the subsequent wavelet compression potential drops.

Much of the research in this field is in progress and it is too early to say wheather the problems mentioned above will be solved or not. However, there is no doubt that wavelet analysis has earned its place as an important alternative to the Fourier analysis – only the scope of its applicability remains to be settled.

# Chapter 4

# Multiscale analysis of the Fredholm type integral equations

**Overview.** In this chapter we present the application of periodized wavelets for solution of the Fredholm type integral equations. The efficiency of the wavelet approach was demonstrated on several illustrative examples. We show that the application of PHW is capable to be compared with other numerical methods.

## 4.1 Introduction

We are concerned with solutions of the Fredholm type integral equation of the second kind which is defined as follows:

$$f(x) = \lambda \int_a^b K(x,t)\, f(t)\, dt + g(x)\ , \tag{4.1.1}$$

where $a$ and $b$ denote finite or infinite numbers, $K(x,t) \in L^2([a;b] \times [a;b])$, $g(x) \in L^2[a;b]$. The number $\lambda$ is called the characteristic value of integral equation (4.1.1) if there exist nontrivial solutions of the corresponding homogeneous equation ($g(x) \equiv 0$). Nontrivial solutions are called eigenfunctions of the integral equation corresponding to the characteristic value $\lambda$.

There exist two different methods to solve an IE numerically: expand the unknown function into a series of orthogonal basis functions, and to reduce the equation to

simultaneous ones with respect to the expansion coefficients. The second approach employs the trapezoidal formula for integration. All these approaches may be used with advantage for suitable types of problem but, in general, they suffer from various well understood limitations.

Analytical solution of IE can be obtained only in exceptional cases. Therefore there were developed many special methods, such us the quadrature method, degenerate kernel approximation, method of least squares (the description of these methods is given in Appendix C), the Bateman method [67] etc. There exist many papers which have obtained important results (e.g. [18, 21, 50, 51]). For solving this kind of bounded problems, C.A. Micchelli and Y. Xu [50] constructed orthonormal multiwavelet bases on bounded domain. In reference [18] interpolating wavelets were constructed on invariant sets. These multiwavelets on bounded domains were studied for efficient solution of IEs. The multiscale representation of integral operators based on these wavelets lead to linear systems with sparse coefficients matrices whose condition numbers are bounded. These sparse representations form the base of fast numerical algorithms for solution of such IEs.

The approach we take in the thesis, employs harmonic and periodized wavelets. The main advantage of our approach over the existing wavelet methods is that the wavelet expansion coefficients can be computed analytically. In addition, the computational cost of our approach is low and the accuracy is high. It is necessary to emphasize that the application of wavelets plays a special place in the modern computational methods thanks to quick convergence of a series of wavelets and the possibility to get the solution with any given approximation error.

One of the early investigations on the wavelet approach for solution of IEs belongs to Beylkin et al. [6]. Another efforts to the solution of this problem are presented in references [42, 48, 79]. The interest to the wavelet approach for solution of IEs is popular nowadays [44].

The most part of the existing research programs is devoted to solution of the Fredholm and Volterra type integral equations. The Galerkin and collocation methods

are mainly used in such research programs.

There were applied different wavelet bases. Besides the well-known Daubechies wavelets, many other wavelets have been used, such as the Haar wavelets [42, 43, 44] and CAS-wavelets [81] etc.

In our opinion, the attention to periodic wavelets and their application for solution of IEs is not sufficient. Another advantage of our choice is that PHW are continuous and differentiable everywhere functions.

This chapter shows that PHW, which satisfy the axioms of the MRA, can be used as basis functions in solution of IEs. The main purpose of the present chapter is to propose for numerical solution of IEs a simple method based on PHW. The recommended technique is also applicable with minor changes to the Volterra and integro-differential equations. The error estimation shows that the accuracy of computations is very high even when the scaling parameter is small.

## 4.2 Wavelet-Galerkin method for solution of low-dimensional integral equations

Let us consider the Fredholm type of the second kind, which has the form:

$$f(x) - \int_0^1 K(x,t)f(t)dt = g(x)\,, \quad x \in [0;1] \tag{4.2.1}$$

where $f(x)$, $g(x) \in L^2[0;1]$, and the kernel $K(x,t) \in L^2([0;1] \times [0;1])$. Using the operator

$$(\mathcal{K}f)(x) = \int_0^1 K(x,t)f(t)dt\,, \quad x \in [0;1]$$

it is usually convenient to write the equation as follows:

$$(\mathcal{F} - \mathcal{K})f = g\,.$$

According to the idea of the Galerkin method, we look for an approximate solution $f \in V_N$ of equation (4.2.1) by requiring that the error equation $f_j - \mathcal{K}f_j - g$ must be orthogonal to spaces $\{V_j\}$.

Then we find coefficients $\{a_{j,k}\}$ for

$$\mathcal{P}_{V_j} f(x) = \sum_{k=0}^{2^{j+1}-1} a_{j,k}\psi_{j,k}(x)\,,$$

such that

$$\langle \mathcal{P}_{V_j} f - \mathcal{K}f - g, \psi_{j,k}\rangle_{L^2[0;1]} = 0\,, \quad k = 0, \ldots, 2^{N+1} - 1\,.$$

All that we need, is to solve the following system of linear algebraic equations with the unknown vector $\vec{a}_N = (a_{j,0}, \ldots, a_{2^{N+1}-1})^T$, i.e.

$$(\vec{I} - \vec{A}_N)\vec{a}_N = \vec{g}_N\,,$$

where $\vec{I}$ is the $2^{N+1}$-identity matrix, $\vec{A}_N = \{c_{k,k'}\}$ and $\vec{g_N} = (g_1, \ldots, g_{2^{N+1}-1})^T$ with

$$c_{k,k'} = \int_0^1 \left( \int_0^1 K(x,t)\psi_{N,k'}(t)dt \right) \psi_{N,k}(x)dx$$

$$g_k = \int_0^1 g(x)\psi_{N,k}(x)dx\,.$$

If $f(x) \in L^2[0;1]$, then the error of the projection of the solution on the space of periodic wavelets $V_N^{per}$ can be estimated (3.3.4) as follows: $|\mathcal{P}_{V_N} f(x) - f(x)|_{L^2[0;1]} = \mathcal{O}(2^{-NP})$.

## 4.3 Wavelet-Galerkin method for solution of the Fredholm type integral equations in two dimensions

Integral equations with two variables arise in many problems of mathematical modelling. Solution of two-dimensional IEs by means of regular methods can be reduced

to a system of linear algebraic equations with dense matrices. The present approach suggests the use of wavelets for solution of two-dimensional IE. The novelty consists in the extension of one-dimensional wavelet-Galerkin theory (see e.g. [27]) for integral equations to two dimensions.

Multiscale analysis allows to analyze multidimensional problems. The method of tensor product represents a natural way to the development of multiscale decomposition, which leads to multidimensional wavelets with compact support.

Consider the Fredholm type integral equation of the second kind:

$$\int_a^b \int_a^b K(x,y,u,v)J(u,v)dudv + J(x,y) = f(x,y)\,, \tag{4.3.1}$$

where $f(x,y)$ is the given function, and $J(x,y)$ is an unknown continuous function. Let us suppose that the kernel satisfies the estimation:

$$\left|\frac{\partial^l K(x,y,u,v)}{\partial x^{l_1}\partial y^{l_2}\partial u^{l_3}\partial v^{l_4}}\right| \le \frac{C}{\left(\sqrt{(x-u)^2+(y-v)^2+\alpha^2}\right)^l}\,, \quad l = l_1+l_2+l_3+l_4 \tag{4.3.2}$$

where $\alpha$ is a parameter of Pocklington's equation [20, 69].

The most trivial approach for construction of two-dimensional orthonormal wavelet-basis from one-dimensional basis (2.3.13) consists in the tensor product of two one-dimensional bases [23, 78].

**Proposition.** Let $\psi_{j_1,k_1}(x)$ and $\psi_{j_2,k_2}(y)$ be wavelets defined on $\mathbb{R}$, and assume

$$\psi_{j_1,k_1;j_2,k_2}(x,y) = \psi_{j_1,k_1}(x)\psi_{j_2,k_2}(y)\,,$$

then the system

$$2^{\frac{j_1+j_2}{2}}\psi\left(2^{j_1}x-k_1, 2^{j_2}y-k_2\right)$$

$\forall j_1, j_2$ and $k_1, k_2 \in \mathbb{Z}$ forms orthonormal basis in $L^2(\mathbb{R}\times\mathbb{R})$.

For example, a graph of two-dimensional harmonic wavelet for $j=0$, $k=0$, $l=1$ is shown in Fig. 4.1.

**Remark.** It should be noted that the multivariable wavelet theory is much less developed than the theory of one variable presented in Chapters 1 – 2.

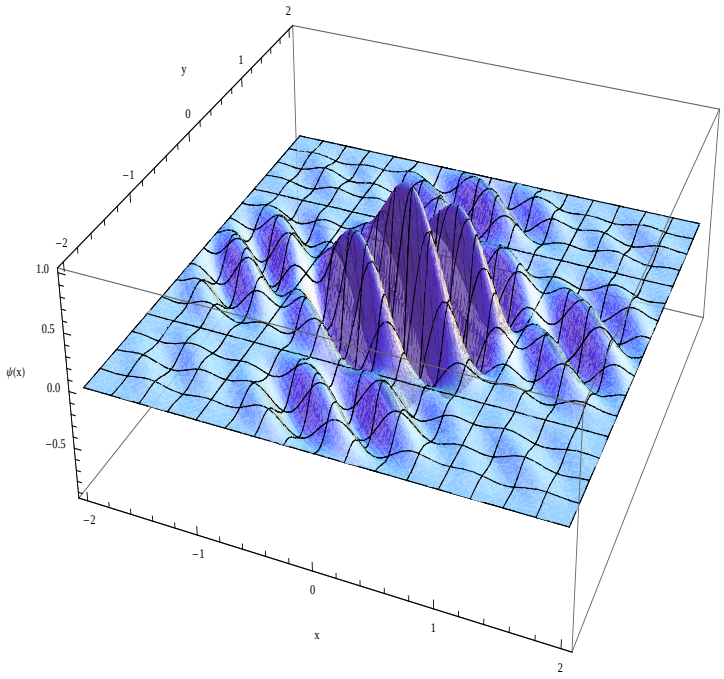


Figure 4.1: *Real part of 2D harmonic wavelet* $\psi_{0,1}^{0,0}(x,y)$.

Under the previous proposition, we can create an orthonormal basis in $W_j$, where we must involve the following family of three products:

$$\varphi_{j,k}(x)\psi_{j,l}(y)\,, \quad \psi_{j,k}(x)\varphi_{j,l}(y)\,, \quad \psi_{j,k}(x)\psi_{j,l}(y)\,,$$

which means that the variation of the scale along the $X$ and $Y$-axes goes with the equal resolution. Since functions $\varphi(x-k)$ constitute an orthonormal basis in $V_0$, the product $\varphi_{j,k}(x)\varphi_{j,l}(y)$ presents orthonormal basis in $V_0$, and formed by $\mathbb{Z}^2$ shifts of a single function $\varphi(x,y)$. Then the family of orthogonal scaling functions and wavelets

is searched in the form

$$J(x,y) = \sum_{i,j=-m+1}^{2^N-1} d_{i,j}\varphi_{N,i}(x)\varphi_{N,j}(y) + \sum_{i=-m+1}^{2^N-1}\sum_{s=1}^{k-N}\sum_{j=-m+1}^{2^{N+s-1}-m} d_{i,j,s}\varphi_{N,i}(x)\psi_{N+s,j}(y) +$$
$$\sum_{j=-m+1}^{2^N-1}\sum_{q=1}^{k-N}\sum_{i=-m+1}^{2^{N+q-1}-m} d_{i,j,q}\varphi_{N,j}(y)\psi_{N+q,i}(x) +$$
$$\sum_{s=1}^{k-N}\sum_{q=1}^{k-N}\sum_{i=-m+1}^{2^{N+q-1}-m}\sum_{j=-m+1}^{2^{N+s-1}-m} c_{i,j,s,q}\psi_{N+q,i}(x)\psi_{N+s,j}(y)\,,$$

and presents a general solution of equation (4.3.1). Functions $\varphi(x)$, $\psi(x)$ are defined by formulae (2.3.16) and (2.3.13). The idea of the wavelet-Galerkin method is presented by the following conditions:

$$\int_a^b\int_a^b\int_a^b\int_a^b K(x,y,u,v)J(u,v)\varphi_{N,r}(x)\varphi_{N,t}(y)dxdydudv +$$
$$\int_a^b\int_a^b J(x,y)\varphi_{N,r}(x)\varphi_{N,t}(y)dxdy =$$
$$\int_a^b\int_a^b f(x,y)\varphi_{N,r}(x)\varphi_{N,t}(y)dxdy\,, \quad -m+1 \le r,t \le 2^N-1\,;$$

$$\int_a^b\int_a^b\int_a^b\int_a^b K(x,y,u,v)J(u,v)\varphi_{N,r}(x)\psi_{N,t}(y)dxdydudv +$$
$$\int_a^b\int_a^b J(x,y)\varphi_{N,r}(x)\psi_{N,t}(y)dxdy =$$
$$\int_a^b\int_a^b f(x,y)\varphi_{N,r}(x)\psi_{N,t}(y)dxdy\,, \quad -m+1 \le r \le 2^N-1\,,$$
$$-m+1 \le t \le 2^{n-1}-m, N+1 \le n \le k;$$

$$\int_a^b \int_a^b \int_a^b \int_a^b K(x,y,u,v)J(u,v)\psi_{N,r}(x)\varphi_{N,t}(y)dxdydudv+$$
$$\int_a^b \int_a^b J(x,y)\psi_{N,r}(x)\varphi_{N,t}(y)dxdy =$$
$$\int_a^b \int_a^b f(x,y)\psi_{N,r}(x)\varphi_{N,t}(y)dxdy\,, \quad -m+1 \le t \le 2^N-1\,,$$
$$-m+1 \le r \le 2^{n-1}-m,\ N+1 \le n \le k;$$

$$\int_a^b \int_a^b \int_a^b \int_a^b K(x,y,u,v)J(u,v)\psi_{N,r}(x)\psi_{N,t}(y)dxdydudv+$$
$$\int_a^b \int_a^b J(x,y)\psi_{N,r}(x)\psi_{N,t}(y)dxdy =$$
$$\int_a^b \int_a^b f(x,y)\psi_{N,r}(x)\psi_{N,t}(y)dxdy\,, \quad -m+1 \le r,t \le 2^{n-1}-m, N+1 \le n \le k$$

where $\psi_{N,t}, \psi_{N,r}, \varphi_{N,t}, \varphi_{N,r}$ are observation wavelets and scaling functions. The set of these conditions represents a system of linear algebraic equations with a squared

Gramian matrix [5] **A**:

$$\begin{pmatrix}
A_{11}^{11} & \dots & A_{1k-N+1}^{11} & A_{11}^{12} & \dots & A_{1k-N+1}^{12} & \dots & A_{1k-N+1}^{1k-N+1} \\
A_{11}^{21} & \dots & A_{1k-N+1}^{21} & A_{11}^{22} & \dots & A_{1k-N+1}^{22} & \dots & A_{1k-N+1}^{2k-N+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
A_{11}^{k-N+11} & \dots & A_{1k-N+1}^{k-N+11} & A_{11}^{k-N+12} & \dots & A_{1k-N+1}^{k-N+12} & \dots & A_{1k-N+1}^{k-N+1k-N+1} \\
A_{21}^{11} & \dots & A_{2k-N+1}^{11} & A_{21}^{12} & \dots & A_{2k-N+1}^{12} & \dots & A_{2k-N+1}^{1k-N+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
A_{21}^{k-N+11} & \dots & A_{2k-N+1}^{k-N+11} & A_{21}^{k-N+12} & \dots & A_{2k-N+1}^{k-N+12} & \dots & A_{k-N+1k-N+1}^{k-N+1k-N+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
A_{k-N+11}^{k-N+11} & \dots & A_{k-N+1k-N+1}^{k-N+11} & A_{k-N+11}^{k-N+12} & \dots & A_{1k-N+1k-N+1}^{k-N+12} & \dots & A_{k-N+1k-N+1}^{k-N+1k-N+1}
\end{pmatrix}$$

It is important to underline that we only consider the sparsity of matrices. The use of classical functions in the Galerkin method yields matrix **A** with nonzero elements or its values are not enough small to consider **A** as a sparse matrix. Then let us estimate elements of matrix **A** if basis functions are wavelets.

$$\begin{aligned}
A_{pq}^{ts} &= \{a_{ijrf}^{ptqs}\}\,, 1 \le p,t,q,s \le k-N+1\,, \\
A_{11}^{11} &= \{a_{ijrf}^{1111}\}\,, -m+1 \le i,j,r,f \le 2^N-1\,, \\
A_{pp}^{pp} &= \{a_{ijrf}^{pppp}\}\,, 2^{N+p-2} \le i,j,r,f \le 2^{N+p-1}-1\,,
\end{aligned}$$

$$\begin{aligned}
a_{ijrf}^{ptqs} = &\int_a^b \int_a^b \int_a^b \int_a^b K(x,y,u,v)\psi_{ij}^{pt}(u,v)\psi_{rf}^{qs}(x,y)dxdydudv + \\
&\int_a^b \int_a^b \psi_{ij}^{pt}(x,y)\psi_{rf}^{qs}(x,y)dxdy\,.
\end{aligned}$$

Consider only those elements of matrix **A**, whose wavelet supports do not intersect. In particular, the term

$$\int_a^b \int_a^b \psi_{ij}^{pt}(x,y)\psi_{rf}^{qs}(x,y)dxdy$$

[5] The set of functions $\psi_{j,k}(x), \psi_{j,l}(y), \varphi_{j,k}(x), \varphi_{j,l}(y) \in L^2(\mathbb{R})$ in an inner product space.

equals to zero, and where $\psi_{ij}^{pt}(x,y)$ is the observation function with scaling (upper indices) and translation (lower indices) parameters.

**Theorem 4.3.1.** *Elements of matrix* $\boldsymbol{A}$ *satisfy the following estimate:*

$$|a_{pqts}^{ijrf}| \le 2^{-\frac{q+s}{2}} \frac{C(2^{-4p}+2^{-4s})}{\left(\rho^2(supp\,\psi_{ij}^{pt},\psi_{rf}^{qs})+\alpha^2\right)^2},$$

*where* $\rho(A,B)$ *is the euclidean distance between sets* $A$ *and* $B$ *on the plane.*

*Proof.* Consider a function:

$$g(x,y) = \int\int_{\operatorname{supp}\psi_{ij}^{pt}} K(x,y,u,v)\psi_{ij}^{pt}(u,v)dudv\,.$$

If $p \neq q$, $s \neq t$ then we must choose $\max(p,q)$ and $\max(s,t)$. Taking into account the approximation properties of wavelets and estimation (4.3.2), suppose $\exists\, N(x,y)$ such that

$$\begin{aligned}|g(x,y)-N(x,y)| &\le \frac{C\left(\sqrt{2^{-2p}+2^{-2s}}\right)^4}{\left(\sqrt{(r-i)^2 2^{2p}+(f-j)^2 2^{2s}+\alpha^2}\right)^4} \\ &\le \frac{C(2^{-2p}+2^{-2s})}{\left(2^{2p}(r-i)^2+2^{2s}(f-j)^2+\alpha^2\right)^2} = \frac{C(2^{-4p}+2^{-4s})}{\left(\rho^2(\operatorname{supp}\psi_{ij}^{pt},\psi_{rf}^{qs})+\alpha^2\right)^2}.\end{aligned}$$

Then, let us estimame

$$\begin{aligned}|a_{ijrf}^{pqts}| &= \left|\int_a^b\int_a^b\int_a^b\int_a^b K(x,y,u,v)\psi_{ij}^{pt}(u,v)\psi_{rf}^{qs}(x,y)dxdydudv\right| \\ &= \left|\int\int_{\operatorname{supp}\psi_{rf}^{qs}}\left(\int\int_{\operatorname{supp}\psi_{ij}^{pt}} K(x,y,u,v)\psi_{ij}^{pt}(u,v)dudv\right)\psi_{rf}^{qs}(x,y)dxdy\right| \\ &\le \frac{C(2^{-4p}+2^{-4s})}{\left(\rho^2(\operatorname{supp}\psi_{ij}^{pt},\psi_{rf}^{qs})+\alpha^2\right)^2}\int\int_{\operatorname{supp}\psi_{rf}^{qs}}\psi_{rf}^{qs}(x,y)dxdy \\ &= 2^{-\frac{q+s}{2}}\frac{C(2^{-4p}+2^{-4s})}{\left(\rho^2(\operatorname{supp}\psi_{ij}^{pt},\psi_{rf}^{qs})+\alpha^2\right)^2},\end{aligned}$$

which completes the proof of our theorem. □

It follows from the obtained estimation that at large scaling parameters, the absolute value of many elements in matrix **A** is small, i.e. the matrix is pseudo-sparse. Therefore, quick solution of the corresponding system of linear algebraic equations can be obtained thanks to pseudo-sparse matrices.

This result can be useful to offload the CPU by operating with quickly convergering series of wavelet coefficients and pseudo-sparse matrices, and thus obtaining additional speedup for no cost.

## 4.4 Collocation method for solution of integral equations

Another way for handling low-dimensional IEs is the collocation method. Following the idea of all projection methods, we involve the operator of orthogonal projection $\mathcal{P}_{V_N^{per}}$ (see Definition 1.2.3). This operator projects a function from the space $L^2[0;1]$ onto the space of periodic wavelets $V_N^{per}$.

In view of Theorem 3.2.1, periodic wavelets form MRA, and $f(x) \in L^2[0;1]$ can be presented as the following series of wavelets with the corresponding coefficients:

$$f(x) = a_0 + \sum_{j=0}^{\infty} \sum_{k=-\infty}^{\infty} \left\{ a_{j,k}\psi_{j,k}(x) + \widetilde{a}_{j,k}\psi^*_{j,k}(x) \right\},$$

In particular, the projection of function $f(x)$ onto the space of PHWs $V_N$ as basis functions is written as follows:

$$\mathcal{P}_{V_N} f(x) = a_0 + \sum_{j=0}^{N-1} \sum_{k=0}^{2^j-1} \left\{ a_{j,k}\psi_{j,k}(x) + \widetilde{a}_{j,k}\psi^*_{j,k}(x) \right\}. \tag{4.4.1}$$

Consider now analytical computations for the partial case, when $N = 1$, which form the idea of our approach. The substitution of $\mathcal{P}_{V_1} f(x)$ into equation (4.1.1) gives us

the following linear equation

$$a_0 + a_{0,0}\psi_{0,0}(x) + \widetilde{a}_{0,0}\psi^*_{0,0}(x) = \int_a^b K(x,t)\{a_0 + a_{0,0}\psi_{0,0}(t) + \widetilde{a}_{0,0}\psi^*_{0,0}(t)\}\,dt + g(x)\,. \tag{4.4.2}$$

Thus, we received equation with unknown coefficients $a_0$, $a_{0,0}$ and $\widetilde{a}_{0,0}$, which can be computed by employing the collocation method. In order to do this, let us define the collocation points. Denote

$$a \le x_1 < x_2 < \ldots < x_n \le b\,.$$

In the wavelet-collocation algorithm a set of collocation points $\{x_j\}$ is defined in such a way that the collocation points of the coarser level of resolution are the subset of the collocation points of the finer level of resolution. In our case, when $N = 1$, we need to choose two collocation points [6]: $x_1$, $x_2 \in [a; b]$, and also recall the equality, which states that if $f(x) : \mathbb{R} \to \mathbb{R}$, then $\widetilde{a}_{j,k} = a^*_{j,k}$ [57, 58]. The substitution of such $\{x_j\}$ into (4.4.2) gives us the following system of algebraic equations

$$a_0 + a_{0,0}\psi_{0,0}(x_j) + \widetilde{a}_{0,0}\psi^*_{0,0}(x_j) = \int_a^b K(x_j,t)\{a_0 + a_{0,0}\psi_{0,0}(t) + \widetilde{a}_{0,0}\psi^*_{0,0}(t)\}dt + g(x_j), \tag{4.4.3}$$

where $j = 1, 2$. The analysis of the system for higher values of $N$ represents a distinct problem, which is beyond the scope of the research. In order to demonstrate our approach, we solve several IEs (See C. Cattani and A. Kudreyko [92]).

### 4.4.1 Implementation of the collocation method

As an example, let us apply the proposed algorithm for equation of the following type

$$f(x) - \int_0^1 \sin(4\pi x + 2\pi t) f(t)dt = \cos(2\pi x) + \frac{1}{2}\sin(4\pi x)\,. \tag{4.4.4}$$

[6] If $[a; b]$ differs from $[0; 1]$, then $[x \times t]$ map must be rescaled.

It is evident that the solution of this equation is: $f(x) = \cos 2\pi x$. One can show that this solution can be obtained by the Fourier method. For example, if we choose $x_1 = 0.5$ and $x_2 = 0.75$ as the collocation points, the solution of system (4.4.3) can be easily found, and it is: $a_0 = 0$; $a_{0,0} = \widetilde{a}_{0,0} = 0.5$. The substitution of these coefficients into equation (4.4.1) for $N = 1$ yields us directly the solution $f(x) = \cos 2\pi x$.

Let us note that we obtained the projection of the solution of equation (4.4.4) onto a finite space with the corresponding basis functions. Wavelet coefficients $\{a_{j,k}\}$ for the successive levels ($N > 1$) are zeros.

The same result was obtained in reference [79], but there periodic Daubechies wavelets were applied as basis functions. The numerical solution, which was obtained has an absolute error of $10^{-15}$ for $N = 10$, which is quite good, but comparing with the Daubechies wavelets, harmonic wavelets are analytically defined, infinitely differentiable and band limited. Thus, it enables us to study, their differentiable properties.

### 4.4.2 Construction of eigenfunctions

Consider the homogeneous Fredholm integral equation

$$f^*(x^*) - \lambda \int_0^{\pi} \cos(x^* + t^*) f^*(t^*) dt^* = 0\,, \tag{4.4.5}$$

and solve it by using the collocation method. Denote the collocation points by

$$a \leq x_1 < x_2 < \ldots < x_l < \ldots \leq b\,.$$

Recalling the decomposition of a real periodic function (4.4.1) on the space of PHW, we have

$$\mathcal{P}_{V_1} f(x) = a_0 + \psi_{0,0}(x) + \widetilde{a}_{0,0}\psi^*_{0,0}(x)$$

for $N = 1$. In addition, for $f(x) : \mathbb{R} \to \mathbb{R}$, we can use the equality $a^*_{0,0} = \widetilde{a}_{0,0}$. In order to deal with $1-$periodic functions, it is convenient to introduce new variables:

$x^* = 2\pi x$ and $t^* = 2\pi t$, and we get a rescaled equation

$$f(x) - 2\pi\lambda \int_0^{0.5} \cos\left[2\pi(x+t)\right] f(t)dt = 0\,, \tag{4.4.6}$$

which we will solve by using the collocation method. The corresponding choice of the collocation points $\{x_l\}$ leads us to a system of linear algebraic equations with the parameter $\lambda$

$$a_0 + a_{0,0}e^{2\pi i x_l} + \widetilde{a}_{0,0}e^{-2\pi i x_l} - 2\pi\lambda \int_0^{1/2} \cos[2\pi(x_l+t)](a_0 + a_{0,0}e^{2\pi i t} + \widetilde{a}_{0,0}e^{-2\pi i t}) = 0\,.$$

Thus, we can find parameters $\lambda_1 = \dfrac{2}{\pi}; \lambda_2 = -\dfrac{2}{\pi}$ and eigenfunctions $f_1^*(x^*) = \cos x^*$, $f_2^*(x^*) = \sin x^*$. Note, that the obtained projection for $N = 1$ coincides with the analytical solution. If we had continued to search for the solution on the successive levels of approximation, the wavelet coefficients $\{a_{j,k}; j \geq 0,\, 0 \leq k \leq 2^j - 1\}$ would be zeros (A. Kudreyko and C. Cattani [91]).

### 4.4.3 MRA for the Fredholm type integral equation

This section illustrates our approach and the property of periodic harmonic wavelets to construct the multiresolution analysis. Consider the following integral equation:

$$\int_0^1 f(t)e^{t\sin x}dt = -\frac{(1 - e^{\sin x})(\sin^3 x + 16\pi^2 \sin x - 8\pi^3 - 2\pi \sin^2 x)}{(\sin^2 x + 8\pi^2)^2 + 4\pi^2 \sin^2 x}\,. \tag{4.4.7}$$

**Remark.** It should be noted that such a special form of equation (4.4.7) is taken only to demonstrate the applicability of periodized harmonic wavelets and multiresolution analysis. In fact, there could be any 1-periodic function from space $L^2[0;1]$.

Let us find the solution on the lowest approximation level, i.e. $N = 1$. Then, taking into account (4.4.1) and (2.3.44), the projection of the solution on the space

of periodic harmonic wavelets is

$$\mathcal{P}_{V_1} f(t) = a_0 + a_{0,0} e^{2\pi i t} + \widetilde{a}_{0,0} e^{-2\pi i t} .$$

If the expression for $\mathcal{P}_{V_1} f(t)$ is substituted into (4.4.7), we obtain

$$\int_0^1 (a_0 + a_{0,0} e^{2\pi i t} + \widetilde{a}_{0,0} e^{-2\pi i t}) e^{t \sin x} dt = X ,$$

where $X$ is the right hand side of (4.4.7). Denote the collocation points as follows: $x_1 = 0$, $x_2 = \pi/6$, $x_3 = \pi/4$. Then we will get a system of linear algebraic equations with respect to wavelet-coefficients

$$\int_0^1 (a_0 + a_{0,0} e^{2\pi i t} + \widetilde{a}_{0,0} e^{-2\pi i t}) e^{t \sin x_l} dt = X_l ,$$

where $l = 1, 2, 3$. If we assume that $f(t) \in \mathbb{R}$, then we can take advantage of the equality $a^*_{j,k} = \widetilde{a}_{j,k}$ [57]. The solution of such system yields us the following coefficients: $a_0 = 0, a_{0,0} = 0.487$, $\widetilde{a}_{0,0} = 0.487$ and the corresponding plot is presented in Fig. 4.1 by a dashed line. In the case if we take $N = 2$, we find

$$\begin{aligned} \mathcal{P}_{V_2} f(t) =& a_0 + a_{0,0}\psi_{0,0}(t) + \widetilde{a}_{0,0}\psi^*_{0,0}(t) \\ &+ a_{1,0}\psi_{1,0}(t) + \widetilde{a}_{1,0}\psi^*_{1,0}(t) + a_{1,1}\psi_{1,1}(t) + \widetilde{a}_{1,1}\psi^*_{1,1}(t). \end{aligned}$$

And the corresponding choice of the collocation points will also give us a system of algebraic equations with respect to the wavelet coefficients. It can be shown that these coefficients are $a_0 = 0, a_{0,0} = \widetilde{a}_{0,0} = 0.5, a_{1,0} = -i/8, \widetilde{a}_{1,0} = i/8, a_{1,1} = -i/8, \widetilde{a}_{1,1} = i/8$. The plot for $N = 2$ is shown in Fig. 4.1 by a solid line. It is obvious that the first approximation represents the raw approximation of the second level. The projection of the solution on the second approximation level represents the exact solution of integral equation (4.4.7).

If we had continued our computations for $N = 3$ etc, the proceeding wavelet coefficients would eventually be zeros (C. Cattani, A. Kudreyko [93]).

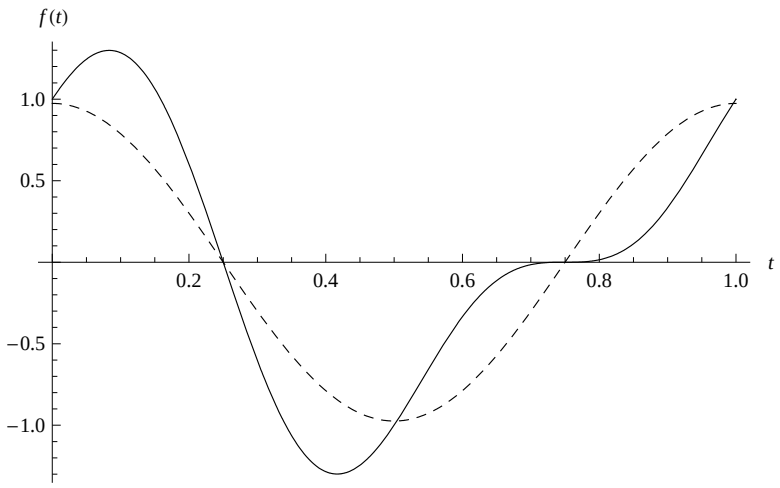


Figure 4.2: *Multiscale solution of equation* (4.4.7) *at* $N = 1$- *dashed line and* $N = 2$ *solid line.*

### 4.4.4 Wavelet-collocation method for integro-differential equations

Consider another type of problem, i.e. an integro-differential equation [22]

$$\begin{cases} y'(x) = \int\limits_0^1 \sin(2\pi t + 4\pi x) y(t) dt - \cos 2\pi x (1 + \sin 2\pi x) - 2\pi \sin 2\pi x; \\ y(0) = 1\,. \end{cases} \tag{4.4.8}$$

Similar to the previous example, we can assume that the solution can be represented in terms of periodic harmonic wavelets (2.3.44), i.e.

$$\mathcal{P}_{V_1} f(t) = a_0 + a_{0,0} e^{2\pi i t} + \widetilde{a}_{0,0} e^{-2\pi i t}\,.$$

The corresponding choice of the collocation points $x_1, x_2, x_3$ gives us a system of linear algebraic equations with respect to $a_0, a_{0,0}, \widetilde{a}_{0,0}$. The values of the wavelet coefficients are $a_0 = 0$; $a_{0,0} = \widetilde{a}_{0,0} = 0.5$; and we can conclude that $\mathcal{P}_{V_1} f(t) = \cos 2\pi t$.

# Conclusion of Chapter 4

In order to verify our approach, we included the results of a number of numerical examples. The advantages of the application of PHW for solution of integral equations are:

1. The resulting function for every level of approximation is analytically defined, which differs our approach from many other methods.

2. The approximation error of our approach is considerably less than it is in other methods.

However, the algorithmic complexity of our approach is proportional to the number of significant coefficients in the wavelet expansion. The approach, that we propose does not claim to be a universal, and it is not considered as the best, we only extended the borders of the application of PWH. Unfortunately, in the theory that of periodic wavelets that we develop does not approximate non-periodic functions nearby bounds, and the choice of the wavelet family for the analysis of data represents more "art" than a routine operation.

Our research is not intended to provide a complete discussion on the subject. We concentrated on the specific properties, which are useful for numerical solutions of integral and differential equations. As it was mentioned in Chapter 3, the current wavelet theory does not give a clear answer about the best choice of a wavelet [60]. The most important criterion in the choice of a wavelet is its belonging to a certain functional space, where it is expected to get a solution. Another criterion is a quick convergence of the wavelet series with respect to the classical methods (e.g. Fourier method). Indeed, there exist a lot of alternative (traditional) methods of solution of PDEs (see Appendix B) as well as integral equations (see Appendix C), and each of these methods has its own disadvantages, which can be equilibrated by using other methods. Wavelet approach represents one of such methods, and it finds its applications with the increasing frequency in many engineering and mathematical problems.

# Main results and conclusions

In this thesis we studied PHW in the context of solution of PDEs and IEs, and the following results were obtained:

1. It was proved that periodic harmonic wavelets satisfy the properties of the MRA. It was also shown that periodic wavelets form a complete basis for periodic functions in $L^2[0;1]$. The obtained results allow us to use PHW in solution to some applied problems.

2. The solution of the Burgers equation was given by means of PHW, and the corresponding connection coefficients were discussed. In particular, it was obtained that the maximum scale can be chosen from the initial condition, and the range of variation of the translation parameter is related with boundary conditions. The model makes it possible to simulate the solution even for small values of the viscosity coefficients. It means that the application of PHW allows us to describe periodic processes with strong gradients.

   The Korteweg-de Vries equation was solved as another example. In this case, the system of PHW does not form a complete basis in a finite interval of a non-periodic function localized in the space function. The lack of such approach for solutions (e.g.) of the KdV equation is that there unavoidably appears the problem of boundary effects.

3. The proposed approximation scheme for a function is also useful in computation of some integral equations. The application of the collocation method with

PHW as basis functions permits to find solution at low scale, which means a quick convergence of the wavelet series.

4. The analytical formula for the computation of the approximation of projection on the space of periodic wavelets was obtained. The error exponentially decreases if the resolution and the number of vanishing moments grow, i.e. $|e_N^{per}| = \mathcal{O}(2^{-NP})$.

5. The application of wavelet functions for two-dimensional integral equations allows to reduce the computational costs and system requirements for the random access memory of a computer.

According to the ISI Web of Knowledge as of March 8, 2011, author's main publications [86, 91, 92] were cited 25 times.

# Appendix A

## A.1 Fourier transform of periodic function

Let $x(t)$ be a periodic function with period $T$, which satisfies the Dirichlet conditions on $(-\pi, \pi)$:

- $x(t)$ is continuous or has a finite number of breakpoints of the first kind;
- the interval $(-\pi, \pi)$ can be divided on a finite number of subintervals, where the function changes monotonously,

as shown in Fig. A.1, then we can express $x(t)$ as an infinite Fourier series

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left( a_k \cos \frac{2\pi kt}{T} + b_k \sin \frac{2\pi kt}{T} \right), \tag{A.1.1}$$

where $\{a_0, a_k, b_k;\ k \geq 1\}$ are constant Fourier coefficients defined by

$$\begin{aligned} a_0 &= \frac{1}{T} \int_{-T/2}^{T/2} x(t)\ dt; \\ a_k &= \frac{2}{T} \int_{-T/2}^{T/2} x(t) \cos \frac{2\pi kt}{T}\, dt; \\ b_k &= \frac{2}{T} \int_{-T/2}^{T/2} x(t) \sin \frac{2\pi kt}{T}\, dt. \end{aligned} \tag{A.1.2}$$

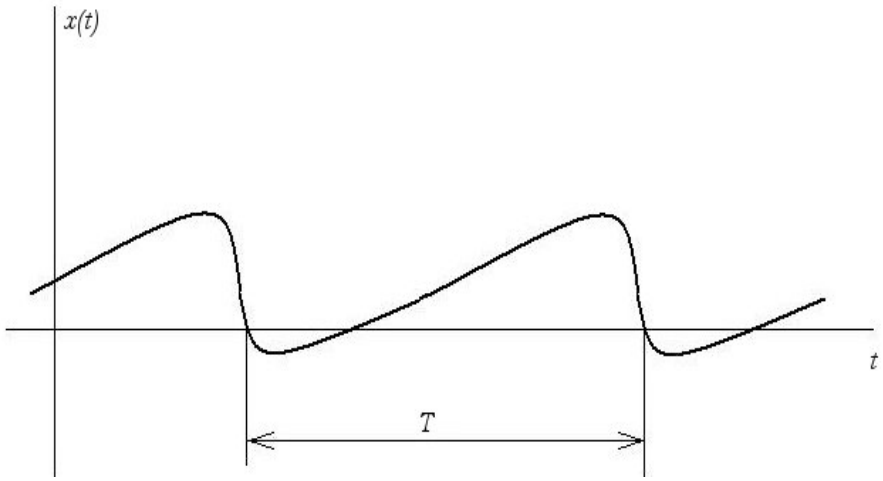


Figure A.1: *Arbitrary periodic function of time*

Suppose that the position of the $t$-axis in Fig. A.1 is adjusted in such a way, that the mean value $\int_{\mathbb{R}} x(t)dt = 0$. Then, according to (A.1.2), the coefficient $a_0$ will be 0.

The frequency of the $k$-th coefficient is

$$\omega_k = \frac{2\pi k}{T}\,. \tag{A.1.3}$$

The spacing between adjacent harmonics is

$$\Delta\omega_k = \frac{2\pi}{T}\,, \tag{A.1.4}$$

and it is evident that when the period $T$ becomes greater, the frequency spacing $\Delta\omega$ becomes smaller, and the Fourier coefficients are getting tightly packed. In the limit, when $T \to \infty$, these harmonics will be merged together.

The substitution of (A.1.2) into (A.1.1) gives

$$x(t) = \sum_{k=1}^{\infty} \left\{ \frac{2}{T} \int_{-T/2}^{T/2} x(t) \cos \frac{2\pi k t}{T}\, dt \right\} \cos \frac{2\pi k t}{T} + \sum_{k=1}^{\infty} \left\{ \frac{2}{T} \int_{-T/2}^{T/2} x(t) \sin \frac{2\pi k t}{T}\, dt \right\} \sin \frac{2\pi k t}{T}$$

for $a_0 = 0$. Further, substituting for $2\pi k/T$ from (A.1.3) and $1/T$ from (A.1.4), we obtain

$$x(t) = \sum_{k=1}^{\infty} \left\{ \frac{\Delta\omega}{\pi} \int_{-T/2}^{T/2} x(t) \cos \omega_k t\, dt \right\} \cos \omega_k t + \sum_{k=1}^{\infty} \left\{ \frac{\Delta\omega}{\pi} \int_{-T/2}^{T/2} x(t) \sin \omega_k t\, dt \right\} \sin \omega_k t\,.$$

When the period $T \to \infty$, then $\Delta\omega \to d\omega$, and the $\sum$ becomes an integral with limits from $\omega = 0$ to $\omega = \infty$. In this case,

$$x(t) = \int_0^{\infty} \frac{d\omega}{\pi} \left\{ \int_{-\infty}^{\infty} x(t) \cos \omega t\, dt \right\} \cos \omega t + \int_0^{\infty} \frac{d\omega}{\pi} \left\{ \int_{-\infty}^{\infty} x(t) \sin \omega t\, dt \right\} \sin \omega t$$

or, denoting

$$\begin{aligned} A(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} x(t) \cos \omega t\, dt; \\ B(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} x(t) \sin \omega t\, dt, \end{aligned} \tag{A.1.5}$$

we obtain

$$x(t) = 2 \int_0^{\infty} A(\omega) \cos \omega t\, d\omega + 2 \int_0^{\infty} B(\omega) \sin \omega t\, d\omega. \tag{A.1.6}$$

The terms $A(\omega)$ and $B(\omega)$ defined by (A.1.5) are the components of the Fourier transform of $x(t)$, and equation (A.1.6) is the representation of $x(t)$ by the inverse or integral Fourier transform.

A Fourier integral may be regarded as the formal limit of the Fourier series when the period is tending to infinity. The reason for introducing this concept is because Fourier integrals indicate the frequency composition of an aperiodic function. Usually, equations (A.1.5) and (A.1.6) are written in the complex form, making use of Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{A.1.7}$$

Defining function $\hat{x}(\omega)$ as

$$\hat{x}\left(\omega\right) \equiv A(\omega) + iB(\omega)\,, \tag{A.1.8}$$

and recalling formulae (A.1.5) and (A.1.7), we have

$$\hat{x}\left(\omega\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x\left(t\right) e^{-i\omega t}\, dt. \tag{A.1.9}$$

Equation (A.1.9) is the formal definition of $\hat{x}(\omega)$ and it is called Fourier transform of $x\left(t\right)$.

In order to put (A.1.6) into a compact form, we must first note from (A.1.5) that $A(\omega)$ is an even function of $\omega$ and $B(\omega)$ is an odd function of $\omega$. The reason for this is that, if the sign of $\omega$ is changed in both formulas (A.1.5), $A(\omega)$ remains the same, but $B(\omega)$ changes its sign. It means that both $A(\omega)\cos\omega t$ and $B(\omega)\sin\omega t$ are even functions of $\omega$ and both remain the same when the sign of $\omega$ is changing. The Fourier integral equation (A.1.6) can be written in the following form:

$$x\left(t\right) = \int_{-\infty}^{\infty} A\left(\omega\right)\cos\omega t\, d\omega + \int_{-\infty}^{\infty} B\left(\omega\right)\sin\omega t\, d\omega. \tag{A.1.10}$$

where $-\infty < \omega < \infty$, and the factor 2 disappears. Thus, the idea of "negative" frequency has introduced, which is only a mathematical trick to simplify equations. Furthermore, since $A(\omega)$ is an even function and $\sin\omega t$ is an odd function of $\omega$, then

$A(\omega)\sin\omega t$ is an odd function. In the same way for term $B(\omega)\cos\omega t$

$$\begin{aligned}\int_{-\infty}^{\infty} A(\omega)\sin\omega t\, d\omega = 0\,;\\ \int_{-\infty}^{\infty} B(\omega)\cos\omega t\, d\omega = 0.\end{aligned} \tag{A.1.11}$$

Integrals (A.1.10) and (A.1.11) can be summated without making any difference to the value of $x(t)$. This can be written as follows:

$$\begin{aligned}x(t) = \int_{-\infty}^{\infty} A(\omega)\cos\omega t\, d\omega + \int_{-\infty}^{\infty} B(\omega)\sin\omega t\, d\omega + \\ i\int_{-\infty}^{\infty} A(\omega)\sin\omega t\, d\omega - i\int_{-\infty}^{\infty} B(\omega)\cos\omega t\, d\omega\,.\end{aligned}$$

Since both integrals in (A.1.11) are equal to zeros, then

$$\begin{aligned}x(t) = \int_{-\infty}^{\infty} \{A(\omega) - iB(\omega)\}\{\cos\omega t + i\sin\omega t\}\, d\omega = \\ \int_{-\infty}^{\infty} \hat{x}(\omega)e^{i\omega t}\, d\omega.\end{aligned} \tag{A.1.12}$$

Equations (A.1.9) and (A.1.12) represent Fourier transform and inverse Fourier transform.

A disadvantage of the Fourier transform is that frequency information can be only extracted for the complete duration of a signal (function) $x(t)$. Since the integral in the equation of the Fourier transform (A.1.9) extends all over time from $-\infty$ to $\infty$, the information it provides arises from the average over the whole length of the signal. If at some point of the lifetime of the function $x(t)$, there is a local oscillation representing a particular feature, this will contribute to the calculated Fourier transform $\hat{x}(\omega)$, but this oscillation on the time-axis will be lost. There is no way to know whether the

value of $\hat{x}(\omega)$ at a particular $\omega$ derives from frequencies present throughout the life of $x(t)$ or during just one or a few selected periods. This disadvantage overcomes in the wavelet analysis [1, 23, 24, 49], which provides an alternative way of breaking a signal down into its constituent parts.

# Appendix B

## B.1 Fourier method

A general difficulty in dealing with PDEs is that their solutions involve more than one independent variable. The method of the Fourier transform allows to reduce a PDE to one independent variable. We commonly try to transform the $x$-dependence through the Fourier transform, provided that the space domain is infinite, i.e. $-\infty < x < \infty$.

Consider a function $u(x,t)$ with $-\infty < x < \infty$, $t \geq 0$, and let

$$\hat{u}(\omega,t) = \int_{-\infty}^{\infty} u(x,t)e^{i\omega t}\,dx$$

be a FT of $u(x,t)$ with respect to $x$. The original function can be reconstructed from the inverse Fourier transform:

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(\omega,t)e^{-i\omega x}\,d\omega\,. \tag{B.1.1}$$

The differentiation of equation (B.1.1) with respect to $x$ and $t$ gives us the most oft-used derivatives:

$$u_x(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (-i\omega)\hat{u}(\omega,t)e^{-i\omega x}\,d\omega\,,$$

$$u_{xx}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (-i\omega)^2\,\hat{u}(\omega,t)e^{-i\omega x}\,d\omega\,,$$

$$u_t(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}_t(\omega,t)e^{-i\omega x}\,d\omega\,,$$

$$u_{tt}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}_{tt}(\omega,t)e^{-i\omega x}\,d\omega\,.$$

## B.2 Finite element and finite difference method

One of the first steps in using finite difference methods is to replace the continuous problem domain by a discrete mesh or a grid. Let $f(x)$ be a function with the single independent variable $x$, where $a \leq x \leq b$. The interval $[a,b]$ is discretized by considering the nodes $a = x_0 < x_1 < \ldots < x_N < x_{N+1} = b$, and we denote $f(x_i)$ by $f_i$. The mesh size is $x_{i+1} - x_i$. But for the simplicity we assume a constant mesh size

$$h = \frac{b-a}{N+1}$$

and

$$x_i = a + ih, \quad i = 0,1,2,\ldots,N+1\,.$$

In two dimensional case function $f(x,y)$ can be specified at a nodal point $(x_i, y_i)$ by $f_{ij}$. The spacing in the $x$ direction is $h_x$ and in the $y$ direction is $h_y$ [56].

The difference approximations for the derivatives can be expanded in Taylor series. The truncation error is the difference between the partial derivative and its finite difference representation.

## B.3 Galerkin and collocation methods

Let us show general ideas with the linear two-point boundary-value problem

$$u''(x) + q(x)u = f(x), \quad 0 \leq x \leq 1, \tag{B.3.1}$$

with

$$u(0) = 0, \quad u(1) = 0\,.$$

Suppose that we look for an approximate solution of (B.3.1) in the form of

$$u(x) \approx \sum_{j=0}^{n} c_j \phi_j(x), \tag{B.3.2}$$

where $\phi_j(x)$ are the basis functions, which satisfy the boundary conditions:

$$\phi_j(0) = \phi_j(1) = 0, \quad j = 0, \ldots, n\,. \tag{B.3.3}$$

If (B.3.3) holds true, then the approximate solution of $u$, given by (B.3.2), satisfies the boundary conditions. A classical example of a set of basis functions, that satisfies (B.3.3) is $\phi_j(x) = \sin j\pi x$, $j = 0, \ldots, n$.

The *collocation method* is a member of the family of methods known as *residual methods*. Let $x_1, x_2, \ldots, x_n$ be $n$ grid points on the interval [0, 1]. Then we can require that the approximate solution satisfies the differential equation in these $n$ points. Thus, for equation (B.3.1) and approximation (B.3.2) we require that

$$\frac{d^2}{dx^2}\left(\sum_{j=1}^{n} c_j \phi_j\,(x)\right)\Bigg|_{x_i} + q\,(x_i) \sum_{j=1}^{n} c_j \phi_j\,(x_i) = f\,(x_i)\,, \quad i = 1, \ldots, n \tag{B.3.4}$$

and we assume that the basis functions at least are twice differentiable. If we carry out the differentiation in (B.3.4) and collect unknown coefficients $c_j$, we obtain

$$\sum_{j=1}^{n} c_j [\phi_j''(x_i) + q(x_i)\phi_j(x_i)] = f(x_i)\,.$$

Thus, we received a system of $n$ linear equations with $n$ unknowns $c_1, c_2 \ldots, c_n$ [31].

The *Galerkin method* is based on the concept of orthogonality of functions. Recall that two vectors $\vec{f}$ and $\vec{g}$ are orthogonal if the inner product satisfies

$$\left(\vec{f}, \vec{g}\right) = \sum_{j=0}^{n} f_j g_j = 0\,.$$

Now suppose that the components of vectors $\vec{f}$ and $\vec{g}$ are the values of two functions $f$ and $g$ at $n$ equally spaced grid points within the interval $[0, 1]$

$$\vec{f} = (f(h),\, f(2h), \ldots, f(nh))\,,$$

where $h = (n+1)^{-1}$ is the grid-point spacing, and similarly for $\vec{g}$. Then the orthogonality relation for discrete $f$ and $g$ is

$$\sum_{j=0}^{n} f(jh)g(jh) = 0\,,$$

and this relation is unchanged if we multiply it by $h \to 0$. Assume, that the functions $f$ and $g$ are integrable, then

$$\int_0^1 f(x)\, g(x)dx = 0\,. \tag{B.3.5}$$

With this motivation we define two functions $f$ and $g$ to be orthogonal on the interval $[0, 1]$ if the relation (B.3.5) holds.

Let the residual function for $u(x)$ be defined by

$$r(x) = u''(x) + q(x)u(x) - f(x), \quad 0 \le x \le 1\,. \tag{B.3.6}$$

If $u(x)$ was exact solution of equation (B.3.1), then the residual function would be identically zero. The Galerkin criterion is to choose $u(x)$ in such a way that, its residual function (B.3.6) must be orthogonal to all the basis functions, i.e. $\phi_1, \phi_2, \ldots, \phi_n$:

$$\int_0^1 r(x)\phi_i(x)\, dx, \quad i = 0, \ldots, n\,. \tag{B.3.7}$$

By substitution (B.3.2) into (B.3.7), we obtain a system of equations, which depends on only the unknown coefficients $\{c_j\}$ [31].

If the number of equations is large and matrices, which apprear in this method are dense, then there appears the main difficulty of this method, which consists in high computational costs.

# Appendix C

## C.1 Integral equations

Integral equations are encountered in various fields of science and numerous applications (in elasticity, plasticity, heat and mass transfer, oscillation theory, fluid dynamics, filtration theory, electrostatics, electrodynamics, biomechanics, game theory, control, queuing theory, electrical engineering, economics, medicine, etc.) [67].

There is a number of classifications of integral equations, which distinguish different kinds of equations. The following classification is the most frequently used

$$\int_a^b K(x,t)\phi(t)dt = f(x)\,; \tag{C.1.1}$$

$$\phi(x) - \lambda \int_a^b K(x,t)\phi(t)dt = f(x)\,; \tag{C.1.2}$$

$$a(x)\phi(x) - \lambda \int_a^b K(x,t)\phi(t)dt = f(x)\,. \tag{C.1.3}$$

The above equations (C.1.1) – (C.1.3) are generally known as Fredholm equations of the first, second, and third kind, respectively. The interval $(a, b)$ may in general be a finite interval, semi-infinite or infinite. If $a(x)$ does not vanish, one can divide (C.1.3) by $a(x)$ and reduce it to (C.1.2). The functions $f(x)$, $a(x)$ and $K(x,t)$ are presumably known functions and the function $\phi(x)$ is unknown. The parameter $\lambda$

could be absorbed by function $K(x,t)$, but more convenient to retain it in the equation, since its role becomes clearer in the study of integral operators. The function $K(x,t)$ is generally known as the *kernel* of the equation (kernel function).

**Remark 1.** The variables $x$ and $t$ may vary in different ranges (e.g., $a \leq t \leq b$ and $c \leq x \leq d$).

**Remark 2.** In general, the case where the limits of integration $a$ and/or $b$ can be infinite is not excluded; however, in this case, the validity of the condition that the kernel $K(x,t)$ is square integrable on the square $S = \{a \leq x \leq b, a \leq t \leq b\}$ is especially significant.

The second class of integral equations is the Volterra equations of the first, second, and third kind, namely

$$\int_a^x K(x,t)\phi(t)dt = f(x)\,; \tag{C.1.4}$$

$$\phi(x) - \lambda \int_a^x K(x,t)\phi(t)dt = f(x)\,; \tag{C.1.5}$$

$$a(x)\phi(x) - \lambda \int_a^x K(x,t)\phi(t)dt = f(x)\,. \tag{C.1.6}$$

One can study these equations as a special case of the Fredholm type IEs. These equations reduce to the corresponding Volterra equations if $K(x,t) = 0$ for $t > x$. Nevertheless, the Volterra equations have many interesting properties which do not emerge from the general theory of the Fredholm equations [35].

## C.2 Quadrature method

In dealing with integral equations, the reduction to solution of systems of algebraic equations, obtained by replacing the integrals with finite sums, is one of the most effective tools. The method of quadratures is related to approximate methods. It is a widespread in practice because it is rather universal with respect to the principle of

constructing algorithms for solving both linear and nonlinear equations. The method is based on the quadrature formula

$$\int_a^b \varphi(x)dx = \sum_{j=1}^{n} A_j \varphi(x_j) + \varepsilon_n[\varphi]\,, \tag{C.2.1}$$

where $\{x_j\}$ are the nodes of the quadrature formula, coefficients $\{A_j\}$ are independent from $\varphi(x)$, and $\varepsilon_n[\varphi]$ is the error of the replacement of integration by summation.

Let us consider the Fredholm integral equation of the second kind

$$y(x) - \lambda \int_a^b K(x,t)y(t)dt = f(x)\,, \quad a \le x \le b \tag{C.2.2}$$

where we assume that $x = x_j\,(i = 1, \ldots, n)$, then we obtain the following relation:

$$y(x_i) - \lambda \int_a^b K(x_i,t)y(t)dt = f(x_i)\,, \quad i = 1, \ldots, n \tag{C.2.3}$$

which represents the basic formula for the method of quadratures. The application of formula (C.2.1) to integral (C.2.3) gives us the following system of equations:

$$y(x_i) - \lambda \sum_{j=1}^{n} A_j K(x_i, x_j) y(x_j) = f(x_i + \lambda \varepsilon_n[y])\,.$$

By neglecting infinitely small term $\lambda\varepsilon_n[y]$ in the latter formula, we obtain a system of linear algebraic equations for approximate values of $y_i$, which is the solution $y(x)$ in the nodes $x_i$

$$y_i - \lambda \sum_{j=1}^{n} A_j K_{ij} y_j = f_i\,, \quad i = 1, \ldots, n \tag{C.2.4}$$

where $K_{i,j} = K(x_i, x_j)$, $f_i = f(x_i)$. The solution of system (C.2.4) gives the values $y_1, \ldots, y_n$, which determine approximate solution of integral equation (C.2.2) in the interval $[a; b]$. Here, for the approximate solution we can take the function obtained by linear interpolation, i.e. the function which coincides with $y_i$ in nodes $x_i$ and

linearly dependent within intervals $[x_i, x_{i+1}]$. Moreover, for the analytic expression of the approximate solution, we can write:

$$\widetilde{y}(x) = f(x) + \lambda \sum_{j=1}^{n} A_j K(x, x_j) y_j \,.$$

## C.3 Degenerate kernel approximation

For approximate solution of the Fredholm integral equation of the second kind

$$y(x) - \int_a^b K(x,t) y(t) dt = f(x)\,, \quad a \le x \le b$$

we assume that functions $f(x)$ and $K(x,t)$ are continious. In this case, the kernel $K(x,t)$ can be replaced by a degenerate one, i.e.

$$K_{(n)}(x,t) = \sum_{k=0}^{n} g_k(x) h_k(t)\,.$$

Let us show several ways on how to perform such replacement. If the kernel is a differentiable function with respect to $x$ on the interval $[a;b]$ sufficientely many times, then for a degenerate kernel $K_{(n)}(x,t)$ we can apply a finite segment of the Taylor series:

$$K_{(n)}(x,t) = \sum_{m=0}^{n} \frac{(x-x_0)^m}{m!} K_x^{(m)}(x_0,t)\,,$$

where $x_0 \in [a;b]$. A similar trick can be used for the case if $K(x,t)$ is differentiable sufficientely many times with respect to $t$ on the interval $[a;b]$.

To construct a degenerate kernel, a finite segment method of the double Fourier series can be used [67]:

$$K_{(n)}(x,t) = \sum_{p=0}^{n} \sum_{q=0}^{n} a_{pq} (x-x_0)^p (t-t_0)^q\,,$$

where

$$a_{pq} = \frac{1}{p!q!} \frac{\partial^{p+q}}{\partial x^p \partial t^q} K(x,t) \bigg|_{x=x_0, t=t_0} \qquad a \le x_0 \le b, \quad a \le t_0 \le b\,.$$

A continious kernel $K(x,t)$ admits approximation by a triginometric polynomial of period $2l$, where $l = b - a$. For instance, we can set

$$K_{(n)}(x,t) = \frac{a_0(t)}{2} + \sum_{k=1}^{n} a_k(t) \cos\left(\frac{\pi k x}{l}\right) ,$$

where $a_k(t)$ are the Fourier coefficients

$$a_k(t) = \frac{2}{l} \int_a^b K(x,t) \cos\left(\frac{\pi k x}{l}\right) dx \, .$$

A similar decomposition can be obtained by interchanging variables $x$ and $t$. One can use other methods of intepolation and approximation of the kernel [67].

## C.4 Method of least squares

Let,

$$\varepsilon[y(x)] \equiv y(x) - \lambda \int_a^b K(x,t) y(t) dt - f(x) = 0 \, . \tag{C.4.1}$$

Suppose that we have the following basis

$$Y_n(x) = \varphi_0(x) + \sum_{i=1}^{n} A_i \varphi_i(x) \, , \tag{C.4.2}$$

where $\{\varphi_i(x)\}$ are given linear independent functions, and $A_1, \ldots, A_n$ are the coefficients. The substitution of (C.4.2) into the left-hand side of (C.4.1), yields us the residual term

$$\varepsilon[Y_n(x)] = \psi_0(x,\lambda) + \sum_{i=1}^{n} A_i \psi_i(x,\lambda) \, ,$$

where $\psi_0(x,\lambda)$ and $\psi_i(x,\lambda)$ are defined by

$$\psi_0(x,\lambda) = \varphi_0(x) - f(x) - \lambda \int_a^b K(x,t) \varphi_0(t) dt \, ;$$

$$\psi_0(x,\lambda) = \varphi_i(x) - f(x) - \lambda \int_a^b K(x,t) \varphi_i(t) dt \, , \quad i = 1, \ldots, n \, .$$

According to the method of least squares, coefficients $\{A_i\}$ can be found from the condition of the minimum of the following integral

$$I = \int_a^b \{\varepsilon[Y_n(x)]\}^2 dx = \int_a^b \left[\psi_0(x,\lambda) + \sum_{i=1}^{n} A_i \psi_i(x,\lambda)\right]^2 dx\,.$$

This requirement leads to a system of algebraic equations

$$\frac{\partial I}{\partial A_j} = 0\,, \quad j = 1, \ldots, n\,.$$

The method of least squares can also be applied for approximate construction of characteristic values and eigenfunctions of the kernel $K(x,t)$.

# Bibliography


[1] N.M.Astafieva. Wavelet analysis: basic theory and some applications, *Physics-Uspekhi* 166, 11 (1996) 1145 – 1170.

[2] P.Auscher. Ondelettes à support compact et conditions aux limites, *J. Funct. Anal.* 11 (1993) 29 – 43.

[3] E.Bacry, S.Mallat, G.Papanicolaou. A wavelet based space-time adaptive numerical method for partial differential equations. Technical report, Courant Institute of Mathematical Sciences, New York, 1990.

[4] V.Barrera-Figueroa, A.Lucas-Bravo, J.López-Bonilla, *Annales Mathematicae et Informaticae* 34 (2007) 17 – 28.

[5] S.Bertoluzza, G.Naldi. A wavelet collocation method for the numerical solution of partial differential equations. *Appl. Comput. Harmon. Anal.* 3 (1996) 1 - 9.

[6] G.Beylkin, R.Coifman, V.Rokhlin. Fast wavelet transforms and numerical algorithms I, *Commun. Pure Appl. Math.* 44 (1991) 141 - 183.

[7] G.Beylkin. On wavelet-based algorithms for solving differential equations, Wavelets - Mathematics and Applications, Chapter 12, pp. 449 - 466. CRC Press, 1994.

[8] A.Boggess, F.J.Narkowich. A First Course in Wavelets with Fourier Analysis, Prentice Hall, 293 p.

[9] Ed Bullmore *et al.* Wavelets and functional magnetic resonance imaging of the human brain, *Neuroimage* 23 (2004) S234 - S249.

[10] C.Cattani. Multiscale Analysis of wave propagation in composite materials, *Mathematical Modelling and Analysis* 8(4) (2003) 267 – 282.

[11] C.Cattani. Harmonic Wavelets towards the Solution of Nonlinear PDE, *Comput. Math. Appl.* 50 (2005) 1191 – 1210.

[12] C.Cattani. Connection coefficients of Shannon wavelets, *Mathematical Modelling and Analysis* 11(2) (2006) 1 – 16.

[13] C.Cattani. Shannon Wavelets Theory, *Math. Probl. Eng.* 2008 Article ID 164808, (2008).

[14] C.Cattani. Wavelet analysis of the localized Weierstrass function, *Vestnik Chelyabinskogo Gosudarstvennogo Universiteta* 8 (148) (2009) 25 – 27. ISSN 1994-2796.

[15] C.Cattani. Harmonic wavelet approximation of random, fractal and high frequency signals, *Telecommun. Syst.* 43 (2010) 207 - 217.

[16] C.Cattani, J.Rushchitsky. Wavelet and wave analysis as applied to materials with micro or nanostructure, World Scientific Publishing Company, 2007, 476 p.

[17] P.Charton, V.Perrier. A Pseudo-Wavelet Scheme for the Two-Dimensional Navier-Stokes equations, *Matematica Aplicada e Computacional* 15(2) (1996) 139 - 160.

[18] Z.Chen, C.A.Micchelli, Y.Xu. A construction of interpolating wavelets on invariant sets, Mathematics of Computation 68 (1999) 1569 – 1587

[19] A.Cichon, S.Borucki, T.Boczar, M.Lorenc. Results of the wavelet analysis of the acoustic emission signals generated by partial discharges in insulation oil of

various temperature and flow speed, *Eur. Phys. J. Spec. Top.* 154 (2008) 31 - 38.

[20] X.Claeys. On the theoretical justification of Pocklington's equation, *Math. Models Methods Appl. Sci.*, 19 (2009) 1325 – 1355.

[21] W.Dahmen, R.Schneider, Y.Xu. Nonlinear functionals of wavelet expansions – adaptive reconstruction and fast evaluation, *Numer. Math.* 86 (2000) 49 – 101.

[22] H.Danfu, S.Xufeng. Numerical solution of integro-differential equations by using CAS wavelet operational matrix of integration, *Appl. Math. Comput.* 194 (2007) 460 - 466.

[23] I.Daubechies. Ten Lectures on Wavelets, SIAM, Philadelphia, 1992, 377 p.

[24] I.M.Dremin, O.V.Ivanov, V.A.Nechitailo. Wavelets and their applications, *Physics-Uspekhi* 171, 5 (2001) 465 – 501.

[25] B.Engquist, S.Osher, S.Zhong. Fast wavelet based algorithms for linear evolution equations, *SIAM J. Sci. Comput.* 15(4) (1994) 755 - 775.

[26] G.Espinosa-Paredes, A.Nunez-Carrera, A.Prieto-Guerrero, M.Cecenas. Wavelet approach for analysis of neutronic power using data of Ringhals stability benchmark, *Nucl. Eng. Des/Fusion* 237 (2007) 1009 - 1015.

[27] W.Fang, Yi.Wang, Y.Xu. An implementation of fast wavelet galerkin methods for integral equations of the second kind, *J. Sci. Comput.* 20(2) (2004) 277 – 302.

[28] J.Frölich, K.Schneider. An adaptive wavelet-vaguelette algorithm for the solution of PDEs. *J. Comput. Phys.* 130 (1997) 174 - 190.

[29] M.Gallegati. Wavelet analysis of stock returns and aggregate economic activity, *Comput. Stat. Data Anal.* 52 (2008) 3061 - 3074.

[30] S.Goedecker, O.Ivanov. Solution of multi-scale partial differential equations using wavelets, *Computers in Physics* 12, Issue 4 (1998) 548 – 555.

[31] G.H.Golub, J.M.Ortega. Scientific Computing and Differential Equations. An Introduction to Numerical Methods, Academic Press, San Diego – New York – Boston, 1992, 337 p.

[32] M.Gulsu, T.Ozis. Numerical solution of Burgers' equation with restrictive Taylor approximation, *Appl. Math. Comput.* 171 (2005) 1192 – 1200.

[33] S.K.Haigh, B.Teymur, S.P.G.Madabhushi, D.E.Newland. Applications of wavelet analysis to the investigation of the dynamic behaviour of geotechnical structures, *Soil Dynam. Earthquake Eng.* 22 (2002) 995 - 1005.

[34] H.Heinrich, G.H.Molla, H.Dickhausb, V.Kolevc, J.Yordanovac, A.Rothenberger. Time-on-task analysis using wavelet networks in an event-related potential study on attention-defcit hyperactivity disorder, *Clin. Neurophysiol.* 112 (2001) 1280 – 1287.

[35] H.Hochstadt. Integral Equations, New York, 1978, 282p.

[36] M.Holmström. Wavelet based methods for time dependent PDE, Ph.D. thesis, Uppsala University, Sweden, 1997.

[37] D.Huybrechs. Multiscale and hybrid methods for the solution of oscillatory integral equations, Ph.D. thesis, Katholieke Universiteit Leuven, Belgium, 2006.

[38] L.Jameson. On the wavelet-optimized finite difference method. Technical Report NASA CR-191601, ICASE Report No. 94-9, 1994.

[39] G.Kaiser. Physical wavelets and radar. *IEEE Antennas and Propagation Magazine*, Feb. 1996.

[40] J.M.Keiser. Wavelet Based Approach to Numerical Solution of Nonlinear Partial Differential Equations. Ph.D. thesis, University of Colorado, 1995.

[41] P.K.Kundu, I.M.Cohen. Fluid Mechanics, Second Edition. San Diego, Academic Press, 2002, 730 p.

[42] U.Lepik, E.Tamme. Application of the Haar Wavelets for Solution of Linear Integral Equations, Dynamical Systems and Applications, Proceedings, 510 July 2004, Antalya, Turkey, pp. 494 – 507.

[43] U.Lepik. Numerical solution of evolution equations by the Haar wavelet method, *Appl. Math. Comput.* 185 (2007) 696 – 704.

[44] U.Lepik. Solving fractional integral equations by the Haar wavelet method, *Appl. Math. Comput.* 214 (2009) 468 - 478.

[45] E.B.Lin, X.Zhou. Connection coefficients on an interval and wavelet solutions of Burgers equation, *J. Comput. Appl. Math.* 135 (2001) 63 – 78.

[46] B.Liu. Adaptive harmonic wavelet transform with applications in vibration analysis, *J. Sound Vib.* 262 (2003) 45 - 64.

[47] Y.Liu, I.T.Cameron, F.Y.Wang. The wavelet-collocation method for transient problems with steep gradients, *Chem. Eng. Sci.* 55 (2000) 1729 – 1734.

[48] Y.Mahmoudi. Wavelet Galerkin method for numerical solution of nonlinear integral equation. *Appl. Math. Comput.* 167 (2005) 1119 – 1129.

[49] S.Mallat. A Wavelet Tour of Signal Processing, École Polytechnique, Paris, Second Edition, Academic Press, 1990.

[50] C.A.Micchelli, Y.Xu. Reconstruction and Decomposition Algorithms for Biorthogonal Multiwavelets, *Multidim. Syst. Sign. P.* 8 (1997) 31 – 69.

[51] C.A.Micchelli, Y.Xu, Y.Zhao. Wavelet Galerkin methods for second-kind integral equations, J. Comput. Appl. Math. 86, (1997) 251 – 270.

[52] T.Morita. Expansion in Harmonic Wavelets of a Periodic Function, *Interdiscipl. Inform. Sci.* 3(1) (1997) 5 – 12.

[53] S.V.Muniandy, I.M.Moroz. Galerkin modeling of the Burgers equation using harmonic wavelets, *Phys. Lett. A* 235 (1997) 352 – 356.

[54] H.Mouri, H.Kubotani. Real-valued harmonic wavelets, *Phys. Lett. A* 201 (1995) 53 – 60.

[55] I.Mustafa. On numerical solutions of one-dimensional nonlinear Burgers' equation and convergence of the decomposition method, *Appl. Math. Comput.* 170 (2005) 76 – 85.

[56] B.Neta. Numerical solution of partial differential equations. Lecture Notes, USA (CA), 2003.

[57] D.E.Newland. Harmonic wavelet analysis, *Proc. R. Soc. Lond. A Math. Phys. Sci.* 443 (1993) 203 – 225.

[58] D.E.Newland. An Introduction to Random Vibrations, Spectral & Wavelet Analysis, Third edition, 1993.

[59] D.E.Newland. Harmonic and Musical Wavelets, *Proc. R. Soc. Lond. A Math. Phys. Sci.* 444 (1994) 605 – 620.

[60] D.E.Newland. Practical Signal Analysis: Do Wavelets Make any Difference? Proc. of 1997 Design Engineering Technical Conferences, September 14 – 17, 1997, Sacramento, California.

[61] D.E.Newland. Harmonic wavelets in vibrations and acoustics, *Phil. Trans. R. Soc. Lond. A* 357 (1999) 2607 – 2625.

[62] T.Ohwada. Cole-Hopf Transformation as numerical tool for the Burgers equation, *Appl. Comput. Math.* 8 (2009) 107 – 113.

[63] V.Perrier, C.Basdevant. Travelling wavelets method. In Proc. of the US-French "Wavelets and Turbulence" Workshop. Princeton, 1991.

[64] M.Pinsky. Introduction to Fourier analysis and wavelets, American Mathematical Society (2009) 376 p.

[65] G.Plonka, M.Tasche. On the Computation of Periodic Spline wavelets, *Appl. Comput. Harmon. Anal.* 2 (1995) 1 – 14.

[66] A.D.Polyanin, V.F.Zaytsev, A.I.Zhurov. Methods for Solving Nonlinear Equations of Mathematical Physics and Mechanics, (*In Russian*), Moscow, FIZMATLIT (2005) 256 p.

[67] A.D.Polyanin, A.V.Manzhirov. Handbook of integral equations, CRC Press, London-New York-Washington D.C. (1998) 796 p.

[68] T.Regińska. Solving the sideways heat equation by a waveletGalerkin method, *Inverse Probl.* 13 (1997) 1093 - 1106.

[69] B.P.Rynne. On the well-posedness of Pocklington's equation for a straight wire antenna and convergence of numerical solution, *Journal of Electromagnetic Waves and Applications* 14 (2000) 1489 – 1503.

[70] M.Skopina. Wavelet Approximation of Periodic Functions, *J. Approx. Theory* 104 (2000) 302 – 329.

[71] H.Szu, B.Tefler, J.Garcia. Wavelet Transforms and Neural Networks for Compression and Recognition, *Neural Netw.* 9 (1996) 695 – 708.

[72] C.Torrence, G.P.Compo. A Practical Guide to Wavelet Analysis, *Bulletin of the American Meteorological Society* 79 (1998) 61 – 78.

[73] M.Vetterli, J. Kovacević. Wavelets and Subband Coding, Prentice Hall PTR, Englewood Cliffs, New Jersey, 2007.

[74] D.F.Walnut. An Introduction to Wavelet Analysis, Boston – Basel – Berlin, Birkhauser, 2002.

[75] A.M.Wazwaz. The tanh-coth method for solitons and kink solutions for nonlinear parabolic equations, *Appl. Math. Comput.* 188 (2007) 1467 – 1475.

[76] J.Waldén. Wavelet Solvers for Hyperbolic PDEs. Ph.D. thesis, Department of Scientific Computing, Uppsala University, 1996.

[77] D.Wen-Shan, Y.Hong-Juan, Shi Yu-Ren. An exact solution of Fisher equation and its stability, *Chinese Physics* 15(7) (2006) 1414 – 1417.

[78] P.Wojtaszczyk. A Mathematical introduction to Wavelets, London Mathematical Society Student Texts, 1997, 276 p.

[79] J.Xiao, L.Wen, D.Zhang. Solving second kind Fredholm integral equations by periodic wavelet Galerkin method, *Appl. Math. Comput.* 175 (2006) 508 - 518.

[80] J.C.Xu, W.C.Shann. Galerkin-wavelet methods for two point boundary value problems, *Numer. Math.* 63 (1992) 123 - 144.

[81] S.Yousefi, A.Banifatemi. Numerical solution of Fredholm integral equations by using CAS wavelets, *Appl. Math. Comput.* 183 (2006) 458 - 463.

[82] A.A.Kudreyko. Wavelet approach towards the Burgers problems using the Daubechie's wavelets: Proceedings of All-Russian conference of young physicists, Taganrog (2007) p. 46.

[83] A.A.Kudreyko, C.Cattani, I.I.Latypov. Wavelet approach towards heat propagation in the infinite body, *Proceedings of All-Russian conference in computer technology and numerical methods*, Birsk (2007) 75 – 78.

[84] C.Cattani, A.A.Kudreyko. Wavelet approach towards the Korteweg-de Vries Problems, *Proceedings of All-Russian conference of young physicists and young researchers*, Ufa (2008) p. 43.

[85] C.Cattani, A.A.Kudreyko. Multiscale Analysis of the Fisher Equation, *ICCSA 2008*, Part I, LNCS 5072 (2008) 1171 – 1180. Springer-Verlag Berlin Heidelberg.

[86] C.Cattani, A.A.Kudreyko. On the Discrete Harmonic Wavelet Transform, *Math. Probl. Eng.* 2008, Article ID 687318 (2008).

[87] C.Cattani, A.A.Kudreyko. Application of harmonic wavelets towards solution of the Fredholm equations, *Proceedings of All-Russian School-Conference for PhD students and young researchers*, Ufa (2008) 143 – 146.

[88] A.A.Kudreyko. On the wavelet-approach towards solution of the Fredholm integral equations of the second kind, *Scientific Notes of the Ufa State Pedagogical University* 10 (2009) 62 – 66.

[89] C.Cattani, A.A.Kudreyko. Harmonic wavelet solution of the Korteweg-de Vries Equation, *Vestnik Chelyabinskogo Gosudarstvennogo Universiteta* 8(146) (2009) 28 – 31. ISSN 1994-2796.

[90] A.A.Kudreyko. Harmonic wavelets in solution of PDEs, *Vestnik Pomorskogo Gosudarstvennogo Universiteta* 1, (2010) 88 – 92. ISSN 1728-7340.

[91] C.Cattani, A.A.Kudreyko. Application of periodized harmonic wavelets towards solution of eigenvalue problems for integral equation, *Math. Probl. Eng.* 2010 Article ID 570136 (2010).

[92] C.Cattani, A.A.Kudreyko. Harmonic wavelet method towards solution of the Fredholm type integral equations of the second kind. *Appl. Math. Comput.* 215 (2010) 4164 - 4171.

[93] C.Cattani, A.Kudreyko. Application of wavelet-basis for solution of the Fredholm type integral equations, *ICCSA 2010*, Part II, LNCS 6017 (2010) 155 - 163. Springer-Verlag Berlin Heidelberg 2010.

[94] A.Kudreyko. Application of circular harmonic wavelets towards solution of the Fredholm type integral equations. *Proc. of the International Conference "DAYS on DIFFRACTION 2009"* Saint-Petersburg (2009) 122 - 128.

# Bibliography