Geometries associated with von Dyck groups

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Contents

Introduction

This thesis contains a theoretical result in the field of group actions on combinatorial structures, which is an important area of research, bringing forth some thirty publications per year. Besides its applications, mostly found in cryptography and image–recognition, such a discipline casts interesting bridges between the abstract theory of groups and more "visible" objects, like the graphs. On the theoretical side, in the last three decades, the deepest contributions have been given, among others, by M. D. E. Conder and his collaborators (see, e.g., [14] and references therein).

But what is the advantage of associating a geometric structure to a given group G? If G is of a "tamed" kind, i.e., it is either finite, Abelian, solvable, or equipped with some additional structure like, e.g., that of a Lie group, then there are no advantages at all. In this thesis I will rather consider a class of groups, which is rather "wild", in the sense that it comprises infinite, discrete and solvable groups and it was allegedly introduced by W. von Dyck at the turn of the eighteenth century [57], and I will study some natural combinatorial geometric structures they act upon. My purpose is to show that even extremely elementary structures can play a nontrivial role in the understanding of such groups.

I would like to stress, from the very beginning, that all the theoretical reasonings in this thesis are carried out in terms of abstract groups. By an "abstract group" I mean a group without any additional structure whatsoever which is not, a priori, realised as the transformation groups of some other mathematical entity. Such kind of groups are classically introduced through a *presentation*. In symbols, a presentation looks like $G = \langle S \mid R\rangle$, and it means that G can be obtained from the *free group* over the set S, i.e., the "largest"—so to speak—group admitting S as a set of generators, by factoring out the elements belonging to the normal subgroup generated by R.

In spite of its rather formal and abstract flavour, the notion of a presentation is heavy with geometrical implications. At the end of the eighteen century, A. Cayley introduced a combinatorial geometric structure, which nowadays is known to be a *coloured graph*, associated, in a functorial way, to a given presentation $G = \langle S \mid R\rangle$ of the group G. Such a graph, which I will denote by $\Gamma(G, S)$, goes under the name of *Cayley graph*, and it has the remarkable property of being *regular* (or *homogeneous*) with respect to G, i.e., the original group G act freely and transitively on it. It is a true marvel that such a rudimentary construction can be put at the foundations of some straightforward yet far–reaching observations:

- the so–called *word problem* (a still open problem) for a group $G = \langle S \mid R\rangle$ is equivalent to the constructability (i.e., the possibility of defining it through a recursive function) of $\Gamma(G, S)$ [35];
- if $\Gamma(G, S)$ can be embedded into a *surface*, then it makes sense to attach to the group G a typical geometric property, namely, that of the *genus* [58];
- the geometric properties of the surface (e.g., its *compactness*) $\Gamma(G, S)$ is embedded into may be used to check the *finiteness* of G [56, 20].

In this thesis, I basically discovered a link—a *duality*, to be more precise—between such a classical construction as the Cayley graph and another interesting (though perhaps less known) way of linking a graph to a group, which is the *incidence graph* [36] associated with the so–called *coset geometry*.

Its main result stems from a subtle yet unfairly forgotten theorem, formulated by G. Sabidussi in 1958, establishing that $\Gamma(G, S)$ is the unique, up to isomorphisms, edge–coloured graph on which the original group G acts vertex–transitively [46]. So, since the incidence graph of the coset geometry of G carries a natural edge–transitive G–action, and it is naturally vertex–coloured, I was led to suspect that the incidence graph of the coset geometry is, in fact, the same thing as the Cayley graph, provided that—roughly speaking—"vertices are replaced with edges". This thesis contains a rigorous proof of such a result, complemented by all the necessary preliminaries, and some (envisaged) applications and perspectives.

The role played herewith by the von Dyck group $D(n, n, n)$ is that of a tool to better explain such a vertex–to–edge duality. Indeed, for $n > 3$ the von Dyck groups $D(n, n, n)$ belong to a class of (infinite, discrete and not solvable) groups known as *Fuchsian groups*, which are the discrete and finitely generated subgroup of the three–dimensional Lie group $\mathrm{PSL}\,(2, \mathbb{R})$ of the isometries of the hyperbolic plane $\mathbb{H}$. The Fuchsian groups naturally act on a tangible geometric entity such as $\mathbb{H}$. Moreover, $D(n, n, n)$ is generated by the orientation–preserving transformations of a regular–triangular *tessellation* $\mathcal{T}_n$, which, once again, is a rather elementary structure of combinatorial character, being essentially a triangulation by regular triangles.

The vertex–to–edge duality, which is a quite general result, admits a nice transparent geometric formulation: in the case of von Dyck groups $D(n, n, n)$ both the Cayley graph and the incidence graph of the coset geometry of $D(n, n, n)$ are inscribed into $\mathcal{T}_n$. More precisely, the latter is the 1–skeleton of $\mathcal{T}_n$, while the former is the 1–skeleton of the so–called *derived tessellation* of $\mathcal{T}_n$. In other words, the groups of von Dyck allow to visualise the vertex–to–edge duality in terms of the most elementary geometric shapes: triangles on a surface.

Let me describe some of the applications and perspectives of the vertex–to–edge duality.

First of all, it makes it evident that the Cayley graph of the von Dyck group is a planar one, which, to my best knowledge, has never been observed before, though a very similar result can be found in [56].

Then, the vertex–to–edge duality immediately allows to recast, in a transparent geometric way, a result proved in 1983 by T.W. Tucker [55], concerning the genus of the factors of $D(n, n, n)$. I must mention that among the factors of $D(n, n, n)$ there are the famous free Burnisde groups with two generators $B(2, n)$, whose importance also pushed me to look for a recursive way to enumerate the elements of $D(n, n, n)$.

This is probably the most important consequence of the vertex–to–edge duality I have managed to discover so far: it allows to recursively enumerate the edges (called *cliques*, as in the theory of coloured graphs) of the incidence graph of the coset geometry and, hence, the elements of $D(n, n, n)$. I dubbed this procedure "cliques enumeration algorithm", and there are indications that it may be useful for attacking the celebrated Burnside problem in the still unsolved cases with two generators. Indeed, in view of the natural surjection between $D(n, n, n)$ and $B(2, n)$, which makes it possible to associate to the latter some of the geometric features of the former, the finiteness of $B(2, n)$ can be a consequence of the finiteness of a suitable "quotient tessellation" of $\mathcal{T}_n$, with respect to the kernel of the projection $D(n,n,n) \to B(2,n)$.

Incidentally, this solves the word problem for the Burnisde groups with two generators.

I must stress that, in one form or another, any existing algorithm to check the finiteness of $B(2,n)$ implements a mechanism to enumerate words, and, due to the presence of relations, the lexicographic way is not necessarily the cheapest one (see [28] and references therein). On the other hand, the cliques enumeration algorithm I proposed allows to "avoid" relations and to produce exactly once each element of the group, so that it may be computationally more advantageous. I have written the cliques enumeration algorithm by using the Wolfram Mathematica™computer algebra software, but, due to the unavoidable computational complexity, I have managed to test it only for $n \leq 4$. Nevertheless, it seems that, especially in comparison with the existing techniques, the cliques enumeration algorithm has the "aesthetic" merit of providing an unified approach to the problem of the finiteness of $B(2,n)$, for arbitrary n, in sharp contrast with the methods applied so far to each particular situation.

Structure of the thesis

This thesis is pivoted on the central Chapter 7, which contains the main result on the vertex–to–edge duality. The previous chapters merely pave the way for it.

Chapter 1 begins with a minimal refreshment of group–theoretic notions, and then introduces the special classes of groups considered in this thesis, namely the triangle groups, the von Dyck groups, and the Burnside groups. Chapter 2 has a similar structure, but it deals with graph theory, by focusing on such less known (though classical) notions as coloured graphs, duality, and morphisms. It also introduces the first main gadget of this thesis: the Cayley graph. The second one, the coset geometry, is introduced in Chapter 3, which also gives some perspective on combinatorial geometric structures. This marks the end of the survey on the discrete structures I will need.

Continuous structures, namely constant–curvature surfaces, are dealt with in Chapter 4, with a special emphasis on the hyperbolic cases, which are the most interesting ones, since they allow to formulate the nontrivial and still open problems. All the models of the hyperbolic plane are reviewed, as well as the Möbis transformations and the Fuchsian groups.

The links between combinatorial and continuous structures begin to revel themselves in Chapter 5, where tessellations are introduced, with a due stress on such less known notions as the coloured graph associated to a tessellation and the dual and derived tessellations. Finally, Chapter 6 frames the theorem of G. Sabidussi in the context of groups actions on geometric structures, which is the last and most important theoretical tool I need to formulate the main result.

Some of the corollaries of the vertex–to–edge duality are listed in the last Chapter 8, where the clique enumeration algorithm is explained, and its Wolfram Mathematica™ implementation is appended. More perspective can be found in the Appendix 9, which contains also some side topics.

Acknowledgements

I am grateful to Prof. A. Vinogradov, for introducing me some of the topics discussed in this work, and to my advisor Prof G. Sparano, who assisted me in academic research.

I would also like to thank P. Longobardi and C. Sica for sharing their expertise in group theory. I am thankful to E. Ferrara Dentice for relevant suggestions and to the anonymous referee of the paper [37].

I also appreciated the usefulness of the MathOverflow forum, through which experts like M. Sapir, M. Newman and H. Wilton let me know the state–of–the–art of the computational approaches to the Burnside problem (http://mathoverflow.net/questions/119082/can-the-finiteness-of-a-burnside-group-with-two-generators-be-checked-algorithmi).

I am much obliged to my co–author G. Moreno, for his help and support during the completion of the Ph. program. My thanks go also to the University of Salerno, University of Opava, and to my friends and colleagues for creating a stimulating and nice atmosphere for my research.

Last, I wish to thank my family, who always believed in me.

CHAPTER 1

Introduction to finitely generated groups

1.1. Preliminaries to group theory

In this chapter I will review some general notions of group theory and then I will focus on the groups of special interest for this thesis, namely the groups with two generators and, in particular, the triangle groups, the von Dyck groups and the Burnside groups. For a more complete reference on group theory, I suggest D. Robinson's book [44]. More information on groups of von Dyck can be found in [57, 34]. Finally, concerning the Burnside problem, look [2, 30].

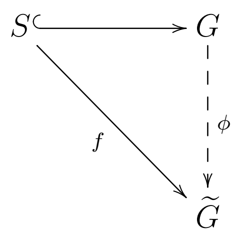
To better appreciate the main result, I will reformulate, in a concise way, the classical proofs of the finiteness of $B(2,3)$ and $B(2,4)$. These proofs are contained in [12, 54].

1.1.1. Presentations, generators, relations. One of the methods for defining a group is through a presentation.

A group $G := (G, \cdot_G)$ has the *presentation*

$$G = \langle S \mid R \rangle \tag{1}$$

if any element of G can be written as a product of powers of some of elements of the set S (called "generators"), keeping the relations among them given by the set R. More precisely, the group G is isomorphic to the quotient of a free group on S by the normal subgroup generated by the relations R. The group $G = \langle S \rangle$, without any relation among its generators, is called the *free group*. This means that, given any function f from S to another group $\widetilde{G}$, there exists a unique homomorphism $\phi\colon G \to \widetilde{G}$ such that the diagram



commutes.

Example 1. Consider the free group G_1 generated by one generator $\star$. Of course, $G_1 = \langle \star \rangle = \mathbb{Z}$. Let $G_2 = \mathbb{Z}_6 = \{[0], [1], \ldots, [5]\}$ and

$$f\colon \{\star\} \to \mathbb{Z}_6, \quad \star \mapsto [1].$$

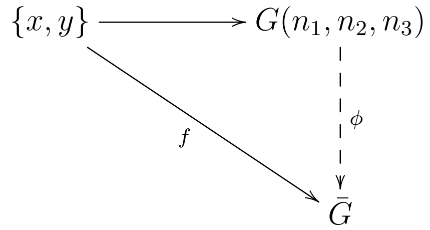
Then the unique homomorphism extending f is

$$\phi\colon \mathbb{Z} \to \mathbb{Z}_6, \quad n \mapsto [n].$$

Observe that $\ker\phi = 6\mathbb{Z} = \{6n \mid n \in \mathbb{Z}\}$ and $\frac{\mathbb{Z}}{\ker\phi} = \langle \star \mid \star^6 = 1 \rangle \cong \mathbb{Z}_6$.

This example shows, that the cyclic group of order n has the presentation $\langle a \mid a^n = 1\rangle$. For simplification it can written $\langle a \mid a^n = 1\rangle := \langle a \mid a^n\rangle$.

EXAMPLE 2. Consider the group $\langle x, y \mid x^{n_1} = y^{n_2} = (xy)^{n_3}\rangle =: G(n_1, n_2, n_3)$. Of course, this group is not free, but is the "free-est" in the class of the groups with 2 generators fulfilling all the relations of $G(n_1, n_2, n_3)$. More precisely, let $\overline{G} = \langle \bar{x}, \bar{y} \mid R\rangle$, where $R \supseteq \{\bar{x}^{n_1} = \bar{y}^{n_2} = (\bar{x}\bar{y})^{n_3}\}$: for such groups the universal property for $G(n_1, n_2, n_3)$ implies the existence of a unique epimorphism ϕ, such that



commutes, where $f(x) := \bar{x}$ and $f(y) := \bar{y}$.

DEFINITION 1. *S is a* Borel–free *set of generators if* $\bigcap\limits_{s\in S}\langle s\rangle = 1$, *and* $\langle\, s \,\rangle \neq 1$ *for all* $s \in S$.

1.1.2. Products of groups: interior, exterior, (semi)direct. Let $(G, \cdot_G)$ and $(H, \cdot_H)$ be groups. The structure $(G \times H, \cdot_{G\times H})$, where the elements of $G \times H$ are the ordered pairs (g, h), with $g \in G, h \in H$, and

$$(g_1, h_1) \cdot_{G\times H} (g_2, h_2) = (g_1 \cdot_G g_2, h_1 \cdot_H h_2),\ \forall g_1, g_2 \in G, \forall h_1, h_2 \in H,$$

is called the *(exterior) direct product of the groups G and H.*
It is easy to notice that the operation above introduces a group structure on $G \times H$. The identity element is $(1_G, 1_H)$, where 1_G and 1_H are the identity elements of G and H, respectively. The inverse of an element $(g, h) \in G \times H$ is $(g^{-1}, h^{-1}) \in G \times H$, where g^{-1} is the inverse of g in G, and h^{-1} is the inverse of h in H.

In the additive notation, the direct product of G and H is called the *direct sum* and it is denoted $G \oplus H$.

Let $(K, \cdot_K)$ be a group, $G, H \leqslant K$. The set $GH := \{g \cdot_K h \mid g \in G, h \in H\} \subseteq K$ is the *interior product* of the subgroups G and H. GH is a group with the operation inherited from K if and only if G and H permute, i.e., $GH = HG$.

Let G and N be groups, and let $\phi : G \to \mathrm{Aut}(N)$ be a group homomorphism. The *(exterior) semidirect product of the groups G and N with respect to ϕ* is obtained by introducing on the set $N \times G := \{(n, g) \mid n \in N, g \in G\}$ the operation defined in following way:

$$(n_1, g_1)(n_2, g_2) := (n_1\phi_{g_1}(n_2), g_1 g_2).$$

The resulting group is denoted by $N \rtimes_\phi G$. Its identity element is $(1_N, 1_G)$, where $1_N, 1_G$ are the identity elements in N and G, respectively, and the inverse is given by $(n, g)^{-1} = (\phi_{d^{-1}}(n^{-1}), d^{-1})$.
Pairs $(n, 1_G)$ form a normal subgroup (isomorphic to N) of the above defined group, while pairs $(1_N, g)$ form a subgroup isomorphic to G.

Let K be a group and $N \trianglelefteq K$. In this case, any $G \leqslant K$ acts on N by the conjugation, i.e., I can set $\phi(g)(n) = n^g$. The group NG is called the *interior product of groups N and G*, and it is denoted by $N \rtimes G$, if and only if G is a complement of N in K, i.e., $K = NG$ (equivalently: $K = GN$) and $N \cap G = 1_K$. Notice that $N \rtimes G \cong N \rtimes_\phi G$.

EXAMPLE 3. Every *dihedral group* of rank $2n$ is an interior semidirect product: $\mathbf{D}_{2n} = \mathbb{Z}_n \ltimes \mathbb{Z}_2$.

EXAMPLE 4. The isometry group of the n-dimensional Euclidean space, denoted by $\mathrm{E}(n)$, is the semidirect product of the rotation group $\mathrm{SO}(n)$ with the (abelian) group $\mathbb{R}^n$ of the translations: $\mathrm{E}(n) = \mathrm{SO}(n) \ltimes \mathbb{R}^n$. A group K is said to be the *direct (inner)* product of its subgroups G and H, if[1] $K = \langle G, H \rangle$ and $G \cap H = 1$.

1.1.3. Properties of groups. Let $G = (G, \cdot_G)$ be a group and $g_1, g_2 \in G$. Recall that the *commutator* is given by

$$[g_1, g_2] = g_1^{-1} \cdot_G g_2^{-1} \cdot_G g_1 \cdot_G g_2.$$

Let H, K be subgroups of G. Define the subgroup generated by all the commutators $[h, k]$ with $h \in H, k \in K$ by

$$[H, K] = \langle [h, k] \mid h \in H, k \in K \rangle .$$

$[H, K]$ is called the *commutator subgroup* or the *derived subgroup*.

A group G where any two elements commute (i.e., their commutator is equal to the neutral element of the group) is called an *abelian group*. More precisely, if $g_1 \cdot_G g_2 = g_2 \cdot_G g_1$ holds for any elements $g_1, g_2 \in G$ then G is an abelian group.

G is called *meta–abelian* if its commutator subgroup is abelian.

The *center* $Z(G)$ of a group G is the set of elements which commute with every element of the group,

$$Z(G) = \{z \in G \mid \forall g \in G, zg = gz\}.$$

DEFINITION 2. *The* lower central series $(\gamma_i(G))$ *(for $i \geq 1$) is the chain of subgroups of the group G defined by*

$$\gamma_1(G) = G$$

and

$$\gamma_{i+1}(G) = [\gamma_i(G), G] \quad \textit{for } i \geq 1.$$

DEFINITION 3. *A group G is* nilpotent *if $\gamma_{c+1}(G) = 1$ for some c. The least such c is the* nilpotency class *of G.*

1.2. The von Dyck and the triangle groups

Let $a, b, c \geq 2$ be integers and define

$$\lambda := \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

DEFINITION 4. *The group*

$$\triangle(a, b, c) = \langle l, m, n \mid l^2 = m^2 = n^2 = (lm)^a = (mn)^b = (ln)^c = 1\rangle,$$

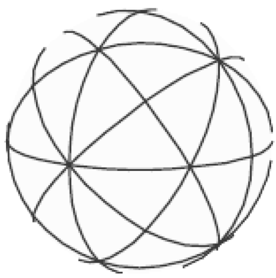
is called the triangle group [3].

The name "triangle group" comes from the fact that the elements l, m, n can be thought of as the reflections by the sides of a fixed triangle (henceforth called the *basic triangle*) with angles $\frac{\pi}{a}, \frac{\pi}{b}, \frac{\pi}{c}$. Notice that the product of the reflections by two adjacent sides is a rotation.

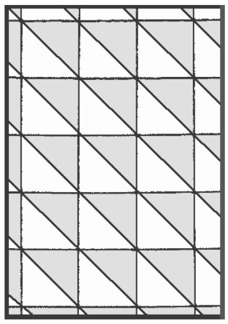
[1] Notice that $\langle \cdot \rangle$ is used here with a different meaning than in (1), namely it denotes the subgroup generated by a subset.

The triangle groups determine a tessellation (see Chapter 5, Definition 44) of a suitable surface by taking the reflections of the basic triangle. The sum of the angles of the triangle establishes the type of the surface by the Gauss–Bonnet theorem ([49]). So, three cases can be distinguished:

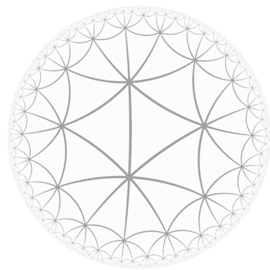
- if $\lambda > 0$ then $\triangle(a, b, c)$ is spherical (the tessellation can be represented on a sphere),
- if $\lambda = 0$ then $\triangle(a, b, c)$ is planar (the tessellation can be represented on a plane) and
- if $\lambda < 0$ then $\triangle(a, b, c)$ is hyperbolic (the tessellation can be represented on a hyperbolic plane).



$D(2,3,4)$



$D(2,4,4)$



$D(4,4,4)$

Notice that the spherical triangle groups are finite (in view of the compactness of the sphere).

The triangle group $\triangle(a, b, c)$ contains a subgroup of index 2, belonging to the class introduced in Example 2, denoted by $D(a, b, c)$.

Definition 5. *The subgroup*

$$D(a, b, c) := \langle x, y \mid x^a = y^b = (xy)^c = 1 \rangle$$

is called the von Dyck group [57].

Geometrically, the elements x, y and xy can be seen as the rotations around the vertices of the basic triangle by angles $\frac{2\pi}{a}$, $\frac{2\pi}{b}$ and $\frac{2\pi}{c}$ respectively.
As before, the von Dyck groups are classified according to the geometry of the corresponding tessellation. In particular, a hyperbolic von Dyck group is a Fuchsian group [5], a concept which will be introduced later on (see 4.4).

1.3. The Burnside Groups

One of the oldest and most interesting challenges of group theory was introduced by W. Burnside in 1902 the, so-called Burnside problem. It asks whether a finitely generated periodic group must necessarily be a finite group. It had a huge impact on the development of the theory of groups and, despite the passage of years, it is still of interest to professionals, especially to those who deal with the combinatorial theory of groups and groups with various finiteness conditions.

1.3.1. Periodic groups and the exponent of a group.

Definition 6. *A group G is called* periodic *or* torsion *if*

$$\forall g \in G\ \exists n \in N \mid g^n = 1.$$

In other words, each element of G has finite order. All finite groups are periodic. The exponent of a periodic group G is the least common multiple, provided it exists, of the orders of the elements of G.

DEFINITION 7. *A group G is called* periodic of bounded exponent *if*

$$\exists n \in N \mid g^n = 1 \ \forall g \in G.$$

The minimal such n is the exponent *of G.*

1.3.2. Formulations of the Burnside problem, and state of the art. Let $F_m = \langle\{1, \ldots, m\}\rangle$ be the free group of rank m, and F_m^n its normal subgroup generated by all the g^n's with $g \in F_m$.

DEFINITION 8. *The quotient group*

$$B(m, n) := F_m / F_m^n$$

is called the m-generated Burnside group with m generators of exponent n.

In other words, $B(m, n)$ is a group with m distinguished generators in which the identity $g^n = 1$ holds for all the elements $g \in B(m, n)$, and which is the "largest" group satisfying these requirements.

The Burnside problem is usually stated as follows: "For which values of m and n is $B(m, n)$ a finite group?"
Of course, for $m = 1$, $B(1, n)$ is the cyclic group of order n. In the case $m = 2$, so far it is known that

- the order of $B(2, 3)$ is equal to 27, established by F. Levi and van B. L. der Waerden (1933),
- the order of $B(2, 4)$ is equal to 2^{12}, established by J. J. Tobin (1954),
- the order of $B(2, 6)$ is equal to $2^{28}3^5$, established by P. Hall (1958).

These cases, even if they seem closely related, have been solved with completely different methods and the techniques used in one case by no means facilitate the answer to the next. Computations are generally rather lengthy, although the basic ideas can be easily understood, and the problem itself has a very "friendly"formulation: think about the first unsolved case, asking basically whether "every group generated by two elements, where each element satisfies the equation $x^5 = 1$, is finite."

1.3.3. The variety of the Burnside groups. In this subsection I will rigorously prove that $B(m, n)$ is the "largest" one with m generators of exponent n. To this end, I will work in the context of the varieties of groups [**31**, **20**].

DEFINITION 9. *The variety* $\texttt{Bur}_n$ *of equation $x^n = 1$ is called the* variety of Burniside groups *(of order n).*

Regard $\texttt{Bur}_n$ as a category.

LEMMA 1. *For any set S of rank m, there is a free object over S in* $\texttt{Bur}_n$*.*

PROOF. Let F_m be the free group on S, and F_m^n be the (normal) subgroup generated by its n^{th} powers. Then

$$B(m, n) = \frac{F_m}{F_m^n} \tag{2}$$

is the desired free object (over S, understood as a subset of $B(m,n)$). Indeed, if $B \in \mathtt{Bur}_n$, and $f : S \to B$ is a map, then there is a unique group homomorphism $\phi : B(m,n) \to B$ extending f. □

REMARK 1. The free object of $\mathtt{Bur}_n$, with m generators, defined by (2), is called the *free Burnside group* of order n with m generators, and it is precisely the group from Definition 8.

I will need the corollary below in the sequel. It shows that S is a Borel–free set of generators of $B(m,n)$ (see Definition 1).

COROLLARY 1. *If* $s, s' \in S$, *with* $s \neq s'$, *then* $\langle s \rangle \cap \langle s' \rangle = 1$.

PROOF. The abelian group $A := \oplus_{s \in S} \langle\, s\, \rangle$ is an object of $\mathtt{Bur}_n$: hence, the map $f : S \to A$, which acts on S as the identity, can be extended to a group homomorphism $\phi : B(m,n) \to A$. Observe that $\phi|_{\langle\, s\, \rangle}$ is injective, i.e., $\ker \phi|_{\langle\, s\, \rangle} = 1$, for any $s \in S$. On the other hand, if $s' \neq s$, then $\phi|_{\langle\, s\, \rangle}(\langle\, s\, \rangle \cap \langle\, s'\, \rangle) = \langle\, s\, \rangle \cap \langle\, s'\, \rangle = 1$, i.e., the intersection $\langle\, s\, \rangle \cap \langle\, s'\, \rangle$ is a subgroup of the identical subgroup $\ker \phi|_{\langle\, s\, \rangle}$, and as such it must be identical as well.

Observe that none of the $\langle\, s\, \rangle$'s can be identical, otherwise a map $f : S \to \mathbb{Z}_n$ sending s to 1 could not be extended to a group homomorphism ($\mathbb{Z}_n$ is an element of $\mathtt{Bur}_n$). □

1.3.4. The finiteness of $B(2,3)$: an algebraic proof. The departing point to check the finiteness of $B(2,n)$ is the expression

$$w = x^{a_1} y^{b_1} x^{a_2} y^{b_2} \cdots x^{a_l} y^{b_l} \tag{3}$$

of an its generic element w. In one form or another, all existing algorithms to attack the Burnside problem entail manipulating the right–hand side of (3) in order to move all the instances of x (resp., y), say, on the first (resp., second) leftmost position. Performing this by brute–force inevitably brings about a surfeit of commutators:

$$w = x^{\sum a_i} y^{\sum b_i} \cdot \text{(binary) commutators} \cdot \text{ternary commutators} \cdot \cdots . \tag{4}$$

Seemingly, (4) makes (3) no better at all: nevertheless, it reveals the role of the nilpotency of $B(2,n)$. Indeed, if $B(2,n)$ is nilpotent (see Definition 3), its class sets an upper bound to the multiplicity of the commutators appearing in (4). For instance, the fact that $B(m,2)$ is abelian (proved by W. Burnside in 1902 [12]), i.e., nilpotent of class 1, implies that no commutators whatsoever appear in (4). In 1933, F. Levi and B. L. Van der Waerden independently proved that $B(m,3)$ is nilpotent of class 3, and this means that the expression (4) contains commutators of multiplicity not greater than 3.

When $m = 2$, the nil–potency class is 2, i.e., $B = B(2,3)$ is meta–abelian (see Section 1.1.3), or, in other words, B' is central (see Section 1.1.3) in B. This means that all the commutators arising from the above manipulation of (3) can be collected in its rightmost position. In order to better appreciate the geometrical finiteness result later on (Chapter 8), I review here the classical proof of the finiteness of $B(2,3)$.

Let G be a group of exponent three. This means that, in particular, $g^{-1} = g^2$, for any $g \in G$.

PROPOSITION 1 ([32]). *Let $g, h \in G$. Then*

$$[g, h^2] = [h, g]. \tag{5}$$

PROOF. Observe that $1 = (gh)^3 = ghghgh$ implies

$$ghg = h^{-1}g^{-1}h^{-1} = h^2g^2h^2, \quad g, h \in G. \tag{6}$$

Equation (7) below is an identity, since its both sides coincide with the same element $h^2gh^2g^2h^2$:

$$h^2(gh^2g)gh^2 = h^2g^2(g^2h^2g^2)h^2. \tag{7}$$

Hence, by applying (6) to the parenthesized expressions in (7), one obtains

$$h^2(hg^2h)gh^2 = h^2g^2(hgh)h^2,$$

i.e., $g^2hgh^2 = h^2g^2hg$, which is precisely $[g, h^2] = [h, g]$, and (5) holds true. □

COROLLARY 2. *Every subgroup of G generated by two elements is meta–abelian.*

PROOF. Let $H := \langle h, g \rangle$ be a two–generators subgroup. Relation (5) from Proposition 1 immediately gives

$$\begin{aligned} [g, h^2] &= [g, h]^2, && (8)\\ [g^2, h] &= [g, h]^2, && (9)\\ [g^2, h^2] &= [g, h], && (10) \end{aligned}$$

showing that H' is generated by $[g, h]$. Then, combining (8) with the general group–theoretic identity $[g, h^2] = [h, g]^{h^2}$, one obtains $[h, g] = [h, g]^{h^2}$, i.e.,

$$[h, g, h^2] = 1.$$

Similarly, from (9) one proves that

$$[h, g, g^2] = 1.$$

Hence, the subgroup generated by h^2 and g^2, which is H again, commutes with H'. □

LEMMA 2 ([20, 26]). *Any element $w \in B$ can be uniquely written as*

$$w = x^a y^b [x, y]^c, \quad a, b, c \in \{0, 1, 2\}. \tag{11}$$

In particular, $|B| = 27$.

PROOF. B is metabelian by Corollary 2. In particular, the normal subgroup B' is abelian, and the sequence

$$\mathbb{Z}_3 \equiv B' = \langle [x, y] \rangle \longrightarrow B \longrightarrow \frac{B}{B'} = \langle xB', yB' \rangle \equiv \mathbb{Z}_3^2$$

is exact, thus showing (11).

□

In the modern classification of groups, precisely 5 groups of order p^3 can be found, and only two of them are non–abelian: they have exponent p or p^2, with p prime number [44]. Hence, for $p = 3$, $B(2,3)$ is the non–abelian one with exponent 3, which is also the unique non–abelian metabelian group with two generators and exponent 3, possessing a cyclic derived subgroup (see Section 1.1.3), and, as such, it is presented as

$$B(2,3) = \langle\, x, y \mid x^3 = y^3 = [x,y,x] = [x,y,y] = 1 \,\rangle.$$

1.3.5. The finiteness of $B(2,4)$: an algebraic proof. The first results about the finiteness of $B(2,4)$ were given already by W. Burnside in his 1902 paper [12], in which he claims that $B(2,4)$ is finite of order $\leq 2^{12}$. Next, in 1940, I. N. Sanov [47] proved that $B(m,4)$ is finite, by using the following Lemma 3.

Lemma 3. [35] *Let B be a group of exponent 4, $D \leq B$ an its finite subgroup, and $c \in B$ an element such that $c^2 \in D$, and $\langle\, c, D \,\rangle = B$. Then B is finite.*

Proof. Put $d := |D|$. Observe that, since $B = \langle\, c, D \,\rangle$, every element $b \in B$ can be written in the form

$$b = P_1 c Q_1 c P_2 c Q_2 c \dots P_s c Q_s c P_{s+1} c Q_{s+1}, \tag{12}$$

where $P_i, Q_i \in D$ for $1 \leq i \leq s+1$ and all of them are different from identity, except possibility for P_1 or Q_{s+1}. It is enough to show that, in the expression (12) of b, the number of occurrences $2s+1$ of the factor c is bounded by a fixed number. Precisely, I will show that $s \leq d$ by applying to the representation of b a transformation which reduces the occurrences of the factor c if $s > d$.
Notice that $c^3 = c^{-1}$ (since $c \in B$) and, for any $R \in D$, $(R^{-1}c^{-1})^4 = 1$, so that

$$cRc = R^{-1}c^{-1}R^{-1}c^{-1}R^{-1} = R^{-1}c \cdot c^2 R^{-1} c \cdot c^2 R^{-1} = R^{-1} c \tilde{R} c \tilde{R}, \tag{13}$$

where $\tilde{R} = c^2 R^{-1} \in D$. Now, let me use (13) to transform the representation (12) as follows:

$$\cdots cP_{l+k-1}cQ_{l+k-1}cP_{l+k}(cQ_{l+k}c)\cdots$$
$$\cdots cP_{l+k-1}cQ_{l+k-1}(cP_{l+k}Q_{l+k}^{-1}c)UcU\cdots$$
$$\cdots cP_{l+k-1}(cQ_{l+k-1}Q_{l+k}P_{l+k}^{-1}c)VcWcU\cdots$$
$$\cdots c(P_{l+k-1}P_{l+k}Q_{l+k}^{-1}Q_{l+k-1}^{-1})cXcYcWcU\cdots,$$

where $U, \cdots Y \in D$. In this way I do not change the number of the factors c, but I just show a new representation of b in which two subsequent elements are separated by an element $S_{l,k}$ (or $S_{l,k}^{-1}$) of D defined below,

$$S_{l,k} = P_l P_{l+1} \dots P_{l+k} Q_{l+k}^{-1} \dots Q_l^{-1},$$

where $l > 0, k \geq 0, l + k \leq s$. On the other hand, notice, that any $S_{l,k} \in D$ could be extended to $S_{1,k}$, $k = 0, \dots, s-1$, therefore if $s > d$, which is the order of D, then at least two element, say $S_{1,k}$ and $S_{1,l}$ must be equal, i.e

$$P_1 \dots P_{k+1} Q_{k+1}^{-1} \dots Q_1^{-1} = P_1 \dots P_{l+1} Q_{l+1}^{-1} \dots Q_1^{-1}$$

and assuming $k < l$, I obtain

$$S_{k+2,l-k-1} = P_{k+2} \dots P_{l+1} Q_{l+1}^{-1} \dots Q_{k+2}^{-1} = 1. \tag{14}$$

Now to finish proof, it is just enough to use (14) in the appropriate representation of b:

$$\dots Q_{k+1} c S_{k+2,l-k-1} c P_{l+1} \cdots = \dots Q_{k+1} \underbrace{c^2}_{\in D} P_{l+1} \dots .$$

□

Then finiteness of $B(2,4)$ is straightforward. The group $\langle\, x, y^2 \,\rangle$, its subgroup $\langle\, x \,\rangle$, and the element y^2 fulfills the hypotheses of Lemma 3: hence, $\langle\, x, y^2 \,\rangle$ is finite. Then apply again Lemma 3 to the group $B(2,4)$, its subgroup $\langle\, x, y^2 \,\rangle$, and the element y, and obtain the desired result.

CHAPTER 2

Elements of graph theory

Graph theory is a growing area in mathematical research, and has so large specialized vocabulary that is used the same word by different authors with different meanings. In this chapter I undertake the necessary task of introducing just some of the basic notations for graphs which will be an useful tool later on. A very good compendium of knowledge in this subject is the book [18], and also I can suggest [23]. I will deal with coloured graphs and in particular I will focus on the Cayley graph, which is so pivotal in this thesis (look, for example, [35]).

2.1. Basic definitions

I recall here the basic definitions and results from the standard graph theory.

2.1.1. Graphs, sub–graphs, morphsims.

DEFINITION 10. *A* graph *is an ordered pair* $\Gamma = (V, E)$*, where*

- V *is a non empty set, whose elements are called* vertices*,*
- E *is a family of 2-elements subsets of* V*, called* edges*.*

To avoid ambiguity, as a graph I mean a *simple graph* which is a graph without edges connected at both ends to the same vertex (loop) and having no more then one edge between any two different vertices. So, $E \subseteq \{\{u, v\} : u, v \in V, u \neq v\}$.

DEFINITION 11. *If* E *is made of ordered pairs of vertices, then such a graph is named* directed graph *or* digraph*.*

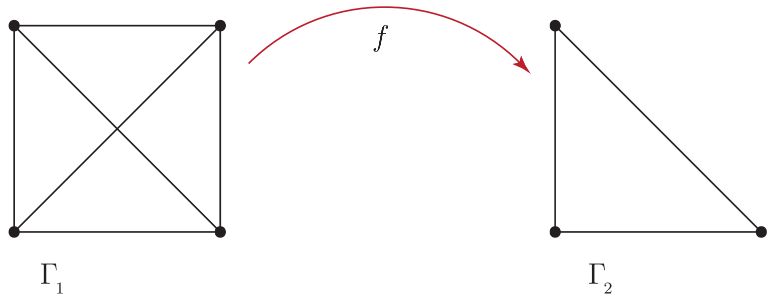
The *degree* of a vertex $v \in V$ in Γ is the number of edges to which v is *incident*, i.e., the edges which touch v. If all vertices of Γ have the same degree then Γ is called *regular*. The number of vertices of a graph Γ is its *order* and it is denoted as $|\Gamma|$. A graph can be finite, infinite, countable and so on, according to its order. A *bipartite graph* (or *bigraph*) is a graph whose vertices can be divided into two disjoint sets, such that edges never connect vertices of the same set.

DEFINITION 12. *A* subgraph of a graph $\Gamma = (V, E)$ *is a graph* $\Gamma_s = (V_s, E_s)$ *where* $V_s \subseteq V$ *and* $E_s \subseteq E$*.*

We say that two vertices are *adjecent* if they are linked by an edge.

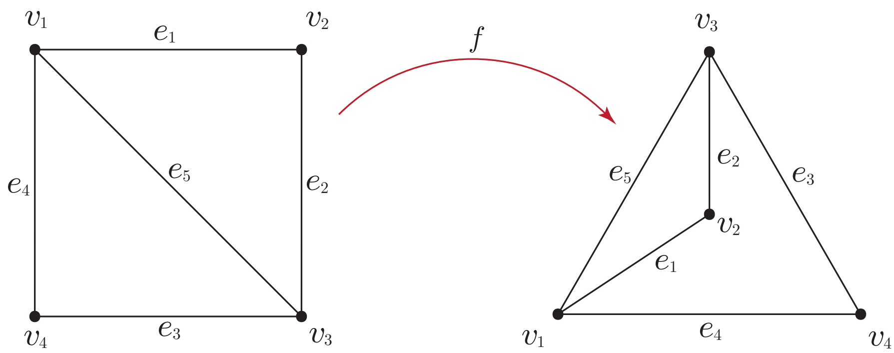
DEFINITION 13. *Let* $\Gamma_1 = (V_1, E_1)$ *and* $\Gamma_2 = (V_2, E_2)$ *be graphs. A map* $f \colon V_1 \to V_2$ *is called a* morphism *of graphs if*

- $f(V_1) \subseteq V_2$, $f(E_1) \subseteq E_2$,
- *if* $u, v \in V_1$ *are adjacent in* Γ_1 *then* $f(u)$ *and* $f(v)$ *are adjacent in* Γ_2*.*



Definition 14. *An* isomorphism *of graphs* $\Gamma_1 = (V_1, E_1)$ *and* $\Gamma_2 = (V_2, E_2)$ *is a bijection* $f\colon V_1 \to V_2$ *such that any* $u, v \in V_1$ *are adjacent in* Γ_1 *iff* $f(u)$ *and* $f(v)$ *are adjacent in* Γ_2.

Remark 2. An isomorphism is an invertible morphism.

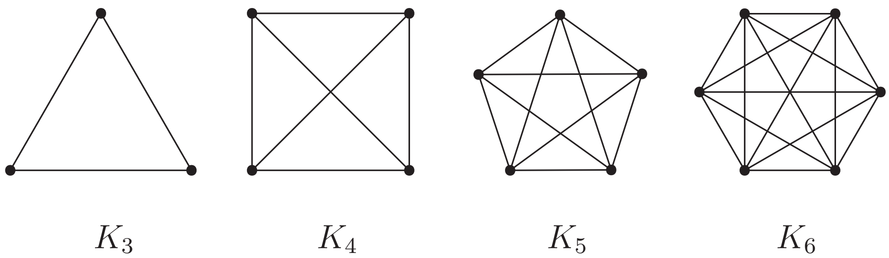


Definition 15. *An isomorphism from a graph* $\Gamma = (V, E)$ *to itself is called an* automorphism.

Under the operation of composition, the family of all automorphisms of a graph Γ forms a group $\mathrm{Aut}\,(\Gamma)$ called the *automorphism group* of a Γ.

Definition 16. *A graph is called* complete *if every pair of vertices is adjacent.*

The complete graph of n vertices is usually denoted by K_n. Observe that K_n is unique, up to isomorphism. In other words, if two complete graphs have the same number of vertices, then they are isomorphic.



2.1.2. Paths. Let $\Gamma_n := (V, E)$ be a nonempty graph of the form

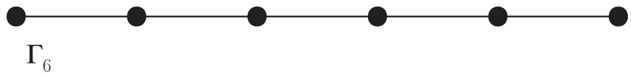
$$V = \{v_0, v_1, \dots, v_n\}, \quad E = \{v_0v_1, v_1v_2, \dots, v_{k-1}v_n\},$$

where v_i are all distinct.

Definition 17. *A graph morphism* $\gamma\colon \Gamma_n \to \Gamma$ *is called a* path *in* Γ.

Let $u, v \in V$. *If a path* f *exists, such that* $\gamma(0) = u$ *and* $\gamma(n) = v$*, then we say that* u *and* v *are connected by a path. The number of edges of a path is its* length.

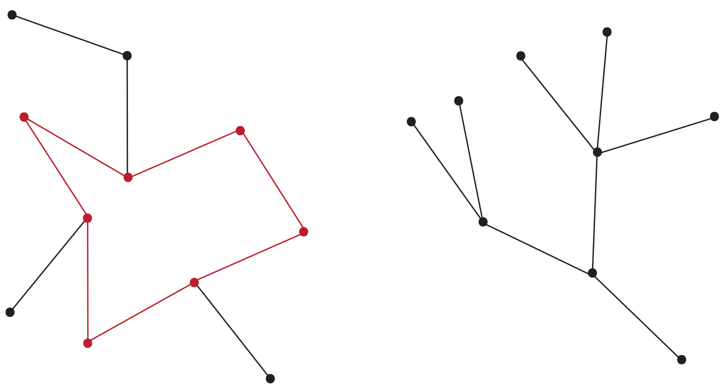
EXAMPLE 5. Let $V = \{0, 1, \ldots, n\}$ and $E = \{\{i-1, i\} \ : \ i = 1, 2, \ldots, n\}$. $\Gamma_n \overset{\text{def.}}{=} (V, E)$ is a path graph of length n.



DEFINITION 18. *If any pair of vertices of* Γ *are connected by a path (resp., a unique path), then* Γ *is called* connected *(resp., a* tree*).*

If no pair of vertices of Γ is connected by a path, Γ is *totally disconnected*. In other words, for a totally disconnected graph $\Gamma = (V, E)$, $E = \emptyset$.

DEFINITION 19. *A path* $\gamma \colon \Gamma_n \to \Gamma$ *which contain at least 3 edges and* $\gamma(0) = \gamma(n)$ *is a called a* cycle*. A cycle is* simple *if* $\gamma(i) = \gamma(j)$ *if and only if* $\{i, j\} = \{0, n\}$.



In other words, a simple cycle is a cycle without self–intersections. Observe that a bigraph does not contain any odd–length cycle.

2.2. Planar, dual graph and vertex–to–edge duality

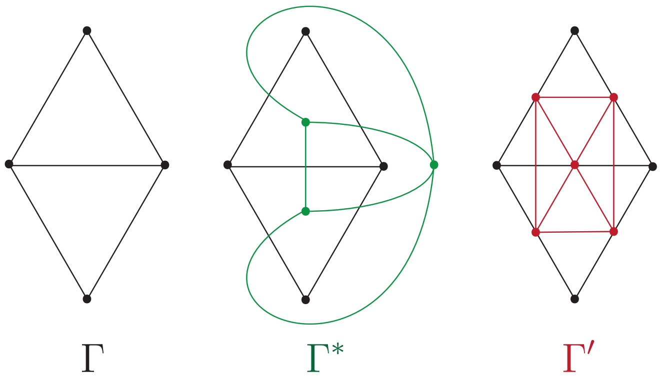
A graph is called *planar* if it can be embedded in the plane, i.e., roughly speaking, drawn without crossing edges. The edges of the graph divide the plane into regions called the *faces* of the planar graph. Euler's famous formula gives the relationship between the number of vertices, edges, and faces of a connected planar graph. It says that if n is the number of vertices, e the number of edges and f the number of faces of a graph, including the exterior face, then

$$n - e + f = 2.$$

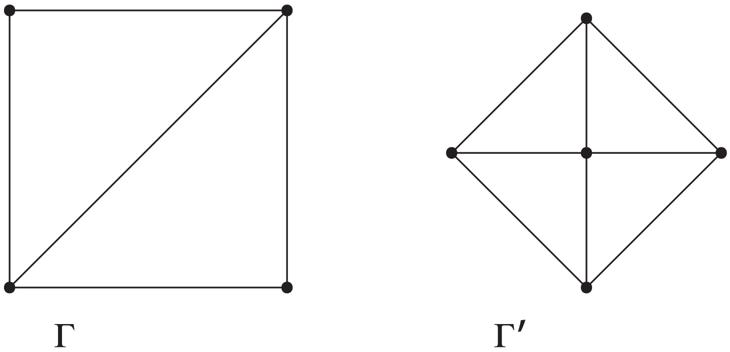
Given a planar graph Γ, it can be form another plane graph called the *dual graph* Γ^*. The vertices of Γ^* correspond to the faces of Γ, with each vertex being placed in the corresponding face. Every edge e of Γ gives rise to an edge of Γ^* joining the two faces of Γ that contain e. The name "dual graph" is not coincidental: it can be shown that, if Γ is connected, then $(\Gamma^*)^*$ is isomorphic to Γ. The reader should bear in mind that the notion of a dual graph is not related to the *vertex–to–edge duality*, which is described below.

Given a graph Γ, it is also possible to construct another graph Γ' in such a way that each edge of Γ represents a vertex of Γ' and two vertices of Γ' are adjacent if and only if their corresponding edges share a common endpoint (i.e., they are incident) in Γ.

Definition 20. *Graphs Γ' and Γ constructed as above are linked by a* vertex–to–edge duality.

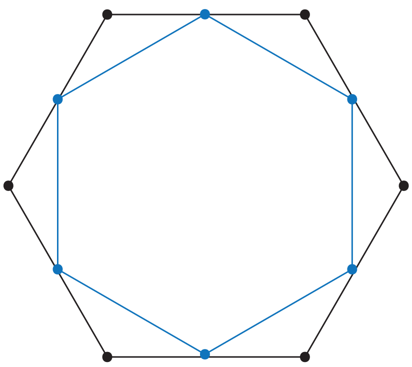


Sometimes the graph Γ' is called the *line graph* or the *derived graph* of Γ.



Example 6. The path graph Γ_n from Example 5, has a derived graph isomorphic to the shorter path graph Γ_{n-1}.

Taking the derived graph twice does not return the original graph unless the derived graph of the graph Γ is isomorphic to Γ itself. In fact, the only connected graph that is isomorphic to its derived graph the a cycle graph.



2.3. Coloured graphs and their morphisms

I will review the notion of a *coloured graph* and some related mathematical gadgets (see, e.g., [18]), stressing that some details, especially the definition of a *clique*, may vary according to the source.

2.3.1. Coloured graphs, colouring function, chromatic number.

DEFINITION 21. *Let* $\Gamma = (V, E)$ *be a graph,* C *a nonempty finite set,* $\chi : V \to C$ *a surjective map such that*

$$\{v_1, v_2\} \in E \Rightarrow \chi(v_1) \neq \chi(v_2). \tag{15}$$

The pair (Γ, χ) *is a* coloured graph*. Elements of* C *are called* colours*:* $\chi(v)$ *is the* colour *of the vertex* $v \in V$*, and* χ *is the* colouring function*. The* rank *of* (Γ, χ) *is the number of colours, i.e.,* $|C|$*.*

REMARK 3. Definition above gives the notion of a vertex–coloured graph. The reader can easily guess how to obtain the definition of an *edge–coloured graph*.

Roughly speaking, a coloured graph is a graph where a colour has been assigned to each vertex, in such a way that two vertices of the same colour never form an edge. Plainly, a given graph Γ can be coloured in different ways: the minimum rank of all coloured graphs of the form (Γ, χ) is the *chromatic number* of Γ. In order to formalize these "different ways" to colour a graph, it is useful to introduce morphisms. I warn the reader that a pair (Γ, χ) will be identified with Γ in non ambiguous contexts.

2.3.2. Morphisms of coloured graphs, regular morphisms, colour shufflings. Let now (Γ_i, χ_i) be a coloured graph, $i = 1, 2$, and $F : \Gamma_1 \to \Gamma_2$ be a graph morphism (see Definition 13). Roughly speaking, F is a morphism of coloured graphs if it preserves the colouring.

DEFINITION 22. F *is a* morphism of coloured graphs *if a map* $f : C_1 \to C_2$ *exists, such that the diagram*

$$\begin{array}{ccc} \Gamma_1 & \xrightarrow{F} & \Gamma_2 \\ \downarrow{\scriptstyle \chi_1} & & \downarrow{\scriptstyle \chi_2} \\ C_1 & \xrightarrow{f} & C_2 \end{array} \tag{16}$$

is commutative. If f *is a bijection, then the morphism* F *is called* regular*. A regular morphism of the form* (id_Γ, f) *is called a* colour shuffling*.*

2.3.3. Categories of coloured graphs. Together with morphisms, coloured graphs form the *category of coloured graphs*, denoted by `C-Gra`. For any $\Gamma \in$ `C-Gra`, the monoid $\mathrm{Mor}\,(\Gamma, \Gamma)$ contains the distinguished subgroup $\mathrm{Aut}\,(\Gamma)$ made of regular morphisms of the form (F, id_C), where $F : \Gamma \to \Gamma$ is a graph automorphism.

In those contexts where properties of coloured graphs do not depend on the actual choice of colours, it is convenient to work with a category where the colourings are defined only up to permutations. This is achieved as follows. First, take the sub–class of coloured graphs of a fixed rank m: together with regular morphisms, it forms a (not full) sub–category of `C-Gra`. Then, in the so–obtained category, identify two morphisms if they differ by a colour shuffling. The result is a new category, which is called the *category of* m*–colours graphs* and denoted by $\texttt{C-Gra}_m$: an its object may be thought of as a coloured graph Γ of rank m whose vertices are coloured only up to permutations.

2.3.4. Sub–graphs and cliques. A sub–graph $\Gamma_0 \subset \Gamma$, can be understood as a sub–object in $\texttt{C-Gra}_m$ only if the canonical inclusion $\iota : \Gamma_0 \to \Gamma$ gives rises to a morphism: in particular, Γ_0 must possess all the colours of Γ. For example, if Γ_0 is a *complete sub–graph* (see Definition 16), it cannot be considered as a sub–object unless it has exactly m vertices.

DEFINITION 23. *A complete sub–graph* $\Gamma_0 \subset \Gamma$ *with* m *vertices is called a* clique.

Notice that some authors (e.g., [33]) use the term "clique" to denote a complete sub–graph: according, the above definition should read "clique with m vertices", resulting in a pointless overload of the notation.

Hence, in the category $\texttt{C-Gra}_m$, a clique Γ_0 is a sub–object of Γ. It is worth observing that any morphism $F : \Gamma \to \Gamma'$, being regular, is clique–preserving (i.e., $F(\Gamma_0)$ is a clique in Γ'), and as such it can be restricted to the sub–object Γ_0.

Denote by $\mathfrak{C}(\Gamma)$ the (possibly empty) set of all the cliques of Γ.

REMARK 4. Let $\mathfrak{C}(\Gamma) \neq \emptyset$. For similar reasons as above, any $F \in \mathrm{Aut}\,(\Gamma)$ is clique–preserving (F is a graph automorphism), and as such it induces a bijection $\mathfrak{C}(F)$ on $\mathfrak{C}(\Gamma)$. Correspondence $\mathfrak{C} : F \mapsto \mathfrak{C}(F)$ is a group homomorphsim between $\mathrm{Aut}\,(\Gamma)$ and the group of permutations of $\mathfrak{C}(\Gamma)$.

2.4. Cayley graph

There are many graph–theoretical constructions that can be associated to a group. In this section I will focus on the Cayley graph, which is fundamental for this thesis. For other examples of such constructions, look the Appendix (section 9.1).

Prominent method to make more interesting the study of a group G is to give it some geometry. One of the most fundamental ways to provide such an extra geometric structure is to specify a list of generators S of the group G. In fact, the Cayley graph is an edge–coloured digraph (see Definition 11) whose vertices can be thought of as the elements of some finitely generated group G. More precisely, if S is any generating set for G, then the Cayley graph of G with respect to S, which I will denote by $\Gamma(G, S)$, has a directed edge labeled by s from the initial vertex g to the terminal vertex gs for all $g \in G$ and for all $s \in S$.

Here it follows the formal definition. Suppose that G is a group and S is an its generating set.

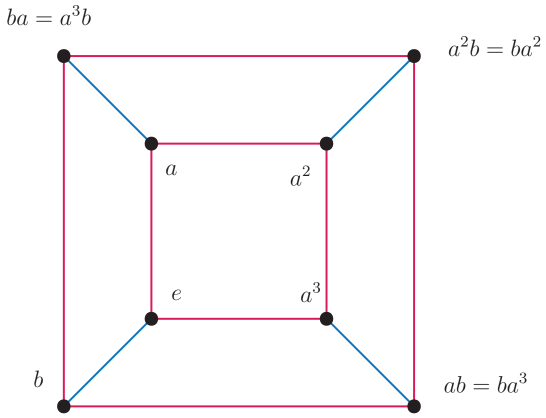
DEFINITION 24. *The* Cayley graph $\Gamma := \Gamma(G, S)$ *is a coloured directed graph (see Definition 21), constructed as follows:*

- *each element* $g \in G$ *is assigned to a vertex: the vertex set* V *of* Γ *is identified with* G,
- *each generator* $s \in S$ *is assigned to a colour* $c_s \in C$*: the colour set* C *is identified with* S,
- *for any* $g \in G, s \in S$*, the vertices corresponding to the elements* g *and* gs *are joined by a directed edge of colour* $c_s \in C$*. Thus the edge set* E *consists of pairs of the form* (g, gs)*, with* $s \in S$ *providing the colour.*

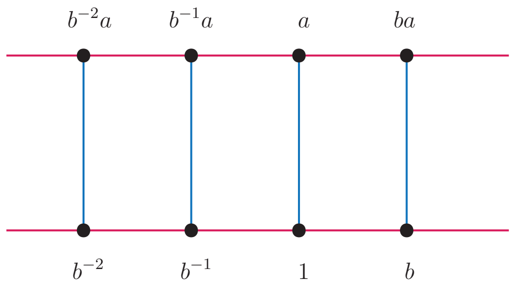
.

Some crucial properties of the Cayley graphs are that they are connected (see Definition 18), regular with respect to edge colour and direction, vertex–transitive (see Section 6.3 later on), and that the group of automorphism $\Gamma(G, S)$ (see Definition 15), which preserve edge colour and directions is isomorphic to G.

Let look at the example of $\mathbf{D}_4 = \langle a, b \mid a^4 = b^2 = e,\ ab = ba^3 \rangle$ (defined in Example 3), with $S = \{a, b\}$ and c_a = red and c_b = blue. On right, I picked the Cayley graph of infinite dihedral group $\mathbf{D}_\infty = \langle a, b \mid a^2 = e,\ aba = b^{-1} \rangle$ which is isomorphic to semidirected product (see Section 1.1.2) of $\mathbb{Z}$ and $\mathbb{Z}_2$.

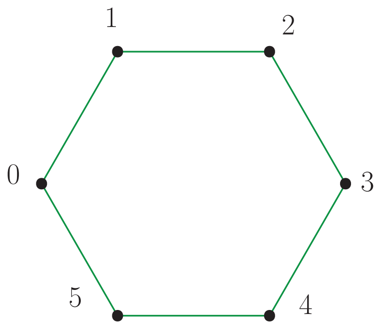


$\boldsymbol{D}_4 = \langle a, b \rangle$

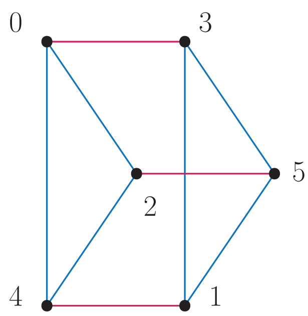


$\boldsymbol{D}_\infty = \langle a, b \rangle$

It is important to stress that the Cayley graph depends on the choice of the generating set, i.e., to the same group G there can be associated different Cayley graphs.



Cayley graph of $(\mathbb{Z}_6, \{[1]\})$



Cayley graph of $(\mathbb{Z}_6, \{[2], [3]\})$

The Cayley graph associated to a given group G and generating set S reveals much about the group. Every path in the graph corresponds to a *word* in the generators, and every cycle indicate relations between the elements of S. I conclude this chapter with the following fundamental theorem.

THEOREM 1. *(See e.g. [9]) The problem of constructing the Cayley graph of a given presentation is equivalent to solving the so–called* word problem *for it .*

CHAPTER 3

Coset geometry

I will start this chapter by acquainting the reader with the idea of a *tolerance space*. This notion was formally introduced in 1962 by E. C. Zeeman [60] as an useful mean of investigating the geometry of the visual perception, appreciated and exploited in computer science. It gave rise to a large–scale studying, which nowadays is the area of *near sets* [42]. Even if I will not use tolerance spaces here, they can help to understand the motivating article [56] of this thesis and justify the step I am going to take in the direction of *coset geometry*.
Next, I will follow [10] for the definitions of an *incidence structure* and of an incidence geometry. An incidence structure consists of a set of elements, a symmetric relation on these elements, and a type function from the set of elements to an index set (meaning that every element has a "type"). The cornerstone of this chapter is a special kind of incidence structure, the so–called *coset geometry*.

3.1. Tolerance spaces

The relations which share the same formal properties with the *similarity relations* of perception were firstly considered by H. Poincaré [43] and are nowadays, after E. C. Zeeman, called *tolerance relations*.

3.1.1. Tolerance relations.

Definition 25. *A* tolerance *on a set S is a binary relation $\sim$*

- *reflexive and*
- *symmetric.*

The structure $(S, \sim)$ is called a *tolerance space.*
Denote by $\mathcal{P}(X)$ the set of the parts of X.

Example 7. Let X be a set, and $S \subseteq \mathcal{P}(X)$. Then $A \sim B \overset{\text{def.}}{\Leftrightarrow} A \cap B \neq \emptyset$, $A, B \in S$, is a tolerance in S.

Example 8. Let $\epsilon > 0$ and let (S, d) be a metric space. Then in S there is the tolerance $x \sim y \overset{\text{def.}}{\Leftrightarrow} d(x, y) < \epsilon$, $x, y \in S$.

For any equivalence relation on a set X, the set of its equivalence classes is a partition of X. Conversely, from any partition S of X, we can define an equivalence relation on X by setting $x \sim y \Leftrightarrow x$ and y belongs to the same element of S. Thus the notions of an equivalence relation and of a partition are essentially equivalent. The same happens for tolerance relations on X and coverings of X.

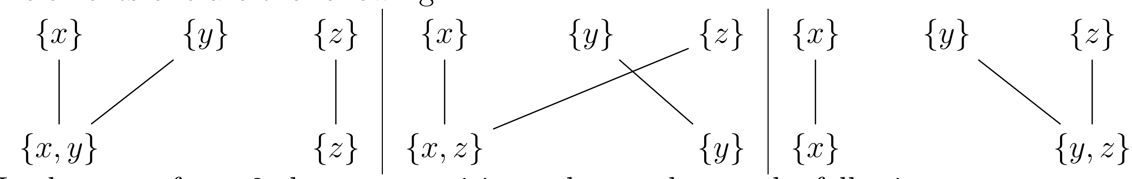
Example 9. Let X be a topological space with a fixed covering S. Then $x \sim y \overset{\text{def.}}{\Leftrightarrow} x, y$ are contained in one element of S, is a tolerance in X.

Definition 26. *Let* $\sim_1, \sim_2$ *be two equivalence relations on* X*. The tolerance* $\sim$*:*
$x \sim y \Leftrightarrow \begin{cases} x \sim_1 y \\ \text{or} \\ x \sim_2 y \end{cases}$ *is called* the tolererance generated *by two equivalence relations.*

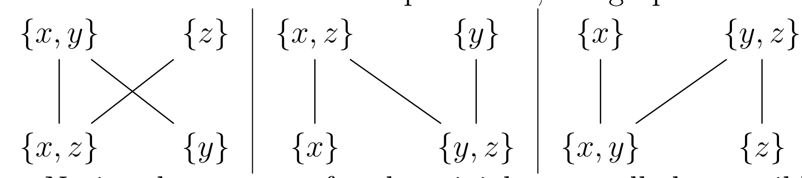
Two elements of the covering associated with the tolerance generated by two equivalence relations are called of the same *type*, if they belong to the same partition.

3.1.2. Graphs associated with tolerance spaces. To any covering S of the set X it can be associate a graph $\Gamma_\sim = (V, E)$ in such a way that any element of S corresponds to a vertex $v \in V$ and any tolerance relation corresponds to an edge. If S is a partition (which is a particular case of a covering) the graph is totally disconnected (see Section 2.1.2), i.e., made only of isolated vertices.
The graph associated with a tolerance generated by two equivalence relations is a bigraph (see Section 2.1).

Example 10. Let consider the 3-elements set $\{x, y, z\}$. The possible partitions are:
1) the discrete one: $\{\{x\}, \{y\}, \{z\}\}$ • • • ,
2) 2-elements partitions: $\{\{x, y\}, \{z\}\}$ • • ,
$\{\{x, z\}, \{y\}\}$ • • ,
$\{\{x\}, \{y, z\}\}$ • • ,
3) and the trivial one: $\{\{x, y, z\}\}$ • .
The graphs associated with the tolerance generated by the discrete partitions and a 2-elements one are the following:

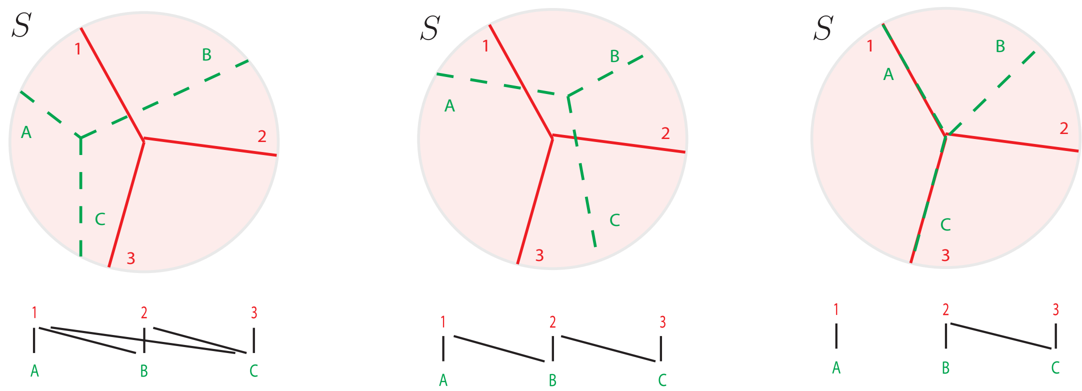


In the case of two 2-elements partitions, the graphs are the following:



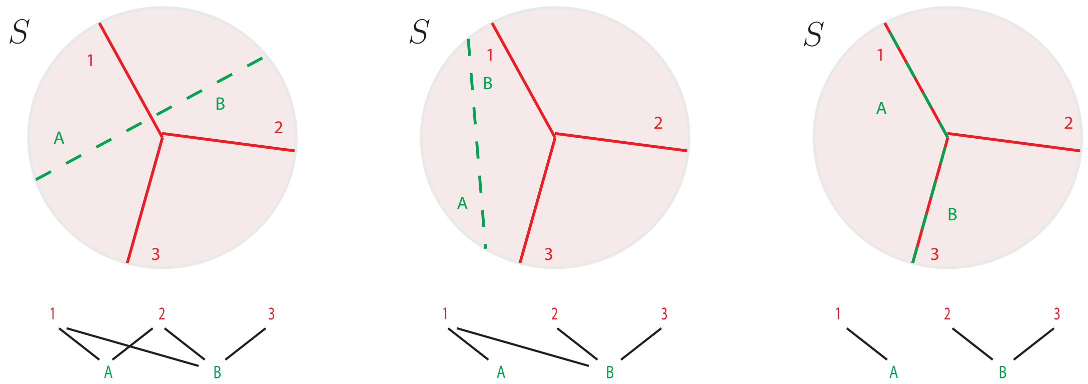
Notice that, except for the trivial cases, all the possible combinations are displayed above.

Example 11. Two 3-elements partitions on the same set S can give rise to many non–isomorphic graphs, some of which are depicted hereafter:



I mark different types of elements in different colours.
Even a 3-elements partition, combined with a 2-elements one, give rise to several

non–isomorphic graphs:



Theorem 2 below shows how a topological property of the graph $\Gamma_\sim$ reflects the relationship between two partitions of X. It is the first original result of this thesis.

Let X be a set, and $\mathcal{H} = \{H_i\}_{i\in I}$ and $\mathcal{K} = \{K_j\}_{j\in J}$ two finite partitions of X. Consider the tolerance $\sim$ generated by them (see Definition 26), and the bigraph $\Gamma_\sim = (V, E)$ corresponding to $\sim$.

THEOREM 2. *$\Gamma_\sim$ is disconnected if and only if a subset of X exists, which is simultaneously the union of elements from $\mathcal{H}$ and from $\mathcal{K}$, i.e.,*

$$G_\sim \textit{ disconnected} \Leftrightarrow \exists A \subsetneq X \ : \ A = \bigcup_{m\in M} H_{i_m} = \bigcup_{l\in L} K_{j_l}.$$

PROOF. Notice that between the elements of type $\mathcal{H}$ or those of type $\mathcal{K}$ there cannot be any edges.

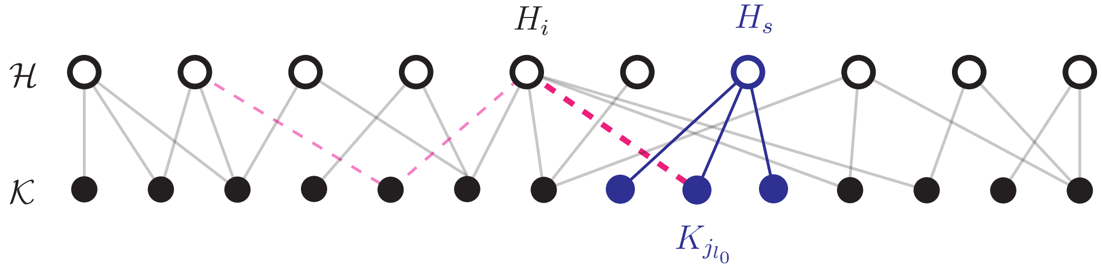
$\Leftarrow$

Let now suppose that there exists an union of elements from the first partition which is the union of some elements from the other one. In particular, it can be one element, say $H_s \in \mathcal{H}$, i.e., $H_s = \bigcup_{l\in L} K_{j_l}, K_{j_l} \in \mathcal{K}$. Toward an absurd, suppose that the graph associated with $(S, \sim)$ is connected, i.e., a path exists between any pair of its vertices (see Section 2.1.2). In particular, any K_{j_l}, with $l \in L$ (blue filled circles in the figure below), must be connected by a path γ (rose dotted line) with all elements $H_i \in \mathcal{H}$ and, in particular, with $i \neq s$. Since elements of the same type cannot be connected, γ must contain an edge between some $K_{j_{l_0}}$, with $l_0 \in L$, and H_i.

In other words, an $m \in X$ exists, such that $m \in K_{j_{l_0}} \cap H_i$. Since $H_s = \bigcup_{l\in L} K_{j_l}$, I find

$$m \in H_s \cap H_i,$$

but elements of $\mathcal{H}$ are not connected.



$\Rightarrow$

Let now suppose that the graph $\Gamma_\sim = (V, E)$ associated with $(S, \sim)$ is disconnected. It means, that a proper, connected subgraph exists, i.e.,

$$\exists \Gamma'_\sim = (V', E') \ : \ \Gamma'_\sim \subsetneq \Gamma_\sim.$$

Of course there cannot exist any edge between the elements of $\Gamma'_\sim$ and $\Gamma_\sim \setminus \Gamma'_\sim$. Since $\Gamma'_\sim$ is proper,

$$\exists V' = \{H_i, K_j, i \in I', j \in J'\} \text{ where } I' \subsetneq I \text{ or } J' \subsetneq J.$$

Consider the case $I' \subsetneq I$ (the other one being completely analogous) and define $A \overset{\text{def.}}{=} \bigcup_{i \in I'} H_i \neq X$. Then,

$$\Big((\forall a \in A \ \exists K_{j_a} \ : \ a \in K_{j_a}) \Rightarrow (A \cap K_{j_a} \neq \emptyset) \Rightarrow j_a \in J' \Big) \Rightarrow A \subseteq \bigcup_{j \in J'} K_j$$
$$\Leftrightarrow \bigcup_{i \in I'} H_i \subseteq \bigcup_{j \in J'} K_j$$

Now let $B \overset{\text{def.}}{=} \bigcup_{j \in J'} H_i$. Then,

$$\Big((\forall b \in B \ \exists H_{i_b} \ : \ b \in H_{i_b}) \Rightarrow (B \cap H_{i_b} \neq \emptyset) \Rightarrow i_b \in I' \Big) \Rightarrow B \subseteq \bigcup_{i \in I'} H_i$$
$$\Leftrightarrow \bigcup_{j \in J'} K_j \subseteq \bigcup_{i \in I'} H_i.$$

So,

$$\bigcup_{j \in J'} K_j = \bigcup_{i \in I'} H_i$$

□

3.2. Incidence Geometries

I will start with the definition of an *incidence system* — also called a *pregeometry*. For this, I followed F. Buekenhout [10, 11]. Originally, the incidence geometry was proposed as a generalisation of projective geometry, but later it also found remarkable applications to problems of image recognition [10, 60, 41, 50, 56, 21]

Definition 27. *Let T be a four–tuple $(X, *, t, I)$ where*

- *X is a set whose elements are called the* elements *of T,*
- *I is a set whose elements are called the* types *of T,*
- *t is a map from X to I called the* type function,
- *$*$ is a symmetric and reflexive relation on X, called the* incidence relation *of T (if $x_1 * x_2$ then x_1 and x_2 are called* incident*).*

If the restriction of t to any flag *(set of pairwise incident elements of T) of T is an injection then T is called an* incidence system *over I. The cardinality of I is called the* rank *of T.*

Observe that X is the disjoint union of all the $X_i := t^{-1}(i)$, with $i \in I$, and so any two elements of the same type cannot be incident.

If $A \subseteq X$, then A is of type $t(A)$ and of rank equal to the cardinality of $t(A)$. A flag of type I is called *chamber* of T.

Notice that $*$ is a tolerance relation (see Definition 25) and similarly, like in tolerance space it is possible to associate a graph, whose vertices are the elements of X, and two vertices are connected by edge whenever they are incident in T.

DEFINITION 28. *The graph constructed as above is called the* incidence graph *of T and it is denoted by* $(\Gamma, *)$.

Let $T = (X, *, t, I)$ be a pregeometry and let $\mathcal{B}$ be a partition of X such that, for each $B \in \mathcal{B}$, all the elements of B have the same type. The *quotient pregeometry* of T with respect to $\mathcal{B}$ is the pregeometry whose elements are the parts of $\mathcal{B}$ and two parts B_1 and B_2 are incident if there exists $x \in B_1$ and $y \in B_2$ such that $x * y$. The type function is inherited from T.

To somehow "tame" the concept of an incidence structure it is necessary to add a few more regularity conditions.

DEFINITION 29. *A* geometry *over I is a pregeometry T over I in which any maximal flag is a chamber.*

Observe that the above condition is equivalent to the fact that any nonmaximal flag can be extended to at least one chamber.

REMARK 5. There exists a forgetful functor from the category of geometries to the category of tolerance spaces.

3.3. Coset geometry

The construction of geometries via cosets goes back to the work of J. Tits [53]. Given a group G and a family of subgroups $(G_i)_{i \in I}$ it is possible to define a pregeometry $T := T(G, \{G_i, i \in I\})$ over I as follows. The elements of type i, for $i \in I$, are the right cosets $G_i g$. Two elements $G_i g$ and $G_j h$ are incident if and only if $G_i g \cap G_j h$ is nonempty.

PROPOSITION 2 (J. Tits). *Let G be a group and let $(G_i)_{i \in I}$ be a collection of subgroups of G and let*

- $X = \bigcup_{i \in I} G/G_i = \{G_i g \mid i \in I, g \in G\}$,
- $t : G_i g \in X \mapsto i \in I$,
- $* = \{(G_i g, G_j h) \mid G_i g \cap G_j h \neq \emptyset\}$.

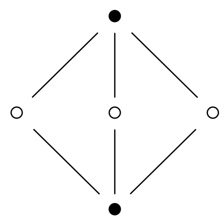
Then T is a pregeometry having a chamber.

Such a pregeometry is denoted by $T(G, \{G_i, i \in I\})$, but I will refer to it simply by T. It is worth observing that the chamber mentioned above is precisely $\{G_i, i \in I\}$.

DEFINITION 30. *If T is a geometry, it is called a* coset geometry.

EXAMPLE 12. Let $G = \mathbb{Z}_2 \oplus \mathbb{Z}_3 =< x, y | x^2 = y^3 = 1, xy = yx >$, $G_1 = \mathbb{Z}_2$, $G_2 = \mathbb{Z}_3$. Then,

$$\begin{aligned} G_1 = \mathbb{Z}_2 = \{(0,0), (1,0)\}, \quad & G/G_1 = \{G_1, (0,1) + G_1, (0,2) + G_1\}, \\ G_2 = \mathbb{Z}_3 = \{(0,0), (0,1)(0,2)\}, \quad & G/G_2 = \{G_2, (1,0) + G_2\}. \end{aligned}$$



I use different symbols to describe different types of elements: little circles for the cosets with respect to G_1 and full little circles for the cosets with respect to G_2.

Proposition 3. *The coset geometry $T = T(G, \{G_i, i \in I\})$ is* connected *if and only $\{G_i, i \in I\}$, taken together, generate the whole group G.*

Coset geometries can reveal many interesting properties. However, since they will not be used in this thesis I will skip them, suggesting the interested reader to consult the book [15].

The second original result of this thesis, due to Proposition 3, is a specialization of Theorem 2 to the case of a coset geometry. Let G be a group and H, K two its subgroups. Then $T(G, \{H, K\})$ is also a tolerance space (see Remark 5) and it can be considered the corresponding incidence graph Γ (see Definition 28).

Theorem 3. *Γ is disconnected i.e., $X = \bigcup_{i \in I} Hg_i = \bigcup_{j \in J} K\bar{g}_j \subsetneq G$ exists, if and only if X is a union of right cosets of the subgroup $S =< H, K >$.*

Proof. $\Leftarrow$
Obviously H and K are subgroup of S, therefore $S = \bigcup_{l \in L} Ha_l = \bigcup_{m \in M} Kb_m$, for some opportune sets L and M. Consequently, any union of left cosets of S is union of left cosets of H and union of left cosets of K.
$\Rightarrow$
Let $X = \bigcup_{i \in I} Hg_i = \bigcup_{j \in J} K\bar{g}_j$. Firstly I prove that, for any $x \in X$ and any $h \in H$, it is true that $hx \in X$. Indeed, $x \in Hg_i$ for some suitable g_i, therefore $Hx = Hg_i \subseteq X$. Analogously I have $kx \in X$ for any $x \in X$, $k \in K$, since $x \in K\bar{g}_j$ for some opportune $\bar{g}_j$ and therefore $Kx = K\bar{g}_j \subseteq X$.
For any $s \in S$ I have $s = h_1k_1 \cdots h_nk_n$ with $h_1, h_2, \cdots, h_n \in H$, $k_1, k_2, \cdots k_n \in K$. I will show that $sx \in X$ for any $x \in X$, $s \in S$. From the fact that $x \in X$ it follows that $k_nx \in X$, so $h_nk_nx \in X$. So the statement follows easily by induction on n. In particular $Sg_i \subseteq X$ for any $i \in I$, so that $\bigcup_{i \in I} Sg_i \subseteq X$ and from $X = \bigcup_{i \in I} Hg_i \subseteq \bigcup_{i \in I} Sg_i$ it follows that $X = \bigcup_{i \in I} Sg_i$, as it was expected. □

CHAPTER 4

Geometry of hyperbolic plane

The hyperbolic (sometimes improperly called non–Euclidean) geometry is the study of the spaces of constant negative curvature. It reveals many interesting features: some are similar to those of the Euclidean geometry but some are quite different, like the absence of the parallel postulate. In particular, it has a very rich group of isometries, allowing a huge variety of crystallographic symmetry patterns. To fully understand the subject, I will start with some basic surface theory, then I will introduce different models of the hyperbolic geometry, geodesics and such an important concept here like that of a *triangle*. A good source of information about the hyperbolic space is the book [**13**].

4.1. Preliminaries of theory of surfaces

I will recall here the notions of the fundamental forms, which are extremely important and useful in determining the metric properties of a surface, such as geodesics, area and curvature.

4.1.1. First and second fundamental forms.

DEFINITION 31. *A 2–dimensional smooth manifold (without boundary) M is called a* surface*.*

A surface is *compact* (resp., *connected*), if such is the underlying topological space. Let M be a surface, $p \in M$, and $\phi : I \to M$ a smooth curve through p.

DEFINITION 32. *The tangent vector to ϕ at p is called a* tangent vector *to M at p. The set of all tangent vectors to M at p is called the* tangent space *and it is denoted by T_pM.*

passing

EXAMPLE 13. Suppose that the surface $M \subseteq \mathbb{R}^3$ has the following parametrization:

$$\boldsymbol{x}(u,v) = (x(u,v), y(u,v), z(u,v)),$$

where $(u,v) \in U \subset \mathbb{R}^2$, U is open, and x, y, z are differentiable functions.

Let me introduce the following notation:

$$x_u = \frac{\partial x}{\partial u}(u,v)\,, \qquad y_u = \frac{\partial y}{\partial u}(u,v)\,, \qquad z_u = \frac{\partial z}{\partial u}(u,v)\,,$$
$$x_v = \frac{\partial x}{\partial v}(u,v)\,, \qquad y_v = \frac{\partial y}{\partial v}(u,v)\,, \qquad z_v = \frac{\partial z}{\partial v}(u,v)\,.$$

Then, for any $(u,v) \in U$, the vector–valued functions $\boldsymbol{x}_u = (x_u, y_u, z_u)$ and $\boldsymbol{x}_v = (x_v, y_v, z_v)$ give rise to a basis $\{\boldsymbol{x}_u(u,v), \boldsymbol{x}_v(u,v)\}$ of the tangent space $T_{\boldsymbol{x}(u,v)}M$. Let $M \subseteq \mathbb{R}^n$ be a surface.

DEFINITION 33. *The scalar product*

$$\mathrm{I}_p := \langle\ ,\ \rangle_p : T_pM \times T_pM \to \mathbb{R}\,,$$

obtained by restricting the standard Euclidean scalar product on $\mathbb{R}^n$*, is called the* first fundamental form *of the surface* M *at the point* p*.*

If $(U, \boldsymbol{x})$ is a chart on M, then the standard basis of T_pM is $\left\{ \frac{\partial}{\partial x^i}\Big|_p \right\}$. Hence, if $M \subseteq \mathbb{R}^n$, one can compute $g_{ij} = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle$, where $\langle \ , \ \rangle$ is the standard scalar product (from $\mathbb{R}^n$). From the symmetry of the scalar product, it follows that $g_{ij} = g_{ji}$. So, the scalar product on $\mathbb{R}^n$ induces a Riemann metric on M, which corresponds to the bilinear form whose matrix is $\|g_{ij}\|$, $1 \leq i, j \leq 2$.

REMARK 6. The above–defined quadratic form, determined by the matrix $\|g_{ij}\|$, coincides with the first fundamental form given by Definition 33.
It is common usage to denote such a matrix by

$$\|g_{ij}\| \equiv \left\| \begin{array}{cc} E & F \\ F & G \end{array} \right\| .$$

EXAMPLE 14. The torus with major radius R and minor radius r can be defined parametrically by

$$\begin{aligned} x(u,v) &= (R + r\cos v)\cos u \ , \\ y(u,v) &= (R + r\cos v)\sin u \ , \\ z(u,v) &= r\sin v \ . \end{aligned}$$

Then, the first fundamental form of the torus is given by

$$\begin{aligned} g_{11}((u,v)) &= (R - r\sin(v))^2 + r^2\cos^2(v) \ , \\ g_{12}((u,v)) &= g_{21}((u,v)) = 0 \ , \\ g_{22}((u,v)) &= (R + r\cos(v))^2 \ , \end{aligned}$$

for all $(u, v) \in \mathbb{R}^2$.

Now I have a tool to define the length of a curve on a surface.
Let $\gamma : [a, b] \to U \subset M$ be a differentiable curve. The length L of γ is equal to

$$L(\gamma) = \int_a^b \langle \gamma'(t), \gamma'(t) \rangle^{\frac{1}{2}} \ \mathrm{d}t = \int_a^b \left(\sum_{i,j}^{2} \left(g_{ij}(\gamma(t))\gamma_i'(t)\gamma_j'(t) \right) \right)^{\frac{1}{2}} \mathrm{d}t \, .$$

Let me denote by $|\boldsymbol{x}| := \langle \boldsymbol{x}, \boldsymbol{x} \rangle^{\frac{1}{2}}$ the *norm* of $\boldsymbol{x} \in T_pM$.

DEFINITION 34. *The curve* $\gamma = \gamma(t)$ *is* parametrised by the arch length *if* $|\gamma'(t)| = 1$ *for all* $t \in [a, b]$.

The angle θ between two curves $\gamma_1(t)$, $\gamma_2(t)$ intersecting at $t = t_0$ on the surface is given by

$$\cos\theta = \frac{\langle \gamma_1'(t_0), \gamma_2'(t_0) \rangle}{|\gamma_1'(t_0)||\gamma_2'(t_0)|} .$$

Since coordinate curves always have tangents $\boldsymbol{x}_u$ and $\boldsymbol{x}_v$, so the angle between these curves is

$$\cos\theta = \frac{\langle x_u, x_v \rangle}{|x_u||x_v|} = \frac{F}{\sqrt{EG}} \tag{17}$$

It is also possible to define the area by using the first fundamental form.

DEFINITION 35. *The* area *of a domain* $\boldsymbol{x}(U) \subset \mathbb{R}^3$ *of the surface parametrised by* $\boldsymbol{x}$ *is defined by*

$$\int_U \sqrt{EG - F^2}\,\mathrm{d}u\,\mathrm{d}v\,. \tag{18}$$

Let $f : M \longrightarrow N$ be a map between two surfaces, and I^M and I^N be the first fundamental forms of M and N, respectively. Denote by f_* the push–forward (or differential) of f, i.e., the linear map from T_pM to $T_{f(p)}N$ given, in the standard bases, by the Jacobian of f.

DEFINITION 36. *If*

$$\mathrm{I}^N_{f(p)}\left(f_*\boldsymbol{x}, f_*\boldsymbol{y}\right) = \mathrm{I}^M_p\left(\boldsymbol{x}, \boldsymbol{y}\right)$$

for all $\boldsymbol{x}, \boldsymbol{y} \in T_pM$ *and* $p \in M$*, then* f *is called an* isometry *from* M *to* N *(or of* M *if* $M = N$*).*

The second fundamental form relates to the way the surface sits in $\mathbb{R}^3$, though it is not independent on the first fundamental form, which is a mean to measure lengths and areas.
Given a surface $\boldsymbol{x}(u, v)$, and moving it by a parameter t along its normal vectors, one obtains the following one–parameter family of surfaces:

$$\boldsymbol{R}(u, v, t) = \boldsymbol{x}(u, v) - t\boldsymbol{n}(u, v)\,,$$

with

$$\boldsymbol{R}_u = \boldsymbol{x}_u - t\boldsymbol{n}_u \text{ and } \boldsymbol{R}_v = \boldsymbol{x}_v - t\boldsymbol{n}_v.$$

Now, having a first fundamental form $Edu^2 + 2Fdudv + Gdv^2$ depending on t it is possible calculate

$$\frac{1}{2}\frac{\partial}{\partial t}(Edu^2+2Fdudv+Gdv^2)|_{t=0} = -(\boldsymbol{x}_u\cdot\boldsymbol{n}_u du^2+(\boldsymbol{x}_u\cdot\boldsymbol{n}_v+\boldsymbol{x}_v\cdot\boldsymbol{n}_u)dudv+\boldsymbol{x}_v\cdot\boldsymbol{n}_v dv^2).$$

The right–hand side is the *second fundamental form.* From this point of view, it is clearly the same type of object as the first fundamental form—a quadratic form on the tangent space.

REMARK 7. Since $\boldsymbol{n}$ is orthogonal to $\boldsymbol{x}_u$ and $\boldsymbol{x}_v$,

$$0 = (\boldsymbol{x}_u \cdot \boldsymbol{n})_u = \boldsymbol{x}_{uu} \cdot \boldsymbol{n} + \boldsymbol{x}_u \cdot \boldsymbol{n}_u,$$

and similarly

$$0 = \boldsymbol{x}_{uv} \cdot \boldsymbol{n} + \boldsymbol{x}_u \cdot \boldsymbol{n}_v$$

$$\cdots\ ,$$

and, since $\boldsymbol{x}_{uv} = \boldsymbol{x}_{vu}$, then $\boldsymbol{x}_u \cdot \boldsymbol{n}_v = \boldsymbol{x}_v \cdot \boldsymbol{n}_u$.

DEFINITION 37. *The* second fundamental form *of a surface is the expression*

$$\mathrm{II} := Ldu^2 + 2Mdudv + Ndv^2\,, \tag{19}$$

where $L = \boldsymbol{x}_{uu} \cdot \boldsymbol{n}$, $M = \boldsymbol{x}_{uv} \cdot \boldsymbol{n}$, $N = \boldsymbol{x}_{vv} \cdot \boldsymbol{n}$.

4.1.2. Geodesics. Geodesics are the curves on a surface which are the analogues of the straight lines in the plane.

DEFINITION 38. *A* geodesic *on a surface X is a curve $\gamma(s)$ on X such that t' (where t is the unit tangent to γ) is normal to the surface.*

To find the geodesics, in general one needs to solve a nonlinear system of ordinary differential equations.

PROPOSITION 4. *A curve $\gamma(s) = (u(s), v(s))$ on a surface parametrised by arc length (see Definition 34) is a geodesic if and only if*

$$\frac{d}{ds}(Eu' + Fv') - \frac{1}{2}(E_u u'^2 + 2F_u u'v' + G_u v'^2)\,, \tag{20}$$

$$\frac{d}{ds}(Fu' + Gv') - \frac{1}{2}(E_v u'^2 + 2F_v u'v' + G_v v'^2). \tag{21}$$

It is clear from the definition above that the geodesics only depend on the first fundamental form, so that the geodesics can be defined for abstract surfaces. Moreover an *isometry* (see Definition 36) takes geodesics to geodesics.

4.1.3. Gauss curvature. In familiar parlance, the *principal curvatures* of a surface at a point p are the minimum and the maximum of all the curvatures of the curves passing through p. They correspond to the eigenvalues of the second fundamental form.

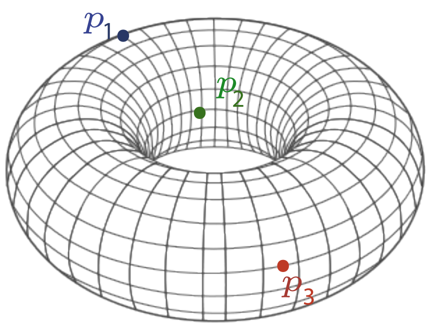
DEFINITION 39. *The* Gaussian curvature $K(p)$ *of a surface at a point p is the product of the principal curvatures, k_1 and k_2, at p, i.e.,*

$$K(p) = k_1(p) \cdot k_2(p)\,.$$

The Gaussian curvature of a surface in $\mathbb{R}^3$ can be also calculated using the first and the second fundamental forms:

$$K(p) = \frac{\det \mathrm{II}_p}{\det \mathrm{I}_p}\,.$$

EXAMPLE 15. Consider the torus depicted below.

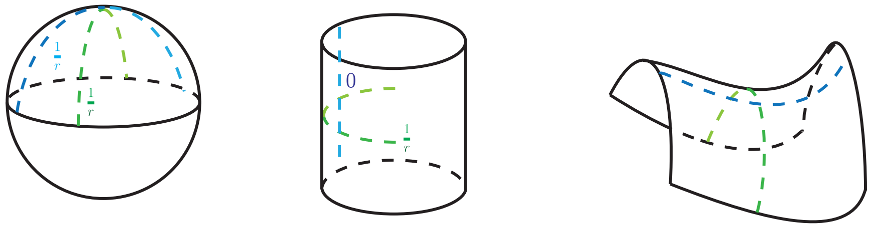


Notice that in the point p_3 there is a positive Gaussian curvature $K(p_3) = \frac{1}{rR}$, in p_2 it is negative, $K(p_2) = -\frac{1}{rR}$, and in p_3, $K(p_1) = 0$.

4.1.4. Surfaces of constant Gauss curvature. A surface S has *constant Gauss curvature*, if K is constant on S.

The unique compact surface with constant positive Gauss curvature K is the sphere of radius $\frac{1}{\sqrt{K}}$. For the Euclidean space, the Gaussian curvature is $K = 0$, but also for such *flat surfaces* of revolution as the cylinder and the cone (in both cases, only one principal curvature is zero, whereas they both vanish in the Euclidean space).

The so–called *Lobachevsky plane* is an example of a surface of negative Gauss curvature.



Sphere of radius r $K > 0$ | Cylinder of radius r $K = 0$ | "Saddle" $K < 0$

4.1.5. The group of rigid transformations. Because of their geometric nature, the bijections of the plane into itself are called *transformations*. Those of them which preserve distances and angles are called *rigid motions*. Obviously, the identity transformation is a rigid motion, since it carries every point of the plane into itself, and the composition of rigid motions is again a rigid motion. This pushes to define the *group of rigid transformation* as the set off all rigid motions, with composition as the group operation and the identity map as the identity element. It is denoted by $\mathrm{E}(2)$ and it is the semidirect product of $\mathrm{SO}(2)$ and $\mathbb{R}^2$ (see Example 4). Let me mention some specific rigid motions.

A *translation* τ of the Euclidean plane is a rigid motion which sends any segment PP' to another segment QQ' of the same length and direction, with $Q = \tau(P)$ and $Q' = \tau(P')$. Let me state the following propositions, whose proof can be found in [**51**].

Proposition 5. *For any two points A and B there is a unique translation, denoted by τ_{AB}, that carries A onto B.*

Proposition 6. *If A, B, C are any three points, then $\tau_{BC} \circ \tau_{AB} = \tau_{AC}$.*

Proposition 7. *The inverse of translation τ_{AB} is the translation τ_{BA}.*

Let (C, α) be a pair, where C is a point and α an oriented angle. The *rotation* $R_{C,\alpha}$ is the map that associates to any point P the unique point P' such that $\|CP\| = \|CP'\|$ and $\angle PCP' = \alpha$. Of course, for any C, $R_{C,0}$ is identity map. The inverse of the rotation $R_{C,\alpha}$ is the rotation $R_{C,-\alpha}$ for any (C, α).

4.2. Möbius transformations

Let $a, b, c, d \in \mathbb{C}$ be complex numbers.

Definition 40. *The transformation*

$$f(z) := \frac{az+b}{cz+d}, \quad \text{where } |a| + |c| > 0, \; ad - bc \neq 0, \tag{22}$$

is called a Möbius transformation.

If $c = 0$ then the transformation is linear. If $c \neq 0$ and $a = 0$ then the transformation is a so–called *inversion*.

Any Möbius transformation is equivalent to a sequence of simpler transformations. To show this, I need the following transformations:

$$f_1(z) = z + \frac{d}{c}\,, \qquad \text{(translation by } d/c)\,,$$
$$f_2(z) = \frac{1}{z}\,, \qquad \text{(inversion and reflection with respect to the real axis)}\,,$$
$$f_3(z) = \frac{bc - ad}{c^2} z\,, \qquad \text{(homotethy and rotation)}\,,$$
$$f_4(z) = z + \frac{a}{c}\,, \qquad \text{(translation by } a/c)\,.$$

Composing the above transformations, one gets

$$f_4 \circ f_3 \circ f_2 \circ f_1(z) = f(z) = \frac{az + b}{cz + d}\,.$$

From this decomposition arise that Möbius transformations carry over all non–trivial properties of the circle inversion. For example, the preservation of angles boils down to prove that the circle inversion preserves the angles, since the other types of transformations are the dilation and the isometries (translation, reflection, rotation), which trivially preserve the angles.
The existence of the inverse of a Möbius transformation and its explicit formula are easily derived by the composition of the inverse functions of the simpler transformations. That is, just define the maps g_1, g_2, g_3, g_4 such that each g_i is the inverse of f_i. Then the composition

$$g_1 \circ g_2 \circ g_3 \circ g_4(z) = f^{-1}(z) = \frac{dz - b}{-cz + a}$$

gives a formula for the inverse of f, which is again a Möbius transformation.
Furthermore, it can be easily checked that the composition of two transformations, $f_1(z) := \frac{a_1 z + b_1}{c_1 z + d_1}$ and $f_2(z) := \frac{a_2 z + b_2}{c_2 z + d_2}$, is again a Möbius transformation, namely:

$$f_2 \circ f_1(z) = \frac{(a_1 a_2 + b_2 c_1)z + a_2 b_1 + b_2 d_1}{(a_1 c_2 + c_1 d_2)z + b_1 c_2 + d_1 d_2}\,.$$

On the other hand, $f \circ I = I \circ f = f$, where I is the identity map.

Corollary 3. *The set of all the Möbius transformations is a group with respect to the composition, isomorphic to* $\mathrm{PSL}\,(2, \mathbb{C})$.

It worth stressing that $\mathrm{PSL}\,(2, \mathbb{C})$ does not admits such an easy decomposition as that of $\mathrm{E}(2)$ (see Section 4.1.5 before).

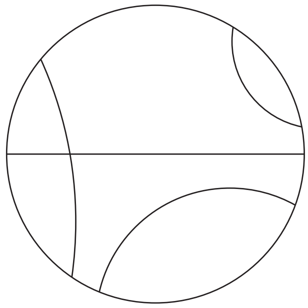
4.3. Models

In order to visualise the hyperbolic geometry, its necessary to resort to a model. The most commonly used are the *Klein model*, the *Lorentz model*, the *Poincaré disk* and the *upper half–plane* model.

The *Klein model* is a model in which the points are represented by the points in the interior of the n–dimensional unit ball (or unit disk, in two dimensions) and the lines are represented by the chords, straight line segments with endpoints on the boundary sphere (or circle, in two dimensions). This model has the advantage of simplicity, but the disadvantage that the angles in the hyperbolic plane are distorted.

The *Lorentz model* or *hyperboloid model* employs a 2–dimensional hyperboloid of revolution (which is made of two sheets, even if only one is needed) embedded in the 3–dimensional Minkowski space. This model has direct applications to special relativity, as the Minkowski 3–space is a model for the space–time, suppressing one spatial dimension.

In the *Poincaré disk* the underlying space is the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Every hyperbolic line in $\mathbb{D}$ is the intersection of $\mathbb{D}$ with a circle in the extended complex plane perpendicular to the unit circle bounding $\mathbb{D}$. Moreover, every such intersection is a hyperbolic line. A curiosity is that this model appears in the artworks of M. C. Escher depicting the tessellation of the hyperbolic plane.



Lines in $\mathbb{D}$

Even if the Poincaré disk and the upper half–plane model are in fact almost equivalent, I will focus on the latter.

4.3.1. The upper half–plane model. The *upper half–plane model* (also know as *Lobachevsky plane* or Poincaré half–plane), is the upper half–plane $\mathbb{H} = \{z \in \mathbb{C} \mid \Im z > 0\}$, with the Poincaré metric $ds = \frac{|dz|}{\Im z}$. This simply says that all the complex numbers whose imaginary part is strictly positive, i.e., with $z = x + yi$, then $b > 0$, are points here, and that the metric is $(ds)^2 = \frac{(dx)^2+(dy)^2}{y^2}$. Thus the hyperbolic length of a vector v at a point $z \in \mathbb{H}$ is just its Euclidean length divided by the height $\Im z$ of its location z. According, all the distances widen when approaching the x–axis.

Seemingly, there are two different types of hyperbolic lines, both defined in terms of Euclidean objects in $\mathbb{C}$. One is the intersection of the half–plane with an Euclidean line in the complex plane perpendicular to the real axis $\mathbb{R}$. The other one is the intersection of $\mathbb{H}$ with an Euclidean circle with centre on the real axis $\mathbb{R}$. Two hyperbolic lines in $\mathbb{H}$ are parallel if they are disjoint.
The notion of an angle in $\mathbb{H}$ is the one inherited from $\mathbb{C}$, so that the angle between two curves is the angle between their tangent lines. A circle (i.e., the curve made of the points which are equidistant from a central point) with centre $(x, y) \in \mathbb{H}$ and radius $R \in \mathbb{R}$ is modeled by a circle with centre $(x, y \cosh R)$ and radius $y \sinh R$.

4.3.1.1. *Geodesics and isometries.* H. Poincaré (1881) realized that the isometries of the half–plane model are exactly the Möbius transformation (as defined in 4.2)

$$z \mapsto w = \frac{az + b}{cz + d}, \tag{23}$$

with $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$. This makes it particularly easy to study such isometries and to perform computations.

For instance, if I substitute

$$w = \frac{az + b}{cz + d} \quad \text{and} \quad dw = \left(\frac{a}{cz - d} - \frac{c(az + b)}{(cz + d)^2}\right) dz = \frac{ad - bc}{(cz + d)^2} dz$$

into

$$\frac{dx^2+dy^2}{y^2}=\frac{4\,|dw|}{|w-\bar{w}|}\,,$$

I get

$$\frac{4(ad-bc)\,|dz|^2}{|(az+b)(c\bar{z}+d)(a\bar{z}+b)(cz+d)|^2}=\frac{4(ad-bc)\,|dz|^2}{|(ad-bc)(z-\bar{z})|^2}=\frac{4\,|dz|^2}{|(z-\bar{z})|^2}=\frac{dx^2+dy^2}{y^2}\,.$$

Thus the Möbius transformation $z \longmapsto w$ is an isometry of $\mathbb{H}$. And so is the transformation $z \mapsto -\bar{z}$, and hence the composition

$$z \mapsto \frac{b-a\bar{z}}{d-c\bar{z}} \tag{24}$$

is also an isometry from of $\mathbb{H}$. In fact, (23) and (24) encompass all the isometries of $\mathbb{H}$.

Below I describe all the geodesics of $\mathbb{H}$.

PROPOSITION 8. *The geodesics in the upper half–plane model are either the vertical lines, or the arcs of the circles with centre on* $\mathbb{R}\cup\infty=\partial\mathbb{H}$.

PROOF. First, I will find the geodesic equations, which can be easily solved, since $E=G=1/y^2$ and $F=0$, and these are independent on x. The first geodesics equation (20)

$$\frac{d}{ds}(Eu'+Fv')-\frac{1}{2}(E_u u'^2+2F_u u'v'+G_u v'^2)$$

becomes

$$\frac{d}{ds}\left(\frac{x'}{y^2}\right)=0\,,$$

and so

$$x'=cy^2. \tag{25}$$

Knowing that the parametrisation is by arc length in these equations (see Proposition 4), one has

$$\frac{x'^2+y'^2}{y^2}=1\,. \tag{26}$$

If $c=0$ one gets $x=const.$, which is a vertical line. If now $c\neq 0$, then from (25) and (26) one obtains

$$\frac{dy}{dx}=\sqrt{\frac{y^2-c^2y^4}{c^2y^4}}\,,$$

and then

$$\frac{cydy}{\sqrt{1-c^2y^2}}=dx\,,$$

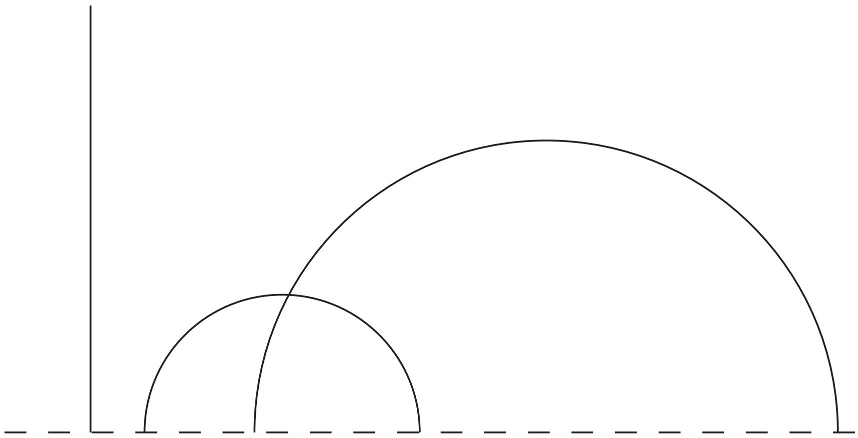
which integrates directly to

$$-\frac{1}{c}\sqrt{1-c^2y^2}=x-a\,,$$

i.e.,

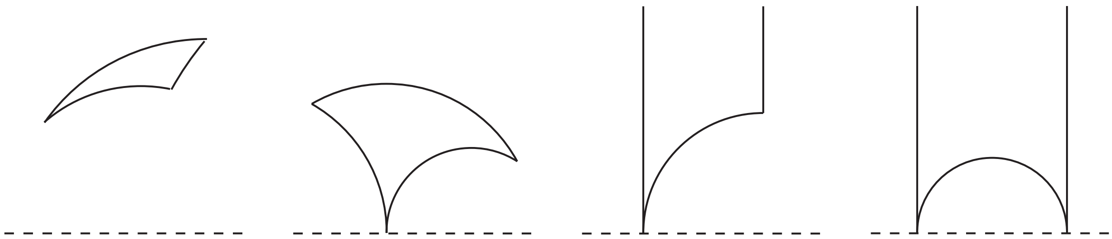
$$(x-a)^2+y^2=\frac{1}{c^2}\,,$$

which is a semicircle with centre on the real axis. □



4.3.1.2. *Hyperbolic triangles.* As in the Euclidean space, the triangles are the most fundamental polygons to understand. The first thing to notice here is that the sum of the angles of a hyperbolic triangle cannot exceed π. Of course, the hyperbolic angles (expressed in terms of tangents, see (17)) are the same as the Euclidean angles, since their first fundamental form satisfies $E = G$ and $F = 0$ (see 4.1.1).

Suppose that 3 points in the hyperbolic plane $\mathbb{H}$ are given. A *triangle* with these points as vertices is the set of three geodesic segments (i.e., the part of a geodesic between two given points on it) with these three points as endpoints. An *ideal triangle* is one with all the three vertices on the boundary $\partial\mathbb{H}$.



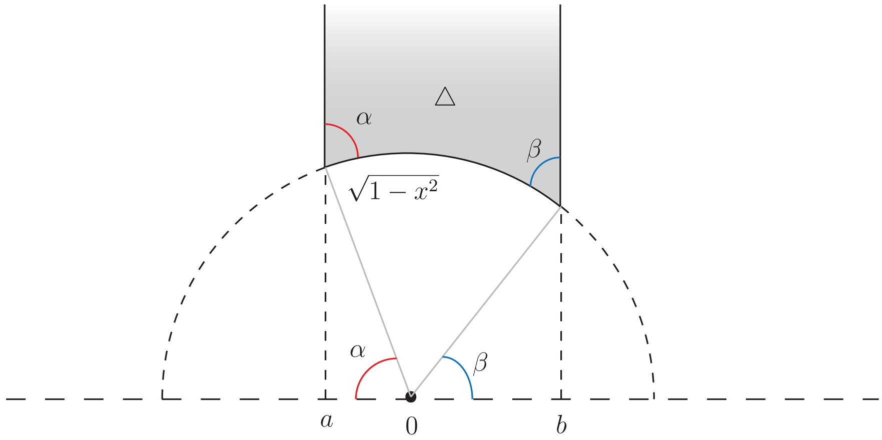
DEFINITION 41. *The* defect *of a triangle is the difference between π and the sum of its interior angles.*

The area of a triangle is bounded and cannot exceed π. Even more, it is exactly equal to its defect.

THEOREM 4 (Gauss–Bonnet theorem for a hyperbolic triangle). *Let $\triangle$ be a hyperbolic triangle with internal angles α, β and γ. Then*

$$\text{Area}_{\mathbb{H}}(\triangle) = \pi - (\alpha + \beta + \gamma)\,. \tag{27}$$

REMARK 8. The Gauss–Bonnet formula (27) implies that the area of a hyperbolic triangle is at most π. The only way in that the area of a hyperbolic triangle can be equal to π is that all the internal angles are equal to zero, i.e., the triangle is an ideal one.

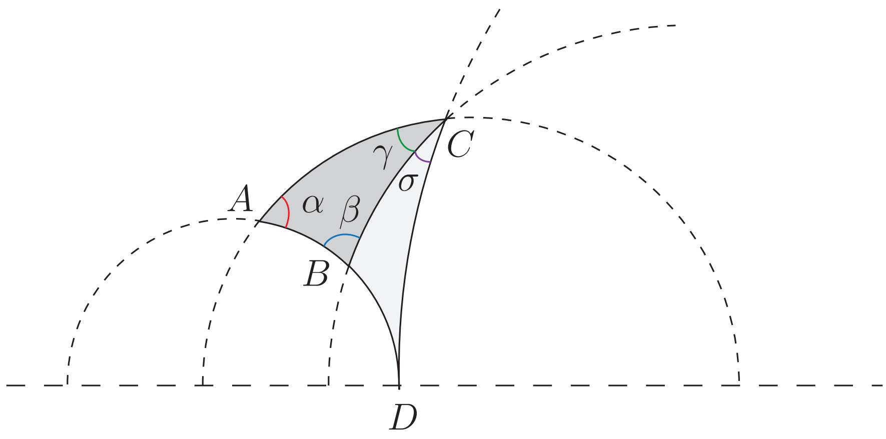


Proof of Theorem 4. Let $\triangle$ be a hyperbolic triangle with internal angles α, β and γ.

Firstly, I will consider the case when at least on of the vertices of $\triangle$ belongs to $\partial\mathbb{H}$, i.e., the angle at this vertex is 0, and by a Möbius transformation it can be mapped to ∞ without changing the area (equivalently, the angles) of $\triangle$. By applying the Möbius transformation $z \mapsto z + b$, for a suitable b, the centre of the circle joining the other two vertices goes to the origin of $\mathbb{C}$ and, moreover, applying the transformation $z \mapsto kz$, for an appropriate k, its radius becomes 1. Hence, following (18),

$$
\begin{aligned}
\mathrm{Area}_{\mathbb{H}}(\triangle) &= \iint_{\triangle} \frac{1}{y^2} dx dy \\
&= \int_a^b \int_{\sqrt{1-x^2}}^{\infty} \frac{1}{y^2} dy dx \\
&= -\int_a^b \left. \frac{1}{y} \right|_{\sqrt{1-x^2}}^{\infty} dx \\
&= \int_b^a \frac{1}{\sqrt{1-x^2}} dx \\
&= \int \frac{-\cos\theta}{\cos\theta} d\theta, \quad \text{replacing } x = \cos\theta \\
&= -\int_{\pi-\alpha}^{\beta} 1 d\theta \\
&= \pi - (\alpha + \beta)\,.
\end{aligned}
$$

This proves (27) for a triangle with one of the vertices lying on $\partial\mathbb{H}$.
To deal with the case in which the triangle $\triangle$ does not have any of the vertices A, B, C at infinity, the trick is to prolong the geodesic segment connecting A and B until it intersects the real axis at a point, say D, and draw the geodesic segment from C to D, as illustrated below.



Now it is just enough to notice that the area of triangle ABC is equal exactly the difference of the areas of ACD and BCD. So,

$$\begin{aligned}\mathrm{Area}_{\mathbb{H}}(\triangle) =& \mathrm{Area}_{\mathbb{H}}(ABC) = \mathrm{Area}_{\mathbb{H}}(ACD) - \mathrm{Area}_{\mathbb{H}}(BCD)\\ =& \pi - (\alpha + (\gamma + \sigma)) - (\pi - ((\pi - \beta) + \sigma))\\ =& \pi - \alpha - \gamma - \sigma - \beta + \sigma\\ =& \pi - (\alpha + \beta + \gamma)\,.\end{aligned}$$

□

4.4. Fuchsian groups

As I noticed in the previous section, the collection of all Möbius transformations of $\mathbb{H}$ forms a group, called the *Möbius group*. If I require that the coefficients a, b, c, d of the Möbius transformations must be real numbers with $ad - bc = 1$, then I obtain a subgroup of the Möbius group denoted by $\mathrm{PSL}\,(2,\mathbb{R})$. Observe that $\mathrm{PSL}\,(2,\mathbb{R})$ is a three–dimensional real Lie group contained into the three–dimensional complex Lie group $\mathrm{PSL}\,(2,\mathbb{C})$ (see Corollary 3).

Definition 42. *A* Fuchsian group *is a discrete subgroup of* $\mathrm{PSL}\,(2,\mathbb{R})$.

Remark 9. Observe that a Fuchsian group can be considered as a zero–dimensional Lie group (if it is countable), so that the Lie algebra methods are of no use here.

Since $\mathrm{PSL}\,(2,\mathbb{R})$ is, by definition, the group of isometries of the hyperbolic plane $\mathbb{H}$, also a Fuchsian group can be regarded as a group acting on $\mathbb{H}$ (see Section 6.1 later on for the notion of a group action).

CHAPTER 5

Tessellations of constant curvature surfaces

Everybody knows how a chessboard look like: this is a simple example of a *regular tessellation* of the plane by squares. But how to define, in general, a tessellation? Intuitively, very informally, it is just a covering of the plane by a countable family of closed subsets, with no gaps nor overlappings. Even if such a concept seems very elementary, it can help to visualise other objects closely related to it, and its properties may bring forth new results.

I will deal with tessellations by regular polygons even if, in principle, a tessellation can be also made of different geometric, and more artistic, shapes like animals and other natural objects (think of the work of M. C. Escher [19]). The more natural tessellation is the one of the Euclidean plane but, obviously, the very concept of a tessellation can be generalized to any space having a group of isomorphisms with the same properties of rigid motions (see e.g., [5]).

5.1. Basic definitions

From now on, the symbol $\mathbb{S}$ will denote a constant–curvature surface, i.e., $\mathbb{S} = S^2, \mathbb{R}^2$, or $\mathbb{H}$ (see 4.1.4). From now on, by a *line* in $\mathbb{S}$ I simply mean a geodesic (see 4.1.2).

DEFINITION 43. *A* polygon $P \subset \mathbb{S}$ *is a closed region which is the union of an open set* P' *and* $\partial P'$*, the boundary of* P'*, such that* $\partial P'$ *is a finite union of line segments, called* edges *(appropriate to the space where* P' *sits in), satisfying the following properties:*

- *two edges are either disjoint or intersect each other in exactly one point, which is an endpoint for both edges,*
- *the point of intersection of two edges, which is called a* vertex*, is an element of exactly two edges.*

The *1–skeleton* of a polygon P is the set of vertices and edges of P. Since it coincides with the boundary of P, I will denote it by ∂P

DEFINITION 44. *A* tessellation $\mathcal{T}$ *of a surface* $\mathbb{S}$ *is a subdivision of the surface into polygonal* tiles t_i*, such that the tiles have the following properties:*

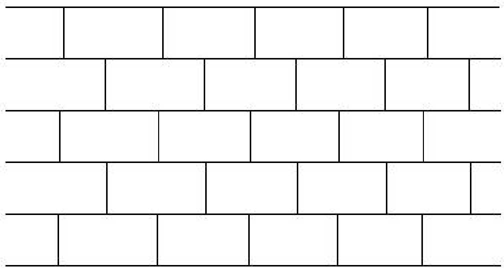
- *if* t_i *and* t_j *are not the same then only one of the following is true:*
  - * $t_i \cap t_j = \emptyset$,
  - * $t_i \cap t_j = \{v\}$*, where* v *is a point in* $\mathbb{S}$*,*
  - * $t_i \cap t_j = e$*, where* e *is a line segment in* $\mathbb{S}$*,*
- *given any point* $p \in \mathbb{S}$*, there is at least one tile* t_i *such that* $p \in t_i$*.*

DEFINITION 45. *If the tessellation* $\mathcal{Q}$ *is made of the same tiles of* $\mathcal{T}$*, then it is called a* sub–tessellation*.*

I will write $\mathcal{Q} \subseteq \mathcal{T}$ to mean that $\mathcal{Q}$ is a sub–tessellation of $\mathcal{T}$. Attributes like *connected*, *convex*, *finite*, etc., should be self–explanatory in the context of tessellations, and I will not insist on them

DEFINITION 46. *A tessellation is* regular *if the tiles are all the same shape.*

In other words, $\mathcal{T}$ is regular if, given any two tiles t_i and t_j, there exists an isometry $T_{i,j}$ of $\mathbb{S}$ (see Definition 36) such that $T_{i,j}(t_i) = t_j$.



Definition 44 allows for some awkward and inconvenient tessellations, like the one displayed above, which I would like to avoid.

DEFINITION 47. *A tessellation in* standard form *is a tessellation with the following properties:*

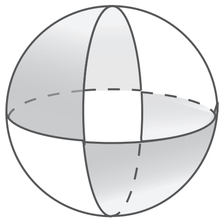
- *if t_i and t_j are not the same then only one of the following is true:*
  * $t_i \cap t_j = \emptyset$,
  * $t_i \cap t_j = \{v\}$, *where v is a point in $\mathbb{S}$, and v is a vertex of both t_i and t_j,*
  * $t_i \cap t_j = e$, *where e is a line segment in $\mathbb{S}$, and e is an entire edge of both t_i and t_j,*
- *given any point $p \in \mathbb{S}$, there is at least one tile t_i such that $p \in t_i$:*
  * *if p is in exactly one tile, then p is in the interior of a tile,*
  * *if p is in exactly two tiles, then p is in an edge,*
  * *if p is in more than two tiles, then p is a vertex.*

DEFINITION 48. *The* 1–skeleton *of a tessellation is the union of the boundaries of all its tiles.*

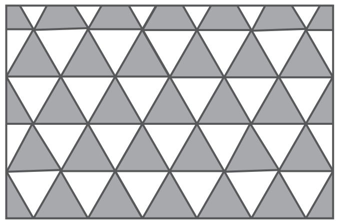
REMARK 10. The 1–skeleton of a tessellation $\mathcal{T}$ should not be confused with the *boundary* $\partial\mathcal{T}$, in the strict topological sense, of $\mathcal{T}$, which is an its subset (unless there is only one tile). In order to avoid the inception of a new symbol, I will keep using $\partial\mathcal{T}$ also for the 1–skeleton of $\mathcal{T}$.

From now on I will deal only with regular tessellations in standard form. The basic one, for my purposes, will be a tessellation by regular triangles of angle $\frac{\pi}{n}$, denoted $\mathcal{T}_n$.

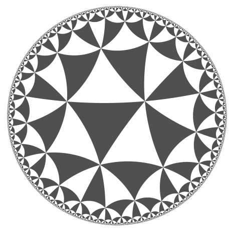
Obviously, $\mathcal{T}_2$ exist only on S^2, $\mathcal{T}_3$ only on $\mathbb{R}^2$, and all the $\mathcal{T}_n$'s, with $n > 3$, exist only on $\mathbb{H}$.



$\mathcal{T}_2$



$\mathcal{T}_3$



$\mathcal{T}_4$

Similarly, like with the von Dyck group (see Section 1.2), suppose that there is a distinguished triangle $\triangle_0 \in \mathcal{T}$, called *basic*, and a distinguished vertex $O \in \triangle_0$ called the *centre* (of the tessellation). Due to the homogeneity (see Remark 12) of the whole structure, such choices are by no means restrictive.

In cthe ase of $\mathcal{T}_n$ I will denote the basic triangle by $\triangle_n$. The *basic* $2n$*–gon* P_n of a tessellation $\mathcal{T}_n$ is the union of all the triangles obtained by rotating the basic triangle $\triangle_n$ around the centre O of the tessellation.

5.2. The coloured graph associated with a tessellation

LEMMA 4. *A unique* 3*–colours*[1] *complete graph* $\partial\mathcal{T}_n \in \texttt{C-Gra}_3$ *exists whose vertices* $V_{\partial\mathcal{T}_n}$ *(resp., edges* $E_{\partial\mathcal{T}_n}$*) are the vertices (resp., edges) of* $\mathcal{T}_n$*.*

PROOF. The existence of the graph is obvious. The colouring function $\chi : V_{\partial\mathcal{T}_n} \to \{\bullet, \bullet, \bullet\}$ can be arbitrarily defined on the vertices of the basic triangle. Suppose now that χ has been defined on a subgraph $\Gamma' \subset \partial\mathcal{T}_n$, and let $\triangle \in \mathcal{T}_n$ a triangle with two vertices in Γ': then there is a unique way to define χ on $\triangle$. Since $\partial\mathcal{T}_n$ is an object in $\texttt{C-Gra}_3$, it is enough to define χ up to chromatic permutations. □

Starting from the basic $2n$–gon P_n, I can define a unique $2n$–gonal tessellation $\mathcal{P}_n$ of $\mathbb{S}$.

DEFINITION 49. P_n *is the* basic $2n$–gon *of* $\mathcal{P}_n$*, and* O *its* centre*.*

Observe that O is not a vertex of P_n. Now I define the unique 2–colours (i.e., bipartite, see Section 2.1) subgraph $\partial\mathcal{P}_n \subseteq \partial\mathcal{T}_n$, by taking only the vertices and the edges which belong to $\mathcal{P}_n$. Observe that only $\mathcal{T}_2$ is finite, consisting of 8 triangles; hence $\mathcal{P}_2$ consists of 2 squares, namely the upper and the lower hemisphere of S^2.

Therefore, on a constant–curvature surface $\mathbb{S}$ there is a 2–colours graph $\partial\mathcal{P}_n$ (see Remark 10), whose edges are geodesic segments of a fixed length, which originated as the 1–skeleton of a regular $2n$–gonal tessellation $\mathcal{P}_n$, whose tiles, in turn, were obtained by the (regular) triangle tiles of the regular triangular tessellation $\mathcal{T}_n$. This last observation is a compulsory step to make the von Dyck group $D(n, n, n)$ (see 1.2) act on the 2–colour graph $\partial\mathcal{P}_n$.

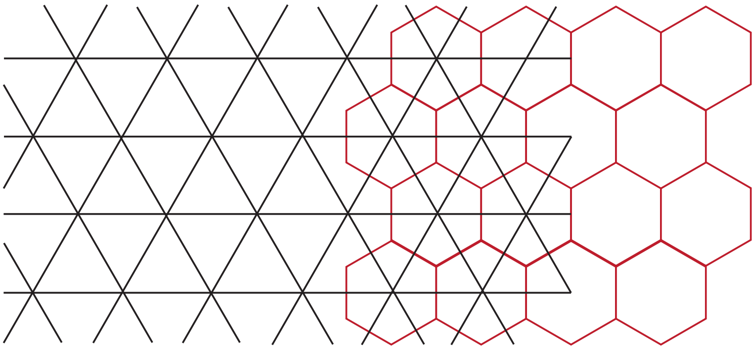
It is worth mentioning that $\partial\mathcal{P}_n$ is made of *cycles* consisting of $2n$ vertices, of alternating colours (they correspond exactly to the boundaries of the tiles of $\mathcal{P}_n$): a transformation (in the sense of graph automorphism, see Definition 15) of $\partial\mathcal{P}_n$

[1] In the category–theoretic sense explained in 2.3.3.

cannot be realized as a rotation of an angle $\frac{\pi}{n}$, since it would switch the vertex colours.

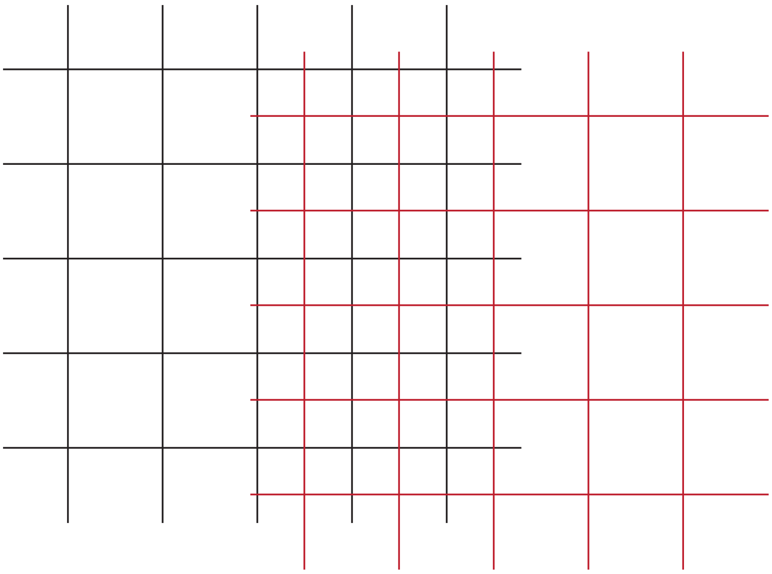
5.3. Dual tessellation

The *dual* $\mathcal{T}^*$ of a regular tessellation $\mathcal{T}$ is obtained by taking the centre of each polygon as a vertex and by joining the centres of adjacent polygons.



Similarly, like in graph theory (see 2.2), the dual to a dual tessellation is once again the initial tessellation, i.e., $(\mathcal{T}^*)^* = \mathcal{T}$.

As it can be seen from the figure above, the triangular and the hexagonal tessellations are dual to each other, and the square tessellation depicted below is self–dual.



The dual to a tessellation by regular convex polygons is again a regular tessellation.

Instead of such a standard duality, I will consider a construction inspired by graph theory (see Section 2.2 and especially Definition 20). Given a regular polygonal tessellation $\mathcal{T}$, there is a unique tessellation $\mathcal{T}'$ whose vertices are the middle points of the edges of the tiles of $\mathcal{T}$, and $\mathcal{T}'$ is called the *derived tessellation*.

Notice that in the case of a $2n$–gonal tessellation $\mathcal{P}_n$ (see Definition 49), P'_n is made both of $2n$–gons and n–gons, i.e., it is not *regular* in the sense of Definition 46 above.

5.4. Examples

To describe a tessellation it is often used the so–called "Schläfli symbol", $\{p, q\}$, where q is the number of p–gons on every vertex of the tessellation.
For the Euclidean plane, the measure of the interior angle of a regular polygon is $(1 - 2/p) \cdot \pi$, so that it comes naturally the equality

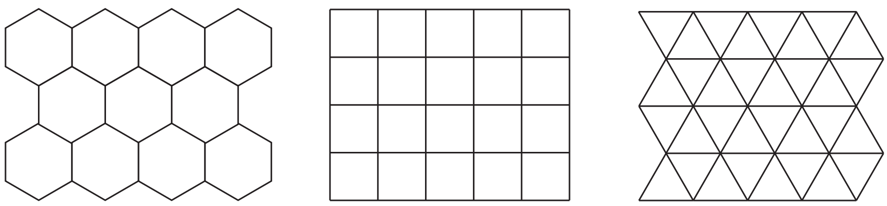
$$(1 - \frac{2}{p}) \cdot \pi = \frac{2\pi}{q},$$

whence

$$(p - 2)(q - 2) = 4.$$

The only possible factorisations are

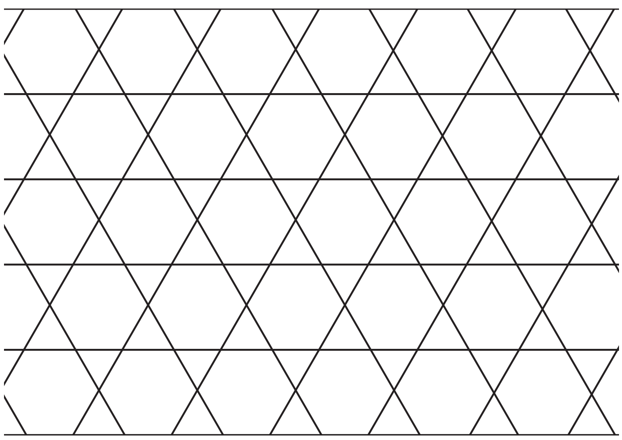
$$\begin{aligned} 4 =& 4 \cdot 1 = (6 - 2)(3 - 2) \Rightarrow \{6, 3\}\,, \\ =& 2 \cdot 2 = (4 - 2)(4 - 2) \Rightarrow \{4, 4\}\,, \\ =& 1 \cdot 4 = (3 - 2)(6 - 2) \Rightarrow \{3, 6\}\,. \end{aligned}$$



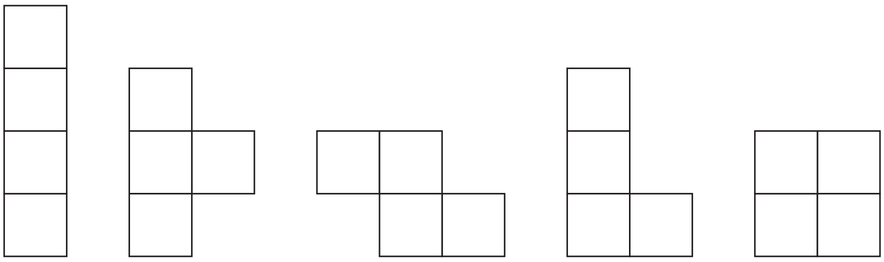
{6,3} {4,4} {3,6}

Therefore, in the Euclidean plane, there are only three regular tessellations (made of hexagons, squares, and triangles). It is worth mentioning that the dual (in sense of standard duality) to the regular tessellation $\{p, q\}$ is exactly $\{q, p\}$.

A *semiregular tessellation* is a tessellation of the plane by two or more convex regular polygons such that the same polygons in the same order surround each polygon vertex. In this sense, the derived tessellation is semiregular.



There are interesting problems that can be formulated in terms of regular tessellations. For instance, let me consider the polygons that have been created from a set of four connected congruent squares. They are called *tetrominoes*, and it can be proved that there are exactly five tetrominoes. One can try to establish which tetromino can be used to tessellate the plane.



A popular use of tetrominoes can be found in the videogame “Tetris”.

CHAPTER 6

Group actions on geometric structures

For a better understanding of the structure of a group G it is sometimes useful to study an *action* of G on an appropriate geometric object. This is the idea which will be developed in the present and in the following sections.

A *group action* can be thought of as an extension of the idea of the symmetry group of a set X, in which every element of the group "acts" like a bijective transformation (or "symmetry") of X, without being identified with that transformation. This allows for a more comprehensive description of the symmetries of an object, such as a polygon, by letting the same group act on several different sets of features, such as the set of the vertices and the set of the edges of a polygon and related structures, like regular tessellations and graphs.

6.1. Group actions

Let me recall here some basic notions concerning group actions.

6.1.1. Group actions, orbits, stabilizers. Let G be a group, e its identity element, and X a set. Denote by $(S_X, \circ)$ the group of the permutations of the set X.

DEFINITION 50. *A group homomorphism*

$$G \xrightarrow{\sigma} S_X$$

is called a representation *of the group G on the set X.*

DEFINITION 51. *A* (left) group action *of G on X is a binary operation*

$$G \times X \to X\,,$$
$$(g, x) \mapsto g * x\,,$$

such that

(1) $g * (h * x) = (gh) * x\ , \quad \forall g, h \in G\,,\ \forall x \in X,$

(2) $e * x = x\ , \quad \forall x \in X.$

It can be verified that the representations of G on X are in 1–1 correspondence with the group actions of G on X.

REMARK 11. If G acts on X, then the latter is sometimes called a G–set (or G–space, or a G–structure).

EXAMPLE 16. The automorphism group of a graph (see Definition 15) acts on the set of the vertices of the graph.

Definition 52. *The image of the map*

$$G \to X,$$
$$g \mapsto g * x,$$

is called the orbit of x *and it is denoted by* $G * x$*, or simply by* Gx*.*

Definition 53. *The set of all the orbits of* X *under the action of* G *is written as* X/G *, and it is called the* orbit space*. The natural map*

$$\begin{array}{ccc} X & \longrightarrow & X/G\,, \\ x & \longmapsto & Gx\,, \end{array}$$

is called the canonical projection*.*

Remark 12. If X/G consists of only one orbit, i.e., $X = Gx$, for some $x \in X$, then X is, in some contexts, called a *(G–)homogeneous space*.

Definition 54. *The subset*

$$G_x := \{g \in G \ : \ g * x = x\} \subseteq G$$

is called the stabilizer *of* x*.*

Obviously, G_x is a subgroup of G.

Notice that, if two elements $x, y \in X$ belong to the same orbit, then their stabilisers G_x and G_y are isomorphic (they are actually conjugate). More precisely, if $y = g * x$ then $G_y = gG_xg^{-1}$. About such elements it can be said that they have the same *type* of orbit.

Let now X and Y be two G–sets, and let $f : X \longrightarrow Y$ be a map between them.

Definition 55. f *is* (G–)equivariant *if* $f(g * x) = g * f(x)$ *for all* $x \in X$*.*

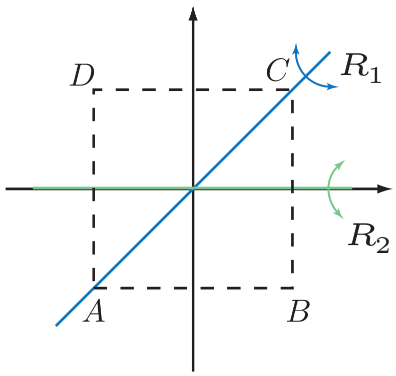
6.1.2. Types of groups actions: transitive, free, regular, effective. There are several types of actions. The action of G on X is called

- *transitive* if $\forall x, y \in X \ \exists g \in G : g * x = y$;
- *free* if $\forall x \in X \ g * x = x \Rightarrow g = e, \ g \in G$;
- *regular* if it is both transitive and free;
- *effective* if $\forall g \neq e \in G \ \exists x \in X : g * x \neq x$.

Equivalently, an action is free if and only of all the stabilizers are trivial. Similarly, it is regular if and only if $\forall x, y \in X \ \exists! g \in G$ such that $g * x = y$. Notice that if an action is free, then it is also effective, but not vice–versa. Intuitively, an action is effective if different elements of G corresponds to different permutations of X.

Example 17. Let $X \subseteq \mathbb{R}^2$ be the square, i.e., $X = \{A, B, C, D\}$, and $G = \mathbb{Z}_2 \simeq \{e, a\}$. Consider the following two representations of G on X:

(1) $\sigma_1 \colon a \mapsto R_1$, the reflection with respect to the diagonal of the first quadrant,
(2) $\sigma_2 \colon a \mapsto R_2$, the reflection with respect to the x–axis.



Observe that

(1) in this case $G_A = \{e, a\}$, $G_B = \{e\}$, $G_C = \{e, a\}$, $G_D = \{e\}$,
(2) in this case $G_A = G_B = G_C = G_D = \{e\}$, since the reflection R_2 has no fixed points.

So, σ_2 is free, while σ_1 is not. Nevertheless both actions are effective, since $G_A \cap G_B \cap G_C \cap G_D = \{e\}$.

6.2. Group action on graphs

Now I will examine the case when X possess more geometric structure.

Let Γ be a (coloured) graph (see Chapter 2, specially Section 2.3). The group of automorphisms of Γ, which was denoted by $\mathrm{Aut}(\Gamma)$ (see Definition 15), can also be referred to as the *symmetry group* of Γ, just to stress its geometric flavour. Accordingly, the elements of $\mathrm{Aut}(\Gamma)$ are called the *transformations* of Γ.

From now on, the generic group G will be denoted by the symbol D instead, in view of its impending assimilation into the von Dyck group (Definition 5).

Definition 56. *A group homomorphism $i : D \to \mathrm{Aut}(\Gamma)$ is called an* action *of D on Γ.*

Such an action is vertex–transitive (–free) if D acts transitively (–freely) on the set V of the vertices of Γ. Similarly, if D acts transitively (–freely) on the set E of the edges of Γ, it is said that such an action is edge–transitive (–free).

6.2.1. Clique–free and clique–transitive group actions on coloured graphs, quotient graphs.

I shall need the notion of a *clique* introduced earlier in 2.3.4, and the group homomorphism $\mathfrak{C}$ (see Remark 4).

From now on, the generic group G will be denoted by instead, in view of its impending assimilation into the von Dyck group (Definition 5).

Definition 57. *An action i of D on Γ is* clique–transitive *(resp.,* clique–free*) if $\mathfrak{C}(\Gamma) \neq \emptyset$ and the action $\mathfrak{C} \circ i$ of D on $\mathfrak{C}(\Gamma)$ is transitive (resp., free).*

If D acts on (Γ, χ), then the *quotient coloured graph* $\left(\frac{\Gamma}{D}, \widetilde{\chi}\right)$ is defined as follows: $V_{\frac{\Gamma}{D}} := \frac{V_\Gamma}{D}$,

$$(Dv_1, Dv_2) \in E_{\frac{\Gamma}{D}} \Leftrightarrow \exists d_1, d_2 \in D \mid (d_1 v_1, d_2 v_2) \in E_\Gamma, \tag{28}$$

and $\widetilde{\chi}(Dv) := \chi(v)$.

Next result makes use of the `C-Gra` category (see 2.3.3).

LEMMA 5. *The canonical projection* $\pi : V_\Gamma \to V_{\frac{\Gamma}{D}}$ *(see Definition 53) is a morphism in* C-Gra. *Moreover, if* $\widetilde{v_1}$ *and* $\widetilde{v_2}$ *are connected by an edge in* $\frac{\Gamma}{D}$, *then any* $v_1 \in \pi^{-1}(\widetilde{v_1})$ *is connected by an edge with an element of* $\pi^{-1}(\widetilde{v_2})$.

PROOF. By (28), π is manifestly edge–preserving and colour–preserving: as such, it defines a morphism in C-Gra (see Definition 22). Let now $(\widetilde{v_1}, \widetilde{v_2}) \in E_{\frac{\Gamma}{D}}$. Then, $v_1' \in \pi^{-1}(\widetilde{v_1})$ and $v_2 \in \pi^{-1}(\widetilde{v_2})$ exist, such that $(v_1', v_2) \in E_\Gamma$. But D acts on Γ by graph automorphisms, and transitively on $\pi^{-1}(\widetilde{v_1})$: so, $v_1 = dv_1'$, with $d \in D$, and hence $(v_1, dv_2) = d(v_1', v_2) \in E_\Gamma$. □

REMARK 13. Observe that π induces a map $\pi_* : \mathfrak{C}(\Gamma) \to \mathfrak{C}\left(\frac{\Gamma}{D}\right)$, which is, in general, not surjective. Let $\widetilde{c} \in \mathfrak{C}\left(\frac{\Gamma}{D}\right)$ be a clique of $\frac{\Gamma}{D}$, and $V_{\widetilde{c}} = \{\widetilde{v_1}, \ldots, \widetilde{v_m}\}$ be the set of its vertices. Then, by Lemma 5, it is possible to show that $v_j \in \pi^{-1}(\widetilde{v_j})$ exists, for $j = 1, \ldots, m$, such that v_1 forms an edge with $v_2, \ldots, v_m$. So, $V_c := \{v_1 \ldots, v_m\}$ does not, in general, determines a clique $c \in \mathfrak{C}(\Gamma)$ unless $m = 2$. Even if in this thesis I am mainly interested in the case $m = 2$, future generalizations may require higher values of m: then, it will be mandatory to find extra conditions on the action of D, guaranteeing the surjectivity of π_*.

6.2.2. A result on clique–free and clique–transitive actions on coloured graphs. Observe that a clique–transitive action does not need to be edge–transitive and it is *never* vertex–transitive, since vertices of different colours cannot be mapped one into another by an element of Aut (Γ). Hence, if D acts clique–transitively, the quotient graph $\frac{\Gamma}{D}$ reduces to a single clique, which, in general, contain more than one edge (and, hence, more than two vertices!). The third original result of this thesis is contained in Lemma 6, whose purpose is that of providing a solid graph–theoretical background to the main result.

LEMMA 6. *Let* D *be a group acting clique–freely and clique–transitively on* Γ, *and* $K \trianglelefteq D$ *be a normal subgroup of* D. *Then,*

(1) D *is* D*–equivariantly (see Definition 55) identified with the set* $\mathfrak{C}(\Gamma)$ *of the cliques of* Γ,
(2) if a clique $c_0 \in \mathfrak{C}(\Gamma)$ *is fixed, then* $\mathfrak{C}(\Gamma)$ *has a group structure isomorphic to* D, *whose identity element is* c_0.

Moreover, if π_* *is surjective, then*

(3) the factor group $\frac{D}{K}$ *acts clique–freely and clique–transitively on the quotient graph* $\frac{\Gamma}{K}$,
(4) $\frac{D}{K}$ *is* $\frac{D}{K}$*–equivariantly identified with the quotient set* $\frac{\mathfrak{C}(\Gamma)}{K} = \mathfrak{C}\left(\frac{\Gamma}{K}\right)$.

PROOF. By Definition 57, D acts freely and transitively on the set $\mathfrak{C}(\Gamma)$: hence (1) and (2) are valid. Moreover, since $\frac{D}{K}$ acts freely and transitively on $\frac{\mathfrak{C}(\Gamma)}{K}$, the group $\frac{D}{K}$ is $\frac{D}{K}$–equivariantly identified with the set $\frac{\mathfrak{C}(\Gamma)}{K}$. In other words, in order to prove (4), it is enough to prove (3).

To this end, let $\widetilde{c} \in \mathfrak{C}\left(\frac{\Gamma}{K}\right)$ be a clique of $\frac{\Gamma}{K}$, and write it as $\widetilde{c} = \pi_*(c)$. Then the desired clique–free and clique–transitive action of $\frac{D}{K}$ on $\frac{\Gamma}{K}$ is defined by

$$\begin{aligned} \frac{D}{K} \times V_{\frac{\Gamma}{K}} &\longrightarrow V_{\frac{\Gamma}{K}}, \\ (dK, \pi(v)) &\longmapsto \pi(dv). \end{aligned} \tag{29}$$

First, I check correctness of (29): if $(d'K, \pi(v')) = (dK, \pi(v))$, then $d' = dk$ and $v' = k_1 v$, with $k, k_1 \in K$, so that $\pi(d'v') = \pi(dkk_1v) = \pi(k_2dv) = \pi(dv)$, where $k_2 \in K$ is such that $(kk_1)^{d^{-1}} = k_2$.

Observe now that (29) is edge–preserving. Indeed, by Lemma 5, if $(\pi(v_1), \pi(v_2)) \in E_{\frac{\Gamma}{K}}$, then it is possible to assume that $(v_1, v_2) \in E_\Gamma$. But K acts by graph automorphisms, so that $d(v_1, v_2) = (dv_1, dv_2) \in E_\Gamma$ as well, and then $(\pi(dv_1), \pi(dv_2)) \in E_{\frac{\Gamma}{K}}$.

Let me verify clique–transitivity. Given two cliques $\widetilde{c}, \widetilde{c_1} \in \mathfrak{C}\left(\frac{\Gamma}{K}\right)$, write them as $\widetilde{c} = \pi_*(c)$, $\widetilde{c_1} = \pi_*(c_1)$, for $c, c_1 \in \mathfrak{C}(\Gamma)$. Then, by clique–transitivity of D–action, $c_1 = dc$, for some d, and hence $dK * \widetilde{c} = \widetilde{c_1}$.

Finally, if $dK * \widetilde{c} = \widetilde{c}$, then $V_{\widetilde{c}} = \{\pi(dv_1), \ldots, \pi(dv_m)\} = \{\pi(v_1), \ldots, \pi(v_m)\}$, i.e., $\pi(dv_j) = \pi(v_j)$ for all $j = 1, \ldots, m$, since v_j and dv_j have the same colour: hence $d \in K$. □

6.3. Action of a group on its Cayley graph: the theorem of Sabidussi

In 1958 G. Sabidussi proved that the Cayley graph $\Gamma(G, S)$ (see Definition 24) is, in a sense, the unique edge–directed and edge–coloured graph (see 2.1) on which G acts in a vertex–regular and vertex–transitive way [46]. Strictly speaking, Sabidussi's theorem was originally introduced as a necessary and sufficient condition for a graph to be a Cayley graph, but I will exploit it here in the above "universal sense". The main result of this thesis is essentially an its corollary, and this is the reason why I present and adapted version of its proof here.

THEOREM 5 (G. Sabidussi, 1958). *A edge–directed and edge–coloured connected graph Γ is a Cayley graph of a group G if and only if it admits a free and transitive action of G on Γ.*

PROOF. ($\Rightarrow$) Regardless of the choice of the generating set S, the group G acts on the Cayley graph $\Gamma(G, S)$ by left multiplication. By definition, such an action is free and transitive (see Section 6.1.2).
($\Leftarrow$) Suppose there is a free and transitive action of G on Γ. In this case, I can identify any element $g \in G$ with a unique vertex $v_g \in V_\Gamma$ of Γ. In particular, there will be an element $v_e \in V_\Gamma$ corresponding to the identity element e of G.

Let $\chi : E_\Gamma \longrightarrow C$ be the colouring map, which is by definition surjective, and $\chi' : C \longrightarrow E_\Gamma$ an its section (i.e., a right inverse). In other words, for any color $c \in C$ I can choose an edge $\chi'(c) = \{x_c, y_c\}$ whose colour is c. Because of the transitivity of the G–action, there is a $g_c \in G$ such that $y_c = x_c g_c$ and, since the action is also free, such a g_c is also unique. So, there is an injective map

$$\begin{aligned} \chi'' : C &\longrightarrow G, \\ c &\longmapsto g_c. \end{aligned}$$

Define $S := \chi''(C)$ and observe that S is a generating set for G in view of the connectedness of Γ. Indeed, any $g \in G$ corresponds to a vertex $V_g \in V_\Gamma$, which is joined to v_e by a path γ. By transcribing the colours $c_1, c_2, \ldots, c_n$ appearing along γ, one immediately sees that $g = \chi''(c_1)\chi''(c_2) \cdot \cdots \cdot \chi''(c_n)$.

Let $\{v_g, v_h\} \in E_\Gamma$ be an edge, and $c = \chi(\{v_g, v_h\})$ its color. By definition, $\chi''(\chi(\{v_g, v_h\})) = g_c$, which is an element $g_c \in S \subseteq G$ such that $\{x_c, x_c g_c\}$ is an edge of Γ of colour c.

So, I have found a correspondence

$$\begin{aligned} \Gamma &\longrightarrow \Gamma(G,S)\,, \\ V_\Gamma \ni v_g &\longmapsto g \in G\,, \\ E_\Gamma \ni \{v_g, v_h\} &\longmapsto \{g, g\chi''(\chi(\{v_g, v_h\}))\}\,, \end{aligned}$$

which is the desired isomorphisms. Indeed, it is by construction one–to–one on the vertices and colour–preserving, and I just need to show that adjacent vertices of Γ are mapped into adjacent vertices of $\Gamma(G,S)$, i.e., that

$$h = g\chi''(\chi(\{v_g, v_h\})) \Leftrightarrow g^{-1}h = \chi''(\chi(\{v_g, v_h\}))\,. \tag{30}$$

By definition,

$$\chi''(\chi(\{v_g, v_h\})) = g_c \text{ such that } \{x_c, x_c g_c\} \text{ has colour } c\,, \tag{31}$$

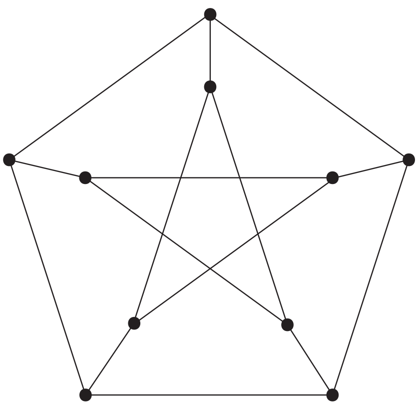
so that it remains to prove that

$$g_c = g^{-1}h\,, \tag{32}$$

and this follows from the fact that, by definition of a coloured graph (see Definition 21 and Remark 3), two edges with the same colour and a common vertex must be the same. Indeed, I have the two c–coloured edges $\{x_c, x_c g_c\}$ and $\{(x_c g^{-1})g, (x_c g^{-1})h)\}$, with the first vertex in common. Then, the desired equality (32) is a consequence of the fact that the second vertices must coincide as well.

□

To show the non–triviality of Theorem 5, let me just point out that not every graph with a transitive group action is the Cayley graph of a group. A good counterexample is the Petersen graph [27] depicted below, which is the most famous highly symmetric graph which is not a Cayley graph.



CHAPTER 7

Main result: the vertex–to–edge duality between $\Gamma(n,n,n)$ and $T(n,n,n)$

7.1. Introduction

Recall that $D(a,b,c) = \langle\, x, y \mid x^a = y^b = (xy)^c = 1 \,\rangle$ is the von Dyck group (see Definition 5), $\Gamma(a,b,c) = \Gamma(D(a,b,c), \{x,y\})$ is its Cayley graph corresponding to the generating set $\{x,y\}$ (see Definition 24), and $T(a,b,c) := T(D(a,b,c), \{H,K\})$ is the rank–two coset geometry determined by the subgroups $H := \langle\, x \,\rangle$ and $K := \langle\, y \,\rangle$ (see Definition 30). The latter will be regarded as a two–coloured (i.e., bipartite) graph (see Sections 2.1 and 2.3, as well as Definition 21).

In this chapter I will prove the following (original) statements.

(1) The natural action of $D(a,b,c)$ on the graph $T(a,b,c)$ (see Section 6.2) is edge–regular and edge–transitive.

(2) There is a $D(a,b,c)$-equivariant bijection b between the set $V_{\Gamma(a,b,c)}$ of the vertices of $\Gamma(a,b,c)$ and the set $E_{T(a,b,c)}$ of edges of $T(a,b,c)$.

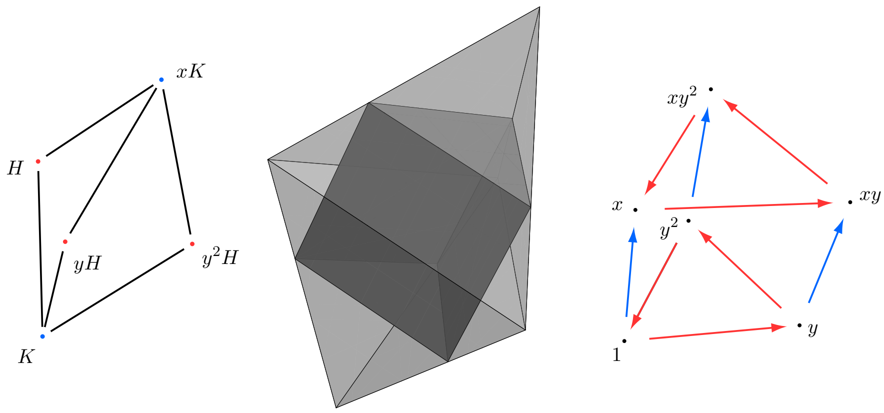
(3) If $I(a,b,c) \subseteq E^2_{T(a,b,c)}$ denotes the set of incident pairs of edges of $T(a,b,c)$, then there is a map $\psi : I(a,b,c) \longrightarrow H \cup K$ such that the vertices d_1 and d_2 of $\Gamma(a,b,c)$ are connected by an x–coloured (resp., y-coloured) oriented edge if and only if $\psi(b(v_1), b(v_2)) = x$ (resp., $= y$).

(4) There are tessellations $\mathcal{T}(a,b,c)$ and $\mathcal{T}'(a,b,c)$ of a constant curvature surface (see Chapter 5), such that the 1–skeleton (see Section 5.1) of the former, supplied with a natural (vertex–)colouring, coincides with $T(a,b,c)$ and the 1–skeleton of the latter, supplied with a natural edge–colouring and edge–orientation, identifies with $\Gamma(a,b,c)$.

The purpose of (1) is merely to pave the way for (2). The meaning of (2) is that $T(a,b,c)$ and $\Gamma(a,b,c)$ are linked by a vertex–to–edge duality (see Section 2.2), while (3) implies that all of the information about $\Gamma(a,b,c)$ is already contained in $T(a,b,c)$, and this is the main result of this thesis. Finally, (4) tells precisely that the passage from $T(a,b,c)$ to $\Gamma(a,b,c)$ can be done by means of manipulations of tessellations; in particular, both $T(a,b,c)$ and $\Gamma(a,b,c)$ are planar graphs.

The vertex–to–edge duality between T and Γ is artistically rendered in the figure below, for the group $\mathbb{Z}_6 = \langle\, x, y \mid x^2 = y^3 = [x,y] = 1 \,\rangle$, and its subgroups H and K

defined as before.



The coset geometry $T := T(\mathbb{Z}_6, \{H, K\})$ is the graph appearing on the left. It consists of three red vertices, H, yH, y^2H, and two blue ones, K, xK; an edge joins the cosets with nonempty intersection (i.e., *incident*, in the sense of Section 3.2). The Cayley graph $\Gamma := \Gamma(\mathbb{Z}_6, \{x, y\})$ is displayed[1] on the right, and it consists of six (uncoloured) vertices, corresponding to the six elements of $\mathbb{Z}_6$: now there is a blue oriented edge (i.e., a blue arrow) to indicate that the head of the arrow is obtained by acting by x on its tail, and a red one when the action is that of y.

The main result of this thesis, which is the vertex–to–edge duality between Γ and T, is illustrated in the center of the figure. Indeed, if the prism Γ is fit into the double pyramid T, in such a way that the vertices of the former touch the middle points of the edges of the latter, then the natural $\mathbb{Z}_6$–action on T determines that on Γ and vice–versa, thus clarifying the meaning of the statements (2) and (3).

7.2. Von Dyck groups as symmetry groups of regular tessellations

As in Section 5.1, the symbol $\mathbb{S}$ will denote either the sphere, the (real) plane $\mathbb{R}^2$, or the hyperbolic plane $\mathbb{H}$.

I denote by $\mathcal{T}_{a,b,c}$ the regular tessellation of $\mathbb{S}$ whose basic triangle $\triangle_0$ has internal angles equal to $\frac{\pi}{a}$, $\frac{\pi}{b}$ and $\frac{\pi}{c}$. Of course, if the basic triangle is equiangular it can used the symbol $\mathcal{T}_{n,n,n}$ or $\mathcal{T}_n$ (see Section 5.1). Recall that, depending on the sum of the values a, b, c, the surface $\mathbb{S}$ must be elliptic, planar, or hyperbolic in order to accommodate $\mathcal{T}_{a,b,c}$ [**16**, **5**, **25**, **45**].

Keep also in mind that the triangle group $\Delta(a, b, c)$ (see Definition 4) acts transitively on $\mathcal{T}_{a,b,c}$, whereas $D(a, b, c)$ (see Definition 5) acts on it by orientation–preserving isometries. The vertices of $\triangle_0$ will be labeled by A, B and O, $\frac{\pi}{a}$ (resp., $\frac{\pi}{b}$, $\frac{\pi}{c}$) being the inner angle at A (resp., B, O), in such a way that the sequence of vertices (A, B, O) occurs counterclockwise.

In Lemma 4 I showed that $\partial\mathcal{T}_{a,b,c}$ is a 3–coloured graph, with the colouring set given by the vertices of $\triangle_0$. Borrowing the terminology of Tits geometries, I say that a vertex $v \in \partial\mathcal{T}_{a,b,c}$ is of *A–type* if its colour is A (and similarly for B and O). A tile $t_i \in \mathcal{T}_{a,b,c}$ is *positively oriented* if running counterclockwise through its vertices, their colours appear in the same order as in $\triangle_0$.

Lemma 7. *$D(a, b, c)$ acts transitively on the subset $\mathcal{T}^+_{a,b,c} \subseteq \mathcal{T}_{a,b,c}$ of positively–oriented tiles.*

[1]Notice that, not to overload the picture, the loops corresponding to the action of x have been replaced by a unique (blue) arrow.

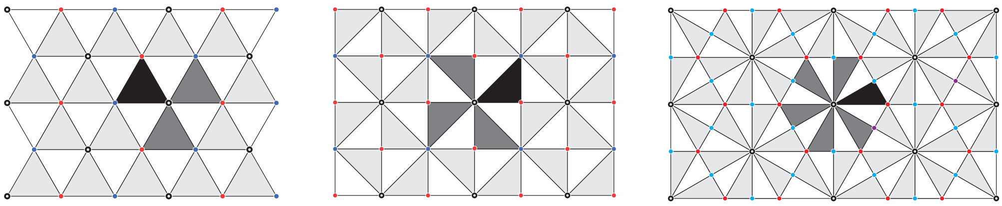
Proof. Follows from the fact that the generator x (resp., y) corresponds to the rotation of $\mathcal{T}_{a,b,c}$ around the vertex A of $\triangle_0$ by an angle $\frac{2\pi}{a}$ (resp., around the vertex B of T_0 by an angle $\frac{2\pi}{b}$) and, by composing such rotations, one can move $\triangle_0$ to any other positively-oriented tile. □

Lemma 7 above provides the necessary geometric interpretation of $D(a,b,c)$, which will make some proofs more transparent. Lemma 8 below shows that $\{x, y\}$ is a Borel–free set of generators (see Definition 1).

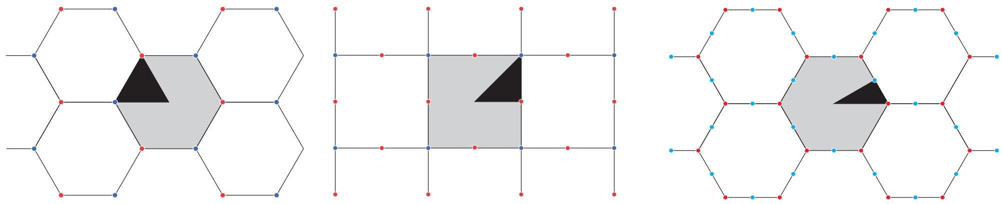
Lemma 8. *The intersection $H \cap K$ is trivial.*

Proof. Suppose that $x^r = y^s \neq 1$: then the two tiles $x^r\triangle_0$ and $y^s\triangle_0$ must be the same. However, the former is obtained by rotating $\triangle_0$ around A, and as such it must contain A itself while the latter must contain B. Since an edge of $\partial\mathcal{T}_{a,b,c}$ determines a unique positively-oriented tile, $x^r\triangle_0$ and $y^s\triangle_0$ must be the same and coincide with $\triangle_0$, which is the unique element of $\mathcal{T}^+_{a,b,c}$ containing the edge (A, B), which is a contradiction. □

In analogy with the construction given in Section 5.2, perform now a full rotation of $\triangle_0$ around its vertex O: the result is a $2c$-sided polygon P_c whose boundary is a 2-coloured (i.e., bipartite) subgraph of $\partial\mathcal{T}_{a,b,c}$ (see the first row of figure below).



$\mathcal{T}_{3,3,3}$ $\mathcal{T}_{2,4,4}$ $\mathcal{T}_{2,3,6}$

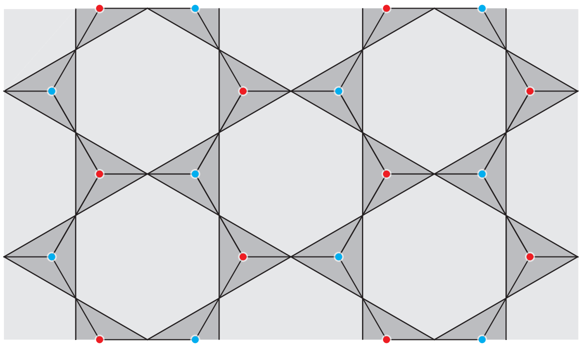


$\mathcal{P}_{3,3,3}$ $\mathcal{P}_{2,4,4}$ $\mathcal{P}_{2,3,6}$

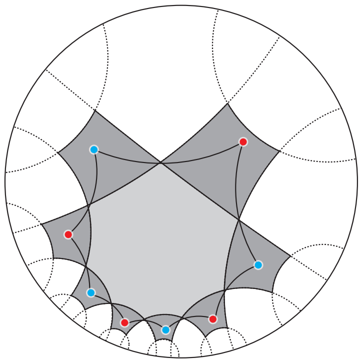
Then there exists a $2c$-polygonal tessellation $\mathcal{P}_{a,b,c}$ of $\mathbb{S}$, such that any of its tiles is the union of the $2c$ tiles of $\mathcal{T}_{a,b,c}$ with a common vertex of type O. Hence, its 1-skeleton $\partial\mathcal{P}_{a,b,c}$ is a bipartite graph and $V_{\partial\mathcal{P}_{a,b,c}}$ is the subset of $\partial\mathcal{T}_{a,b,c}$ consisting of the vertices of type either A or B. Since the tiles of $\mathcal{P}_{a,b,c}$ are in one-to-one correspondence with the O-type vertices of $\partial\mathcal{T}_{a,b,c}$, the von Dyck group $D(a,b,c)$ acts transitively[2] on $\mathcal{P}_{a,b,c}$ as well, making it into a regular tessellation.

Example 18. The derived tessellations (see Section 5.3) of the planar tessellation $\mathcal{P}_{3,3,3}$ (or simply $\mathcal{P}_3$) and the hyperbolic tessellation $\mathcal{P}_{4,4,4}$ (or simply $\mathcal{P}_4$) are shown below, overlying the original tessellations, and their (two-coloured) 1-skeletons.

[2]But not freely: the stabilizer of P_0 is the subgroup generated by xy.



$\mathcal{P}'_3$



$\mathcal{P}'_4$

The tessellation $\mathcal{P}'_3$ is known as a *trihexagonal tessellation* [25].

Observe that, for $n = 3$, $\partial\mathcal{P}'$ is precisely the *derived graph* [6] of $\partial\mathcal{P}$, while for $n > 3$ it is an its subgraph, but with the same vertices, referred to as "medial" by some authors [24] (the explanation of such a phenomenon is that the triangle is the unique polygon which is also a complete graph). The von Dyck group acts transitively both on the set of $2n$–gonal tiles of $\mathcal{P}'$ and on the n–gonal ones.

7.3. The coset geometry of von Dyck groups

Recall that the coset geometry

$$T(a,b,c) = \frac{D(a,b,c)}{H} \bigcup \frac{D(a,b,c)}{K} \tag{33}$$

is the union of all H–cosets and K–cosets of $D(a,b,c)$ (see Definition 30).

For my purposes, it is more convenient to regard $T(a,b,c)$ as a bipartite graph (see Section 2.1), whose vertices are the elements of $T(a,b,c)$, and two vertices are connected by an edge if the corresponding cosets have nontrivial intersection; this is precisely the construction of the *intersection graph* associated to an incidence relation [36] (see Definition 28). Moreover, $T(a,b,c)$ is equipped with a natural colouring: the colour (or type) of the H– (resp., K–) cosets is "H" (resp., "K").

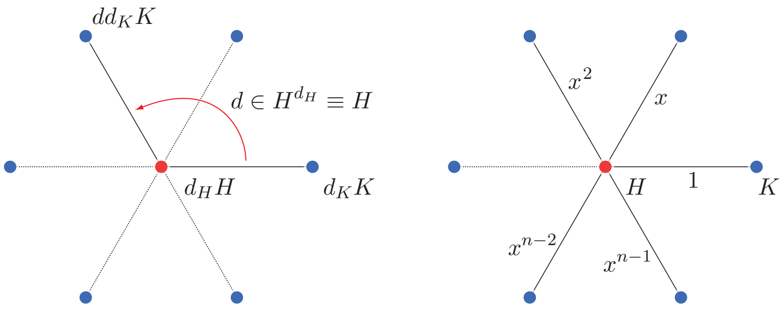
Since $D(a,b,c)$ acts naturally on the two quotient sets appearing at the right hand side of (33), the coset geometry $T(a,b,c)$ is equipped with a natural $D(a,b,c)$–action. Furthermore, $D(a,b,c)$ acts by coloured graph transformations (see Section 6.2), since it sends edges to edges and preserves the type of a coset. The edge (H,K) will be referred to as the *basic edge* (or clique, see Definition 23) I shall write an edge as a pair, where the first entry is always of type H.

Consider an edge $(d_H H, d_K K) \in E_{T(a,b,c)}$, act on it by an element $d \in D(a,b,c)$, and suppose that the resulting edge $d(d_H H, d_K K) = (dd_H H, dd_K K)$ has the same H–vertex as the original one, i.e., $(d_H H, d_K K)$ has been "rotated" around its vertex $d_H H$. This means that $dd_H H = d_H H$, i.e., d stabilizes $d_H H$ and, as such, it belongs to H^{d_H}, which is identified with H via conjugation. This proves the next result.

LEMMA 9. *All the edges of $T(a,b,c)$ having a common H–type (resp., K–type) vertex, say $d_H H$ (resp., $d_K K$), can be obtained by acting on a fixed one by H^{d_H} (resp., K^{d_K}).*

Observe that $D(a,b,c)$ acts edge–transitively on $T(a,b,c)$. Indeed, any edge $e := (d_H H, d_K K)$ can be written as $e = d_H e'$, with $e' = (H, d_H^{-1} d_K K)$ and e' can in turn be obtained by acting on the basic edge by an element of H (Lemma 9). Below

I show how the subgroup H^{d_H} acts transitively on the tree (see Definition 18) of the edges hinged at $d_H H$.

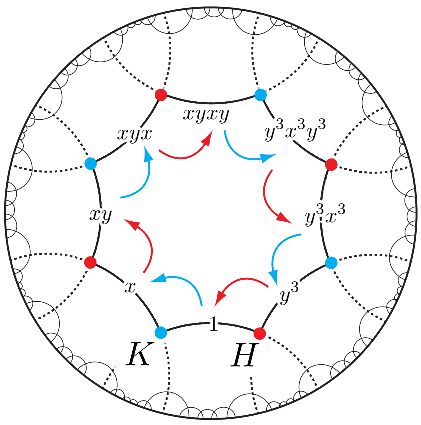


The action clearly behaves like a rotation, which is an indication of the planar character of the coset geometry.

From Lemma 8 it follows also that the action is edge–regular: indeed, in view of the edge–transitivity, the stabilizer of e is conjugate to the stabilizer of the basic edge, which is the trivial intersection $H \cap K$.

Corollary 4 (Statement (1)). *$D(a, b, c)$ acts edge–regularly and edge–transitively on $T(a, b, c)$; in particular, there is a unique way to label the edges of $T(a, b, c)$ by the elements of $D(a, b, c)$ in such a way that the basic edge is labeled by 1.*

Figure below helps to visualize Corollary 4. The 1–skeleton of the hyperbolic tessellation $\mathcal{P}_4$ is labeled by the elements of $D(4, 4, 4)$. The arrows represent the action of the generators of $D(4, 4, 4)$: they turn out to constitute the boundary of a tile of $\mathcal{P}'_4$, corresponding to a cycle of xy in the Cayley graph.



7.4. The Cayley graph of von Dyck groups

Thanks to Theorem 5, the Cayley graph $\Gamma(a, b, c)$ is, in a sense, the unique edge–oriented and edge–coloured graph on which $D(a, b, c)$ acts in a vertex–regular and vertex–transitive way [46] (recall that, in view of Theorem 3, the graph $T(a, b, c)$ is connected and hence Sabidussi's Theorem 5 can be applied here). In Corollary 4 I proved that $D(a, b, c)$ acts on the (vertex–)coloured graph $T(a, b, c)$ in an edge–regular and edge–transitive way: this led me to suspect that $\Gamma(a, b, c)$ and $T(a, b, c)$ can be obtained one from the other by—roughly speaking—"replacing edges with vertices", thus obtaining the main result of this thesis.

I recall that, by definition, $V_{\Gamma(a,b,c)} = D(a, b, c)$ and that for any $d \in D(a, b, c)$ there is a unique edge, call it $b(d)$, of $T(a, b, c)$ which is labeled by d according to

Corollary 4. In other words, the map

$$\begin{array}{rcl} V_{\Gamma(a,b,c)} & \stackrel{b}{\longrightarrow} & E_{T(a,b,c)} \\ d & \longmapsto & b(d) \end{array} \tag{34}$$

is bijective.

COROLLARY 5 (Statement (2)). *The map b defined by (34) is $D(a,b,c)$–equivariant.*

PROOF. In order to prove that $b(d'd) = d'b(d)$, it suffices to observe that if an edge e of $T(a,b,c)$ is labeled by d, then the edge $d'e$ is labeled by $d'd$. □

If one tries to reconstruct $\Gamma(a,b,c)$ out of $T(a,b,c)$, then Corollary 9 allows to obtain the vertices of the former out of the edges of the latter. It remains to describe the edges of $\Gamma(a,b,c)$. Also recall that $\Gamma(a,b,c)$ is edge–coloured and directed, so that not only its edges but their colour and direction must be recovered as well.

Lemma 9, together with the edge–regularity of the $D(a,b,c)$–action (see Corollary 4), guarantees the existence of a map

$$\psi : I(a,b,c) \longrightarrow H \cup K \tag{35}$$

assigning to any pair of incident edges (e_1, e_2) of $T(a,b,c)$ which have a common H–type (resp., K–type) vertex, the unique element $x^r \in H$ (resp., $y^s \in K$) such that the label of e_2 is the label of e_1 multiplied by x^r (resp., y^s). The existence of such an element is clear from the tree displayed at the right of figure on page 58: x^r is just the "ratio"[3] of e_2 by e_1.

COROLLARY 6 (Statement (3)). *There is an x-coloured (resp., y-coloured) directed edge from the vertex d_1 to the vertex d_2 of $\Gamma(a,b,c)$ if and only if $\psi(b(d_1), b(d_2))$ equals x (resp., y).*

PROOF. Let us consider the x–coloured case only. If d_1 is connected to d_2 by a directed edge in the Cayley graph $\Gamma(a,b,c)$, it means that $d_2 = d_1x$. By the definition (34) of the map b, $e_1 := b(d_1)$ is the edge (d_1H, d_1K), while $e_2 := b(d_2)$ is the edge (d_1H, d_2K). Hence, $(e_1, e_2) \in I(a,b,c)$ and, as such, the map ψ defined by (35) can be applied to it: the result is $\psi(e_1, e_2) = x$, to be interpreted as the "ratio" $\frac{d_2}{d_1}$. The converse is also true, due to the bijectivity of b. □

Figure at page 55 helps to visualise the above proof. Let $d_1 = 1$ and $d_2 = x$ be vertices of the Cayley graph (displayed at the right): there is an x–coloured directed edge between them (blue arrow). By the vertex–to–edge duality, portrayed in the center, this edge of the Cayley graph is responsible for the fact that the edge $e_2 := (H, xK)$ of the coset geometry (left) is obtained from $e_1 := (H, K)$ by acting on it by x. Formally, the datum x can be recovered as $\psi(e_1, e_2)$, since x is precisely the "ratio" $\frac{e_2}{e_1}$.

[3] More precisely, $r = n_2 - n_1$, where x^{n_1} (resp., x^{n_2}) corresponds to e_1 (resp., e_2).

7.5. The duality between the Cayley graph and the coset geometry in the context of tessellations

In this final section I prove the statement (4), thanks to which the previous ones acquire more perspective.

First, I establish a correspondence between the bipartite graphs $\partial\mathcal{P}_{a,b,c}$ and $T(a,b,c)$: define a map

$$\begin{array}{rcl} E_{T(a,b,c)} & \longrightarrow & E_{\partial\mathcal{P}_{a,b,c}} \\ d(H,K)\equiv d & \longmapsto & d(A,B), \end{array} \tag{36}$$

where the identification $d(H,K)\equiv d$ is due to Corollary 4 and (A,B) is one of the sides of the basic triangle $\triangle_0$, defined in Section 7.2. Since $\mathcal{P}_{a,b,c}$ is a regular tessellation, the map (36) is surjective. It is also injective since an orientation–preserving isometry which fixes the segment (A,B) must be identical. Hence, the abstractly defined graph $T(a,b,c)$ is the 1–skeleton of a concrete tessellation of $\mathbb{S}$. On this surface, the duality between the Cayley graphs and the coset geometry discussed in Section 7.4 can be recast in terms of the tessellation $\mathcal{P}_{a,b,c}$ and its derived tessellation.

By the definition of derived tessellation (see Section 7.2), there is a bijection

$$\begin{array}{rcl} E_{\partial\mathcal{P}(a,b,c)} & \xrightarrow{\mu} & V_{\partial\mathcal{P}'(a,b,c)}, \\ d & \longmapsto & \mu(d), \end{array}$$

where $\mu(d)$ is the middle point of the edge $d(A,B)$, and the edges of $\partial\mathcal{P}(a,b,c)$ are identified with the elements of $D(a,b,c)$ via (36). Hence, $\mu\circ b$ establishes a one–to–one correspondence between the vertices of $\Gamma(a,b,c)$ and those of $\partial\mathcal{P}(a,b,c)$. Recall that the vertices $\mu(d_1)$ and $\mu(d_2)$ form an edge e in $\partial\mathcal{P}'(a,b,c)$ if and only if $d_1(A,B)$ and $d_2(A,B)$ are incident and belong to the same tile of $\mathcal{P}(a,b,c)$ (see figure at page 57). Since $\partial\mathcal{P}(a,b,c)$ is a bipartite graph, the edge e can be given the same colour of the vertex $v:=d_1(A,B)\cap d_2(A,B)$; moreover, the edge e can be directed from $\mu(d_1)$ to $\mu(d_2)$ if the rotation sending $d_1(A,B)$ to $d_2(A,B)$ within the tile they belong to, is counterclockwise, and vice-versa.[4] Suppose that this rotation is counterclockwise and that v is of type A: then, in view of the correspondence (36), the edges $d_1(H,K)$ and $d_2(H,K)$ have the H–type vertex in common and $d_2=d_1x$, since, by definition, the inner angles of a tiles of $\mathcal{P}(a,b,c)$ at its A–type vertices are $\frac{2\pi}{a}$. Hence, in view of Corollary 6, there is a directed A–coloured edge from $\mu(d_1)$ to $\mu(d_2)$ in $\partial\mathcal{P}'(a,b,c)$ if and only if there is a directed x–coloured edge between d_1 and d_2 in $\Gamma(a,b,c)$.

[4]In figure at page 58 such a tile is the hyperbolic basic octagon P_0, and (A,B) is the edge labeled by "1"; the vertex v is a vertex of ∂P_0, and d_1 (resp., d_2) is the word coming before (in a clockwise sense) v (resp., after it). A red (or blue) arrow, running counterclockwise, rotates the corresponding edge $\mu(d_1)$ on $\mu(d_2)$: hence, in the Caley graph, there is an oriented red (or blue) edge from d_1 to d_2. The eight vertices cycle made by the red and blue arrows corresponds precisely to the element xy of order four.

CHAPTER 8

Applications and perspectives

8.1. the polygonal enlargement algorithm

I obtain here a nice consequence of the results established in the previous chapter. It can be formulated as follows.

COROLLARY 7 (Cliques enumeration algorithm). *A bijection $d : \mathbb{N}_0 \to D(n,n,n)$ exists, which can be defined recursively.*

As the name "cliques enumeration algorithm" shows, in this section, I will focus on cliques, rather than graphs. In this perspective, I will sow that Corollary 4 can be generalised to any group with a Borel–free set of generators (see Definition 1).

PROPOSITION 9. *Let S be a Borel–free set of generators of a group D, and define $T(D) := \bigcup_{s\in S} \frac{D}{\langle s\rangle}$. Then,*

(1) $(T(D),\chi)$ is a coloured graph (see Definision 21) with coloured function χ,
(2) D acts clique–transitively and clique–freely on $(T(D),\chi)$ (see Definition 57),
(3) any element $d \in D$ acts on cliques by a sequence of rotations.

PROOF. As the disjoint union of the left D–cosets $\frac{D}{\langle s\rangle}$, $s \in S$, the set of vertices $V_{T(D)}$ inherits a left D–structure (see Remark 11). Observe also that D acts transitively on each subset of vertices $\frac{D}{\langle s\rangle}$, i.e., it is colour–preserving, and that the sub–graph $c_0 := \{\langle s\rangle \mid s \in S\}$ is a clique (see Definition 23); indeed, any two its vertices are connected by an edge, and any vertex outside c_0 is of the form $d\cdot\langle s\rangle$, for some $s \in S$ and $d \in D \smallsetminus \langle s\rangle$ and, as such, it cannot be connected by an edge to $\langle s\rangle$, since they have the same colour s. Finally, the action of D sends edges to edges, since two nontrivially intersecting left cosets are mapped into nontrivially intersecting left cosets, i.e., $D \leq \mathrm{Aut}\,(T(D))$.

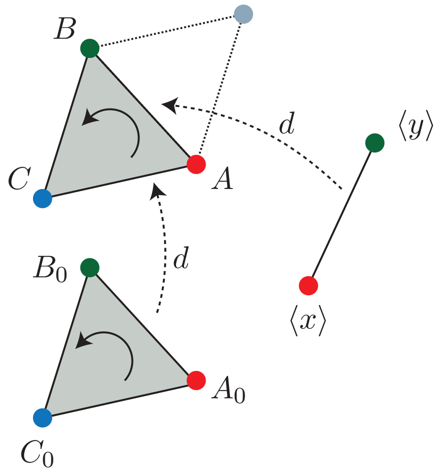
Prove that any clique c is of the form $d\cdot c_0$. Take a vertex $d\cdot\langle s\rangle \in c$ and observe that $d^{-1}\cdot c$ is a clique containing $\langle s\rangle$ (Remark 4). By Definition 23 it must coincide with c_0. Hence, D acts clique–transitively. Finally, $d \in D$ stabilizes c_0 if and only if it stabilizes all its vertices, i.e., it belongs to $\bigcap_{s\in S}\langle s\rangle = 1$.

It remains to be observed that $d = s$ acts on c_0 by a rotation around it vertex $\langle s\rangle$. Let now $d = \underline{d}s$, an suppose that $\underline{d}$ acts on c_0 by a sequence of rotations: then, $d\cdot c_0 = (\underline{d}s\underline{d}^{-1})\cdot(\underline{d}\cdot c_0)$ is obtained from c_0 by the *same* sequence of rotations, followed by a rotation around the vertex $\underline{d}\cdot\langle s\rangle$, since $\underline{d}\langle s\rangle\underline{d}^{-1}$ is precisely the stabilizer of $\underline{d}\cdot\langle s\rangle$. □

The results of previous Chapter 7 can be paraphrased in terms of cliques as follows. Recall that $\mathcal{P}_n$ is $2n$–gonal tessellation, introduced in Section 5.2.

COROLLARY 8. *The von Dyck group $D(n,n,n)$ acts clique–transitively and clique–freely on the 2–colours graph $\partial(\mathcal{P}_n)$. Moreover, $\partial(\mathcal{P}_n)$ is canonically and $D(n,n,n)$–equivariantly identified with $T(D(n,n,n),\{x,y\})$.*

Proof. $D(n,n,n)$ acts on the 3–colours graph $\partial\mathcal{T}_n$ (see Lemma 4), and the stabiliser of the basic triangle is trivial, i.e., the action is clique–free. Even if the action is not clique–transitive, two cliques of the same *orientation* are always moved one to another by an element of $D(n,n,n)$ (see Lemma 7). It remains to observe that a clique of $\partial\mathcal{P}_n$ determines a unique oriented clique of $\partial\mathcal{T}_n$: it is enough to choose a fixed sequence of colours, say red–green–blue and, to any clique of $\partial\mathcal{P}_n$, i.e., an edge with red–green ends, associate the unique (out of the two adjacent triangles) clique of $\partial\mathcal{T}_n$ whose blue vertex makes the resulting sequence red–green–blue run counterclockwise. Figure below shows two triangles facing the same clique (A, B): only the coloured one has positive orientation.



By Proposition 9, any clique c of $T(D(n,n,n),\{x,y\})$ is uniquely written as $c = (d\langle\, x\,\rangle, d\langle\, y\,\rangle)$, with $d \in D(n,n,n)$. Let (A,B,C) the oriented clique of $\partial\mathcal{T}_n$ determined by $A = d\langle\, x\,\rangle$, and $B = d\langle\, y\,\rangle$: hence, (A,B,C) can be uniquely written as $d'(A_0,B_0,C_0)$, where $A_0 = \langle\, x\,\rangle$, and $B_0 = \langle\, y\,\rangle$ (see figure above). It follows that $d' = d$. But the oriented clique (A,B,C) is uniquely determined by the clique (A,B) of $\partial\mathcal{P}_n$: correspondence $c = (d\langle\, x\,\rangle, d\langle\, y\,\rangle) \leftrightarrow (A,B) = (dA_0, dB_0)$ is manifestly one–to–one and $D(n,n,n)$–equivariant.

□

By Lemma 6, $D(n,n,n)$ can be identified with the set of cliques $\mathfrak{C}(\partial\mathcal{P}_n)$, and all its factors with a quotient of $\mathfrak{C}(\partial\mathcal{P}_n)$. In Section 8.3 I will show that among the factors of $D(n,n,n)$ there are the free Burnside groups with two generators. The generalization of such a picture to the cases with more generators is a challenging task. It is worth stressing that only for $n = 2$ the set of cliques $\mathfrak{C}(\partial\mathcal{P}_n)$ is finite.

From now on, the generator x (resp. the generator y, the product $r := xy$) will be identified with the $\frac{2\pi}{n}$ rotation of $\mathcal{P}_n$ around the vertex H (resp., the vertex K, the center O), where (O,H,K) is a fixed orientation of the basic triangle (Lemma 7). I call $c_0 = (H,K)$ the *basic clique* (and it is the basic edge in Section 7.3), and I will identify the element $w \in D(n,n,n)$ with the clique $wc_0 = (wH, wK)$.

By Posposition 9, $D(n,n,n)$ acts on $\partial\mathcal{P}_n$ by a sequence of rotations.

Remark 14. $D(n,n,n)$ acts transitively on the polygons of $\mathcal{P}_n$, since these are the "minimal cycles", but the stabilizers are the cyclic subgroups of order n conjugate to $\langle\, r\,\rangle$.

REMARK 15. When a cycle Γ is oriented, any its clique $c \in \mathfrak{C}(\Gamma)$ inherits an orientation: c is said to be *positively* (resp., *negatively*) *oriented* if, running accordingly to the positive orientation of Γ, the x–coloured vertex comes after the y–coloured one (resp., before). Observe that any cycle must be composed of an even number of cliques: of these, half is positively oriented, and the other half is negatively oriented.

REMARK 16. Denote by $\left[\frac{i}{2}\right]$ the integer part of $\frac{i}{2}$, and by $[i]_{2\mathbb{Z}}$ the parity of i. Then the map

$$i \in \{0, 1, \ldots, 2n-1\} \longmapsto d(i) := r^{\left[\frac{i}{2}\right]} x^{[i]_{2\mathbb{Z}}} \cdot c_0 \in \mathfrak{C}(\partial P_n),$$

is a bijection.

PROPOSITION 10 (Polygonal enlargement). *Let $\mathcal{Q} \subseteq \mathcal{P}_n$ be a sub–tessellation (see Definition 45) such that its boundary (see Remark 10) is a simple cycle (see Definition 19) $\Gamma \subseteq \partial\mathcal{P}_n$, consisting of N cliques. Define $\widetilde{\mathcal{Q}}$ as the larger sub–tessellation obtained from $\mathcal{Q}$ by adding all the $2n$–gons which intersect it, and call $\widetilde{\Gamma}$ its boundary. Assume both Γ and $\widetilde{\Gamma}$ to be oriented clockwisely. Then*

(1) *for any positively oriented clique $w_0 \in \mathfrak{C}(\Gamma)$, unique integers $i_1, \ldots, i_N \in \{1, \ldots, n-1\}$ exist, such that the sequence*

$$w_0, w_0 x^{i_1}, w_0 x^{i_1} y^{i_2}, \ldots, w_0 x^{i_1} y^{i_2} \cdots x^{i_{N-1}} y^{i_N} = w_0 \tag{37}$$

run clockwisely through all the cliques of Γ;

(2) *the number of $2n$–gons contained in $\widetilde{\mathcal{Q}} \smallsetminus \mathcal{Q}$ is*

$$N(n-1) - \sum_{k=1}^{N} i_k; \tag{38}$$

(3) *by replacing in (37) each positively (resp., negatively) oriented clique $w = w_0 \cdots y^{i_{k-1}}$ (resp., $v = w_0 \cdots x^{i_k}$) according to the following rules*

$$w \leftrightarrow (\underbrace{wx^{n-2}d(2), \ldots, wx^{n-2}d(2n-1), \ldots,}_{\text{not present for } n-2=i_k} wx^{i_k}d(2), \ldots, wx^{i_k}d(2n-2)), \tag{39}$$

$$v \leftrightarrow (\underbrace{vy^{n-1}d(1), \ldots, vy^{n-1}d(2n-2), \ldots,}_{\text{not present for } n-1=i_{k+1}+1} vy^{i_{k+1}+1}d(1), \ldots, vy^{i_{k+1}+1}d(2n-3)), \tag{40}$$

one obtains the (positively oriented) sequence of cliques of $\widetilde{\Gamma}$.

PROOF. The cliques of Γ are of alternating orientation (see Remark 15), and any its clique can be obtained by another by a sequence of rotations (see Proposition 9 (3)). In particular, the clique coming next to w (which is, as such, negatively oriented) is obtained by a rotation around the x–coloured vertex of w, hence is uniquely written as wx^{i_k} (see the proof of point (3) of Proposition 9). By iteration, one obtains the sequence (37), thus proving (1).

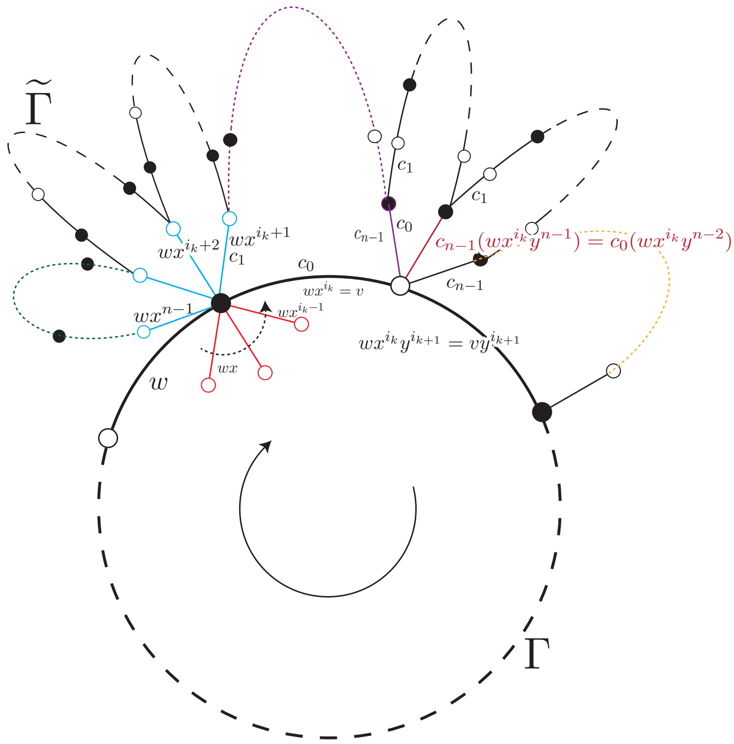


Figure above shows the n cliques pivoted at the x–colour vertex of the (positively oriented) clique w. Starting from w itself and running counterclockwisely, I meet the internal cliques $wx, \dots, wx^{i_k-1}$ (highlighted in red), then the next (negatively oriented) clique $v = wx^{i_k}$ of Γ. It is convenient (since they will form the cycle $\widetilde{\Gamma}$) to list clockwisely the remaining $n - i_k - 1$ cliques, i.e., those outside $\mathcal{Q}$, namely

$$wx^{n-1}, wx^{n-2}, \dots, wx^{i_k+1}$$

(highlighted in blue).

Now the $2n$–gon of $\mathcal{P}_n$ determined by the two consecutive cliques wx^{n-2}, wx^{n-1}, is precisely $wx^{n-2}P_n$; continuing until the $2n$–gon $wx^{i_k}P_n$, I obtain exactly $n-i_k-1$ $2n$–gons. The cliques $wx^{n-2} = wx^{n-2}d(0)$ and $wx^{n-1} = wx^{n-2}d(1)$ of $wx^{n-2}P_n$ will be inside $\widetilde{\mathcal{Q}}$: hence, the remaining ones, i.e., $wx^{n-2}d(2), \dots, wx^{n-2}d(2n-1)$, are those which will contribute to $\widetilde{\Gamma}$ (highlighted in green). Repeating this for $n-3, \dots, i_k+1$, explains the underbraced part of (39). The last $2n$–gon, i.e., $wx^{i_k}P_n$ has three cliques inside $\widetilde{\mathcal{Q}}$, namely $wx^{i_k}d(2n-1)$, $wx^{i_k}d(0) = wx^{i_k}$ (which belongs to $\mathcal{Q}$), and $wx^{i_k}d(1)$: hence, the ones belonging to $\widetilde{\Gamma}$ are $wx^{i_k}d(2), \dots, wx^{i_k}d(2n-2)$ (highlighted in purple), and this explains the remaining part of (39).

Let me pass to a negatively oriented clique v (for simplicity, $v = wx^{i_k}$). The (clockwise) list of the cliques outside $\mathcal{Q}$, obtained by rotating v around its y–vertex, is

$$vy^{n-1}, vy^{n-2}, \dots, vy^{i_{k+1}+1}$$

Again I obtain $n - i_k - 1$ $2n$–gons. So, any vertex of Γ intersects exactly $n - i_k - 1$ $2n$–gons lying outside $\mathcal{Q}$: hence (38) holds true and (2) is proved.

Now observe that the $2n$–gon determined the two consecutive cliques vy^{n-2}, vy^{n-1} is precisely $vy^{n-1}P_n$: indeed, the former is $vy^{n-1}d(2n-1)$ and the latter is $vy^{n-1}d(0)$. Hence, the remaining cliques $vy^{n-1}d(1), \dots, vy^{n-1}d(2n-2)$ are those which will contribute to $\widetilde{\Gamma}$. Repeating this for $n-2, \dots, i_{k+1}+2$, one obtains the underbraced part of (40). The last $2n$–gon, which is $vy^{i_{k+1}+1}P_n$, base three cliques inside $\widetilde{\mathcal{Q}}$: $vy^{i_{k+1}+1}d(2n-2)$, $vy^{i_{k+1}+1}d(2n-1) = vy^{i_{k+1}}$ (which belongs to $\mathcal{Q}$), and $vy^{i_{k+1}+1}d(0)$. The remaining ones, $vy^{i_{k+1}+1}d(1), \ldots, vy^{i_{k+1}+1}d(2n-3)$ explain the last part of (40) and are highlighted orange in figure at page 64.

Observe that in the Euclidean case ($n = 3$), an hexagon intersecting $\mathcal{Q}$ must be adjacent to it: hence, the underbraced parts of (39) and (40) must be suppressed. □

The meaning of the numbers i_k's is that $2\pi\frac{i_k}{n}$ represents the internal angle of Γ at its k^{th} vertex (according to the sequence (37)): on this concern, it is worth observing that the algorithm of Proposition 10 does not requires $\mathcal{Q}$ to be convex, i.e., large values of i_k are allowed (though not useful for the present purposes). In the Euclidean case, $i_k = 2$ is already a "large" value: hence no hexagon at all will be added by the algorithm; there are only three hexagon around any vertex, and if one is "inside" $\mathcal{Q}$ (i.e., $i_k = 1$), the remaining two ones must be "outside", and both must have an edge in common with $\mathcal{Q}$ (whence the disappearance of the underbraced parts of (39) and (40), which would correspond to an hexagon having just one vertex in common with $\mathcal{Q}$).

Proof of Corollary 7. For $n = 2$, the group $D(n, n, n)$ is finite (see Section 1.2), so there is nothing to prove.

Let $n \geq 3$. Define d on $N_1 := \{0, 1, \ldots, 2n-1\}$ as in Remark 16. Let $\widetilde{P_n}$ the enlargement of P_n, constructed as in Proposition 10, and use the current definition of d to extend it to $N_1 \cup N_2$, where $N_2 \subseteq \mathbb{N}_0 \smallsetminus N_1$ is in one–to–one correspondence with the set of cliques $\mathfrak{C}\left(\partial\widetilde{P_n}\right) \smallsetminus \mathfrak{C}(\partial P_n)$. In order to fill up the whole $\mathbb{N}_0$, I choose N_2 in such a way that, together with N_1, it forms an interval.

It remains to be noticed that the boundary of all successive enlargements of T_n are always regular, i.e., Proposition 10 can be used to recursively define the bijection d between $\mathbb{N}_0 = \cup_{j=1}^{\infty} N_j$ and $\mathfrak{C}(\partial\mathcal{P}_n) = \cup_{j=1}^{\infty}(\mathfrak{C}(\partial P_n^{(j+1)}) \smallsetminus \mathfrak{C}(\partial P_n^{(j)}))$, where $P_n^{(j+1)} = \widetilde{P_n^{(j)}}$ and $P_n^{(1)} = P_n$. □

Let now $n > 3$, and consider the j^{th} enlarged polygon $P_n^{(j)}$ defined in Corollary 7.

Corollary 9. *The elements $d(3)^j$ and $d(2n-3)^j$ belong to $\mathfrak{C}\left(\partial P_n^{(j)}\right) \smallsetminus \mathfrak{C}\left(\partial P_n^{(j-1)}\right)$, for all $j > 1$, and $[d(3)^n, d(2n-3)^n] \neq 1$.*

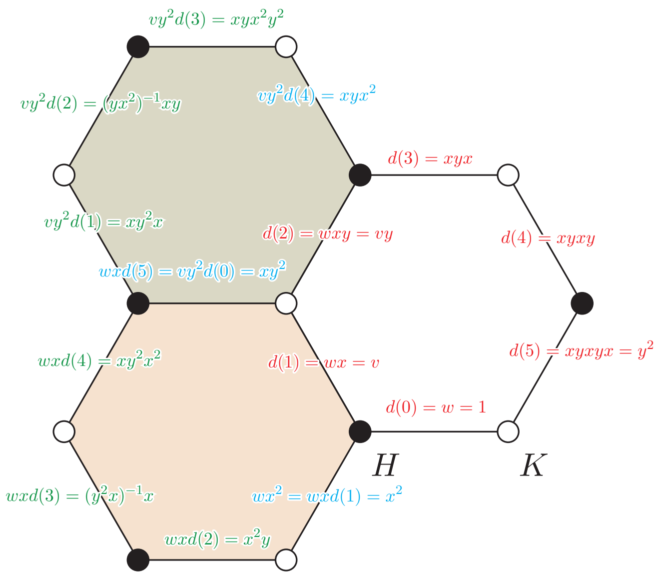
Proof. The first statement is a direct consequence of Proposition 10. Notice that the second part of (39), i.e., $wx^{i_k}d(2), \ldots, wx^{i_k}d(2n-2)$ can be rewritten as $vd(2), \ldots, vd(2n-2)$. Obviously, $d(3)^1 = d(3)$ (resp. $d(2n-2)^1$) belong to $\mathfrak{C}(\partial P_n)$.

The second statement is checked by explicitly computing the Möbius transformations (see Section 4.2) associated with $d(3)^j = (xyx)^j$ and $d(2n-3)^j = ((xy)^{n-2}x)^j$. What is worth to be noticed is that it would be impossible to calculate the commutator $[d(3)^n, d(2n-3)^n]$ in standard algebraic way, since it is an exponential function of n. Thanks to the geometric interpretation, I can apply the formula for the n^{th} (and $-n^{\text{th}}$) power of unimodular matrix (which is based on Chebyshev polynomials of the second kind, see, e.g., [8]) to the elements uvu and $v^{-1}u^{-1}v^{-1} = (uv)^{n-2}u$, where

$$u = \begin{pmatrix} \cos\frac{\pi}{n} & -\sin\frac{\pi}{n} \\ \sin\frac{\pi}{n} & \cos\frac{\pi}{n} \end{pmatrix} \text{ and } v = \begin{pmatrix} \cos\frac{\pi}{n} & -\beta\sin\frac{\pi}{n} \\ (\sin\frac{\pi}{n})/\beta & \cos\frac{\pi}{n} \end{pmatrix}$$

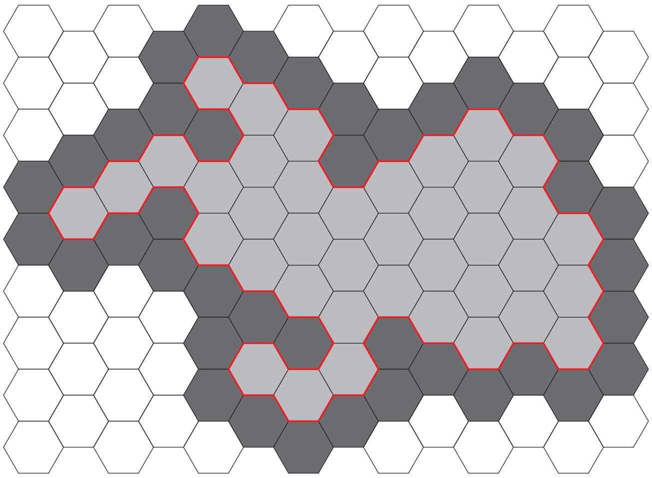
are a counterclockwise rotation matrices through an angle $2\pi/n$ about vertex $x = i$ and $y = \beta i$ respectively. Remember that on hyperbolic plane the length of the edges (of regular polygons), i.e., β is depending on the choice of the interior angles, so of n, therefore it can be computed from the equation $\beta + \frac{1}{\beta} = \frac{2\cos\frac{\pi}{n}}{1-\cos\frac{\pi}{n}}$. Now is easy to quantify the sought–for commutator. Since the output is not in an affordable form, it is enough to consider for example only the second entry of it and notice that it is function of n always growing different from zero for any $n \geq 4$.

$\square$



Above figure shows how the cliques enumeration algorithm works for $n = 3$. The white hexagon is P_3, whose perimeter $\Gamma := \partial P_3$ is described by the map d (see Remark 14). Notice that the six cliques of Γ (in red colour) constitute a six–entries string of the form (37). Then the two coloured hexagons are obtained by a polygonal enlargement (see Proposition 10), the pink (resp., grey) one corresponding to the clique w (resp., v). Their green coloured cliques are precisely those obtained by replacing w (resp., v) by (39) (resp., (10)). The blue cliques are those belonging to $\mathfrak{C}\left(\partial\widetilde{P_3}\right) \smallsetminus \mathfrak{C}(\partial P_3)$, but not to the enlarged perimeter $\widetilde{\Gamma}$.

EXAMPLE 19. The enlargement of the sub-tessellation $\mathcal{Q} \subseteq \mathcal{P}_3$ (soft grey) is accomplished by adding all the hexagons (hard gray) which have a vertex lying on the (not convex) polygon $\Gamma = \partial\mathcal{Q}$ (red line).



The number of added hexagons depend on the internal angles of Γ.

8.2. The genus of the group

In the previous Chapter 7 I have showed the relationship between two abstract procedures to associate a graph to the von Dyck group, in such a way that the group acts on it: undoubtedly, among them the Cayley graph is a much more broadly exploited construction, being linked to the important notion of the *genus of a group* (i.e., the smallest genus of a surface where its Cayley graph can be embedded), introduced in 1972 by A.T. White [58]. On the other hand, except for some sparse and marginal papers, there are no remarkable group theoretic applications of the theory of coset geometries.

Now the coset geometry can be regarded as the 1–skeleton of a tessellation on which the von Dyck group acts transitively, and as such it is linked to the notion of the *strong symmetric genus* of a group (i.e., the smallest genus of a surface on which the group acts by orientation–preserving isometries [55]). It should be stressed that, all constructions being $D(a, b, c)$–equivariant, the results obtained descend to the factors of $D(a, b, c)$ which, as observed by P.M. Neumann in 1973, constitute in fact a very large family [38]. In particular, I can recast (by using perhaps a simpler method) a result of T. W. Tucker [55].

Corollary 10 (T. W. Tucker, 1983). *Let $\frac{\mathbb{S}}{K}$ be a compact surface. Then the Cayley graph of $\frac{D(a,b,c)}{K}$, with respect to the generating set $\{xK, yK\}$, is embedded into $\frac{\mathbb{S}}{K}$ in a $\frac{D(a,b,c)}{K}$-invariant way.*

Remark 17. In spite of the name "duality" used before, the passage from a tessellation to its derivative cannot be easily inverted. An easy consequence of Corollary 10 is that the genus of $\frac{D(a,b,c)}{K}$ is bounded by its strong symmetric genus: a procedure to recover a tessellation out of its derivative would allow to prove the converse as well.

8.3. The Burnside group as a quotient of the von Dyck group

Constructing the Cayley graph of a group is the same as solving the word problem for it (see Section 9.2). The duality between the Cayley graph and the coset geometry discussed in Chapter 7 may provide an effective way to construct the Cayley graph of the von Dyck groups, as well as of its factors. Consider, for example, the groups $D(n, n, n)$, which they are important since they cover the free Burnside groups $B(2, n)$ (see Definition 8), as noticed by some authors (see, e.g., [56, 20]). In this case, $\mathcal{P}_n$ (see Definition 49) is made of regular $2n$–gons, and the whole tessellation can be constructed algorithmically by means of subsequent "enlargements" of the basic polygon P_0 (see Corollary 7).

Corollary 7 becomes more interesting when it descends to the factors of $D(n, n, n)$, for instance, $B(2, n)$. It is well–known that the latter can be obtained by factoring the former by the n^{th} powers subgroup $K_n := D(n, n, n)^n$. In 1986 A.M. Vinogradov proposed an algorithmic way to check the finiteness of the Fuchsian $B(2, n)$'s, i.e., those with $n > 3$, based on the computation of a fundamental domain for K_n in the hyperbolic plane [56] (see also [48]).[1] In turn, to run such an algorithm, it is necessary to effectively list the elements of K_n: an enumeration of the elements of the subgroup K_n based on the result of Corollary 7 will be certainly more efficient than the standard lexicographic method.

Indeed, from Corollary 9 it immediately follows that K_n is infinite and not Abelian for $n > 3$, and this is the last original result of this thesis (see Corollary 11 later on), showing once again how geometric methods can effectively lead to algebraic results.

Example 20 (A toy model: $B(2, 3)$). Up to isomorphisms, there are only 5 groups of order 27 [44], and a unique one which is meta–abelian without being abelian, has two generators, has exponent 3, and also possesses a cyclic derived subgroup:

$$B(2, 3) = \langle\, x, y \mid x^3 = y^3 = [x, y, x] = [x, y, y] = 1 \,\rangle. \tag{41}$$

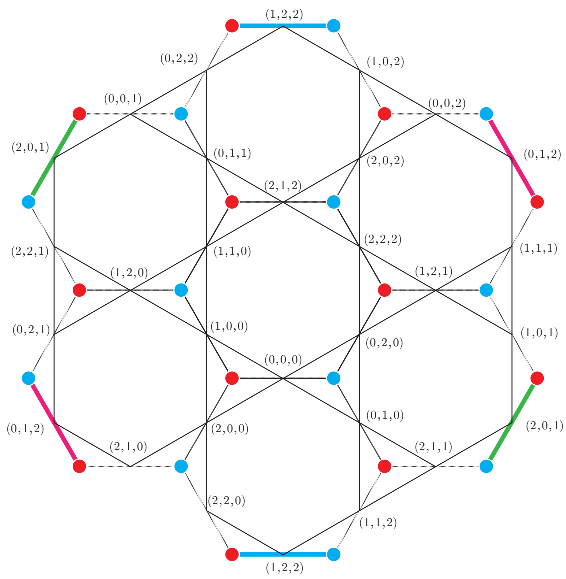
The group (41) is the free Burnside[2] group and (see [20, 26]) any of its element w can be written as

$$w = x^a y^b [x, y]^c, \quad a, b, c \in \{0, 1, 2\}. \tag{42}$$

Figure below displays the coset geometry of $B(2, 3)$ which, as a quotient of $\partial\mathcal{P}_3$, can be embedded in a domain in $\mathbb{R}^2$ with some identifications on its boundary; each edge is labeled by the corresponding element of the group, according to the three parameters (42). The derived tessellation $\mathcal{T}_3'$ (see Section 5.1 and Section 5.3) is also shown: the triangular cycles correspond to the generators x and y, while the hexagonal ones correspond to their product xy.

[1] The planar case is useful for understanding the behavior of the algorithm: in Figure 69 a fundamental domain of $B(2, 3)$ is represented as the intersection of the three bands in the plane bounded by the lines passing through the edges with the same colour. The algorithm produces exactly such bands, and stops when the intersection becomes a closed polygon which, in this toy model, occurs after three steps.

[2] A nice and exhaustive review on Burnside problem can be found in Section 6.8 of H.S. Coxeter's book [16].



The coset geometry of $B(2,3)$ is overlaid by its Cayley graph. This is a geometric evidence of the finiteness of $B(2,3)$: there are exactly 27 edges in the coset geometry, which correspond to the 27 vertices of the Cayley graph.

Corollary 11. *K_2 is trivial, K_3 is the infinite abelian subgroup generated by the cubes of the three translations xy^2, y^2x and xyx, while K_n is an (infinite) non–Abelian group for all $n > 3$.*

Proof. $D(2,2,2)$ is abelian of exponent 2, so $K_2 = 1$. As a subgroup of rigid motions of $\mathbb{R}^2$ (see Section 1.2 and also Section 7.2), any element $d \in D(3,3,3)$ can be written as rt, where r is a rotation and t a translation. It follows that $d^3 = r^3 t^{r^2} t^r t$, i.e., d^3 is a composition of translations. Hence, $K_3 \leq \mathbb{R}^2$ is the lattice made of the centers of a regular hexagonal tiling of thrice the size as the basic hexagon of $\mathcal{P}_3$, and as such it is generated by the three vectors xy^2, y^2x and xyx.

When $n > 3$, Corollary 9 shows that $d(3)^n$ and $d(2n-3)^n$ are aperiodic non commuting elements of K_n, which is then infinite and not abelian. □

8.3.1. Final remarks. I worked with the von Dyck group just because the absence of reflections makes everything easier; the passage to the full triangle group requires more care, but it can be done relying on standard techniques of double coverings.

8.4. Computational experiments

The methods and the techniques used to solve the Burnside problem are incomparable: just look at the proofs of the finiteness of $B(2,3)$ and $B(2,3)$ in Sections 1.3.4 and 1.3.5 respectively. Some proofs, like this of the infiniteness of the free Burnside groups of a large odd exponent, originally proved by S.I. Adian and P.S. Novikov [40] in 1968 using combinatorial methods take several hundreds of pages!

However, recently T. Delzant and M. Gromov [17] provided a new proof of it by a geometrical approach of small cancellation theory (see [22]).

Here I present an algorithm, implemented in Wolfram Mathematica™, which determines the fundamental domain of the Burnside group with 2 generators of order n.

8.4.1. A toy model: $n = 3$. The group $B(2,3)$ is a quotient of the planar von Dyck group (see Definition 5). This fact make it easier to write an algorithm establishing the fundamental domain for this group, since I deal just with Euclidean plane.

As a first thing, I define the matrices of rotation corresponding to the two generators. I have to underline that one of the vertices of the basic triangle is the point $(0, i)$.

```
u = TransformationMatrix[AffineTransform[{RotationMatrix[2π/3], {0, 0}}]]

v = TransformationMatrix[
  Composition[AffineTransform[{RotationMatrix[2π/3], {1, 0}}],
   AffineTransform[{IdentityMatrix[2], {-1, 0}}]]]
```

The next step is to create the list of the elements of K_n (see Section 8.3):

```
n = 3;
l = List[List[1]]
For[k = 2, k < n, k++,
  l = Append[l, List[k]]
 ];
lpulita = l;
For[k = 1, k ≤ 4, k++,
  L = Length[l];
  M = Length[Last[l]];
  For[i = 1, i ≤ L, i++,
   a = l[[i]]; la = Length[a];
   If[la == M,
    t = Total[a];
    For[s = 1, s < n, s++,
     l = Append[l, Append[a, s]];
     If[Mod[t + s, n] == 0,
      lpulita = Append[lpulita, Append[a, s]]
     ]
    ]
   ]
  ]

 ];
l = Drop[lpulita, {1, n - 1}]
```

And realize this elements as transformations of the plane:

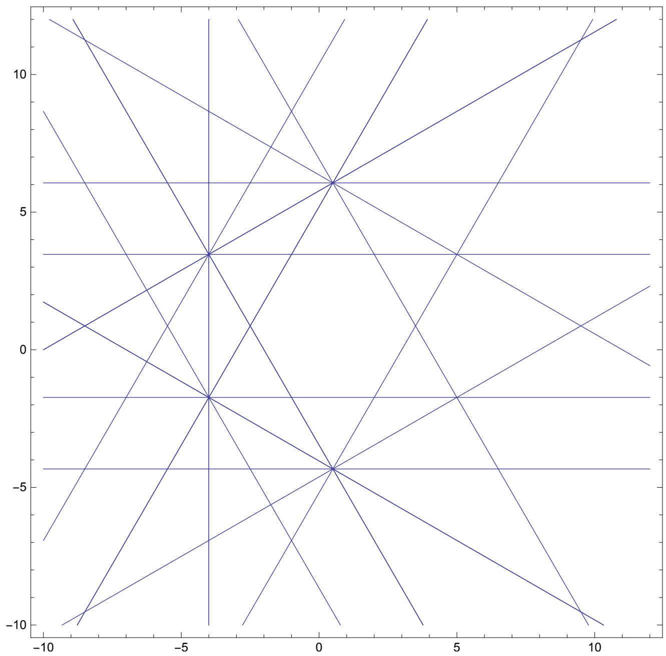
```
W = List[];
For[i = 1, i ≤ Length[l], i++,
 Z = {{1, 0, 0}, {0, 1, 0}, {0, 0, 1}};
 a = l[[i]];
 For[j = 1, j ≤ Length[a], j++,
  If[Mod[j, 2] == 0,
   Z = Z.MatrixPower[v, (a[[j]])],
   Z = Z.MatrixPower[u, (a[[j]])]
  ]
 ];
 W = Append[W, MatrixPower[Z, 3] // Simplify]
]
```

Then I need to calculate the axis of the line segment joining the point $(0, i)$ with its image after the transformations from the list above.

```
For[i = 1, i ≤ Length[W], i++, Y = TransformationFunction[W[[i]]][X];
 f = Append[f, (Y[[2]] - X[[2]]) y ==
     - (Y[[1]] - X[[1]]) x + (Y[[2]]^2 - X[[2]]^2 + Y[[1]]^2 - X[[1]]^2)/2]]
f

f = List[]
For[i = 1, i ≤ Length[W], i++, Y = TransformationFunction[W[[i]]][X];
 f = Append[f, (Y[[2]] - X[[2]]) y +
     (Y[[1]] - X[[1]]) x - (Y[[2]]^2 - X[[2]]^2 + Y[[1]]^2 - X[[1]]^2)/2 > 0]]
```

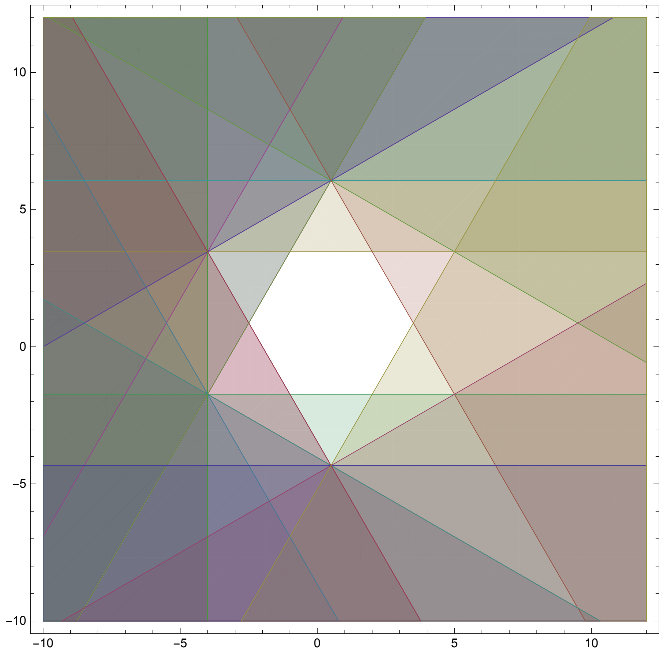
As a result I obtain a closed polygon —hexagon — in which the fundamental domain is contained:

```
ContourPlot[f == 0, {x, -10, 12}, {y, -10, 12}]
```



Finally, to obtain a for better visualisation of the result, I outline it with coloured half–planes:

```
RegionPlot[f, {x, -10, 12}, {y, -10, 12}]
```



8.4.2. The general case. The hyperbolic case starts from the order $n = 4$, on which I base on here. Of course, n can be freely chosen. It should be taken into account that, as n passes from 3 to 4 in $B(2, n)$ the order of $B(2, n)$ increases from 27 to 4096!

Like in the Euclidean case, I fix one vertex of the basic triangle on the point $(0, i)$. Since there exists a close dependency between the angle of a given triangle (by setting number n) and the other two vertices of the triangle (see Proof at the page 65), it useful to find them automatically:

```
n = 4;
ε = Pi / n;
ϰ = (2 * Cos[ε]) / (1 - Cos[ε]);

s = FullSimplify[Solve[β + 1/β == ϰ, β]];
b1 = β /. s[[1]];
b2 = β /. s[[2]];

If [1 > b1 > 0 ,  β = N[b1],
  β = N[b2]]
```

And then I define the matrices of rotation:

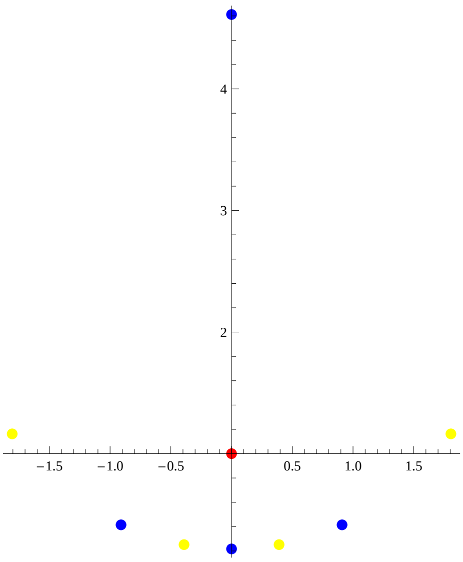
```
u = {{Cos[ε], -Sin[ε]}, {Sin[ε], Cos[ε]}} // Simplify;
v = {{Cos[ε], -β * Sin[ε]}, {Sin[ε] / β, Cos[ε]}} // Simplify;

R = {-1/2 (-1 + β) Tan[ε], 1/2 Sqrt[(1 + β)^2 - (-1 + β)^2 Sec[ε]^2]}
B = {0, β}
Rlist = List[R];
Blist = List[B];
```

The next step is not all-important for the algorithm, but it just help to depict the procedure. It calculates the points of the first "stripe", i.e., first rotation of one vertex of the triangle around the other:

```
For[i = 1, i < 4, i++,
 z = Last[Rlist][[1]] + I * Last[Rlist][[2]];
 w = (u[[1]][[1]] * z + u[[1]][[2]]) / (u[[2]][[1]] * z + u[[2]][[2]]) // Simplify;
 R2 = {Re[w], Im[w]};
 Rlist = Append[Rlist, R2];
 z = Last[Blist][[1]] + I * Last[Blist][[2]];
 w = (u[[1]][[1]] * z + u[[1]][[2]]) / (u[[2]][[1]] * z + u[[2]][[2]]) // Simplify;
 R2 = {Re[w], Im[w]};
 Blist = Append[Blist, R2]
]
```

The image of this is:

```
Graphics[{
  {PointSize[Large], Red, Point[{0, 1}]},
  {PointSize[Large], Blue, Point[Blist]}, {PointSize[Large], Yellow, Point[Rlist]}},
 Axes →
  True]
```



Then I create a module `htri` which takes a list of three points and return a list of three equations in unknowns x and y, representing the hyperbolic lines passing through the input points:

```
htri[P_] :=
 Module[{f = List[], R = {0, 0}, S = {0, 0}, A = 0, r = 0, i = 1},
  For[i = 1, i < 4, i++,
   R = P[[i]];
   If[i == 3,
    S = P[[1]],
    S = P[[i + 1]]
   ];
   If[R[[1]] == S[[1]],
    f = Append[f, x - R[[1]]],
    A = {(R[[1]]^2 - S[[1]]^2 + R[[2]]^2 - S[[2]]^2) / (2 (R[[1]] - S[[1]])), 0};
    r = (R - A).(R - A);
    f = Append[f, FullSimplify[(x - A[[1]])^2 + y^2 - r]]
   ];
  ];
  f
 ]
```

Then — once again just for visualization — the points of first stripe I have to rewrite in a list of three points sets,

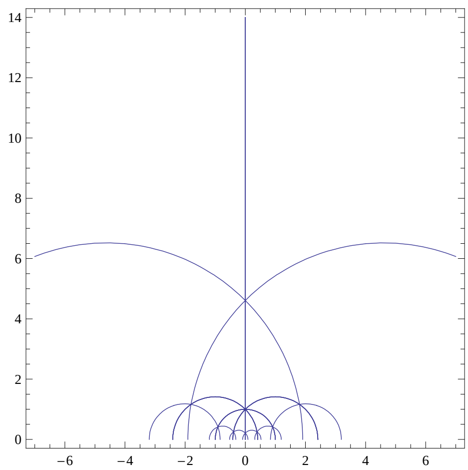
```
Tlist = List[];
For[i = 1, i ≤ Length[Rlist], i++,
 Tlist = Append[Tlist, {{0, 1}, Blist[[i]], Rlist[[i]]}];
 If[i < Length[Rlist],
  Tlist = Append[Tlist, {{0, 1}, Rlist[[i + 1]], Blist[[i]]}],
  Tlist = Append[Tlist, {{0, 1}, Rlist[[1]], Blist[[i]]}]]
]
```

and apply this list to the module htri,

```
f = List[];
For[i = 1, i < Length[Tlist] + 1, i++,
  f = Append[f, htri[Tlist[[i]]]]
 ];
fp = List[];
For[i = 1, i < Length[f] + 1, i++,
  fp = Append[fp, f[[i]][[1]]];
  fp = Append[fp, f[[i]][[2]]];
  fp = Append[fp, f[[i]][[3]]]

 ];
```

so, it can be seen all the triangles of the first strip:

```
ContourPlot[fp == 0, {x, -7, 7}, {y, 0, 14}]
```



The next steps, are analogues to the Euclidean case, i.e.,creating the list of elements of K_n:

```
l = List[List[1]]
For[k = 2, k < n, k++,
  l = Append[l, List[k]]
 ];
lpulita = l;
For[k = 1, k ≤ 3, k++,
  L = Length[l];
  M = Length[Last[l]];
  For[i = 1, i ≤ L, i++,
   a = l[[i]]; la = Length[a];
   If[la == M,
    t = Total[a];
    For[s = 1, s < n, s++,
     l = Append[l, Append[a, s]];
     If[Mod[t + s, n] == 0,
      lpulita = Append[lpulita, Append[a, s]]
     ]
    ]
   ]
  ]

 ];
l = Drop[lpulita, {1, n - 1}]
```

And realizing them as the transformations of the plane:

```
W = List[];
For[i = 1, i ≤ Length[l], i++,
 Z = {{1, 0}, {0, 1}};
 a = l[[i]];
 For[j = 1, j ≤ Length[a], j++,
  If[Mod[j, 2] == 0,
   Z = Z.MatrixPower[v, (a[[j]])],
   Z = Z.MatrixPower[u, (a[[j]])]
  ]
 ];
 If[MatrixPower[Z, n] != {{1, 0}, {0, 1}},
  W = Append[W, MatrixPower[Z, n] // Simplify]
 ]]
```

Then program calculate the axis of the line segment joining the point $(0, i)$ with its image after transformations from the list above.

```
fd = List[];
For[i = 1, i ≤ Length[W], i++, Y = W[[i]];
 aa = (Y[[1]][[2]] Y[[2]][[2]] + Y[[1]][[1]] Y[[2]][[1]]) / (Y[[2]][[1]]^2 + Y[[2]][[2]]^2) // Simplify;
 bb = 1 / (Y[[2]][[1]]^2 + Y[[2]][[2]]^2) // Simplify;
 fd = Append[fd, (bb - 1) x^2 + (bb - 1) y^2 + 2 aa * x + bb - aa^2 - bb^2 > 0 // Simplify ] ]
```

Unfortunately, the overall process is still "too heavy" for a standard PC, and I was not able to get the expected result.

CHAPTER 9

Appendix

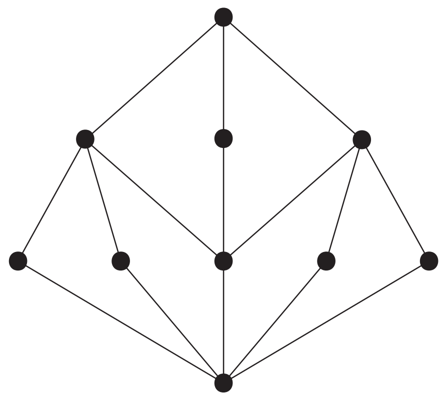
9.1. A survey on existing group–to–graph correspondences

The backbone of this thesis is the idea of associating a graph to a group, in order to obtain a visual perspective on the group itself. I tried to show that, in certain cases, such point of view can actually indicate an advantageous way to prove old and new algebraic results. The huge literature revolving around this idea gives a feeling that the number of ways in which a graph can be associated to a group is limited just by one's imagination. Let me just recall here a few of them, which are probably the most successful ones.

9.1.1. Schreier coset graph. This is a particular case of a Cayley graph (see 2.4). Such a graph is associated to a group G, a generating set $S \subseteq G$, and a subgroup $H \leqslant G$. The difference with the standard Cayley graph is that the vertices here are the right cosets of H. For more information, look for example [52].

9.1.2. Graph of maximal subgroups. Another way, which can be found, e.g., in the paper [29] by M. Herzog, P. Longobardi and M. Maj, is to associate to a finitely generated group a graph in which the maximal subgroups of a given group are the vertices, and where two vertices are connected by an edge if the corresponding subgroups intersect each other non–trivially.

9.1.3. Hasse diagram. Such a diagram [59] is usually associated to a finite partially ordered set. Since the set of all subgroups of a given group can be seen as a partially ordered set, it is very natural to associate vertices to subgroups. Furthermore, if one subgroup is contained into another, with no intermediate subgroup properly between them, then an edge connects them, as it can be seen from the example of $\mathbf{D}_4$ (the dihedral group, see Example 3). The whole $\mathbf{D}_4$ group has 8 permutations and this has 9 subgroups:3 subgroups with 4 permutations, 5 subgroups with 2 permutations, 1 trivial identity permutation.



9.1.4. Non–cyclic graph. Another related graph is the non–cyclic graph which is obtained by connecting vertex one with another if and only if the subgroup generated by them is not cyclic and then removing isolated vertices. This graph has been studied for example in [1].

9.1.5. Conjugacy graph. This is an example where the vertices are not elements, but conjugacy classes of the group instead. It is a graph related to conjugacy classes, where two vertices are connected if their cardinalities are not coprime. More about it can be found in [7].

9.1.6. Graph of groups. A somehow different construction involving groups and graphs is the so–called a **graph of groups** [4] which is an object consisting of a collection of groups indexed both by the vertices and the edges of a given graph, together with a family of monomorphisms from the edge groups into the vertex groups.

Besides these examples, there are of course many more associations. It can be very useful to understand which properties these associated graphs possess and how these properties are related to the algebraic properties of the group (or viceversa).

9.2. The word problem for groups

The problem of deciding whether a word in a group G defines the identity element (or equivalently whether two words define the same element) is the first of the famous three problems formulated by Max Dehn in 1911. These problems are important for presentation theory, as well as for its applications.
Let a group $G = \langle S, R \rangle$ be defined by means of a given presentation (see 1.1.1). For an arbitrary word W in the generators, the problem is to decide in a finite number of steps whether W defines the identity element of G, or not.
The word problem has a simple characterization in terms of the Cayley graph. A word $W \in G$ labels a path starting at the identity and ending at the value of the word. Evidently, a word represents the identity if and only if such a cycle is closed. In other words, it holds the next fundamental theorem, which added some perspective to my thesis.

Theorem 6. *$G = \langle S, R \rangle$ has soluble word problem if and only if there is an algorithm capable of constructing any finite portion of $\Gamma(G, S)$.*

Proof. See, e.g., [39]. (See also [35]). □

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