A problem in the Theory of Groups and a question related to Fibonacci-Like sequences

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Abstract

This thesis is composed of four chapters, two of which, Chapter 3 and Chapter 4, contain original results. In Chapter 1 we recall some basic notions and establish some of the notation and terminology which will be used in the sequel. For example, we recall some useful results about the class X of groups which are isomorphic to their non-abelian subgroups. Every group of this class is infinite and 2-generated. This class of groups has been studied by H. Smith and J. Wiegold([34]). They proved that every insoluble X-group is centre-by-simple and they gave a complete characterization of soluble X-groups. Then we recall some results about finitely generated groups which are isomorphic to their non-trivial normal subgroups. In particular, we will use the result proved by J.C. Lennox, H. Smith and J. Wiegold in [17], for which if G is a finitely generated infinite group that is isomorphic to all its non-trivial normal subgroups and which contains a proper normal subgroup of finite index, then $G \simeq \mathbb{Z}$.

Given a group G, a subgroup K of G is said to be a *derived subgroup* or *commutator subgroup* in G if $K = H'$ where H' is the derived subgroup of H, with H subgroup of G. Recently, many authors have been interested in studying the set of derived subgroups in the lattice of all subgroups.

Let $C(G)$ denote the set of all derived subgroups in G:

$$C(G) = \{H' | H \leq G\}.$$

The influence of $C(G)$ on the structure of the group G has been studied by many authors. For example, F. de Giovanni and D.J.S. Robinson in[8] and M. Herzog, P. Longobardi and M. Maj in[14], have investigated groups G for which $C(G)$ is finite. In particular, they proved that if G is locally graded, then $C(G)$ is finite if and only if G' is finite.

Let n be a positive integer and let D_n denote the class of groups with n isomorphism types of derived subgroups. Clearly D_1 is the

class of all abelian groups and a group G belongs to D_2 if and only if G is non abelian and $H' \simeq G'$ whenever H is a non abelian subgroup of G. P. Longobardi, M. Maj, D.J.S. Robinson and H. Smith in [18] focused their attention on groups in D_2 and described in a precise way some large classes of D_2-groups.

In Chapter 2 we recall some results about D_2-groups.

In this thesis we analyse a dual problem. Let $B(G)$ denote the set of the central factors of all subgroups of a group G:

$$B(G) = \{\frac{H}{Z(H)} | H \leq G\}$$

and let B_n denote the class of groups for which the elements of $B(G)$ fall into at most n isomorphism classes, where n is a positive integer. Clearly B_1 is the class of abelian groups and G is a B_2-group if and only if every subgroup of G is abelian or $\frac{H}{Z(H)} \simeq \frac{G}{Z(G)}$ for all non abelian subgroups H of G.

In Chapter 3 of this thesis we study B_2-groups. For example, it is possible to see that if G is a group where $\frac{G}{Z(G)}$ is elementary abelian of order p^2, with p prime, then G is in B_2. Moreover, if G is a group with $\frac{G}{Z(G)} \simeq \mathbb{Z} \times \mathbb{Z}$, then $G \in B_2$. Groups in B_2 can be very complicated, in fact a non abelian group whose proper subgroups are all abelian is also a B_2-group and so Tarski Monster Groups, infinite simple groups with all proper subgroups abelian, whose existence was proved by A.Yu. Ol'shankii in 1979, are B_2-groups. First we proved some elementary results for B_2-groups. For example it may be seen that the class of B_2-groups is closed under the formation of subgroups and not closed under the formation of homomorphic images but if G is a nilpotent group in B_2 then $\frac{G}{S} \in B_2$, for any $S \leq Z(G)$. In addition, $\frac{G}{Z(G)}$ is 2-generated for every G in B_2 and if G is also nilpotent, then $\frac{G}{Z(G)}$ is abelian. Then we analyse nilpotent B_2-groups and we prove that if G is non abelian, then G is a nilpotent group in B_2 if and only if either $\frac{G}{Z(G)}$ is elementary abelian of order p^2, where p is a prime, or $\frac{G}{Z(G)}$ is the direct product of two infinite cyclic groups. We also study locally finite groups in B_2 and we show that if G is locally finite, then G is in B_2 if and only if $G = Z(G)H$ where H is finite, minimal non abelian. Then we study soluble groups. We show that if G is a soluble non nilpotent group in B_2, then G is metabelian and in this hypothesis we prove that $Z(\frac{G}{Z(G)}) = 1$, $G = A < x >$, for a suitable x in G and a normal abelian subgroup A of G, and every non abelian subgroup of $\frac{G}{Z(G)}$ is isomorphic to $\frac{G}{Z(G)}$. Finally, we analyse the case of non soluble B_2-groups and we prove that they do not satisfy the *Tits alternative*, i.e. soluble by finite groups or groups that contain a free subgroup of rank 2. Up to this point none of the special types of B_2-groups we have analysed has involved *Tarski groups*. But in this last case we have proved that if G is a non soluble B_2 group and G' satisfies the minimal condition on subgroups, then $\frac{G}{Z(G)}$ is simple, minimal non abelian, every soluble subgroup of G is abelian and if N is a normal subgroup of G, then either $N \leq Z(G)$ or $G' \leq N$. In particular $\frac{G}{Z(G)}$ is a Tarski group.

In Chapter 4 we show a result about Fibonacci-like sequences, obtained in collaboration with Professor Giovanni Vincenzi. This results appear in a published paper, *Fibonacci-like sequences and generalized Pascal's triangle*. We have studied the properties pertaining to diagonals of generalized Pascal's triangles $T(k_1, k_2)$ created using two complex numbers. We have also introduced a particular Fibonacci-like sequence $\{H_n\}_{n\in\mathbb{N}}$ whose seeds are the complex numbers considerated above. As in the case of Pascal's triangle, we have found a relationship between the Fibonacci sequence $\{F_n\}_{n\in\mathbb{N}}$ and the sequence $\{D_n\}_{n\in\mathbb{N}}$ of diagonals we have created.

In particular we have proved that the sequence $\{D_n\}_{n\in\mathbb{N}}$ of the numbers which arise when we consider the diagonals of a generalized $T(k_1, k_2)$ is recursive and that the following relationship holds:

Theorem Let k_1 and k_2 be complex numbers. Let $\{D_n\}_{n\in\mathbb{N}}$ be the associate sequence to the generalized Pascal's triangle $T(k_1, k_2)$ and $\{H_n\}_{n\in\mathbb{N}}$ be the Fibonacci-like sequence of seeds k_1 and k_2. Then the following identity holds:

$$H_n - D_n = F_{n-3}(k_2 - k_1), \forall n \in \mathbb{N}.$$

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