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MV-ALGEBRAS, GROTHENDIECK TOPOSES AND APPLICATIONS

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Résumé en français

Cette thèse est une contribution au programme de recherche ‘topos comme ponts’ introduit dans [12], qui vise à développer le potentiel unifiant de la notion de topos de Grothendieck comme un moyen pour relier entre elles différentes théories mathématiques via des invariants topos-théoriques. La méthodologie générale, qui y est précisée, est ici appliquée pour étudier des équivalences catégoriques déjà connues d’intérêt particulier dans le domaine des logiques multi-valuées et aussi pour en produire de nouvelles. Le contenu original de la thèse est inclus dans [21], [20] et [22].

Topos de Grothendieck

La notion de *topos* a été introduite par A. Grothendieck au début des années 1960 dans sa reformulation de la théorie des faisceaux pour la géométrie algébrique. Il étudia les faisceaux non seulement sur des espaces topologiques, mais également sur des *sites*, c’est-à-dire des catégories dotées d’une topologie soi-disant de Grothendieck. Il définit les topos (de Grothendieck) comme des catégories équivalentes à une catégorie de faisceaux sur un site. Puisque de nombreuses propriétés classiques des espaces topologiques peuvent être naturellement formulées en termes des propriétés des catégories des faisceaux associées, les topos de Grothendieck peuvent être considérés comme des ‘espaces généralisés’.

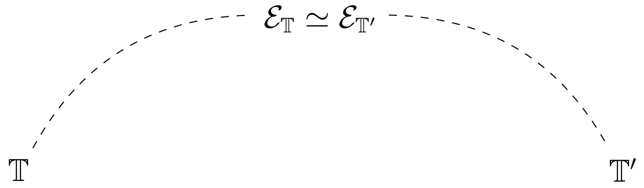
Plus tard, W. Lawvere et M. Tierney observèrent que les topos peuvent être également considérés comme des ‘univers mathématiques généralisés’ où

la plupart des constructions familières, qu'on effectue habituellement avec les ensembles, peuvent être reproduites, comme les produits, les coproduits, et cetera. En fait, les topos de Grothendieck sont assez riches en termes de structure catégorique pour y considerer, à leur intérieur, des modèles de toute sorte de théorie du premier ordre.

A la fin des années soixante-dix, l'école de Montréal de logique catégorique, qui comprend notamment M. Makkai, G. Reyes et A. Joyal, introduisit le concept de *topos classifiant* d'une théorie géométrique (c'est-à-dire une théorie sur une signature du premier ordre dont les axiomes sont des séquents formés par des formules construites à partir de formules atomiques en utilisant uniquement des conjonctions finitaires, disjonctions infinitaires et quantifications existentielles). Ils ajoutèrent de cette manière un troisième point de vue sur les topos. En fait, ils prouvèrent que toute théorie géométrique $\mathbb{T}$ a, à équivalence catégorique près, un unique *topos classifiant* $\mathcal{E}_{\mathbb{T}}$, qui est un topos de Grothendieck contenant un *modèle universel* $U_{\mathbb{T}}$ de $\mathbb{T}$, où universel signifie que tous les autres modèles de $\mathbb{T}$ dans tous les autres topos de Grothendieck $\mathcal{E}$ sont, à isomorphisme près, l'image par (l'image inverse de) un unique morphisme de topos de $\mathcal{E}$ à $\mathcal{E}_{\mathbb{T}}$. Réciproquement, tous les topos de Grothendieck peuvent être considérés comme les topos classifiant d'une théorie géométrique. Il est possible que deux théories mathématiques distinctes aient, à équivalence catégorique près, le même topos classifiant ; dans ce cas, les théories sont dites *Morita-équivalentes*. Pourtant, les topos de Grothendieck peuvent non seulement être considérés comme des espaces généralisés ou des univers généralisés, mais aussi comme des théories, considérées à équivalence de Morita près.

Cette troisième incarnation de la notion de topos est devenue la base de la méthodologie 'topos comme ponts' introduite par O. Caramello dans [12] et développée dans les dernières années. L'existence de différentes représentations du même topos de Grothendieck, donné par exemple par différents sites de définition ou par des théories Morita-équivalentes, permet de transférer

des informations et des résultats d'une représentation à l'autre en utilisant des invariants topos-théoriques sur ce topos comme des 'machines' traductrices.



La puissance de cette technique réside dans le fait qu'un invariant topos-théorique donné peut se manifester de manière complètement différente en termes de différents sites de définition du même topos. On peut alors établir au moyen de ces caractérisations des relations logiques ou des équivalences entre des propriétés ou des constructions complètement différentes en rapport à divers sites. Un exemple remarquable de l'application de cette technique est l'interprétation topos-théorique de la construction de Fraïssé, en théorie des modèles, établie dans [18].

La théorie des topos a déjà été appliquée avec succès dans le cadre des logiques multi-valuées pour établir des représentations en termes de faisceaux de classes notables de MV-algèbres, par exemple dans les travaux de E. J. Dubuc et Y. Poveda ([30]) et de J. L. Castiglione, M. Menni et W. J. Botero ([23]). D'autres représentations en termes des faisceaux ont été établies par A. Filipoiu et G. Georgescu ([32]), et par A. R. Ferraioli et A. Lettieri ([31]).

L'innovation de cette thèse est d'utiliser des méthodes topos-théoriques afin d'obtenir, d'une part des nouveaux résultats de nature à la fois logique et algébrique, et d'autre part des aperçus conceptuels sur des sujets centraux dans le domaine des MV-algèbres, qui ne sont pas visibles avec des méthodes classiques. Nous obtenons ces nouveaux résultats en étudiant les topos classifiants de remarquables théories de MV-algèbres et en appliquant la technique des ponts à des équivalences de Morita entre ces théories et des théories appropriées des groupes abéliens réticulés.

Logiques multi-valuées et MV-algèbres

Motivé par le fait que la logique classique ne peut pas décrire des situations qui admettent plus de deux résultats, J. Łukasiewicz introduisit en 1920 une logique à trois valeurs en ajoutant aux traditionnelles valeurs de vérité 0 et 1, interprétées comme "absolu faux" et "absolu vrai" un troisième degré de vérité entre eux. Plus tard, il présenta de nouvelles généralisations avec n valeurs de vérité (ou même un nombre dénombrable ou continu).

La classe des MV-algèbres a été introduite en 1958 par C. C. Chung (cf. [24] et [25]) afin de fournir une sémantique algébriques pour la logique propositionnelle multi-valuée de Łukasiewicz. Comme cette logique est une généralisation de la logique classique, les MV-algèbres sont une généralisation des algèbres de Boole (ceux-ci peuvent être caractérisées comme les MV-algèbres idempotentes).

Après leur introduction dans le contexte de la logique algébrique, les MV-algèbres devinrent des objets d'intérêt indépendant et de nombreuses applications dans différents domaines des mathématiques ont été trouvés. Les plus remarquables sont en analyse fonctionnelle (cf. [39]), en la théorie des groupes abéliens réticulés (cf. [39] et [28]) et en la théorie de la probabilité généralisée (cf. Chapitres 1 et 10 de [41] pour un aperçu général).

Dans la littérature plusieurs équivalences entre des catégories de MV-algèbres et des catégories de groupes abéliens réticulés (ℓ-groupes) peuvent être trouvées. Nous rappelons les plus importantes :

- l'*equivalence de Mundici* (cf. [39]) entre la catégorie totale des MV-algèbres et la catégorie des ℓ-groupes avec unité forte ;
- l'*equivalence de Di Nola et Lettieri* (cf. [28]) entre la catégorie des MV-algèbres parfaites (c'est-à-dire MV-algèbres générées par leur radical) et la catégorie totale des ℓ-groupes.

Nous observons que ces équivalences catégoriques peuvent être considérées comme des équivalences entre des catégories de modèles sur les ensembles de certaines théories géométriques et nous montrons que ces théories sont Morita-équivalentes, autrement dit, il y a une équivalence catégorique entre leurs catégories de modèles à l'intérieur de tout topos de Grothendieck $\mathcal{E}$, naturellement dans $\mathcal{E}$.

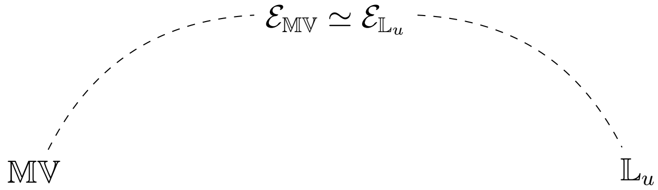
De cette façon, nous obtenons :

- une équivalence de Morita entre la théorie $\mathbb{MV}$ des MV-algèbres et la théorie $\mathbb{L}_u$ des ℓ-groupes avec unité forte (cf. Chapitre 3) ;
- une équivalence de Morita entre la théorie $\mathbb{P}$ des MV-algèbres parfaites et la théorie $\mathbb{L}$ des ℓ-groupes (cf. Chapitre 4).

Nous montrons ensuite que l'équivalence de Morita résultante de l'équivalence de Di Nola-Lettieri est seulement une parmi toute une classe des équivalences de Morita, que nous établissons entre des théories des MV-algèbres locales dans des variétés propres des MV-algèbres et des extensions appropriées de la théorie des ℓ-groupes (cf. Chapitre 5).

Conséquences de l'équivalence de Morita entre $\mathbb{MV}$ et $\mathbb{L}_u$

Une conséquence immédiate de l'équivalence de Morita résultante de l'équivalence de Mundici est le fait que le théorie (infinitaire) des ℓ-groupes avec unité forte est de type préfaisceau. Ceci provient du transfert de la propriété invariante d'être un topos de préfaisceaux à travers l'équivalence de Morita. Rappelons qu'une théorie est de type préfaisceau si son topos classifiant est équivalent à un topos de préfaisceaux. Toute théorie algébrique finie, et plus généralement, toute théorie cartésienne, est de type préfaisceau ; ainsi, cette propriété est transférée à partir de la théorie des MV-algèbres à $\mathbb{L}_u$. On s'intéresse aux théories de type préfaisceau car elles bénéficient de propriétés remarquables, dont certaines sont rappelées dans la Section 1.5, qui ne sont pas satisfaites pour toute théorie géométrique.

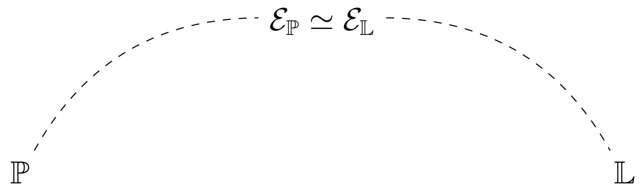


Des modifications de l'invariant considéré au niveau du topos classifiant donne lieu à d'autres résultats. Par exemple, par le Théorème de Dualité de [11] (qui établit une bijection entre les sous-topos du topos classifiant d'une théorie géométrique donnée et les quotients de cette théorie), l'invariant donné par la propriété d'être un sous-topos induit une bijection entre les quotients de la théorie $\mathbb{MV}$ et ceux de la théorie $\mathbb{L}_u$. Il est intéressant de souligner que ce résultat ne peut être déduit de l'équivalence de Mundici. Rappelons qu'un quotient d'une théorie est une extension sur la même signature obtenue en ajoutant des nouveaux axiomes. A partir d'un quotient de $\mathbb{MV}$, on obtient le quotient correspondant de $\mathbb{L}_u$ en traduisant chaque axiom dans le language des ℓ-groupes avec unité forte en utilisant l'interprétation de la théorie $\mathbb{MV}$ à la théorie $\mathbb{L}_u$ établie dans la Section 3.3. Cependant, comme nous avons prouvé dans la même section, il n'y a pas d'interprétation dans la direction contraire qui rendrait trivial la bijection entre les quotients. Si on considère maintenant la propriété invariante des objets des topos d'être irréductibles, on obtient une caractérisation logique des ℓ-groupes finiment présentables avec unité forte. Ils sont les ℓ-groupes avec unité forte correspondant aux MV-algèbres finiment présentées par l'équivalence de Mundici. Plus précisément, nous montrons que ces groupes peuvent être caractérisés comme les ℓ-groupes pointés finiment présentés $\mathcal{G}$ avec élément distinctif v qui est une unité forte pour $\mathcal{G}$, ou, équivalentement, comme les ℓ-groupes présentés par une formule qui est irréductible par rapport à la théorie des ℓ-groupes avec unité forte. Ce dernier résultat est utilisé dans la Section 3.7.2 pour décrire un méthode pour obtenir une axiomatisation d'un quotient de $\mathbb{MV}$ qui correspond à un quotient donné de la théorie $\mathbb{L}_u$. Enfin, nous établissons une forme de compacité et de complétude pour $\mathbb{L}_u$, obtenue à partir des propriétés invariantes du topos classifiant de $\mathbb{MV}$ (donc de $\mathbb{L}_u$) d'avoir un objet terminal compact et d'avoir assez de points.

Enfin, comme cas particulier de cette équivalence de Morita, nous obtenons une version en termes de faisceaux de l'équivalence de Mundici valable pour tout espace topologique X, naturellement dans X.

Conséquences de l'équivalence de Morita entre $\mathbb{P}$ et $\mathbb{L}$ et de l'étude du topos classifiant de $\mathbb{P}$

Comme dans le cas de l'équivalence de Mundici, l'équivalence de Morita résultante de l'équivalence de Di Nola-Lettieri implique une théorie algébrique, c'est-à-dire la théorie $\mathbb{L}$ des ℓ-groupes. Ainsi, la propriété d'être de type préfaisceau est transférée à la théorie cohérente $\mathbb{P}$ des MV-algèbres parfaites. Alors que les deux théories ne sont pas bi-interprétables, d'autres applications de la technique des ponts conduit à trois niveaux différents de bi-interprétabilité entre des classes particulières de formules : formules irréductibles, énoncés géométriques et imaginaires.



Les formules irréductibles pour la théorie $\mathbb{P}$ sont celles qui présentent les MV-algèbres parfaites finiment présentables, c'est-à-dire les algèbres qui correspondent aux ℓ-groupes finiment présentés par l'équivalence de Di Nola-Lettieri. Elles constituent l'analogue pour la théorie $\mathbb{P}$ des formules cartésiennes dans la théorie des MV-algèbres. En fait, même si la catégorie $\mathbb{P}$-mod(**Set**) n'est pas une variété, elle est générée par ses objets finiment présentables puisque la théorie $\mathbb{P}$ est de type préfaisceau classifiée par le topos $[\text{f.p.}\mathbb{P}\text{-mod}(\mathbf{Set}), \mathbf{Set}]$. Nous établissons aussi une bi-interprétabilité

entre la théorie des groupes abéliens réticulés et une théorie cartésienne $\mathbb{M}$ axiomatisant les cÃťnes positifs de ces groupes, que nous utilisons dans la Section 4.5.2 pour obtenir une reformulation plus simple de l'équivalence de Di Nola-Lettieri et dans la Section 4.5.3 pour décrire les bi-interprétations partielles entre $\mathbb{L}$ et $\mathbb{P}$. Cette bi-interprétation entre $\mathbb{M}$ et $\mathbb{L}$ donne en particulier une autre description du groupe de Grothendieck associé à un modèle $\mathcal{M}$ de $\mathbb{M}$ comme un sous-ensemble, au lieu d'un quotient comme dans la définition classique, du produit $\mathcal{M} \times \mathcal{M}$.

Ensuite, on étudie en détail le topos classifiant de la théorie des MV-algèbres parfaites. Ce topos est représenté comme un sous-topos du topos classifiant de la théorie algébrique axiomatisant la variété générée par la MV-algèbre de Chang. Cette étude met en lumière la relation entre ces deux théories, en conduisant notamment à un théorème de représentation pour les algèbres finiment générées (resp. finiment présentées) dans la variété de Chang comme produits finis des MV-algèbres parfaites finiment générées (resp. finiment présentées). Il est intéressant de noter que ce résultat, contrairement à la plupart des théorèmes de représentation disponibles dans la littérature, est entièrement constructif. Parmi les autres aperçus, on mentionne une caractérisation des MV-algèbres parfaites correspondant aux groupes abéliens réticulés finiment présentés par l'équivalence de Di Nola-Lettieri comme les objets finiment présentés de la variété de Chang qui sont des MV-algèbres parfaites, et la propriété que la théorie axiomatisant la variété de Chang prouve tous les séquents cartésiens (en particulier, toutes les identitées algébriques) qui sont valables dans tous les MV-algèbres parfaites.

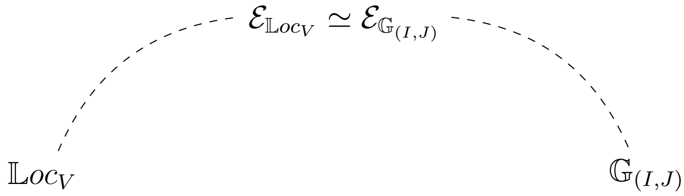
On revisite ensuite le théorème de représentation obtenu par l'analyse du topos classifiant de $\mathbb{P}$ du point de vue des produits sous-directes des MV-algèbres parfaites, pour en obtenir une preuve concrète. Nous montrons aussi que toute MV-algèbre dans la variété de Chang est un produit sous-direct faible des MV-algèbres parfaites. Ces résultats ont des liens étroits avec la littérature existante sur les produits booléens faibles des MV-algèbres. De plus,

dans le domaine des MV-algèbres dans la variété de Chang, nous généralisons la caractérisation de Lindenbaum-Tarski des algèbres booléennes qui sont isomorphes à ensembles des parties comme algèbres booléennes atomiques complètes, en obtenant une caractérisation intrinsèque des MV-algèbres dans la variété de Chang qui sont des produits arbitraires des MV-algèbres parfaites. Ces résultats montrent que la variété de Chang constitue un cadre MV-algébrique particulièrement naturel qui étend la variété des algèbres booléennes.

Enfin, nous transférons les théorèmes de représentation mentionnés ci-dessus pour les MV-algèbres dans la variété de Chang en termes des MV-algèbres parfaites dans le contexte des ℓ-groupes avec unité forte et, en généralisant des résultats dans [2], nous montrons que la théorie des MV-algèbres pointés et parfaites est Morita-équivalente à la théorie des groupes abéliens réticulés avec unité forte (donc à celle des MV-algèbres).

Équivalences de Morita pour MV-algères locales dans les variétés propres des MV-algèbres

Compte-tenu du fait que la classe des MV-algèbres parfaites est l'intersection de la classe des MV-algèbres locales avec une variété propre des MV-algèbres spécifique, c'est-à-dire la variété de Chang, il est naturel de se demander ce qui se passe si on remplace cette variété avec une variété des MV-algèbres arbitraire. Nous montrons que 'globalement', c'est-à-dire en considérant l'intersection avec toute la variété des MV-algèbres, la théorie des MV-algèbres locales n'est pas de type préfaisceau, alors que si on se limite à une sous-variété propre V, la théorie des MV-algèbres locales, indiquée par le symbole $\mathbb{L}oc_V$, est de type préfaisceau. En outre, nous montrons que ces théories sont Morita-équivalentes aux théories appropriées qui étendent la théorie des ℓ-groupes. Plus précisément, si $V = V(\{S_i\}_{i\in I}, \{S_j^{\omega}\}_{j\in J})$ (pour des sous-ensembles finis $I, J \subseteq \mathbb{N}$), on a une théorie $\mathbb{G}_{(I,J)}$ qui est Morita-équivalente à la théorie $\mathbb{L}oc_V$ et qui est écrite sur la signature obtenue à partir de celle des ℓ-groupes en ajoutant un symbole de constante et des prédicats propositionnels correspondant aux éléments de I et J. Les catégories de modèles sur les ensembles de ces théories ne sont en général pas algébriques comme dans le cas des MV-algèbres parfaites ; cependant, dans la Section 5.5.2, nous caractérisons les variétés V tale que on a algébricité précisément comme celles qui peuvent être générées par une seule chaîne. Toute les équivalences de Morita contenues dans cette nouvelle classe ne sont pas triviales, c'est-à-dire elles ne surgisent pas à partir des bi-interprétations, comme nous le démontrons dans la Section 5.5.1.



Des méthodes topos-théoriques sont utilisées ici pour obtenir des résultats à la fois logiques et algébriques. Plus précisément, nous présentons deux axiomatisations (non-constructivement) équivalentes pour la théorie des MV-algèbres locales dans une sous-variété propre arbitraire V et nous étudions les topologies de Grothendieck qui leur sont associées comme quotients de la théorie algébrique $\mathbb{T}_V$ axiomatisant V. La sous-canonicité de la topologie de Grothendieck associée à la première axiomatisation assure que le cartesianisation de la théorie des MV-algèbres locales en V est la théorie $\mathbb{T}_V$. Il est intéressant de noter que ce résultat ne provient pas d'un théorème de représentation des algèbres dans V comme produits sous-directs ou sections globales des faisceaux des modèles de la théorie des MV-algèbres locales dans V, ce qui le qui rendrait trivial. Pour vérifier la provabilité d'un séquent cartésien dans la théorie $\mathbb{T}_V$, on est donc réduit à la vérifier dans la théorie des MV-algèbres locales dans V. En l'utilisant, nous prouvons facilement que le radical de toute MV-algèbre en V est défini par une équation, que on utilise pour présenter la deuxième axiomatisation. Cette dernier axiomatisation a comme remarquable propriété que la topologie de Grothendieck associée est rigide. Cela on permet de conclure que la théorie des MV-algèbres locales en V est de type préfaisceau. L'équivalence des deux axiomatisations et l'égalité des topologies de Grothendieck associées qui en résulte, produit en particulier une représentation de chaque MV-algèbre finiment présentée dans V comme un produit fini des MV-algèbres locales. Ceci généralise le résultat de représentation obtenu pour les MV-algèbres finiment présentées dans la variété de Chang comme produits finis des MV-algèbres parfaites. La théorie des MV-algèbres simples (dans le sens des algèbres universels) est strictement liée à la théorie des MV-algèbres locales ; en effet, une MV-algèbre $\mathcal{A}$ est locale si et seulement si le quotient $\mathcal{A}/Rad(\mathcal{A})$ est une MV-algèbre simple. Cette théorie partage de nombreuses propriétés avec la théorie des MV-algèbres locales : globalement elle n'est pas de type préfaisceau, mais elle l'est si on limite à une sous-variété propre arbitraire. D'autre part, alors que la théorie des MV-algèbres simples de rang fini est de type préfaisceau (car elle coïncide avec la théorie géométrique des chaînes finies), la théorie des MV-algèbres locales de rang fini ne l'est pas, comme nous le prouvons dans la Section 5.2.3.

***

En résumé, dans cette thèse nous utilisons des techniques topos-théoriques afin d'étudier des équivalences de Morita obtenues 'en soulevant' des équivalences catégoriques qui sont déjà connues dans la littérature des MV-algèbres et d'en établir des nouvelles. Cela montre que, comme il a déjà été argumenté dans [12], la théorie des topos est un outil puissant pour découvrir des nouvelles équivalences en Mathématiques et pour examiner celles qui sont connues.

Les principaux thèmes abordés dans cette thèse sont les suivantes :

- théories de type préfaisceau ;

- équivalences de Morita et bi-interprétations ;
- MV-algèbres et groupes abéliens réticulés ;
- résultats de représentation pour classes de MV-algèbres ;
- cartesianisations pour quotients de $\mathbb{MV}$.

Une attention particulière est posée sur le caractère constructif des résultats ; nous indiquons avec le symbole * les points où l'axiome du choix est utilisé.

Structure de la thèse

La thèse est organisée en cinq chapitres.

Chapitre 1. Dans ce chapitre, nous rappelons les notions les plus importantes et les résultats sur la théorie des topos. Nous nous concentrons principalement sur la technique des 'topos comme ponts' et sur les notions de topos classifiant et de théorie de type préfaisceau.

Chapitre 2. Dans ce chapitre, nous introduisons les classes de MV-algèbres qui sont étudiées dans la thèse, c'est-à-dire les MV-algèbres parfaites, locales et simples. De plus, nous établissons quelques résultats préliminaires sur les quotients respectifs de $\mathbb{MV}$. Par exemple, nous montrons que la théorie des MV-algèbres locales et la théorie des MV-algèbres simples ne sont pas de type préfaisceau. De plus,nous introduisons deux axiomatisations équivalentes pour la théorie des MV-algèbres parfaites et nous montrons que le radical de tout MV-algèbre dans la variété de Chang est définissable par une équation. Ce résultat est nécessaire pour définir le radical d'un modèle de la théorie des MV-algèbres parfaits dans un topos de Grothendieck arbitraire puisque la définition classique du radical n'est pas constructif. Nous dérivons aussi le fait que le radical ne peut pas être défini par une formule géométrique dans toute la classe des MV-algèbres comme une conséquence du fait que la classe des MV-algèbres semi-simple ne peut pas être axiomatisée d'une manière géométrique.

Chapitre 3. Dans ce chapitre, nous montrons que la théorie des MV-algèbres est Morita-équivalente à (mais pas bi-interprétables avec) celui des groupes abéliens réticulés avec unité forte. Cela généralise l'équivalence bien connue établie par Mundici entre les catégories de modèles sur les ensembles des deux théories, et permet de transférer des propriétés et des résultats à travers elles en utilisant les méthodes de la théorie des topos. Nous discutons plusieurs applications, y compris une version en termes de faisceaux de l'équivalence de Mundici et une correspondance biunivoque entre les extensions géométriques des deux théories.

Chapitre 4. Nous établissons, en généralisant l'équivalence catégorique de Di Nola-Lettieri, une équivalence de Morita entre la théorie des groupes abéliens réticulés et celui des MV-algèbres parfaites. De plus, après avoir observé que les deux théories ne sont pas bi-interprétables dans le sens classique du terme, nous identifions, en tenant compte des invariants topos-théoriques appropriées sur leurs topos classifiant communs, trois niveaux de bi-interprétabilité pour des catégories particulières des formules : formules irréductibles, énoncés géométriques et imaginaires. Enfin, en étudiant le topos classifiant de la théorie des MV-algèbres parfaites, nous obtenons des résultats différents sur sa syntaxe et sa sémantique et aussi en relation avec la théorie cartésienne de la variété générée par la MV-algèbre de Chang. Ces résultats incluent une représentation concrète pour les modèles finement générées de cette dernière théorie comme produits finis de MV-algèbres parfaites. Nous mentionnons également une équivalence de Morita entre la théorie des groupes abéliens réticulés et celui des monoids cancellatives abéliens réticulés avec élément minimal.

Chapitre 5. Dans ce chapitre, nous étudions les quotients de la théorie géométrique des MV-algèbres locales, en particulier ceux qui axiomatisent la classe des MV-algèbres locales dans une sous-variété propre. Nous mon-

trons que chacun de ces quotients est une théorie de type préfaisceau qui est Morita-équivalente à une extension de la théorie des groupes abéliens réticulés. L'équivalence de Di Nola-Lettieri est obtenue à partir de l'équivalence de Morita pour le quotient axiomatisant les MV-algèbres locales dans la variété de Chang, c'est-à-dire les MV-algèbres parfaites. Nous établissons au passage un certain nombre de résultats d'intérêt indépendant, y compris un traitement constructif du radical pour les MV-algèbres locales dans une variété propre des MV-algèbres fixée et un théorème de représentation des algèbres finiment présentables dans une telle variété comme produits finis des MV-algèbres locales.

Introduction

This thesis is a contribution to the research program 'toposes as bridges' introduced in [12], which aims at developing the unifying potential of the notion of Grothendieck topos as a means for relating different mathematical theories to each other through topos-theoretic invariants. The general methodology outlined therein is applied here to study already existing categorical equivalences of particular interest arising in the field of many-valued logics and also to produce new ones. The original content of the disseration is contained in [21], [20] and [22].

Grothendieck toposes

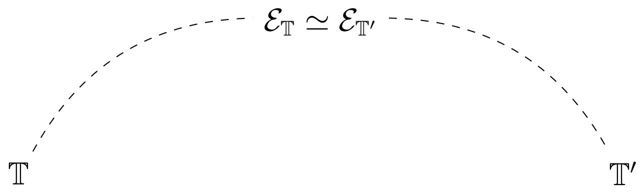
The notion of *topos* was introduced by A. Grothendieck in the early 1960s in his reformulation of sheaf theory for algebraic geometry. He considered sheaves not only on topological spaces but on *sites*, i.e., categories endowed with a so-called Grothendieck topology. He defined (Grothendieck) toposes as categories which are equivalent to a category of sheaves on a site. Since many classical properties of topological spaces can be naturally formulated as properties of the associated categories of sheaves, Grothendieck toposes can be regarded as 'generalized spaces'.

Later, W. Lawvere and M. Tierney realized that toposes can also be considered as 'generalized mathematical universes' where one can reproduce most of the familiar constructions that one is used to perform among sets, like products, coproducts, and so on. In fact, Grothendieck toposes are rich enough in terms of categorical structure to make it possible to consider models of any kind of first-order theory inside them.

At the end of the seventies, the Montréal school of categorical logic, notably including M. Makkai, G. Reyes and A. Joyal, introduced the concept of *classifying topos* of a geometric theory (i.e., a theory over a first-order signature whose axioms are sequents that involve formulas built from atomic ones by only using finitary conjunctions, infinitary disjunctions and existential quantifications). They added in this way a third viewpoint on toposes to the already mentioned ones. Indeed, they proved that every geometric theory $\mathbb{T}$ has a unique, up to categorical equivalence, classifying topos $\mathcal{E}_{\mathbb{T}}$, that is a Grothendieck topos containing a *universal model* $U_{\mathbb{T}}$ of $\mathbb{T}$, universal in the sense that any other model of $\mathbb{T}$ in any other Grothendieck topos $\mathcal{E}$ is, up to isomorphism, the image of this model under (the inverse image of) a unique morphism of toposes from $\mathcal{E}$ to $\mathcal{E}_{\mathbb{T}}$. Vice versa, every Grothendieck topos can be regarded as the classifying topos of a geometric theory. It is possible that two distinct mathematical theories have the same, up to categorical equivalence, classifying topos; in this case we say that the theories are *Morita-equivalent*. Thus, Grothendieck toposes can not only be regarded as generalized spaces or generalized universes, but also as theories, considered up to Morita-equivalence.

This third incarnation of the notion of topos became the basis of the methodology ‘toposes as bridges’ introduced by O. Caramello in [12] and developed throughout the last years. The existence of different representations of the same Grothendieck topos, given for instance by different sites of definition or by Morita-equivalent theories, allows to transfer information and results from one representation to the other by using topos-theoretic invariants on that topos as translating ‘machines’.

The power of this technique lies in the fact that a given topos-theoretic invariant can manifest itself in completely different ways in terms of different sites of definition for the same topos. One can then establish by means



of these site characterizations logical relationships or equivalences between completely different-looking properties or constructions pertaining to different sites. A remarkable example of the application of this technique is the topos-theoretic interpretation of Fraïssé's construction in Model theory established in [18].

Topos theory has already been successfully applied in the context of many-valued logics for establishing sheaf representations for notable classes of MV-algebras, for instance in the work of E. J. Dubuc and Y. Poveda ([30]) and J. L. Castiglione, M. Menni and W. J. Botero ([23]). Further sheaf representations were established by A. Filipoiu and G. Georgescu ([32]), and A. R. Ferraioli and A. Lettieri ([31]).

The innovation of this thesis is that we use topos-theoretic methods in order to obtain new results and conceptual insights, of both logical and algebraic nature, on central topics in the field of MV-algebras, which are not visible with classical methods. We obtain these new results by investigating the classifying toposes of notable theories of MV-algebras and by applying the bridge technique to Morita-equivalences between such theories and suitable theories of lattice-ordered abelian groups.

Many-valued logics and MV-algebras

Motivated by the fact that classical logic cannot describe situations that admit more than two outcomes, in 1920 J. Łukasiewicz introduced a three-valued logic by adding to the traditional truth values 0 and 1, interpreted as "absolute false" and "absolute true", a third degree of truth between them. Later, he presented further generalizations with n truth values (or even a countable or a continuous number of them).

The class of MV-algebras was introduced in 1958 by C. C. Chung (cf. [24] and [25]) in order to provide an algebraic semantics for Łukasiewicz multi-valued propositional logic. As this logic is a generalization of classical logic, MV-algebras are a generalization of boolean algebras (these can be characterized as the idempotent MV-algebras).

After their introduction in the context of algebraic logic, MV-algebras became objects of independent interest and many applications in different areas of Mathematics were found. The most notable ones are in functional analysis (cf. [39]), in the theory of lattice-ordered abelian groups (cf. [39] and [28]) and in the field of generalized probability theory (cf. Chapters 1 and 10 of [41] for a general overview).

Several equivalences between categories of MV-algebras and categories of lattice-ordered abelian groups (ℓ-groups, for short) can be found in the literature, the most important ones being the following:

- *Mundici's equivalence* (cf. [39]) between the whole category of MV-algebras and the category of ℓ-groups with strong unit;
- *Di Nola-Lettieri's equivalence* (cf. [28]) between the category of perfect MV-algebras (i.e., MV-algebras generated by their radical) and the whole category of ℓ-groups.

We observe that these categorical equivalences can be seen as equivalences between categories of set-based models of certain geometric theories and we prove that these theories are Morita-equivalent, i.e., there is a categorical equivalence between their categories of models inside any Grothendieck topos $\mathcal{E}$, naturally in $\mathcal{E}$.

In this way we obtain:

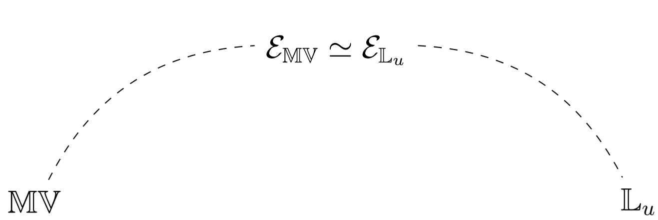
- a Morita-equivelence between the theory $\mathbb{MV}$ of MV-algebras and the theory $\mathbb{L}_u$ of ℓ-groups with strong unit (cf. Chapter 3);

- a Morita-equivalence between the theory $\mathbb{P}$ of perfect MV-algebras and the theory $\mathbb{L}$ of ℓ-groups (cf. Chapter 4).

We then show that the Morita-equivalence arising from Di Nola-Lettieri's equivalence is just one of a whole class of Morita-equivalences that we establish between theories of local MV-algebras in proper varieties of MV-algebras and appropriate extensions of the theory of ℓ-groups (cf. Chapter 5).

Consequences of the Morita-equivalence between $\mathbb{MV}$ and $\mathbb{L}_u$

An immediate consequence of the Morita-equivalence arising from Mundici's equivalence is the fact that the (infinitary) theory of ℓ-groups with strong unit is of presheaf type. This arises from the process of transferring the invariant property of being a presheaf topos across the Morita-equivalence. Recall that a theory is of presheaf type if its classifying topos is equivalent to a topos of presheaves. Every finitary algebraic theory, and more generally, every cartesian theory, is of presheaf type; thus, this property is transferred from the theory of MV-algebras to $\mathbb{L}_u$. We are interested in theories of presheaf type since they enjoy many remarkable properties, some of them recalled in Section 1.5, that do not hold for any geometric theory.



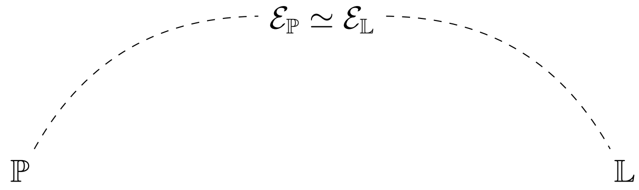
Changing the invariant considered at the level of the classifying topos gives rise to further results. For instance, the invariant given by the property to be a subtopos induces, by the Duality Theorem of [11] (which establishes a bijection between the subtoposes of the classifying topos of a given geometric theory and the quotients of this theory), a bijection between the quotients of the theory $\mathbb{MV}$ and those of the theory $\mathbb{L}_u$. It is worth to stress that this result cannot be deduced from Mundici's equivalence. Recall that a quotient of a theory is an extension over the same signature obtained by adding new axioms. Starting from a quotient of $\mathbb{MV}$, we get the corresponding quotient of $\mathbb{L}_u$ by translating every axiom in the language of ℓ-groups with strong unit by using the interpretation from the theory $\mathbb{MV}$ to the theory $\mathbb{L}_u$ established in Section 3.3. However, as proved in the same section, there is no interpretation in the converse direction that would make trivial the bijection between the quotients. If we consider now the invariant property of objects of toposes to be irreducible we get a logical characterization of the finitely presentable ℓ-groups with strong unit. They are the ℓ-groups with strong unit corresponding to the finitely presented MV-algebras under Mundici's equivalence. Specifically, we show that such groups can be characterized as the finitely presented pointed ℓ-groups $\mathcal{G}$ with a distinguishing element v which is a strong unit for $\mathcal{G}$, or, equivalently, as the ℓ-groups presented by a formula which is irreducible with respect to the theory of ℓ-groups with strong unit. This last result is used in Section 3.7.2 to describe a method for obtaining an axiomatization of the quotient of $\mathbb{MV}$ corresponding to a given quotient of the theory $\mathbb{L}_u$. Lastly, we establish a form of compactness and completeness for $\mathbb{L}_u$, obtained from the invariant properties of the classifying topos of $\mathbb{MV}$ (whence of $\mathbb{L}_u$) to have a compact terminal object and to have enough points.

Finally, as a particular instance of this Morita-equivalence, we obtain a sheaf-theoretic version of Mundici's equivalence valid for any topological space X, naturally in X.

Consequences of the Morita-equivalence between $\mathbb{P}$ and $\mathbb{L}$ and of the study of the classifying topos of $\mathbb{P}$

As in the case of Mundici's equivalence, the Morita-equivalence arising from Di Nola-Lettieri's equivalence involves an algebraic theory, namely the theory

$\mathbb{L}$ of ℓ-groups. Thus, the property to be of presheaf type is transferred to the coherent theory $\mathbb{P}$ of perfect MV-algebras. Whilst the two theories are not classically bi-interpretable, further applications of the bridge technique lead to three different levels of bi-interpretability between particular classes of formulas: irreducible formulas, geometric sentences and imaginaries.



Irreducible formulas for the theory $\mathbb{P}$ are the ones that present the finitely presentable perfect MV-algebras, that is the algebras which correspond to the finitely presented ℓ-groups via Di Nola-Lettieri's equivalence. They constitute the analogue for the theory $\mathbb{P}$ of cartesian formulas in the theory of MV-algebras. Indeed, even though the category $\mathbb{P}$-mod(**Set**) is not a variety, it is generated by its finitely presentable objects since the theory $\mathbb{P}$ is of presheaf type classified by the topos [f.p.$\mathbb{P}$-mod(**Set**), **Set**]. We also establish a bi-interpretability between the theory of lattice-ordered abelian groups and a cartesian theory $\mathbb{M}$ axiomatizing the positive cones of these groups, which we use in Section 4.5.2 to obtain a simpler reformulation of Di Nola-Lettieri's equivalence and in Section 4.5.3 to describe the partial bi-interpretations between $\mathbb{L}$ and $\mathbb{P}$. This bi-interpretation between $\mathbb{M}$ and $\mathbb{L}$ provides in particular an alternative description of the Grothendieck group associated with a model $\mathcal{M}$ of $\mathbb{M}$ as a subset, rather than a quotient as in the classical definition, of the product $\mathcal{M} \times \mathcal{M}$.

Next, we study in detail the classifying topos of the theory of perfect MV-algebras, representing it as a subtopos of the classifying topos of the algebraic theory axiomatizing the variety generated by Chang's MV-algebra. This investigation sheds light on the relationship between these two theories, notably leading to a representation theorem for finitely generated (resp.

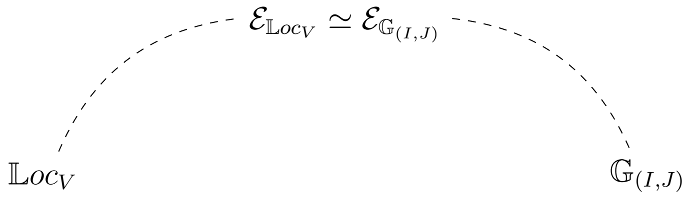
finitely presented) algebras in Chang's variety as finite products of finitely generated (resp. finitely presented) perfect MV-algebras. It is worth to note that this result, unlike most of the representation theorems available in the literature, is fully constructive. Among the other insights, we mention a characterization of the perfect MV-algebras which correspond to finitely presented lattice-ordered abelian groups via Di Nola-Lettieri's equivalence as the finitely presented objects of Chang's variety which are perfect MV-algebras, and the property that the theory axiomatizing Chang's variety proves all the cartesian sequents (in particular, all the algebraic identities) which are valid in all perfect MV-algebras.

We then revisit the representation theorem obtained through the analysis of the classifying topos of $\mathbb{P}$ from the point of view of subdirect products of perfect MV-algebras, obtaining a concrete proof of it. We also show that every MV-algebra in Chang's variety is a weak subdirect product of perfect MV-algebras. These results have close ties with the existing literature on weak boolean products of MV-algebras. Moreover, we generalize to the setting of MV-algebras in Chang's variety the Lindenbaum-Tarski characterization of boolean algebras which are isomorphic to powersets as the complete atomic boolean algebras, obtaining an intrinsic characterization of the MV-algebras in Chang's variety which are arbitrary products of perfect MV-algebras. These results show that Chang's variety constitutes a particularly natural MV-algebraic setting extending the variety of boolean algebras.

Finally, we transfer the above-mentioned representation theorems for the MV-algebras in Chang's variety in terms of perfect MV-algebras into the context of ℓ-groups with strong unit and, generalizing results in [2], we show that a theory of pointed perfect MV-algebras is Morita-equivalent to the theory of lattice-ordered abelian groups with a distinguished strong unit (whence to that of MV-algebras).

Morita-equivalences for local MV-algebras in proper varieties of MV-algebras

In light of the fact that the class of perfect MV-algebras is the intersection of the class of local MV-algebras with a specific proper variety of MV-algebras, namely Chang's variety, it is natural to wonder what happens if we replace this variety with an arbitrary variety of MV-algebras. We prove that 'globally', i.e., considering the intersection with the whole variety of MV-algebras, the theory of local MV-algebras is not of presheaf type, while if we restrict to any proper subvariety V, the theory of local MV-algebras, indicated with the symbol $\mathbb{L}oc_V$, is of presheaf type. Furthermore, we show that these theories are Morita-equivalent to suitable theories expanding the theory of ℓ-groups. More specifically, if $V = V(\{S_i\}_{i\in I}, \{S_j^{\omega}\}_{j\in J})$ (for finite subsets $I, J \subseteq \mathbb{N}$) we have a theory $\mathbb{G}_{(I,J)}$ which is Morita-equivalent to the theory $\mathbb{L}oc_V$ and which is written over the signature obtained from that of ℓ-groups by adding a constant symbol and propositional predicates corresponding to the elements of I and J. The categories of set-based models of these theories are not in general algebraic as in the case of perfect MV-algebras; however, in Section 5.5.2 we characterize the varieties V for which we have algebraicity as precisely those which can be generated by a single chain. All the Morita-equivalences contained in this new class are non-trivial, i.e., they do not arise from bi-interpretations, as we prove in Section 5.5.1.



Topos-theoretic methods are used here to obtain both logical and algebraic results. Specifically, we present two (non-constructively) equivalent axiomatizations for the theory of local MV-algebras in an arbitrary proper subvariety V, and we study the Grothendieck topologies associated with them as quotients of the algebraic theory $\mathbb{T}_V$ axiomatizing V. The subcanonicity of the Grothendieck topology associated with the first axiomatization ensures that the cartesianization of the theory of local MV-algebras in V is the theory $\mathbb{T}_V$. It is worth to note that this result does *not* arise from a representation theorem of the algebras in V as subdirect products or global sections of sheaves of models of the theory of local MV-algebras in V, something that would make this trivial. To verify the provability of a cartesian sequent in the theory $\mathbb{T}_V$, we are thus reduced to checking it in the theory of local MV-algebras in V. Using this, we easily prove that the radical of every MV-algebra in V is defined by an equation, which we use to present the second axiomatization. This latter axiomatization has the notable property that the associated Grothendieck topology is rigid. This allows us to conclude that the theory of local MV-algebras in V is of presheaf type. The equivalence of the two axiomatizations and the consequent equality of the associated Grothendieck topologies yields in particular a representation result of every finitely presented MV-algebra in V as a finite product of local MV-algebras. This generalizes the representation result obtained for the finitely presented MV-algebras in Chang's variety as finite products of perfect MV-algebras.

Strictly related to the theory of local MV-algebras is the theory of simple (in the sense of universal algebra) MV-algebras; indeed, an MV-algebra $\mathcal{A}$ is local if and only if the quotient $\mathcal{A}/\mathrm{Rad}(\mathcal{A})$ is a simple MV-algebra. This theory shares many properties with the theory of local MV-algebras: globally it is not of presheaf type but it has this property if we restrict to an arbitrary proper subvariety. On the other hand, while the theory of simple MV-algebras of finite rank is of presheaf type (as it coincides with the geometric theory of finite chains), the theory of local MV-algebras of finite rank is not, as we prove in Section 5.2.3.

* * *

Summarizing, in this thesis we use topos-theoretic techniques to study Morita-equivalences obtained by 'lifting' categorical equivalences which are already known in the literature of MV-algebras and also to establish new ones. This shows that, as it was already argued in [12], topos theory is indeed a powerful tool for discovering new equivalences in Mathematics, as well as for investigating known ones.

The main themes addressed in this thesis are the following:

- theories of presheaf type;
- Morita-equivalences and bi-interpretations;
- MV-algebras and lattice-ordered abelian groups;
- representation results for classes of MV-algebras;
- cartesianizations for quotients of $\mathbb{MV}$.

A particular attention is posed on the constructiveness of the results; we indicate with the symbol * the points where we use the axiom of choice.

Structure of the thesis

The thesis is organized in five chapters.

Chapter 1. In this chapter we recall the most important notions and results on topos theory. We mostly focus on the technique of 'toposes as bridge' that we apply throughout the thesis and on notions of classifying topos and of theory of presheaf type.

Chapter 2. In this chapter we introduce the classes of MV-algebras that are studied in the thesis, namely perfect, local and simple MV-algebras. Morever, we establish some preliminary results on the respective quotients of $\mathbb{MV}$. For instance, we prove that the theory of local MV-algebras and the theory of simple MV-algebras are not of presheaf type. Further, we introduce two equivalent axiomatizations for the theory of perfect MV-algebras and we show that the radical of every MV-algebra in Chang's variety is definable by an equation. This result is necessary for defining the radical of a model of the theory of perfect MV-algebras in an arbitrary Grothendieck topos as the classical definition of the radical is not constructive. We also derive the fact that the radical cannot be defined by a geometric formula in the whole class of MV-algebras as a consequence of the fact that the class of semisimple MV-algebras cannot be axiomatized in a geometric way.

Chapter 3. In this chapter we show that the theory of MV-algebras is Morita-equivalent to (but not bi-interpretable with) to that of lattice-ordered abelian groups with strong unit. This generalizes the well-known equivalence between the categories of set-based models of the two theories established by Mundici, and allows to transfer properties and results across them by using the methods of topos theory. We discuss several applications, including a sheaf theoretic version of Mundici's equivalence and a bijective correspondence between the geometric theory extensions of the two theories.

Chapter 4. We establish, generalizing Di Nola and Lettieri's categorical equivalence, a Morita-equivalence between the theory of lattice-ordered abelian groups and that of perfect MV-algebras. Further, after observing that the two theories are not bi-interpretable in the classical sense, we identify, by considering appropriate topos-theoretic invariants on their common classifying topos, three levels of bi-interpretability holding for particular classes of formulas: irreducible formulas, geometric sentences and imaginaries. Lastly, by investigating the classifying topos of the theory of perfect MV-algebras, we obtain various results on its syntax and semantics also in relation to the cartesian theory of the variety generated by Chang's MV-algebra, including a concrete representation for the finitely generated models of the latter theory as finite products of perfect MV-algebras. Among the results established on the way, we mention a Morita-equivalence between the theory of lattice-ordered abelian groups and that of cancellative lattice-ordered abelian monoids with bottom element.

Chapter 5. In this chapter we study quotients of the geometric theory of local MV-algebras, in particular those which axiomatize the class of local MV-algebras in a proper subvariety. We show that each of these quotients is a theory of presheaf type which is Morita-equivalent to an expansion of the theory of lattice-ordered abelian groups. Di Nola-Lettieri's equivalence is recovered from the Morita-equivalence for the quotient axiomatizing the local MV-algebras in Chang's variety, that is the perfect MV-algebras. We establish along the way a number of results of independent interest, including a constructive treatment of the radical for local MV-algebras in a fixed proper variety of MV-algebras and a representation theorem of the finitely presentable algebras in such a variety as finite products of local MV-algebras.

Chapter 1

Topos-theoretic background

In this section we recall the most important notions and results on topos theory. For a succinct introduction to this subject we refer the reader to [19]; classical references are [38] and [35].

1.1 Grothendieck toposes

The notion of Grothendieck topology on a category was introduced by A. Grothendieck as a categorical generalization of the classical concept of topology on a set. Here the attention is focused on open sets and on covering families of open sets, i.e., families of open subsets of a given open set U whose union coincides with U. In this generalization the objects of a category take the place of the open sets and covering families become families of arrows with the same codomain which have to satisfy appropriate properties. The formal definition is the following.

Definition 1.1.1. Given a small category $\mathcal{C}$, a *sieve* on an object c of $\mathcal{C}$ is a set S of arrows with codomain c such that $f \circ g \in S$ whenever $f \in S$ and g is composable with f. A *Grothendiek topology* on $\mathcal{C}$ is a function J which assigns to each object $c \in \mathcal{C}$ a collection $J(c)$ of sieves on c in such a way that

(i) (maximality axiom) the maximal sieve $\{f \mid cod(f) = c\}$ is in $J(c)$, for every $c \in \mathcal{C}$;

(ii) (stability axiom) if $S \in J(c)$, then $h^*(S) \in J(d)$ for any morphism $h : d \to c$, where with the symbol $h^*(S)$ we mean the sieve whose morphisms are the pullbaks along h of the morphisms in S;

(iii) (transitivity axiom) if $S \in J(c)$ and R is a sieve on c such that $h^*(R) \in J(d)$ for all $h : d \to c$ in S, then $R \in J(c)$.

The sieves $S \in J(c)$ are called the J-*covering* sieves.

A *site* is a pair $(\mathcal{C}, J)$ consisting of a small category $\mathcal{C}$ and a Grothendieck topology J on $\mathcal{C}$.

Definition 1.1.2. (a) A *presheaf* on a category $\mathcal{C}$ is a functor $P : \mathcal{C}^{\text{op}} \to$ **Set**.

(b) A *sheaf* on $(\mathcal{C}, J)$ is a presheaf $P : \mathcal{C}^{\text{op}} \to \mathbf{Set}$ on $\mathcal{C}$ such that, for every J-covering sieve $S \in J(c)$ and every family $\{x_f \in P(dom(f)) \mid f \in S\}$ such that $P(g)(x_f) = x_{f \circ g}$ for any $f \in S$ and any arrow g in $\mathcal{C}$ composable with f, there exists a unique element $x \in P(c)$ such that $x_f = P(f)(x)$ for all $f \in S$.

(c) The category $\mathbf{Sh}(\mathcal{C}, J)$ of sheaves on the site $(\mathcal{C}, J)$ has as objects the sheaves on $(\mathcal{C}, J)$ and as arrows the natural transformations between them, regarded as functors $\mathcal{C}^{\text{op}} \to \mathbf{Set}$.

(d) A *Grothendieck topos* is a category that is of the form $\mathbf{Sh}(\mathcal{C}, J)$, up to categorical equivalence.

Definition 1.1.3. Let $(\mathcal{C}, J)$ be a site and I be a set of objects of $\mathcal{C}$. If for any arrow $f : a \to b$ in $\mathcal{C}$ such that $b \in I$ then $a \in I$, we say that I is an *ideal*. If further for any J-covering sieve S on an object c of $\mathcal{C}$ such that $dom(f) \in I$ for all $f \in S$ then $c \in I$, we say that I is a J-*ideal*.

The J-ideals on $\mathcal{C}$ correspond bijectively to the subterminal objects of the topos $\mathbf{Sh}(\mathcal{C}, J)$.

Given a site $(\mathcal{C}, J)$, a sieve S on an object c of $\mathcal{C}$ is said to be *J-closed* if for every arrow f with codomain c, $f^*(S) \in J(dom(f))$ implies $f \in S$. If the representable functor $Hom_{\mathcal{C}}(-, c)$ is a J-sheaf then the J-closed sieves on c are in natural bijection with the subobjects of $Hom_{\mathcal{C}}(-, c)$ in the topos $\mathbf{Sh}(\mathcal{C}, J)$.

Definition 1.1.4 (pp. 542 Section C2.1 [35])**.** A sieve R on an object U of $\mathcal{C}$ is called *effective-epimorphic* if it forms a colimit cone under the diagram consisting of the domains of all morphisms in R and all the morphisms over U. A Grothendieck topology is said to be *subcanonical* if all its covering sieves are effective-epimorphic, i.e., every representable functor is a J-sheaf.

Definition 1.1.5. Let $(\mathcal{C}, J)$ be a site.

(a) We say that an object $c \in \mathcal{C}$ is *J-irreducible* if the only J-covering sieve on c is the maximal sieve.

(b) We say that J is *rigid* if for every object c of $\mathcal{C}$, the set of arrows from J-irreducible objects of $\mathcal{C}$ generates a J-covering sieve.

As it follows from the definition, a Grothendieck topos can have more sites of definition. With the Comparison Lemma we can find new sites of definition starting from a given one.

Definition 1.1.6. Let $(\mathcal{C}, J)$ be a site and $\mathcal{D}$ be a full subcategory of $\mathcal{C}$. This category $\mathcal{D}$ is called *J-dense* if for every object of c the sieve generated by the family of arrows to c from objects in $\mathcal{D}$ is a J-covering.

Lemma 1.1.7 (**Comparison Lemma**, Theorem C2.2.3 [35])**.** *Given $\mathcal{C}$ and $\mathcal{D}$ as above, the toposes $\mathbf{Sh}(\mathcal{C}, J)$ and $\mathbf{Sh}(\mathcal{D}, J_{|\mathcal{D}})$ are categorical equivalent, where $J_{|\mathcal{D}}$ is the Grothendieck topology on $\mathcal{D}$ induced by J and defined by: $S \in J_{|\mathcal{D}}(d)$ if and only if $\bar{S} \in J(d)$, where $\bar{S}$ is the sieve in $\mathcal{C}$ generated by the arrows in S.*

We are interested in looking at Grothendieck toposes as classifying toposes for geometric theories. In this case the suitable class of morphisms among toposes is the class of the geometric morphisms.

Definition 1.1.8. A *geometric morphism* $f : \mathcal{F} \to \mathcal{E}$ of toposes is a pair of adjoint functors $f_* : \mathcal{F} \to \mathcal{E}$ and $f^* : \mathcal{E} \to \mathcal{F}$ such that the lef adjoint f^*, called the *inverse image functor*, preserves finite limits.

Given $\mathcal{E}$ and $\mathcal{F}$ two Grothendieck toposes, we indicate with the symbol $\mathbf{Geom}(\mathcal{E}, \mathcal{F})$ the category of geometric morphisms between them.

1.2 Geometric logic and categorical semantics

Let Σ be a first-order signature consisting of a set of sorts, a set of function symbols and a set of relation symbols. A *context* is a finite list $\vec{x} = x_1 \ldots, x_n$ of distinct variables and it is said to be *suitable* for a formula ϕ over Σ if all the free variables of ϕ occur in it. A *formula-in context* is an expression of the form $\phi(\vec{x})$, where ϕ is a formula over Σ and $\vec{x}$ is a suitable context for it.

Definition 1.2.1. (a) The set of *atomic formulas* over Σ is the smallest set closed under relations $R(t_1, \ldots\, .\ t_n)$ and equalities $(t = s)$, where $t_1, \ldots, t_n$, t, s are Σ-terms and R is a Σ-relation symbol.

(b) The set of *Horn formulas* over Σ is the smallest set containing the set of atomic formulas and closed under truth and finitary conjunctions.

(c) The set of *regular formulas* over Σ is the smallest set containing the set of atomic formulas and closed under truth, finitary conjunctions and existential quantifications.

(d) The set of *coherent formulas* over Σ is the smallest set containing the set of regular formulas and closed under false and finitary disjunctions.

(e) The set of *first-order formulas* over Σ is the smallest set containing the set of coherent formulas and closed under implications, negations, existential and universal quantifications.

(f) The set of *geometric formulas* over Σ is the smallest set containing the set of coherent formulas and closed under infinitary disjunctions.

A *sequent* over a signature Σ is an expression of the form $\phi \vdash_{\vec{x}} \psi$, where ϕ and ψ are formulas over Σ and $\vec{x}$ is a context suitable for both of them. In first-order logic a sequent $\phi \vdash_{\vec{x}} \psi$ expresses the same idea of $(\forall x_1) \dots (\forall x_n)(\phi \rightarrow \psi)$. A sequent $(\phi \vdash_{\vec{x}} \psi)$ is Horn (resp. regular, coherent, first-order, geometric) if both ϕ and ψ are Horn (resp. regular, coherent, first-order, geometric) formulas.

Definition 1.2.2. A *theory* over a signature Σ is a set $\mathbb{T}$ of sequents over Σ whose elements are called the (non-logical) axioms of $\mathbb{T}$.

- A theory $\mathbb{T}$ is *algebraic* if its signature Σ has a single sort and no relation symbols (apart from equality) and its axioms are all of the form $\top \vdash_{\vec{x}} \phi$, where ϕ is an atomic formula $(s = t)$ and $\vec{x}$ is its canonical context.

- A theory $\mathbb{T}$ is *Horn* (resp. *regular*, *coherent*, *geometric*) if all the sequents in $\mathbb{T}$ are Horn (resp. regular, coherent, geometric).

- A regular theory $\mathbb{T}$ is *cartesian* if its axioms can be well-ordered in such a way that each axiom is cartesian relative to the sub-theory consisting of all the axioms preceding it in the ordering, in the sense that all the existential quantifications which appear in the given axiom are provably unique relative to that sub-theory.

- A *propositional theory* is a theory over a *propositional signature* which has no sorts, whence any function symbols, and the only relation symbols are atomic propositions. Propositional theories are used for describe subsets of a given structure with particular properties.

Example 1.2.3. (a) The theory of poset is a Horn theory. It has one sort A, one relation symbol $\leq \rightarrowtail A \times A$ and no function symbols. The axioms are:

- $\top \vdash_x x \leq x$;
- $x \leq y \wedge y \leq x \vdash_{x,y} x = y$;
- $x \leq y \wedge y \leq z \vdash_{x,y,z} x \leq z$.

By adding function symbols and appropriate axioms we can axiomatize ordered algebraic structures.

(b) The theory of torsion abelian groups is an example of a theory that is geometric but not first-order. Indeed, we need of an infinitary disjunction to express the property of the groups to have torsion,

$$\top \vdash_x \bigvee_{1<n} (nx = 0) \ .$$

(c) In the opposite direction, the theory of metric space offers an example of a theory which is infinitary first-order but not geometric. The signature consists of one sort A and a family of relation symbols $R_\varepsilon \rightarrowtail A \times A$ indexed by positive real numbers ε. The interpretation of the predicate $R_\varepsilon(x, y)$ is "the distance between x and y is strictly less than ε". Among the axioms we have the following one

$$\bigwedge_{0<\varepsilon} R_\varepsilon(x, y) \vdash_{x,y} x = y$$

that requires an infinitary conjunction.

(d) For examples of propositional theories let us consider a $\wedge$-semilattice L. For each $a \in L$ we have a 0-ary predicate R_a that has the meanig "$a \in R$", with R a subset of L. With these symbols we can describe the theory of filters of L. The axioms of this theory are the following sequents:

- $\top \vdash R_1$, where 1 is the top element of L;
- $R_a \vdash R_b$, for every $a \leq b$ in L;
- $R_a \wedge R_b \vdash R_{a\wedge b}$, for every $a, b \in L$.

This is a cartesian theory. If L is a lattice, with the same signature we can also write the theory of prime filters of L. Further, if L is a complete lattice we can axiomatize the theory of complete prime filters of L. These theories are respectively coherent and geometric.

These three examples of propositional theories are standard in the sense that any cartesian (resp. coherent, geometric) propositional theory is Morita-equivalent[1] to the theory of filters (resp. prime filters, completely prime filters) of a $\wedge$-semilattice (resp. lattice, complete lattice) (cf. Remark D1.4.14 [35]).

To each of the fragments of first-order logic introduced above, we can naturally associate a *deduction system*.

Definition 1.2.4. • The *structural rules* consist of the *identity axiom*

$$(\phi \vdash_{\vec{x}} \phi),$$

the *substitution rule*

$$\frac{(\phi \vdash_{\vec{x}} \psi)}{(\phi[\vec{s}/\vec{x}] \vdash_{\vec{y}} \psi[\vec{s}/\vec{x}])},$$

where $\vec{y}$ is any string of variables including all the variables occurring in the string of terms $\vec{s}$, and the *cut rule*

$$\frac{(\phi \vdash_{\vec{x}} \psi)\ (\psi \vdash_{\vec{x}} \chi)}{(\phi \vdash_{\vec{x}} \chi)}\ .$$

- The *equality rules* consist of the axioms

[1] The notion of Morita-equivalent theories will be introduced in Section 1.4.

$$(\top \vdash_x x = x) \quad ((\vec{x} = y \wedge \vec{\phi}) \vdash_{\vec{z}} \phi[\vec{y}/\vec{x}]),$$

where $\vec{x}$ and $\vec{y}$ are contexts of the same length and type and $\vec{z}$ is a context containing $\vec{x}$, $\vec{y}$ and the free variables of ϕ.

- The *rules for finite conjunction* are the axioms

$$(\phi \vdash_{\vec{x}} \top) \quad ((\phi \wedge \psi) \vdash_{\vec{x}} \phi) \quad ((\phi \wedge \psi) \vdash_{\vec{x}} \psi),$$

and the rule

$$\frac{(\phi \vdash_{\vec{x}} \psi) \; (\phi \vdash_{\vec{x}} \chi)}{(\phi \vdash_{\vec{x}} \psi \wedge \psi)} \, .$$

- The *rules for finite disjunction* are the axioms

$$(\bot \vdash_{\vec{x}} \phi) \quad (\phi \vdash_{\vec{x}} (\phi \vee \psi)) \quad (\psi \vdash_{\vec{x}} (\phi \vee \psi)),$$

and the rule

$$\frac{(\phi \vdash_{\vec{x}} \psi) \; (\chi \vdash_{\vec{x}} \psi)}{((\phi \vee \psi) \vdash_{\vec{x}} \chi)} \, .$$

- The *rules for infinitary conjunction* (resp. *disjunction*) are the infinitary analogues of the rules for finite conjunction (resp. disjunction).

- The *rules for implication* consist of the double rule

$$\frac{\phi \wedge \psi \vdash_{\vec{x}} \chi}{\overline{\psi \vdash_{\vec{x}} (\phi \Rightarrow \chi)}}$$

- The *rules for existential quantification* consist of the double rule

$$\frac{\phi \vdash_{\vec{x},y} \psi}{\overline{((\exists y)\phi \vdash_{\vec{x}} \psi)}}$$

provided that y is not free in ψ.

- The *rules for universal quantification* consist of the double rule

$$\frac{\phi \vdash_{\vec{x},y} \psi}{\overline{(\phi \vdash_{\vec{x}} (\forall y)\psi)}}$$

- The *distributive axiom* is

$$((\phi \wedge (\psi \vee \chi)) \vdash_{\vec{x}} ((\phi \wedge \psi) \vee (\phi \wedge \chi))) .$$

- The *Frobenius axiom* is

$$((\phi \wedge (\exists y)\psi) \vdash_{x} (\exists y)(\phi \wedge \psi),$$

  where y is a variable not in the context $\vec{x}$.

Definition 1.2.5. We can distinguish fragments of first-order logic by adding to the structural and equality rules the ones specified as follows.

- *Algebraic logic*: no additional rules.
- *Horn logic*: finite conjunction.
- *Regular logic*: finite conjunction, existential quantification and Frobenius axiom.
- *Coherent logic*: finite conjunction, finite disjunction, existential quantification, distributive axiom and Frobenius axiom.
- *Geometric logic*: finite conjunction, infinitary disjunction, existential quantification, infinitary distributive axiom, Frobenius axiom.

We say that a sequent σ is *provable* in an algebraic (Horn, regular, coherent, geometric) theory $\mathbb{T}$ if there exists a derivation of σ relative to $\mathbb{T}$ in the appropriate fragment of first-order logic.

Definition 1.2.6. A *quotient* of a geometric theory $\mathbb{T}$ over a signature Σ is a geometric theory $\mathbb{T}'$ over Σ such that every geometric sequent over Σ which is provable in $\mathbb{T}$ is provable in $\mathbb{T}'$.

One can define the notion of models of a geometric theory $\mathbb{T}$ in a Grothendieck topos $\mathcal{E}$ generalizing the definition of tarskian models of a first-order theory in **Set**.

Definition 1.2.7. Let $\mathcal{E}$ be a topos and Σ be a (possibly multi-sorted) first-order signature. A Σ-*structure* M in $\mathcal{E}$ is specified by the following data:

(i) a function assigning to each sort A of Σ, an object MA of $\mathcal{E}$. This function is extended to finite strings of sorts by defining $M(A_1, \ldots, A_n) = MA_1 \times \cdots \times MA_n$ (and setting $M([])$, where $[]$ denotes the empty string, equal to the terminal object 1 of $\mathcal{E}$);

(ii) a function assigning to each function symbol $f : A_1 \ldots A_n \to B$ in Σ an arrow $Mf : M(A_1, \ldots, A_n) \to MB$ in $\mathcal{E}$;

(iii) a function assigning to each relation symbol $R \rightarrowtail A_1 \ldots A_n$ in Σ a subobject $MR \rightarrowtail M(A_1, \ldots, A_n)$ in $\mathcal{E}$.

The Σ-structures in $\mathcal{E}$ are the objects of a category $\Sigma\text{-str}(\mathcal{E})$ whose arrows are the Σ-*structure homomorphisms*. Such homomorphisms $h : M \to N$ are specified by a collection of arrows $h_A : MA \to NA$ in $\mathcal{E}$, indexed by the sorts of Σ and satisfying the following two conditions:

(i) For each function symbol $f : A_1 \cdots A_n \to B$ in Σ, the diagram

$$\begin{array}{ccc} M(A_1, \ldots, A_n) & \xrightarrow{Mf} & MB \\ {\scriptstyle h_{A_1} \times \cdots \times h_{A_n}} \downarrow & & \downarrow {\scriptstyle h_B} \\ N(A_1, \ldots, A_n) & \xrightarrow[Nf]{} & NB \end{array}$$

commutes;

(ii) For each relation symbol $R \rightarrowtail A_1 \cdots A_n$ in Σ, there is a commutative diagram in $\mathcal{C}$ of the form

$$\begin{array}{ccc} MR & \longrightarrow & M(A_1, \ldots, A_n) \\ \downarrow & & \downarrow h_{A_1} \times \cdots \times h_{A_n} \\ NR & \longrightarrow & N(A_1, \ldots, A_n) \end{array}$$

Let $\mathcal{E}$ and $\mathcal{F}$ be toposes. Any functor $T : \mathcal{E} \to \mathcal{F}$ which preserves finite products and monomorphisms induces a functor Σ-str(T) : Σ-str$(\mathcal{E}) \to \Sigma$-str$(\mathcal{F})$ in the obvious way.

Until now we have interpreted function and relation symbols in a Σ-structure. Terms and formulas can be interpreted as well.

Definition 1.2.8. Let M be a Σ-structure in $\mathcal{E}$. If $\{\vec{x} \,.\, t\}$ is a term-in-context over Σ (with $\vec{x} = x_1, \ldots, x_n, x_i : A_i (i = 1, \ldots, n)$ and $t : B$) then its interpretation in M, indicated with the symbol $[\![\vec{x} \,.\, t]\!]_M$, is an arrow

$$[\![\vec{x} \,.\, t]\!]_M : M(A_1, \ldots, A_n) \to MB$$

in $\mathcal{E}$ defined recursively by the following clauses.

(a) If t is a variable, it is necessarily x_i for some unique $i \leq n$, and then $[\![\vec{x} \,.\, t]\!]_M = \pi_i$, the i-th product projection.

(b) If t is $f(t_1, \ldots, t_m)$ (where $t_i : C_i$), then $[\![\vec{x} \,.\, t]\!]_M$ is the composite

$$M(A_1, \ldots, A_n) \xrightarrow{([\![\vec{x} \,.\, t_1]\!]_M, \ldots, [\![\vec{x} \,.\, t_m]\!]_M)} M(C_1, \ldots, C_n) \xrightarrow{Mf} MB$$

Definition 1.2.9. Let M be a Σ-structure in M. Any formula $\phi(\vec{x})$ over Σ is interpretable as a subobject $[\![\vec{x}\,.\,\phi]\!]_M \rightarrowtail M(A_1,\ldots,A_n)$. This interpretation is defined recursively on the structure of the formula.

- If $\phi(\vec{x})$ is $R(t_1,\ldots,t_m)$, where R is a relation symbol (of type $B_1,\ldots,B_m$), then $[\![\vec{x}\,.\,\phi]\!]_M$ is the pullback

$$\begin{array}{ccc} [\![\vec{x}\,.\,\phi]\!]_M & \longrightarrow & MR \\ \downarrow & & \downarrow \\ M(A_1,\ldots,A_n) & \xrightarrow[([\![\vec{x}\,.\,t_1]\!]_M,\ldots,[\![\vec{x}\,.\,t_n]\!]_M)]{} & M(B_1,\ldots,B_n) \end{array}$$

- If $\phi(\vec{x})$ is $(s = t)$, where s and t are terms of sort B, then $[\![\vec{x}\,.\,\phi]\!]_M$ is the equalizer of $[\![\vec{x}\,.\,s]\!]_M, [\![\vec{x}\,.\,t]\!]_M : M(A_1,\ldots,A_n) \to MB$.

- If $\phi(\vec{x})$ is $\top$, then $[\![\vec{x}\,.\,\phi]\!]_M$ is the top element of $Sub_{\mathcal{E}}(M(A_1,\ldots,A_n))$.

- If $\phi(\vec{x})$ is $(\psi \wedge \chi)(\vec{x})$, then $[\![\vec{x}\,.\,\phi]\!]_M$ is the pullback

$$\begin{array}{ccc} [\![\vec{x}\,.\,\phi]\!]_M & \longrightarrow & [\![\vec{x}\,.\,\psi]\!]_M \\ \downarrow & & \downarrow \\ [\![\vec{x}\,.\,\chi]\!]_M & \longrightarrow & M(A_1,\ldots,A_n) \end{array}$$

- If $\phi(\vec{x})$ is $\bot$, then $[\![\vec{x}\,.\,\phi]\!]_M$ is the bottom element of $Sub_{\mathcal{E}}(M(A_1,\ldots, A_n))$.

- If $\phi(\vec{x})$ is $(\psi \vee \chi)(\vec{x})$, then $[\![\vec{x}\,.\,\phi]\!]_M$ is the union of the subobjects $[\![\vec{x}\,.\,\psi]\!]_M$ and $[\![\vec{x}\,.\,\chi]\!]_M$.

- If $\phi(\vec{x})$ is $((\exists y)\psi)(\vec{x})$, where y is of sort B, then $[\![\vec{x} \,.\, \phi]\!]_M$ is the image of the composite

$$[\![\vec{x}, y \,.\, \psi]\!]_M \longrightarrow M(A_1, \ldots, A_n, B) \xrightarrow{\pi} M(A_1, \ldots, A_n)$$

  where π is the product projection on the first n factors.

- If $\phi(\vec{x})$ is $(\bigvee_{i \in I} \phi_i)(\vec{x})$ then $[\![\vec{x} \,.\, \phi]\!]_M$ is the union of the subobjects $[\![\vec{x} \,.\, \phi_i]\!]_M$.

Definition 1.2.10. Let M be a Σ-structure in a topos $\mathcal{E}$.

(a) If $\sigma = (\phi \vdash_{\vec{x}} \psi)$ is a first-order sequent over Σ, we say that σ is *satisfied in* M (and write $M \models \sigma$) if $[\![\vec{x} \,.\, \phi]\!]_M \leq [\![\vec{x} \,.\, \psi]\!]_M$ in the lattice $Sub_{\mathcal{E}}(M(A_1, \ldots, A_n))$ of subobjects of $M(A_1, \ldots, A_n)$ in $\mathcal{E}$.

(b) If $\mathbb{T}$ is a geometric theory over Σ, we say that M is a *model* of $\mathbb{T}$ (and write $M \models \mathbb{T}$) if all the axioms of $\mathbb{T}$ are satisfied in M.

(c) We write $\mathbb{T}$-mod($\mathcal{E}$) for the full subcategory of Σ-str($\mathcal{E}$) whose objects are the models of $\mathbb{T}$.

Lemma 1.2.11 (Lemma D1.2.13 [35]). *Let $T : \mathcal{E} \to \mathcal{F}$ be a cartesian (resp. regular, coherent, Heyting, geometric) functor between toposes; let M be a Σ-structure in $\mathcal{E}$ and let σ be a sequent over Σ. If $M \models \sigma$ in $\mathcal{E}$ then Σ-str$(T)(M) \models \sigma$ in $\mathcal{F}$. Then converse implication holds if T is conservative.*

Sometime it is possible to determine models of a theory $\mathbb{T}$ in a topos $\mathcal{E}$ by regarding at the models of this theory in a more familiar topos. The following theorem gives examples of this operation.

Theorem 1.2.12 (Corollary D1.2.14 [35]). *Let $\mathbb{T}$ be a geometric theory over a signature Σ. Then for every small category $\mathcal{C}$, a Σ-structure M in $[\mathcal{C}, \mathbf{Set}]$ is a $\mathbb{T}$-model if and only if each $ev_c(M)$, $c \in \mathcal{C}$, is a model of $\mathbb{T}$ in* **Set**, *where ev_c denotes the functor 'evalute at c'. Indeed we have an isomorphism $\mathbb{T}$-mod$([\mathcal{C}, \mathbf{Set}]) \simeq [\mathcal{C}, \mathbb{T}\text{-}mod(\mathbf{Set})]$.*

For any topological space X, a Σ-structure M in $\mathbf{Sh}(X)$ is a $\mathbb{T}$-model if and only if $x^*(M)$, $x \in X$, is a $\mathbb{T}$-model in $\mathbf{Set}$, where $x^* : \mathbf{Sh}(X) \to \mathbf{Set}$ is the stalk functor associated with x (i.e., the inverse image functor along x regarded as a continuous map $1 \to X$).

1.2.1 The internal language of a topos

A Grothendieck topos has all small limits and colimits, as well as exponentials and a subobject classifier. It can thus be considered as a mathematical universe in which one can perform all the usual set-theoretic constructions. More specifically, one can attach to any topos $\mathcal{E}$ a canonical signature $\Sigma_{\mathcal{E}}$, called its *internal language*, having a sort $\ulcorner A \urcorner$ for each object A of $\mathcal{E}$, a function symbol $\ulcorner f \urcorner : \ulcorner A_1 \urcorner \ldots \ulcorner A_n \urcorner \to \ulcorner B \urcorner$ for each arrow $f : A_1 \times \cdots \times A_n \to B$ in $\mathcal{E}$ and a relation symbol $\ulcorner R \urcorner \rightarrowtail \ulcorner A_1 \urcorner \ldots \ulcorner A_n \urcorner$ for each subobject $R \rightarrowtail A_1 \times \cdots \times A_n$ in $\mathcal{E}$. There is a tautological $\Sigma_{\mathcal{E}}$-structure $\mathcal{S}_{\mathcal{E}}$ in $\mathcal{E}$ obtained by interpreting each $\ulcorner A \urcorner$ as A, each $\ulcorner f \urcorner$ as f and each $\ulcorner R \urcorner$ as R. For any object $A_1, \ldots, A_n$ of $\mathcal{E}$ and any first-order formula $\phi(\vec{x})$ over $\Sigma_{\mathcal{E}}$, where $\vec{x} = (x_1^{\ulcorner A_1 \urcorner}, \ldots, x_n^{\ulcorner A_n \urcorner})$, the expression $\{\vec{x} \in A_1 \times \cdots \times A_n \mid \phi\}$ can be given a meaning, namely the interpretation of the formula $\phi(\vec{x})$ in the $\Sigma_{\mathcal{E}}$-structure $\mathcal{S}_{\mathcal{E}}$. Since the logic of a topos is in general intuitionistic, any formal proof involving first-order sequents over the signature $\Sigma_{\mathcal{E}}$ will be valid in the structure $\mathcal{S}_{\mathcal{E}}$ provided that the law of excluded middle or any other non-constructive principles are not employed in it. This allows to prove results concerning objects and arrows in the topos by arguing constructively in a set-theoretic fashion. We shall exploit this fact at various points.

An example of reformulations of basic properties of sets in the internal language of a topos is provided by the following proposition.

Proposition 1.2.13 (Lemma D1.3.11 [35]). *Let $\mathcal{E}$ be a topos. The following statements hold*

(i) $f : A \to A$ is the identity arrow if and only if $(\top \vdash_x \ulcorner f \urcorner(x) = x)$ holds in $S_{\mathcal{E}}$.

(ii) $f : A \to C$ in the composite of $g : A \to B$ and $h : B \to C$ if and only if $(\top \vdash_x \ulcorner f \urcorner(x) = \ulcorner h \urcorner(\ulcorner g \urcorner(x)))$ holds in $S_{\mathcal{E}}$.

(iii) $f : A \to B$ is monic if and only if $(\ulcorner f \urcorner(x) = \ulcorner f \urcorner(x') \vdash_x x = x')$ holds in $S_{\mathcal{E}}$.

(iv) $f : A \to B$ is an epimorphism if and only if $(\top \vdash_x (\exists x)(\ulcorner f \urcorner(x) = y))$ holds in $S_{\mathcal{E}}$.

(v) A is a terminal object if and only if the sequents $(\top \vdash (\exists x)\top)$ and $(\top \vdash_{x,x'} (x = x'))$ hold in $S_{\mathcal{E}}$.

1.3 Classifying toposes

Definition 1.3.1. Let $\mathbb{T}$ be a geometric theory over a signature Σ. A *classifying topos* of $\mathbb{T}$ is a Grothendieck topos $\mathcal{E}_{\mathbb{T}}$ such that for any Grothendieck topos $\mathcal{E}$ we have an equivalence of categories

$$\mathbf{Geom}(\mathcal{E}, \mathcal{E}_{\mathbb{T}}) \simeq \mathbb{T}\text{-mod}(\mathcal{E})$$

natural in $\mathcal{E}$, i.e., for any geometric morphism $f : \mathcal{E} \to \mathcal{F}$ we have a commutative square

$$\begin{array}{ccc}
\mathbf{Geom}(\mathcal{F}, \mathcal{E}_{\mathbb{T}}) & \xrightarrow{\simeq} & \mathbb{T}\text{-mod}(\mathcal{F}) \\
{\scriptstyle - \circ f^*} \downarrow & & \downarrow {\scriptstyle \mathbb{T}\text{-mod}(f^*)} \\
\mathbf{Geom}(\mathcal{E}, \mathcal{E}_{\mathbb{T}}) & \xrightarrow{\simeq} & \mathbb{T}\text{-mod}(\mathcal{E})
\end{array}$$

In other words, there is a model $U_{\mathbb{T}}$ of $\mathbb{T}$ in $\mathcal{E}_{\mathbb{T}}$, called the *universal model* of $\mathbb{T}$, characterized by the universal property that any model M in a Grothendieck topos $\mathcal{E}$ can be obtained, up to isomorphism, as a pullback $f^*(U_{\mathbb{T}})$ of the model $U_{\mathbb{T}}$ along the inverse image f^* of a unique (up to isomorphism) geometric morphism from $\mathcal{E}$ to $\mathcal{E}_{\mathbb{T}}$.

Clearly, a classifying topos of a given geometric theory $\mathbb{T}$ is unique up to categorical equivalence.

Classifying toposes for geometric theories can be canonically built by the construction of syntactic sites.

Definition 1.3.2. Let $\mathbb{T}$ be a geometric theory over a signature Σ and $\phi(\vec{x})$ and $\psi(\vec{y})$ be two formulas in $\mathbb{T}$, where $\vec{x}$ and $\vec{y}$ are context of the same type and length. We say that these formulas are *α-equivalent* if $\psi(\vec{y})$ is obtained from $\phi(\vec{x})$ by an *acceptable renaming*, i.e., every free occurrence of x_i is replaced by y_i in ϕ and each x_i is free for y_i in ϕ. We write $\{\vec{x} \,.\, \phi\}$ for the α-equivalence class of the formula $\phi(\vec{x})$. The *geometric syntactic category* $\mathcal{C}_{\mathbb{T}}$ of $\mathbb{T}$ has as objects the geometric formulas-in-context $\{\vec{x} \,.\, \phi\}$ and as arrows between $\{\vec{x} \,.\, \phi\}$ and $\{\vec{y} \,.\, \psi\}$ the $\mathbb{T}$-provable equivalence classes $[\theta]$ of geometric formulas $\theta(\vec{x}, \vec{y})$, where $\vec{x}$ and $\vec{y}$ are disjoint contexts, which are $\mathbb{T}$-*provably functional* from $\{\vec{x} \,.\, \phi\}$ to $\{\vec{y} \,.\, \psi\}$, i.e., such that the sequents

- $(\theta \vdash_{\vec{x},\vec{y}} (\phi \wedge \psi))$
- $(\theta \wedge \theta[\vec{z}/\vec{y}] \vdash_{\vec{x},\vec{y},\vec{z}} (\vec{z} = \vec{y}))$
- $(\phi \vdash_{\vec{x}} (\exists \vec{y})\theta)$

are provable in $\mathbb{T}$.

We shall say that two geometric formulas-in-context $\{\vec{x} \,.\, \phi\}$ and $\{\vec{y} \,.\, \psi\}$, where $\vec{x}$ and $\vec{y}$ are disjoint, are $\mathbb{T}$-*equivalent* if they are isomorphic objects in the syntactic category $\mathcal{C}_{\mathbb{T}}$, that is, if there exists a geometric formula $\theta(\vec{x}, \vec{y})$ which is $\mathbb{T}$-provably functional from $\{\vec{x} \,.\, \phi\}$ to $\{\vec{y} \,.\, \psi\}$ and which moreover satisfies the property that the sequent $(\theta \wedge \theta[\vec{x'}/\vec{x}] \vdash_{\vec{x},\vec{x'},\vec{y}} \vec{x} = \vec{x'})$ is provable in $\mathbb{T}$.

We can equip the geometric category $\mathcal{C}_{\mathbb{T}}$ with its *canonical coverage*, consisting of all sieves generated by small *covering families*, i.e., families of the form $\{[\vec{x}_i, \vec{y} \,.\, \theta_i] \mid i \in I\}$, where $[\theta_i]$ are arrows from $\{\vec{x}_i \,.\, \phi_i\}$ to $\{\vec{y} \,.\, \psi\}$ in $\mathcal{C}_{\mathbb{T}}$ and the sequent $\psi \vdash_{\vec{y}} \bigvee_{i \in I}(\exists \vec{x}_i \theta_i)$ is provable in $\mathbb{T}$. We denote by $J_{\mathbb{T}}$ this topology.

The topos $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$ satisfies the universal property of the classifying topos for $\mathbb{T}$. By this it follows that every geometric theory has a classifying topos. The following theorem states that also the converse is true.

Theorem 1.3.3 (Makkai-Reyes-Joyal, 1970s)**.** *Every geometric theory has a classifying topos; conversely, every Grothendieck topos is the classifying topos of a geometric theory, albeit not canonically.*

Proof. Given a geometric theory, the Grothendieck topos $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$ is the classifying topos of $\mathbb{T}$.

Vice versa, given a Grothendieck topos $\mathbf{Sh}(\mathcal{C}, J)$, by Diaconescu's Theorem we have

$$\mathbf{Geom}(\mathcal{E}, \mathbf{Sh}(\mathcal{C}, J)) \simeq \mathbf{Flat}_J(\mathcal{C}, \mathcal{E})$$

naturally in $\mathcal{E}$. We construct a theory $\mathbb{T}_J^{\mathcal{C}}$ whose models in any Grothendieck topos $\mathcal{E}$ are precisely the J-continuous flat functors from $\mathcal{C}$ to $\mathcal{E}$. Hence we have the thesis. □

We can construct the syntactic sites (and the resulting classifying toposes) also for smaller fragments of first-order logic by choosing appropriate families of formulas for objects and arrows. More precisely, let $\mathbb{T}$ be a regular (resp. cartesian, coherent) theory; the synctactic category $\mathcal{C}_{\mathbb{T}}^{reg}$ (resp. $\mathcal{C}_{\mathbb{T}}^{cart}$, $\mathcal{C}_{\mathbb{T}}^{coh}$) has as objects regular (resp. cartesian, coherent) formulas-in-context and has as arrows equivalence classes of $\mathbb{T}$-provably functional regular (resp. coherent) formulas. The syntatic topology is the trivial one both for regular and cartesian theories, while for coherent theories we require that the covering families are finite.

The following theorem shows that the invariant notion of subtopos admits a natural logical counterpart. Recall that a *subtopos* of a topos $\mathcal{E}$ is the domain of a geometric inclusion $\mathcal{F} \to \mathcal{E}$ (i.e., of a geometric morphism $\mathcal{F} \to \mathcal{E}$ whose direct image functor is full and faithful).

Theorem 1.3.4 (**Duality Theorem**, Theorem 3.6 [11])**.** *Let $\mathbb{T}$ be a geometric theory over a signature Σ. Then the assignment sending a quotient of $\mathbb{T}$ to its classifying topos defines a bijection between the quotients of $\mathbb{T}$ (considered up to the equivalence which identifies two quotients precisely when they prove the same geometric sequents over their signature)and the subtoposes of the classifying topos $\mathcal{E}_{\mathbb{T}}$ of $\mathbb{T}$.*

This theorem associates to every quotient of $\mathbb{T}$ a certain Grothendieck topology J over the syntactic category $\mathcal{C}_{\mathbb{T}}$ which includes the topology $J_{\mathbb{T}}$ and which is defined by the additional axioms of the quotient. Thus, the classifying topos of the quotient is the subtopos of $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$ whose objects are the sheaves with respect to the topology J. Conversely, each subtopos of $\mathcal{E}_{\mathbb{T}}$ is of the form $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J)$, where J is a Grothendieck topology containing $J_{\mathbb{T}}$. The quotient of $\mathbb{T}$ classified by this topos has as axioms all the sequents of the form $\psi \vdash_{\vec{y}} (\exists \vec{x})\theta$, where $[\theta]$ is a morphism $[\theta] : \{\vec{x} \,.\, \phi\} \to \{\vec{y} \,.\, \psi\}$ in $\mathcal{C}_{\mathbb{T}}$ generating a J-covering sieve.

1.4 Toposes as 'bridges'

The 'bridge technique' was introduced by Olivia Caramello in her Ph.D. thesis and deeply developed in her works. For an introduction to this topic see [19].

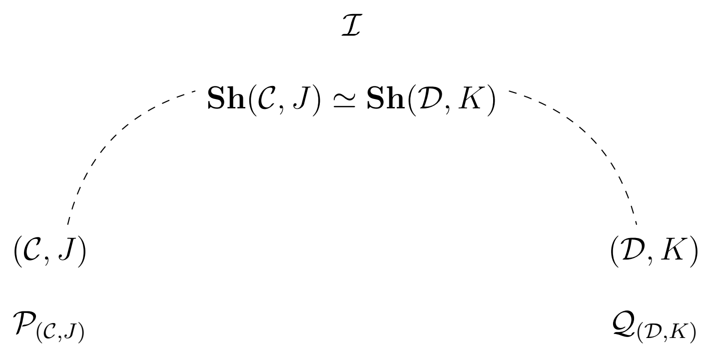
This technique is based on the possibility of representing Grothendieck toposes by means of different sites of definition. These different sites can be considered as different worlds (e.g. theories, categories, etc.) that are linked by the common Grothendieck topos. This topos can thus act as a bridge for transferring information from one world to the other. For instance, let us suppose that $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$ are two sites of definition of the same Grothendieck topos $\mathcal{E}$ and let $\mathcal{I}$ be a topos-theoretic invariant, i.e., a property or a construction on toposes that is stable under categorical equivalences. If we can find equivalences of the type

the topos $\mathcal{E}$ satisfies $\mathcal{I}$ 'if and only if' the site $(\mathcal{C}, J)$ satisfies $\mathcal{P}_{(\mathcal{C},J)}$

the topos $\mathcal{E}$ satisfies $\mathcal{I}$ 'if and only if' the site $(\mathcal{D}, K)$ satisfies $\mathcal{Q}_{(\mathcal{D},K)}$

where $\mathcal{P}_{(\mathcal{C},J)}$ and $\mathcal{Q}_{(\mathcal{D},K)}$ are properties of the sites $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$, then we immediately obtain the logic equivalence between $\mathcal{P}_{(\mathcal{C},J)}$ and $\mathcal{Q}_{(\mathcal{D},K)}$. These properties can be very different in spite of the fact that they are manifestations of the same topos-theoretic invariant $\mathcal{I}$. For example, as shown in [10], the property of a topos to be De Morgan specializes, on a presheaf topos, to the property of the underlying category to satisfy the right Ore condition and on the topos of sheaves on a topological space to the property of the space to be extremely disconnected. In [13] Caramello provided a general method for obtaining bijective site characterizations for 'geometric' invariants of toposes. Indeed, that paper gives a metatheorem furnishing sufficient conditions for a topos-theoretic invariant to have bijective site characterizations holding for large classes of sites.

We can construct bridges even if the relation between toposes is not an equivalence but the property that we are considering is stable under this relation. The advantages of working with equivalences is that every property written in categorical language is automatically invariant with respect to categorical equivalences.



The existence of different sites of definition for the same topos translates, at the logical level, into the existence of different geometric theories classified by the same topos.

Definition 1.4.1. Two geometric theories $\mathbb{T}$ and $\mathbb{T}'$ are said to be *Morita-equivalent* if they have equivalent classifying toposes, equivalently, if they have equivalent categories of models in every Grothendieck topos $\mathcal{E}$, naturally in $\mathcal{E}$, that is for each Grothendieck topos $\mathcal{E}$ there is an equivalence of categories

$$\tau_{\mathcal{E}} : \mathbb{T}\text{-mod}(\mathcal{E}) \to \mathbb{T}'\text{-mod}(\mathcal{E})$$

such that for any geometric morphism $f : \mathcal{F} \to \mathcal{E}$ the following diagram commutes (up to isomorphism):

$$\begin{array}{ccc} \mathbb{T}\text{-mod}(\mathcal{E}) & \xrightarrow{\tau_{\mathcal{E}}} & \mathbb{T}'\text{-mod}(\mathcal{E}) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \mathbb{T}\text{-mod}(\mathcal{F}) & \xrightarrow[\tau_{\mathcal{F}}]{} & \mathbb{T}'\text{-mod}(\mathcal{F}) \end{array}$$

Morita-equivalences can be seen as the ‘decks’ of our bridges, whose ‘arches’ are given by site characterizations.

Remark 1.4.2. (a) Let us suppose that $\mathbb{T}$ and $\mathbb{S}$ are two geometric theories whose categories of models in the category **Set** are categorically equivalent. If this categorical equivalence is established by only using constructive logic and geometric constructions (i.e., finite limits and arbitrary colimits), then the semantic equivalence can be lifted to a Morita-equivalence between the theories $\mathbb{T}$ and $\mathbb{S}$. Indeed, a Grothendieck topos can be seen as a generalized universe of sets where we can work only with constructive principles. Further, the request that the constructions are geometric assure that the naturality condition is satisfied.

(b) Two cartesian theories are Morita-equivalent if and only if they have equivalent categories of models in **Set**. Indeed, categorical equivalences between categories of set-based models always restrict to categorical equivalences between the categories of finitely presentable models. The dual of these categories, with the trivial topology, are sites of definition for the classifying toposes of cartesian theories. More generally, this is true for any pair of theories of presheaf type (cf. Section 1.5).

(c) Different sites of definition of a given Grothendieck topos can be interpreted as Morita-equivalent theories.

Trivial examples of Morita-equivalent theories are given by bi-interpretable theories.

Definition 1.4.3. Let $\mathbb{T}$ and $\mathbb{S}$ be geometric (cartesian, regular, coherent) theories. An *interpretation* (resp. a *bi-interpretation*) of $\mathbb{T}$ in $\mathbb{S}$ is a geometric (cartesian, regular, coherent) functor (resp. an equivalence) $I : \mathcal{C}_{\mathbb{T}} \to \mathcal{C}_{\mathbb{S}}$ between their geometric (cartesian, regular, coherent) syntactic categories. We say that $\mathbb{T}$ is *interpretable* (resp. *bi-interpretable*) in $\mathbb{S}$ if there exists an interpretation (resp. a bi-interpretation) of $\mathbb{T}$ in $\mathbb{S}$.

If two theories are bi-interpretable then by definition their syntactic categories are equivalent whence they are classified by the same topos, in other words, they are Morita-equivalent. Of course, the most interesting examples of Morita-equivalences are the ones that do not arise from bi-interpretations.

We recall that for any geometric theory $\mathbb{T}$ and geometric category $\mathcal{C}$, we have a categorical equivalence

$$\mathbf{Hom}_{geom}(\mathcal{C}_{\mathbb{T}}, \mathcal{C}) \simeq \mathbb{T}\text{-mod}(\mathcal{C})^2$$

natural in $\mathcal{C}$, one half of which sends any model M of $\mathbb{T}$ in $\mathcal{C}$ to the geometric functor $F_M : \mathcal{C}_{\mathbb{T}} \to \mathcal{C}$ assigning to any object $\{\vec{x} \,.\, \phi\}$ of the syntactic category $\mathcal{C}_{\mathbb{T}}$ its interpretation $[\![\vec{x} \,.\, \phi]\!]_M$ in $\mathcal{C}$. Under this equivalence, an interpretation I of a theory $\mathbb{T}$ in a theory $\mathbb{S}$ corresponds to a model of $\mathbb{T}$ in the category $\mathcal{C}_{\mathbb{S}}$. Thus, for any geometric category $\mathcal{C}$, an interpretation I of $\mathbb{T}$ in $\mathbb{S}$ induces a functor

$$s_I^{\mathcal{C}} : \mathbb{T}\text{-mod}(\mathcal{C}) \to \mathbb{S}\text{-mod}(\mathcal{C})$$

defined by the following commutative diagram:

$$\begin{array}{ccc} \mathbf{Hom}_{geom}(\mathcal{C}_{\mathbb{T}}, \mathcal{C}) & \simeq & \mathbb{T}\text{-mod}(\mathcal{C}) \\ {\scriptstyle - \circ I}\downarrow & & \downarrow{\scriptstyle s_I^{\mathcal{C}}} \\ \mathbf{Hom}_{geom}(\mathcal{C}_{\mathbb{S}}, \mathcal{C}) & \simeq & \mathbb{S}\text{-mod}(\mathcal{C}) \end{array}$$

Analogous results hold for cartesian, regular and coherent theories.

1.5 Theories of presheaf type

By definition, a *theory of presheaf type* is a geometric theory whose classifying topos is (equivalent to) a topos of presheaves.

This class contains all the finitary algebraic (and, more generally, all the cartesian) theories as well as many other interesting, even infinitary, theories, such as the theory of lattice-ordered abelian groups with a distinguished

[2] The category $\mathbf{Hom}_{geom}(\mathcal{C}_{\mathbb{T}}, \mathcal{C})$ is the category of geometric functors from $\mathcal{C}_{\mathbb{T}}$ to $\mathcal{C}$.

strong unit considered in [21] or the theory of algebraic extensions of a base field considered in [17].

In this section we recall some fundamental results on this class of geometric theories. For a comprehensive investigation, containing various kinds of characterization theorems, we refer the reader to [17].

Definition 1.5.1. ([33]) Let $\mathbb{T}$ be a geometric theory. A model M of $\mathbb{T}$ in **Set** is *finitely presentable* if the representable functor $Hom(M, -) : \mathbb{T}$-mod(**Set**) $\to$ **Set** preserves filtered colimits.

As shown in [16], the classifying topos of a theory of presheaf type $\mathbb{T}$ can be canonically represented as the functor category [f.p.$\mathbb{T}$-mod(**Set**), **Set**], where f.p.$\mathbb{T}$-mod(**Set**) is the full subcategory of $\mathbb{T}$-mod(**Set**) of finitely presentable $\mathbb{T}$-models. From this representation it follows that two theories of presheaf type are Morita-equivalent if and only if the categories of set-based models are equivalent since categorical equivalences always restrict to equivalence between the full subcategories of finitely presentable objects.

Definition 1.5.2. ([11]) Let $\mathbb{T}$ be a geometric theory over a one-sorted signature Σ and $\phi(\vec{x}) = \phi(x_1, \ldots, x_n)$ be a geometric formula over Σ. We say that a $\mathbb{T}$-model M in **Set** is *finitely presented* by $\phi(\vec{x})$ (or that $\phi(\vec{x})$ *presents* M) if there exists a string of elements $(a_1, \ldots, a_n) \in M^n$, called *generators* of M, such that for any $\mathbb{T}$-model N in **Set** and any string of elements $(b_1, \ldots, b_n) \in [\![\vec{x} \, . \, \phi]\!]_N$, there exists a unique arrow $f : M \to N$ in $\mathbb{T}$-mod(**Set**) such that $f(a_i) = b_i$ for $i = 1, \ldots, n$.

This definition can be clearly generalized to multi-sorted theories.

The two above-mentioned notions of finitely presentability of a model coincide for cartesian theories (cf. pp. 882-883 [35]). More generally, as shown in [14], they coincide for all theories of presheaf type.

Definition 1.5.3. Let $\mathbb{T}$ be a geometric theory over a signature Σ and $\{\vec{x} \, . \, \phi\}$ a geometric formula-in-context over Σ. Then $\{\vec{x} \, . \, \phi\}$ is said to be $\mathbb{T}$-*irreducible* if for any family $\{[\theta_i] \mid i \in I\}$ of classes of $\mathbb{T}$-provably functional geometric formulas $[\theta_i(\vec{x_i}, \vec{x})]$ from $\{\vec{x_i} . \phi_i\}$ to $\{\vec{x} . \phi\}$ such that $\phi \vdash_{\vec{x}} \bigvee_{i \in I} (\exists \vec{x_i})\theta_i$ is provable in $\mathbb{T}$, there exist $i \in I$ and a class $[\theta'(\vec{x}, \vec{x_i})]$ of $\mathbb{T}$-provably functional geometric formulas from $\{\vec{x} . \phi\}$ to $\{\vec{x_i} . \phi_i\}$ such that $\phi \vdash_{\vec{x}} (\exists \vec{x_i})(\theta' \wedge \theta_i)$ is provable in $\mathbb{T}$.

We indicate with the symbol $\mathcal{C}_{\mathbb{T}}^{\text{irr}}$ the full subcategory of $\mathcal{C}_{\mathbb{T}}$ on $\mathbb{T}$-irreducible formulas. Notice that a formula $\{\vec{x} . \phi\}$ is $\mathbb{T}$-irreducible if and only if it is $J_{\mathbb{T}}$-irreducible as an object of the syntactic category $\mathcal{C}_{\mathbb{T}}$ of $\mathbb{T}$ (in the sense of Definition 1.1.5).

Theorem 1.5.4 (cf. Theorem 3.13 [14])**.** *Let $\mathbb{T}$ be a geometric theory. Then $\mathbb{T}$ is of presheaf type if and only if the syntactic topology $J_{\mathbb{T}}$ on $\mathcal{C}_{\mathbb{T}}$ is rigid.*

Theorem 1.5.5 (Corollary 3.15 [14])**.** *Let $\mathbb{T}$ be a geometric theory over a signature Σ. Then $\mathbb{T}$ is classified by a presheaf topos if and only if there exists a collection $\mathcal{F}$ of $\mathbb{T}$-irreducible formulas-in-context over Σ such that for every geometric formula $\{\vec{y} . \psi\}$ over Σ there exist objects $\{\vec{x_i} . \phi_i\}$ in $\mathcal{F}$, as i varies in I, and classes of $\mathbb{T}$-provably functional geometric formulas $[\theta_i(\vec{x_i}, \vec{y})]$ from $\{\vec{x_i} . \phi_i\}$ to $\{\vec{y} . \psi\}$ such that $\psi \vdash_{\vec{y}} \bigvee_{i \in I} (\exists \vec{x_i})\theta_i$ is provable in $\mathbb{T}$.*

Theorem 1.5.6 (Theorem 4.3 [14])**.** *Let $\mathbb{T}$ be a theory of presheaf type over a signature Σ. Then*

(i) Any finitely presentable $\mathbb{T}$-model in **Set** *is presented by a $\mathbb{T}$-irreducible geometric formula $\{\vec{x} . \phi\}$ over Σ;*

(ii) Conversely, any $\mathbb{T}$-irreducible geometric formula $\{\vec{x} . \phi\}$ over Σ presents a $\mathbb{T}$-model.

*In particular, the category f.p.$\mathbb{T}$-mod(**Set**)^{op} is equivalent to the full subcategory $\mathcal{C}_{\mathbb{T}}^{irr}$ of the geometric syntactic category $\mathcal{C}_{\mathbb{T}}$ of $\mathbb{T}$ on the $\mathbb{T}$-irreducible formulas.*

We know by Duality Theorem that, given a geometric theory $\mathbb{T}$, each quotient $\mathbb{S}$ of $\mathbb{T}$ is associated with a Grothendieck topology $J_{\mathbb{T}}^{\mathbb{S}}$ defined on $\mathcal{C}_{\mathbb{T}}$ such that the topos of sheaves on the site $(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}}^{\mathbb{S}})$ is the classifying topos of $\mathbb{S}$. If $\mathbb{T}$ is a theory of presheaf type we have a semantical representation of its classifying topos, hence we have a semantical description of the Grothendieck topology associated with any quotient of $\mathbb{T}$. In details, let σ be an axiom of $\mathbb{S}$ which we can express in the following normal form

$$\phi \vdash_{\vec{x}} \bigvee_{i \in I} (\exists \vec{y_i})\theta_i,$$

where $[\theta_i] : \{\vec{y_i} \,.\, \psi_i\} \to \{\vec{x} \,.\, \phi\}$ is an arrow in $\mathcal{C}_{\mathbb{T}}$ for each $i \in I$ and $\{\vec{x} \,.\, \phi\}$, $\{\vec{y_i} \,.\, \psi_i\}$ are $\mathbb{T}$-irreducible formulas, hence they present $\mathbb{T}$-models M_ϕ and M_{ψ_i}. The interpretation of each arrow $[\theta_i]$ in M_{ψ_i} is the graph of a map $[\![\vec{y_i} \,.\, \psi_i]\!]_{M_{\psi_i}} \to [\![\vec{x} \,.\, \phi]\!]_{M_{\psi_i}}$. By definition of M_ϕ, these maps induce homomorphisms $s_i : M_\phi \to M_{\psi_i}$. Let us call S_σ the sieve in f.p.$\mathbb{T}$-mod$(\mathbf{Set})^{op}$ generated by these homomorphisms $\{s_i\}_{i \in I}$. The Grothendieck topology associated with $\mathbb{S}$ is hence the topology generated by the sieves S_σ, for each axiom σ of $\mathbb{S}$, i.e., the closure of the sieves S_σ under pushouts and finite multicompositions. We refer to pp. 68 [11] for this construction.

Among the notable properties of theories of presheaf type we mention a strong form of definability.

Theorem 1.5.7 (**Definabilty Theorem**, Corollary 3.2 [15])**.** *Let $\mathbb{T}$ be a theory of presheaf type over a one-sorted signature and suppose that we are given, for every finitely presentable* **Set***-model M of $\mathbb{T}$, a subset R_M of M^n in such a way that every $\mathbb{T}$-model homomorphism $h : M \to M$ maps R_M into R_N. Then there exists a geometric formula-in-context $\{\vec{x} \,.\, \phi\}$ such that $R_M = [\![\vec{x} \,.\, \phi]\!]_M$ for each finitely presentable $\mathbb{T}$-model M.*

This theorem generalized to multi-sorted theories.

Remark 1.5.8. (a) The proof of the Definability Theorem in [15] also shows that, for any two geometric formulas $\{\vec{x} \,.\, \phi\}$ and $\{\vec{x} \,.\, \psi\}$ over the signature of $\mathbb{T}$, every assignment $M \to f_M : [\![\vec{x} \,.\, \phi]\!]_M \to [\![\vec{y} \,.\, \psi]\!]_M$ (for finitely presentable $\mathbb{T}$-models M) which is natural in M is definable by a $\mathbb{T}$-provably functional formula $\theta(\vec{x}, \vec{y})$ from $\{\vec{x} \,.\, \phi\}$ to $\{\vec{x} \,.\, \psi\}$.

(b) If the property R of tuples $\vec{x}$ of elements of set-based $\mathbb{T}$-models as in the statement of the theorem is also preserved by filtered colimits of $\mathbb{T}$-models then we have $R_M = [\![\vec{x} \,.\, \phi]\!]_M$ for each set-based $\mathbb{T}$-model M, that is R is definable by the formula $\{\vec{x} \,.\, \phi\}$.

(c) If $\mathbb{T}$ is coherent and the property R is not only preserved but also reflected by arbitrary $\mathbb{T}$-model homomorphisms then the formula $\{\vec{x} \,.\, \phi\}$ in the statement of the theorem can be taken to be coherent and $\mathbb{T}$-boolean (in the sense that there exists a coherent formula $\{\vec{x} \,.\, \psi\}$ in the same context such that the sequents $(\phi \vdash \psi \vdash_{\vec{x}} \bot)$ and $(\top \vdash_{\vec{x}} \phi \vee \psi)$ are provable in $\mathbb{T}$). Indeed, the theorem can be applied both to the property R and to the negation of it yielding two geometric formulas $\{\vec{x} \,.\, \phi\}$ and $\{\vec{x} \,.\, \psi\}$ such that $(\phi \vdash \psi \vdash_{\vec{x}} \bot)$ and $(\top \vdash_{\vec{x}} \phi \vee \psi)$ are provable in $\mathbb{T}$. Hence, since every geometric formula is provably equivalent to a disjunction of coherent formulas and $\mathbb{T}$ is coherent, we can suppose ϕ and ψ to be coherent without loss of generality (cf. [14]).

In the sequel we list some results about theories of presheaf type and finitely presentable objects.

Proposition 1.5.9. *Let $\mathbb{T}$ be a theory of presheaf type. The category $\mathbb{T}$-mod*(**Set**) *is the ind-completion of the category f.p.$\mathbb{T}$-mod*(**Set**).

Recall that the *inductive completion*, or ind-completion, of a category $\mathcal{C}$ is the closure of that category under filtered colimits obtained by formally adding them.

Theorem 1.5.10 (Theorem 7.9 [17]). *Let $\mathbb{T}$ be a theory of presheaf type and $\mathbb{T}'$ be a sub-theory (i.e., a theory of which $\mathbb{T}$ is a quotient) of $\mathbb{T}$ such that every set-based model of $\mathbb{T}'$ admits a representation as a structure of global sections of a model of $\mathbb{T}$. Then every finitely presentable model of $\mathbb{T}$ is finitely presented as a model of $\mathbb{T}'$.*

This theorem shows the importance of sheaf representation as a way to understand if a theory is of presheaf type or not.

Theorem 1.5.11 (Theorem 6.26 [15]). *Let $\mathbb{T}'$ be a quotient of a theory of presheaf type $\mathbb{T}$ corresponding to a Grothendieck topology J on the category f.p.$\mathbb{T}$-mod$(\mathbf{Set})^{op}$ under the Duality Theorem. Suppose that $\mathbb{T}'$ is itself of presheaf type. Then every finitely presentable $\mathbb{T}'$-model is finitely presentable also as a $\mathbb{T}$-model if and only if the topology J is rigid.*

The following theorem provides a method for constructing theories of presheaf type whose category of finitely presented models is equivalent to a given small category of structures.

Theorem 1.5.12 (Theorem 6.29 [17]). *Let $\mathbb{T}$ be a theory of presheaf type and $\mathcal{A}$ be a full subcategory of f.p.$\mathbb{T}$-mod$(\mathbf{Set})$. Then the $\mathcal{A}$-completion $\mathbb{T}'$ of $\mathbb{T}$ (i.e., the set of all geometric sequents over the signature of $\mathbb{T}$ which are valid in all models in $\mathcal{A}$) is of presheaf type classified by the topos $[\mathcal{A}, \mathbf{Set}]$; in particular, every finitely presentable $\mathbb{T}'$-model is a retract of a model in $\mathcal{A}$.*

Under appropriate conditions, it is possible to give an axiomatization for the theories as in Theorem 1.5.12:

Theorem 1.5.13 (cf. Theorem 6.32 [17]). *Let $\mathbb{T}$ be a theory of presheaf type over a signature Σ with one sort and $\mathcal{K}$ a full subcategory of the category of finitely generated and finitely presented (with respect to the same generators) $\mathbb{T}$-models. Then the following sequents, added to the axioms of $\mathbb{T}$, yield an axiomatization of the theory $\mathbb{T}_{\mathcal{K}}$ classified by the topos $[\mathcal{K}, \mathbf{Set}]$ (where we* *denote by $\mathcal{P}$ the set of geometric formulas over Σ which presents a $\mathbb{T}$-model in $\mathcal{K}$):*

(i) The sequent

$$(\top \vdash_{[]} \bigvee_{\phi(\vec{x})\in\mathcal{P}} (\exists \vec{x})\phi(\vec{x}));$$

(ii) For any formulas $\phi(\vec{x})$ and $\psi(\vec{y})$ in $\mathcal{P}$, where $\vec{x} = (x_1, \ldots, x_n)$ and $\vec{y} = (y_1, \ldots, y_m)$, the sequent

$$(\phi(\vec{x}) \wedge \psi(\vec{y}) \vdash_{\vec{x},\vec{y}} \bigvee_{\chi(\vec{z})\in\mathcal{P}, t_1(\vec{z}),\ldots,t_n(\vec{z}), s_1(\vec{z}),\ldots,s_m(\vec{z})} (\exists \vec{z})(\chi(\vec{z}) \wedge \bigwedge_{i\in\{1,\ldots,n\}, j\in\{1,\ldots,m\}} (x_i = t_i(\vec{z}) \wedge y_j = s_j(\vec{z})))),$$

where the disjunction is taken over all the formulas $\chi(\vec{z})$ in $\mathcal{P}$ and all the sequences of terms $t_1(\vec{z}), \ldots, t_n(\vec{z})$ and $s_1(\vec{z}), \ldots, s_m(\vec{z})$ and such that, denoting by $\vec{\xi}$ the set of generators of the model $M_{\{\vec{z}.\chi\}}$ finitely presented by the formula $\chi(\vec{z})$, $t_1(\vec{\xi}), \ldots, t_n(\vec{\xi}) \in [\![\vec{x} \,.\, \phi]\!]_{M_{\{\vec{z}.\chi\}}}$ and $s_1(\vec{\xi}), \ldots, s_n(\vec{\xi}) \in [\![\vec{x} \,.\, \psi]\!]_{M_{\{\vec{z}.\chi\}}}$;

(iii) For any formulas $\phi(\vec{x})$ and $\psi(\vec{y})$ in $\mathcal{P}$, where $\vec{x} = (x_1, \ldots, x_n)$ and $\vec{y} = (y_1, \ldots, y_m)$, and any terms $t_1(\vec{y}), s_1(\vec{y}) \ldots, t_n(\vec{y}), s_n(\vec{y})$, the sequent

$$(\bigwedge_{i\in\{1,\ldots,n\}} (t_i(\vec{y}) = s_i(\vec{y}) \wedge \phi(t_1/x_1, \ldots, t_n/x_n) \wedge \phi(s_1/x_1, \ldots, s_n/x_n) \wedge \psi(\vec{y}) \vdash_{\vec{y}} \bigvee_{\chi(\vec{z})\in\mathcal{P}, u_1(\vec{z}),\ldots,u_m(\vec{z})} ((\exists \vec{z})(\chi(\vec{z}) \wedge \bigwedge_{j\in\{1,\ldots,m\}} (y_j = u_j(\vec{z}))),$$

where the disjunction is taken over all the formulas $\chi(\vec{z})$ in $\mathcal{P}$ and all the sequences of terms $u_1(\vec{z}), \ldots, u_m(\vec{z})$ such that, denoting by $\vec{\xi}$ the set of generators of the model $M_{\{\vec{z}.\chi\}}$ finitely presented by the formula $\chi(\vec{z})$, $(u_1(\vec{\xi}), \ldots, u_m(\vec{\xi})) \in [\![\vec{y} \,.\, \psi]\!]_{M_{\{\vec{z}.\chi\}}}$ and $t_i(u_1(\vec{\xi}), \ldots, u_m(\vec{\xi})) = s_i(u_1(\vec{\xi}), \ldots, u_m(\vec{\xi}))$ in $M_{\{\vec{z}.\chi\}}$ for all $i \in \{1, \ldots, n\}$;

(iv) The sequent

$(\top \vdash_x \bigvee_{\chi(\vec{z})\in\mathcal{P}, t(\vec{z})} (\exists\vec{z})(\chi(\vec{z}) \wedge x = t(\vec{z})))$, *where the disjunction is taken over all the formulas* $\chi(\vec{z})$ *in* $\mathcal{P}$ *and all the terms* $t(\vec{z})$*;*

(v) For any formulas $\phi(\vec{x})$ *and* $\psi(\vec{y})$ *in* $\mathcal{P}$*, where* $\vec{x} = (x_1, \ldots, x_n)$ *and* $\vec{y} = (y_1, \ldots, y_m)$*, and any terms* $t(\vec{x})$ *and* $s(\vec{y})$*, the sequent*

$$(\phi(\vec{x}) \wedge \psi(\vec{y}) \wedge t(\vec{x}) = s(\vec{y}) \vdash_{\vec{x},\vec{y}} \bigvee_{\chi(\vec{z})\in\mathcal{P}, p_1(\vec{z}),\ldots,p_n(\vec{z}), q_1(\vec{z}),\ldots,q_m(\vec{z})} (\exists\vec{z})(\chi(\vec{z}) \wedge \bigwedge_{i\in\{1,\ldots,n\}, j\in\{1,\ldots,m\}} (x_i = p_i(\vec{z}) \wedge y_j = q_j(\vec{z})))),$$

where the disjunction is taken over all the formulas $\chi(\vec{z})$ *in* $\mathcal{P}$ *and all the sequences of terms* $p_1(\vec{z}), \ldots, p_n(\vec{z})$ *and* $q_1(\vec{z}), \ldots, q_m(\vec{z})$ *such that, denoting by* $\vec{\xi}$ *the set of generators of the model* $M_{\{\vec{z}.\chi\}}$ *finitely presented by the formula* $\chi(\vec{z})$*,* $(p_1(\vec{\xi}), \ldots, p_n(\vec{\xi})) \in [\![\vec{x} \,.\, \phi]\!]_{M_{\{\vec{z}.\chi\}}}$ *and* $(q_1(\vec{\xi}), \ldots, q_m(\vec{\xi})) \in [\![\vec{y} \,.\, \psi]\!]_{M_{\{\vec{z}.\chi\}}}$ *and* $t(p_1(\vec{\xi}), \ldots, p_n(\vec{\xi})) = s(q_1(\vec{\xi}), \ldots, q_m(\vec{\xi}))$ *in* $M_{\{\vec{z}.\chi\}}$*.*

The following theorem shows that adding sequents of a certain kind to a theory of presheaf type gives a theory that is still of presheaf type.

Theorem 1.5.14 (Theorem 6.28 [17])**.** *Let* $\mathbb{T}$ *be a theory of presheaf type over a signature* Σ*. Then any quotient* $\mathbb{T}'$ *of* $\mathbb{T}$ *obtained by adding sequents of the form* $(\phi \vdash_{\vec{x}} \bot)$*, where* $\phi(\vec{x})$ *is a geometric formula over* Σ*, is classified by the topos* $[\mathcal{T}, \mathbf{Set}]$*, where* $\mathcal{T}$ *is the full subcategory of f.p.*$\mathbb{T}$*-mod*$(\mathbf{Set})$ *on the* $\mathbb{T}'$*-models.*

Given a quotient of a geometric theory, it can be interesting to study its cartesianization.

Definition 1.5.15. Let $\mathbb{T}$ be a geometric theory over a signature Σ. The *cartesianization* $\mathbb{T}_c$ of $\mathbb{T}$ is the cartesian theory consisting of all the cartesian sequents over Σ which are provable in $\mathbb{T}$. Equivalently, it is the biggest cartesian theory over Σ of which $\mathbb{T}$ is a quotient.

Recall that a cartesian theory is always a theory of presheaf type; hence, the cartesianization of a theory that is not of presheaf type gives a good presheaf type approximation of this theory. The following remark shows the importance of sheaf representations in determining the cartesianization of a given geometric theory.

Remark 1.5.16. Let $\mathbb{T}$ be a cartesian theory and $\mathbb{S}$ be a geometric theory such that every set-based model of $\mathbb{T}$ is represented as the structure of global sections of an $\mathbb{S}$-model in a topos of sheaves over a topological space (or a locale). Then, the cartesianization of $\mathbb{S}$ is the theory $\mathbb{T}$. Indeed, the global sections functor is cartesian and hence preserves the validity of cartesian sequents. So every cartesian sequent that is provable in S is provable in every set-model of T. Since every cartesian theory (more generally, every theory of presheaf type) is complete with respect to its set-based models, our claim follows.

Chapter 2

The geometric theory of MV-algebras

In this chapter, after recalling some basic definitions and results on the theory of MV-algebras, we study some of its quotients. In particular, we focus our attention on the class of perfect MV-algebras, local MV-algebras and simple MV-algebras and on the geometric theories axiomatizing these classes. Whilst the theory of local MV-algebras and of simple MV-algebras are not of presheaf type, we will see in Chapter 4 that this property is satisfied for the theory of perfect MV-algebras. We observe also that not every proper subclass of MV-algebras can be axiomatized in a geometric way. This is the case for the class of semimple MV-algebras; indeed, every quotient of the theory of MV-algebras which contains among its set-based models every semisimple MV-algebras is equivalent to the theory of MV-algebras. For the background on MV-algebras, we refer to [26]. The results on perfect, local and simple MV-algebras are contained in [20] and [22].

2.1 Preliminary results

Definition 2.1.1. An *MV-algebra* is a structure $\mathcal{A} = (A, \oplus, \neg, 0)$, where $\oplus$ is a binary function symbol, $\neg$ is a unary function symbol and 0 is a constant, satisfying the following axioms:

MV.1 $\top \vdash_{x,y,z} x \oplus (y \oplus z) = (x \oplus y) \oplus z$;

MV.2 $\top \vdash_{x,y} x \oplus y = y \oplus x$;

MV.3 $\top \vdash_{x} x \oplus 0 = x$;

MV.4 $\top \vdash_{x} \neg\neg x = x$;

MV.5 $\top \vdash_{x} x \oplus \neg 0 = \neg 0$;

MV.6 $\top \vdash_{x,y} \neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

We can define the 'geometric theory' $\mathbb{MV}$ of MV-algebras. The signature Σ_{MV} consits of one sort, a constant symbol 0, a binary function symbol $\oplus$ and a unary function symbol $\neg$. The theory $\mathbb{MV}$ is a geometric theory over the signature Σ_{MV} whose axioms are the sequents MV.1-MV.6 of the definition of MV-algebra.

One can define on $\mathcal{A}$ the following derived operations:

- $x \odot y := \neg(\neg x \oplus \neg y)$,
- $x \ominus y = x \odot \neg y$,
- $\sup(x, y) := (x \odot \neg y) \oplus y$,
- $\inf(x, y) := (x \oplus \neg y) \odot y$,
- $1 := \neg 0$.

We write $x \leq y$ if $\inf(x, y) = x$; this relation defines a partial order relation on $\mathcal{A}$ called *natural order*. In the sequel we will use the notations inf or $\wedge$ and sup or $\vee$ to indicate respectively the infimum and the supremun of two or more elements in an MV-algebra. If the natural order is total we say that $\mathcal{A}$ is an *MV-chain*.

Lemma 2.1.2 (cf. Lemma 1.1.2 [26]). *Let $\mathcal{A}$ be an MV-algebra and $x, y \in A$. Then the following conditions are equivalent:*

(i) $\neg x \oplus y = 1$*;*

(ii) $x \odot \neg y = 0$*;*

(iii) there is an element $z \in A$ such that $x \oplus z = y$;

(iv) $x \leq y$*.*

We write nx for $x \oplus \cdots \oplus x$ (n times) and x^n for $x \odot \cdots \odot x$ (n times). The least integer for which $nx = 1$ is called the *order* of x. When such an integer exists, we denote it by $ord(x)$ and we say that x has *finite order*; otherwise we say that x has *infinite order* and we write $ord(x) = \infty$.

Boolean algebras are particular examples of MV-algebras. Moreover, every MV-algebra $\mathcal{A}$ has a *boolean kernel* $B(\mathcal{A})$ given by the set of *boolean elements*, that are idempotent elements with respect to the sum. Every boolean element distinct from 1 has infinite order.

Example 2.1.3. Let $[0, 1]$ be the unit interval of real numbers. Consider the operations

- $x \oplus y := \min\{1, x + y\}$,
- $\neg x := 1 - x$.

The structure $([0, 1], \oplus, \neg, 0)$ is an MV-algebra. We shall refer to it as to the *standard MV-algebra*; in fact, Chang proved in [25], non-constructively, that this algebra generates the variety of MV-algebras.

Let $\mathcal{A}$ and $\mathcal{B}$ be MV-algebras. A fuction $h : \mathcal{A} \to \mathcal{B}$ is an *MV-homomorphism* if, for all $x, y \in A$:

H.1 $h(0) = 0$;

H.2 $h(x \oplus y) = h(x) \oplus h(y)$;

H.3 $h(\neg x) = \neg h(x)$.

Congruences relations on an MV-algebra $\mathcal{A} = (A, \oplus, \neg, 0)$ can be identified with its *ideals*, i.e., non-empty subsets I of A satisfying the conditions:

I.1 $0 \in I$;

I.2 if $x \in I, y \in A$ and $y \leq x$, then $y \in I$;

I.3 if $x, y \in I$, then $x \oplus y \in I$.

An ideal I is *prime* if it is proper, i.e., it does not coincide with the whole algebra, and for each $x, y \in A$, either $x \ominus y \in I$ or $y \ominus x \in I$. An ideal is *maximal* if it is proper and no proper ideal strictly contains it.

On every MV-algebra $\mathcal{A}$ we can define the *distance function* $d : A \times A \to A$

$$d(x, y) = (x \ominus y) \oplus (y \ominus x)$$

that allows to establish the bijection between congruences and ideals of $\mathcal{A}$.

Proposition 2.1.4 (cf. Proposition 1.2.6 [26]). *Let I be an ideal of an MV-algebra $\mathcal{A}$.Then the binary relation $\equiv_I$ definied on A by*

$$x \equiv_I y \text{ if and only if } d(x, y) \in I$$

is a congruence relation. Conversely, if $\equiv$ is a congruence relation, then $\{x \in A \mid x \equiv 0\}$ is an ideal of $\mathcal{A}$.

The *radical* $\mathrm{Rad}(\mathcal{A})$ of an MV-algebra $\mathcal{A}$ is either defined as the intersection of all the maximal ideals of $\mathcal{A}$ (and as $\{0\}$ if $\mathcal{A}$ is the trivial algebra in which $0 = 1$) or as the set of infinitesimal elements (i.e., those elements $x \neq 0$ such that $kx \leq \neg x$ for every $k \in \mathbb{N}$) plus 0. The coradical $\neg\mathrm{Rad}(\mathcal{A})$ is the set of elements such that their negation is in the radical. The first definition of the radical immediately implies that it is an ideal (as it is intersection of ideals), but it requires the axiom of choice to be consistent. The second definition is instead constructive but it does not show that the radical is an ideal. In Section 2.2 we will prove that if we restrict to the variety generated by the so-called Chang's algebra then the radical is defined by an equation, and we will show constructively that it is an ideal. In Section 5.2 we shall see that this result generalizes to the case of an arbitrary proper subvariety of MV-algebras. However, it is not possible to define the radical by a geometric formula in the variety of MV-algebras. This is a consequence of the fact that the class of *semisimple MV-algebras*, i.e., the MV-algebras whose radical is equal to $\{0\}$, cannot be axiomatized in a geometric way over the signature Σ_{MV}.

Proposition* 2.1.5. *The class of semisimple MV-algebras does not admit a geometric axiomatization over the signature of the theory of MV-algebras.*

Proof. We know that the theory $\mathbb{MV}$ of MV-algebras is of presheaf type; hence, every MV-algebra is a filtered colimit of finitely presented MV-algebras. Now, every finitely presented MV-algebra is semisimple (cf. Theorem 3.6.9 [26][1]) so, if the class of semisimple MV-algebras admitted a geometric axiomatization over the signature of the theory $\mathbb{MV}$, every MV-algebra would be semisimple (recall that the categories of set-based models of a geometric theories are closed by filtered colimits). Since this is clearly not the case, our thesis follows. □

[1]This result is not constructive.

Corollary* 2.1.6. *There is not any geometric formula $\{x \,.\, \phi\}$ such that, for every MV-algebra $\mathcal{A}$, its interpretation in $\mathcal{A}$ is equal to Rad($\mathcal{A}$).*

Proof. If there existed a geometric formula $\{x \,.\, \phi\}$ which defines the radical for every MV-algebra then the semisimple MV-algebras would be precisely the set-based models of the quotient of $\mathbb{MV}$ obtained by adding the sequent

$$(\phi \vdash_x x = 0),$$

contradicting Proposition 2.1.5. □

Chang proved a foundamental representation result for MV-algebras.

Theorem* 2.1.7 (Chang's Subdirect Representation Theorem, Lemma 3 [25]**).** *Every MV-algebra is a subdirect product of MV-chains.*

In this representation result, which uses the axiom of choice, every MV-algebra is represented as a subdirect product of its quotients with respect to prime ideals. This is the reformulation, in the theory of MV-algebras, of Birkhoff's Subdirect Representation Theorem in universal algebras which states that every algebras is a subdirect product of irreducible algebras. Indeed, MV-chains are subdirect irreducible MV-algebras.

The theory of MV-algebras is algebraic whence of presheaf type. This means that we have a very simple representation of its classifying topos that is given by the topos of presheaves over the dual category of its finitely presented models. Observe that in this case the finitely presented models of $\mathbb{MV}$ coincide with the finitely presented algebras, in the sense of universal algebra, in the variety of MV-algebras, i.e., quotients of free MV-algebras over a finite number of generators with respect to a finitely generated ideal.

A model of the theory $\mathbb{MV}$ in a category $\mathcal{E}$ with finite products (in particular, a Grothendieck topos) consists of an object M, interpreting the unique sort of the signature Σ_{MV}, an arrow $M\oplus : M \times M \to M$ in $\mathcal{E}$ interpreting the binary operation $\oplus$, an arrow $M\neg : M \to M$ in $\mathcal{E}$ interpreting the unary operation $\neg$ and a global element $M0 : 1 \to M$ of M in $\mathcal{E}$ (where 1 is the terminal object of $\mathcal{E}$) interpreting the constant 0. In the sequel, we shall omit the indication of the model M in the notation for the operations and the constant.

2.2 Perfect MV-algebras

Boolean algebras are always semisimple but this is no longer true for MV-algebras. Indeed, there are classes of MV-algebras where the radical plays an important role.

Definition 2.2.1. An MV-algebra $\mathcal{A} = (A, \oplus, \neg, 0)$ is said to be *perfect* if $\mathcal{A}$ is non-trivial (i.e., $A \neq \{0\}$ or equivalently $1 \neq 0$) and $A = Rad(\mathcal{A}) \cup \neg Rad(\mathcal{A})$.

The set $\neg Rad(\mathcal{A})$ is called the *coradical* of $\mathcal{A}$ and it is also denoted by $Corad(\mathcal{A})$.

Chang's MV-algebra C is the prototype of perfect MV-algebras, in the sense that it is a perfect MV-algebra and every perfect MV-algebra is contained in the variety $V(C)$, called *Chang's variety*, generated by it (cf. Theorem 2.2.2(5)). It is defined on the following infinite set of formal symbols

$$C = \{0, c, \ldots, nc, \ldots, 1 - nc, \ldots, 1 - c, 1\}$$

with the following operations:

- $$x \oplus y := \begin{cases} (m+n)c & \text{if } x = nc \text{ and } y = mc \\ 1-(n-m)c & \text{if } x = 1-nc,\ y = mc \text{ and } 0 \lneq m \lneq n \\ 1 & \text{if } x = 1-nc,\ y = mc \text{ and } 0 \lneq n \leq m \\ 1 & \text{if } x = 1-nc,\ y = 1-mc \end{cases}$$
- $\neg x := 1 - x$

Chang's MV-algebra is a particular example of Komori chain (cf. Section 5.2). In the literature these chains are indicated with the symbol S_n^ω and Chang's MV-algebra is the Komori chain with $n = 1$. In the following we will use the latter notation.

Let **Perfect** be the class of perfect MV-algebras and **Local** be the class of *local* MV-algebras, i.e., the class of MV-algebras that have only one maximal ideal (this class will be deeply studied in the following section). The following theorem resumes some of the most relevant results about perfect MV-algebras.

Theorem* 2.2.2 (Proposition 5 [3]). *The following hold.*

(1) The only finite perfect MV-algebra is $\{0, 1\}$.

(2) Every nonzero element in a perfect MV-algebra $\mathcal{A} \neq B(\mathcal{A})$*, where* $B(\mathcal{A})$ *is the set of idempotent elements with respects to the sum (boolean elements), generates a subalgebra isomorphic to the Chang MV-algebra* S_1^ω.

(3) Sudirectly irreducible MV-algebras in $V(S_1^\omega)$ *are all perfect MV-chains.*

(4) $V(\textbf{Perfect}) = V(S_1^\omega)$.

(5) $\textbf{Perfect} = V(S_1^\omega) \cap \textbf{Local}$.

(6) $\mathcal{A} \in V(S_1^\omega)$ *if and only if every* $x \in A$, $2x^2 = (2x)^2$.

(7) $\mathcal{A}$ *is perfect if and only if it is generated by* $Rad(\mathcal{A})$.

(8) if $\mathcal{A}$*, then* $x \in Rad(\mathcal{A})$ *if and only if* $ord(x) = \infty$.

(9) **Perfect** *is closed under homomorphic images and subalgebras.*

In [3] the authors proved, by using the axiom of choice, that the class of perfect MV-algebras is first-order definable.

Proposition* 2.2.3 (cf. Proposition 6 [3]). *Let* $\mathcal{A}$ *be an MV-algebra. The following are equivalent.*

(i) $\mathcal{A}$ *is perfect.*

(ii) $\mathcal{A}$ *satisfies the sequents*

σ.1 $\top \vdash_x 2x^2 = (2x)^2$,

σ.2 $2x = x \vdash_x x = 0 \vee x = 1$.

The sequent $\sigma.1$ characterizes the variety $V(S_1^\omega)$ (cf. Theorem 2.2.2 (6)) while the sequent $\sigma.2$ expresses the property that every perfect MV-algebra has only 0 and 1 as boolean elements.

Let us indicate with the symbol $\mathbb{C}$ the quotient of the theory $\mathbb{MV}$ obtained by adding the following sequents:

C. $\top \vdash_x 2x^2 = (2x)^2$.

Non-constructively, the models of this theory in **Set** coincide with the algebras in the variety $V(S_1^\omega)$.

Moreover, it can be provable that all the algebras in $V(S_1^\omega)$ satisfy the sequent

$$\top \vdash_x 2(2x)^2 = (2x)^2,$$

expressing the property that every element of the form $(2x)^2$ is a boolean element. Indeed, as observed in [28], Chang's Subdirect Representation Theorem allows to embed every algebra in $V(S_1^\omega)$ in a direct product of (totally ordered) perfect MV-algebras and in each perfect MV-algebras this sequent is trivially satisfied (cf. Claim 1 in the proof of Theorem 5.8 [28]). In Proposition 3.7.4 we will show that this sequent is also constructively provable in $\mathbb{C}$.

Lemma 2.2.4. *The sequent*

$$\gamma_n : (2^n x = 1 \vdash_x 2x = 1)$$

is provable in $\mathbb{C}$.

In particular, every element of finite order of an MV-algebra in $\mathbb{C}$*-mod*(**Set**) *has order at most* 2.

Proof. By the proof of Theorem 3.9 [24], for each natural number n, the sequent $\chi_n : (nx^2 = 1 \vdash_x 2x = 1)$ is provable in the theory $\mathbb{MV}$.

First, let us prove that the sequent $(\top \vdash_x 2^n x^2 = (2^n x)^2)$ is provable in $\mathbb{C}$ by induction on n. For $n = 1$, it is a tautology. For $n{>}1$, we argue (informally) as follows. We have $2^n x^2 = 2(2^{n-1}x^2) = 2((2^{n-1}x)^2) = (2(2^{n-1}x))^2 = (2^n x)^2$, where the second equality follows from the induction hypothesis and the third follows from the axiom C.

Now, $(2^n x = 1 \vdash_x 2x = 1)$ is provable in $\mathbb{C}$ since $2^n x = 1$ implies $(2^n x)^2 = 1$. But $(2^n x)^2 = 2^n x^2$. So $2^n x^2 = 1$ whence, by sequent χ_{2^n}, $2x = 1$, as required. □

Remark 2.2.5. Let $\mathcal{A}$ be a perfect MV-algebra. For any $x \in \neg Rad(\mathcal{A})$ not equal to 1, the order of x is equal to 2. Indeed, as we have already observed, every perfect MV-algebra is in the variety $V(S_1^\omega)$ and the coradical of a perfect MV-algebra contains only elements of finite order. Hence our claim follows from Lemma 2.2.4.

Lemma 2.2.6. *The following sequent is provably entailed by the non-triviality axiom* $(0 = 1 \vdash \bot)$ *in the theory* $\mathbb{C}$*:*

$$\alpha : (x = \neg x \vdash_x \bot).$$

Proof. Given a non-trivial $\mathbb{C}$-model $\mathcal{A}$, suppose that there is an $x \in A$ such that $x = \neg x$; thus, $x \oplus x = 1$. By axiom C, we have that $x^2 \oplus x^2 = (x \oplus x)^2$; but

$(x \oplus x)^2 = 1$

$x^2 \oplus x^2 = (\neg(\neg x \oplus \neg x)) \oplus (\neg(\neg x \oplus \neg x)) = 0 \oplus 0 = 0$

This is a contradiction since $\mathcal{A}$ is non-trivial. □

Lemma 2.2.7. *The sequent* α *holds in every perfect MV-algebra.*

Proof. This result trially follows from previous lemma since every perfect MV-algebra is non-trivial and in $\mathbb{C}$-mod(**Set**). $\square$

The geometric theory $\mathbb{P}$ of perfect MV-algebras is the quotient of the theory $\mathbb{C}$ where we add the following sequents:

P.1 $x \oplus x = x \vdash_x x = 0 \vee x = 1$;

P.2 $x = \neg x \vdash_x \bot$.

Theorem 2.2.8. *The family of sequents* $\{C,P.1\}$ *is provably equivalent in the geometric theory* $\mathbb{MV}$ *to the family of sequents* $\{C, \beta\}$*, where*

$$\beta : (\top \vdash_x x \leq \neg x \vee \neg x \leq x) \ .$$

Proof. Let us show that β and C entail sequent P.1. Given x such that $x \oplus x = x$ (equivalently $x \odot x = x$, cf. Theorem 1.16 [24]), we know from the sequent β that $x \leq \neg x$ or $\neg x \leq x$. Recall that $x \leq y$ iff $\neg x \oplus y = 1$ iff $x \odot \neg y = 0$. Hence, if $x \leq \neg x$ then $x \odot x = 0$ whence $x = 0$. On the other hand, if $\neg x \leq x$ then $x \oplus x = 1$ whence $x = 1$. This proves sequent P.1.

Conversely, let us show that the family of sequents {C and P.1 entails β. Given x, the element $2x^2$ is boolean, while by sequent C, $2x^2 = (2x)^2$. Sequent P.1 thus implies that either $2x^2 = 0$ or $(2x)^2 = 1$. But $2x^2 = 0$ clearly implies $x^2 = 0$, which is equivalent to $x \leq \neg x$, while $(2x)^2 = 1$ implies $2x = 1$, which is equivalent to $\neg x \leq x$. $\square$

The radical of every set-based model of $\mathbb{C}$ is definable by a first-order formula, as shown by the following more general result.

Proposition 2.2.9. *Let* $\mathcal{A}$ *be a set-based model of* $\mathbb{C}$*. Then*

$$Rad(\mathcal{A}) = \{x \in A \mid x \leq \neg x\}.$$

Proof. We shall verify that the sequent $(\top \vdash_x x^2 \oplus x^2 = (x \oplus x)^2)$ entails the sequents $(x \leq \neg x \vdash_x nx \leq \neg x)$ for each $n \in \mathbb{N}$. This will imply our thesis by soundness. It clearly suffices to prove this for n of the form 2^k for some k. Now, $x \leq \neg x$ if and only if $x^2 = 0$ and $2^k x \leq \neg x$ if and only if $(2^k x) \odot x = 0$. But $x \leq (2^k x)$, whence $(2^k x) \odot x \leq (2^k x) \odot (2^k x) = (2^k x)^2 = 2^k x^2$, where the last equality follows from the proof of Lemma 2.2.4. So $(2^k x) \odot x = 0$ if $x^2 = 0$, as required. □

The following lemma gives a list of sequents that are provable in the theory $\mathbb{P}$ and which therefore hold in every perfect MV-algebra by soundness.

Lemma 2.2.10. *The following sequents are provable in* $\mathbb{P}$*:*

(i) $(x \leq \neg x \wedge y \leq x \vdash_{x,y} y \leq \neg y)$*;*

(ii) $(\neg z \leq z \vdash_z \neg z^2 \leq z^2)$*;*

(iii) $(z \leq \neg z \vdash_z 2z \leq \neg 2z)$*;*

(iv) $(z^2 \leq \neg z^2 \vdash_z z \leq \neg z)$*;*

(v) $(x \leq \neg x \wedge y \leq \neg y \vdash_{x,y} \sup(x, y) \leq \neg \sup(x, y))$*;*

(vi) $(x \leq \neg x \wedge y \leq \neg y \vdash_{x,y} \inf(x, y) \leq \neg \inf(x, y))$*;*

(vii) $(x \leq \neg x \wedge y \leq \neg y \vdash_{x,y} x \oplus y \leq \neg(x \oplus y))$*;*

(viii) $(\neg x \leq x \wedge \neg y \leq y \vdash_{x,y} x \oplus y = 1)$*;*

(ix) $(x \leq \neg x \wedge \neg y \leq y \vdash_{x,y} x \leq y)$*.*

Proof. In the proof of this lemma we shall make an extensive use of the equivalent definitions of the natural order given by Lemma 2.1.2.

(i) Given $x \leq \neg x$ and $y \leq x$, we have that:

$$y \leq x \Rightarrow y \odot y \leq x \odot x \Rightarrow y \odot y = 0 \Leftrightarrow y \leq \neg y.$$

(ii) If $\neg z \leq z$, from axiom C and from identities that are provable in the theory of MV-algebras we have that:

$$0 = (2\neg z)^2 = (\neg z \oplus \neg z) \odot (\neg z \oplus \neg z) =$$
$$= (\neg z^2) \odot (\neg z^2) =$$
$$= \neg(z^2 \oplus z^2),$$

which means that $\neg z^2 \leq z^2$.

(iii) Given $z \leq \neg z$, we want to prove that $2z \leq \neg(2z)$. But this is equivalent to $(2z)^2 = 0$, which follows from $z^2 = 0$ (which is equivalent to $z \leq \neg z$) since $2z^2 = (2z)^2$ by axiom C.

(iv) Given $z^2 \leq \neg z^2$, by axiom β either $z \leq \neg z$ or $\neg z \leq z$. If $\neg z \leq z$, by point (ii) $\neg z^2 \leq z^2$ whence $\neg z^2 = z^2$. But from Lemma 2.2.7 we know that it is false, whence $z \leq \neg z$.

(v) Given $x \leq \neg x$ and $y \leq \neg y$, we have already observed that $x^2 = 0$ and $y^2 = 0$. From this it follows that $\sup(x, y)^3 = 0$ whence $\sup(x, y)^4 = 0$. Indeed, by using the identity $x \odot \sup(y, z) = \sup(x \odot y, x \odot z)$ (cf. Lemma 1.1.6(i) [26]), we obtain that:

$$\sup(x, y)^3 = \sup(x, y) \odot \sup(x, y) \odot \sup(x, y) =$$
$$= \sup(x, y) \odot \sup(\sup(x, y) \odot x, \sup(x, y) \odot y) =$$
$$= \sup(x, y) \odot \sup(\sup(x^2, x \odot y), \sup(x \odot y, y^2)) =$$
$$= \sup(x, y) \odot \sup(x \odot y, x \odot y) =$$
$$= \sup(x, y) \odot (x \odot y) =$$
$$= \sup(x^2, x \odot y) \odot y =$$
$$= x \odot y \odot y = 0.$$

If $\sup(x, y)^4 = 0$, then $\sup(x, y)^2 \leq \neg(\sup(x, y)^2)$. By point (iv) we thus have that $\sup(x, y) \leq \neg \sup(x, y)$, as required.

(vi) Given $x \leq \neg x$ and $y \leq \neg y$, since $\inf(x, y) \leq x, y$, the thesis follows from point (i).

(vii) Given $x \leq \neg x$ and $y \leq \neg y$, we know from points (iii) and (v) that $2\sup(x, y) \leq \neg(2\sup(x, y))$. But $x \oplus y \leq 2\sup(x, y)$, whence the thesis follows from point (i).

(viii) Given $\neg x \leq x$ and $\neg y \leq y$, by point (vi) we have that:

$$\neg x \oplus \neg y \leq \neg(\neg x \oplus \neg y) \Leftrightarrow$$
$$\neg(x \odot y) \leq (x \odot y) \Leftrightarrow$$
$$(x \odot y) \oplus (x \odot y) = 1.$$

But $(x \odot y) \oplus (x \odot y) = 1$ implies $x \oplus y = 1$ (cf. Theorem 3.8 [24]), as required.

(ix) Given $x \leq \neg x$ and $\neg y \leq y$, by point (viii) we have that

$$\neg x \oplus y = 1 \Leftrightarrow x \leq y,$$

as required. □

Remark 2.2.11. It will follow from Proposition 4.6.2 that each of the sequents in the statement of the previous lemma is provable in the theory $\mathbb{C}$. This means that it is possible to prove in a constructive way that the radical is an ideal.

In the following we will see that every finitely presentable perfect MV-algebra (i.e., a finitely presentable $\mathbb{P}$-model) is finitely presentable as an algebra in Chang's variety, that is, as a model of $\mathbb{C}$. Conversely every finitely presentable model of $\mathbb{C}$ which is perfect is finitely presentable as a model of $\mathbb{P}$. However, a finitely presentable perfect MV-algebra is not finitely presentable as an $\mathbb{MV}$-model. Indeed, Chang's algebra is finitely presented as a model of

$\mathbb{P}$ by the formula $\{x \,.\, x \leq \neg x\}$ but it is not finitely presentable as an MV-algebra since we have already recall that every such algebra is semisimple, hence the only finitely presentable MV-algebra that is also perfect is $\{0, 1\}$.

Proposition 2.2.12. *Every finitely presentable perfect MV-algebra is finitely presentable as an algebra in* $\mathbb{C}$*-mod*(**Set**) *(but not necessarily as an MV-algebra).*

Proof. This follows immediately from the fact that for any MV-algebra $\mathcal{B}$, the MV-subalgebra of $\mathcal{B}$ generated by $\mathrm{Rad}(\mathcal{B})$ is perfect, and the construction of the radical is preserved by filtered colimits in $\mathbb{C}$-mod(**Set**) since it is definable in this variety by the geometric formula $\{x \,.\, x \leq \neg x\}$ (cf. Proposition 2.2.9). □

As for MV-algebras, we can consider models of $\mathbb{P}$ in any Grothendieck topos $\mathcal{E}$. For any perfect MV-algebra $\mathcal{A} = (A, \oplus, \neg, 0)$ in $\mathcal{E}$, we define by using the internal language the subobject

$$Rad(\mathcal{A}) = \{x \in A \mid x \leq \neg x\} \rightarrowtail A,$$

and we call it the *radical* of $\mathcal{A}$. Observe that, in the theory $\mathbb{C}$, the formula $x \leq \neg x$ is equivalent to the equation $(2x)^2 = 0$. Indeed, by Lemma 2.1.2, $x \leq \neg x$ if and only if $x^2 = 0$. From the axiom C the claim follows. Similarly, we define the subobject

$$\neg Rad(\mathcal{A}) = \{x \in A \mid \neg x \leq x\} \rightarrowtail A,$$

and we call it the *coradical* of $\mathcal{A}$.

Notice that the union of the subobjects $Rad(\mathcal{A})$ and $\neg Rad(\mathcal{A})$ is precisely the interpretation in $\mathcal{A}$ of the formula $\{x \,.\, x \leq \neg x \vee \neg x \leq x\}$. In particular, a perfect MV-algebra is generated by its radical also in an arbitrary Grothendieck topos $\mathcal{E}$.

2.3 Local MV-algebras

In the previous section we have already intoduced the class of local MV-algebras. In this section we descrive this class in details.

Local MV-algebras admits several equivalent geometrical axiomatizations some of which can be deduced from the following proposition.

Proposition* 2.3.1 ([27]). *For any MV-algebra $\mathcal{A}$, the following are equivalent.*

(a) *For any $x \in \mathcal{A}$, $ord(x) < \infty$ or $ord(\neg x) < \infty$.*

(b) *For any $x, y \in \mathcal{A}$, $x \odot y = 0$ implies $x^n = 0$ or $y^n = 0$ for some $n \in \mathbb{N}$.*

(c) *For any $x, y \in \mathcal{A}$, $ord(x \oplus y) < \infty$ implies $ord(x) < \infty$ or $ord(y) < \infty$.*

(d) *$\{x \in \mathcal{A} : ord(x) = \infty\}$ is a proper ideal in $\mathcal{A}$.*

(e) *$\mathcal{A}$ has only one maximal ideal.*

(f) *For any $x \in \mathcal{A}$, there is an integer $n \geq 1$ such that $(nx)^2 \in \{0, 1\}$.*

(g) *$Rad(\mathcal{A})$ is a prime ideal.*

(h) *$A = Rad(\mathcal{A}) \cup Fin(\mathcal{A}) \cup \neg Rad(\mathcal{A})$, where $Fin(\mathcal{A})$ is the set of finite elements.*

In Proposition 3.7 [27], the authors proved that the class of local MV-algebras is a universal class; however, the proof of that result is not constructive. For defining the geometric theory of local MV-algebras, we shall use the characterization given by Proposition 2.3.1(a).

Definition 2.3.2. The geometric theory $\mathbb{L}oc$ of local MV-algebras is the quotient of the theory $\mathbb{MV}$ obtained by adding the axioms

$$(\top \vdash_x \bigvee_{k \in \mathbb{N}} (kx = 1 \vee k(\neg x) = 1));$$

$$(0 = 1 \vdash \bot) .$$

Particular examples of local MV-algebras are perfect MV-algebras, that are all contained in Chang's variety. However, there are local MV-algebras that are not contained in any proper subvariety; for instance, every infinite simple MV-algebra is local and generates the whole variety of MV-algebras.

In [30], Dubuc and Poveda provided a representation for the whole class of MV-algebras as global sections of a sheaf of MV-chains on a topological space. The proof of this result, as it is presented there, relies on Chang's Subdirect Representation Theorem, hence on the axiom of choice; however, in [23] the authors give a constructive proof of this result. We use this representation to calculate the cartesianization of the theory $\mathbb{L}oc$.

Theorem 2.3.3. *The cartesianization of the theory of local MV-algebras is the theory of MV-algebras.*

Proof. Every MV-chain is a local MV-algebra (in every Grothendieck topos): indeed, in such an algebra for any x, either $x \leq \neg x$ (whence $2(\neg x) = 1$) or $\neg x \leq x$ (whence $2x = 1$). So, by Dubuc-Poveda's representation theorem, every MV-algebra is isomorphic to the algebra of global sections of a sheaf of local MV-algebras on a locale (meaning a model of the theory $\mathbb{L}oc$ in a localic topos). Remark 1.5.16 thus implies our thesis. □

Proposition* 2.3.4. *The theory $\mathbb{L}oc$ of local MV-algebras is not of presheaf type.*

Proof. Let us suppose that the theory $\mathbb{L}oc$ is of presheaf type. Then $\mathbb{L}oc$ and its cartesianization $\mathbb{MV}$ satisfy the hypothesis of Theorem 1.5.10 whence every finitely presentable $\mathbb{L}oc$-model is finitely presented as $\mathbb{MV}$-model. So every local MV-algebra is a filtered colimit of local finitely presented MV-algebras. But every finitely presented MV-algebra is semisimple and every local semisimple MV-algebra is simple, i.e., it has exactly two ideals (see Proposition 2.3 [5]). So every local MV-algebra is a filtered colimit of simple MV-algebras, whence it is a simple MV-algebra. Since this is clearly not the case, the theory of local MV-algebras is not of presheaf type. □

Remark 2.3.5. With the same arguments used for the theory of local MV-algebras it is possible to prove that the theory of MV-chains is not of presheaf type and that its cartesianization is the theory of MV-algebras.

2.4 Simple MV-algebras

Strictly related to the theory of local MV-algebras is the theory of *simple MV-algebras*, i.e., those algebras which have no non-trivial ideals. Indeed, an algebra is local if and only if its quotient with respect to the radical is a simple MV-algebra.

By Theorem 3.5.1 [26], an MV-algebra is simple if and only if every element is equal to 0 or it has finite order. We use this characterization to define the theory of simple MV-algebras.

Definition 2.4.1. The geometric theory $\mathbb{S}imple$ of simple MV-algebras is the quotient of the theory $\mathbb{MV}$ obtained by adding the following sequent:

$$S : (\top \vdash_x \bigvee_{n\in\mathbb{N}} x = 0 \vee nx = 1) .$$

Theorem 2.4.2. *The theory $\mathbb{S}imple$ of simple MV-algebras is not of presheaf type.*

Proof. To prove this result we will use the categorical equivalence between the category of MV-algebras and the category of lattice-ordered abelian groups, called Mundici's equivalence. This equivalence will be described in details in the following chapter.

We will show that the property of an element to be determined by "Dedekind sections" relative to an irrational number is not definable in all simple MV-algebras by a geometric formula over the language of $\mathbb{S}imple$ even though it is preserved by homomorphisms and filtered colimits of $\mathbb{S}imple$-models. This will imply, by the definability theorem for theories of presheaf type (cf. Corollary 3.2 [15]), that the theory $\mathbb{S}imple$ is not of presheaf type.

Given an irrational number $\xi \in [0,1]$, this is approximated from above and below by rational numbers. Notice that $\xi \leq \frac{n}{m}$ if and only if $m\xi \leq n1$ and $\frac{n}{m} \leq \xi$ if and only if $n1 \leq m\xi$. Let us define $S_\xi = \{\frac{n}{m} \in \mathbb{Q} \cap [0,1] \mid \frac{n}{m} \geq \xi\}$ and $I_\xi = \{\frac{n}{m} \in \mathbb{Q} \cap [0,1] \mid \frac{n}{m} \leq \xi\}$. Consider the property P_ξ of an element x of a simple MV-algebra $\mathcal{A}$ defined by:

$$x \text{ satisfies } P_\xi \Leftrightarrow \forall(\frac{n}{m}) \in S_\xi, mx \leq nu \text{ and } \forall(\frac{n}{m}) \in I_\xi, nu \leq m\xi,$$

where the conditions on the right-hand side are expressed in terms of the lattice-ordered abelian group with strong unit corresponding to the MV-algebra $\mathcal{A}$ under Mundici's equivalence. Notice that, since P_ξ is expressible in the language of lattice-ordered abelian groups with strong unit by means of an infinitary conjunction of geometric formulas, it is preserved by homomorphisms and filtered colimits of simple ℓ-groups; it thus follows from Mundici's equivalence that the same property, referred to an element of an MV-algebra, is preserved by homomorphisms and filtered colimits of simple MV-algebras.

Let us suppose that this property is definable in the theory $\mathbb{S}imple$ by a geometric formula $\phi(x)$, which we can put in the following normal form:

$$\bigvee_{i \in I} \exists \vec{y_i} \psi_i(x, \vec{y_i}),$$

where $\psi_i(x, \vec{y_i})$ are Horn formulas over the signature of MV-algebras. This means that for every simple MV-algebra $\mathcal{A}$ and every $a \in A$

$$\mathcal{A} \models_a \phi(x) \Leftrightarrow a \text{ satisfies } P_\xi \ .$$

We can take in particular $\mathcal{A}$ equal to the standard MV-algebra $[0,1]$.

Now, every Horn formula $\psi_i(x, \vec{y_i})$ is a finite conjunction of formulas of the form $t_i^j(x, \vec{y_i}) = 1$, where t_i^j is a term over the signature of $\mathbb{MV}$. Since in the theory of MV-algebras $\inf(x,y) = x \odot (\neg x \oplus y) = 1$ if and only if $x = 1$ and $y = 1$, we can suppose without loss of generality that $\psi_i(x, \vec{y_i})$ is a formula of the form $t_i(x, \vec{y_i}) = 1$, where t_i is a term over the signature of $\mathbb{MV}$.

Thus, an element $a \in A$ satisfies $\phi(x)$ if and only if there exists $i \in I$ and elements $(y_1, \ldots, y_k)$ such that $t_i(a, y_1, \ldots, y_k) = 1$. Now, if $A = [0, 1]$ then $t_i^{-1}(1)$ is a rational polyhedron (cf. Corollary 2.10 [41]), whence it either consists of a single point whose coordinates are all rational numbers or contains infinitely many solutions with a different first coordinate.

We can thus conclude that the propriety P_ξ is not definable by a geometric formula and that the theory $\mathbb{S}imple$ is not of presheaf type. $\square$

Chapter 3

Generalization of Mundici's equivalence

In [39] D. Mundici presented a categorical equivalence between the category $\mathbf{MV}$ of MV-algebras and MV-homomorphisms and the category $\mathbf{L}_u$ of lattice-ordered abelian groups with strong unit and appropriate homomorphisms.

In this chapter we interpret Mundici's equivalence as an equivalence of categories of set-based models of two geometric theories, namely the algebraic theory of MV-algebras and the theory of lattice-ordered abelian groups with strong unit, and we show that this equivalence generalizes over arbitrary Grothendieck topos yielding a Morita-equivalence between the two theories. Further, applications of the bridge technique are explored. The results of this chapter are contained in [21].

3.1 Lattice-ordered abelian groups

Definition 3.1.1 ([6])**.** A *lattice-ordered abelian group* (ℓ-group, for brevity) is a structure $\mathcal{G} = (G, +, -, \leq, 0)$ such that $(G, +, -, 0)$ is an abelian group, $(G, \leq)$ is a lattice-ordered set and the following *translation invariance prop-*

erty holds:

$$\text{for any } x, y, z \in G \qquad x \leq y \text{ implies } x + z \leq y + z\ .$$

Any pair of elements x and y of an ℓ-group has a supremum, indicated by $\sup(x, y)$, and an infimum, indicated by $\inf(x, y)$. We also write nx for $x + \cdots + x$, n-times[1]. For each element x of an ℓ-group, one can define the *positive part* x^+, the *negative part* x^-, and the *absolute value* $|x|$ as follows:

1. $x^+ := \sup(0, x)$;

2. $x^- := \sup(0, -x)$;

3. $|x| := x^+ + x^- = \sup(x, -x)$.

Recall that, for every $x \in G$, $x = x^+ - x^-$.

Example 3.1.2. A simple example of an ℓ-group is given by the group of integers with the natural order $(\mathbb{Z}, +, \leq)$.

Given an ℓ-group $\mathcal{G}$ with a distinguished element u, u is said to be a *strong unit* for $\mathcal{G}$ if the following properties are satisfied:

- $u \geq 0$;

- for any positive element x of $\mathcal{G}$ there is a natural number n such that $x \leq nu$.

We shall refer to ℓ-groups with strong unit simply as ℓ-u groups.

Example 3.1.3. The structure $(\mathbb{R}, +, -, 0, \leq)$ is clearly an ℓ-group. Further, any strictly positive element of $\mathbb{R}$ is a strong unit, $\mathbb{R}$ being archimedean.

[1]Observe that we use the same notation nx both for the sum in an MV-algebra and in an ℓ-group. It will be clear from the context to which sum we are referring to.

An *ℓ-homomorphim* $h : \mathcal{G} \to \mathcal{H}$ is a homomorphism of groups that preserves the lattice-order structure. If both $\mathcal{G}$ and $\mathcal{H}$ are ℓ-u groups we say that h is an ℓ-u homomorphism if it preserves also the strong unit.

Let Σ_ℓ be the first-order signature consisting of a relation symbol $\leq$, of a constant 0, and of function symbols $+$, $-$, inf and sup formalizing the ℓ-group operations. We denote by $\mathbb{L}$ the geometric theory of ℓ-groups, whose axioms are the following sequents:

L.1 $\top \vdash_{x,y,z} x + (y + z) = (x + y) + z$;

L.2 $\top \vdash_x x + 0 = x$;

L.3 $\top \vdash_x x + (-x) = 0$;

L.4 $\top \vdash_{x,y} x + y = y + x$;

L.5 $\top \vdash_x x \leq x$;

L.6 $(x \leq y) \wedge (y \leq x) \vdash_{x,y} x = y$;

L.7 $(x \leq y) \wedge (y \leq z) \vdash_{x,y,z} x \leq z$;

L.8 $\top \vdash_{x,y} \inf(x, y) \leq x \wedge \inf(x, y) \leq y$;

L.9 $z \leq x \wedge z \leq y \vdash_{x,y,z} z \leq \inf(x, y)$;

L.10 $\top \vdash_{x,y} x \leq \sup(x, y) \wedge y \leq \sup(x, y)$;

L.11 $x \leq z \wedge y \leq z \vdash_{x,y,z} \sup(x, y) \leq z$;

L.12 $x \leq y \vdash_{x,y,t} t + x \leq t + y$.

Extending the signature Σ_ℓ by adding a new constant symbol u, we can define the theory of ℓ-u groups $\mathbb{L}_u$, whose axioms are L.1-L.12 plus

L_u.1 $\top \vdash u \geq 0$;

$\mathrm{L}_u.2$ $x \geq 0 \vdash_x \bigvee_{n \in \mathbb{N}} (x \leq nu)$.

A model of $\mathbb{L}$ (respectively of $\mathbb{L}_u$) in $\mathcal{E}$ is called an ℓ-group in $\mathcal{E}$ (resp. an ℓ-u group).

An ℓ-group in $\mathcal{E}$ is a structure $\mathcal{G} = (G, +, -, \leq, \inf, \sup, 0)$ in $\mathcal{E}$ which satisfies the axioms L.1-L.12. Note that such a structure consists of an object G in the topos $\mathcal{E}$ and arrows (resp. subobjects) in the topos interpreting the function (resp. the relation) symbols of the signature Σ_ℓ:

- $+ : G \times G \to G$;
- $- : G \to G$;
- $\leq \rightarrowtail G$;
- $\inf : G \times G \to G$;
- $\sup : G \times G \to G$;
- $0 : 1 \to G$.

Remark 3.1.4. An ℓ-group in $\mathcal{E}$ is an ℓ-group in the traditional sense if $\mathcal{E} = \mathbf{Set}$.

3.2 Equivalence in Set

Theorem* 3.2.1 (Mundici, 1965)**.** *There is a categorical equivalence between the category* **MV** *of MV-algebras and the cateogory* $\mathbf{L}_u$ *of ℓ-u groups where the arrows are the respective morphisms.*

The proof of this result given in [26] is not constructive as it relies on Chang's Subdirect Representation Theorem and on Birkhoff's Representation Theorem ([7]) applied to pointed ℓ-groups (i.e., every pointed ℓ-groups is a subdirect product of totally ordered pointed ℓ-groups).

To prove this theorem, Mundici constructed two functors and he proved that they are one the categorical inverse of the other. In particular, let $\mathcal{G} = (G, +, -, \leq, \inf, \sup, 0, u)$ be an ℓ-u group, we set

$$[0, u] = \{x \in G \,.\, 0 \leq x \leq u\}$$

and, for each $x, y \in [0, u]$ we define the following operations:

$$x \oplus y := \inf(u, (x + y)), \qquad \neg x := u - x.$$

The structure $\Gamma(\mathcal{G}) := ([0, u], \oplus, \neg, 0)$ is the MV-algebra associated with the ℓ-u group $\mathcal{G}$.

Any homomorphism h of ℓ-u groups preserves the unit interval. The operations $\oplus$ and $\neg$ being defined in term of the operations of $\mathcal{G}$, the homomorphism h preserves them. Hence, the restriction of h to the unit interval is an MV-algebra homomorphism and we can set $\Gamma(h) := h|_{[0,u]}$. Thus, we have a functor $\Gamma : \mathbf{L}_u \to \mathbf{MV}$, called in the literature *Mundici's functor*.

In the converse direction, let $\mathcal{A} = (A, \oplus, \neg, 0)$ be an MV-algebra. A countable sequence $\mathbf{a} = (a_1, a_2, \ldots, a_n, \ldots)$ of elements of A is said to be *good* if $a_i \oplus a_{i+1} = a_i$, for each $i \in \mathbb{N}$, and there is a natural number n such that $a_r = 0$ for any $r > n$. For any pair of good sequences $\mathbf{a}$ and $\mathbf{b}$, one defines their sum $\mathbf{a} + \mathbf{b}$ as the sequence $\mathbf{c}$ whose components are

$$c_i = a_i \oplus (a_{i-1} \odot b_1) \oplus \cdots \oplus (a_1 \odot b_{i-1}) \oplus b_i.$$

Let M_A be the set of good sequences of $\mathcal{A}$. We can endow this set with a structure of abelian monoid where the sum is defined as before and the neutral element is the good sequence $(0) = (0, 0, \ldots, 0, \ldots)$ whose components are all equal to 0. We indicate this structure with the symbol $M_{\mathcal{A}}$. The natural order in $\mathcal{A}$ induces a partial order relation $\leq$ in this monoid, given by:

$$\mathbf{a} \leq \mathbf{b} \text{ if and only if } a_i \leq b_i, \text{ for every } i \in \mathbb{N}.$$

Mundici proved, by using Chang's Subdirect Representation Theorem, that this order admits inf and sup (for any pair of good sequences) which are given by:

$$\inf(\mathbf{a},\mathbf{b}) = (\inf(a_1,b_1),\dots,\inf(a_n,b_n),\dots),$$

$$\sup(\mathbf{a},\mathbf{b}) = (\sup(a_1,b_1),\dots,\sup(a_n,b_n),\dots)\ .$$

From the lattice-ordered abelian monoid $M_{\mathcal{A}}$ one can build an ℓ-group $\mathcal{G}_{\mathcal{A}}$, by adding formal inverses to the elements of $M_{\mathcal{A}}$ (mimicking the construction of $\mathbb{Z}$ from $\mathbb{N}$). The elements of this ℓ-group are equivalence classes $[x,y]$ of pairs of elements x,y of the monoid. The constant $u=[(1),(0)]$, where with the symbol (a) we mean the good sequence $(a,0,\dots,0,\dots)$, is a strong unit for the group.

This construction is clearly functorial from the category **MV** to the category $\mathbf{L}_u$ and we call L this functor. The behaviour of the functor L with respect to homomorphisms is defined in the following way. Let $\mathcal{A}$ and $\mathcal{B}$ be two MV-algebras and $h:\mathcal{A}\to\mathcal{B}$ be an MV-homomorphism. If $\mathbf{a}=(a_1,a_2,\dots)$ is a good sequence of $\mathcal{A}$, then $h(\mathbf{a})=(h(a_1),h(a_2),\dots)$ is a good sequence of $\mathcal{B}$. Let $h^*: M_{\mathcal{A}}\to M_{\mathcal{B}}$ be defined by:

$$\mathbf{a}\in M_{\mathcal{A}}\to h(\mathbf{a})\in M_{\mathcal{B}}.$$

It is possible to prove that h^* is both a monoid homomorphism and a lattice homomorphism. We can further define the map

$$\bar{h}[\mathbf{a},\mathbf{b}]\in\mathcal{G}_{\mathcal{A}}\to[h^*(\mathbf{a}),h^*(\mathbf{b})]\in\mathcal{G}_{\mathcal{B}},$$

that is an ℓ-u homomorphism.

Theorem 3.2.2 (Theorem 2.4.5 [26]). *Let $\mathcal{A}$ be an MV-algebra. The correspondence*

$$\varphi_{\mathcal{A}}: a\in A\to[(a),(0)]\in\Gamma(L(A))$$

definies an isomorphism from the MV-algebra $\mathcal{A}$ to the MV-alegebra $\Gamma(L(\mathcal{A}))$. Further, for every MV-homomorphism $h : \mathcal{A} \to \mathcal{B}$ the following diagram commutes

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\quad h \quad} & \mathcal{B} \\ \downarrow{\scriptstyle \varphi_{\mathcal{A}}} & & \downarrow{\scriptstyle \varphi_{\mathcal{B}}} \\ \Gamma(L(\mathcal{A})) & \xrightarrow[\Gamma(L(h))]{} & \Gamma(L(\mathcal{B})) \end{array}$$

The previous theorem describes the natural transformation from the functor $\Gamma \circ L$ to the identity functor on the category of MV-algebras. A little more complicated is the description of the natural trasformation from the functor $L \circ \Gamma$ to the identity functor on the category of ℓ-u groups.

Lemma 3.2.3 (Lemma 7.1.3 [26])**.** *Let $\mathcal{G}$ be an ℓ-u group and $\mathcal{A} = \Gamma(\mathcal{G})$. For each $0 \leq a \in G$ there is a unique good sequence $g(a) = (a_1, \ldots, a_n)$ of elements of A such that $a = a_1 + \cdots + a_n$.*

The correspondence $a \to g(a)$ defines an injective map from the positive cone $\mathcal{G}^+$ of $\mathcal{G}$ to the monoid of good sequences $M_{\mathcal{A}}$. This map is both a monoid isomorphism and a lattice isomorphism[2].

Theorem* 3.2.4 (Corollary 7.1.6 [26])**.** *Given an ℓ-u group $\mathcal{G}$, the correspondence*

$$\psi_{\mathcal{G}} : a \in G \to [g(a^+), g(a^-)] \in L(\Gamma(G))$$

is an ℓ-u group isomorphism. Further, for every ℓ-u homomorphism $h : \mathcal{G} \to \mathcal{H}$ the following diagram commutes

[2]The proof of this result uses Birkhoff's Representation Theorem.

$$
\begin{array}{ccc}
\mathcal{G} & \xrightarrow{\quad h \quad} & \mathcal{H} \\
\Big\downarrow {\scriptstyle \psi_{\mathcal{G}}} & & \Big\downarrow {\scriptstyle \psi_{\mathcal{H}}} \\
L(\Gamma(\mathcal{G})) & \xrightarrow[\quad L(\Gamma(h)) \quad]{} & L(\Gamma(\mathcal{H}))
\end{array}
$$

3.3 From models of $\mathbb{L}_u$ to models of $\mathbb{MV}$

Let $\mathcal{E}$ be a topos and $\mathcal{G} = (G, +, -, \leq, \inf, \sup, 0, u)$ be a model of $\mathbb{L}_u$ in $\mathcal{E}$.

Mundici's construction of the functor $\Gamma : \mathbb{L}_u\text{-mod}(\mathbf{Set}) \to \mathbb{MV}\text{-mod}(\mathbf{Set})$ can be immediately generalized to any topos by using the internal language. Specifically, we define the interval $[0, u]$, where u is the strong unit, as the subobject of G

$$[0, u] := [\![x \in G \;.\; 0 \leq x \leq u]\!]_{\mathcal{G}},$$

where the expression '$0 \leq x \leq u$' is an abbreviation of the formula $(0 \leq x) \wedge (x \leq u)$.

We can define arrows

$$\oplus : [0, u] \times [0, u] \to [0, u]$$

$$\neg : [0, u] \to [0, u]$$

in $\mathcal{E}$ again by using the internal language, as follows:

$$x \oplus y = \inf(u, x + y),$$

$$\neg x = u - x.$$

The structure $\Gamma_{\mathcal{E}}(\mathcal{G}) := ([0, u], \oplus, \neg, 0)$ is a model of the $\mathbb{MV}$ in $\mathcal{E}$ (cf. Corollary 3.3.2). Further, the definition of the structure $\Gamma(\mathcal{G})$ with a first-order formula suggests that the theory $\mathbb{MV}$ is interpretable in $\mathbb{L}_u$.

Theorem 3.3.1. *The theory* $\mathbb{MV}$ *is interpretable in the theory* $\mathbb{L}_u$*, but not bi-interpretable.*

Proof. As remarked above, defining an interpretation of $\mathbb{MV}$ in $\mathbb{L}_u$ is equivalent to defining a model of $\mathbb{MV}$ in the syntactic category $\mathcal{C}_{\mathbb{L}_u}$. Let us consider the object $A := \{x \,.\, 0 \leq x \leq u\}$ of $\mathcal{C}_{\mathbb{L}_u}$ and the following arrows in $\mathcal{C}_{\mathbb{L}_u}$:

- $\oplus := [x, y, z \,.\, z = \inf(u, x + y)] : A \times A \to A$;

- $\neg := [x, z \,.\, z = u - x] : A \to A$;

- $0 := [x \,.\, x = 0] : 1 \to A$.

We have a Σ_{MV}-structure $\mathcal{A} = (A, \oplus, \neg, 0)$ in $\mathcal{C}_{\mathbb{L}_u}$. The following sequents are provable in $\mathbb{L}_u$:

(i) $(0 \leq x, y, z \leq u \vdash_{x,y,z} x \oplus (y \oplus z) = (x \oplus y) \oplus z)$;

(ii) $(0 \leq x, y \leq u \vdash_{x,y} x \oplus y = y \oplus x)$;

(iii) $(0 \leq x \leq u \vdash_x x \oplus 0 = 0)$;

(iv) $(0 \leq x \leq u \vdash_x \neg\neg x = x)$;

(v) $(0 \leq x \leq u \vdash_x x \oplus \neg 0 = \neg 0)$;

(vi) $(0 \leq x, y \leq u \vdash_{x,y} \neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x)$.

The proofs of these facts are straightforward. For instance, to prove sequent (ii), we observe that $x \oplus y = \inf(u, x + y) = \inf(u, y + x) = y \oplus x$, where the second equality follows from axiom 4 of the theory $\mathbb{L}_u$.

The validity of the axioms of the theory $\mathbb{MV}$ in the structure $\mathcal{A}$ is equivalent to provability of the sequents (i)-(vi) in the theory $\mathbb{L}_u$. Hence, the structure $\mathcal{A}$ is a model of $\mathbb{MV}$ in $\mathcal{C}_{\mathbb{L}_u}$.

This proves that $\mathbb{MV}$ is interpretable in $\mathbb{L}_u$.

Suppose that there exists a bi-interpretation $J : \mathcal{C}_{\mathbb{L}_u} \to \mathcal{C}_{\mathbb{MV}}$. This induces a functor $s_J^{\mathbf{Set}} : \mathbb{MV}\text{-mod}(\mathbf{Set}) \to \mathbb{L}_u\text{-mod}(\mathbf{Set})$ which is part of a categorical equivalence and which therefore reflects isomorphisms.

Let $\mathcal{M}$ be an MV-algebra, $\mathcal{N} := s_J^{\mathbf{Set}}(\mathcal{M})$ and $\{\vec{y} \,.\, \psi\} := J(\{x \,.\, \top\})$. We have that $F_{\mathcal{N}} \cong F_{\mathcal{M}} \circ J$. Hence:

$$F_{\mathcal{N}}(\{x \,.\, \top\}) \cong F_{\mathcal{M}}(\{\vec{y} \,.\, \psi\}),$$

$$[\![x \,.\, \top]\!]_{\mathcal{N}} \cong [\![\vec{y} \,.\, \psi]\!]_{\mathcal{M}},$$

$$\mathcal{N} \cong [\![\vec{y} \,.\, \psi]\!]_{\mathcal{M}} \,.$$

If $\mathcal{M}$ is a finite MV-algebra then we have that $[\![\vec{y} \,.\, \psi]\!]_{\mathcal{M}} \subseteq \mathcal{M}^n$ (for some n) is finite as well; thus, the ℓ-group $\mathcal{N}$ is finite. By Corollary 1.2.13 [6], every ℓ-group is without torsion; hence, every non-trivial ℓ-group is infinite. It follows that $\mathcal{N} = s_J(\mathcal{M})$ is trivial for any finite MV-algebra $\mathcal{M}$. Since the functor $s_J^{\mathbf{Set}}$ reflects isomorphisms and there are two non-isomorphic finite MV-algebras, we have a contradiction. □

Corollary 3.3.2. *The structure $\Gamma_{\mathcal{E}}(\mathcal{G})$ is a model of $\mathbb{MV}$ in $\mathcal{E}$.*

Proof. By Theorem 3.3.1, there is an interpretation I of $\mathbb{MV}$ in $\mathbb{L}_u$ whence a functor $s_I^{\mathcal{E}} : \mathbb{L}_u\text{-mod}(\mathcal{E}) \to \mathbb{MV}\text{-mod}(\mathcal{E})$. By definition of I, this functor sends any $\mathbb{L}_u$-model $\mathcal{G}$ to the structure $\Gamma(\mathcal{G})$. Hence $\Gamma(\mathcal{G})$ is a model of $\mathbb{MV}$ in $\mathcal{E}$. □

Let $h : \mathcal{G} \to \mathcal{G}'$ be a homomorphism between models of $\mathbb{L}_u$ in $\mathcal{E}$. Since h preserves the unit and the order relation, it restricts to a morphism between the unit intervals $[0, u_{\mathcal{G}}]$ and $[0, u_{\mathcal{G}'}]$. This restriction is an MV-algebra homomorphism since h clearly preserves the operations $\oplus$ and $\neg$. Thus Γ defines a functor from $\mathbb{L}_u\text{-mod}(\mathcal{E})$ to $\mathbb{MV}\text{-mod}(\mathcal{E})$.

Remarks 3.3.3. (a) The interpretation functor I defined above extends the assignment from MV-terms to ℓ-group terms considered at pp. 43 of [26];

(b) The functor I sends every formula-in-context $\{\vec{x}\ .\ \phi\}$ in $\mathcal{C}_{\mathbb{MV}}$ to a formula in the same context $\vec{x}$ over the signature of $\mathbb{L}_u$. This can be proved by an easy induction on the structure of geometric formulas by using the fact that, by definition of I, for any formula-in-context $\{\vec{x}\ .\ \phi\}$ over the signature of $\mathbb{MV}$, the formula $I(\{\vec{x}\ .\ \phi\})$ is equal to the interpretation of the formula $\{\vec{x}\ .\ \phi\}$ in the internal MV-algebra $\mathcal{A} = (A, \oplus, \neg, 0)$ in $\mathcal{C}_{\mathbb{L}_u}$ defined above. In particular, for any geometric sequent $\sigma = (\phi \vdash_{\vec{x}} \psi)$ over the signature of $\mathbb{MV}$ and any Grothendieck topos $\mathcal{E}$, the sequent $I(\sigma) := I(\{\vec{x}\ .\ \phi\}) \vdash_{\vec{x}} I(\{\vec{x}\ .\ \psi\})$ is valid in a unital ℓ-group $\mathcal{G}$ in $\mathcal{E}$ if and only if σ is valid in the associated MV-algebra $[0, u_{\mathcal{G}}]$.

3.4 From models of $\mathbb{MV}$ to models of $\mathbb{L}_u$

More delicate is the generalization of the other functor of Mundici's equivalence which involves the concept of good sequence.

Let $\mathcal{E}$ be a Grothendieck topos, with its unique geometric morphism $\gamma_{\mathcal{E}} : \mathcal{E} \to \mathbf{Set}$ to the topos of sets.

In **Set** the object of all sequences with values in a given set A can be identified with the exponential $A^{\mathbb{N}}$ (where $\mathbb{N}$ is the set of natural numbers). This construction can be generalized to any topos; indeed, we can consider the object $A^{\gamma^*_{\mathcal{E}}(\mathbb{N})}$ in $\mathcal{E}$. As the functor $A^- : \mathcal{E} \to \mathcal{E}^{op}$ has a right adjoint, it preserves coproducts. Therefore, since $\gamma^*_{\mathcal{E}}(\mathbb{N}) = \bigsqcup_{n\in\mathbb{N}} \gamma^*_{\mathcal{E}}(1)$, the object $A^{\gamma^*_{\mathcal{E}}(\mathbb{N})}$ is isomorphic to $\prod_{n\in\mathbb{N}} A$, and $A^n \cong A^{\gamma^*_{\mathcal{E}}(I_n)}$, where I_n is the n-element set $\{1, \dots, n\}$. From this observation we see that the construction of the object of sequences $A^{\gamma^*_{\mathcal{E}}(\mathbb{N})}$ is not geometric; however, as we shall see below, the construction of the subobject of good sequences associated with an MV-algebra in a topos is geometric.

Let $\mathcal{A} = (A, \oplus, \neg, 0)$ be a model of $\mathbb{MV}$ in $\mathcal{E}$. We need to define the subobject of good sequences of $A^{\gamma^*_{\mathcal{E}}(\mathbb{N})}$.

We shall argue informally as we were working in the classical topos of sets, but all our constructions can be straightforwardly formalized in the internal language of the topos $\mathcal{E}$.

Definition 3.4.1. We say that $\mathbf{a} = (a_1, \ldots, a_n) \in A^n$ is a *n-good sequence* if

$$a_i \oplus a_{i+1} = a_i, \text{ for any } i = 1, \ldots, n-1.$$

Let $s_n : GS_n \to A^n$ be the subobject $\{\mathbf{a} \in A^n \mid \mathbf{a} \text{ is a } n\text{-good sequence of } A^n\}$ (for any $n \in \mathbb{N}$).

Any n-good sequence can be completed to an infinite good sequence by adding an infinite tail of zeros. Anyway, n-good sequences for different natural numbers n can give rise to the same infinite good sequence. Indeed, if $\mathbf{a} \in GS_m$ and $\mathbf{b} \in GS_n$, with $m \leq n$, are of the form $\mathbf{a} = (a_1, \ldots, a_m)$ and $\mathbf{b} = (a_1, \ldots, a_m, 0, \ldots, 0)$, then the completed sequences coincide.

This observation shows that we can realize the subobject of good sequences on $\mathcal{A}$ as a quotient of the coproduct $\coprod_{n \in \mathbb{N}} GS_n$ by a certain equivalence relation, which can be specified as follows (below we shall denote by $\chi_m : GS_m \to \coprod_{n \in \mathbb{N}} GS_n$ the canonical coproduct injections).

For each $m \leq n$, consider the arrow $\pi_{m,n} : A^m \to A^n$ which sends an m-sequence $\mathbf{a}$ to the n-sequence whose first m components are those of $\mathbf{a}$ and the others are 0. Notice that if $m = n$ then $\pi_{m,n}$ is the identity on A^m. As the image of a m-good sequence under $\pi_{m,n}$ is a n-good sequence, the arrows $\pi_{m,n} : A^m \to A^n$ restrict to the subobjects s_m and s_n, giving rise to arrows:

$$\xi_{m,n} : GS_m \to GS_n,$$

for each $m \leq n$.

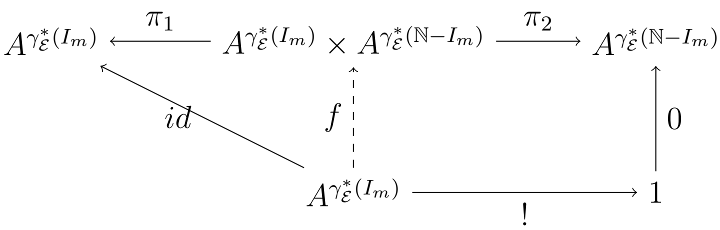
By using internal language we next define the following relation on the coproduct $\coprod_{n \in \mathbb{N}} GS_n$: for any $(\mathbf{a}, \mathbf{b}) \in \coprod_{n \in \mathbb{N}} GS_n \times \coprod_{n \in \mathbb{N}} GS_n$ we stipulate that $\mathbf{a} R \mathbf{b}$ iff

$$\bigvee_{m \leq n} [(\exists \mathbf{a}' \in GS_m)(\exists \mathbf{b}' \in GS_n)(\chi_m(\mathbf{a}') = \mathbf{a} \wedge \chi_n(\mathbf{b}') = \mathbf{b} \wedge \xi_{m,n}(\mathbf{a}') = \mathbf{b}')]$$
$$\bigvee_{n \leq m} [(\exists \mathbf{a}' \in GS_m)(\exists \mathbf{b}' \in GS_n)(\chi_m(\mathbf{a}') = \mathbf{a} \wedge \chi_n(\mathbf{b}') = \mathbf{b} \wedge \xi_{n,m}(\mathbf{b}') = \mathbf{a}')].$$

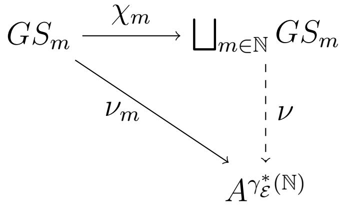
It is immediate to check that this is an equivalence relation; in fact, R can be characterized as the equivalence relation on the coproduct $\bigsqcup_{n\in\mathbb{N}} GS_n$ generated by the family of arrows $\{\xi_{m,n} \mid m \leq n\}$. Roughly speaking, the relation R identifies finite good sequences that differ only in the number of zeros in the final components. In fact, the quotient $(\bigsqcup_{m\in\mathbb{N}} GS_m)/R$ can alternatively be characterized as the (directed) colimit of the functor $\xi : \mathbb{N} \to \mathcal{E}$ (where $\mathbb{N}$ is considered as a preorder category) sending any $n \in \mathbb{N}$ to the object GS_n and any arrow $m \leq n$ in $\mathbb{N}$ to the arrow $\xi_{m,n} : GS_m \to GS_n$.

Let us now show how to realize the quotient $(\bigsqcup_{m\in\mathbb{N}} GS_m)/R$, which is our candidate for the object of good sequences associated with the MV-algebra $\mathcal{A}$, as a subobject of the object $A^{\gamma^*_{\mathcal{E}}(\mathbb{N})}$ of 'all sequences' on A.

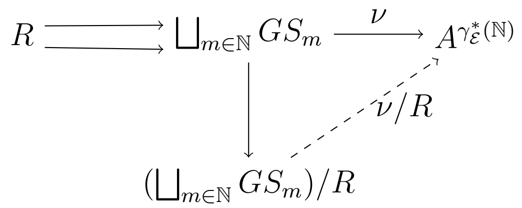
Let us define an arrow f from $A^{\gamma^*_{\mathcal{E}}(I_m)}$ to $A^{\gamma^*_{\mathcal{E}}(\mathbb{N})} \cong A^{\gamma^*_{\mathcal{E}}(I_m)} \times A^{\gamma^*_{\mathcal{E}}(\mathbb{N}-I_m)}$ by setting the first component equal to the identity on $A^{\gamma^*_{\mathcal{E}}(I_m)}$ and the second equal to the composition of the unique arrow $A^{\gamma^*_{\mathcal{E}}(I_m)} \to 1$ with the arrow $0 : 1 \to A^{\gamma^*_{\mathcal{E}}(\mathbb{N}-I_m)}$ induced at each components by the zero arrow $\mathcal{A}0 : 1 \to A$ of the MV-algebra $\mathcal{A}$.



This arrow is clearly monic. By composing with $s_m : GS_m \to A^m \cong A^{\gamma^*_{\mathcal{E}}(I_m)}$ we thus obtain a monomorphism $\nu_m : GS_m \to A^{\gamma^*_{\mathcal{E}}(\mathbb{N})}$. These arrows determine, by the universal property of the coproduct, an arrow $\nu : \bigsqcup_{m\in\mathbb{N}} GS_m \to A^{\gamma^*_{\mathcal{E}}(\mathbb{N})}$:



This arrow ν coequalizes the two natural projections corresponding to the relation R; hence, we have a unique factorization $\nu/R : (\bigsqcup_{m\in\mathbb{N}} GS_m)/R \rightarrowtail A^{\gamma^*_{\mathcal{E}}(\mathbb{N})}$ of ν through the quotient $(\bigsqcup_{m\in\mathbb{N}} GS_m)/R$.



Lemma 3.4.2. *The arrow ν/R is monic.*

Proof. By using the internal language, if $[\mathbf{a}], [\mathbf{b}] \in (\bigsqcup_{m\in\mathbb{N}} GS_m)_{/R}$ then there exist $m, n \in \mathbb{N}$, $\mathbf{a}' \in GS_m$ and $\mathbf{b}' \in GS_n$ such that $\mathbf{a} = \chi_m(\mathbf{a}')$ and $\mathbf{b} = \chi_m(\mathbf{b}')$; so $\nu(\mathbf{a}) = \nu(\mathbf{b})$ if and only if $\nu_m(\mathbf{a}') = \nu_n(\mathbf{b}')$; but this clearly holds if and only if either $n \leq m$ and $\xi_{n,m}(\mathbf{b}') = \mathbf{a}'$ or $m \leq n$ and $\xi_{m,n}(\mathbf{a}') = \mathbf{b}'$, either of which implies that $\mathbf{a}R\mathbf{b}$ (i.e., $[\mathbf{a}] = [\mathbf{b}]$), as required. □

The subobject just defined admits natural descriptions in terms of internal language of the topos.

Proposition 3.4.3. *Let $\mathcal{A} = (A, \oplus, \neg, 0)$ be a model of $\mathbb{MV}$ in $\mathcal{E}$. Then the following monomorphisms to $A^{\gamma^*_{\mathcal{E}}(\mathbb{N})}$ are isomorphic:*

(i) $\nu/R : (\bigsqcup_{m\in\mathbb{N}} GS_m)/R \rightarrowtail A^{\gamma^*_{\mathcal{E}}(\mathbb{N})}$;

(ii) $[\![\mathbf{a} \in A^{\gamma^*_{\mathcal{E}}(\mathbb{N})} \,.\, \bigvee_{n\in\mathbb{N}}((\exists \mathbf{a}' \in S_n)(\mathbf{a} = \chi_m(\mathbf{a}')))]\!] \rightarrowtail A^{\gamma^*_{\mathcal{E}}(\mathbb{N})}$.

We call the resulting subobject the subobject of good sequences *of the* $\mathbb{MV}$-*algebra $\mathcal{A}$, and denote it by the symbol $GS_{\mathcal{A}}$.*

Proof. According to the semantics of the internal language, the second subobject is given by the union of all the subobjects $\nu_m : GS_m \to A^{\gamma^*_{\mathcal{E}}(\mathbb{N})}$. This union is clearly isomorphic to the image of the arrow ν, which is isomorphic to the arrow ν/R, as the latter arrow is monic and the canonical projection $(\bigsqcup_{m\in\mathbb{N}} GS_m) \to (\bigsqcup_{m\in\mathbb{N}} GS_m)/R$ is epic. This proves the isomorphism between the first subobject and the second. □

Notice that in the case $\mathcal{E} = \mathbf{Set}$, our definition of subobject of good sequences specializes to the classical one.

Let us now proceed to define an abelian monoid structure on the object $GS_{\mathcal{A}}$, by using the internal language of the topos $\mathcal{E}$.

Consider the term $\mathbf{a} \in A^{\gamma^*_{\mathcal{E}}(\mathbb{N})}$, and denote by a_i the term $\mathbf{a}(\gamma^*_{\mathcal{E}}(\varepsilon_i))$, where $\varepsilon_i : 1 = \{*\} \to \mathbb{N}$ (for any $i \in \mathbb{N}$) is the function in **Set** defined by: $\varepsilon_i(*) := i$ (the object 1 is the terminal object in **Set**). We can think of the a_i as the *components* of $\mathbf{a}$.

We set

$$M_{\mathcal{A}} = (GS_{\mathcal{A}}, +, \leq, \sup, \inf, 0),$$

where the operations and the relation are defined as follows (by using the internal language): for any $\mathbf{a}, \mathbf{b} \in GS_{\mathcal{A}}$,

- the sum $\mathbf{a} + \mathbf{b}$ is given by the sequence $\mathbf{c} \in GS_{\mathcal{A}}$ whose components are $c_i := a_i \oplus (a_{i-1} \odot b_i) \oplus \cdots \oplus (a_1 \odot b_{i-1}) \oplus b_i$;
- $\sup(\mathbf{a}, \mathbf{b})$, where $\sup(\mathbf{a}, \mathbf{b})_i := \sup(a_i, b_i)$;
- $\inf(\mathbf{a}, \mathbf{b})$, where $\inf(\mathbf{a}, \mathbf{b})_i := \inf(a_i, b_i)$;
- $\mathbf{a} \leq \mathbf{b}$ if and only if $\inf(\mathbf{a}, \mathbf{b}) = \mathbf{a}$, equivalently if there exists $\mathbf{c} \in GS_{\mathcal{A}}$ such that $\mathbf{a} + \mathbf{c} = \mathbf{b}$;
- $0 = (0)$, i.e., $0_i = 0$ for every $i \in \mathbb{N}$.

Mundici proved that this is a lattice-ordered abelian monoid in the case $\mathcal{E}$ equal to **Set**. In the following proposition we prove that this is the case for an arbitrary $\mathcal{E}$.

Proposition 3.4.4. *Let $\mathcal{A}$ be a model of $\mathbb{MV}$ in $\mathcal{E}$. Then $M_{\mathcal{A}}$ is a well-defined structure, i.e., all the operations are well-defined, and the axioms of the theory $\mathbb{L}$, except for the axiom L.3, hold in $M_{\mathcal{A}}$. Furthermore, the structure $M_{\mathcal{A}}$ satisfies the cancellation property, i.e., if $\mathbf{a} + \mathbf{b} = \mathbf{a} + \mathbf{c}$ then $\mathbf{b} = \mathbf{c}$, for any $\mathbf{a}, \mathbf{b}, \mathbf{c} \in GS_{\mathcal{A}}$.*

Proof. As shown in [26], all the required properties can be deduced from the validity of certain Horn (whence cartesian) sequents written in the signature of the theory $\mathbb{MV}$ for all MV-algebras in the given algebra $\mathcal{A}$. For instance, the associativity property can be deduced from the validity of the following sequent:

$$\mathbf{a} \in GS_n \wedge \mathbf{b} \in GS_m \wedge \mathbf{c} \in GS_k \vdash_{a_i,b_j,c_l} \bigwedge_{1\leq t\leq n+m+k} ((\mathbf{a}+\mathbf{b})+\mathbf{c})_t = (\mathbf{a}+(\mathbf{b}+\mathbf{c}))_t,$$

where $((\mathbf{a}+\mathbf{b})+\mathbf{c})_t = d_t \oplus (d_{t-1} \odot c_1) \oplus \cdots \oplus (d_1 \odot c_{t-1}) \oplus c_t$,

$(\mathbf{a}+(\mathbf{b}+\mathbf{c}))_t = a_t \oplus (a_{t-1} \odot f_1) \oplus \cdots \oplus (a_1 \odot f_{t-1}) \oplus f_t$,

$\mathbf{d} := \mathbf{a}+\mathbf{b} \quad \mathbf{f} := \mathbf{b}+\mathbf{c}$.

These cartesian sequents can be easily verified to hold for all MV-chains (cf. [26]) whence they are provable in the cartesianization of the theory of MV-chains. By Remark 2.3.5 we know that this cartesianization is the theory of MV-algebras; thus, these sequents hold in every MV-algebra in a Grothendieck topos. □

In order to make the given lattice-ordered abelian monoid into a lattice-ordered abelian group, we mimick the construction of $\mathbb{Z}$ from $\mathbb{N}$, as is done in [26]. Specifically, for any lattice-ordered abelian monoid M satisfying the cancellation property in a topos $\mathcal{E}$, the corresponding lattice-ordered abelian group is obtained as the quotient of $M \times M$ by the equivalence relation $\sim$ defined, by using the internal language, as: $(a, b) \sim (c, d)$ if and only if $a + d = b + c$. This is essentially the construction of the Grothendieck group $G(M)$ from an abelian monoid M. The operations and the order on this structure are defined in the obvious well-known way, again by using the internal language. In particular, in the case of the ℓ-group $\mathcal{G}_{\mathcal{A}}$ corresponding to the monoid $\mathcal{M}_{\mathcal{A}}$ associated with an MV-algebra $\mathcal{A}$, they are defined as follows:

- *addition*: $[\mathbf{a}, \mathbf{b}] + [\mathbf{c}, \mathbf{d}] := [\mathbf{a}+\mathbf{c}, \mathbf{b}+\mathbf{d}]$;

- *subtraction*: $-[\mathbf{a}, \mathbf{b}] := [\mathbf{b}, \mathbf{a}]$;

- $[\mathbf{a}, \mathbf{b}] \leq [\mathbf{c}, \mathbf{d}]$ if and only if $\mathbf{a} + \mathbf{d} \leq \mathbf{c} + \mathbf{b}$, equivalently if and only if there exists $\mathbf{e} \in GS_{\mathcal{A}}$ such that $[\mathbf{c}, \mathbf{d}] - [\mathbf{a}, \mathbf{b}] = [\mathbf{e}, (0)]$;

- $\sup([\mathbf{a}, \mathbf{b}], [\mathbf{c}, \mathbf{d}]) := [\sup(\mathbf{a} + \mathbf{d}, \mathbf{c} + \mathbf{b}), \mathbf{b} + \mathbf{d}]$;

- $\inf([\mathbf{a}, \mathbf{b}], [\mathbf{c}, \mathbf{d}]) := [\inf(\mathbf{a} + \mathbf{d}, \mathbf{c} + \mathbf{b}), \mathbf{b} + \mathbf{d}]$;

- *zero element*: $[(0), (0)] \in G_{\mathcal{A}}$,

where the symbol (0) indicates the sequence all whose components are zero.

Let us moreover define $u := [(1), (0)]$, where the symbol (1) indicates the sequence whose first component is 1 and the others are 0.

Proposition 3.4.5. *The structure $\mathcal{G}_{\mathcal{A}}$, equipped with the element $u := [(1), (0)]$ as a unit, is a model of $\mathbb{L}_u$ in $\mathcal{E}$.*

Proof. We have already observed that $\mathcal{G}_{\mathcal{A}}$ is a pointed ℓ-group. It remains to prove that u is a strong unit in $\mathcal{G}_{\mathcal{A}}$. It is clear that $u \geq 0$. Reasoning in the internal language we can work in $\mathcal{E}$ as we were in **Set** whence we can say that if $0 \leq [\mathbf{a}, \mathbf{b}] \in G_{\mathcal{A}}$ then there exists $\mathbf{c} \in GS_{\mathcal{A}}$ such that $[\mathbf{a}, \mathbf{b}]$ is equal to $[\mathbf{c}, (0)]$, thus there is a natural number m such that $\mathbf{c} \in GS_m$; then $mu = [1^m, (0)] \geq [\mathbf{c}, (0)]$, where $1^m = (1, \dots, 1, 0, 0, \dots, 0, \dots)$ is the good sequence having the first m components equal to 1 and the others equal to 0. □

The assignment $\mathcal{A} \to \mathcal{G}_{\mathcal{A}}$ is clearly functorial; we thus obtain a functor

$$L_{\mathcal{E}} : \mathbb{MV}\text{-mod}(\mathcal{E}) \to \mathbb{L}_u\text{-mod}(\mathcal{E}),$$

with $L_{\mathcal{E}}(\mathcal{A}) := \mathcal{G}_{\mathcal{A}}$ for any MV-algebra $\mathcal{A}$ in $\mathcal{E}$.

3.5 The Morita-equivalence between $\mathbb{L}_u$ and $\mathbb{MV}$

In the previous sections we have built two functors

$$L_{\mathcal{E}} : \mathbb{MV}\text{-mod}(\mathcal{E}) \to \mathbb{L}_u\text{-mod}(\mathcal{E}),$$

$$\Gamma_{\mathcal{E}} : \mathbb{L}_u\text{-mod}(\mathcal{E}) \to \mathbb{MV}\text{-mod}(\mathcal{E}),$$

which generalize to an arbitrary topos $\mathcal{E}$ the classical functors of Mundici's equivalence. We want to prove that these two functors are one the categorical inverse of the other. We have remarked several times that their construction is geometrical, i.e., only finite limits and arbitrary colimits are involved. This implies that, if there exists, the cateogorical equivalence given by $\Gamma_{\mathcal{E}}$ and $L_{\mathcal{E}}$ is natural in $\mathcal{E}$.

Proposition 3.5.1. *For every $\mathcal{A} = (A, \oplus, \neg, 0) \in \mathbb{MV}$-mod($\mathcal{E}$), the arrows $\varphi_{\mathcal{A}} : a \in \mathcal{A} \to [(a), (0)] \in \Gamma_{\mathcal{E}}(\mathcal{G}_{\mathcal{A}})$ are isomorphisms* natural in $\mathcal{A}$. *In other words, they are the components of a natural isomorphism from the identity functor on $\mathbb{MV}$-mod($\mathcal{E}$) to $\Gamma_{\mathcal{E}} \circ L_{\mathcal{E}}$.*

Proof. Let us argue in the internal language of the topos $\mathcal{E}$. The arrow $\varphi_{\mathcal{A}}$ is clearly a monic homomorphism of MV-algebras. By definition of the order $\leq$ on $\mathcal{G}_{\mathcal{A}}$, we have that $[(0), (0)] \leq [\mathbf{a}, \mathbf{b}] \leq [(1), (0)]$ if and only if there exists $c \in A$ such that $[\mathbf{a}, (0)] = [(c), (0)]$. Hence, $\varphi_{\mathcal{A}}$ is epic. It is immediate to verify that $\varphi_{\mathcal{A}}$ preserves $\oplus$ and $\neg$, and that for any homomorphism $h : \mathcal{A} \to \mathcal{B}$ of MV-algebras in $\mathcal{E}$, the following square commutes:

$$\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\quad h \quad} & \mathcal{B} \\
\Big\downarrow{\scriptstyle \varphi_{\mathcal{A}}} & & \Big\downarrow{\scriptstyle \varphi_{\mathcal{B}}} \\
\Gamma_{\mathcal{E}}(L_{\mathcal{E}}(\mathcal{A})) & \xrightarrow[\Gamma_{\mathcal{E}}(L_{\mathcal{E}}(h))]{} & \Gamma_{\mathcal{E}}(L_{\mathcal{E}}(\mathcal{B}))
\end{array}$$

Thus, the arrows $\varphi_{\mathcal{A}}$ yields a natural isomorphism $\varphi : 1 \to \Gamma_{\mathcal{E}} \circ L_{\mathcal{E}}$. □

Proposition* 3.5.2. *For every* $\mathcal{G} = (G, +, -, \inf, \sup, 0, u) \in \mathbb{L}_u$*-mod*$(\mathcal{E})$*, there is an isomorphism* $\phi_{\mathcal{G}} : L(\Gamma(\mathcal{G})) \to \mathcal{G}$*,* natural in $\mathcal{G}$*. In other words, the isomorphisms* $\phi_{\mathcal{G}}$ *are the components of a natural isomorphism from* $L_{\mathcal{E}} \circ \Gamma_{\mathcal{E}}$ *to the identity functor on* $\mathbb{L}_u$*-mod*$(\mathcal{E})$*.*

Proof. By using the internal language, we can easily generalize the definition of the assignment g (cf. the proof of Lemma 7.1.3 [26]) and the proof that the generalization of map $\phi_{\mathcal{G}}$ is injective and surjective. It remains to prove that g is a unital ℓ-homomorphism. Letting G^+ denote the positive cone of $\mathcal{G}$, it is enough to show that the inverse arrow

$$f_{\mathcal{G}} : (a_1, \dots, a_n) \in M_{\Gamma(\mathcal{G})} \to a = a_1 + \dots + a_n \in G^+,$$

is a unital ℓ-homomorphism. This amounts to proving:

(i) $f_{\mathcal{G}}(\mathbf{a} + \mathbf{b}) = f_{\mathcal{G}}(\mathbf{a}) + f_{\mathcal{G}}(\mathbf{b})$;

(ii) $f_{\mathcal{G}}(\inf(\mathbf{a}, \mathbf{b})) = \inf(f_{\mathcal{G}}(\mathbf{a}), f_{\mathcal{G}}(\mathbf{b}))$;

(iii) $f_{\mathcal{G}}(\sup(\mathbf{a}, \mathbf{b})) = \sup(f_{\mathcal{G}}(\mathbf{a}), f_{\mathcal{G}}(\mathbf{b}))$;

(iv) $f_{\mathcal{G}}((u)) = u$.

By definition of $f_{\mathcal{G}}$, property (iv) holds. Properties (i)-(iii) can be expressed in terms of the validity in the group $\mathcal{G}$ of certain Horn sequents written in the signature of the theory $\mathbb{L}$ of pointed ℓ-groups. For instance, property (i) can be expressed by the sequent

$$(\mathbf{a} \in GS_n \wedge \mathbf{b} \in GS_m \bigwedge_{1 \leq n}(0 \leq a_i \leq u) \bigwedge_{1 \leq j \leq m}(0 \leq b_j \leq u) \vdash_{a_i, b_j} c_1 + \dots + c_{n+m} = a_1 + \dots + a_n + b_1 + \dots + b_n),$$

where $\mathbf{c} = (c_1, \dots, c_{n+m}) := \mathbf{a} + \mathbf{b}$.

Now, Mundici's proof of Lemma 7.1.5 [26] shows that these sequents hold in any totally ordered pointed abelian group whence, by Birkoff's classical result that every pointed ℓ-group is a subdirect product of totally ordered groups, in every model of $\mathbb{L}$ in **Set**, if one assumes the axiom of choice[3]. The completeness theorem for cartesian theories (Theorem D1.5.1 [35]) thus allows us to conclude that these sequents are provable in $\mathbb{L}$ whence valid in $\mathcal{G}$.

Hence, the function $f_{\mathcal{G}}$ is a unital ℓ-homomorphism, which induces an isomorphism $\phi_{\mathcal{G}} : L_{\mathcal{E}}(\Gamma_{\mathcal{E}}(\mathcal{G})) \to \mathcal{G}$, for any $\mathcal{G}$.

It is immediate to see that for any homomorphism $h : \mathcal{G} \to \mathcal{H}$ the square below commutes:

$$\begin{array}{ccc}
L_{\mathcal{E}}(\Gamma_{\mathcal{E}}(\mathcal{G})) & \xrightarrow{L_{\mathcal{E}}(\Gamma_{\mathcal{E}}(h))} & L_{\mathcal{E}}(\Gamma_{\mathcal{E}}(\mathcal{H})) \\
\downarrow{\scriptstyle \phi_{\mathcal{G}}} & & \downarrow{\scriptstyle \phi_{\mathcal{H}}} \\
\mathcal{G} & \xrightarrow[h]{} & \mathcal{H}
\end{array}$$

We can thus conclude that the $\phi_{\mathcal{G}}$ are the components of a natural isomorphism from $L \circ \Gamma$ to the identity functor on $\mathbb{L}_u$-mod$(\mathcal{E})$, as required. □

Remark 3.5.3. If $\mathcal{E} = \mathbf{Set}$, the function $\phi_{\mathcal{G}}$ is the inverse of function $\psi_{\mathcal{G}}$ defined in Section 3.2.

We have built, for every Grothendieck topos $\mathcal{E}$, an equivalence of categories

$$\mathbb{MV}\text{-mod}(\mathcal{E}) \simeq \mathbb{L}_u\text{-mod}(\mathcal{F})$$

given by functors

$$L_{\mathcal{E}} : \mathbb{MV}\text{-mod}(\mathcal{E}) \to \mathbb{L}_u\text{-mod}(\mathcal{E})$$

[3] In the field of MV-algebras there is a constructive representation theorem in terms of totally ordered algebras for the whole class of MV-algebras, namely Dubuc-Poveda's sheaf representation. However, we were not able to find in the literature an analogous constructive representation for unital ℓ-groups.

and

$$\Gamma_{\mathcal{E}} : \mathbb{L}_u\text{-mod}(\mathcal{E}) \to \mathbb{MV}\text{-mod}(\mathcal{E})$$

generalizing the functors of Mundici's equivalence.

To prove that the theories $\mathbb{MV}$ and $\mathbb{L}_u$ are Morita-equivalent, it remains to show that this equivalence is natural in $\mathcal{E}$, that is for any geometric morphism $f : \mathcal{F} \to \mathcal{E}$, the following diagrams commute:

$$\begin{array}{ccc}
\mathbb{L}_u\text{-mod}(\mathcal{E}) & \xrightarrow{\Gamma_{\mathcal{E}}} & \mathbb{MV}\text{-mod}(\mathcal{E}) \\
{\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\
\mathbb{L}_u\text{-mod}(\mathcal{F}) & \xrightarrow[\Gamma_{\mathcal{F}}]{} & \mathbb{MV}\text{-mod}(\mathcal{F})
\end{array}$$

$$\begin{array}{ccc}
\mathbb{MV}\text{-mod}(\mathcal{E}) & \xrightarrow{L_{\mathcal{E}}} & \mathbb{L}_u\text{-mod}(\mathcal{E}) \\
{\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\
\mathbb{MV}\text{-mod}(\mathcal{F}) & \xrightarrow[L_{\mathcal{F}}]{} & \mathbb{L}_u\text{-mod}(\mathcal{F})
\end{array}$$

The commutativity of these diagrams follows from the fact that all the constructions that we used to build the functors $\Gamma_{\mathcal{E}}$ and $L_{\mathcal{E}}$ are *geometric* (i.e., only involving finite limits and colimits) whence preserved by the inverse image functors of geometric morphisms.

We have therefore proved the following

Theorem* 3.5.4. *The functors $L_{\mathcal{E}}$ and $\Gamma_{\mathcal{E}}$ defined above yield a Morita-equivalence between the theories $\mathbb{MV}$ and $\mathbb{L}_u$. In particular, $\mathcal{E}_{\mathbb{MV}} \simeq \mathcal{E}_{\mathbb{L}_u}$.*

Remarks 3.5.5. (a) We have observed that the theories $\mathbb{MV}$ and $\mathbb{L}_u$ are not bi-interpretable (in the sense that the geometric syntactic categories $\mathcal{C}_{\mathbb{MV}}$ and $\mathcal{C}_{\mathbb{L}_u}$ are not equivalent). On the other hand, we have just proved that the ∞-pretopos completions $\mathcal{E}_{\mathbb{MV}}$ of $\mathcal{C}_{\mathbb{MV}}$ and $\mathcal{E}_{\mathbb{L}_u}$ of $\mathcal{C}_{\mathbb{L}_u}$ are equivalent (by

Proposition D3.1.12 [35], the classifying topos $\mathcal{E}_{\mathbb{T}}$ of a geometric theory is equivalent to the ∞-pretopos completion of the geometric syntactic category $\mathcal{C}_{\mathbb{T}} \hookrightarrow \mathcal{E}_{\mathbb{T}}$ of $\mathbb{T}$). Now, the objects of the ∞-pretopos completion $\mathcal{E}_{\mathbb{T}}$ of the syntactic category $\mathcal{C}_{\mathbb{T}}$ of a geometric theory $\mathbb{T}$ are formal quotients of infinite coproducts of objects of $\mathcal{C}_{\mathbb{T}}$ by equivalence relations in $\mathcal{E}_{\mathbb{T}}$ (cf. the proof of Proposition D1.4.12(iii) [35]). In our particular case, the object G of $\mathcal{E}_{\mathbb{MV}}$ which corresponds to the object $\{x \,.\, \top\}$ of $\mathcal{E}_{\mathbb{L}_u}$ under the equivalence $\mathcal{E}_{\mathbb{MV}} \simeq \mathcal{E}_{\mathbb{L}_u}$ of Theorem 3.5.4 can be described as follows. For any natural number $n \geq 1$, let $\phi_n(x_1, \dots, x_n)$ be the formula $\bigwedge\limits_{i \in \{1,\dots,n-1\}} x_i \oplus x_{i+1} = x_i$ over Σ_{MV} asserting that $(x_1, \dots, x_n)$ is a n-good sequence, and let R be the equivalence relation on the coproduct $\coprod\limits_{n\geq 1} \phi_n(x_1, \dots, x_n)$ defined in section 3.4. Then G is isomorphic to the formal quotient of the product $(\coprod\limits_{n\geq 1} \phi_n(x_1, \dots, x_n))/R \times (\coprod\limits_{n\geq 1} \phi_n(x_1, \dots, x_n))/R$ by the equivalence relation used for defining the Grothendieck group associated with a cancellative abelian monoid. From this representation of G, it is straightforward to derive an expression for G as a formal quotient of an infinite coproduct of formulas in $\mathcal{C}_{\mathbb{MV}}$.

(b) We could have alternatively proved that the classifying toposes $\mathcal{E}_{\mathbb{MV}}$ and $\mathcal{E}_{\mathbb{L}_u}$ are equivalent by first showing that the theories $\mathbb{MV}$ and $\mathbb{L}_u$ are of presheaf type (i.e., classified by a presheaf topos) and then appealing to the classical Mundici's equivalence (the fact that the theory $\mathbb{MV}$ is classified by a presheaf topos is straightforward, it being algebraic, while the fact that $\mathbb{L}_u$ is of presheaf type can be proved by using the methods of [17], cf. Section 8.7 therein). Indeed, we have remarked in Section 1.5 that two theories of presheaf type are Morita-equivalent if and only if they have equivalent categories of set-based models.

3.6 Sheaf-theoretic Mundici's equivalence

For every Grothendieck topos $\mathcal{E}$ we have just defined a categorical equivalence between the category of models of $\mathbb{L}_u$ in $\mathcal{E}$ and the category of models of $\mathbb{MV}$ in $\mathcal{E}$, which is natural in $\mathcal{E}$. By specializing this result to toposes $\mathbf{Sh}(X)$ of sheaves on a topological space X, we shall obtain a sheaf-theoretic generalization of Mundici's equivalence.

The category of models of the theory $\mathbb{MV}$ in the topos $\mathbf{Sh}(X)$ is isomorphic to the category $\mathbf{Sh}_{\mathbb{MV}}(X)$ whose objects are the sheaves F on X endowed with an MV-algebra structure on each set $F(U)$ (for an open set U of X) in such a way that the maps $F(i_{U,V}) : F(U) \to F(V)$ corresponding to inclusions of open sets $i_{U,V} : V \subseteq U$ are MV-algebra homomorphisms, and whose arrows are the natural transformations between them which are pointwise MV-algebra homomorphisms. Indeed, the evaluation functors $ev_U : \mathbf{Sh}(X) \to \mathbf{Set}$ (for each open set U of X) preserve finite limits whence preserve and jointly reflect models of the theory $\mathbb{MV}$ (cf. Theorem 1.2.12).

The category of models of $\mathbb{L}_u$ in $\mathbf{Sh}(X)$ is isomorphic to the category $\mathbf{Sh}_{\mathbb{L}_u}(X)$ whose objects are the sheaves F on X endowed with a structure of pointed ℓ-group on each set $F(U)$ (for an open set U of X) in such a way that the maps $F(i_{U,V}) : F(U) \to F(V)$ corresponding to inclusions of open sets $i_{U,V} : V \subseteq U$ are ℓ-group unital homomorphisms and for each point x of X the canonically induced ℓ-group structure on the stalks F_x is an ℓ-group with strong unit, and whose arrows are the natural transformations between them which are pointwise ℓ-group homomorphisms. Indeed, the stalk functors $(-)_x : \mathbf{Sh}(X) \to \mathbf{Set}$ (for each point x of X) preserve and jointly reflect models of the theory $\mathbb{L}_u$ (cf. Theorem 1.2.12).

The two functors $\Gamma_{\mathbf{Sh}(X)}$ and $L_{\mathbf{Sh}(X)}$ defining the equivalence can be described as follows: $\Gamma_{\mathbf{Sh}(X)}$ sends any sheaf F in $\mathbf{Sh}_{\mathbb{L}_u}(X)$ to the sheaf $\Gamma_{\mathbf{Sh}(X)}(F)$ on X sending every open set U of X to the MV-algebra given by the unit interval in the ℓ-group $F(U)$, and it acts on arrows in the obvious way. In the converse direction, $L_{\mathbf{Sh}(X)}$ assigns to any sheaf G in $\mathbf{Sh}_{\mathbb{MV}}(X)$ the sheaf $L_{\mathbf{Sh}(X)}(G)$ on X whose stalk at any point $x \in X$ is equal to the ℓ-group corresponding via Mundici's equivalence to the MV-algebra G_x.

The naturality in $\mathcal{E}$ of our Morita-equivalence implies in particular that the resulting equivalence

$$\tau_X : \mathbf{Sh}_{\mathbb{MV}}(X) \simeq \mathbf{Sh}_{\mathbb{L}_u}(X)$$

is natural in X. Indeed, any continuous map $f : X \to Y$ induces a geometric morphism $\mathbf{Sh}(f) : \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$ such that $\mathbf{Sh}(f)^*$ is the inverse image functor on sheaves along f (cf. Chapter II of [38]). In particular, by taking X to be the one-point space, we obtain that, at the level of stalks, τ_X acts as the classical Mundici's equivalence (indeed, the geometric morphism $\mathbf{Set} \to \mathbf{Sh}(X)$ corresponding to a point $x : 1 \to X$ of X has as inverse image precisely the stalk functor at x).

Summarizing, we have the following result.

Corollary 3.6.1. *Let X be a topological space. Then, with the above notation, we have a categorical equivalence*

$$\tau_X : \mathbf{Sh}_{\mathbb{MV}}(X) \simeq \mathbf{Sh}_{\mathbb{L}_u}(X)$$

sending any sheaf F in $\mathbf{Sh}_{\mathbb{L}_u}(X)$ to the sheaf $\Gamma_{\mathbf{Sh}(X)}(F)$ on X sending every open set U of X to the MV-algebra given by the unit interval in the ℓ-group $F(U)$, and any sheaf G in $\mathbf{Sh}_{\mathbb{MV}}(X)$ to the sheaf $L_{\mathbf{Sh}(X)}(G)$ in $\mathbf{Sh}_{\mathbb{L}_u}(X)$ whose stalk at any point x of X is the ℓ-group corresponding to the MV-algebra G_x under Mundici's equivalence.

The equivalence τ_X is natural in X, *in the sense that for any continuous map $f : X \to Y$ of topological spaces, the diagram*

$$\begin{array}{ccc} \mathbf{Sh}_{\mathbb{MV}}(Y) & \xrightarrow{\tau_Y} & \mathbf{Sh}_{\mathbb{L}_u}(Y) \\ \downarrow{\scriptstyle j_f} & & \downarrow{\scriptstyle i_f} \\ \mathbf{Sh}_{\mathbb{MV}}(X) & \xrightarrow{\tau_X} & \mathbf{Sh}_{\mathbb{L}_u}(X) \end{array}$$

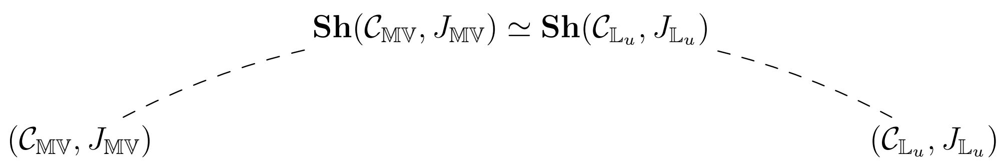
commutes, where $i_f : \mathbf{Sh}_{\mathbb{MV}}(Y) \to \mathbf{Sh}_{\mathbb{MV}}(X)$ *and* $j_f : \mathbf{Sh}_{\mathbb{L}_u}(Y) \to \mathbf{Sh}_{\mathbb{L}_u}(X)$ *are the inverse image functors on sheaves along* f*.*

Moreover, τ_X *acts, at the level of stalks, as the classical Mundici's equivalence.*

There is a vast literature on sheaf representations of ℓ-groups and MV-algebras. Already [6] has section on this topic. Via the Γ functor, all the theory was transplanted to MV-algebras by several people. We have already recalled the work of Dubuc and Poveda in [30]. Another example of sheaf representation for MV-algebras is given by Filipoiu and Georgescu in [32].

3.7 Applications of the bridge technique

The Morita-equivalence between the theories $\mathbb{MV}$ and $\mathbb{L}_u$ established above allows us to transfer properties and results between the two theories according to the 'bridge technique' of [12]. More specifically, for any given topos-theoretic invariant T one can attempt to build a 'bridge' yielding a logical relationship between the two theories by T.



3.7.1 Correspondence between geometric extensions

A *subtopos* of a given topos is an isomorphism class of geometric inclusions to that topos. The Duality Theorem gives a bijection between the subtoposes of the classifying topos of a given theory and the quotients of this theory. By the Morita-equivalence, the theories $\mathbb{MV}$ and $\mathbb{L}_u$ have the same classifying topos whence there is a bijection between the quotients of these theories. More specifically, we have the following theorem.

Theorem 3.7.1. *Every quotient of the theory* $\mathbb{MV}$ *is Morita-equivalent to a quotient of the theory* $\mathbb{L}_u$ *and conversely. These Morita-equivalences are the restrictions of the one between* $\mathbb{MV}$ *and* $\mathbb{L}_u$ *of Theorem 3.5.4.*

This theorem would be trivial if the two theories $\mathbb{MV}$ and $\mathbb{L}_u$ were bi-interpretable, but we proved that this is not the case. The unifying power of the notion of classifying topos allows us to obtain a syntactic result by arguing semantically.

Given this result, it is natural to wonder whether there exists an effective means for obtaining, starting from a given quotient of either $\mathbb{L}_u$ or $\mathbb{MV}$, an explicit axiomatization of the quotient corresponding to it as in the theorem. The interpretation functor $I : \mathcal{C}_{\mathbb{MV}} \to \mathcal{C}_{\mathbb{L}_u}$ induces the equivalence of classifying toposes $\mathbf{Sh}(\mathcal{C}_{\mathbb{MV}}, J_{\mathbb{MV}}) \simeq \mathbf{Sh}(\mathcal{C}_{\mathbb{L}_u}, J_{\mathbb{L}_u})$ (cf. Remark 3.3.3). We thus obtain the following result.

Proposition 3.7.2. *Let* $\mathbb{S}$ *be a quotient of the theory* $\mathbb{MV}$*. Then the quotient of* $\mathbb{L}_u$ *corresponding to* $\mathbb{S}$ *as in Theorem 3.7.1 can be described as the quotient* $I(\mathbb{T})$ *of* $\mathbb{L}_u$ *obtained by adding all the sequents of the form* $I(\sigma)$ *where* σ *ranges over all the axioms of* $\mathbb{S}$*.*

Let K be a class of MV-algebras that can be axiomatized by a quotient $\mathbb{T}$ of $\mathbb{MV}$; then the class K' of unital ℓ-groups that corresponds to K by Mundici's equivalence is axiomatized by the quotient $I(\mathbb{T})$ of $\mathbb{L}_u$. Examples of such classes are given by all subvarieties of MV-algebras as well as the class of perfect MV-algebras (cf. [3]), the class of local MV-algebras (cf. [27]) and many others.

For example, in [4] the authors characterized the unital ℓ-groups corresponding to perfect MV-algebras via Mundici's equivalence, by introducing the notion of antiarchimedean ℓ-groups which arises by translating the axioms of the theory $\mathbb{P}$ via the interpretation functor I.

Definition 3.7.3. An ℓ-u group $\mathcal{G} = (G, u)$ is said to be *antiarchimedean* if it satisfies the following sequents:

Ant.1 $(0 \leq x \leq u \vdash_x \sup(0, 2\inf(2x, u) - u) = \inf(u, 2\sup(2x - u, 0));$

Ant.2 $(0 \leq x \leq u \wedge \inf(2x, u) = x \vdash_x x = 0 \vee x = u);$

Ant.3 $(0 \leq x \leq u \wedge (x = u - x) \vdash_x \bot).$

We denote with $\mathbb{L}_{Chang}$ and $\mathbb{A}nt$ respectively the quotients $\mathbb{L}_u \cup \{\text{Ant.1}\}$ and $\mathbb{L}_u \cup \{\text{Ant.1,Ant.2,Ant.3}\}$. It follows from Theorem 3.7.2 that $\mathbb{L}_{Chang}$ and $\mathbb{A}nt$ are Morita-equivalent to the theories $\mathbb{C}$ and $\mathbb{P}$.

In the sequel we will use the interpretation functor I to prove sequents in the theory of MV-algebras, or in some of its quotients, by proving its translation into the theory of ℓ-u groups, or into the appropriate Morita-equivalent quotient. By this method we establish for instance the following proposition.

Proposition 3.7.4. *The following sequent is provable into the theory* $\mathbb{C}$*,*

$$(\top \vdash_x 2(2x)^2 = (2x)^2) .$$

Proof. We will reason into the the theory $\mathbb{L}_{Chang}$ that is Morita-equivalent to $\mathbb{C}$. Observe that the sequent Ant.1 is equivalent to

$$(0 \leq x \leq u \vdash_x \sup(0, u + \inf(0, 4x - 2u)) = \inf(u, \sup(0, 4x - 2u))),$$

and further to

$$(0 \leq x \leq u \vdash_x u \leq \sup(0, 4x - 2u) + \sup(0, 2u - 4x)) .$$

Now, this implies the sequent

$$(0 \leq x \leq u \vdash_x \inf(u, \sup(0, 4x - 2u)) = \inf(u, 2\sup(0, 4x - 2u))),$$

(since if $u \leq \sup(0, 4x-2u)+\sup(0, 2u-4x)$ then $\inf(u, 2\sup(0, 4x-2u)) \leq \inf(\sup(0, 4x - 2u) + \sup(0, 2u - 4x), 2\sup(0, 4x - 2u)) = \sup(0, 4x - 2u) + \inf(\sup(0, 2u - 4x), \sup(0, 4x - 2u)) = \sup(0, 4x - 2u))$ and hence our thesis.

□

The sequent proved in the previous proposition expresses the property that in every set-based $\mathbb{C}$-model the elements of the form $(2x)^2$ are boolean. This fact will be important for the generalization of Di Nola-Lettieri's equivalence into a Morita-equivalence obtained in Chapter 4.

3.7.2 Finitely presented ℓ-groups with strong unit

The theory $\mathbb{L}_u$ can be regarded as a quotient of the theory $\mathbb{L}$ where we enlarge the signature including a constant symbol. Being $\mathbb{L}$ an algebraic (hence, of presheaf type) theory, by the Duality Theorem, the classifying topos for the theory $\mathbb{L}_u$ can be represented in the form $\mathbf{Sh}(\text{f.p.}\mathbb{L}\text{-mod}(\mathbf{Set})^{\text{op}}, J) \hookrightarrow [\text{f.p.}\mathbb{L}\text{-mod}(\mathbf{Set}), \mathbf{Set}]$ for a unique topology J (recall that every Horn theory is classified by the topos of covariant set-valued functors on its category of finitely presentable models [8]). One can then naturally pose the question as to whether the equivalence of classifying toposes

$$\mathbf{Sh}(\text{f.p.}\mathbb{L}\text{-mod}(\mathbf{Set})^{\text{op}}, J) \simeq [\text{f.p.}\mathbb{MV}\text{-mod}(\mathbf{Set}), \mathbf{Set}]$$

is induced by a morphism of sites $\text{f.p.}\mathbb{MV}\text{-mod}(\mathbf{Set}) \to \text{f.p.}\mathbb{L}\text{-mod}(\mathbf{Set})$, that is, if the ℓ-groups corresponding to finitely presented MV-algebras are all finitely presented as pointed ℓ-groups. The answer to this question is positive and it is provided in the following.

The next result is probably known by specialists but we give a proof as we have not found one in the literature.

Proposition 3.7.5. *An ℓ-u group is finitely presentable iff it is finitely presentable as pointed ℓ-group.*

Proof. Recall that the absolute value $|x|$ of an element x of a pointed ℓ-group $(G, +, -, \leq, \inf, \sup, 0)$ is the element $\sup(x, -x)$, that $|x| \geq 0$ for all $x \in G$, that $|x| = |-x|$ for all $x \in G$ and that the triangular inequality $|x + y| \leq |x| + |y|$ holds for all $x, y \in G$. These properties easily imply that for any pointed ℓ-group $\mathcal{G} := (G, u)$ with generators $x_1, \ldots, x_n$, if for every $i \in \{1, \ldots, n\}$ there exists a natural number k_i such that $|x_i| \leq k_i u$ then the unit u is strong for $\mathcal{G}$ (one can prove by induction on the structure $t_{\mathcal{G}}(x_1, \ldots, x_n)$ of the elements of G). Now, it is immediate to see that for any finitely presented pointed ℓ-group (G, u), any ℓ-group with strong unit (H, v) and any ℓ-group unital homomorphism $f : (G, u) \to (H, v)$ there exists an ℓ-group with strong unit (G', u') and an ℓ-group unital homomorphisms $h : (G, u) \to (G', u')$ and $g : (G', u') \to (H, v)$ such that $f = g \circ h$. Indeed, given generators $x_1, \ldots, x_n$ for G, since v is a strong unit for H there exists for each $i \in \{1, \ldots, n\}$ a natural number k_i such that $|f(x_i)| \leq k_i v$; it thus suffices to take G' equal to the quotient of G by the congruence generated by the relations $|x_i| \leq k_i u$ for $i \in \{1, \ldots, n\}$ and $u' = u$.

The fact that every homomorphism from a finitely presented pointed ℓ-group to a pointed ℓ-group (H, v) factors through a homomorphism from an ℓ-group with strong unit to (H, v) clearly implies that every ℓ-group with strong unit can be expressed as a filtered colimit of ℓ-groups with strong unit which are finitely presented as pointed ℓ-groups. Since a retract of a finitely presented ℓ-group is again finitely presented, we can conclude that every ℓ-group with strong unit which is finitely presentable in the category of ℓ-groups with strong unit is finitely presented as pointed ℓ-group. This concludes the proof of the proposition, as the other direction is trivial. □

For various other class of unital ℓ-groups one may conjecture that a similar result holds, mutatis mutandis, but in this paper we focus our attention only on this class.

We shall now proceed to give a syntactic description of the category of finitely presentable ℓ-groups with strong unit.

By the Comparison Lemma if $\mathbb{T}$ is classified by a presheaf topos then its classifying topos $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$ is equivalent to the topos $[{\mathcal{C}_{\mathbb{T}}^{irr}}^{\text{op}}, \mathbf{Set}]$, where $\mathcal{C}_{\mathbb{T}}^{irr}$ is the full subcategory of $\mathcal{C}_{\mathbb{T}}$ on the $\mathbb{T}$-irreducible formulas whence that this latter category is dually equivalent to the category of finitely presentable models of $\mathbb{T}$ via the equivalence sending any such formula $\{\vec{x}\ .\ \phi\}$ to the model of $\mathbb{T}$ which it presents. In fact, this model corresponds to the geometric functor $\mathcal{C}_{\mathbb{T}} \to \mathbf{Set}$ represented by the formula $\{\vec{x}\ .\ \phi\}$ whence it admits the following syntactic description: its underlying set is given by $Hom_{\mathcal{C}_{\mathbb{T}}}(\{\vec{x}\ .\ \phi\}, \{z\ .\ \top\})$ and the order and operations are the obvious ones.

On the other hand, the category of finitely presented MV-algebras is well-known to be dual to the algebraic syntactic category $\mathcal{C}^{alg}_{\mathbb{MV}}$ of the theory $\mathbb{MV}$ (cf. Section 1.2). It follows that the category $\mathcal{C}^{alg}_{\mathbb{MV}}$ is equivalent to the category of rational polyhedra equipped with $\mathbb{Z}$-*maps*, i.e., continuous G_n-equidissections, where $G_n = \mathrm{GL}(n, \mathbb{Z}) \ltimes \mathbb{Z}^n$ is the n-dimensional affine group over the integers (cf. Section 4 in [37]).

We can thus conclude that, even though the theories $\mathbb{MV}$ and $\mathbb{L}_u$ are not bi-interpretable, there exists an equivalence of categories between the category $\mathcal{C}^{alg}_{\mathbb{MV}}$ and the category $\mathcal{C}^{irr}_{\mathbb{L}_u}$, as it is proved in the following theorem.

Theorem 3.7.6. *With the notation above, we have an equivalence of categories*

$$\mathcal{C}^{alg}_{\mathbb{MV}} \simeq \mathcal{C}^{irr}_{\mathbb{L}_u}$$

representing the syntactic counterpart of the equivalence of categories

$$f.p.\mathbb{MV}\text{-}mod(\mathbf{Set}) \simeq f.p.\mathbb{L}_u\text{-}mod(\mathbf{Set})\ .$$

The former equivalence is the restriction to $\mathcal{C}^{alg}_{\mathbb{MV}}$ of the interpretation of the theory $\mathbb{MV}$ into the theory $\mathbb{L}_u$ defined in Section 3.3.

Proof. In view of the arguments preceding the statement of the theorem, it remains to prove that the syntactic equivalence $\mathcal{C}^{alg}_{\mathbb{MV}} \simeq \mathcal{C}^{irr}_{\mathbb{L}_u}$ induced by the equivalence of classifying toposes is the restriction of the interpretation functor I defined in Section 3.4. For any ℓ-group with strong unit $\mathcal{G} = (G, u)$, the interpretation of the formula $I(\{\vec{x}\ .\ \phi\})$ in $\mathcal{G}$ is by definition of I in natural bijection with the interpretation of the formula $\{\vec{x}\ .\ \phi\}$ in the MV-algebra $[0, u_{\mathcal{G}}]$, which is in turn in bijection with the MV-algebra homomorphisms $A \to [0, u_{\mathcal{G}}]$ whence, by Mundici's equivalence, with the ℓ-group unital homomorphisms $\mathcal{G}_{\mathcal{A}} \to \mathcal{G}$. This means that if $\{\vec{x} \, . \, \phi\}$ presents an MV-algebra, i.e., it is algebric , then the formula $I(\{\vec{x} \, . \, \phi\})$ present the corresponding ℓ-u group, i.e., it is $\mathbb{L}_u$-irreducibile. □

Remarks 3.7.7. (a) Since the syntactic equivalence of Theorem 3.7.6 is the restriction of the interpretation of the theory $\mathbb{MV}$ into the theory $\mathbb{L}_u$, the semantic equivalence is just the restriction of Mundici's equivalence. Thus, the finitely presentable $\mathbb{L}_u$-models are precisely the unital ℓ-groups which correspond to finitely presented MV-algebras via Mundici's equivalence.

(b) Different approaches to finitely presentable ℓ-u groups can be found in the literature. For instance, in [40] the ℓ-u groups corresponding via the Γ functor to the finitely presented MV-algebras are geometrically characterized as the principal quotients of the ℓ-u groups $\mathcal{M}([0,1]^n, \mathbb{R})$ of piecewise linear real-valued functions f over $[0,1]^n$ (for some natural number n). In [9] they are intrinsically characterized as the ℓ-u groups which are generated by an abstract Schauder basis over their maximal spectral space. Notice also that, by Proposition 3.7.5, the class of finitely presented pointed ℓ-groups whose distinguished element is a strong unit coincides with the two above-mentioned classes of unital ℓ-groups.

Remark 3.7.8. The formula-in-context $\{x \, . \, \top\}$ is clearly not $\mathbb{L}_u$-irreducible, and in fact we proved in Section 3.4 that it is not in the image of the interpretation functor $I : \mathcal{C}_{\mathbb{MV}} \to \mathcal{C}_{\mathbb{L}_u}$.

As a corollary of Theorem 3.7.6 and Proposition 3.7.5, we obtain the following result.

Corollary 3.7.9. *The finitely presentable ℓ-groups with strong unit are exactly the finitely presentable pointed ℓ-groups which are presented by a $\mathbb{L}_u$-*

irreducible formula. The ℓ-group presented by such a formula $\{\vec{x} \,.\, \phi\}$ has as underlying set the set $Hom_{\mathcal{C}^{irr}_{\mathbb{L}_u}}(\{\vec{x} \,.\, \phi\}, \{z \,.\, \top\})$ and as order and operations the obvious ones.

Proof. If we consider $\mathbb{L}$ as an algebraic theory (i.e., without the predicate $\leq$, which can be defined in terms of the operation inf), we have a canonical isomorphism $Hom_{\mathcal{C}^{irr}_{\mathbb{L}_u}}(\{\vec{x} \,.\, \phi\}, \{z \,.\, \top\}) \cong Hom_{\mathcal{C}^{alg}_{\mathbb{L}}}(\{\vec{x} \,.\, \phi\}, \{z \,.\, \top\})$; that is, for any $\mathbb{L}_u$-provably functional formula $\theta(\vec{x}, z) : \{\vec{x} \,.\, \phi\} \to \{z \,.\, \top\}$ there exists a term $t(\vec{x})$ over the signature of $\mathbb{L}_u$ such that the sequent $(\theta(\vec{x}, z) \dashv\vdash_{\vec{x},z} z = t(\vec{x}))$ is provable in $\mathbb{L}_u$. □

Thanks to Theorem 3.7.6, we can now describe a method for obtaining an axiomatization of the quotient of $\mathbb{MV}$ corresponding to a given quotient of the theory $\mathbb{L}_u$ as in Theorem 3.7.1 (recall that the converse direction was already addressed to in Theorem 3.7.2). Indeed, since the classifying toposes of $\mathbb{MV}$ (resp. of $\mathbb{L}_u$) can be represented in the form $[{\mathcal{C}^{alg}_{\mathbb{MV}}}^{\text{op}}, \mathbf{Set}]$ (resp. in the form $[{\mathcal{C}^{irr}_{\mathbb{L}_u}}^{\text{op}}, \mathbf{Set}]$), by the Duality Theorem, the quotients of $\mathbb{MV}$ (resp. of $\mathbb{L}_u$) are in bijective correspondence with the Grothendieck topologies on the category $\mathcal{C}^{alg}_{\mathbb{MV}}$ (resp. on the category $\mathcal{C}^{irr}_{\mathbb{L}_u}$); as the categories $\mathcal{C}^{alg}_{\mathbb{MV}}$ and $\mathcal{C}^{irr}_{\mathbb{L}_u}$ are equivalent, the Grothendieck topologies on the two categories correspond to each other bijectively through this equivalence, yielding the desired correspondence between the quotient theories. Specifically, any Grothendieck topology K on $\mathcal{C}^{irr}_{\mathbb{L}_u}$ corresponds to the quotient $\mathbb{L}_u^K$ of $\mathbb{L}_u$ consisting of all the geometric sequents of the form $(\psi \vdash_{\vec{y}} \bigvee_{i \in I} (\exists \vec{x_i})\theta_i)$, where $\{\vec{y} \,.\, \psi\}$ and the $\{\vec{x_i} \,.\, \phi_i\}$ are all $\mathbb{L}_u$-irreducible formulas, the $\{\vec{x_i}, \vec{y} \,.\, \theta_i\}$ are geometric formulas over the signature of $\mathbb{L}_u$ which are $\mathbb{L}_u$-provably functional from $\{\vec{x_i} \,.\, \phi_i\}$ to $\{\vec{y} \,.\, \psi\}$ and the sieve generated by the family of arrows $\{[\theta_i] : \{\vec{x_i} \,.\, \phi_i\} \to \{\vec{y} \,.\, \psi\} \mid i \in I\}$ in $\mathcal{C}^{irr}_{\mathbb{L}_u}$ generates a K-covering sieve. Conversely, every quotient $\mathbb{S}$ of $\mathbb{L}_u$ is syntactically equivalent to a quotient of this form (cf. Theorem 1.5.5); the Grothendieck topology associated with it is therefore the topology on the category $\mathcal{C}^{irr}_{\mathbb{L}_u}$ generated by the (sieves generated by the) families of arrows $\{[\theta_i] : \{\vec{x_i} \,.\, \phi_i\} \to \{\vec{y} \,.\, \psi\} \mid i \in I\}$ in $\mathcal{C}^{irr}_{\mathbb{L}_u}$ such that the sequent $(\psi \vdash_{\vec{y}} \bigvee\limits_{i \in I} (\exists \vec{x_i})\theta_i)$ is provable in $\mathbb{S}$. An analogous correspondence holds for the theory $\mathbb{MV}$ (with the $\mathbb{L}_u$-irreducible formulas being replaced by the formulas in $\mathcal{C}^{alg}_{\mathbb{MV}}$). The quotient of $\mathbb{MV}$ corresponding to a given quotient $\mathbb{S}$ of $\mathbb{L}_u$ is thus the quotient of $\mathbb{MV}$ corresponding to the Grothendieck topology on $\mathcal{C}^{alg}_{\mathbb{MV}}$ obtained by transferring the Grothendieck topology on $\mathcal{C}^{irr}_{\mathbb{L}_u}$ associated with $\mathbb{S}$ along the equivalence $\mathcal{C}^{alg}_{\mathbb{MV}} \simeq \mathcal{C}^{irr}_{\mathbb{L}_u}$ of Theorem 3.7.6. See Chapter III of [38] for more details on Grothendieck topology.

By using this technique we can for instance identify the quotient of the theory $\mathbb{MV}$ corresponding to the quotient of the theory $\mathbb{L}_u$ given by (totally ordered) archimedean groups with strong unit. Recall that a totally ordered group is said to be *archimedean* if for any strictly positive elements x and y there exists a natural number n such that $x \leq ny$. The fact that such groups correspond to simple MV-algebras via Mundici's equivalence is essentially folklore, even though an explicit proof cannot be found in the literature. We can axiomatize totally ordered archimedean groups with strong unit as follows.

Let $\mathbb{A}$ be the geometric theory whose axioms are the ones of $\mathbb{L}_u$ plus the following sequent:

$$\alpha : (0 \leq x \wedge 0 \leq y \vdash_{x,y} (\textstyle\bigvee_{n \in \mathbb{N}} x \leq ny) \vee y = 0).$$

Lemma 3.7.10. *The following sequents are $\mathbb{L}_u$-provably equivalent:*

- $\alpha : (0 \leq x \wedge 0 \leq y \vdash_{x,y} \bigvee_{n \in \mathbb{N}} x \leq ny \vee y = 0)$*;*
- $\beta : (0 \leq x \vdash_{x} (\bigvee_{n \in \mathbb{N}} u \leq nx) \vee x = 0)$*.*

Moreover, each of them $\mathbb{L}_u$-provably implies the sequent

$$\gamma : (\top \vdash_{x,y} x \leq y \vee y \leq x)\ .$$

Proof. Obviously the sequent α implies β. To prove the converse implication, we shall argue informally by using elements. Given $x, y \geq 0$, either or $y = 0$

or $u \leq ny$ for some n. In the first case the conclusion of the sequent α is trivially satisfied, so it remains to consider the second case. By the second axiom of strong unit, there is k_1 such that $x \leq k_1 u$. Thus we have that $x \leq k_1 u \leq k_1 ny$, as required.

The sequent γ is clearly provably equivalent to the sequent $(\top \vdash_x x \geq 0 \vee x \leq 0)$. We shall prove that this latter sequent is implied by the sequent α. Again, we shall argue informally by using elements. Given x, we can consider the two elements $x^+ = \sup(x, 0)$ and $x^- = \sup(0, -x)$. Clearly, $x^+, x^- \geq 0$. Sequent α implies that either $x^- = 0$ (equivalently, $x \geq 0$) or $x^+ \leq nx^-$ for some n. Let us prove by induction on n that $x^+ \leq nx^-$ implies $x \leq 0$. If $n = 0$ then $x^+ \leq 0$, that is $x \leq 0$. For a $n \geq 1$, $x^+ \leq (n-1)x^- + x^-$ is equivalent to the condition $x = x^+ - x^- \leq (n-1)x^-$. Now, from the fact that $(n-1)x^- \geq 0$ it follows that $x^+ = \sup(x, 0) \leq (n-1)x^-$, which implies by the induction hypothesis $x \leq 0$, as desired.

Changing the role of x^+ and x^- we have that $x \geq 0$. Thus we prove that given x, it is always comparable with 0. □

To calculate the quotient of $\mathbb{MV}$ corresponding to $\mathbb{A}$, let us use the simpler axiomatization of $\mathbb{A}$ over $\mathbb{L}_u$ given by sequent β. We have to 'decompose' all the formulas appearing in β as disjunctions involving $\mathbb{L}_u$-irreducible formulas. The formula $\{x \,.\, x \geq 0\}$ is clearly not $\mathbb{L}_u$-irreducible, but thanks to the second axiom of strong unit, we can 'decompose' it as the disjunction $\bigvee_{n \in \mathbb{N}} 0 \leq x \leq nu$. Now, each of the fourmulas $\{x \,.\, 0 \leq x \leq nu\}$ is $\mathbb{L}_u$-irreducible, isomorphic in the syntactic category $\mathcal{C}_{\mathbb{L}_u}$ of $\mathbb{L}_u$ to the formula $\{x \,.\, 0 \leq x \leq u\}$. Indeed, it is easy to prove that the following sequents are provable in $\mathbb{L}_u$:

No torsion:

$$(nx = 0 \vdash_x x = 0),$$

for all $n \in \mathbb{N}$.

Riesz decomposition property:

$(0 \leq x, y, z \wedge x \leq y+z \vdash_{x,y,z} (\exists x_1)(\exists x_2)(0 \leq x_1 \leq y \wedge 0 \leq x_2 \leq z \wedge x = x_1+x_2))$.

These sequents ensure that the arrows $\theta(x, x') : \{x . 0 \leq x \leq u\} \to \{x' . 0 \leq x' \leq nu\}$ and $\theta'(x', x) : \{x' . 0 \leq x' \leq nu\} \to \{x . 0 \leq x \leq u\}$ in the syntactic category $\mathcal{C}_{\mathbb{L}_u}$ given by $\theta(x, x') := (x' = nx)$ and $\theta'(x', x) := (nx = x')$ are well-defined and the inverse to one another. Since the formula $\{x . 0 \leq x \leq u\}$ is $\mathbb{L}_u$-irreducible (it being the image of the algebraic formula $\{x . \top\}$ under the interpretation functor I), it follows that all the $\mathbb{L}_u$-equivalent fourmulas $\{x . 0 \leq x \leq nu\}$ are $\mathbb{L}_u$-irreducible as well.

Axiom β thus becomes $\mathbb{L}_u$-provably equivalent to the set of sequents

$$\beta'_n : (0 \leq x \leq nu \vdash_x \bigvee_{m \in \mathbb{N}} (u \leq mx \wedge 0 \leq x \leq nu) \vee (x = 0 \wedge 0 \leq x \leq nu))$$

(for $n \in \mathbb{N}$).

The fourmulas $\{x . u \leq mx\}$ are $\mathbb{L}_u$-irreducible since they are the images of the algebraic fourmulas $\{x . u = mx\}$ in the theory $\mathbb{MV}$. It follows that the fourmulas $\{x . u \leq mx \wedge 0 \leq x \leq nu\}$ are all $\mathbb{L}_u$-irreducible as they are finite conjunctions of $\mathbb{L}_u$-irreducible fourmulas (recall that the functor I is cartesian). Similarly, one proves that all the fourmulas $\{x . x = 0 \wedge 0 \leq x \leq nu\}$ are $\mathbb{L}_u$-irreducible. To obtain an axiomatization of $\mathbb{A}$ having the required form for making the translation to the theory $\mathbb{MV}$, we observe that the subobjects $\{x' . u \leq mx' \wedge 0 \leq x' \leq nu\} \rightarrowtail \{x' . 0 \leq x' \leq nu\}$ correspond to the subobjects $\{x . u \leq mnx \wedge 0 \leq x \leq u\} \rightarrowtail \{x . 0 \leq x \leq u\}$ under the above-mentioned isomorphism $\{x . 0 \leq x \leq u\} \cong \{x' . 0 \leq x' \leq nu\}$. From the fact that $u \leq nx$ implies $u \leq mnx$ (if $x \geq 0$) for any non-zero $m \in \mathbb{N}$, it follows that each sequent β'_n is $\mathbb{L}_u$-provably equivalent to the sequent β'_1, which clearly translates into the sequent $S : (\top \vdash_x (\bigvee_{n \in \mathbb{N}} nx = 1) \vee x = 0)$ over the signature of $\mathbb{MV}$. We have observed in Section 2.4 that the quotient of the theory $\mathbb{MV}$ obtained by adding the sequent S is the theory $\mathbb{S}imple$ of simple MV-algebras.

Summarizing, we have the following result.

Theorem 3.7.11. *The theories $\mathbb{S}imple$ of simple MV-algebras and $\mathbb{A}$ of Archimedean ℓ-u groups are Morita-equivalent.*

3.7.3 Geometric compactness and completeness for $\mathbb{L}_u$

The Morita-equivalence between the geometric theory $\mathbb{L}_u$ of ℓ-groups with strong unit and the algebraic theory $\mathbb{MV}$ of MV-algebras implies a form of compactness and completeness for the theory $\mathbb{L}_u$, properties which are *a priori* not expected as this theory is infinitary. To prove this, we need some preliminaries.

A *point* of a Grothendieck topos $\mathcal{E}$ is a geometric morphism $\mathbf{Set} \to \mathcal{E}$. A topos $\mathcal{E}$ is said to have *enough points* if the class of all the inverse image functors $\mathcal{E} \to \mathbf{Set}$ is jointly surjective (that is, if their inverse image functors jointly reflect isomorphisms).

Lemma 3.7.12. *The classifying topos of a theory of presheaf type has enough points.*

Proof. Let $\mathbb{T}$ be a theory of presheaf type and $[\mathcal{C}^{op}, \mathbf{Set}]$ its classifying topos. We can consider the following family of points

$$\mathcal{I} := \{f_c : \mathbf{Set} \to [\mathcal{C}^{op}, \mathbf{Set}] \mid c \in \mathcal{C}\},$$

where the inverse image functor $f_c^* : [\mathcal{C}^{op}, \mathbf{Set}] \to \mathbf{Set}$ is the evaluation map, i.e., $f_c^*(F) := F(c)$, for any $F \in [\mathcal{C}^{op}, \mathbf{Set}]$. This family $\mathcal{I}$ is jointly surjective since for any pair of distinct arrows $\alpha, \beta : F \to G$ in $[\mathcal{C}^{op}, \mathbf{Set}]$, there is $c \in \mathcal{C}$ such that $f_c^*(\alpha) \neq f_c^*(\beta)$. □

Theorem 3.7.13. *(i) For any geometric sequent σ over the signature $\Sigma_{\mathbb{L}_u}$, σ is valid in all abelian ℓ-groups with strong unit in* **Set** *if and only if it is provable in the theory $\mathbb{L}_u$;*

(ii) *For any geometric sentences ϕ_i over the signature $\Sigma_{\mathbb{L}_u}$, $(\top \vdash \bigvee_{i \in I} \phi_i)$ is provable in $\mathbb{L}_u$ (equivalently by (i), every abelian ℓ-group with strong unit in* **Set** *satisfies at least one of the ϕ_i) if and only if there exists $i \in I$ such that the sequent $(\top \vdash \phi_i)$ is provable in $\mathbb{L}_u$ (equivalently by (i), every abelian ℓ-group with strong unit in* **Set** *satisfies ϕ_i).*

Proof. (i) This follows from the fact that every theory classified by a presheaf topos has enough (finitely presentable) set-based models since its classifying topos has enough points (cf. Lemma 3.7.12).

(ii) This follows from the fact that the formula $\{[] \, . \, \top\}$ is $\mathbb{L}_u$-irreducible, since it presents the ℓ-u group $(\mathbb{Z}, 1)$. □

Remark 3.7.14. Notice that, whilst the formula $\{[] \, . \, \top\}$ is $\mathbb{L}_u$-irreducible, the formula $\{x \, . \, \top\}$ is not (cf. Remark 3.7.8). In fact, the analogue of Theorem 3.7.13(ii) for such formula no longer holds. This represents a substantial difference with the theory $\mathbb{MV}$, in which both the formulae $\{[] \, . \, \top\}$ and $\{x \, . \, \top\}$ (as well as any formula of the form $\{\vec{x} \, . \, \top\}$) are irreducible.

Chapter 4

Generalization of Di Nola-Lettieri's equivalence

In 1994, A. Di Nola and A. Lettieri established a categorical equivalence between the category of perfect MV-algebras and that of lattice-ordered abelian groups (cf. [28]). Perfect MV-algebras form an interesting class of MV-algebras, which is directly related to the important problem of incompleteness of first-order Łukasiewicz logic; indeed, the subalgebra of the Lindenbaum algebra of first-order Łukasiewicz logic generated by the classes of formulas which are valid but not provable is a perfect MV-algebra (cf. [1]).

As for Mundici's equivalence, we show in this chapter that Di Nola-Lettieri's equivalance can be lifted to a Morita-equivalence between the theory of perfect MV-algebras and that of lattice-ordered abelian groups. Further information are deduced by applying the technique of toposes as bridges and by computing the classifying topos of the theory $\mathbb{P}$. The results of this chapter are contained in [20].

4.1 Equivalence in Set

Let $\mathcal{G}$ be an ℓ-group and $\mathbb{Z} \times_{\text{lex}} \mathcal{G}$ be the lexicographic product of the ℓ-group $\mathbb{Z}$ of integers with $\mathcal{G}$. This is again an ℓ-group, whose underlying set is the cartesian product $\mathbb{Z} \times G$, whose group operations are defined pointwise and whose order relation is given by the lexicographic order. The element $(1, 0)$ is a strong unit of $\mathbb{Z} \times_{\text{lex}} \mathcal{G}$; hence, we can consider the MV-algebra $\Sigma(\mathcal{G}) := \Gamma(\mathbb{Z} \times_{\text{lex}} \mathcal{G}, (1, 0))$, where Γ is the Mundici's functor. By definition of lexicographic order, we have that:

$$\Sigma(G) = \{(0, x) \in \Gamma(\mathbb{Z} \times G) \mid x \geq 0\} \cup \{(1, x) \in \Gamma(\mathbb{Z} \times G) \mid x \leq 0\},$$

where $\Sigma(G)$ is the underlying set of $\Sigma(\mathcal{G})$. This MV-algebra is perfect; indeed, $\{(0, x) \in \Gamma(\mathbb{Z} \times G) \mid x \geq 0\}$ is the radical and $\{(1, x) \in \Gamma(\mathbb{Z} \times G) \mid x \leq 0\}$ is the coradical. If $h : \mathcal{G} \to \mathcal{G}'$ is an ℓ-homomorphism, the function

$$h^* : (m, g) \in \mathbb{Z} \times_{\text{lex}} \mathcal{G} \to (m, h(g)) \in \mathbb{Z} \times_{\text{lex}} \mathcal{G}$$

is a unital ℓ-homomorphism. We set $\Sigma(h) = h^*|_{\Gamma(\mathbb{Z} \times_{\text{lex}} \mathcal{G})}$. It is easily seen that Σ is a functor.

In the converse direction, let $\mathcal{A}$ be a perfect MV-algebra.

Lemma* 4.1.1. *For every MV-algebra $\mathcal{A}$, the structure*

$$(Rad(\mathcal{A}), \oplus, \leq, \inf, \sup, 0)$$

is a cancellative lattice-ordered abelian monoid[1].

Proof. As an ideal of $\mathcal{A}$, the radical is a lattice-ordered abelian monoid. It is also cancellative (see Lemma 3.2 [28]). □

From a cancellative lattice-ordered abelian monoid $\mathcal{M}$ we can canonically define its Grothendieck ℓ-group. We have already used this technique

[1]This result is not constructive because in general the proof that the radical is an ideal requires the axiom of choice.

to define the ℓ-group associated with the monoid of good sequences of an MV-algebra. This general method will yield a bi-interpretation between the theory of ℓ-groups and that of their positive cones in Section 4.5.1. Let $\Delta(\mathcal{A})$ be the ℓ-group built from $Rad(\mathcal{A})$ by using this construction. Any MV-homomorphism $f : \mathcal{A} \to \mathcal{A}'$ between perfect MV-algebras preserves the radical, the MV-operations and the natural order. Thus, the MV-homomorphism f induces by restriction a homomorphism between the associated lattice-ordered abelian monoids, which in turn can be extended to a homomorphism between the corresponding ℓ-groups, as follows:

$$\Delta(f) : [x, y] \in \Delta(\mathcal{A}) \to [f(x), f(y)] \in \Delta(\mathcal{A}').$$

It is easy to prove that Δ is a functor.

The functors Σ and Δ are categorical inverses to each other, i.e., $\Sigma(\Delta(\mathcal{A})) \cong \mathcal{A}$ and $\Delta(\Sigma(\mathcal{G})) \cong \mathcal{G}$ for every perfect MV-algebra $\mathcal{A}$ and every ℓ-group $\mathcal{G}$, naturally in $\mathcal{A}$ and $\mathcal{G}$. The maps that define the isomorphisms are the following:

$$\alpha_{\mathcal{A}} : a \in \mathcal{A} \to (0, [a, 0]) \in \Sigma(\Delta(\mathcal{A}));$$

$$\beta_{\mathcal{G}} : a \in \mathcal{G} \to [(0, a^+), (0, a^+ - a)] \in \Delta(\Sigma(\mathcal{G})).$$

Theorem 4.1.2 (Di Nola-Lettieri, 1994). *There is a categorical equivalence between the category* **P** *of perfect MV-algebras and the category* **L** *of ℓ-groups with the respective morphisms.*

4.2 From models of $\mathbb{L}$ to models of $\mathbb{P}$

In every Grothendieck topos $\mathcal{E}$ there is an object generalizing the set of integers which we call $\mathbb{Z}_{\mathcal{E}}$. This object is the coproduct $\coprod_{z\in\mathbb{Z}} 1$ of $\mathbb{Z}$ copies of the terminal object 1 of the topos; we denote by $\{\chi_z : 1 \to \coprod_{z\in\mathbb{Z}} 1 \mid z \in \mathbb{Z}\}$ the canonical coproduct arrows. This object is precisely the image of $\mathbb{Z}$ under the inverse image functor $\gamma_{\mathcal{E}}^*$ of the unique geometric morphism $\gamma_{\mathcal{E}} : \mathcal{E} \to \mathbf{Set}$. The ℓ-group structure with strong unit of $\mathbb{Z}$ induces an ℓ-group structure with strong unit on $\mathbb{Z}_{\mathcal{E}}$, since $\gamma_{\mathcal{E}}^*$ preserves it. In particular the total order relation $\leq$ on $\mathbb{Z}$ induces a total order relation on $\mathbb{Z}_{\mathcal{E}}$ which we indicate, abusing notation, also with the symbol $\leq$. Note that $\mathbb{Z}_{\mathcal{E}}$ is a decidable object of $\mathcal{E}$, being the image under $\gamma_{\mathcal{E}}$ of a decidable object, i.e., the equality relation on it is complemented. This allows to define the strict order $<$ as the intersection of $\leq$ with the complement of the equality relation.

Let $\mathcal{G}$ be an ℓ-group in $\mathcal{E}$. The *lexicographic product* $\mathbb{Z}_{\mathcal{E}} \times_{\text{lex}} \mathcal{G}$ of $\mathbb{Z}_{\mathcal{E}}$ and $\mathcal{G}$ is an ℓ-group whose underlying object is the product $\mathbb{Z}_{\mathcal{E}} \times G$, whose group operations are defined componentwise and whose order relation is defined by using the internal language as follows:

$$(a, x) \leq (b, y) \text{ iff } (a<b) \vee (a = b \wedge x \leq y).$$

Note that the infimum and the supremum of two "elements" are given by:

$$\inf((a, x), (b, y)) = \begin{cases} (a, x) & \text{if } a<b \\ (b, y) & \text{if } b<a \\ (a, \inf(x, y)) & \text{if } a = b \end{cases}$$

$$\sup((a, x), (b, y)) = \begin{cases} (a, x) & \text{if } a>b \\ (b, y) & \text{if } b>a \\ (a, \sup(x, y)) & \text{if } a = b \end{cases}$$

The generalized element $\langle \chi_1, 0 \rangle : 1 \to \mathbb{Z}_{\mathcal{E}} \times_{\text{lex}} \mathcal{G}$ yields a strong unit for the ℓ-group $\mathbb{Z}_{\mathcal{E}} \times_{\text{lex}} \mathcal{G}$, which we denote, abusing notation, simply by $(1, 0)$.

Proposition 4.2.1. *The lexicographic product $\mathbb{Z}_{\mathcal{E}} \times_{lex} \mathcal{G}$ is an ℓ-group and $(1, 0)$ is a strong unit for it.*

Proof. It is easy to see that $\mathbb{Z}_{\mathcal{E}} \times_{\text{lex}} \mathcal{G}$ satisfies the axioms L.1-L.12. For instance, given $(a, x), (b, y) \in \mathbb{Z}_{\mathcal{E}} \times G$, we have that:

- $(a,x)+(b,y)=(a+b,x+y)=$ by definition of sum

  $=(a+b,y+x)=$ by L.4 in $\mathbb{Z}_{\mathcal{E}}$ and $\mathcal{G}$

  $=(b,y)+(a,x)$;

- $(a,x)+(0,0)=$

  $=(a+0,x+0)=(a,x)$ by L.2 in $\mathbb{Z}_{\mathcal{E}}$ and $\mathcal{G}$.

Thus L.2 and L.4 hold. In a similar way it can be shown that the other axioms of $\mathbb{L}$ hold. Finally, we have to prove that $(1,0)$ is a strong unit, i.e., that it satisfies the axioms $L_u.1$ and $L_u.2$. By definition of order in $\mathbb{Z}_{\mathcal{E}} \times_{\text{lex}} \mathcal{G}$, we have that $(1,0) \geq (0,0)$, thus $L_u.1$ holds. Given $(a,x) \geq (0,0)$, this means that $a \geq 0$. From axiom $L_u.2$ applied to $\mathbb{Z}_{\mathcal{E}}$ we know that $\bigvee_{n\in\mathbb{N}} a \leq n1$. Therefore $\bigvee_{n\in\mathbb{N}} (a,x) \leq n(1,0)$. Thus, $L_u.2$ holds too. □

We set $\Sigma_{\mathcal{E}}(\mathcal{G}) := \Gamma_{\mathcal{E}}(\mathbb{Z}_{\mathcal{E}} \times_{\text{lex}} \mathcal{G},(1,0))$, where $\Gamma_{\mathcal{E}}$ is the unit interval functor from $\mathbb{L}_u$-mod$(\mathcal{E})$ to $\mathbb{MV}$-mod$(\mathcal{E})$ introduced in Section 3.3. The structure $\Sigma_{\mathcal{E}}(\mathcal{G})$ is thus an MV-algebra in $\mathcal{E}$ whose underlying object is $\Sigma_{\mathcal{E}}(G)=\{(a,x)\in \mathbb{Z}_{\mathcal{E}} \times G \mid (0,0) \leq (a,x) \leq (1,0)\}$.

Proposition 4.2.2. *The MV-algebra $\Sigma_{\mathcal{E}}(\mathcal{G})$ in $\mathcal{E}$ is perfect.*

Proof. By Theorem 2.2.8, it suffices to prove that $\Sigma_{\mathcal{E}}(\mathcal{G})$ satisfies axioms C, P.2 and β. Clearly, $\Sigma_{\mathcal{E}}(\mathcal{G})$ satisfies β and P.2 if and only if it is the disjoint union of its radical and its coradical. Let us prove this by steps:

Claim 1. $\Sigma_{\mathcal{E}}(G)=\{(0,x)\in \mathbb{Z}_{\mathcal{E}} \times G \mid x \geq 0\} \cup \{(1,x)\in \mathbb{Z}_{\mathcal{E}} \times G \mid x \leq 0\}$;

Claim 2. $Rad(\mathbb{Z}_{\mathcal{E}} \times_{\text{lex}} \mathcal{G}) = \{(0,x)\in \mathbb{Z}_{\mathcal{E}} \times G \mid x \geq 0\}$;

Claim 3. $\neg Rad(\mathbb{Z}_{\mathcal{E}} \times_{\text{lex}} \mathcal{G}) = \{(1,x)\in \mathbb{Z}_{\mathcal{E}} \times G \mid x \leq 0\}$.

We shall argue informally in the internal language of the topos $\mathcal{E}$ to prove these claims.

Claim 1. Given $(a, x) \in \Sigma_{\mathcal{E}}(G)$, we have to prove that it belongs to $\{(0, x) \in \mathbb{Z}_{\mathcal{E}} \times G \mid x \geq 0\}$ or $\{(1, x) \in \mathbb{Z}_{\mathcal{E}} \times G \mid x \leq 0\}$. Recall that $(0, 0) \leq (a, x) \leq (1, 0)$. This implies that $0 \leq a \leq 1$. In $\mathbb{Z}$ the following sequent holds

$$0 \leq a \leq 1 \vdash_a (a = 0) \vee (a = 1).$$

This is a geometric sequent; thus, it holds in $\mathbb{Z}_{\mathcal{E}}$ too. If $a = 0$, we have that $(0, 0) \leq (a, x)$ whence $0 \leq x$. This implies that $(a, x) \in \{(0, x) \in \mathbb{Z}_{\mathcal{E}} \times G \mid x \geq 0\}$. If instead $a = 1$ we have that $(a, x) \leq (1, 0)$, whence $x \leq 0$ and $(a, x) \in \{(1, x) \in \mathbb{Z}_{\mathcal{E}} \times G \mid x \leq 0\}$.

Claim 2. Given $(a, x) \in \Sigma_{\mathcal{E}}(G)$, if $(a, x) \in Rad(\mathbb{Z}_{\mathcal{E}} \times_{\text{lex}} \mathcal{G})$ then

$$(0, 0) \leq (a, x) \leq (1, 0);$$
$$(a, x) \leq \neg(a, x) = (1 - a, -x).$$

It follows that $(a, x) = (0, x)$ with $x \geq 0$. Conversely, for any $x \geq 0$, $(0, x) \leq \neg(0, x) = (1, -x)$; thus, $(0, x) \in Rad(\mathbb{Z}_{\mathcal{E}} \times_{\text{lex}} \mathcal{G})$.

Claim 3. The proof is analogous to that of Claim 2.

To conclude our proof, it remains to show that $\Sigma_{\mathcal{E}}(\mathcal{G})$ satisfies axiom C. This is straightforward, using the decomposition of the algebra as the disjoint union of its radical and coradical. □

Let $h : \mathcal{G} \to \mathcal{G}'$ be an ℓ-homomorphism in $\mathcal{E}$. We define the following arrow in $\mathcal{E}$ by using the internal language:

$$h^* : (a, x) \in \mathbb{Z}_{\mathcal{E}} \times G \to (a, h(x)) \in \mathbb{Z}_{\mathcal{E}} \times G'.$$

This is trivially an ℓ-homomorphism which preserves the strong unit $(1, 0)$. We set $\Sigma_{\mathcal{E}}(h) := \Gamma(h^*) = h^*|_{\Gamma(\mathbb{Z}_{\mathcal{E}} \times G)}$.

Proposition 4.2.3. *$\Sigma_{\mathcal{E}}$ is a functor from $\mathbb{L}$-mod($\mathcal{E}$) to $\mathbb{P}$-mod($\mathcal{E}$).*

Proof. This easily follows from the fact that $\Gamma_{\mathcal{E}}$ is a functor. □

4.3 From models of $\mathbb{P}$ to models of $\mathbb{L}$

Lemma 4.3.1. *Let $\mathcal{A}$ be a $\mathbb{P}$-model in $\mathcal{E}$. The structure $(Rad(\mathcal{A}), \oplus, \leq, \inf, \sup, 0)$ is a cancellative lattice-ordered abelian monoid in $\mathcal{E}$, i.e., it is a model in $\mathcal{E}$ of the theory whose axioms are L.1-L.12 (except axiom L.3) plus*

C. $(x + a = y + a \vdash_{x,y,a} x = y)$.

Proof. From Lemma 2.2.10(i)-(ii)-(iii)-(v)-(vi)-(vii) it follows that $Rad(\mathcal{A})$ is a lattice-ordered abelian monoid. Given $x, y, a \in Rad(\mathcal{A})$ such that $x \oplus a = y \oplus a$, we have that:

$$\neg a \odot (x \oplus a) = \neg a \odot (y \oplus a) \Leftrightarrow \inf(\neg a, x) = \inf(\neg a, y).$$

Lemma 2.2.10(ix) thus implies that $x = y$. This completes the proof. □

Let us call $\Delta_{\mathcal{E}}(\mathcal{A})$ the Grothendieck group associated with the monoid $Rad(\mathcal{A})$. The constant, the order relation and the operations on $\Delta_{\mathcal{E}}(\mathcal{A})$ are defined by using the internal language of the topos $\mathcal{E}$ as in Section 3.4.

Any MV-homomorphism $h : \mathcal{A} \to \mathcal{A}'$ between perfect MV-algebras preserves the natural order, thus $h(Rad(\mathcal{A})) \subseteq Rad(\mathcal{A}')$. Hence the arrow $h^* := h|_{Rad(\mathcal{A})} : Rad(\mathcal{A}) \to Rad(\mathcal{A}')$ is a lattice-ordered monoid homomorphism. We set

$$\Delta_{\mathcal{E}}(h) : (x, y) \in \Delta_{\mathcal{E}}(\mathcal{A}) \to [h^*(x), h^*(y)] \in \Delta_{\mathcal{E}}(\mathcal{A}').$$

Proposition 4.3.2. *$\Delta_{\mathcal{E}}$ is a functor from $\mathbb{P}$-mod($\mathcal{E}$) to $\mathbb{L}$-mod($\mathcal{E}$).*

Proof. This follows by a straightforward computation. □

4.4 The Morita-equivalence between $\mathbb{P}$ and $\mathbb{L}$

In the previous sections we have defined, for each Grothendieck topos $\mathcal{E}$, two functors:

$$\Sigma_{\mathcal{E}} : \mathbb{L}\text{-mod}(\mathcal{E}) \to \mathbb{P}\text{-mod}(\mathcal{E});$$

$$\Delta_{\mathcal{E}} : \mathbb{MV}\text{-mod}(\mathcal{E}) \to \mathbb{L}\text{-mod}(\mathcal{E}).$$

Theorem 4.4.1. *For every Grothendieck topos $\mathcal{E}$, the categories $\mathbb{P}$-mod($\mathcal{E}$) and $\mathbb{L}$-mod($\mathcal{E}$) are naturally equivalent.*

Proof. We have to define two natural isomorphisms

$$\alpha : 1_{\mathbb{L}} \to \Delta_{\mathcal{E}} \circ \Sigma_{\mathcal{E}},$$

$$\beta : 1_{\mathbb{P}} \to \Sigma_{\mathcal{E}} \circ \Delta_{\mathcal{E}},$$

where $1_{\mathbb{L}}$ and $1_{\mathbb{P}}$ are, respectively, the identity functors on the categories $\mathbb{L}$-mod($\mathcal{E}$) and $\mathbb{P}$-mod($\mathcal{E}$).

Let $\mathcal{G} = (G, +, -, \leq, \inf, \sup, 0)$ be an ℓ-group in $\mathcal{E}$. Let $\alpha_{\mathcal{G}} : \mathcal{G} \to (\Delta_{\mathcal{E}} \circ \Sigma_{\mathcal{E}})(\mathcal{G})$ be the arrow defined by using the internal language of the topos as follows:

$$\alpha_{\mathcal{G}} : g \in G \to [(0, g^+), (0, g^+ - g)] \in \Delta_{\mathcal{E}}(\Sigma_{\mathcal{E}}(G)).$$

Claim 1. $\alpha_{\mathcal{G}}$ is monic. Indeed, for any elements $g_1, g_2 \in G$ such that $\alpha_{\mathcal{G}}(g_1) = \alpha_{\mathcal{G}}(g_2)$, we have that $(0, g_1^+) + (0, g_2^+ - g_2) = (0, g_2^+) + (0, g_1^+ - g_1)$, whence $g_1 = g_2$. The monicity of $\alpha_{\mathcal{G}}$ thus follows from Proposition 1.2.13(iii).

Claim 2. $\alpha_{\mathcal{G}}$ is epic. Given $[(0, g_1), (0, g_2)] \in \Delta_{\mathcal{E}}(\Sigma_{\mathcal{E}}(G))$, the element $g_1 - g_2$ satisfies $\alpha_{\mathcal{G}}(g_1 - g_2) = [(0, (g_1 - g_2)^+), (0, (g_1 - g_2)^+ - (g_1 - g_2))] = [(0, g_1), (0, g_2)]$. Proposition 1.2.13(iv) thus implies that $\alpha_{\mathcal{G}}$ is an epimorphisms.

Claim 3. $\alpha_{\mathcal{G}}$ preserves $+$ and $-$. This follows by direct computation.

Claim 4. $\alpha_{\mathcal{G}}$ preserves inf and sup. Given $g_1, g_2 \in G$,

$$\alpha_{\mathcal{G}}(\sup(g_1, g_2)) = [(0, \sup(g_1, g_2)^+), (0, \sup(g_1, g_2)^+ - \sup(g_1, g_2))];$$

$$\sup(\alpha_{\mathcal{G}}(g_1), \alpha_{\mathcal{G}}(g_2)) = [(0, g_1^+ + g_2^+), \inf((0, (g_1^+ + g_2^+ - g_2), (g_2^+ + g_1^+ - g_1)))].$$

Now, the sequent

$$\top \vdash_{g_1,g_2} \sup(g_1,g_2)^+ + \inf((g_1^+ + g_2^+ - g_2),(g_2^+ + g_1^+ - g_1)) = \\ g_1^+ + g_2^+ + \sup(g_1,g_2)^+ - \sup(g_1,g_2)$$

is provable in $\mathbb{L}$, hence it holds in every $\mathbb{L}$-model by soundness. This ensures that $\alpha_{\mathcal{G}}$ preserves sup. In a similar way it can be shown that $\alpha_{\mathcal{G}}$ preserves inf.

By Claims 1-4 the arrow $\alpha_{\mathcal{G}}$ is an isomorphism in $\mathbb{L}$-mod$(\mathcal{E})$. Further, it is easy to prove that for any ℓ-homomorphism $h : \mathcal{G} \to \mathcal{G}'$ in $\mathcal{E}$, the following square commutes:

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{\alpha_{\mathcal{G}}} & \Delta_{\mathcal{E}} \circ \Sigma_{\mathcal{E}}(\mathcal{G}) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \Delta_{\mathcal{E}} \circ \Sigma_{\mathcal{E}}(h)} \\ \mathcal{G}' & \xrightarrow[\alpha_{\mathcal{G}'}]{} & \Delta_{\mathcal{E}} \circ \Sigma_{\mathcal{E}}(\mathcal{G}') \end{array}$$

We set α equal to the natural isomorphism whose components are the $\alpha_{\mathcal{G}}$ (for every ℓ-group $\mathcal{G}$).

In the converse direction, let $\mathcal{A}$ be a perfect MV-algebra in $\mathcal{E}$. Recall that $A = Rad(\mathcal{A}) \cup \neg Rad(\mathcal{A})$ and that the sequent C holds in $\mathcal{A}$. We define the following arrow by using the internal language

$$\beta_{\mathcal{A}} : x \in A \to \begin{cases} (0,[x,0]) & \text{for } x \in Rad(\mathcal{A}) \\ (1,[0,\neg x]) & \text{for } x \in \neg Rad(\mathcal{A}) \end{cases} \in \Sigma_{\mathcal{E}}(\Delta_{\mathcal{E}}(\mathcal{A}))$$

Let us prove that $\beta_{\mathcal{A}}$ preserves $\oplus$. Given $x, y \in A$, we can distinguish three cases:

Case i. $x, y \in Rad(\mathcal{A})$. By direct computation it follows at once that $\beta_{\mathcal{A}}(x \oplus y) = \beta_{\mathcal{A}}(x) \oplus \beta_{\mathcal{A}}(y)$.

Case ii. $x, y \in \neg Rad(\mathcal{A})$. From Lemma 2.2.10(viii) we have that $x \oplus y = 1$; thus $\beta_{\mathcal{A}}(x \oplus y) = (1, [0, 0])$. On the other hand, $\beta_{\mathcal{A}}(x) = (1, [0, \neg x])$ and $\beta_{\mathcal{A}}(y) = (1, [0, \neg y])$, whence $\beta_{\mathcal{A}}(x), \beta_{\mathcal{A}}(y) \in \neg Rad(\Sigma_{\mathcal{E}}(\Delta_{\mathcal{E}}(\mathcal{A})))$ and $\beta_{\mathcal{A}}(x) \oplus \beta_{\mathcal{A}}(y) = (1, [0, 0])$.

Case iii. $x \in Rad(\mathcal{A}), y \in \neg Rad(\mathcal{A})$. In a similar way we obtain that $\beta_{\mathcal{A}}(x \oplus y) = \beta_{\mathcal{A}}(x) \oplus \beta_{\mathcal{A}}(y)$.

The fact that $\beta_{\mathcal{A}}$ preserves $\neg$ and is both monic and epic is clear. We can thus conclude that $\beta_{\mathcal{A}}$ is an isomorphism.

It is clear that if $h : \mathcal{A} \to \mathcal{A}'$ is an MV-homomorphism then following square commutes:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\beta_{\mathcal{A}}} & \Sigma_{\mathcal{E}}(\Delta_{\mathcal{E}}(\mathcal{A})) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \Sigma_{\mathcal{E}}(\Delta_{\mathcal{E}}(h))} \\ \mathcal{A}' & \xrightarrow[\beta_{\mathcal{A}'}]{} & \Sigma_{\mathcal{E}}(\Delta_{\mathcal{E}}(\mathcal{A}')) \end{array}$$

Thus, we have a natural isomorphism β whose components are the arrows $\beta_{\mathcal{A}}$ (for every perfect MV-algebra $\mathcal{A}$). □

Note that all the constructions that we used to define the functors $\Sigma_{\mathcal{E}}$ and $\Delta_{\mathcal{E}}$ are geometric. Hence, the categorical equivalence proved in the last theorem is natural in the topos $\mathcal{E}$. This implies that the classifying toposes $\mathcal{E}_{\mathbb{P}}$ and $\mathcal{E}_{\mathbb{L}}$ are equivalent, i.e., that the theories $\mathbb{P}$ and $\mathbb{L}$ are Morita-equivalent. Summarizing, we have the following

Theorem 4.4.2. *The functors $\Delta_{\mathcal{E}}$ and $\Sigma_{\mathcal{E}}$ yield a Morita-equivalence between the coherent theory $\mathbb{P}$ of perfect MV-algebras and the cartesian theory $\mathbb{L}$ of lattice-ordered abelian groups.*

4.5 Interpretability between $\mathbb{L}$ and $\mathbb{P}$

We proved that the theories $\mathbb{P}$ and $\mathbb{L}$ are Morita-equivalent by establishing a categorical equivalence between the categories of models of these two theories in any Grothendieck topos $\mathcal{E}$, naturally in $\mathcal{E}$. This result would be trivial if the theories were bi-interpretable. In this section we show that this is not the case, i.e., the theories $\mathbb{P}$ and $\mathbb{L}$ are not bi-intepretable in a global sense. Nevertheless, if we consider particular categories of formulas we have three different levels of bi-interpretability. To better understand these partial levels of bi-interpretation we pass through an intermediary Morita-equivalence.

4.5.1 The bi-interpretability between the theory of ℓ-groups and that of their positive cones

We establish a Morita-equivalence involving the theory $\mathbb{L}$ that we will be useful in particular for obtaining an explicit description of the partial bi-interpretability between the theories $\mathbb{P}$ and $\mathbb{L}$ relating $\mathbb{P}$-irreducible formualas and $\mathbb{L}$-cartesian fourmulas. This stems from the observation that the ℓ-groups arising in the context of MV-algebras as the counterparts of MV-algebras via Mundici's functor, as well as those which correspond to perfect MV-algebras under Di Nola and Lettieri's equivalence, are determined by their positive cones. One can naturally axiomatize the monoids arising as the positive cones of such groups in such a way as to obtain a theory which is Morita-equivalent to (in fact, bi-interpretable in) that of ℓ-groups.

Specifically, let Σ_M be the one-sorted first-order signature consisting of three function symbols $+$, $\inf$, $\sup$, a constant symbol 0 and a derivable relation symbol: $x \leq y$ iff $\inf(x, y) = x$. Over this signature we define the theory $\mathbb{M}$, obtained from that of abelian partially-ordered monoids by adding the following sequents:

M.1 $(x \leq y \vdash_{x,y,t} t + x \leq t + y)$;

M.2 $(x + y = x + z \vdash_{x,y,z} y = z)$;

M.3 $(\top \vdash_{x} 0 \leq x)$;

M.4 $(x \leq y \vdash_{x,y} (\exists z)x + z = y)$.

We call $\mathbb{M}$ the theory of *cancellative subtractive lattice-ordered abelian monoids with bottom element*. This theory is cartesian; indeed, by the cancellation property, the existential quantification of the last axiom is provably unique. In the latter sequent the unique element z satisfying $x + z = y$ will be denoted by $y - x$.

Notice that the sequent $(x+z \leq y+z \vdash_{x,y,z} x \leq y)$ is provable in $\mathbb{M}$. From this it easily follows that the sequent $(\top \vdash_{a,b,c} \inf(a, b) + c = \inf(a+c, b+c))$ is also provable in $\mathbb{M}$.

The models of $\mathbb{M}$ are particular lattice-ordered abelian monoids. We shall prove that $\mathbb{M}$ is the theory of positive cones of ℓ-groups.

Let $\mathcal{M} = (M, +, \leq, \inf, \sup, 0)$ be a model of $\mathbb{M}$ in an arbitrary Grothendieck topos $\mathcal{E}$. The lattice-ordered Grothendieck group $G(\mathcal{M})$ associated with $\mathcal{M}$ has as underlying object the quotient of $M \times M$ under the following equivalent relation: $(x, y) \sim (h, k)$ if and only if $x + k = y + h$. This equivalence relation, as well as the operations and the order relation below, is defined by using the internal language of the topos. The operations are defined as usual (cf. Section 3.4).

Notice that, for every perfect MV-algebra $\mathcal{A}$, $\Delta(\mathrm{Rad}(\mathcal{A}))$ is the lattice-ordered Grothendieck group $G(\mathrm{Rad}(\mathcal{A}))$ associated with $\mathrm{Rad}(\mathcal{A})$, where the latter is regarded as a model of $\mathbb{M}$.

Theorem 4.5.1. *The theories* $\mathbb{M}$ *and* $\mathbb{L}$ *are Morita-equivalent.*

Proof. We need to prove that the categories of models of the two theories in any Grothendieck topos $\mathcal{E}$ are equivalent, naturally in $\mathcal{E}$.

Let $\mathcal{E}$ be a Grothendieck topos. We can define two functors.

- $T_{\mathcal{E}} : \mathbb{M}\text{-mod}(\mathcal{E}) \to \mathbb{L}\text{-mod}(\mathcal{E})$. For any monoid $\mathcal{M}$ in $\mathbb{M}\text{-mod}(\mathcal{E})$ we set $T_{\mathcal{E}}(\mathcal{M})$ to be the Grothendieck group $G(\mathcal{M})$. For an $\mathbb{M}$-model homomorphism $f : \mathcal{M} \to \mathcal{N}$, we set $T_{\mathcal{E}}(f)$ equal to the function $f^* : G(\mathcal{M}) \to G(\mathcal{N})$ defined by using the internal language of the topos $\mathcal{E}$ as $f^*([x, y]) = [f(x), f(y)]$.

- $R_{\mathcal{E}} : \mathbb{L}\text{-mod}(\mathcal{E}) \to \mathbb{M}\text{-mod}(\mathcal{E})$. For every ℓ-group $\mathcal{G}$ in $\mathbb{L}\text{-mod}(\mathcal{E})$, its positive cone is trivially a model of $\mathbb{M}$. We set $R_{\mathcal{E}}(\mathcal{G}) = (G^+, +, \leq, \inf, \sup, 0)$, where $+, \leq, \inf, \sup$ are the restrictions to the positive cone of $\mathcal{G}$ of the operations and of the order of $\mathcal{G}$. Since every ℓ-homomorphism preserves the order, we can set $R_{\mathcal{E}}(g) = g|_{G^+}$.

These two functors are categorical inverses to each other. Indeed, we can define two natural isomorphisms $T_{\mathcal{E}} \circ R_{\mathcal{E}}(\mathcal{G}) \cong \mathcal{G}$ and $R_{\mathcal{E}} \circ T_{\mathcal{E}}(\mathcal{M}) \cong \mathcal{M}$ (for every ℓ-group $\mathcal{G}$ and for every model $\mathcal{M}$ of $\mathbb{M}$ in an arbitrary Grothendieck topos $\mathcal{E}$).

Let $\mathcal{M}$ be a model of $\mathbb{M}$ in $\mathcal{E}$. The arrow $\phi_{\mathcal{M}} : \mathcal{M} \to G(\mathcal{M})^+$ with $\phi_{\mathcal{M}}(x) := [x, 0]$ is an isomorphism.

- $\phi_{\mathcal{M}}$ is injective: given $x, y \in M$, $[x, 0] = [y, 0]$ iff $x = y$.

- $\phi_{\mathcal{M}}$ is surjective: given $[x, y] \in G(\mathcal{M})^+$, this means that

$$[0,0] \leq [x,y] \Leftrightarrow \mathrm{Inf}([0,0],[x,y]) = [0,0] \Leftrightarrow [\inf(x,y), y] = [0,0] \Leftrightarrow \inf(x,y) = y \Leftrightarrow y \leq x.$$

  By axiom M.4, there exists $z \in M$ such that $x = z + y$. Thus, $[x, y] = [z, 0] = \phi_{\mathcal{M}}(z)$.

- $\phi_{\mathcal{M}}$ preserves $+$: given $x, y \in M$, $\phi_{\mathcal{M}}(x) + \phi_{\mathcal{M}}(y) = [x, 0] + [y, 0] = [x + y, 0] = \phi_{\mathcal{M}}(x + y)$.

In a similar way we can prove that $\phi_{\mathcal{M}}$ preserves the other ℓ-group operations whence the order relation.

Let $\mathcal{G}$ be a model of $\mathbb{L}$ in $\mathcal{E}$. The arrow $\chi_{\mathcal{G}} : G \to G(R_{\mathcal{E}}(\mathcal{G}))$ with $\chi_{\mathcal{G}}(g) := [g^+, g^-]$ is an isomorphism.

- $\chi_{\mathcal{G}}$ is injective: given $g, h \in G$ such that $[g^+, g^-] = [h^+, h^-]$, we have

$$g^+ + h^- = g^- + h^+ \text{ iff } g^+ - g^- = h^+ - h^- \text{ iff } g = h.$$

- $\chi_{\mathcal{G}}$ is surjective: given $[x, y] \in G(R_{\mathcal{E}}(\mathcal{G}))$, there exists $g = x - y$ in G.

$$[g^+, g^-] = [x, y] \text{ iff } g^+ + y = g^- + x \text{ iff } g^+ - g^- = x - y.$$

  Thus, $\chi_{\mathcal{G}}(g) = [x, y]$.

- $\chi_{\mathcal{G}}$ preserves $+$: given $g, h \in G$, we have that $\chi_{\mathcal{G}}(g+h) = [(g+h)^+, (g+h)^-]$ and $\chi_{\mathcal{G}}(g) + \chi_{\mathcal{G}}(h) = [g^+ + h^+, g^- + h^-]$. These two elements are equal iff

$$(g+h)^+ + g^- + h^- = (g+h)^- + g^+ + h^+ \text{ iff}$$
$$(g+h)^+ - (g+h)^- = g^+ - g^- + h^+ - h^- \text{ iff } g + h = g + h.$$

- $\chi_{\mathcal{G}}$ preserves $-$: given $g \in G$. We have that $\chi_{\mathcal{G}}(-g) = [(-g)^+, (-g)^-]$ and $-\chi_{\mathcal{G}}(g) = [g^-, g^+]$. These two elements are equal iff

$$(-g)^+ + g^+ = (-g)^- + g^- \text{ iff } (-g)^+ - (-g)^- = g^- - g^+ \text{ iff } -g = -g.$$

It is easy to check that $\chi_{\mathcal{G}}$ is a homomorphism.

Finally, the categorical equivalence just established is natural in $\mathcal{E}$; indeed, all the constructions that we have used are geometric. □

To this pair of Morita-equivalent theories we can apply the bridge technique. Since the theories $\mathbb{M}$ and $\mathbb{L}$ are cartesian, they are both of presheaf type. In this case, an interesting invariant to consider is the notion of irreducible objects of the classifying topos.

Remark 4.5.2. For any two Morita-equivalent theories of presheaf type $\mathbb{T}$ and $\mathbb{T}'$, the equivalence of classifying toposes $[\mathcal{C}_{\mathbb{T}}^{\mathrm{irr}\,\mathrm{op}}, \mathbf{Set}] \simeq [\mathcal{C}_{\mathbb{T}'}^{\mathrm{irr}\,\mathrm{op}}, \mathbf{Set}]$ restricts to the full subcategories $\mathcal{C}_{\mathbb{T}}^{\mathrm{irr}}$ and $\mathcal{C}_{\mathbb{T}'}^{\mathrm{irr}}$ of irreducible objects, yielding an equivalence

$$\mathcal{C}_{\mathbb{T}}^{\mathrm{irr}} \simeq \mathcal{C}_{\mathbb{T}'}^{\mathrm{irr}} .$$

Applying this to our theories, we obtain a categorical equivalence

$$\mathcal{C}_{\mathbb{M}}^{\mathrm{irr}} \simeq \mathcal{C}_{\mathbb{L}}^{\mathrm{irr}},$$

which we can explicitly describe as follows. Since both the theories $\mathbb{M}$ and $\mathbb{L}$ are cartesian, we have natural equivalences $\mathcal{C}_{\mathbb{M}}^{\mathrm{irr}} \simeq \mathcal{C}_{\mathbb{M}}^{\mathrm{cart}}$ and $\mathcal{C}_{\mathbb{L}}^{\mathrm{irr}} \simeq \mathcal{C}_{\mathbb{L}}^{\mathrm{cart}}$. In fact, the $\mathbb{T}$-irreducible formulas for a cartesian theory $\mathbb{T}$ are precisely the $\mathbb{T}$-cartesian ones (up to isomorphism in the syntactic category).

Recall that for any cartesian theory $\mathbb{T}$ and cartesian category $\mathcal{C}$, we have a categorical equivalence

$$\mathbf{Cart}(\mathcal{C}_{\mathbb{T}}^{\mathrm{cart}}, \mathcal{C}) \simeq \mathbb{T}\text{-mod}(\mathcal{C}),$$

where $\mathbf{Cart}(\mathcal{C}, \mathcal{D})$ is the category of cartesian functors between cartesian categories $\mathcal{C}$ and $\mathcal{D}$. In the category $\mathcal{C}_{\mathbb{L}}^{\mathrm{cart}}$ there is a canonical model of $\mathbb{L}$ given by the structure $G_{\mathbb{L}} = (\{x \,.\, \top\}, +, -, \leq, \inf, \sup, 0)$. It is immediate to see that we can restrict the operations $+$, inf and sup on $G_{\mathbb{L}}$ to the subobject $\{x \,.\, x \geq 0\}$ of $\{x \,.\, \top\}$. The resulting structure $(\{x \,.\, x \geq 0\}, +, \leq, \inf, \sup, 0)$ is a model U of $\mathbb{M}$ in $\mathcal{C}_{\mathbb{L}}$.

In the converse direction, consider the syntactic category $\mathcal{C}_{\mathbb{M}}^{\mathrm{cart}}$ of $\mathbb{M}$ and the canonical model $M_{\mathbb{M}} = (\{y \,.\, \top\}, +, \leq, \inf, \sup, 0)$ of $\mathbb{M}$ in it. The ℓ-group associated with a model $\mathcal{M}$ of $\mathbb{M}$ in an arbitrary Grothendieck topos $\mathcal{E}$ via the Morita-equivalence described above is the Grothendieck group of $\mathcal{M}$, whose elements, we recall, are equivalence classes $[x, y]$ of pairs of elements of $\mathcal{M}$. Given a pair of elements (x, y) of $M_{\mathbb{M}}$, consider $\inf(x, y)$; since $\inf(x, y) \leq x$ and $\inf(x, y) \leq y$, by axiom M.4 there exist exactly two elements u, v such that $x = \inf(x, y) + u$ and $y = \inf(x, y) + v$. These elements clearly satisfy $[x, y] = [u, v]$; moreover, $\inf(u, v) = 0$. Indeed,

$$\inf(u, v) + \inf(x, y) = \inf(u + \inf(x, y), v + \inf(x, y)) = \inf(x, y),$$

whence $\inf(u, v) = 0$.

Note that the pair (u, v) does not depend on the equivalence class of (x, y). Indeed, if $[x, y] = [u', v']$ and $\inf(u', v') = 0$ then $x + v' = y + u'$ and the following identities hold:

$$\inf(x, y) + u' = \inf(x + u', y + u') = \inf(x + u', x + v') = x + \inf(u', v') = x,$$

which implies that $u = u'$. In an analogous way we can prove that $v = v'$.

This allows us to choose the pair (u, v) defined above as a canonical representative for the equivalence class $[x, y]$ in $G(M_{\mathbb{M}})$.

We are thus led to consider the following structure in $\mathcal{C}_{\mathbb{M}}^{\text{cart}}$:

- underlying object: $\{(u, v) \ . \ \inf(u, v) = 0\}$;
- sum: $[z + v + b = t + u + a \wedge \inf(z, t) = 0] : \{(u, v) \ . \ \inf(u, v) = 0\} \times \{(a, b) \ . \ a \wedge b = 0\} \to \{(z, t) \ . \ z \wedge t = 0\}$;
- opposite: $[a = v \wedge b = u] : \{(u, v) \ . \ \inf(u, v) = 0\} \to \{(a, b) \ . \ \inf(a, b) = 0\}$;
- zero: $[u = 0, v = 0] : \{[] \ . \ \top\} \to \{(u, v) \ . \ \inf(u, v) = 0\}$;
- Inf : $[z + u + b = t + \inf(u + b, v + a) \wedge \inf(z, t) = 0] : \{(u, v) \ . \ \inf(u, v) = 0\} \times \{(a, b) \ . \ \inf(a, b) = 0\} \to \{(z, t) \ . \ \inf(z, t) = 0\}$;
- Sup : $[z + u + b = t + \sup(u + b, v + a) \wedge \inf(z, t) = 0] : \{(u, v) \ . \ \inf(u, v) = 0\} \times \{(a, b) \ . \ \inf(a, b) = 0\} \to \{(z, t) \ . \ \inf(z, t) = 0\}$.

It can be easily seen that this structure is a model V of $\mathbb{L}$ inside $\mathcal{C}_{\mathbb{M}}^{\text{cart}}$.

Let $F_U : \mathcal{C}_{\mathbb{M}}^{\text{cart}} \to \mathcal{C}_{\mathbb{L}}^{\text{cart}}$ and $F_V : \mathcal{C}_{\mathbb{L}}^{\text{cart}} \to \mathcal{C}_{\mathbb{M}}^{\text{cart}}$ be the cartesian functors respectively induced by the models U and V. For every object $\{\vec{x} \ . \ \phi\}$ of

$\mathcal{C}_{\mathbb{M}}$, $F_U(\{\vec{x}\ .\ \phi\}) := \{\vec{x}\ .\ \phi \wedge \vec{x} \geq 0\}$. The functor F_V admits the following inductive definition:

- $F_V(\{\vec{y}\ .\ \top\}) := \{(\vec{u}, \vec{v})\ .\ \inf(\vec{u}, \vec{v}) = 0\}$;
- $F_V((\{\vec{y}, \vec{x})\ .\ \vec{y} + \vec{x}\}) := \{(\vec{u}, \vec{v}, \vec{a}, \vec{b})\ .\ (\vec{u}, \vec{v}) + (\vec{a}, \vec{b}) \wedge \inf(\vec{u}, \vec{v}) = 0 \wedge \inf(\vec{a}, \vec{b}) = 0\}$;
- $F_V(\{\vec{y}\ .\ -\vec{y}\}) := \{(\vec{v}, \vec{u})\ .\ \inf(\vec{v}, \vec{u}) = 0\}$;
- $F_V(\{(\vec{y}, \vec{x})\ .\ \mathrm{Inf}(\vec{y}, \vec{x})\}) := \{(\vec{u}, \vec{v}, \vec{a}, \vec{b})\ .\ (\inf(\vec{u}, \vec{a}), \inf(\vec{v}, \vec{b})) \wedge \inf(\vec{u}, \vec{v}) = 0 \wedge \inf(\vec{a}, \vec{b}) = 0\}$;
- $F_V(\{(\vec{y}, \vec{x})\ .\ \mathrm{Sup}(\vec{y}, \vec{x})\}) := \{(\vec{u}, \vec{v}, \vec{a}, \vec{b})\ .\ (\sup(\vec{u}, \vec{a}), \sup(\vec{v}, \vec{b})) \wedge \inf(\vec{u}, \vec{v}) = 0 \wedge \inf(\vec{a}, \vec{b}) = 0\}$.

Let us now proceed to show that the functors F_U and F_V are categorical inverses to each other.

Claim 1. The formulas $\{x\ .\ \top\}$ and $\{(u, v)\ .\ \inf(u, v) = 0\}$ are isomorphic in $\mathcal{C}_{\mathbb{M}}^{\mathrm{cart}}$.

To see this, consider the following arrow in $\mathcal{C}_{\mathbb{M}}$:

$$[u = x, v = 0] : \{x\ .\ \top\} \to \{(u, v)\ .\ \inf(u, v) = 0 \wedge (u, v) \geq (0, 0)\}.$$

All the "elements" of the object $\{(u, v)\ .\ \inf(u, v) = 0 \wedge (u, v) \geq (0, 0)\}$ are of the form $(u, 0)$; indeed, $(u, v) \geq (0, 0)$ iff $\inf((0, 0), (u, v)) = (0, 0)$, and this means that $v = \inf(u, v) = 0$. It follows that the arrow just defined is an isomorphism.

Claim 2. The formulas $\{x\ .\ \top\}$ and $\{(u, v)\ .\ (u \wedge v = 0) \wedge u \geq 0 \wedge v \geq 0\}$ are isomorphic in $\mathcal{C}_{\mathbb{L}}^{\mathrm{cart}}$.

To see this, consider the following arrow in $\mathcal{C}_{\mathbb{L}}$:

$$[u = x^+, v = x^-] : \{x \,.\, \top\} \to \{(u, v) \,.\, (u \wedge v = 0) \wedge u \geq 0 \wedge v \geq 0\}.$$

It is well-defined because $\inf(x^+, x^-) = 0$. In addition, taken u, v such that $\inf(u, v) = 0$, we can consider $x = u - v$. We have that:

$$u = (u - v)^+ \text{ and } v = (u - v)^- \text{ iff}$$

$$v + u = v + (u - v)^+ \text{ and } v + u = u + (u - v)^- \text{ iff}$$

$$v + u = v + (\sup((u - v), 0)) \text{ and } v + u = u + (\sup((v - u), 0)) \text{ iff}$$

(Proposition 1.2.2 [6])

$$v + u = \sup((v + u - v), (v + 0)) \text{ and } v + u = \sup((u + v - u), (u + 0))$$

iff

$$v + u = \sup(u, v) \text{ and } v + u = \sup(v, u).$$

But the sequent $\top \vdash_{x,y} x + y = \sup(x, y) + \inf(x, y)$ is provable in $\mathbb{L}$ (cf. Proposition 1.2.6 [6]). Hence, the arrow $[u = x^+, v = x^-]$ is an isomorphism, as required.

From Claim 1 and Claim 2 it follows at once that the functors F_U and F_V are categorical inverses to each other.

Summarizing, we have the following result.

Proposition 4.5.3. *The functors $F_U : \mathcal{C}_{\mathbb{M}}^{cart} \to \mathcal{C}_{\mathbb{L}}^{cart}$ and $F_V : \mathcal{C}_{\mathbb{L}}^{cart} \to \mathcal{C}_{\mathbb{M}}^{cart}$ defined above form a categorical equivalence. Being the theories $\mathbb{M}$ and $\mathbb{L}$ both cartesian, this means that they are bi-interpretable.*

Lemma 4.5.4. *The ℓ-Grothendieck group $G(\mathcal{M})$ associated with a model of $\mathcal{M}$ of $\mathbb{M}$ in* **Set** *satisfies the following universal property:*

(*) there exists an ℓ-monoid homomorphism $i : \mathcal{M} \to G(\mathcal{M})$ of models such that for every ℓ-monoid homomorphism $f : \mathcal{M} \to \mathcal{H}$, where H is an ℓ-group, there exists a unique ℓ-group homomorphism $g : G(\mathcal{M}) \to \mathcal{H}$ such that $f = g \circ i$.

Proof. Set $i : \mathcal{M} \to G(M)$ equal to the function $i(x) = [x, 0]$. This is an ℓ-monoid homomorphism since it is the composite of the ℓ-monoid isomorphism $\phi_{\mathcal{M}} : \mathcal{M} \to G(\mathcal{M})^{+}$ considered in the proof of Theorem 4.5.1 with the inclusion $G(\mathcal{M})^{+} \hookrightarrow G(\mathcal{M})$, which is an ℓ-monoid homomorphism since the ℓ-monoid structure on $G(\mathcal{M})^{+}$ is induced by restriction of that on $G(\mathcal{M})$.

Given an ℓ-monoid homomorphism $f : \mathcal{M} \to \mathcal{H}$, where H is an ℓ-group, in order to have $g \circ i = f$, we are forced to define g as $g : [x, y] \in G(\mathcal{M}) \to f(x) - f(y) \in \mathcal{H}$. This is clearly a well-defined group homomorphism. It remains to show that it also preserves the lattice structure.

- g preserves Inf: $g(\mathrm{Inf}([x, y], [h, k])) = g([\inf(x + k, y + h), y + k]) = \inf(f(x+k), f(y+h)) - f(y+k) = \inf(f(x+k) - f(y+k), f(y+h) - f(y+k)) = \inf(f(x) - f(y), f(h) - f(k)) = \mathrm{Inf}(g([x, y]), g([h, k]))$.
- g preserves Sup: the proof is analogous to that for Inf. □

Proposition 4.5.5. *The functors F_U and F_V correspond to the functors $T_{\mathbf{Set}}$ and $R_{\mathbf{Set}}$ under the canonical equivalences $\mathcal{C}_{\mathbb{M}}^{cart} \simeq f\,.\,p\,.\,\mathbb{M}\text{-}mod(\mathbf{Set})^{op}$ and $\mathcal{C}_{\mathbb{L}}^{cart} \simeq f\,.\,p\,.\,\mathbb{L}\text{-}mod(\mathbf{Set})^{op}$:*

$$
\begin{array}{ccc}
\mathcal{C}_{\mathbb{L}}^{cart} & \simeq & f.p.\mathbb{L}\text{-}mod(\mathbf{Set})^{op} \\
F_U \uparrow & & \uparrow T_{\mathbf{Set}} \\
\mathcal{C}_{\mathbb{M}}^{cart} & \simeq & f.p.\mathbb{M}\text{-}mod(\mathbf{Set})^{op}
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{C}_{\mathbb{L}}^{cart} & \simeq & f.p.\mathbb{L}\text{-}mod(\mathbf{Set})^{op} \\
F_V \downarrow & & \downarrow R_{\mathbf{Set}} \\
\mathcal{C}_{\mathbb{M}}^{cart} & \simeq & f.p.\mathbb{M}\text{-}mod(\mathbf{Set})^{op}
\end{array}
$$

Proof. Since $T_{\mathbf{Set}}$, $R_{\mathbf{Set}}$ and F_U, F_V are respectively categorical inverses to each other, it is sufficient to prove that the diagram on the left-hand side commutes (up to natural isomorphism).

From Lemma 4.5.4 it follows that if $\mathcal{N}$ is a model of $\mathbb{M}$ presented by a formula $\{\vec{x}\,.\,\phi\}$ in $\mathbb{M}$, then the model $T_{\mathbf{Set}}(\mathcal{N})$ of $\mathbb{L}$ is presented by the formula $\{\vec{x}\,.\,\phi \wedge \vec{x} \geq 0\}$, that is by the image of the object $\{\vec{x}\,.\,\phi\}$ under the functor F_U. This immediately implies our thesis. □

Remark 4.5.6. From Proposition 4.5.5 it follows in particular that $\mathbb{N} \times \mathbb{N}$, as a model of $\mathbb{M}$, is presented by the formula $\{(u, v) \,.\, \inf(u, v) = 0\}$. Indeed, $\mathbb{Z} \times \mathbb{Z}$ is presented as an $\mathbb{L}$-model by the formula $\{x \,.\, \top\}$, it being the free ℓ-group on one generator (namely, $(1, -1)$), whence $\mathbb{N} \times \mathbb{N} = R_{\mathbf{Set}}(\mathbb{Z} \times \mathbb{Z})$ is presented by the formula $F_V(\{x \,.\, \top\}) = \{(u, v) \,.\, \inf(u, v) = 0\}$; a pair of generators is given by $((1, 0), (0, 1))$.

4.5.2 Di Nola-Lettieri's equivalence for monoids

The equivalence between ℓ-groups and cancellative subtractive lattice-ordered abelian monoids with bottom element obtained in the last section can be used to rewrite Di-Nola-Lettieri's equivalence in a simpler form.

Indeed, composing Di Nola-Lettieri's equivalence

$$\mathbb{P}\text{-mod}(\mathcal{E}) \to \mathbb{L}\text{-mod}(\mathcal{E})$$

in a topos $\mathcal{E}$ with the equivalence

$$\mathbb{M}\text{-mod}(\mathcal{E}) \to \mathbb{L}\text{-mod}(\mathcal{E})$$

yields an equivalence

$$\mathbb{P}\text{-mod}(\mathcal{E}) \to \mathbb{M}\text{-mod}(\mathcal{E})$$

which admits the following simple description. A perfect MV-algebra $\mathcal{A}$ is sent to its radical $\text{Rad}(\mathcal{A})$, while a monoid $\mathcal{M}$ in $\mathbb{M}\text{-mod}(\mathcal{E})$ to the MV-algebra $\mathcal{A}_{\mathcal{M}}$ defined as follows: the underlying object of $\mathcal{A}_{\mathcal{M}}$ is the coproduct $M \sqcup M$ in $\mathcal{E}$, where M is the underlying object of $\mathcal{M}$, the zero is the zero of the first copy of M, the negation operation $\neg$ is the swapping of the two copies of M and the sum operation $\oplus$ is given for " elements" lying in the same copy of M by the monoid sum in M and for elements x_1, x_2 lying in different copies of M by the element $\sup(x, y) - x$ in the second copy of M. Indeed, for any ℓ-group $\mathcal{G}$, the sum of two "elements" $(0, x)$ and $(1, y)$ in the MV-algebra $\Sigma(\mathcal{G})$ is given by $\inf((1, x+y), (1, 0)) = (1, \inf(x+y, 0)) = (1, -(\sup(y-x, 0))) = (1, -(\sup(x, y) - x))$.

Moreover, this description of Di Nola-Lettieri's equivalence in the language of $\mathbb{M}$ allows to obtain an explicit description of the formulas in the language of MV-algebras which present the finitely presentable perfect MV-algebras (equivalently, the $\mathbb{P}$-irreducible formulas).

4.5.3 Partial levels of bi-intepretation

We proved that the theories $\mathbb{P}$ and $\mathbb{L}$ are Morita-equivalent by establishing a categorical equivalence between the categories of models of these two theories in any Grothendieck topos $\mathcal{E}$, naturally in $\mathcal{E}$. This result would be trivial if the theories were bi-interpretable. In this section we show that this is not the case, i.e., the theories $\mathbb{P}$ and $\mathbb{L}$ are not bi-intepretable in a global sense. Nevertheless, if we consider particular categories of formulas we have three different levels of bi-interpretability.

Theorem 4.5.7. *The theory $\mathbb{L}$ is interpretable in the theory $\mathbb{P}$ but not bi-interpretable.*

Proof. By Lemma 2.2.10, the object $\{x \,.\, x \leq \neg x\}$ of $\mathcal{C}_{\mathbb{P}}$ has the structure of a cancellative lattice-ordered abelian monoid with bottom element, and therefore defines a model M of the theory $\mathbb{M}$ inside the category $\mathcal{C}_{\mathbb{P}}$. This induces a geometric functor $\text{Rad} : \mathcal{C}_{\mathbb{M}} \to \mathcal{C}_{\mathbb{P}}$, that is an interpretation of the theory $\mathbb{M}$ in the theory $\mathbb{P}$. Composing this functor with $F_V : \mathcal{C}_{\mathbb{L}}^{\text{cart}} \to \mathcal{C}_{\mathbb{M}}^{\text{cart}}$ of Proposition 4.5.3, we obtain a cartesian functor $\mathcal{C}_{\mathbb{L}}^{cart} \to \mathcal{C}_{\mathbb{P}}$, which corresponds to a model of $\mathbb{L}$ in $\mathcal{C}_{\mathbb{P}}$ whose underlying object is the formula-in-context $\{(u, v) \,.\, \inf(u, v) = 0 \wedge u \leq \neg u \wedge v \leq \neg v\}$, and hence to an interpretation functor $\mathcal{C}_{\mathbb{L}} \to \mathcal{C}_{\mathbb{P}}$.

Suppose now that $\mathbb{P}$ and $\mathbb{L}$ were bi-interpretable. Then there would be in particular an interpretation functor

$$J : \mathcal{C}_{\mathbb{P}} \to \mathcal{C}_{\mathbb{L}},$$

inducing a functor

$$s_J : \mathbb{L}\text{-mod}(\mathbf{Set}) \to \mathbb{P}\text{-mod}(\mathbf{Set}).$$

Notice that if $\mathcal{M}$ is a $\mathbb{L}$-model in **Set** and $\mathcal{N} = s_J(\mathcal{M})$, we would have that $F_{\mathcal{N}} = F_{\mathcal{M}} \circ J$.

Now, let $\mathcal{M}$ be the trivial model of $\mathbb{L}$ in **Set**, that is the model whose underlying set is $\{0\}$, $\mathcal{N} = s_J(\mathcal{M})$ and $J(\{\vec{x} \,.\, \top\}) = \{\vec{x} \,.\, \psi\}$. We would have

$$F_{\mathcal{N}}(\{\vec{x} \,.\, \top\}) \cong F_{\mathcal{M}}(\{\vec{x} \,.\, \psi\}),$$

$$[\![\vec{x} \,.\, \top]\!]_{\mathcal{N}} \cong [\![\vec{x} \,.\, \psi]\!]_{\mathcal{M}},$$

$$N \cong [\![\vec{x} \,.\, \psi]\!]_{\mathcal{M}} \subseteq M^n \cong M.$$

Hence the domain of $\mathcal{N}$ would be contained in $\{0\}$. But we know from Theorem 2.2.2(i) that the only finite perfect MV-algebra is the one whose underlying set is $\{0, 1\}$. This is a contradiction. □

Even though, as we have just seen, the theories of perfect MV-algebras and of ℓ-groups are not bi-interpretable in the classical sense, the Morita-equivalence between them, combined with the fact that both theories are of presheaf type, guarantees that there is a bi-interpretation between them holding at the level of irreducible formulas (cf. Remark 4.5.2). More specifically, the following result holds.

Theorem 4.5.8. *The categories of irreducible formulas of the theories $\mathbb{P}$ of perfect MV-algebras and $\mathbb{L}$ of ℓ-groups are equivalent.*

In particular, the functor $\mathcal{C}_{\mathbb{L}}^{irr} = \mathcal{C}_{\mathbb{L}}^{cart} \to \mathcal{C}_{\mathbb{P}}$ given by the composite of the functor $F_V : \mathcal{C}_{\mathbb{L}}^{cart} \to \mathcal{C}_{\mathbb{M}}^{cart}$ of Proposition 4.5.3 with the restriction to $\mathcal{C}_{\mathbb{M}}^{cart}$ of the functor $Rad : \mathcal{C}_{\mathbb{M}} \to \mathcal{C}_{\mathbb{P}}$ yields a categorical equivalence

$$\mathcal{C}_{\mathbb{L}}^{irr} = \mathcal{C}_{\mathbb{L}}^{cart} \simeq \mathcal{C}_{\mathbb{P}}^{irr}.$$

□

Remarks 4.5.9. (a) By Theorem 1.5.6, the semantical counterpart of the equivalence of Theorem 4.5.8 is the categorical equivalence between the categories of finitely presented models of the two theories in **Set**. In symbols f.p.$\mathbb{P}$-mod(**Set**) $\simeq$ f.p.$\mathbb{L}$-mod(**Set**). The finitely presentable perfect MV-algebras are thus the images of the finitely presented ℓ-groups under Di Nola-Lettieri's equivalence.

(b) It follows from Theorem 4.5.8 that the $\mathbb{P}$-irreducible formulas are precisely, up to isomorphism in the syntactic category, the ones that come from the $\mathbb{M}$-cartesian formulas via the functor Rad : $\mathcal{C}_{\mathbb{M}} \to \mathcal{C}_{\mathbb{P}}$. For instance, the formula $\{x \,.\, 2x = x\}$ is not $\mathbb{P}$-irreducible, while the formula $\{x, y \,.\, x \leq \neg x \wedge y \leq \neg y\}$ is. Notice that the $\mathbb{P}$-irreducible fourmulas are the analogues for the theory $\mathbb{P}$ of cartesian fourmulas in the theory of MV-algebras, since they are the fourmulas which present the finitely presentable models of the theory. In fact, even though the category $\mathbb{P}$-mod(**Set**) is not a variety, the theory $\mathbb{P}$ is of presheaf type classified by the topos [f.p.$\mathbb{P}$-mod(**Set**), **Set**] (cf. Section 4.6).

(c) We saw in Proposition 2.2.12 that every algebra in f.p.$\mathbb{P}$-mod(**Set**) is finitely presentable as an algebra in f.p.$\mathbb{C}$-mod(**Set**). If $\mathcal{A}$ is a finitely presentable perfect MV-algebra presented by a $\mathbb{P}$-irreducible geometric formula $\{\vec{x} \,.\, \phi\}$, with $\vec{x} = (x_1, \ldots, x_n)$ then this MV-algebra is finitely presented as an MV-algebra in $\mathbb{C}$-mod(**Set**) by the formula $\{\vec{x} \,.\, \phi \wedge x_1 \leq \neg x_1 \wedge \cdots \wedge x_n \leq \neg x_n\}$. Indeed, for any MV-algebra $\mathcal{B}$ in $\mathbb{C}$-mod(**Set**) and any tuple $\vec{y} \in [\![\vec{x} \,.\, \phi \wedge x_1 \leq \neg x_1 \wedge \cdots \wedge x_n \leq \neg x_n]\!]_{\mathcal{B}}$, $y_1, \ldots, y_n \in \text{Rad}(\mathcal{B})$. Now, the MV-subalgebra of $\mathcal{B}$ generated by Rad($\mathcal{B}$) is perfect, whence there exists a unique MV-algebra homomorphism $f : \mathcal{A} \to \langle \text{Rad}(\mathcal{B}) \rangle \hookrightarrow \mathcal{B}$ such that $f(\vec{x}) = \vec{y}$.

Changing the invariant property to consider on the classifying topos of the theories $\mathbb{P}$ and $\mathbb{L}$, we uncover another level of bi-interpretability. Specifically, the invariant notion of subterminal object of the classifying topos yields a categorical equivalence between the full subcategories of $\mathcal{C}_{\mathbb{P}}$ and $\mathcal{C}_{\mathbb{L}}$ on the geometric sentences. Recall that a geometric sentence is a geometric formula without any free variables. For any geometric theory $\mathbb{T}$, the subterminal objects of its classifying topos $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$ can be exactly identified with the geometric sentences over the signature of $\mathbb{T}$, considered up to the following equivalence relation: $\phi \sim_{\mathbb{T}} \psi$ if and only if $(\phi \vdash_{[]} \psi)$ and $(\psi \vdash_{[]} \phi)$ are provable in $\mathbb{T}$.

Since the theories $\mathbb{P}$ and $\mathbb{L}$ are Morita-equivalent, we thus obtain the following result.

Theorem 4.5.10. *There is a bijective correspondence between the classes of geometric sentences of* $\mathbb{P}$ *and of* $\mathbb{L}$*.*

We can explicitly describe this correspondence by using the bi-interpretation between irreducible formulas provided by Theorem 4.5.8 and the concept of ideal on a category presented in Section 1.1.

Lemma 4.5.11. *Let* $\mathbb{T}$ *be a theory of presheaf type and*

$$A = \{\mathbb{T}\text{-}\textit{classes of geometric sentences}\},$$
$$B = \{\textit{ideals of } \mathcal{C}_{\mathbb{T}}^{irr}\}.$$

There is a canonical bijection between A *and* B*.*

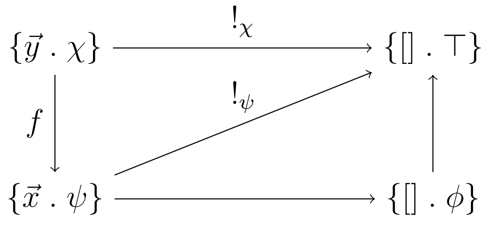
Proof. For any object $\{\vec{x} . \psi\} \in \mathcal{C}_{\mathbb{T}}^{irr}$ there is a unique arrow

$$!_{\psi} : \{\vec{x} . \psi\} \to \{[] . \top\}$$

in $\mathcal{C}_{\mathbb{T}}$, where $\{[] . \top\}$ is the terminal object of $\mathcal{C}_{\mathbb{T}}$. Given $\{[] . \phi\} \in A$, we set

$$I_{\phi} := \{\{\vec{x} . \psi\} \in \mathcal{C}_{\mathbb{T}}^{\text{irr}} \mid \; !_{\psi} \text{ factors through } \{[] . \phi\} \rightarrowtail \{[] . \top\}\} .$$

This is an ideal of $\mathcal{C}_{\mathbb{T}}^{\mathrm{irr}}$. Indeed, if $\{\vec{x} . \psi\} \in I_\phi$ and $f : \{\vec{y} . \chi\} \to \{\vec{x} . \psi\}$ is an arrow in $\mathcal{C}_{\mathbb{T}}^{\mathrm{irr}}$, the commutativity of the following diagram guarantees that the arrow $!_\chi$ factors through $\{[] . \phi\}$, i.e., that $\{\vec{y} . \chi\} \in I_\psi$:



The assignment $\phi \to I_\phi$ defines a map $f : A \to B$.

In the converse direction, suppose that $I \in B$. For any $\{\vec{x} . \psi\} \in I$, the arrow $!_\psi$ factors through the subobject $\{[] . (\exists \vec{x})\psi(\vec{x})\} \rightarrowtail \{[] . \top\}$. We can consider the union $\{[] . \phi_I\} \rightarrowtail \{[] . \top\}$ of these subobjects for all the objects in I. In other words, we set ϕ_I equal to the ($\mathbb{T}$-class of) the formula $\bigvee_{\{\vec{x}.\psi\} \in I} (\exists \vec{x})\psi(\vec{x})$.

The assignment $I \to \phi_I$ defines a map $g : B \to A$.

The verification the assignments $\phi \to I_\phi$ and $I \to \phi_I$ are inverse to each other is straightforward. □

Remark 4.5.12. Applying Lemma 4.5.11 to the theory $\mathbb{P}$ of perfect MV-algebras and to the theory $\mathbb{L}$ of ℓ-groups we obtain two bijections:

1 {$\mathbb{P}$-classes of sentences} $\simeq$ {ideals of $\mathcal{C}_{\mathbb{P}}^{\mathrm{irr}}$};

2 {$\mathbb{L}$-classes of sentences} $\simeq$ {ideals of $\mathcal{C}_{\mathbb{L}}^{\mathrm{irr}}$}.

From these bijections and Theorem 4.5.8 we obtain a bijection between the $\mathbb{P}$-classes of geometric sentences and the $\mathbb{L}$-classes of geometric sentences.

The following proposition provides a characterization of the $\mathbb{P}$-equivalence classes of geometric sentences in terms of the theory $\mathbb{C}$.

Proposition 4.5.13. *The $\mathbb{P}$-equivalence classes of geometric sentences are in natural bijection, besides with the ideals on f.p.$\mathbb{P}$-mod*$(\mathbf{Set})^{op}$ *(cf. Lemma*

4.5.11), with the $J_{\mathbb{P}}$*-ideals on f.p.*$\mathbb{C}$*-mod*$(\mathbf{Set})^{op}$*, that is with the sets* S *of finitely presented algebras in* $\mathbb{C}$*-mod*$(\mathbf{Set})$ *such that for any homomorphism* $f : \mathcal{A} \to \mathcal{B}$ *in f.p.*$\mathbb{C}$*-mod*$(\mathbf{Set})$*,* $\mathcal{A} \in S$ *implies* $\mathcal{B} \in S$ *and for any* $\mathcal{A} \in \mathbb{C}$*-mod*$(\mathbf{Set})$ *and any boolean element* a *of* $\mathcal{A}$*,* $\mathcal{A}/(a) \in S$ *and* $\mathcal{A}/(\neg a) \in S$ *imply* $\mathcal{A} \in S$*.*

Proof. The thesis follows immediately from the fact that the subterminal objects of the topos $\mathbf{Sh}(\text{f.p.}\mathbb{C}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{\mathbb{P}})$ can be naturally identified with the $J_{\mathbb{P}}$-ideals on the category f.p.$\mathbb{C}$-mod$(\mathbf{Set})^{\text{op}}$ (cf. Section 1.1). □

Since the theories $\mathbb{P}$ of perfect MV-algebras and $\mathbb{L}$ of ℓ-groups are both coherent, we have a third level of bi-interpretability between them.

Let $\mathbb{T}$ be a coherent theory; starting from its coherent syntactic category $\mathcal{C}_{\mathbb{T}}^{\text{coh}}$, we can construct the category $\mathcal{C}_{\mathbb{T}}^{\text{eq}}$ of *imaginaries* of $\mathbb{T}$ (also called the *effective positivization of* $\mathcal{C}_{\mathbb{T}}^{\text{coh}}$) by adding formal finite coproducts and coequalizers of equivalence relations in $\mathcal{C}_{\mathbb{T}}^{\text{coh}}$.

Theorem 4.5.14 (Theorem D3.3.7 [35])**.** *Let* $\mathbb{T}$ *be a coherent theory. Then the category* $\mathcal{C}_{\mathbb{T}}^{eq}$ *is equivalent to the full subcategory of its classifying topos of the coherent objects.*

From Theorem 4.5.14 it follows that, if two coherent theories are Morita-equivalent, then the respective categories of imaginaries are equivalent. Notice that the topos-theoretic invariant used in this application of the 'bridge' technique is the notion of coherent object. Specializing this to our Morita-equivalence between $\mathbb{P}$ and $\mathbb{L}$ yields the following result.

Theorem 4.5.15. *The effective positivizations of the syntactic categories of the theories* $\mathbb{P}$ *and* $\mathbb{L}$ *are equivalent:*

$$\mathcal{C}_{\mathbb{P}}^{eq} \simeq \mathcal{C}_{\mathbb{L}}^{eq}.$$

Remark 4.5.16. It is natural to wonder whether we can give an explicit description of this equivalence. Consider the functor $F : \mathcal{C}_{\mathbb{L}} \to \mathcal{C}_{\mathbb{P}}$ given by the composition of the functor Rad : $\mathcal{C}_{\mathbb{M}} \to \mathcal{C}_{\mathbb{P}}$ with the functor $F_V : \mathcal{C}_{\mathbb{L}} \to \mathcal{C}_{\mathbb{M}}$ corresponding to the model V of $\mathbb{L}$ in $\mathcal{C}_{\mathbb{M}}$ introduced in Section 4.5.1. The formal extension $F^{\text{eq}} : \mathcal{C}_{\mathbb{L}}^{\text{eq}} \to \mathcal{C}_{\mathbb{P}}^{\text{eq}}$ of F is part of a categorical equivalence whose other half is the functor $F_Z : \mathcal{C}_{\mathbb{P}}^{\text{eq}} \to \mathcal{C}_{\mathbb{L}}^{\text{eq}}$ induced by the model Z of $\mathbb{P}$ in $\mathcal{C}_{\mathbb{L}}^{\text{eq}}$ defined as follows. Recall that, for any ℓ-group $\mathcal{G}$ in **Set**, the corresponding perfect MV-algebra is given by $\Gamma(\mathbb{Z} \times_{\text{lex}} \mathcal{G})$. Now, this set is isomorphic to the coproduct $G^+ \sqcup G^-$, where G^+ and G^- are respectively the positive and the negative cone of the ℓ-group $\mathcal{G}$. The model Z has as underlying object in $\mathcal{C}_{\mathbb{L}}^{\text{eq}}$ the coproduct $\{x \,.\, x \leq 0\} \sqcup \{x \,.\, x \geq 0\}$, whereas the operations and the order relation are defined as follows:

- $\oplus : (\{x \,.\, x \leq 0\} \sqcup \{x' \,.\, x' \geq 0\}) \times (\{y \,.\, y \leq 0\} \sqcup \{y' \,.\, y' \geq 0\}) \cong$
  $\{x, y \,.\, x \leq 0 \wedge y \leq 0\} \sqcup \{u, v \,.\, u \leq 0 \wedge v \geq 0\} \sqcup \{w, p \,.\, w \geq 0 \wedge p \leq 0\} \sqcup$
  $\{q, r \,.\, q \geq 0 \wedge r \geq 0\} \to \{\alpha \,.\, \alpha \leq 0\} \sqcup \{\beta \,.\, \beta \geq 0\}$
  is given by $[x, y \,.\, \alpha = 0] \sqcup [u, v \,.\, \alpha = \inf(u + v, 0)] \sqcup$
  $[w, p \,.\, \alpha = \inf(w + p, 0)] \sqcup [q, r \,.\, \beta = q + r]$;
- $\neg : \{x \,.\, x \leq 0\} \sqcup \{x' \,.\, x' \geq 0\} \to \{y \,.\, y \leq 0\} \sqcup \{y' \,.\, y' \geq 0\}$
  is given by $[y = -x'] \sqcup [y' = -x]$;
- $0 : \{[] \,.\, \top\} \to \{\alpha \,.\, \alpha \leq 0\} \sqcup \{\beta \,.\, \beta \geq 0\}$ is given by $[\beta = 0]$.

4.6 The classifying topos for perfect MV-algebras

Recall that the theory $\mathbb{P}$ of perfect MV-algebras is a quotient of the theory $\mathbb{C}$ of MV-algebras in Chang's variety. From the Duality Theorem we know that the classifying topos of $\mathbb{P}$ can be represented as a subtopos $\mathbf{Sh}(\text{f.p.}\mathbb{C}\text{-mod}(\mathbf{Set})^{op}, J_{\mathbb{P}})$ of the classifying topos $[\text{f.p.}\mathbb{C}\text{-mod}(\mathbf{Set}), \mathbf{Set}]$ of $\mathbb{C}$, where $J_{\mathbb{P}}$ is the Grothendieck topology associated with the quotient $\mathbb{P}$.

By Theorem 1.5.11 it follows that the topology $J_{\mathbb{P}}$ is rigid, since by Proposition 2.2.12 f.p.$\mathbb{P}$-mod(**Set**) $\subseteq$ f.p.$\mathbb{C}$-mod(**Set**). Moreover, from the remark following Theorem 6.26 [17] we know that the $J_{\mathbb{P}}$-irreducible objects are precisely the objects of the category f.p.$\mathbb{P}$-mod(**Set**). In particular, the classifying topos of $\mathbb{P}$ is equivalent to the presheaf topos [f.p.$\mathbb{P}$-mod(**Set**), **Set**].

We can describe the Grothendieck topology $J_{\mathbb{P}}$ explicitly as follows (cf. Section 1.3 for the standard method for calculating the Grothendieck topology associated with a quotient of a theory of presheaf type). Recall that the theory $\mathbb{P}$ of perfect MV-algebras is obtained from $\mathbb{C}$ by adding the axioms

P.1 $(x \oplus x = x \vdash_x x = 0 \vee x = 1)$;

P.2 $(x = \neg x \vdash_x \bot)$;

or equivalently,

P.1' $(\inf(x, \neg x) = 0 \vdash_x x = 0 \vee x = 1)$;

P.2 $(x = \neg x \vdash_x \bot)$.

The axioms P.1' and P.2 generate two cosieves $S_{P.1'}$ and $S_{P.2}$ in f.p.$\mathbb{C}$-mod(**Set**), and consequently two sieves in f.p.$\mathbb{C}$-mod$(\mathbf{Set})^{\text{op}}$. The topology $J_{\mathbb{P}}$ on f.p.$\mathbb{C}$-mod$(\mathbf{Set})^{\text{op}}$ is generated by these sieves. Specifically:

- the cosieve $S_{P.1'}$ is generated by the canonical projections

$$p_1 : Free_x/(\inf(x, \neg x)) \to Free_x/(x),$$
$$p_2 : Free_x/(\inf(x, \neg x)) \to Free_x/(\neg x)\ ,$$

  where $Free_x$ is the one-generated free algebra in Chang's variety;

- the cosieve $S_{P.2}$ is the empty one on the trivial algebra in Chang's variety.

The cotopology induced by $J_{\mathbb{P}}$ on the category f.p.$\mathbb{C}$-mod(**Set**) is thus generated by the empty cosieve on the trivial algebra and the finite 'multi-compositions' of the pushouts of the generating arrows of the cosieve $S_{P.1'}$ along arbitrary homomorphisms in f.p.$\mathbb{C}$-mod(**Set**). We can describe these pushouts explicitly. Let $f : Free_x/(\inf(x, \neg x)) \to \mathcal{A}$ be an MV-homomorphism in f.p.$\mathbb{C}$-mod(**Set**); then the pushouts of the generating arrows of $S_{P.1'}$ along f are: $f_1 : \mathcal{A} \to \mathcal{A}/(a)$ and $f_2 : \mathcal{A} \to \mathcal{A}/(\neg a)$, where $a = f([x]) \in A$ satisfies $\inf(a, \neg a) = 0$.

We shall say that an MV-algebra $\mathcal{A}$ is a *weak subdirect product* of a family $\{\mathcal{A}_i \mid i \in I\}$ of MV-algebras if the arrows $\mathcal{A} \to \mathcal{A}_i$ are jointly injective (equivalently, jointly monic).

Note that every weak subdirect product of finitely presented perfect MV-algebras is in $\mathbb{C}$-mod(**Set**). Indeed, perfect MV-algebras are in $\mathbb{C}$-mod(**Set**) and the identities that define this variety are preserved by weak subdirect products. It is natural to wonder if the converse is true, that is if every algebra in $\mathbb{C}$-mod(**Set**) is a weak subdirect product of finitely presented perfect MV-algebras. We shall prove in the following that the answer is affirmative.

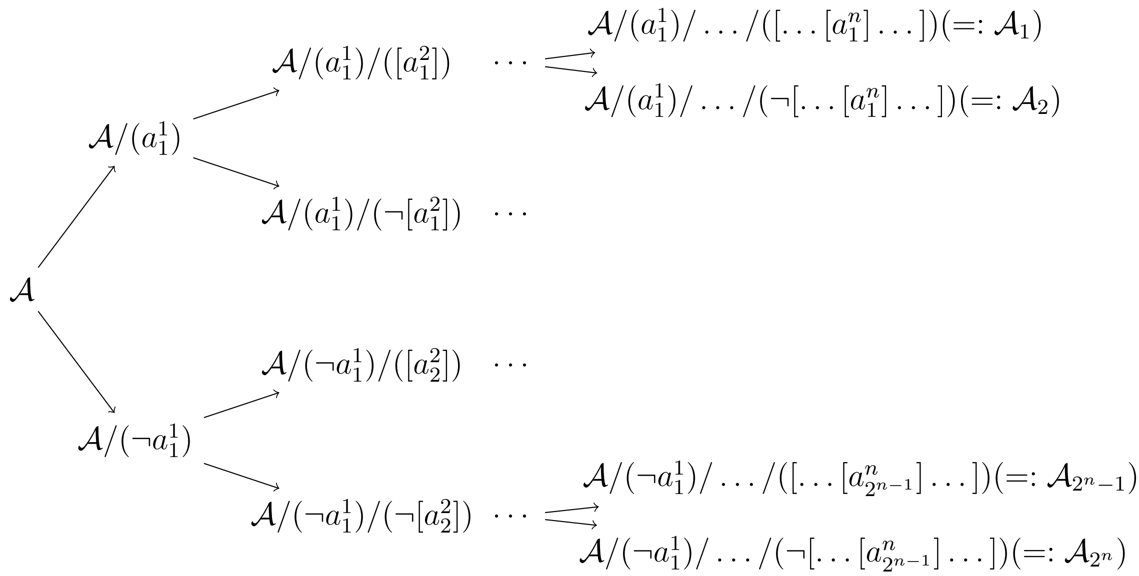
Theorem 4.6.1. *Every finitely presented non-trivial MV-algebra in* $\mathbb{C}$*-mod*(**Set**) *is a direct product of a finite family of finitely presented perfect MV-algebras. In fact, the topology* $J_{\mathbb{P}}$ *is subcanonical.*

Proof. Let $\mathcal{A} \in \mathbb{C}$-mod(**Set**) be a finitely presented non-trivial MV-algebra. This algebra satisfies the axiom P.2 (cf. Lemma 2.2.6); thus, the only non-trivial $J_{\mathbb{P}}$-coverings of $\mathcal{A}$ are those which contain a cosieve generated by finite multicompositions of the pushouts of p_1 and p_2.

Now, the Pushout-Pullback Lemma (Lemma 7.1 [30]) asserts that for any MV-algebra $\mathcal{A}$ and any elements $x, y \in \mathcal{A}$, the following pullback diagram is also a pushout:

$$\begin{array}{ccc} \mathcal{A}/(\inf(x,y)) & \longrightarrow & \mathcal{A}/(y) \\ \downarrow & & \downarrow \\ \mathcal{A}/(x) & \longrightarrow & \mathcal{A}/(\sup(x,y)) \end{array} .$$

Note that if $\inf(x, y) = 0$ then $\sup(x, y) = x \oplus y$; in particular, for any boolean element x of $\mathcal{A}$, $\mathcal{A}$ is the product of $\mathcal{A}/(x)$ and $\mathcal{A}/(\neg x)$. The same reasoning can be repeated for every pair of arrows in the diagram below, which represents a $J_{\mathbb{P}}$-covering of $\mathcal{A}$.



It follows that the MV-algebra $\mathcal{A}$ is the direct product of the $\mathcal{A}_i$.

Since $J_{\mathbb{P}}$ is rigid and the $J_{\mathbb{P}}$-irreducible objects are the finitely presented perfect MV-algebras, there is a $J_{\mathbb{P}}$-covering of $\mathcal{A}$ such that all the $\mathcal{A}_i$ are finitely presented perfect MV-algebras.

Finally, we observe that for any boolean element x of an MV-algebra $\mathcal{A}$, there is a unique arrow $\mathcal{A}/(x) \to \mathcal{A}/(\neg x)$ over $\mathcal{A}$ if and only if $x = 0$, whence the sieve generated by the family $\{\mathcal{A} \to \mathcal{A}/(x), \mathcal{A} \to \mathcal{A}/(\neg x)\}$ is effective epimorphic in f.p.$\mathbb{C}$-mod$(\mathbf{Set})^{\text{op}}$ if and only if $\{\mathcal{A} \to \mathcal{A}/(x), \mathcal{A} \to \mathcal{A}/(\neg x)\}$ is a product diagram in f.p.$\mathbb{C}$-mod$(\mathbf{Set})$.

This proves our statement. □

The following results are consequences of the subcanonicity of the topology $J_{\mathbb{P}}$.

Proposition 4.6.2. *The theory $\mathbb{C}$ axiomatizing Chang's variety $V(S_1^\omega)$ coincides with the cartesianization of the theory $\mathbb{P}$ of perfect MV-algebras. That is, for any $\mathbb{C}$-cartesian sequent $\sigma = (\phi \vdash_{\vec{x}} \psi)$, σ is provable in $\mathbb{C}$ (equivalently, valid in all algebras in $V(S_1^\omega)$) if and only if it is provable in $\mathbb{P}$ (that is, valid in all perfect MV-algebras).*

Moreover, for any $\mathbb{C}$-cartesian fourmulas $\{\vec{x} \,.\, \phi\}$ and $\{\vec{x} \,.\, \psi\}$ and a geometric formula $\theta(\vec{x}, \vec{y})$, θ is $\mathbb{P}$-provably functional from $\{\vec{x} \,.\, \phi\}$ to $\{\vec{x} \,.\, \phi\}$ if and only if it is $\mathbb{C}$-provably functional from $\{\vec{x} \,.\, \phi\}$ to $\{\vec{x} \,.\, \psi\}$.

Proof. The theory $\mathbb{C}$ is algebraic, hence it is of presheaf type. By Corollary D3.1.2 [35], the universal model $U_{\mathbb{C}}$ of $\mathbb{C}$ in its classifying topos [f.p.$\mathbb{C}$-mod(**Set**), **Set**] is given by $\mathrm{Hom}_{\text{f.p.}\mathbb{C}\text{-mod}(\mathbf{Set})}(F, -)$, where F is the free $\mathbb{C}$-algebra on one generator. Since $J_{\mathbb{P}}$ is subcanonical, the model $U_{\mathbb{C}}$ is also a universal model of $\mathbb{P}$ in the topos **Sh**(f.p.$\mathbb{C}$-mod(**Set**)$^{\text{op}}$, $J_{\mathbb{P}}$) (cf. Lemma 2.1 [15]). Now, given a geometric theory $\mathbb{T}$, a geometric sequent over its signature is provable in $\mathbb{T}$ if and only if it is satisfied in its universal model $U_{\mathbb{T}}$ (cf. Theorem D1.4.6 [35]). From this the first part of the proposition follows at once.

The second part follows from the fact that the canonical functor $r : \mathcal{C}_{\mathbb{C}}^{\text{cart}} \simeq$ f.p.$\mathbb{C}$-mod(**Set**)$^{\text{op}} \to$ **Sh**(f.p.$\mathbb{C}$-mod(**Set**)$^{\text{op}}$, $J_{\mathbb{P}}$) is full and faithful since the topology $J_{\mathbb{P}}$ is subcanonical. Recalling from Theorem 2.2 [15] that, given the universal model U of a geometric theory $\mathbb{T}$ in its classifying topos $\mathcal{E}_{\mathbb{T}}$, for any geometric formulas $\{\vec{x} \,.\, \phi\}$ and $\{\vec{y} \,.\, \psi\}$ over the signature of $\mathbb{T}$, the arrows $[\![\vec{x} \,.\, \phi]\!]_U \to [\![\vec{y} \,.\, \psi]\!]_U$ in $\mathcal{E}_{\mathbb{T}}$ correspond exactly to the $\mathbb{T}$-provably functional fourmulas from $\{\vec{x} \,.\, \phi\}$ to $\{\vec{y} \,.\, \psi\}$, the thesis follows immediately. □

Proposition 4.6.3. *The following definability properties of the theory $\mathbb{P}$ in relation to the theory $\mathbb{C}$ hold:*

(i) *Every property P of tuples $\vec{x}$ of elements of perfect MV-algebras which is preserved by arbitrary MV-algebra homomorphisms and by filtered colimits of perfect MV-algebras is definable by a geometric formula $\{\vec{x} . \phi\}$ over the signature of $\mathbb{P}$. For any two geometric fourmulas $\{\vec{x} . \phi\}$ and $\{\vec{x} . \psi\}$ over the signature of $\mathbb{P}$, every assignment $M \to f_M : [\![\vec{x} . \phi]\!]_M \to [\![\vec{y} . \psi]\!]_M$ (for finitely presented perfect MV-algebras M) which is natural in M is definable by a $\mathbb{P}$-provably functional formula $\theta(\vec{x}, \vec{y})$ from $\{\vec{x} . \phi\}$ to $\{\vec{x} . \psi\}$.*

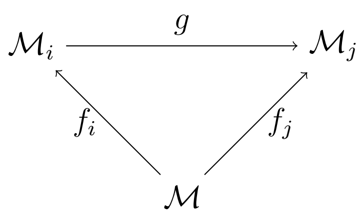
(ii) *The properties P of tuples $\vec{x}$ of elements of perfect MV-algebras which are preserved by arbitrary homomorphisms and filtered colimits of perfect MV-algebras are in natural bijection with the properties Q of tuples $\vec{x}$ of elements of algebras in $\mathbb{C}$-mod(**Set**) which are preserved by filtered colimits of algebras in $\mathbb{C}$-mod(**Set**) and such that for any finitely presented algebra $\mathcal{A}$ in $\mathbb{C}$-mod(**Set**) and any boolean element a of $\mathcal{A}$, the canonical projections $\mathcal{A} \to \mathcal{A}/(a)$ and $\mathcal{A} \to \mathcal{A}/(\neg a)$ jointly reflect Q.*

Proof. The first part of the theorem follows from Theorem 1.5.7 in light of the fact that $\mathbb{P}$ is of presheaf type.

The second part follows from the fact that the properties P of tuples $\vec{x} = (x_1, \ldots, x_n)$ of elements of perfect MV-algebras which are preserved by arbitrary MV-algebra homomorphisms and by filtered colimits of perfect MV-algebras correspond precisely to the subobjects of $U \times \cdots \times U$ in the classifying topos of $\mathbb{P}$, where U is a universal model of $\mathbb{P}$ inside it. But, as we have observed above in the proof of Proposition 4.6.2, the universal model $U_{\mathbb{C}} = \mathrm{Hom}_{\mathrm{f.p.}\mathbb{C}\text{-}\mathrm{mod}(\mathbf{Set})}(F, -)$ (where F is the free $\mathbb{C}$-algebra on one generator), of $\mathbb{C}$ in its classifying topos is also a universal model of $\mathbb{P}$ in its classifying topos $\mathbf{Sh}(\mathrm{f.p.}\mathbb{C}\text{-}\mathrm{mod}(\mathbf{Set})^{\mathrm{op}}, J_{\mathbb{P}})$. Now, the subobjects of $U \times \cdots \times U$ in $\mathbf{Sh}(\mathrm{f.p.}\mathbb{C}\text{-}\mathrm{mod}(\mathbf{Set})^{\mathrm{op}}, J_{\mathbb{P}})$ are precisely the $J_{\mathbb{P}}$-closed sieves on $F \times \cdots \times F$ in $\mathrm{f.p.}\mathbb{C}\text{-}\mathrm{mod}(\mathbf{Set})^{\mathrm{op}}$ (cf. Section 1.1). From this our thesis follows at once. □

The following proposition provides an explicit reformulation of the subcanonicity property of the Grothendieck topology $J_{\mathbb{P}}$.

Proposition 4.6.4. *Let $\mathcal{M}$ be a finitely presented algebra in Chang's variety $\mathbb{C}$-mod(**Set**) and $\{\vec{x} \, . \, \phi\}$ a $\mathbb{C}$-cartesian formula. For any family of tuples $\vec{a_i} \in [\![\vec{x} \, . \, \phi]\!]_{\mathcal{M}_i}$ indexed by the MV-homomorphisms $f_i : \mathcal{M} \to \mathcal{M}_i$ from $\mathcal{M}$ to finitely presented perfect MV-algebras $\mathcal{M}_i$ such that for any MV-homomorphism $g : \mathcal{M}_i \to \mathcal{M}_j$ such that $g \circ f_i = f_j$, $g(\vec{a_i}) = \vec{a_j}$, there exists a unique tuple $\vec{a} \in [\![\vec{x} \, . \, \phi]\!]_{\mathcal{M}}$ such that $f_i(\vec{a}) = \vec{a_i}$ for all i.*



Proof. This immediately follows from the subcanonicity of the topology $J_{\mathbb{P}}$ (cf. Theorem 4.6.1) in view of the equivalence $\mathcal{C}_{\mathbb{C}}^{\text{cart}} \simeq$ f.p.$\mathbb{C}$-mod(**Set**)$^{\text{op}}$. □

We can give a more explicit description of a family of finitely presented perfect MV-algebras $\{\mathcal{A}_1, \dots, \mathcal{A}_m\}$ such that the family of arrows $\{\mathcal{A} \to \mathcal{A}_i \mid i \in \{1, \dots m\}\}$ as in the proof of Theorem 4.6.1 generates a $J_{\mathbb{P}}$-covering sieve.

Lemma 4.6.5. *Let $\mathcal{A}$ be an MV-algebra in $\mathbb{C}$-mod(**Set**) generated by elements $\{x_1, \dots, x_n\}$. Then the boolean kernel $B(\mathcal{A})$ of $\mathcal{A}$ is finitely generated by the family $\{(2x_1)^2, \dots, (2x_n)^2\}$.*

Proof. From Proposition 3.7.4 for every $x \in A$, $(2x)^2 \in B(\mathcal{A})$ and from Theorem 5.12 [28] we know that an MV-algebra $\mathcal{A}$ is in $\mathbb{C}$-mod(**Set**) if and only if $\mathcal{A}/\text{Rad}(\mathcal{A}) \cong B(\mathcal{A})$, where the isomorphism is given by the following map:

$$f : x \in A \to (2x)^2 \in B(\mathcal{A}).$$

If $\mathcal{A}$ is an MV-algebra in $\mathbb{C}$-mod(**Set**) generated by $\{x_1, \ldots, x_n\}$ then the quotient $\mathcal{A}/\mathrm{Rad}(\mathcal{A})$ is generated by $\{[x_1], \ldots, [x_n]\}$; hence, $B(\mathcal{A})$ is generated by the family $\{(2x_1)^2, \ldots, (2x_n)^2\}$. □

Recall that if an MV-algebra $\mathcal{A}$ is finitely presented, then it is finitely generated. Let $\mathcal{A} = \langle x_1, \ldots, x_n \rangle$ be an MV-algebra as in Theorem 4.6.1. From Lemma 4.6.5 it follows that a family of finitely presented perfect MV-algebras that $J_{\mathbb{P}}$-covers $\mathcal{A}$ is given by $\{\mathcal{A}_1, \ldots, \mathcal{A}_{2^n}\}$ (in the notation of Theorem 4.6.1), where $a^i_j = (2x_i)^2$ for all $j = 1, \ldots, 2^{i-1}$. Indeed, the iterated quotients of the previous diagram actually remove every non-trivial boolean element, thus every $\mathcal{A}_k$ is perfect. In fact, this argument shows more generally the following result.

Theorem 4.6.6. *Every finitely generated MV-algebra in* $\mathbb{C}$*-mod*(**Set**) *is a direct product of finitely generated perfect MV-algebras.*

Theorem 4.6.1 can be alternatively deduced from existing theorems on weak boolean products of MV-algebras as follows. First, we need a lemma, clarifying the relationship between finite direct products and weak boolean products of MV-algebras. Recall from [26] that a *weak boolean product* of a family $\{\mathcal{A}_x \mid x \in X\}$ of MV-algebras is a subdirect product $\mathcal{A}$ of the given family, in such a way that X can be endowed with a boolean (i.e., Stone) topology satisfying the following conditions (where $\pi_x : \mathcal{A} \to \mathcal{A}_x$ are the canonical projections):

(i) for all $f, g \in A$, the set $\{x \in X \mid \pi_x(f) = \pi_x(g)\}$ is open in X;

(ii) for every clopen set Z of X and any $f, g \in A$, there exists a unique element $h \in A$ such that $\pi_x(h) = \pi_x(f)$ for all $x \in Z$ and $\pi_x(h) = \pi_x(g)$ for all $x \in X \setminus Z$.

Lemma 4.6.7. *Let $\mathcal{A}$ be a weak boolean product of a finite family $\{\mathcal{A}_x \mid x \in X\}$ of MV-algebras. Then the topology of X is discrete and $\mathcal{A}$ is a finite direct product of the $\mathcal{A}_x$.*

Proof. It is clear that the only boolean topology on finite set is the discrete one. To prove that $\mathcal{A}$ is a finite direct product of the $\mathcal{A}_x$ via the weak boolean product projections π_x, it suffices to verify that for every family $\{z_x\}_{x \in X}$ of elements such that $z_x \in A_x$ for all $x \in X$ there exists an element $h \in A$ such that $\pi_x(h) = z_x$ for all $x \in X$. Since $\mathcal{A}$ is a subdirect product of the $\mathcal{A}_x$, the functions π_x are all surjective. By choosing, for each $x \in X$, an element $a_x \in A$ such that $\pi_x(a_x) = z_x$ and repeatedly applying condition (ii) to such elements (taking Z to be the singletons $\{x\}$ for $x \in X$), we obtain the existence of an element $h \in A$ such that $\pi_x(h) = \pi_x(a_x) = z_x$ for each $x \in X$, as required. □

Now, by Lemma 9.4 [27], every algebra $\mathcal{A}$ in $V(S_1^\omega)$ is quasi-perfect, i.e., it is a weak boolean product of perfect MV-algebras. By Theorem 6.5.2 [26], the indexing set of this boolean product identifies with the set of ultrafilters of the boolean algebra $B(\mathcal{A})$. But by Lemma 4.6.5 the set of ultrafilters of $B(\mathcal{A})$ is finite, and can be identified with the set of atoms of $B(\mathcal{A})$, since $B(\mathcal{A})$ is finitely generated and hence finite. By Lemma 4.6.7, we can then conclude that the given weak boolean product is in fact a finite direct product.

4.7 Weak subdirect products of perfect MV-algebras and a comparison with boolean algebras

Theorem 4.7.1. *Every MV-algebra in* $\mathbb{C}$*-mod*(**Set**) *is a weak subdirect product of (finitely presentable) perfect MV-algebras.*

Proof. Since every MV-algebra in $\mathbb{C}$-mod(**Set**) is a filtered colimit of finitely presented MV-algebras in $\mathbb{C}$-mod(**Set**), it suffices to prove the statement for the finitely presentable MV-algebras in $\mathbb{C}$-mod(**Set**); indeed, an MV-algebra is a weak subdirect product of finitely presentable perfect MV-algebras if and only if the arrows from it to such algebras are jointly monic. But this follows from Theorem 4.6.1. □

Remark 4.7.2. Theorem 4.7.1 represents a constructive version of Lemma 9.6 [27].

It is natural to wonder if one can intrinsically characterize the class of MV-algebras in $\mathbb{C}$-mod(**Set**) which are direct products of perfect MV-algebras. We already know from the discussion above that all the finitely generated MV-algebras in $\mathbb{C}$-mod(**Set**) belong to this class.

The following lemma, which generalizes its finitary version given by Lemmas 6.4.4 and 6.4.5 in [26] as well as the version for complete MV-algebras given by Lemma 6.6.6 in [26], will be useful in this respect. Relevant references on the relationship between direct product decompositions of MV-algebras and boolean elements are [34], [44] and Sections 6.4-5-6 of [26].

Lemma 4.7.3. *Let $\mathcal{A}$ be an MV-algebra. Then the following two conditions are equivalent:*

(i) $\mathcal{A}$ is a direct product of MV-algebras $\mathcal{A}_i$ (for $i \in I$);

(ii) There exists a family $\{a_i \mid i \in I\}$ of boolean pairwise disjoint elements of $\mathcal{A}$ such that every family of elements of the form $\{z_i \leq a_i \mid i \in I\}$ has a supremum $\bigvee_{i\in I} z_i$ in $\mathcal{A}$ and every element a of $\mathcal{A}$ can be expressed (uniquely) in this form.

Proof. Let $\mathcal{A}$ be an MV-algebra that is direct product of a family $\{\mathcal{A}_i \mid i \in I\}$ of MV-algebras. The elements a_i of the MV-algebra $\prod_{i\in I}\mathcal{A}_i$ which are 0 everywhere except at the place i where it is equal to 1 are boolean and satisfy the following properties: they are pairwise disjoint (i.e., $a_i \wedge a_i = 0$ whenever $i \neq i'$), $1 = \bigvee_{i\in I} a_i$, every family of elements of the form $\{z_i \leq a_i \mid i \in I\}$ has a supremum $\bigvee_{i\in I} z_i$ in $\prod_{i\in I}\mathcal{A}_i$ and every element a of $\prod_{i\in I}\mathcal{A}_i$ can be expressed uniquely in this form.

Conversely, suppose that $\{a_i \in \mathcal{A} \mid i \in I\}$ is a set of boolean pairwise disjoint elements of an MV-algebra $\mathcal{A}$ such that every family of elements of the form $\{z_i \leq a_i \mid i \in I\}$ has a supremum $\bigvee_{j\in J} z_j$ in $\mathcal{A}$ and every element a of $\mathcal{A}$ can be expressed uniquely in this form. Then $\mathcal{A}$ is isomorphic to the product of the MV-algebras $(a_i]$ considered in [26] (cf. Corollary 1.5.6) via the canonical homomorphism $\mathcal{A} \to \prod_{i\in I}(a_i]$ (equivalently, by Proposition 6.4.3 [26], to the product of the quotient algebras $\mathcal{A}/(\neg a_i)$ via the canonical projections). Indeed, the canonical homomorphism $\mathcal{A} \to \prod_{i\in I}(a_i]$, which sends any element b of $\mathcal{A}$ to the string $(b \wedge a_i)$ admits as inverse the map sending a tuple (z_i) in $\prod_{i\in I}(a_i]$ to the supremum $\bigvee_{i\in I} z_i$. This can be proved as follows. The composite of the former homomorphism with the latter is clearly the identity, so it remains to prove the converse. Given an element $b \in \mathcal{A}$, we have to prove that $b = \bigvee_{i\in I}(b \wedge a_i)$. Set $b' = \bigvee_{i\in I}(b \wedge a_i)$. Clearly, $b' \leq b$. Now, by our hypothesis, we can decompose b in the form $b = \bigvee_{i\in I} c_i$ where $c_i \leq a_i$ for each i. Now, $c_i \leq b$, whence $c_i \leq a_i \wedge b$ and $b = \bigvee_{i\in I} c_i \leq \bigvee_{i\in I}(b \wedge a_i) = b'$. So $b = b'$, as required. □

Remark 4.7.4. The algebras $\mathcal{A}_i$ as in the first condition are given by the quotients $\mathcal{A}/(\neg a_i)$, while the elements a_i of the product $\prod_{i\in I}\mathcal{A}_i$ satisfying the second conditions are the tuples which are zero everywhere except at the place i where they are equal to 1.

In order to achieve an intrinsic characterization of the MV-algebras $\mathcal{A}$ which are products of perfect MV-algebras, it remains to characterize the elements a_i such that $\mathcal{A}/(\neg a_i)$ is a perfect MV-algebra. Since $\mathcal{A}$ is in $\mathbb{C}$-mod(**Set**), this amounts to requiring that a_i is boolean and for every element x such that $x \wedge \neg x \leq \neg a_i$ (equivalently, $x \wedge \neg x \wedge a_i = 0$), either $x \leq \neg a_i$ (equivalently, $x \wedge a_i = 0$) or $\neg x \leq \neg a_i$ (equivalently, $a_i \leq x$) but not both. We shall call such elements the *perfect elements* of the algebra $\mathcal{A}$.

Summarizing, we have the following result.

Theorem 4.7.5. *For a MV-algebra $\mathcal{A}$, the following conditions are equivalent:*

(i) $\mathcal{A}$ is isomorphic to a direct product of perfect MV-algebras;

*(ii) $\mathcal{A}$ belongs to $\mathbb{C}$-mod(**Set**) and there exists a family of boolean pairwise disjoint perfect elements of $\mathcal{A}$ such that every family of elements of the form $\{z_i \leq a_i \mid i \in I\}$ has a supremum $\bigvee_{j \in J} z_j$ in $\mathcal{A}$ and every element a of $\mathcal{A}$ can be expressed (uniquely) in this form.*

Remark 4.7.6. By Theorem 4.7.1, every finitely generated MV-algebra $\mathcal{A}$ in $\mathbb{C}$-mod(**Set**) satisfies these conditions. In fact, for every finite set $\{x_1, \ldots, x_n\}$ of generators of $\mathcal{A}$, a family of elements satisfying the hypotheses of Lemma 4.7.3 is given by the family of finite meets of the form $u_1 \wedge \cdots \wedge u_n$ where for each i, u_i is either equal to $(2x_i)^2$ or its complement $\neg(2x_i)^2$.

The class of MV-algebras in $\mathbb{C}$-mod(**Set**) naturally generalizes that of boolean algebras (recall that every boolean algebra is an MV-algebra, actually lying in $\mathbb{C}$-mod(**Set**)), with perfect algebras representing the counterpart of the algebra $\{0, 1\}$ and powerset algebras, that is products of the algebra $\{0, 1\}$, corresponding to products of perfect MV-algebras. The class of boolean algebras isomorphic to powersets can be intrinsically characterized, thanks to Lindenbaum-Tarski's theorem, as that of complete atomic boolean algebras. Theorem 4.7.1 represents a natural generalization in this setting of the Stone representation of a boolean algebra as a field of sets, while Theorem 4.7.5 represents the analogue of Lindenbaum-Tarski's Theorem. Note that, as every boolean algebra with n generators is a product of 2^n copies of the algebra $\{0, 1\}$, so every finitely presented algebra in $\mathbb{C}$-mod(**Set**) with n generators is a product of 2^n finitely presented perfect MV-algebras (cf. Theorem 4.6.1). These relationships are summarized in the following table.

Classical context	**MV-algebraic generalization**
Boolean algebra	MV-algebra in $\mathbb{C}$-mod(**Set**)
$\{0, 1\}$	Perfect MV-algebra
Powerset $\cong$ product of $\{0, 1\}$	Product of perfect MV-algebras
Finite boolean algebra	Finitely presentable MV-algebra in $\mathbb{C}$-mod(**Set**)
Complete atomic boolean algebra	MV-algebra in $\mathbb{C}$-mod(**Set**) satisfying the hypotheses of Theorem 4.7.5
Representation Theorem for finite boolean algebras	Theorem 4.6.1
Stone representation for boolean algebras	Theorem 4.7.1
Lindenbaum-Tarski's Theorem	Theorem 4.7.5

4.8 Transferring results for ℓ-groups with strong unit

In this section we transfer some of the representation results that we obtained for MV-algebras in Chang's variety to ℓ-u groups.

Proposition 4.8.1. *Under Mundici's equivalence*

$$\mathbb{MV}\text{-}mod(\mathbf{Set}) \simeq \mathbb{L}_u\text{-}mod(\mathbf{Set}),$$

(i) the injective homomorphisms of MV-algebras correspond precisely to the injective homomorphisms of ℓ-u groups;

(ii) the finitely generated MV-algebras correspond precisely to the finitely generated ℓ-u groups.

Proof. (i) Cf. Lemma 7.2.i(iii) [26].

(ii) It is clear that the MV-algebra corresponding to a finitely generated ℓ-u groups is finitely generated. Conversely, since by point (i) of the proposition the category of MV-algebra and injective homomorphisms between them and the category of ℓ-u groups and injective homomorphisms between them are equivalent and every finitely generated MV-algebra is a finitely presentable object of the former category, every ℓ-u group which corresponds to a finitely generated MV-algebra under Mundici's equivalence is finitely presentable as an object of the category of ℓ-u groups and injective homomorphisms between them. Now, since every ℓ-u group $\mathcal{G}$ is the filtered union of its finitely generated ℓ-u subgroups, if $\mathcal{G}$ is finitely presentable as an object of the category of ℓ-u groups and injective homomorphisms between them then $\mathcal{G}$ is finitely generated. This implies our thesis. □

We are now in the position to transfer the representation results for the MV-algebras in Chang's variety that we obtained in Section 4.6 to the context of ℓ-u groups.

In Section 3.7.1 we have already introduced the quotients $\mathbb{L}_{Chang}$ and $\mathbb{A}nt$ of $\mathbb{L}_u$ which are Morita-equivalent to the quotients $\mathbb{C}$ and $\mathbb{P}$ of $\mathbb{MV}$. Observe, in particular, that the theory $\mathbb{A}nt$ is Morita-equivalent to the theory of lattice-ordered abelian groups $\mathbb{L}$, by Theorem 4.4.2. It follows that an ℓ-u group is antiarchimedean if and only if it is isomorphic to a ℓ-u group of the form $\mathbb{Z} \times_{\text{lex}} \mathcal{G}$, for an ℓ-group $\mathcal{G}$.

In view of Proposition 4.8.1, we immediately obtain the following result, representing the translation of Theorems 4.6.1 and 4.7.1.

Theorem 4.8.2. *Every ℓ-u group which is a model of $\mathbb{L}_{Chang}$ is a weak subdirect product of antiarchimedean ℓ-u groups.*

Every finitely generated (resp. finitely presentable) ℓ-u group which is a model of $\mathbb{L}_{Chang}$ is a finite direct product of antiarchimedean (resp. antiArchimedean finitely presentable) ℓ-u groups.

One could also, by using the same method as that leading to the proof of Theorem 4.7.5, intrinsically characterize the ℓ-u groups which are direct products of antiarchimedean ℓ-u groups.

4.9 A related Morita-equivalence

We conclude this chapter by discussing the relationship between the category of perfect MV-algebras and that of lattice-ordered abelian groups with strong unit. Generalizing the work [2] of Belluce and Di Nola concerning locally archimedean MV-algebras and archimedean ℓ-u groups, we establish a Morita-equivalence between the category of pointed perfect MV-algebras and the category of ℓ-u groups. This will allow us to reinterpret in the context of ℓ-groups the representation results for the MV-algebras in Chang's variety obtained in the last section.

We call a perfect MV-algebra *pointed* if its radical is generated by a single element. This class of algebras can be axiomatized. Let us extend the signature Σ_{MV} by adding a new constant symbol a. We call $\mathbb{P}^*$ the theory over this signature whose axioms are those of $\mathbb{P}$ plus:

P*.1 $(\top \vdash a \leq \neg a)$;

P*.2 $(x \leq \neg x \vdash_x \bigvee_{n \in \mathbb{N}} x \leq na)$.

We shall prove that the theory $\mathbb{P}^*$ is Morita-equivalent to the theory $\mathbb{L}_u$. Indeed we can "restrict" the functors $\Delta_{\mathcal{E}}$ and $\Sigma_{\mathcal{E}}$ respectively to the categories $\mathbb{P}^*$-mod$(\mathcal{E})$ and $\mathbb{L}_u$-mod$(\mathcal{E})$, for every Grothendieck topos $\mathcal{E}$, and show that they are still categorical inverses to each other.

Let $\mathcal{A} = (A, a)$ be a model of $\mathbb{P}^*$ in $\mathcal{E}$. This structure, without the constant a, is a perfect MV-algebra in $\mathcal{E}$. We can thus consider $\Delta_{\mathcal{E}}(\mathcal{A})$ and we know that it is a model of $\mathbb{L}$ in $\mathcal{E}$.

Proposition 4.9.1. *The structure* $(\Delta_{\mathcal{E}}(\mathcal{A}), [a, 0])$ *is a model of* $\mathbb{L}_u$ *in* $\mathcal{E}$.

Proof. We already know that $\Delta_{\mathcal{E}}(\mathcal{A})$ is an ℓ-group in $\mathcal{E}$, so it remains to prove that $[a, 0]$ is a strong unit for it.

- $[a, 0] \geq [0, 0] \Leftrightarrow \inf([a, 0], [0, 0]) = [0, 0] \Leftrightarrow [\inf(a \oplus 0, 0 \oplus 0), 0 \oplus 0] = [0, 0] \Leftrightarrow [0, 0] = [0, 0]$. Thus, $\mathrm{L}_u.1$ holds.

- Given $[x, y] \in \Delta_{\mathcal{E}}(\mathcal{A})$ such that $[x, y] \geq [0, 0]$, we have that $x, y \in \mathrm{Rad}(\mathcal{A})$, i.e., $x \leq \neg x$ and $y \leq \neg y$. By axiom P*.2 we have $\bigvee_{n \in \mathbb{N}} x \leq na$ and $\bigvee_{n \in \mathbb{N}} y \leq ma$. Further, by definition of the order relation in $\Delta_{\mathcal{E}}(\mathcal{A})$

$$[x, y] \geq [0, 0] \Leftrightarrow x \geq y.$$

Thus $\bigvee_{n \in \mathbb{N}} y \leq x \leq na$ and $\bigvee_{n \in \mathbb{N}} [x, y] \leq n[a, 0]$. Therefore $\mathrm{L}_u.2$ holds. □

Let $\mathcal{A} = (A, a)$ and $\mathcal{A}' = (A', a')$ be two models of $\mathbb{P}^*$ in $\mathcal{E}$ and $h : \mathcal{A} \to \mathcal{A}'$ an arrow in $\mathbb{P}^*$-mod$(\mathcal{E})$, i.e., an MV-homomorphism such that $h(a) = a'$. We can consider $\Delta_{\mathcal{E}}(h)$. This is an ℓ-homomorphism satisfying $\Delta_{\mathcal{E}}(h)([a, 0]) = [h(a), 0] = [a', 0]$. So $\Delta_{\mathcal{E}}(h)$ defines an $\mathbb{L}_u$-model homomorphism $(\Delta_{\mathcal{E}}(\mathcal{A}), [a, 0]) \to (\Delta_{\mathcal{E}}(\mathcal{A}'), [a', 0])$. Thus $\Delta_{\mathcal{E}}$ is a functor from $\mathbb{P}^*$-mod$(\mathcal{E})$ to $\mathbb{L}_u$-mod$(\mathcal{E})$.

In the converse direction, let $\mathcal{G} = (G, u)$ be a model of $\mathbb{L}_u$ in $\mathcal{E}$. We know that $\Sigma(\mathcal{G})$ is a model of $\mathbb{P}$ in $\mathcal{E}$.

Proposition 4.9.2. *The structure* $(\Sigma_{\mathcal{E}}(G), (0, u))$ *is a model of* $\mathbb{P}^*$ *in* $\mathcal{E}$.

Proof. It remains to show that this structure satisfies P*.1 and P*.2.

- $\neg(0, u) = (1, u) \geq (0, u)$. Thus, P*.1 holds.

- Let (c, x) be an element of $\Sigma_{\mathcal{E}}(G)$ such that $(c, x) \leq \neg(c, x)$. By Theorem 4.2.2, $(c, x) = (0, y)$ with $y \geq 0$. Thus, by axiom $\mathrm{L}_u.2$, we have $\bigvee_{n \in \mathbb{N}} y \leq nu$. Hence, $\bigvee_{n \in \mathbb{N}} (0, y) \leq n(0, u)$ and P*.2 holds. □

It is easily seen that $\Sigma_{\mathcal{E}}$ is a functor from $\mathbb{L}_u$-mod($\mathcal{E}$) to $\mathbb{P}^*$-mod($\mathcal{E}$), i.e., that $\Sigma_{\mathcal{E}}(h)$ is an MV-homomorphism which preserves the generating element of the radical for every ℓ-unital homomorphism h.

Theorem 4.9.3. *The categories $\mathbb{P}^*$-mod($\mathcal{E}$) and $\mathbb{L}_u$-mod($\mathcal{E}$) are equivalent, naturally in $\mathcal{E}$. Hence the theories $\mathbb{P}^*$ and $\mathbb{L}_u$ are Morita-equivalent.*

Proof. This immediately follows from Theorem 4.4.2 noticing that the isomorphisms $\beta_{\mathcal{A}} : \mathcal{A} \to \Sigma_{\mathcal{E}} \circ \Delta_{\mathcal{E}}(\mathcal{A})$ and $\alpha_{\mathcal{G}} : \mathcal{G} \to \Delta_{\mathcal{E}} \circ \Sigma_{\mathcal{E}}(\mathcal{G})$ defined in the proof of Theorem 4.4.1 satisfy:

$\beta_{\mathcal{A}}(a) = (0, [a, 0])$;

$\alpha_{\mathcal{G}}(u) = [(0, u), (0, 0)]$. □

Remark 4.9.4. From Theorem 3.7.1 we obtain that the theory $\mathbb{P}^*$ is Morita-equivalent to the theory $\mathbb{MV}$.

Chapter 5

Morita-equivalences for theories of local MV-algebras

In the previous chapters we consider well-known equivalences between categories of MV-algebras and categories of ℓ-groups and we proved that they can be lifted to Morita-equivalences. As a consequence of these Morita-equivalences the infinitary theory of ℓ-groups with strong unit and the coherent theory of perfect MV-algebras are of presheaf type. Conversely, in this chapter we establish a new class of Morita, and categorical, equivalences that are unknown by the specialists of MV-algebras by proving first that the theories involved in these equivalences are of presheaf type. In particular, for every proper subvariety V of MV-algebras, we prove that the theory of local MV-algebras in V is Morita-equivalent to an appropriate extensions of the theory of ℓ-groups. Among the Morita-equivalences established here there is also the Morita-equivalence that arises from Di Nola-Lettieri's equivalence. Indeed, as we have already remarked, perfect MV-algebras are exactly the local MV-algebras contained in Chang's variety. The results of this chapter are contained in [22].

5.1 The algebraic theory of a Komori variety

In [36] Komori gave a (non-constructive) complete characterization of the lattice of all subvarieties of the variety of MV-algebras. In particular, he proved that every proper subvariety is generated by a finite number of finite simple MV-algebras $S_m = \Gamma(\mathbb{Z}, m)$ and a finite number of so-called *Komori chains*, i.e., algebras of the form $S_m^\omega = \Gamma(\mathbb{Z} \times_{\text{lex}} \mathbb{Z}, (m, 0))$. We call the varieties of this form *Komori varieties*. In this paper every proper subvariety V is intended to be a Komori variety.

The next result shows that, whilst a Komori variety can be presented by differents sets of generators, the least common multiple of the ranks of the generators is an invariant of the variety.

Proposition 5.1.1. *Let V be a Komori variety such that*

$$\begin{aligned} V &= V(S_{n_1}, \ldots, S_{n_k}, S^\omega_{m_1}, \ldots, S^\omega_{m_s}) \\ &= V(S_{n'_1}, \ldots, S_{n'_h}, S^\omega_{m'_1}, \ldots, S^\omega_{m'_t}) \ . \end{aligned}$$

The numbers $N = l\ .\ c\ .\ m\ .\ \{n_i, m_j \mid i = 1, \ldots, k, j = 1, \ldots, s\}$ and $N' = l\ .\ c\ .\ m\ .\ \{n'_i, m'_j \mid i = 1, \ldots, h, j = 1, \ldots, t\}$ are equal.

Proof. By Theorem 2.1[36] we have that:

$$S_{n'_i} \in V \Rightarrow \text{there exists } n \in \{n_i, m_j \mid i = 1, \ldots, k, j = 1, \ldots, s\} \text{ such that } n'_i \text{ divides } n;$$
$$S^\omega_{m'_j} \in V \Rightarrow \text{there exists } m \in \{m_j \mid j = 1, \ldots, s\} \text{ such that } m'_j \text{ divides } m.$$

This yields that $N' \leq N$. In a similar way we prove that $N \leq N'$; hence, $N = N'$, as required. □

Remark 5.1.2. By means of the same arguments used in the proof of the proposition, one can show that also the maximum of the ranks of the generators of a variety is an invariant. We use nonetheless the l.c.m. as invariant since we want to regard local MV-algebras in a given Komori variety as subalgebras of algebras of a fixed finite rank, which therefore must be a multiple of all the ranks of the generators of the variety (cf. Section 5.2 below).

Note that both the l.c.m. and the maximum are not discriminating invariants, i.e., there exist different varieties with the same associated invariant, for example $V(S_n)$ and $V(S_n^\omega)$ for any $n \in \mathbb{N}$. In [42], Panti identified a discriminating invariant in the concept of *reduced pair*: a pair (I, J) of finite subsets of $\mathbb{N}$ is said to be *reduced* if no $m \in I$ divides any $m_0 \in (I \setminus \{m\}) \cup J$, and no $t \in J$ divides any $t_0 \in J \setminus \{t\}$ (in particular, $I \cap J = \emptyset$).

Di Nola and Lettieri have given in [29] equational axiomatizations for all varieties of MV-algebras. More specifically, they have proved the following

Theorem* 5.1.3 ([29]). *Let* $V = V(S_{n_1}, \dots, S_{n_k}, S^\omega_{m_1}, \dots, S^\omega_{m_s})$, $I = \{n_1, \dots, n_k\}$, $J = \{m_1, \dots, m_s\}$ *and for each* $i \in I$,

$$\delta(i) = \{n \in \mathbb{N} \mid n \geq 1 \text{ and } n \text{ divides } i\} .$$

Then an MV-algebra lies in V *if and only if it is a model of the theory whose axioms are the axioms of* $\mathbb{MV}$ *plus the following:*

$$(\top \vdash_x ((n+1)x^n)^2 = 2x^{n+1}),$$

where $n = max(I \cup J)$*;*

$$(\top \vdash_x (px^{p-1})^{n+1} = (n+1)x^p),$$

for every positive integer $1<p<n$ *such that* p *is not a divisor of any* $i \in I \cup J$*;*

$$(\top \vdash_x (n+1)x^q = (n+2)x^q),$$

for every $q \in \bigcup_{i\in I} \Delta(i, J)$*, where*

$$\Delta(i, J) = \{d \in \delta(i) \setminus \bigcup_{j\in J} \delta(j)\} .$$

The theory obtained by adding to the theory defined in Theorem 5.1.3 the axiom

$$(\top \vdash_x ((n+1)x^n)^2 = 2x^{n+1}),$$

for $n = \text{l.c.m.}(I \cup J)$ will be denoted by $\mathbb{T}_V$, to stress that it consists of all the algebraic sequents which are satisfied by all the algebras in V. This additional axiom is actually classically redundant since it follows from that for $\max(I \cup J)$ as it expresses the property of an MV-chain to have rank $\leq n$ (cf. Lemma 8.4.1 [26]) and is satisfied by all the generators of V. From now on the number n attached to a variety V will always be the invariant defined in Proposition 5.1.1.

In Section 2.2 we studied the theory of perfect MV-algebras, that is, the theory of local MV-algebras in the variety generated by the algebra S_1^ω. We proved that the radical of any MV-algebra in $V(S_1^\omega)$ is defined by the equation

$$(2x)^2 = 0\ .$$

We also proved that all the elements of the form $(2x)^2$ in an algebra in $V(S_1^\omega)$ are boolean.

In light of these results, it is natural to conjecture that for an arbitrary Komori variety with associated invariant n, the radical of an MV-algebra $\mathcal{A}$ in V be defined by the formula $((n+1)x)^2 = 0$, and that the elements of the form $((n+1)x)^2$ be all boolean elements (notice that $n = 1$ in the case of Chang's variety). The following proposition settles the second question in the affirmative and provides the essential ingredients for the proof of the first conjecture which will be achieved in Lemma 5.1.8 below.

Proposition 5.1.4. *Let V be a Komori variety and n the associated invariant. Then the following sequents are provable in the theory $\mathbb{T}_V$:*

(i) $(((n+1)x)^2 = 0 \vdash_x ((n+1)kx)^2 = 0)$, *for every $k \in \mathbb{N}$;*

(ii) $(\top \vdash_x ((n+1)x)^2 \oplus ((n+1)x)^2 = ((n+1)x)^2)$.

Proof. (i) It clearly suffices to prove that $((n+1)x)^2 = 0 \vdash_x ((n+1)2x)^2 = 0)$ is provable in $\mathbb{T}_V$. To show this, let us first prove that the sequent $(((n+1)x)^2 = 0 \vdash_x (n+1)(\neg(2nx)) = 1)$ is provable in $\mathbb{T}_V$.

We can use the interpretation functor from MV-algebras to ℓ-groups with strong unit (as in Section 4.2) to verify the provability of this sequent by arguing in the language of ℓ-u groups. The condition $((n+1)x)^2 = 0$ is equivalent to $2\neg((n+1)x) = 1$ and hence to the condition $2(u - \inf(u,(n+1)x)) \geq u$ in the theory of ℓ-u groups. But $2(u - \inf(u,(n+1)x)) \geq u$ if and only if $\inf(u, 2(n+1)x - u) \leq 0$, which is equivalent, since $u \geq 0$, to the condition $2(n+1)x - u \leq 0$. Multiplying by n, we obtain that $n(2(n+1)x - u) \leq 0$. On the other hand, the condition $(n+1)(\neg(2nx)) = 1$ is equivalent to the condition $(n+1)(u - \inf(u, 2nx)) \geq u$ in the language of ℓ-u groups or, equivalently, to the condition $\inf(u,(n+1)(2nx-u)+u) \leq 0$. Since $(n+1)(2nx - u) + u = n(2(n+1)x - u)$, we are done.

Now that we have proved our sequent, to deduce our thesis, it suffices to show that the sequent $((n+1)(\neg(2nx)) = 1 \vdash_x ((n+1)(2x))^2 = 0)$ is provable in the theory $\mathbb{T}_V$. By writing $\neg(2nx) = \neg(n(2x)) = (\neg(2x))^n$ we see that $(n+1)(\neg(2nx)) = 1$ is equivalent to $(n+1)((\neg(2x))^n) = 1$. The first axiom of $\mathbb{T}_V$ thus yields that $2((\neg(2x))^{n+1}) = 1$; but $(\neg(2x))^{n+1} = \neg((n+1)(2x))$ whence $2\neg((n+1)(2x)) = 1$, that is $((n+1)(2x))^2 = 0$, as required.

(ii) We shall argue as in (i) in the language of ℓ-u groups to show the provability of the given sequent. Let us start reformulating the axiom $(\top \vdash_x ((n+1)x^n)^2 = 2x^{n+1})$ of $\mathbb{T}_V$ in the language of ℓ-u groups. It is easy to see, by means of simple calculations in the theory of ℓ-u groups, that the term $2x^{n+1}$ corresponds to the term $\inf(u, \sup(0, 2((n+1)(x-u)+u)))$, while the term $((n+1)x^n)^2$ corresponds to the term $\sup(0, u + \inf(0, \sup(-2u, -2(n+1)(nu - nx - u) - 2u)))$. Now, $2(n+1)(nx - nu + u) - 2u = 2n(nx + x - nu)$. Let us set $z = 2(nx + x - nu)$. Then $2((n+1)(x-u)+u)) = z$ and $-2(n+1)(nu - nx - u) - 2u = nz$, so the two terms rewrite respectively as $\inf(u, z^+)$ and $\sup(0, u + \inf(0, \sup(-2u, nz)))$.

The sequent in the theory of ℓ-u groups which corresponds to the sequent $(\top \vdash_x ((n+1)x^n)^2 = 2x^{n+1})$ is therefore

$$(0 \leq x \leq u \vdash_x \inf(u, z^+) = \sup(0, u + \inf(0, \sup(-2u, nz)))),$$

where z is an abbreviation for the term $2(nx + x - nu)$. We have to prove that this sequent provably entails the sequent expressing the property that the elements of the form $((n+1)x)^2$ are boolean. Let us first prove that the elements of the form $2x^{n+1}$, that is, of the form $\inf(u, z^+)$ in the language of ℓ-u groups, are boolean. Clearly, this is the case if and only if $\inf(u, 2z^+) \leq z^+$. To show this, we observe that the above-mentioned sequent implies that $z^+ \geq \inf(u, z^+) = \sup(0, u + \inf(0, \sup(-2u, nz))) \geq u + \inf(0, \sup(-2u, nz)) \geq u + \inf(0, nz) = u - nz^-$, in other words $u \leq z^+ + nz^-$. So $\inf(u, 2z^+) \leq \inf(z^+ + nz^-, z^+ + z^+) = z^+ + \inf(nz^-, z^+)$. But $\inf(nz^-, z^+) = 0$ since $0 \leq \inf(nz^-, z^+) \leq \inf(nz^-, nz^+) = n \inf(z^-, z^+) = 0$. This completes the proof that $\inf(u, z^+)$ is a boolean element. Now, we can rewrite the term $((n+1)x)^2$ as $\neg(2(\neg x)^{n+1})$. By the first part of the proof, $2(\neg x)^{n+1}$ is a boolean element. But the negation of a boolean element is still a boolean element, whence $((n+1)x)^2$ is boolean, as required. □

5.1.1 The theory $\mathbb{L}oc^1_V$

To prove that the formula $((n+1)x)^2 = 0$ defines the radical of an MV-algebra in V, it is convenient to regard the theory $\mathbb{T}_V$ as a sub-theory of a theory of which it is the cartesianization and in which computations are easier. A quotient of $\mathbb{T}_V$ satisfying this requirement (cf. Proposition 5.1.6 below) is the theory $\mathbb{L}oc^1_V$ obtained from $\mathbb{T}_V$ by adding the following sequents:

σ_n: $(\top \vdash_x ((n+1)x)^2 = 0 \vee (n+1)x = 1)$;

NT: $(0 = 1 \vdash \bot)$.

We use the notation $\mathbb{L}oc^1_V$ because, as we shall see in Section 5.2 (cf. Proposition 5.2.5), the models of $\mathbb{L}oc^1_V$ in **Set** are precisely the local MV-algebras in V (at least non-constructively).

As a quotient of $\mathbb{T}_V$, the theory $\mathbb{L}oc^1_V$ is associated with a Grothendieck topology J_1 on the category f.p.$\mathbb{T}_V$-mod$(\mathbf{Set})^{\text{op}}$.

Proposition 5.1.5. *The Grothendieck topology associated with* $\mathbb{L}\text{oc}^1_V$ *as a quotient of* $\mathbb{T}_V$ *is subcanonical.*

Proof. The topology J_1 associated with the quotient $\mathbb{L}oc^1_V$ of $\mathbb{T}_V$ can be calculated as follows. The sequent NT produces the empty cocovering on the trivial algebra, while the sequent σ_n produces, for every $\mathcal{A} \in$ f.p.$\mathbb{T}_V$-mod(**Set**), the cosieve generated by finite multicompositions of diagrams of the following form:

$$\begin{array}{ccc} & & \mathcal{A}/(((n+1)x)^2) \\ & \nearrow & \\ \mathcal{A} & & \\ & \searrow & \\ & & \mathcal{A}/(\neg((n+1)x)^2) \end{array}$$

By Proposition 5.1.4(ii), the elements of the form $((n+1)x)^2$ are boolean elements of $\mathcal{A}$. Thus, by the Pushout-Pullback Lemma we have that $\mathcal{A}$ is a direct product of $\mathcal{A}/(((n+1)x)^2)$ and $\mathcal{A}/(\neg((n+1)x)^2)$. We can repeat the same reasoning for each pair of arrows in a finite multicomposition; each J_1-multicomposition thus yields a representation of $\mathcal{A}$ as a direct product of the algebras appearing as codomains of the arrows in it. Finally, for every boolean element x of an MV-algebra $\mathcal{A}$, there is an arrow $\mathcal{A}/(x) \to \mathcal{A}/(\neg x)$ over $\mathcal{A}$ if and only if $x = 0$, whence the sieve in f.p.$\mathbb{T}_V$-mod$(\mathbf{Set})^{\text{op}}$ generated by the family $\{\mathcal{A}/(x) \to \mathcal{A}, \mathcal{A}/(\neg x) \to \mathcal{A}\}$ is effective epimorphic if and only if $\{\mathcal{A} \to \mathcal{A}/(x), \mathcal{A} \to \mathcal{A}/(\neg x)\}$ is a product diagram in f.p.$\mathbb{T}_V$-mod(**Set**). This proves our statement. □

In the sequel we shall refer to multicompositions of diagrams as in the proof of Proposition 5.1.5 as to J_1-multicompositions.

Proposition 5.1.6. *The cartesianization of the theory $\mathbb{L}oc_V^1$ is the theory $\mathbb{T}_V$.*

Proof. Since the theory $\mathbb{T}_V$ is algebraic, its universal model U in its classifying topos $[\text{f.p.}\mathbb{T}_V\text{-mod}(\mathbf{Set}), \mathbf{Set}]$ is of the form $\text{Hom}_{\text{f.p.}\mathbb{T}_V\text{-mod}(\mathbf{Set})}(F, -)$, where F is the free algebra in V on one generator. By Proposition 5.1.5, the topology J_1 is subcanonical; hence, the model U lies in the classifying topos of the theory $\mathbb{L}oc_V^1$ and is, as such, also 'the' universal model of $\mathbb{L}oc_V^1$. Now, given a cartesian sequent σ in the language of MV-algebras, if σ is provable in the theory $\mathbb{L}oc_V^1$, then it is valid in U, regarded as a model in the classifying topos of $\mathbb{L}oc_V^1$. Since $\mathcal{E}_{\mathbb{L}oc_V^1}$ is a subtopos of $\mathcal{E}_{\mathbb{T}_V}$ and the interpretations of cartesian formulas are the same in the two toposes, we have that σ holds also in U regarded as a structure in $\mathcal{E}_{\mathbb{T}_V}$, and hence that σ is provable in $\mathbb{T}_V$. □

5.1.2 Constructive definition of the radical

Proposition 5.1.6 allows us to establish the provability of cartesian sequents over the signature of $\mathbb{MV}$ in the theory $\mathbb{T}_V$ by showing it in the theory $\mathbb{L}oc_V^1$.

Lemma 5.1.7. *The following sequents are provable in the theory $\mathbb{T}_V$:*

(i) $(kx = 1 \vdash_x (n+1)x = 1)$, *for every* $k \in \mathbb{N}$;

(ii) $(((n+1)x)^2 = 0 \wedge y \leq x \vdash_{x,y} ((n+1)y)^2 = 0)$;

(iii) $(((n+1)x)^2 = 0 \vdash_x ((n+1)kx)^2 = 0)$, *for every* $k \in \mathbb{N}$;

(iv) $(((n+1)x)^2 = 0 \vdash_x (kx)^2 = 0)$, *for every* $k \in \mathbb{N}$;

(v) $(((n+1)x)^2 = 0 \vdash_x kx \leq \neg x)$, *for every* $k \in \mathbb{N}$;

(vi) $(((n+1)x)^2 = 0 \wedge ((n+1)y)^2 = 0 \vdash_{x,y} ((n+1)(x \vee y))^2 = 0)$;

(vii) $(((n+1)x)^2 = 0 \wedge ((n+1)y)^2 = 0 \vdash_{x,y} ((n+1)(x \oplus y))^2 = 0)$;

(viii) $((n+1)x \leq \neg x \vdash_x ((n+1)x)^2 = 0)$.

(ix) $(((n+1)\neg x)^2 = 0 \vdash_x 2x = 1)$.

Proof. By Proposition 5.1.6, every cartesian sequent that is provable in $\mathbb{L}oc^1_V$ is also provable in $\mathbb{T}_V$. Since (i)-(ix) are cartesian sequents, it is therefore sufficient to show that they are provable in $\mathbb{L}oc^1_V$. We argue (informally) as follows. First of all, we notice that:

$$((n+1)x)^2 = 0 \Leftrightarrow (n+1)x \leq \neg(n+1)x \; .$$

(i) Let us suppose that $kx = 1$. By σ_n, we know that either $((n+1)x)^2 = 0$ or $(n+1)x = 1$. If $((n+1)x)^2 = 0$ then by Proposition 5.1.4(i), $((n+1)kx)^2 = 0$. But $kx = 1$, whence $1 = 0$, contradicting sequent NT. Therefore $(n+1)x = 1$, as required.

(ii) If $((n+1)x)^2 = 0$ and $y \leq x$, then

$$(n+1)y \leq (n+1)x \leq \neg(n+1)x \leq \neg(n+1)y \; .$$

Therefore $((n+1)y)^2 = 0$, as required.

(iii) See Proposition 5.1.4(i).

(iv) If $((n+1)x)^2 = 0$ then (by (iii)) for any $k \in \mathbb{N}$, $((n+1)kx)^2 = 0$, in other words $(n+1)kx \leq \neg(n+1)kx$. Thus,

$$kx \leq (n+1)kx \leq \neg(n+1)kx \leq \neg kx,$$

whence $(kx)^2 = 0$ (for any $k \in \mathbb{N}$).

(v) If $((n+1)x)^2 = 0$, then (by (iii))

$$(kx)^2 = 0, \text{ for every } k \in \mathbb{N} \; .$$

Thus,

$$kx \leq \neg kx \leq \neg x, \text{ for every } k \in \mathbb{N}\ .$$

(vi) For any x, y, we have that:

$$((n+1)(x \vee y)) = 1 \Leftrightarrow (n+1)x \vee (n+1)y = 1 \Leftrightarrow$$
$$((n+1)x)^2 \vee ((n+1)y)^2 = 1 \text{ (cf. Theorem 3.7 [24])}.$$

If $((n+1)x)^2 = 0$ and $((n+1)y)^2 = 0$ it then follows from sequents σ_n and NT that

$$((n+1)(x \vee y))^2 = 0\ .$$

(vii) If $((n+1)x)^2 = 0$ and $((n+1)y)^2 = 0$ then

$$((n+1)2(x \vee y))^2 = 0 \text{ (by (vi) and (iii))}.$$

Since $x \oplus y \leq 2(x \vee y)$, it then follows from (ii) that $((n+1)(x \oplus y))^2 = 0$.

(viii) Let us suppose that $(n+1)x \leq \neg x$ and $(n+1)x = 1$. By sequent σ_n, this means that $\neg x = 1$ whence $x = 0$. Sequent NT thus implies that $((n+1)x)^2 = 0$.

(ix) If $((n+1)(\neg x))^2 = 0$ then

$$\neg x \leq (n+1)(\neg x) \leq \neg(n+1)(\neg x) \leq x$$

$$\Rightarrow 2x = 1\ .$$

□

Lemma 5.1.8. *Given $\mathcal{A} \in V$, the set $K(\mathcal{A}) = \{x \in A \mid ((n+1)x)^2 = 0\}$ is an ideal of $\mathcal{A}$ and it coincides with the radical of $\mathcal{A}$.*

Proof. By Lemma 5.1.7(ii) and (vii), the set $K(\mathcal{A})$ is a $\leq$-downset and it is closed with respect to the sum. Clearly, it contains 0; thus, it is an ideal of $\mathcal{A}$. By Lemma 5.1.7(v), every element in $K(\mathcal{A})$ is either 0 or an infinitesimal element. Vice versa, if x is an infinitesimal element then in particular $(n+1)x \leq \neg x$, whence $x \in K(\mathcal{A})$ by Lemma 5.1.7(viii). □

Remark 5.1.9. The radical is defined equivalently by the equation $(kx)^2 = 0$, for any $k \geq (n+1)$. Indeed, by Lemma 5.1.7(iv) we have that if $((n+1)x)^2 = 0$ then $(kx)^2 = 0$ for every k. Vice versa, if $(kx)^2 = 0$ with $k \geq n+1$ then

$$(n+1)x \leq kx \leq \neg kx \leq \neg(n+1)x,$$

whence $((n+1)x)^2 = 0$.

Lemma 5.1.10. *Let $\mathcal{A}$ be an MV-algebra in V. Then the structure*

$$(Rad(\mathcal{A}), \oplus, \wedge, \vee, 0)$$

is a cancellative lattice-ordered monoid.

Proof. By Lemma 5.1.8, $\mathrm{Rad}(\mathcal{A})$ is an ideal. Thus, it is a lattice-ordered monoid. It remains to prove that it is cancellative. We shall deduce this as a consequence of the following two claims.

Claim 1. Given $x, y \in \neg\mathrm{Rad}(\mathcal{A})$, $x \oplus y = 1$.

Indeed, if $x, y \in \neg\mathrm{Rad}(\mathcal{A})$, then $\neg x, \neg y \in \mathrm{Rad}(\mathcal{A})$. Thus, $\neg x \oplus \neg y$ is an infinitesimal element by Lemma 5.1.7(iv). Hence, $\neg x \oplus \neg y \leq \neg(\neg x \oplus \neg y)$, equivalently $\neg(x \odot y) \leq x \odot y$. But

$$\neg(x \odot y) \leq x \odot y \Leftrightarrow (x \odot y) \oplus (x \odot y) = 1 \Leftrightarrow \mathrm{ord}(x \odot y) \leq 2,$$

and $\mathrm{ord}(x \odot y) \leq 2$ implies $x \oplus y = 1$ (see Theorem 3.8 [24]).

Claim 2. Given $x \in \mathrm{Rad}(\mathcal{A})$ and $y \in \neg\mathrm{Rad}(\mathcal{A})$, $x \leq y$.

This follows from Claim 1 since $\neg x \oplus y = 1 \Leftrightarrow x \leq y$.

Given $x, y, a \in \mathrm{Rad}(\mathcal{A})$ such that $x \oplus a = y \oplus a$, we clearly have that $\neg a \odot (x \oplus a) = \neg a \odot (y \oplus a)$. But by Proposition 1.1.5 [26] this is equivalent to $\neg a \wedge x = \neg a \wedge y$. By Claim 2, we can thus conclude that $x = y$. □

5.2 Where local MV-algebras meet varieties

In this section we study classes of local MV-algebras in proper subvarieties of the variety of MV-algebras and the theories that axiomatize them.

Definition 5.2.1. Let n be a positive integer. A local MV-algebra $\mathcal{A}$ is said to be of *rank* n if $\mathcal{A}/\mathrm{Rad}(\mathcal{A}) \cong S_n$ (where S_n is the simple n-element MV-algebra) and it is said to be of *finite rank* if $\mathcal{A}$ is of rank n for some integer n.

The generators of Komori varieties are particular examples of local MV-algebras of finite rank.

In [27] it is proved (in a non-constructive way) that every local MV-algebra in a Komori variety is of finite rank.

Definition 5.2.2 ([27])**.** Let I, J be finite subsets of $\mathbb{N}$. We denote by

$$\mathrm{Finrank}(I, J)$$

the class of simple MV-algebras embeddable into a member of $\{S_i \mid i \in I\}$ and of local MV-algebras $\mathcal{A}$ of finite rank such that $\mathcal{A}/\mathrm{Rad}(\mathcal{A})$ is embeddable into a member of $\{S_j \mid j \in J\}$.

Theorem* 5.2.3 (Theorem 7.2 [27])**.** *The class of local MV-algebras contained in the variety $V(\{S_i\}_{i \in I}, \{S_j^{\omega}\}_{j \in J})$ is equal to Finrank(I, J)*[1]*.*

[1] The non-constructive part of this result concerns the fact that the rank of a local MV-algebra in V is finite. On the other hand, the fact that the rank, if finite, divides the rank of one of the generators follows by Theorem 2.3 [36], which is constructive.

The following theorem provides a representation for local MV-algebras of finite rank.

Theorem* 5.2.4 (Theorem 5.5 [27]). *Let $\mathcal{A}$ be a local MV-algebra. Then the following are equivalent:*

(i) $\mathcal{A}$ is an MV-algebra of finite rank n;

(ii) $\mathcal{A} \cong \Gamma(\mathbb{Z} \times_{lex} \mathcal{G}, (n, h))$ where $\mathcal{G}$ is an ℓ-group and $h \in G$.

Using this theorem, one can show that the theory $\mathbb{L}oc^1_V$ introduced in Section 5.1.1 axiomatizes the local MV-algebras in V.

Proposition* 5.2.5. *Let $\mathcal{A}$ be an MV-algebra in V. Then $\mathcal{A}$ is a model of $\mathbb{L}oc^1_V$ if and only if it is a local MV-algebra (i.e., a model of $\mathbb{L}oc$).*

Proof. Let us suppose that $\mathcal{A}$ is a model of the theory $\mathbb{L}oc^1_V$. Given $x \in A$, by sequent σ_n either $(n+1)x = 1$ or $((n+1)x)^2 = 0$. If $(n+1)x = 1$ then the order of x is finite. If $((n+1)x)^2 = 0$ then, by sequent σ_n, $(n+1)\neg x = 1$ as $((n+1)x)^2 = 0$ and $((n+1)\neg x)^2 = 0$ imply by Lemma 5.1.7(vi) that $1 = ((n+1)(x \oplus \neg x))^2 = 0$, contradicting sequent NT.

Conversely, suppose that $\mathcal{A}$ is a local MV-algebra. By Theorems 5.2.3 and 5.2.4, $\mathcal{A} \cong \Gamma(\mathbb{Z} \times_{\text{lex}} \mathcal{G}, (d, h))$, where d divides n. It is easy to verify that the elements of $\text{Rad}(A)$ are precisely those whose first component is 0, while any other element x satisfies the equation $(n+1)x = 1$. So $\mathcal{A}$ is a model of $\mathbb{L}oc^1_V$, as required. □

Remark 5.2.6. The non-constructive part of the proposition is the 'if' direction; the 'only if' part is constructive.

Proposition 5.2.7. *Let $\mathcal{A}$ be a model of $\mathbb{L}oc^1_V$. Then the radical of $\mathcal{A}$ is the only maximal ideal of $\mathcal{A}$.*

Proof. By Lemma 5.1.7, $\text{Rad}(\mathcal{A}) = \{x \in A \mid ((n+1)x)^2 = 0\}$. Let I be an ideal of $\mathcal{A}$. If there exists $x \in I$ such that $(n+1)x = 1$ then I is equal to A.

Otherwise $I \subseteq \mathrm{Rad}(\mathcal{A})$ by sequent σ_n. Thus, the radical is the only maximal ideal of $\mathcal{A}$. □

We shall now proceed to identifying an axiomatization for the local MV-algebras in V which will allow to constructively prove that the Grothendieck topology associated with it is rigid.

We observe that if $\mathcal{A}$ is a local MV-algebra in V of finite rank k and n is the invariant of V defined by Proposition 5.1.1 then the rank of $\mathcal{A}$ divides n. So, by Theorems 5.5 and 5.6 [27], we have embeddings of MV-algebras

$$\mathcal{A} \cong \Gamma(\mathbb{Z} \times_{\mathrm{lex}} \mathcal{G}, (k, g)) \overset{f}{\hookrightarrow} \Gamma(\mathbb{Z} \times_{\mathrm{lex}} \mathcal{G}, (k, 0)) \overset{g}{\hookrightarrow} \Gamma(\mathbb{Z} \times_{\mathrm{lex}} \mathcal{G}, (n, 0))$$

for some ℓ-group $\mathcal{G}$. The embedding f sends an element (m, y) of $\Gamma(\mathbb{Z} \times_{\mathrm{lex}} \mathcal{G}, (k, g))$ to the element $(m, ky - mg)$ of $\Gamma(\mathbb{Z} \times_{\mathrm{lex}} \mathcal{G}, (k, 0))$, while g is the homomorphism of multiplication by the scalar $\frac{n}{k}$. Clearly, both f and g lift to unital group homomorphisms $(\mathbb{Z} \times_{\mathrm{lex}} \mathcal{G}, (k, g)) \to (\mathbb{Z} \times_{\mathrm{lex}} \mathcal{G}, (k, 0))$ and $(\mathbb{Z} \times_{\mathrm{lex}} \mathcal{G}, (k, 0)) \to (\mathbb{Z} \times_{\mathrm{lex}} \mathcal{G}, (n, 0))$.

Identifying $\mathcal{A}$ with its image $g(f(\mathcal{A}))$, we can partition its elements into *radical classes* (i.e., equivalence classes with respect to the relation induced by the radical), corresponding to the inverse images of the numbers $d = 0, \ldots, n$ under the natural projection map $\phi : \mathcal{A} \to \mathbb{Z}$. Note that, regarding S_n as the simple $(n+1)$-element MV-algebra $\{0, 1, \ldots, n\}$, ϕ is an MV-algebra homomorphism $\mathcal{A} \to S_n$. Moreover, we have that $\mathrm{Rad}(\mathcal{A}) = \phi^{-1}(0)$ and that $\phi(a) = \phi(a')$ if and only if $a \equiv_{\mathrm{Rad}(\mathcal{A})} a'$. We shall write $\mathrm{Fin}^n_d(A)$ for $\phi^{-1}(d)$. Notice that this is not really a partition in the strict sense of the term since some of the sets $\phi^{-1}(d)$ could be empty.

We shall see below in this section that these radical classes can be defined by Horn formulas over the signature of $\mathbb{MV}$.

An important feature of these radical classes is that they are compatible with respect to the MV-operations, in the sense that the radical class to which an element $t(x_1, \ldots, x_r)$ obtained by means of a term combination of elements $x_1, \ldots, x_r$ belongs is uniquely and canonically determined by the radical classes to which the elements $x_1, \ldots, x_r$ belong. Indeed, the conditions

$$(x \in \mathrm{Fin}^n_d \wedge y \in \mathrm{Fin}^n_b \vdash_{x,y} x \oplus y \in \mathrm{Fin}^n_{d \oplus b}),$$

(for each $d, b \in \{0, \ldots, n\}$ and where with $d \oplus b$ we indicate the sum in $S_n = \{0, 1, \ldots, n\}$), and

$$(x \in \mathrm{Fin}^n_d \vdash_x \neg x \in \mathrm{Fin}^n_{n-d}),$$

(for each $d \in \{0, \ldots, n\}$) are valid in every MV-algebra $\mathcal{A}$ in V.

Notice that, for a local MV-algebra of finite rank $\mathcal{A}$ in V, neither the three-element partition

$$A = \mathrm{Rad}(\mathcal{A}) \cup \mathrm{Fin}(\mathcal{A}) \cup \neg\mathrm{Rad}(\mathcal{A}),$$

nor the two-element partition

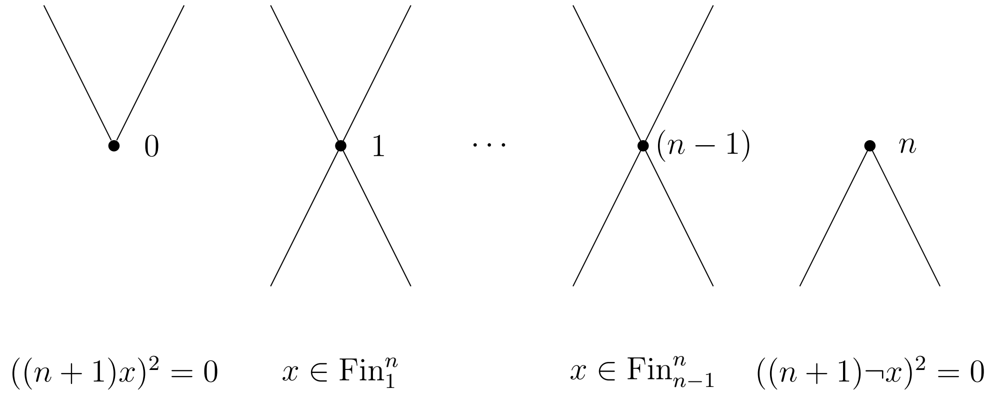
$$A = \mathrm{Rad}(\mathcal{A}) \cup (\mathrm{Fin}(\mathcal{A}) \cup \neg\mathrm{Rad}(\mathcal{A}))$$

satisfy this compatibility property. Indeed, the sum of two elements in $\mathrm{Fin}(\mathcal{A})$ can be in $\mathrm{Fin}(\mathcal{A})$ or in $\neg\mathrm{Rad}(\mathcal{A})$, and the negation of an element in $(\mathrm{Fin}(\mathcal{A}) \cup \neg\mathrm{Rad}(\mathcal{A}))$ can be either in $\mathrm{Rad}(\mathcal{A})$ or in $(\mathrm{Fin}(\mathcal{A}) \cup \neg\mathrm{Rad}(\mathcal{A}))$.

The compatibility property of the partition

$$A \subseteq \bigcup_{d \in \{0,\ldots,n\}} x \in \mathrm{Fin}^n_d(\mathcal{A}),$$

together with the definability of the radical classes by Horn formulas, will be the key for designing an axiomatization for the local MV-algebras in V such that the corresponding Grothendieck topology is rigid.



To the end of obtaining definitions within geometric logic of the predicates $x \in \mathrm{Fin}^n_d$, we recall the following version of Bezout's identity.

Theorem 5.2.8 (Bézout's identity)**.** *Let a and b be natural numbers. Then, denoting by D the greatest common divisor of a and b, there exist exactly one natural number $0 \leq \xi_{(a,b)} \leq \frac{b}{D}$ and one natural number $0 \leq \chi_{(a,b)} \leq \frac{a}{D}$ such that $D = \xi_{(a,b)}a - \chi_{(a,b)}b$.*

Notice that if a divides b then $D = a$ and $\xi_{(a,b)} = 1$, $\chi_{(a,b)} = 0$.

Given $d \in \{1, \dots, n\}$, we set $D = \text{g.c.d.}(d, n)$ and consider the following Horn formula over the signature of $\mathbb{MV}$ (where we write ky for $y \oplus \cdots \oplus y$ k times):

$$\alpha^n_d(x) := (x \equiv^n_{\mathrm{Rad}} \frac{d}{D} D^x_{d,n}) \wedge (\bigwedge_{k=0}^{\frac{n}{D}} \neg k D^x_{d,n} \equiv^n_{\mathrm{Rad}} (\frac{n}{D} - k) D^x_{d,n}),$$

where $\equiv^n_{\mathrm{Rad}}$ is the equivalence relation defined by $z \equiv^n_{\mathrm{Rad}} w$ if and only if $((n+1)d(z,w))^2 = 0$ and $D^x_{d,n}$ is the MV-algebraic term in x obtained in the following way. We would like $D^x_{d,n}$ to be equal to the element $\xi_{(d,n)}x - \chi_{(d,n)}u$ in the unit interval of the ℓ-u group $L(\mathcal{A}) = (\mathbb{Z} \times_{\mathrm{lex}} \mathcal{G}, (k, g))$ corresponding to the MV-algebra $\mathcal{A} = \Gamma(\mathbb{Z} \times_{\mathrm{lex}} \mathcal{G}, (k, g))$ if $\phi(x) = d$. To this end, we show that, given $x \in A$, if $\phi(x) = d$ then the element $\xi_{(d,n)}x - \chi_{(d,n)}u$ belongs to A, that is, $0 \leq \xi_{(d,n)}x - \chi_{(d,n)}u \leq u$ in the group $\mathbb{Z} \times_{\mathrm{lex}} \mathcal{G}$. Indeed, since f and g are unital group homomorphisms, $g(f(\xi_{(d,n)}x - \chi_{(d,n)}u)) = \xi_{(d,n)}g(f(x)) - \chi_{(d,n)}(n,0) = (D,y)$ for some element $y \in G$. Now, there are two cases: either d divides n or d does not divide n. In the first case, $\xi_{(d,n)} = 1$ and $\chi_{(d,n)} = 0$, so $\xi_{(d,n)}x - \chi_{(d,n)}u = x$ and we are done since $x \in A$. In the second case, $D = \text{g.c.d}(d,n)$ is strictly less than d, whence $0 \leq g(f(\xi_{(d,n)}x - \chi_{(d,n)}u)) \leq (n,0) = g(f(u))$ by definition of the lexicografic ordering on $\mathbb{Z} \times_{\text{lex}} \mathcal{G}$; so, since f and g reflect the order, $0 \leq \xi_{(d,n)}x - \chi_{(d,n)}u \leq u$, that is, $\xi_{(d,n)}x - \chi_{(d,n)}u \in A$, as required.

To express $D^x_{d,n}$ as a term in the language of $\mathbb{MV}$, we recall that the elements of the positive cone of the ℓ-u group associated with an MV-algebra $\mathcal{A}$ can be represented as 'good sequences' (in the sense of Section 3.4) of elements of A and that the elements a of A correspond to the good sequences of the form $(a,0,0,\ldots)$. Let us identify x with the good sequence $(x) = (x,0,\ldots,0,\ldots)$ and u with the good sequence $(1) = (1,0,\ldots,0,\ldots)$.

Note that if $a = (a_1,\ldots,a_r)$ and $b = (b_1,\ldots,b_t)$ are two good sequences, we can suppose without loss of generality that $r = t$. Indeed,

$$(a_1,\ldots,a_r) = (a_1,\ldots,a_r,0^m),$$

for every natural number $m \geq 1$.

Let $a = (a_1,\ldots,a_r)$ be a good sequence. With the symbol a^* we indicate the sequence $(a_r,\ldots,a_1)$. Note that this sequence is not necessarily a good sequence.

Proposition 5.2.9 (Proposition 2.3.4 [26])**.** *Let $a = (a_1,\ldots,a_r)$ and $b = (b_1,\ldots,b_r)$ be two good sequences. If $a \leq b$ then there is a unique good sequence c such that $a + c = b$, denoted by $b - a$ and given by:*

$$c = (b_1,\ldots,b_r) + (\neg a_r,\ldots,\neg a_1) = b + (\neg a)^* .$$

We define the term $D^x_{d,n}$ as the first component of the following sequence:

$$\xi_{(d,n)}(x) - \chi_{(d,n)}(1) := \xi_{(d,n)}(1) + (\neg(\chi_{(d,n)}(x)))^* .$$

By the proposition and the above remarks, if $\phi(x) = d$ then $\xi_{(d,n)}(x) - \chi_{(d,n)}(1)$ is actually a good sequence equal to $(D^x_{d,n}, 0, 0, \ldots)$, since $0 \leq \xi_{(d,n)}x - \chi_{(d,n)}u \leq u$.

From now on we abbreviate the formula $\alpha^n_d(x)$ by the expression $x \in \mathrm{Fin}^n_d$; if $d = 0$, we set $x \in \mathrm{Fin}^n_0$ as an abbreviation for the expression $((n+1)x)^2 = 0$. This is justified by the following

Proposition 5.2.10. *Let $\mathcal{A}$ be a local MV-algebra of finite rank in a Komori variety V, and n be the invariant of V as defined in Proposition 5.1.1. Then an element x of A satisfies the formula α^n_d if and only if it belongs to $Fin^n_d(\mathcal{A})$.*

Proof. Let us use the notation introduced before the statement of the Proposition. If $x \in \mathrm{Fin}^n_d(\mathcal{A})$, that is, $\phi(x) = d$, then

$$\phi(D^n_{d,x}) = \xi_{(d,n)}d - \chi_{(d,n)}n = D \ .$$

Thus,

$$\phi(x) = \frac{d}{D}\phi(D^x_{d,n}) \ .$$

Further,

$$\phi(\neg kD^x_{d,n}) = n - kD = (\frac{n}{D} - k)\phi(D^x_{d,n}),$$

for every $k = 0, \ldots, \frac{n}{D}$. So, by the above remarks, x satisfies α^n_d.

Conversely, let $x = (m, g)$ be an element of A (regarded as a subalgebra of $\mathbb{Z} \times_{\mathrm{lex}} \mathcal{G}$ via the embedding $g \circ f$). If x satisfies α^n_d then, since $\phi : \mathcal{A} \to S_n$ is a MV-algebra homomorphism, we have that:

$$m = \frac{d}{D}\phi(D^x_{d,n}) \text{ and } n - k\phi(D^x_{d,n}) = (\frac{n}{D} - k)\phi(D^x_{d,n})$$

in S_n for every $k = 0, \ldots, \frac{n}{D}$. In particular,

$$m = \frac{d}{D}\phi(D^x_{d,n}) \text{ and } n - \phi(D^x_{d,n}) = \frac{n}{D}\phi(D^x_{d,n}) - \phi(D^x_{d,n}) \Rightarrow$$

$$m = \frac{d}{D}\phi(D^x_{d,n}) \text{ and } D = \phi(D^x_{d,n}) \Rightarrow$$

$$m = d \ .$$

Hence, the element x is in $\mathrm{Fin}^n_d(\mathcal{A})$. □

Given an arbitrary MV-algebra $\mathcal{A}$, we use the expression $x \in \text{Fin}_d^n(\mathcal{A})$ as an abbreviation for the condition $x \in [\![x \,.\, \alpha_d^n(x)]\!]_{\mathcal{A}}$. By the proposition, this notation agrees with the other notation $\text{Fin}_d^n(\mathcal{A}) = \phi^{-1}(d)$ introduced above for a local MV-algebra $\mathcal{A}$ in V.

Remark 5.2.11. It is important to notice that, unless n is the rank of $\mathcal{A}$, the condition $x \in \text{Fin}_d^n(\mathcal{A})$ is *not* equivalent to the condition $(\exists y)(y \in \text{Fin}_1^n \wedge x = dy)$. Indeed, $\mathcal{A}$ is only *contained* in $\Gamma(\mathbb{Z} \times_{\text{lex}} \mathcal{G}, (n, 0))$ so $\phi^{-1}(1)$ could for instance be empty.

5.2.1 The theory $\mathbb{L}oc_V^2$

Let us consider the geometric sequent

$$\rho_n : (\top \vdash_x \bigvee_{d=0}^{n} x \in \text{Fin}_d^n),$$

and call $\mathbb{L}oc_V^2$ the quotient of $\mathbb{T}_V$ obtained by adding the sequents ρ_n and NT. This notation is justified by the following

Theorem* 5.2.12. *Let $\mathcal{A}$ be an MV-algebra in V. Then the following conditions are equivalent:*

(i) $\mathcal{A}$ is a local MV-algebra;

(ii) $\mathcal{A}$ is a model of $\mathbb{L}oc_V^2$.

Proof. The direction (i) $\Rightarrow$ (ii) follows from Theorem 5.2.4 and the discussion following it.

To prove the (ii) $\Rightarrow$ (i) direction, we have to verify that if $\mathcal{A}$ is a model of $\mathbb{L}oc_V^2$ then it is local. For this, it suffices to verify, thanks to Proposition 5.2.5, that the theory $\mathbb{L}oc_V^2$ is a quotient of $\mathbb{L}oc_V^1$, in other words that the sequent σ_n is provable in $\mathbb{L}oc_V^2$. We argue informally as follows. If $x \in \text{Fin}_0^n$, then by definition $((n+1)x)^2 = 0$. If $x \in \text{Fin}_d^n$ with $d \neq 0$ then by definition

$$(x \equiv_{\text{Rad}}^n \frac{d}{D} D_{d,n}^x) \wedge \bigwedge_{k=0}^{\frac{n}{D}} (\neg k D_{d,n}^x \equiv_{\text{Rad}}^n (\frac{n}{D} - k) D_{d,n}^x),$$

where $D = \text{g.c.d.}(d, n)$. In particular, taking $k = 0$, we have that

$$\frac{n}{D} D^x_{d,n} \equiv^n_{\text{Rad}} 1 \ .$$

It follows that

$$\frac{n}{D} x \equiv^n_{\text{Rad}} \frac{d}{D} (\frac{n}{D} D^x_{d,n}) \equiv^n_{\text{Rad}} 1 \ .$$

So $\frac{n}{D} x \equiv^n_{\text{Rad}} 1$, whence by Lemma 5.1.7(ix) $2\frac{n}{D} x = 1$, which in turn implies, by Lemma 5.1.7(i), that $(n+1)x = 1$.

This shows that the algebra $\mathcal{A}$ is local. □

As shown by the following theorem, the two axiomatizations $\mathbb{L}oc^1_V$ and $\mathbb{L}oc^2_V$ for the class of local MV-algebras in a Komori variety V are actually equivalent.

Theorem* 5.2.13. *The theory $\mathbb{L}oc^1_V$ is equivalent to the theory $\mathbb{L}oc^2_V$.*

Proof. By Theorem 5.2.12 and Proposition 5.2.5, the theories $\mathbb{L}oc^1_V$ and $\mathbb{L}oc^2_V$ have the same set-based models. Since they are both coherent theories, it follows from the classical (non-constructive) completeness for coherent logic (cf. Corollary D1.5.10 [35]) that they are syntactically equivalent (i.e., any coherent sequent over the signature of $\mathbb{M}V$ which is provable in $\mathbb{L}oc^1_V$ is provable in $\mathbb{L}oc^2_V$ and vice versa). □

Remarks 5.2.14. (a) The non-constructive part of the theorem is the statement that the theory $\mathbb{L}oc^1_V$ is a quotient of $\mathbb{L}oc^2_V$, while the fact that $\mathbb{L}oc^2_V$ is a quotient of $\mathbb{L}oc^1_V$ is fully constructive (cf. the proof of Theorem 5.2.12).

(b) The sequent

$$(((n+1)x)^2 = 0 \vee (n+1)x = 1) \vdash_x \bigvee_{d=0}^{n} x \in \text{Fin}^n_d)$$

is *not* provable in $\mathbb{T}_V$ in general. Indeed, take for instance $V = V(S_4)$ and the element $x := (\frac{1}{2}, \frac{1}{4})$ of the algebra $\mathcal{A} = S_4 \times S_4$ in V. Note that

$\text{Fin}_d(\mathcal{A}) = \{(d, d)\}$ for any d. We clearly have that $(n + 1)x = 1$ but $x \notin \text{Fin}_d(\mathcal{A})$ for all d.

5.2.2 Rigidity of the Grothendieck topology associated with $\mathbb{L}oc_V^2$

In this section, we shall prove that the Grothendieck topology associated with the theory $\mathbb{L}oc_V^2$ as a quotient of the theory $\mathbb{T}_V$ is rigid. From this we shall deduce that the theory $\mathbb{L}oc_V^2$ is of presheaf type and that its finitely presentable models are precisely the local MV-algebras that are finitely presented as models of the theory $\mathbb{T}_V$.

Let us begin by proving that the partition determined by the sequent ρ_n is 'compatible' with respect to the MV-operations. In this respect, the following lemma is useful.

Lemma 5.2.15. *Let $\mathcal{A}$ be an MV-algebra and $(\mathcal{G}, u)$ be the ℓ-group with strong unit corresponding to it via Mundici's equivalence. Then, for any natural number m, an element x of A satisfies the condition $\neg x = (m - 1)x$ in $\mathcal{A}$ if and only if $mx = u$ in $\mathcal{G}$ (where the addition here is taken in the group $\mathcal{G}$). In this case, for every $k = 0, \ldots, m$, $\neg(kx) = (m - k)x$ in $\mathcal{A}$.*

Proof. The MV-algebra $\mathcal{A}$ can be identified with the unit interval $[0, u]$ of the group $\mathcal{G}$. Recall that $x \oplus y = \inf(x + y, u)$, for any $x, y \in A$. Now, $\neg x = (m - 1)x$ in $\mathcal{A}$ if and only if $\inf((m - 1)x, u) = u - x$, equivalently if and only if $\inf(mx, u + x) = u$. Consider the Horn sequent

$$\sigma := (0 \leq x \leq u \wedge \inf(mx, u + x) = u \vdash_x mx = u)$$

in the theory of ℓ-u groups. Let us show that it is provable in the theory of ℓ-u groups.[2] Let us argue informally in terms of elements. Given an element x

[2]In fact, the following proof does not actually use the hypothesis that the element u is a strong unit, but only the fact that $u \geq 0$.

such that $0 \leq x \leq u$, if $\inf(mx, u+x) = u$ then $mx \geq u$, that is, $mx - u \geq 0$. Further,

$$\inf(mx, u + x) = u \Leftrightarrow \inf(mx - u, x) = 0 \Leftrightarrow$$

$$\inf(k(mx - u), kx) = 0, \text{ for every } k \in \mathbb{N} .$$

Since $mx - u \geq 0$, we have that $mx - u \leq k(mx - u)$, for every $k \in \mathbb{N}$. Applying this in the case $k = m$, we obtain that

$$mx - u \leq m(mx - u) \quad \text{and} \quad mx - u \leq mx$$

$$\Rightarrow mx - u \leq \inf(m(mx - u), mx) = 0$$

$$\Rightarrow mx - u = 0$$

$$\Rightarrow mx = u .$$

Now, if $mx = u$ then for any $k = 0, \dots, m$, $\neg(kx) = u - kx = mx - kx = (m - k)x$. This completes the proof of the lemma. □

Remarks 5.2.16. (a) The lemma clearly admits a syntactic formulation in terms of the interpretation functor from MV-algebras to ℓ-u groups defined in Section 3.3.

(b) By the lemma, the formula

$$\alpha_d^n(x) := (x \equiv^n_{\text{Rad}} \frac{d}{D} D^x_{d,n}) \wedge (\bigwedge_{k=0}^{\frac{n}{D}} (\neg k D^x_{d,n} \equiv^n_{\text{Rad}} (\frac{n}{D} - k) D^x_{d,n}))$$

is provably equivalent in $\mathbb{T}_V$ to the simpler formula

$$(x \equiv^n_{\text{Rad}} \frac{d}{D} D^x_{d,n}) \wedge (\neg D^x_{d,n} \equiv^n_{\text{Rad}} (\frac{n}{D} - 1) D^x_{d,n}) .$$

Proposition 5.2.17. *The sequents*

$$(x \in Fin^n_d \wedge y \in Fin^n_b \vdash_{x,y} x \oplus y \in Fin^n_{d\oplus b}) \tag{5.1}$$

(for each $d, b \in \{0, \ldots, n\}$ *and where with* $d \oplus b$ *we indicate the sum in* $S_n = \{0, 1, \ldots, n\}$*) and*

$$(x \in Fin^n_d \vdash_x \neg x \in Fin^n_{n-d}) \tag{5.2}$$

(for each $d \in \{0, \ldots, n\}$*) are provable in the theory* $\mathbb{T}_V$*.*

Proof. Since the theory $\mathbb{T}_V$ is of presheaf type, we can show the provability in $\mathbb{T}_V$ of the sequents of type (5.1) and (5.2) by verifying semantically their validity in every MV-algebra $\mathcal{A}$ in V. In fact, the proposition also admits an entirely syntactic proof; we argue semantically just for the sake of better readability. Since sequents (5.1) and (5.2) involve equalities between radical classes, we reason as we were in the quotient $\mathcal{A}/\mathrm{Rad}(\mathcal{A})$ but, with an abuse of notation, we indicate radical classes by avoiding the standard notation with square brackets.

(1) By definition and Remark 5.2.16(b), we have that

$$x \in \mathrm{Fin}^n_d \Leftrightarrow x = \frac{d}{D} D^x_{d,n} \text{ and } \neg D^x_{d,n} = (\frac{n}{D} - 1) D^x_{d,n},$$
$$y \in \mathrm{Fin}^n_b \Leftrightarrow y = \frac{b}{B} D^y_{b,n} \text{ and } \neg D^y_{b,n} = (\frac{n}{B} - 1) D^y_{b,n},$$

where $D = \text{g.c.d.}(d, n)$ and $B = \text{g.c.d.}(b, n)$.

If $x \in \mathrm{Fin}^n_d(\mathcal{A})$ and $y \in \mathrm{Fin}^n_b(\mathcal{A})$ then Lemma 5.2.15 implies that

$$\frac{n}{D} D^x_{d,n} = u = \frac{n}{B} D^y_{b,n}$$

in the ℓ-u group associated with $\mathcal{A}/\mathrm{Rad}(\mathcal{A})$ (where all the sums are taken in the ℓ-u group). It follows in particular that $\frac{d}{D} D^x_{d,n} \leq u$ and $\frac{b}{B} D^y_{b,n} \leq u$. Now, for any element z of an MV-algebra $\mathcal{M}$ with associated ℓ-u group $L(\mathcal{M})$, if $kz \leq u$ in the group $L(\mathcal{M})$ then the element $kz = z \oplus \cdots \oplus z$ k times, where the sum is taken in the MV-algebra $\mathcal{M}$, coincides with the element $kz = z + \cdots + z$ k times, where the sum is taken in $L(\mathcal{M})$. Thus, we have that:

$$nx = d\frac{n}{D} D^x_{d,n} \text{ and } ny = b\frac{n}{B} D^y_{b,n},$$

where all the sums are taken in the ℓ-u group. This in turn implies that

$$n(x \oplus y) = (d \oplus b)u \ .$$

Indeed, $n(x \oplus y) = \inf(nx+ny, nu) = \inf((d+b)u, nu) = \inf((d+b), n)u$, where the last equality follows from the fact that the order on S_n is total.

By definition, the element $D^{x\oplus y}_{d\oplus b,n}$ is equal to

$$D^{x\oplus y}_{d\oplus b,n} = \xi_{(d\oplus b,n)}(x \oplus y) - \chi_{(d\oplus b,n)}u$$

if this element is in $[0, u]$, where $\xi_{(d\oplus b,n)}$ and $\chi_{(d\oplus b,n)}$ are the Bézout coefficients of the g.c.d. of $(d \oplus b)$ and n, which we call C. To see this, we calculate in the ℓ-group

$$\frac{n}{C}(\xi_{(d\oplus b,n)}(x \oplus y) - \chi_{(d\oplus b,n)}u) = \frac{\xi_{(d\oplus b,n)}}{C}n(x \oplus y) - \frac{\chi_{(d\oplus b,n)}}{C}nu =$$

$$\frac{\xi_{(d\oplus b,n)}}{C}(d \oplus b)u - \frac{\chi_{(d\oplus b,n)}}{C}nu =$$

$$\frac{\xi_{(d\oplus b,n)}(d \oplus b) - \chi_{(d\oplus b,n)}n}{C}u = u,$$

whence in particular $0 \leq \xi_{(d\oplus b,n)}(x \oplus y) - \chi_{(d\oplus b,n)}u \leq u$ since $C \leq n$ and ℓ-groups are torsion-free.

We can thus conclude that

$$\frac{n}{C}D^{x\oplus y}_{d\oplus b,n} = u,$$

whence by Lemma 5.2.15,

$$\neg D^{x\oplus y}_{d\oplus b,n} = (\frac{n}{C} - 1)D^{x\oplus y}_{d\oplus b,n} \ .$$

Finally, from the equality $n(x \oplus y) = (d \oplus b)u$ it follows that

$$n(x \oplus y) = (d \oplus b)\frac{n}{C}D^{x\oplus y}_{d\oplus b,n},$$

whence, since ℓ-groups are torsion-free, we have that

$$x \oplus y = \frac{d \oplus b}{C} D_{d\oplus b,n} \ .$$

So $x \oplus y \in \mathrm{Fin}^n_{d\oplus b}(\mathcal{A})$, as required.

(2) As before, we have that

$$x \in \mathrm{Fin}^n_d \Leftrightarrow x = \frac{d}{D} D^x_{d,n} \text{ and } \neg D^x_{d,n} = (\frac{n}{D} - 1) D^x_{d,n},$$

(where $D = \text{g.c.d.}(d, n)$), which implies that:

$$\frac{n}{D} D^x_{d,n} = u \text{ and } nx = du \ .$$

Thus, if $x \in \mathrm{Fin}^n_d(\mathcal{A})$ then

$$n(\neg x) = n(u - x) = nu - nx = nu - ud = (n - d)u \ .$$

Further, we have that:

$$\frac{n}{D}(\xi_{(n-d,n)}(\neg x) - \chi_{(n-d,n)} u) = \frac{\xi_{(n-d,n)}}{D} n(\neg x) - \frac{\chi_{(n-d,n)}}{D} nu =$$

$$= \frac{\xi_{(n-d,n)}}{D}(n - d)u - \frac{\chi_{(n-d,n)}}{D} nu =$$

$$= \frac{\xi_{(n-d,n)}(n - d) - \chi_{(n-d,n)} n}{D} u = u,$$

where the last equality follows from the fact that $D = g\ .\ c\ .\ d(n-d, d)$. It follows in particular that $\xi_{(n-d,n)} \neg x - \chi_{(n-d,n)} u \in [0, u]$ and hence that

$$\frac{n}{D} D^{\neg x}_{n-d,n} = u \ .$$

By Lemma 5.2.15, this means that $\neg D^{\neg x}_{n-d,n} = (\frac{n}{D} - 1) D^{\neg x}_{n-d,n}$. Finally, since ℓ-groups are torsion-free, we have that:

$$n(\neg x) = (n - d)\frac{n}{D} D^{\neg x}_{n-d,n} \Rightarrow \neg x = \frac{n - d}{D} D^{\neg x}_{n-d,n} \ .$$

Hence, $\neg x \in \mathrm{Fin}^n_{n-d}(\mathcal{A})$, as required. □

In [27], the authors proved, by using the axiom of choice, that every MV-algebra has a greatest local subalgebra (cf. Theorem 3.19 therein). The following proposition represents a constructive version of this result holding for MV-algebras in a Komori variety V.

Proposition 5.2.18. *Let $\mathcal{A}$ be an MV-algebra in a Komori variety V with invariant n. The biggest subalgebra $\mathcal{A}_{loc}$ of $\mathcal{A}$ that is a set-based model of Loc^2_V is given by:*

$$\mathcal{A}_{loc} = \{x \in A \mid x \in Fin^n_d(\mathcal{A}) \textit{ for some } d \in \{0, \dots, n\}\}$$

Proof. We know from Proposition 5.2.17 that $\mathcal{A}_{\text{loc}}$ is a subalgebra of $\mathcal{A}$; trivially, $\mathcal{A}_{\text{loc}}$ is a model of $\mathbb{L}oc^2_V$. Now, let $\mathcal{B}$ be a set-based model of $\mathbb{L}oc^2_V$ that is a subalgebra of $\mathcal{A}$. By Theorem 5.2.12, the algebra $\mathcal{B}$ satisfies the sequent ρ_n; thus, it is contained in $\mathcal{A}_{\text{loc}}$, as required. □

Theorem 5.2.19. *The theory $\mathbb{L}oc^2_V$ is of presheaf type and the Grothendieck topology associated with it as a quotient of the theory $\mathbb{T}_V$ is rigid. In particular, the finitely presentable models of $\mathbb{L}\mathrm{oc}^2_V$ are precisely the models of $\mathbb{L}oc^2_V$ that are finitely presentable as models of the theory $\mathbb{T}_V$.*

Proof. To prove that the theory $\mathbb{L}oc^2_V$ is of presheaf type it is sufficient to show that the topology associated with the quotient $\mathbb{T}_V \cup \{\rho_n\}$ is rigid. Indeed, this implies that the theory $\mathbb{T}_V \cup \{\rho_n\}$ is of presheaf type (cf. Theorem 1.5.11). From Theorems 1.5.11 and 1.5.14 it will then follow that the finitely presentable models of $\mathbb{L}oc^2_V$ are precisely the models of $\mathbb{L}oc^2_V$ that are finitely presentable as models of the theory $\mathbb{T}_V$, and hence (again, by Theorem 1.5.11) that the topology associated with $\mathbb{L}oc^2_V$ as a quotient of $\mathbb{T}_V$ is rigid as well.

Let J'_2 be the topology on f.p.$\mathbb{T}_V$-mod$(\mathbf{Set})^{\text{op}}$ associated with the theory $\mathbb{T}_V \cup \{\rho_n\}$ as a quotient of $\mathbb{T}_V$. Any J'_2-covering sieve contains a finite multicomposition of families of arrows of the following form:

$$
\begin{array}{ccc}
 & & \mathcal{A}/(x \in \mathrm{Fin}_0^n(\mathcal{A})) \\
 & \nearrow & \vdots \\
\mathcal{A} & \longrightarrow & \mathcal{A}/(x \in \mathrm{Fin}_d^n(\mathcal{A})) \\
 & \searrow & \vdots \\
 & & \mathcal{A}/(x \in \mathrm{Fin}_n^n(\mathcal{A}))
\end{array}
$$

where x is any element of A and the expression $(x \in \mathrm{Fin}_d^n(\mathcal{A}))$ denotes the congruence on $\mathcal{A}$ generated by the condition $x \in \mathrm{Fin}_d^n(\mathcal{A})$ (this congruence actually exists since this condition amounts to a finite conjunction of equational conditions in the language of MV-algebras). Indeed, $\mathbb{T}_V$ is an algebraic theory, so each of the quotients $\mathcal{A}/(x \in \mathrm{Fin}_i^n(\mathcal{A}))$ are finitely presentable models of $\mathbb{T}_V$ if $\mathcal{A}$ is. The arrows $\mathcal{A} \to \mathcal{A}/(x \in \mathrm{Fin}_i^n(\mathcal{A}))$ that occur in the above diagram are therefore surjective (as they are canonical projections). It follows that every J_2'-covering sieve contains a family of arrows generating a J_2'-covering sieve (given by a finite multicomposition of diagrams of the above form), all of which are surjective. Thus, given a family of generators for $\mathcal{A}$, if we choose one of them at each step, the resulting multicomposite family will generate a J_2'-covering cosieve and the codomains of all the arrows in it will be generated by elements x each of which is in Fin_d^n for some $d = 0, \ldots, n$. Because of the compatibility property of the partition induced by the sequent ρ_n (cf. Proposition 5.2.17), these algebras are models of the theory $\mathbb{T}_V \cup \{\rho_n\}$ in **Set** whence the topology J_2' is rigid. □

Remark 5.2.20. If $\{\vec{x} \,.\, \phi\}$ is a formula presenting a model of $\mathbb{L}oc_V^2$, where $\vec{x} = (x_1, \ldots, x_k)$, there exists $d_1, \ldots, d_k$ natural numbers such that the following sequent is provable in the theory $\mathbb{L}oc_V^2$:

$$(\phi \vdash_{\vec{x}} x_1 \in \mathrm{Fin}_{d_1}^n \wedge \cdots \wedge x_k \in \mathrm{Fin}_{d_k}^n) \, .$$

This is a consequence of the fact that the formula $\{\vec{x} \,.\, \phi\}$ is $\mathbb{L}oc_V^2$-irreducible.

Remark* 5.2.21. In Theorem 5.2.13 we proved that the theories $\mathbb{L}oc_V^1$ and $\mathbb{L}oc_V^2$ are equivalent. By Duality Theorem, this means that the Grothendieck topologies J_1 and J_2 associated with these theories as quotients of $\mathbb{T}_V$ are equal. By Proposition 5.1.5 and Theorem 5.2.19, these topologies are subcanonical and rigid.

5.2.3 Representation results for finitely presented MV-algebras in a proper subvariety

In Section 4.6 we proved that every finitely presentable MV-algebra in the variety $V(S_1^\omega)$ is a direct product of a finite family of perfect MV-algebras (cf. Theorem 4.6.1 therein). An analogous result holds for the finitely presentable algebras in V. However, there are differences with the case of perfect MV-algebras. Recall that an arbitrary family of generators $\{x_1, \ldots, x_n\}$ for an algebra $\mathcal{A}$ in Chang's variety yields a decomposition of $\mathcal{A}$ as a finite product of perfect MV-algebras: more specifically, $\mathcal{A}$ decomposes as the finite product of algebras arising as the leaves of diagrams obtained from multicompositions of diagrams of the form

$$
\begin{array}{ccc}
 & & \mathcal{A}/((2x)^2) \\
 & \nearrow & \\
\mathcal{A} & & \\
 & \searrow & \\
 & & \mathcal{A}/(\neg(2x)^2)
\end{array}
$$

where at each step one selects as x (the image in the relevant quotient of) one of the generators $\{x_1, \ldots, x_n\}$. This is no longer true for finitely generated algebras in an arbitrary Komori variety; only special sets of generators give the desired decomposition result (cf. Theorem 5.2.22(b) below). For example, let us consider the algebra $\mathcal{A} = S_7 \times S_7$. This is generated by the element $x = (2/7, 3/7)$ and also by the elements $\{x_1 = (1/7, 0), x_2 = (0, 1/7)\}$. The J_1-multicomposition corresponding to the choice of x is the following:

$$\mathcal{A} \begin{array}{l} \nearrow \mathcal{A}/((n+1)x)^2) \cong \{0\} \\ \searrow \mathcal{A}/(\neg((n+1)x)^2) \cong \mathcal{A} \end{array}$$

On the other hand, the second generating system yields a decomposition of $\mathcal{A}$ as a product of local (or trivial) MV-algebras:

$$\mathcal{A} \begin{array}{l} \nearrow \mathcal{A}_1 = \mathcal{A}/((n+1)x_1)^2) = \{0\} \times S_7 \begin{array}{l} \nearrow \mathcal{A}_1/(((n+1)[x_2]_1)^2) \cong \{0\} \\ \searrow \mathcal{A}_1/(\neg((n+1)[x_2]_1)^2) \cong S_7 \end{array} \\ \searrow \mathcal{A}_2 = \mathcal{A}/(\neg((n+1)x_1)^2) \cong S_7 \times \{0\} \begin{array}{l} \nearrow \mathcal{A}_2/(((n+1)[x_2]_2)^2) \cong S_7 \\ \searrow \mathcal{A}_2/(\neg((n+1)[x_2]_2)^2) \cong \{0\} \end{array} \end{array}$$

(where the subscript notation $[...]_i$ means that the given equivalence class is taken in $\mathcal{A}_i$). Indeed, for the first step we have:

$$((n+1)x_1)^2 = (1,0) \Rightarrow ((1,0)) = S_7 \times \{0\} \Rightarrow \mathcal{A}_1 = \mathcal{A}/(S_7 \times \{0\}) \cong \{0\} \times S_7;$$

$$\neg((n+1)x_1)^2 = (0,1) \Rightarrow ((0,1)) = \{0\} \times S_7 \Rightarrow \mathcal{A}_2 = \mathcal{A}/(\{0\} \times S_7) \cong S_7 \times \{0\};$$

while for the second step we have:

$$[x_2]_1 = (0, \frac{1}{7}) \Rightarrow ((n+1)[x_2]_1)^2 = (0,1) \text{ and } \neg((n+1)[x_2]_1)^2 = (0,0)$$

$$\Rightarrow \mathcal{A}_1/(((n+1)[x_2]_1)^2) \cong \{0\} \text{ and } \mathcal{A}_1/(\neg((n+1)[x_2]_1)^2) \cong S_7;$$

$$[x_2]_2 = (0,0) \Rightarrow ((n+1)[x_2]_1)^2 = (0,0) \text{ and } \neg((n+1)[x_2]_1)^2 = (1,0)$$

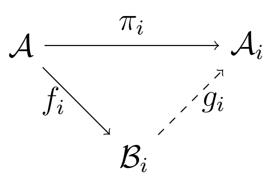
$$\Rightarrow \mathcal{A}_2/(((n+1)[x_2]_2)^2) \cong S_7 \text{ and } \mathcal{A}_2/(\neg((n+1)[x_2]_2)^2) \cong \{0\}\ .$$

Theorem* 5.2.22. [3]

(i) Every finitely presentable non-trivial algebra in V is a finite direct product of finitely presentable local MV-algebras in V;

(ii) Given a set of generators $\{x_1, \ldots, x_m\}$ for $\mathcal{A}$, the J_1-multicomposition obtained by choosing at each step one of the generators gives a representation of $\mathcal{A}$ as a product of local MV-algebras (i.e., the codomains of the arrows in the resulting product diagram are local MV-algebras) if and only if the image of each generator under the projections to the product factors satisfies the sequent ρ_n.

(i) From Theorem 5.2.19 we know that the topology J_2, and hence the topology J_1, is rigid. This means that for every $\mathcal{A}$ in f.p.$\mathbb{T}_V$-mod(**Set**), the family of arrows $f : \mathcal{A} \to \mathcal{B}$, where $\mathcal{B}$ is a local MV-algebra in f.p.$\mathbb{T}_V$-mod(**Set**), generates a J_1-covering sieve S. By definition of the topology J_1, S contains a family of arrows $\{\pi_i : \mathcal{A} \to \mathcal{A}_i \mid i = 1, \ldots, r\}$ obtained by a finite J_1-multicomposition relative to certain elements $x_1, \ldots, x_m \in A$. We know from the proof of Proposition 5.1.5 that $\mathcal{A}$ is the product of the algebras $\mathcal{A}_1, \ldots, \mathcal{A}_r$. It follows that for every $i = 1, \ldots, r$, π_i factors through an arrow $f_i : \mathcal{A} \to \mathcal{B}_i$ whose codomain $\mathcal{B}_i$ is a local MV-algebra:



We know that the π_i are surjective maps; thus, the arrows g_i are surjective too. Thus, for every $i = 1, \ldots, r$, the algebra $\mathcal{A}_i$ is a homomorphic

[3]This theorem requires the axiom of choice to ensure that the topologies J_1 and J_2 coincide.

image of a local MV-algebra and hence it is local. So $\mathcal{A}$ is a finite product of local MV-algebras.

(ii) Suppose that in the decomposition considered in (i) corresponding to a family of generators $\{x_1, \ldots, x_m\}$ for $\mathcal{A}$, the projection of every element x_i in any product factor satisfies the sequent ρ_n. Since every arrow in a J_1-multicomposition is surjective, it sends a family of generators of $\mathcal{A}$ to a family of generators of its codomain. These codomains are thus MV-algebras whose generators satisfy the sequent ρ_n. From Proposition 5.2.17 we can then conclude that these algebras are local MV-algebras. The other direction is trivial. □

Proposition* 5.2.23. *Every algebra $\mathcal{A}$ in f.p.$\mathbb{T}_V$-mod(**Set**) with generators $x_1, \ldots, x_n$ forms a limit cone over the diagram consisting of the algebras appearing as codomains of the arrows in the J_2-multicomposition relative to the generators $x_1, \ldots, x_n$, and all the homomorphisms over $\mathcal{A}$ between them.*

Proof. Our thesis follows from the subcanonicity of the topology J_2, which is given by Proposition 5.1.5 in light of the fact that $J_1 = J_2$. □

Let us now describe an algorithm which, starting from a representation of an algebra $\mathcal{A}$ in V as a finite subproduct of a family of local MV-algebras

$$\mathcal{A} \hookrightarrow \mathcal{A}_1 \times \cdots \times \mathcal{A}_r,$$

produces a decomposition of $\mathcal{A}$ as a finite product of local MV-algebras. This can for instance be applied to the representations of algebras $\mathcal{A}$ in f.p.$\mathbb{T}_V$-mod(**Set**) provided by Proposition 5.2.23.

First, we observe that, given an embedding $f : \mathcal{A} \to \mathcal{B}$ of MV-algebras and an ideal I of $\mathcal{A}$, f yields an embedding $\mathcal{A}/I \to \mathcal{B}/(f(I))$, where $f(I)$ is the ideal of $\mathcal{B}$ generated by the subset $f(I)$. Indeed, every embedding of MV-algebras reflects the order relation since the latter is equationally definable.

Given a representation

$$\mathcal{A} \hookrightarrow \mathcal{A}_1 \times \cdots \times \mathcal{A}_r$$

of an algebra $\mathcal{A}$ in V as a finite subproduct of a family $\{\mathcal{A}_i\}_{i=1}^r$ of local MV-algebras, if $\mathcal{A}$ is not local, there exists $x \in A$ such that x is neither in the radical, nor in the coradical nor it is finite. This means that $x = (x_1, \ldots, x_r)$ has at least a component x_i which is in the radical of $\mathcal{A}_i$ (otherwise x would have finite order) and at least a component x_j which is in the radical of $\mathcal{A}_j$ (otherwise x would belong to $\mathrm{Rad}(\mathcal{A})$). We know from Proposition 5.1.4(ii) that $((n+1)x)^2$ is a boolean element whence it is a sequence of 0 and 1 since boolean elements in local MV-algebras are just the trivial ones. Thus, the ideal generated by $((n+1)x)^2$ in $\mathcal{A}_1 \times \cdots \times \mathcal{A}_r$ is the product of the algebras $\bar{\mathcal{A}}_1 \times \cdots \times \bar{\mathcal{A}}_r$, where

$$\bar{\mathcal{A}}_i = \begin{cases} \mathcal{A}_i \text{ if } ((n+1)x_i)^2 = 1 \\ \{0\} \text{ otherwise} \end{cases}$$

Thus the quotient $\mathcal{B}_1 = \mathcal{A}/(((n+1)x)^2)$ embeds in the finite product of the algebras $\{\mathcal{A}'_i\}_{i=1}^r$ defined by:

$$\mathcal{A}'_i = \begin{cases} \mathcal{A}_i \text{ if } ((n+1)x_i)^2 = 0 \\ \{0\} \text{ otherwise} \end{cases}$$

Similarly, the quotient $\mathcal{B}_2 = \mathcal{A}/(\neg((n+1)x)^2)$ embeds in the finite product of the algebras $\{\mathcal{A}'_i\}_{i=1}^r$ defined by:

$$\mathcal{A}'_i = \begin{cases} \mathcal{A}_i \text{ if } ((n+1)x_i)^2 = 1 \\ \{0\} \text{ otherwise} \end{cases}$$

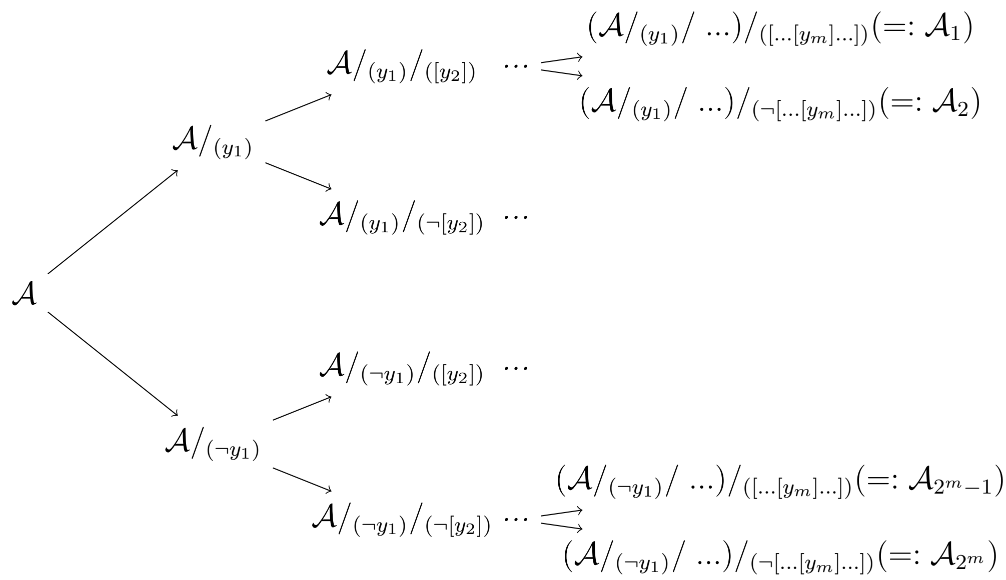
If the number of non-trivial factors of the product in which $\mathcal{B}_1$ embeds is strictly bigger than 1 then $\mathcal{B}_1$ is not local and we repeat the same process, and similarly for the algebra $\mathcal{B}_2$. Since the initial product is finite and the number of non-trivial factors strictly decreases at each step, this process must end after a finite number of steps. This means that after a finite number of iterations of our 'algorithm' the resulting quotients embed into products with only one non-trivial factor and hence are local MV-algebras. The Pushout-Pullback Lemma recalled in the proof of Proposition 5.1.5 thus yields the desired representation of $\mathcal{A}$ as a finite product of local MV-algebras.

We shall now present an alternative approach, based on the consideration of the boolean skeleton of $\mathcal{A}$, to the representation of $\mathcal{A}$ as a finite product of local MV-algebras.

Proposition 5.2.24. *Let $\mathcal{A}$ be an MV-algebra in V and $\{y_1, \dots, y_m\}$ a set of boolean elements of $\mathcal{A}$. Then the following conditions are equivalent:*

(i) The elements $\{y_1, \dots, y_m\}$ generate the boolean skeleton of $\mathcal{A}$;

(ii) The non-trivial algebras $\mathcal{A}_1, \dots, \mathcal{A}_{2^m}$ appearing as terminal leaves of the following diagram are local:



Proof. By the Pushout-Pullback Lemma, we have that

$$\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_{2^m} \ .$$

(i) $\Rightarrow$ (ii) We shall prove that the algebras $\mathcal{A}_1, \ldots, \mathcal{A}_{2^m}$ are local or trivial by showing that their boolean skeleton is contained in $\{0,1\}$. Given $\xi \in B(\mathcal{A}_j)$, with $j \in \{1, \ldots, 2^m\}$, there exists $x \in B(\mathcal{A})$ such that $\pi_j(x) = \xi$ (indeed, we can take x equal to the sequence whose components are all 0 except for the jth-component that is equal to ξ). Since $B(\mathcal{A})$ is generated by $\{y_1, \ldots, y_m\}$, we have that x is equal to $t(y_1, \ldots, y_m)$ for some term t over the signature of the theory $\mathbb{MV}$. Thus,

$$\xi = \pi_j(x) = \pi_j(t(y_1, \ldots, y_m)) = t(\pi_j(y_1), \ldots, \pi_j(y_m)) .$$

By construction, we have that $\pi_j(y_i) \in \{0,1\}$ for every $j = 1, \ldots, 2^m$ and $i = 1, \ldots, m$. Hence, $\xi \in \{0,1\}$ for each $\xi \in B(\mathcal{A}_j)$ for every $j = 1, \ldots, 2^m$, as required.

(ii) $\Rightarrow$ (i) The elements $\{y_1, \ldots, y_m\}$ have the following form:

$$y_1 = (0, \ldots, 0, 1, \ldots, 1),$$

$$y_2 = (0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0, 1, \ldots, 1),$$

$$\vdots$$

$$y_m = (0, 1, 0, 1, \ldots, 0, 1) .$$

By our hypothesis, every $\mathcal{A}_j$ (where $j = 1, \ldots, 2^m$) is either a local or a trivial MV-algebra. Thus, the boolean kernel $B(\mathcal{A})$ is a finite product of subalgebras of $\{0,1\}$. Now, for every $i \in \{1, \ldots, 2^m\}$, the element $e_i = (0, \ldots, 0, 1, 0 \ldots, 0)$, where the 1 is in the position i, is equal to $\inf(\bar{y}_1, \ldots, \bar{y}_m)$, where

$$\bar{y}_k = \begin{cases} y_k \text{ if } (y_k)_i = 1 \\ \neg y_k \text{ otherwise} \end{cases}$$

The elements $e_1, \ldots, e_{2^m}$ are the atoms of $B(\mathcal{A})$. Since they are contained in the algebra generated by $\{y_1, \ldots, y_m\}$, it follows that this algebra coincides with $B(\mathcal{A})$, as required. □

Remarks 5.2.25. (a) It is known that every algebra in a Komori variety is quasilocal, i.e., it is a weak boolean product of local MV-algebras (cf. Section 9 of [27]). Proposition 5.2.24 gives a concrete representation result for the algebras in a Komori variety whose boolean skeleton is finite (recall that every finite product can be seen as a weak boolean product, cf. Section 6.5 of [26]);

(b) The boolean skeleton of an MV-algebra in f.p.$\mathbb{T}_V$-mod(**Set**) is finitely generated as it is finite (by Theorem 5.2.22 or Proposition 5.2.23). Still, this result is non-constructive as it relies on the non-constructive equivalence between the axiomatizations $\mathbb{L}oc^1_V$ and $\mathbb{L}oc^2_V$;

(c) If V is Chang's variety, the boolean skeleton of a finitely generated MV-algebra $\mathcal{A}$ in V is finitely generated since there exists an isomorphism between $\mathcal{A}/\text{Rad}(\mathcal{A})$ and $B(\mathcal{A})$ induced by the following homomorphism:

$$f : x \in \mathcal{A} \to (2x)^2 \in B(\mathcal{A}) \ .$$

The following proposition shows that this cannot be generalized to the setting of an arbitrary Komori variety V.

Proposition* 5.2.26. *Let $\mathcal{A}$ be a local MV-algebras in V. If the map $f : x \in \mathcal{A} \to ((n+1)x)^2 \in B(\mathcal{A})$ is a homomorphism then $\mathcal{A}$ is in Chang's variety.*

Proof. If $\mathcal{A}$ is local then its boolean skeleton is $\{0, 1\} = S_1$. If $f : x \in \mathcal{A} \to ((n+1)x)^2 \in S_1$ is a homomorphism, then we have an induced homomorphism $\bar{f} : \mathcal{A}/\text{Rad}(\mathcal{A}) \to S_1$ (since the radical of S_1 is trivial). Since every local MV-algebra in V has finite rank, we have that $\mathcal{A}/\text{Rad}(\mathcal{A}) \cong S_m$ for some $m \in \mathbb{N}$. The map $\bar{f}$ is thus a homomorphism from S_m to S_1. This clearly implies that $m = 1$. So $\mathcal{A}$ is in Chang's variety (cf. Theorem 5.2.3). □

5.2.4 Local MV-algebras in varieties generated by simple MV-algebras

By Theorem 5.2.3, the local MV-algebras in varieties generated by simple MV-algebras $S_{n_1}, \dots, S_{n_h}$ are just the simple chains that generate the variety and their subalgebras. In particular, the local MV-algebras in a variety $V(S_n)$ generated by a single finite chain are precisely the simple MV-algebras S_k where k divides n.

Let us indicate with $\mathbb{T}_n$ the theory $\mathbb{L}oc^2_{V(S_n)}$ (for each $n \in \mathbb{N}$). It is clear that the theory $\mathbb{L}oc^2_{V(S_{n_1},\dots,S_{n_k})}$ is the infimum of the theories $\mathbb{T}_{n_1}, \dots, \mathbb{T}_{n_k}$ (with respect to the natural ordering between geometric theories over a given signature introduced in [11]). Indeed, the models of this theory are precisely $S_{n_1}, \dots, S_{n_k}$ and their subalgebras, and each of the theories $\mathbb{T}_{n_1}, \dots, \mathbb{T}_{n_k}$ and $\mathbb{L}oc^2_{V(S_{n_1},\dots,S_{n_k})}$ is of presheaf type (by Theorem 5.2.19) whence the validity of a geometric sequent over the signature of $\mathbb{MV}$ in all its set-based models amounts precisely to its provability in it.

So all the theories of the form $\mathbb{T}_{n_1} \wedge \cdots \wedge \mathbb{T}_{n_k}$ are of presheaf type. It is natural to ask if this property still holds for an infinite infimum, i.e., if the theory $\bigwedge_{n\in\mathbb{N}} \mathbb{T}_n$ is also of presheaf type. We shall answer to this question in the affermative in the Section 5.4.

5.3 The geometric theory of local MV-algebras of finite rank

We have studied the theory of local MV-algebras of finite rank contained in a proper variety (i.e., Komori variety) V and we have proved that it is of presheaf type for any V. It is natural to wonder whether the 'global' theory of local MV-algebras of finite rank (with no bounds on their ranks imposed by the fact that they lie in a given variety V) is of presheaf type or not. Note that this theory does not coincide with the theory of local MV-algebras as there exists local MV-algebras that are not of finite rank (for example, every infinite simple MV-algebra). We shall prove in this section that the answer to this question is negative, even though, as we saw in Section 2.4, the theory of finite chains, which is the 'simple' counterpart of this theory, is of presheaf type. The essential difference between these two theories in relation to the property of being of presheaf type is the presence of the infinitesimal elements. Indeed, by definition simple MV-algebras have no infinitesimal elements, while, as we shall see below, it is not possible to capture by a geometric formula the radical of every local MV-algebra of finite rank.

Definition 5.3.1. The geometric theory $\mathbb{F}in\mathbb{R}ank$ of local MV-algebras of finite rank consists of all the geometric sequents over the signature of $\mathbb{MV}$ which are satisfied in every local MV-algebra of finite rank.

Theorem* 5.3.2. *The theory $\mathbb{F}in\mathbb{R}ank$ is not of presheaf type.*

Proof. Let us suppose that the theory $\mathbb{F}in\mathbb{R}ank$ is of presheaf type. We will show that this leads to a contradiction.

First, let us prove that for every proper subvariety V, if $\mathcal{A}$ is a finitely presentable model of the theory $\mathbb{L}oc_V^2$ then $\mathcal{A}$ is a finitely presentable model of the theory $\mathbb{F}in\mathbb{R}ank$. Since the algebra $\mathcal{A}$ is a model of $\mathbb{F}in\mathbb{R}ank$ and by our hypothesis $\mathbb{F}in\mathbb{R}ank$ is of presheaf type, we can represent $\mathcal{A}$ as a filtered colimit of finitely presentable $\mathbb{F}in\mathbb{R}ank$-models $\{\mathcal{A}_i\}_{i\in I}$. For every $i \in I$ we thus have a canonical homomorphism $\mathcal{A}_i \to \mathcal{A}$ and hence an embedding $\mathcal{A}_i/\mathrm{Rad}(\mathcal{A}_i) \to \mathcal{A}/\mathrm{Rad}(\mathcal{A})$. So every $\mathcal{A}_i$ has a rank that divides the rank of $\mathcal{A}$ and hence all the $\mathcal{A}_i$ are contained in $\mathbb{L}oc_V^2$-mod(**Set**) (by Theorem 5.2.3). Since $\mathcal{A}$ is finitely presentable as a model of the theory $\mathbb{L}oc_V^2$, $\mathcal{A}$ is a retract of one of the $\mathcal{A}_i$, whence it is finitely presentable also as a $\mathbb{F}in\mathbb{R}ank$-model.

Next, we show that the radical of every model of $\mathbb{F}in\mathbb{R}ank$ is definable by a geometric formula $\{x \,.\, \phi\}$. The fact that this is true for all the finitely presentable models of $\mathbb{F}in\mathbb{R}ank$ is a consequence of Corollary 3.2 [15]. To prove that it is true for general models of $\mathbb{F}in\mathbb{R}ank$, we have to show that the construction of the radical $\mathcal{A} \to \text{Rad}(\mathcal{A})$ commutes with filtered colimits. To this end, we recall from Section 2.2 that Chang's algebra S_1^ω is finitely presentable as an object of Chang's variety (by the formula $\{x \,.\, x \leq \neg x\}$). By the above discussion, it follows that S_1^ω is finitely presentable as model of $\mathbb{F}in\mathbb{R}ank$, i.e., the functor $\text{Hom}(S_1^\omega, -) : \mathbb{F}in\mathbb{R}ank\text{-mod}(\mathbf{Set}) \to \mathbf{Set}$ preserves filtered colimits. But for any MV-algebra $\mathcal{A}$, $\text{Hom}(S_1^\omega, \mathcal{A}) \cong \text{Rad}(\mathcal{A})$, naturally in $\mathcal{A}$. Therefore the formula $\{x \,.\, \phi\}$ defines the radical of every algebra in $\mathbb{F}in\mathbb{R}ank\text{-mod}(\mathbf{Set})$ and hence it presents the algebra S_1^ω as a $\mathbb{F}in\mathbb{R}ank$-model. It follows that $\{x \,.\, \phi\}$ is $\mathbb{F}in\mathbb{R}ank$-irreducible. Now, the sequent

$$(\phi \vdash_x \bigvee_{n \in \mathbb{N}} ((n+1)x)^2 = 0)$$

is provable in $\mathbb{F}in\mathbb{R}ank$ since it is satisfied by all the local MV-algebras in a proper variety V; the $\mathbb{F}in\mathbb{R}ank$-irreducibility of $\{x \,.\, \phi\}$ thus implies that there exists $n \in \mathbb{N}$ such that the sequent

$$(\phi \vdash_x ((n+1)x)^2 = 0)$$

is provable in $\mathbb{F}in\mathbb{R}ank$. As this is clearly not the case, we have reached a contradiction, as desired. □

5.4 The geometric theory of finite chains

In Section 5.2.4 we introduced the theory $\bigwedge_{n \in \mathbb{N}} \mathbb{T}_n$. This is exactly the geometric theory $\mathbb{F}$ of finite chains, i.e., the theory consisting of all the geometric sequents over the signature of $\mathbb{MV}$ which are satisfied in every finite chain. Indeed a geometric sequent holds in S_n if and only if it is provable in $\mathbb{T}_n$ (since this last theory is of presheaf type), for every $n \in \mathbb{N}$.

Theorem 5.4.1. *The geometric theory of finite chains is of presheaf type.*

Proof. Apply Theorem 1.5.12 to the category of finite chains. □

Corollary 5.4.2. *The finitely presentable models of the theory $\mathbb{F}$ are exactly the finite chains.*

Proof. By Theorem 1.5.12, the finitely presentable models of $\mathbb{F}$ are precisely the retracts of finite chains, i.e., the finite chains (since any retract of a finite chain is trivial). □

We can exhibit the formulas presenting these models.

Lemma 5.4.3. *The finite chain S_n is presented as an MV-algebra by the formula*

$$\{x \,.\, (n-1)x = \neg x\} \,.$$

Proof. The chain $S_n = \Gamma(\mathbb{Z}, n)$ is generated by the element 1, which clearly satisfies the formula in the statement of the lemma. Let $\mathcal{A}$ be an MV-algebra and $y \in [\![x \,.\, (n-1)x = \neg x]\!]_{\mathcal{A}}$. We want to prove that there exists a unique MV-algebra homomorphism f from S_n to $\mathcal{A}$ such that $f(1) = y$. For every $k \in \{0, \ldots, n\}$, we set $f(k) := ky$. By working in the language of the associated ℓ-u groups, it is immediate to see that the map f preserves the sum, the negation and 0 (cf. Lemma 5.2.15). Thus f is a homomorphism and it is clearly the unique homomorphism that satisfies the property $f(1) = y$. □

We indicate the formula $\{x \,.\, (n-1)x = \neg x\}$ with the symbol $\{x \,.\, \phi_n\}$.

Theorem 5.4.4. *The set-based models of the geometric theory $\mathbb{F}$ of finite chains are exactly the (simple) MV-algebras that can be embedded in the algebra $\mathbb{Q} \cap [0,1]$.*

Proof. By Theorem 5.4.1 and Corollary 5.4.2, the theory $\mathbb{F}$ of finite chains is of presheaf type and its finitely presentable models are precisely the finite chains. Hence every model of $\mathbb{F}$ is a filtered colimit of finite chains. Now, for every finite chain S_n there exists a unique homomorphism $f : S_n \to \mathbb{Q} \cap [0,1]$

which assigns 1 to the element $\frac{1}{n}$ in $\mathbb{Q} \cap [0,1]$. Thus, every finite chain can be embedded into the the algebra $\mathbb{Q} \cap [0,1]$. Since the MV-algebra homomorphisms $S_n \to S_m$ correspond precisely to the multiplication by the scalar $\frac{m}{n}$ if n divides m (and do not exist otherwise), it follows from the universal property of colimits that every filtered colimit of finite chains can be embedded into $\mathbb{Q} \cap [0,1]$. Vice versa, every subalgebra of $\mathbb{Q} \cap [0,1]$ is the directed union of all its finitely generated (that is, finite) subalgebras and hence it is a filtered colimit of finite chains. □

Let us now provide an explicit axiomatization for the theory $\mathbb{F}$.

Lemma 5.4.5. *For every $r \in \mathbb{N}$ and any term t in the language of the MV-algebras, the following sequent is provable in $\mathbb{F}$:*

$$(\phi_r(x) \vdash_x \bigvee_{m \in \mathbb{N}} t(x) = mx) \ .$$

Proof. We reason informally by induction on the structure of the term t:

- If $t(x) = x$ then it is clearly true;
- $t(x) = s(x) \oplus q(x)$, by the induction hypothesis there exist $m, k \in \mathbb{N}$ such that $s(x) = mx$ and $q(x) = kx$. Hence, $t(x) = mx \oplus kx = (m \oplus k)x$;
- $t(x) = \neg s(x)$, by the induction hypothesis there exists $m \in \mathbb{N}$ such that $s(x) = mx$. Hence, $t(x) = \neg mx = (r - m)x$ (cf. Lemma 5.2.15). □

Theorem 5.4.6. *The geometric theory $\mathbb{F}$ of finite chains is the theory obtained from $\mathbb{MV}$ by adding the following axiom:*

$$(\top \vdash_x \bigvee_{k,t \in \mathbb{N}} (\exists z)(\phi_k(z) \wedge x = tz)) \ .$$

Proof. By definition of $\mathbb{F}$, the sequent

$$(\top \vdash_x \bigvee_{k,t \in \mathbb{N}} (\exists z)(\phi_k(z) \wedge x = tz))$$

is provable in $\mathbb{F}$ as it is satisfied in every finite chain. On the other hand, this axiom, added to the theory $\mathbb{MV}$, entails that every model homomorphism in any Grothendieck topos is monic; so, applying Theorem 6.32 in [17] in view of Lemmas 5.4.3 and 5.4.5 and Remarks 5.4(b) and 5.8(a) in [17], we obtain that $\mathbb{F}$ can be axiomatized by adding to the theory of MV-algebras the following sequents:

(i) $(\top \vdash_{[]} \bigvee_{n\in\mathbb{N}} (\exists x)(\phi_n(x)))$;

(ii) $(\phi_n(x) \wedge \phi_m(y) \vdash_{x,y} \bigvee_{k,t,s} (\exists z)(\phi_k(z) \wedge x = tz \wedge y = sz))$, where the disjunction is taken over all the natural numbers k and all the terms t and s such that, denoting by ξ the canonical generator of S_k, $t\xi \in [\![x \,.\, \phi_n(x)]\!]_{S_k}$ and $s\xi \in [\![x \,.\, \phi_m(x)]\!]_{S_k}$;

(iii) $(\top \vdash_x \bigvee_{k,t\in\mathbb{N}} (\exists z)(\phi_k(z) \wedge x = tz))$.

Now, axiom (iii) clearly entails axiom (i). Let us show that axiom (ii) is provable in the theory of MV-algebras, equivalently satisfied in every MV-algebra $\mathcal{A}$. Given elements x and y in A which respectively satisfy formulas ϕ_n and ϕ_m, we have by Lemma 5.2.15 that $nx = u$ and $my = u$ in $L(\mathcal{A})$. Set z equal to $ay - bx$ in this group, where a and b are the Bezout coefficients for the g.c.d. of n and m (cf. Theorem 5.2.8), so that $\text{g.c.d.}(n, m) = an - bm$. Let us show that $kz = u$ for $k = \text{l.c.m.}(n, m) = \frac{nm}{\text{g.c.d.}(n,m)}$. Since ℓ-groups are torsion-free, $kz = u$ if and only if $nmz = \text{g.c.d.}(n, m)u$. But $nmz = nm(ay - bx) = na(my) - mb(nx) = (an - bm)u = \text{g.c.d.}(n, m)u$, as required. Since $kz = u$, z is an element of A which by Lemma 5.2.15 satisfies the formula ϕ_k. So by Lemma 5.4.3 there exists an homomorphism $i : S_k \to \mathcal{A}$ sending the canonical generator ξ of S_k to z. Set $t = \frac{k}{n}$ and $s = \frac{k}{m}$. We clearly have that $x = tz$ and $y = sz$. The fact that $t\xi \in [\![x \,.\, \phi_n(x)]\!]_{S_k}$ and $s\xi \in [\![x \,.\, \phi_m(x)]\!]_{S_k}$ follows from these identities observing that i is an embedding of MV-algebras.

To obtain an axiomatization for $\mathbb{F}$ starting from the theory $\mathbb{MV}$ it therefore suffices to add axiom (iii). □

5.5 A new class of Morita-equivalences

In this section we shall introduce, for each Komori variety V axiomatized as above by the algebraic theory $\mathbb{T}_V$, a geometric theory extending that of ℓ-groups which will be Morita-equivalent to the theory $\mathbb{L}oc_V^2$.

We borrow the notation from Section 5.1. We shall work with varieties V generated by simple MV-algebras $\{S_i\}_{i \in I}$ and Komori chains $\{S_j^\omega\}_{j \in J}$. We indicate with the symbol $\delta(I)$ (resp. $\delta(J)$) and $\delta(n)$ the set of divisors of a number in I (resp. in J) and the set of divisors of n.

We observe that, by Theorem 5.2.3, the set-based models of the theory $\mathbb{L}oc_V^2$, that is, the local MV-algebras in V, are precisely the local MV-algebras $\mathcal{A}$ of finite rank $\text{rank}(\mathcal{A}) \in \delta(I) \cup \delta(J)$ such that if $\text{rank}(\mathcal{A}) \in \delta(I) \setminus \delta(J)$ then $\mathcal{A}$ is simple. On the other hand, by Theorem 5.2.4, for any ℓ-group $\mathcal{G}$, element $g \in G$ and natural number k, the algebra $\Gamma(\mathbb{Z} \times_{\text{lex}} \mathcal{G}, (k, g))$ is local of rank k and hence belongs to $\mathbb{L}oc_V^2$-mod(**Set**) if $k \in \delta(I) \cup \delta(J)$ and $\Gamma(\mathbb{Z} \times_{\text{lex}} \mathcal{G}, (k, g))$ is simple in case $\text{rank}(A) \in \delta(I) \setminus \delta(J)$.

In order to obtain an expansion of the theory $\mathbb{L}$ of ℓ-groups which is Morita-equivalent to our theory $\mathbb{L}oc_V^2$, we should thus be able to talk in some way about the ranks of the corresponding algebras inside such a theory. So we expand the signature of $\mathbb{L}$ by taking a 0-ary relation symbol R_k for each $k \in \delta(n)$. The predicate R_k has the meaning that the rank of the corresponding MV-algebra is a multiple of k (notice that we cannot expect the property 'to have rank *equal* to k' to be definable by a geometric formula since it is not preserved by homomorphisms of local MV-algebras in V).

To understand which axioms to put in our theory, the following lemma is useful.

Lemma 5.5.1. *For any ℓ-group $\mathcal{G}$, any element $g \in G$ and any natural number k, the algebra $\Gamma(\mathbb{Z} \times_{lex} \mathcal{G}, (k, g))$ is simple if and only if $\mathcal{G} = \{0\}$.*

Proof. It is clear that $\mathrm{Rad}(\Gamma(\mathbb{Z} \times_{\mathrm{lex}} \mathcal{G}, (k, g))) = \{(0, h) \mid h \geq 0\}$. So $\Gamma(\mathbb{Z} \times_{\mathrm{lex}} \mathcal{G}, (k, g))$ is simple if and only if $G^+ = \{0\}$, that is, if and only if $\mathcal{G} = \{0\}$. □

We add the following axioms to the theory $\mathbb{L}$ of ℓ-groups:

(1) $(\top \vdash R_1)$;

(2) $(R_k \vdash R_{k'})$, for each k' which divides k;

(3) $(R_k \wedge R_{k'} \vdash R_{\mathrm{l.c.m.}(k,k')})$, for any k, k';

(4) $(R_k \vdash_g g = 0)$, for every $k \in \delta(I) \setminus \delta(J)$;

(5) $(R_k \vdash \bot)$, for any $k \notin \delta(I) \cup \delta(J)$.

We also add a constant to our language to be able to name the unit of $\mathbb{Z} \times_{\mathrm{lex}} \mathcal{G}$ necessary to define the corresponding MV-algebra.

Let us denote by $\mathbb{G}_{(I,J)}$ the resulting theory.

Remark 5.5.2. We can equivalently define the theory $\mathbb{G}_{(I,J)}$ by considering a 0-ary relation symbol T_k for each $k \in \delta(I) \cup \delta(J)$ and by adding the following axioms.

(1) $(\top \vdash T_1)$;

(2) $(T_k \vdash T_{k'})$, for each k' which divides k;

(3) $(T_k \wedge T_{k'} \vdash T_{\mathrm{l.c.m.}(k,k')})$, for any k, k' such that $\mathrm{l.c.m.}(k, k') \in \delta(I) \cup \delta(J)$;

(4) $(T_k \vdash_g g = 0)$, for every $k \in \delta(I) \setminus \delta(J)$;

(5) $(T_k \wedge T_{k'} \vdash \bot)$, for any k, k' such that $\mathrm{l.c.m.}(k, k') \notin \delta(I) \cup \delta(J)$.

This theory is clearly bi-interpretable with the previous axiomatization.

Theorem 5.5.3. *The theory $\mathbb{G}_{(I,J)}$ is of presheaf type.*

Proof. Every axiom of $\mathbb{G}_{(I,J)}$, except for the last one, is cartesian. Thus, the theory obtained by adding the axioms (1)-(4) to the theory of ℓ-groups with an arbitrary constant is cartesian and hence of presheaf type. The thesis then follows from Theorem 1.5.14. □

In Section 1.5 we have observed that two theories of presheaf type are Morita-equivalent if and only if they have equivalent categories of set-based models. Thanks to Theorems 5.2.19 and 5.5.3, we can apply this to our theories $\mathbb{G}_{(I,J)}$ and $\mathbb{L}oc^2_V$.

To prove that $\mathbb{L}oc^2_V$ and $\mathbb{G}_{(I,J)}$ have equivalent categories of set-based models, we start by characterizing the set-based models of the latter theory.

Proposition 5.5.4. *The models of $\mathbb{G}_{(I,J)}$ in* **Set** *are triples $(\mathcal{G}, g, R)$, where $\mathcal{G}$ is an ℓ-group, g is an element of G and R is a subset of $\delta(n)$, which satisfy the following properties:*

(i) R is an ideal of $\delta(n)$;

(ii) if $R \subseteq \delta(I) \setminus \delta(J)$, then the ℓ-group $\mathcal{G}$ is the trivial one;

(iii) $R \subseteq \delta(I) \cup \delta(J)$.

Proof. The interpretation of the propositional symbols over the signature of $\mathbb{G}_{(I,J)}$ can be identified with a subset R of $\delta(n)$ satisfying particular properties. Axioms (1)-(3) assert that R is an ideal of $(\delta(n), /)$ (recall that an ideal of a sup-semilattice with bottom element is a lowerset which contains the bottom element and which is closed with respect to the sup operation). Axiom (4) asserts that for any $a \in \delta(I) \setminus \delta(J)$, if $a \in R$ then the group $\mathcal{G}$ is the trivial one. This corresponds to condition (ii). Lastly, axiom (5) asserts that R is contained in $\delta(I) \cup \delta(J)$. □

Lemma 5.5.5. *There is a bijection between the elements of $\delta(I) \cup \delta(J)$ and the ideals of $\delta(n)$ contained in $\delta(I) \cup \delta(J)$.*

Proof. Let k be an element in $\delta(I) \cup \delta(J)$. The ideal $\downarrow k$ generated by k is contained in $\delta(I) \cup \delta(J)$ since this set is a lowerset. On the other hand, given an ideal R of $\delta(n)$ contained in $\delta(I) \cup \delta(J)$, its maximal element, which always exists since R is finite and closed with respect to the least commom multiple, belongs to $\delta(I) \cup \delta(J)$. This correspondence yields a bijection. Indeed, it is easy to prove that

$$R = \downarrow \max(R) \text{ and } \max(\downarrow k) = k$$

for every ideal R of $\delta(n)$ contained in $\delta(I) \cup \delta(J)$ and every $k \in \delta(I) \cup \delta(J)$. □

Remark 5.5.6. By Lemma 5.5.5, a set-based model of the theory $\mathbb{G}_{(I,J)}$ can be identified with a triple $(\mathcal{G}, g, k)$, where $\mathcal{G}$ is a ℓ-group, g is an element of G and k is an element of $\delta(I) \cup \delta(J)$, such that if $k \in \delta(I) \setminus \delta(J)$ then $\mathcal{G}$ is the trivial group.

Let $(\mathcal{G}, g, R)$ and $(\mathcal{H}, h, P)$ be two models of $\mathbb{G}_{(I,J)}$ in **Set**. The $\mathbb{G}_{(I,J)}$-model homomorphisms $(\mathcal{G}, g, R) \to (\mathcal{H}, h, P)$ are pairs of the form (f, i), where f is an ℓ-group homomorphism $\mathcal{G} \to \mathcal{H}$ such that $f(g) = h$ and i is an inclusion $R \subseteq P$.

Theorem* 5.5.7. *Let $V = V(\{S_i\}_{i \in I}, \{S_j^\omega\}_{j \in J})$ be a Komori variety. Then the category of set-based models of the theory $\mathbb{L}oc_V^2$ is equivalent to the category of set-based models of the theory $\mathbb{G}_{(I,J)}$.*

Proof. We shall define a functor

$$M_{(I,J)} : \mathbb{G}_{(I,J)}\text{-mod}(\mathbf{Set}) \to \mathbb{L}oc_V^2\text{-mod}(\mathbf{Set}),$$

and prove that it is a categorical equivalence, i.e., that it is full and faithful and essentially surjective.

Objects: Let $(\mathcal{G}, g, R)$ be a model of $\mathbb{G}_{(I,J)}$ in **Set**. We set

$$M_{(I,J)}(\mathcal{G}, g, R) := \Gamma(\mathbb{Z} \times_{\text{lex}} \mathcal{G}, (\max(R), g)) .$$

By Proposition 5.5.4(iii) and Lemma 5.5.5, $\max(R)$ belongs to $\delta(I) \cup \delta(J)$. By Proposition 5.5.4(ii), if $\max(R) \in \delta(I) \setminus \delta(J)$ then the ℓ-group $\mathcal{G}$ is trivial and $M_{(I,J)}(\mathcal{G}, g, R)$ is a simple MV-algebra. We can thus conclude from Theorem 5.2.3 that the algebra $M_{(I,J)}(\mathcal{G}, g, R)$ lies in V.

Arrows: Let $(\mathcal{G}, g, R)$ and $(\mathcal{H}, h, P)$ be two models of $\mathbb{G}_{(I,J)}$ in **Set** and $(f, i : R \subseteq P)$ a homomorphism between them. Since $R \subseteq P$, $\max(R)$ divides $\max(P)$. We set

$$M_{(I,J)}(f,i) : \Gamma(\mathbb{Z} \times_{\text{lex}} \mathcal{G}, (\max(R), g)) \to \Gamma(\mathbb{Z} \times_{\text{lex}} \mathcal{H}, (\max(P), h))$$
$$(i, x) \mapsto (\frac{\max(P)}{\max(R)} i, f(x))$$

Since $M_{(I,J)}(f,i)$ is the result of applying the functor Γ to a unital ℓ-group homomorphism, it is an MV-algebra homomorphism.

The functoriality of the assignment $(f,i) \to M_{(I,J)}(f,i)$ is clear.

Let us now prove that the functor $M_{(I,J)}$ is full and faithful and essentially surjective. The fact that it is essentially surjective follows at once from Theorems 5.2.4 and 5.2.3 in light of Remark 5.5.6. The fact that it is full and faithful follows from the fact that one can recover any f from $M_{(I,J)}(f,i)$ as the ℓ-group homomorphism induced by the monoid homomorphism

$$G^+ = \text{Rad}(\Gamma(\mathbb{Z} \times_{\text{lex}} \mathcal{G}, (\max(R), g))) \to \text{Rad}(\Gamma(\mathbb{Z} \times_{\text{lex}} \mathcal{H}, (\max(P), h))) = H^+$$

(cf. the proof of Lemma 5.5.1) and that the existence of an MV-algebra homomorphism $\Gamma(\mathbb{Z} \times_{\text{lex}} \mathcal{G}, (\max(R), g))) \to \Gamma(\mathbb{Z} \times_{\text{lex}} \mathcal{H}, (\max(P), h))$ implies that $\text{rank}(\Gamma(\mathbb{Z} \times_{\text{lex}} \mathcal{G}, (\max(R), g))) = \max(R)$ divides $\text{rank}(\Gamma(\mathbb{Z} \times_{\text{lex}} \mathcal{H}, (\max(P), h)) = \max(P)$ and hence that $R \subseteq P$. □

Corollary* 5.5.8. *Let $V = V(\{S_i\}_{i \in I}, \{S_j^\omega\}_{j \in J})$ be a Komori variety. Then the theory $\mathbb{L}oc_V^2$ of local MV-algebras in V and the theory $\mathbb{G}_{(I,J)}$ are Morita-equivalent.* □

5.5.1 Non-triviality of the Morita-equivalences

In Section 4.5.3 we proved that the Morita-equivalence lifting Di Nola-Lettieri's equivalence was non-trivial; in this section we shall see that this is true more generally for all the Morita-equivalences of Corollary 5.5.8.

Let $V = V(\{S_i\}_{i\in I}, \{S_j^\omega\}_{j\in J})$ be a Komori variety. Suppose that we have an interpretation I of $\mathbb{G}_{(I,J)}$ into $\mathbb{L}oc_V^2$. Then the induced functor

$$s_I : \mathbb{G}_{(I,J)}\text{-mod}(\mathbf{Set}) \to \mathbb{L}oc_V^2\text{-mod}(\mathbf{Set})$$

sends the model $\mathcal{M} = (\{0\}, 0, \downarrow 1)$ of the theory $\mathbb{G}_{(I,J)}$ to a model $\mathcal{N}$ of $\mathbb{L}oc_V^2$. If $I(\{x \,.\, \top\}) = \{\vec{y} \,.\, \psi\}$ we have that:

$$F_{\mathcal{N}}(\{x \,.\, \top\}) \cong F_{\mathcal{M}}(\{\vec{y} \,.\, \psi\}),$$

$$N \cong [\![x \,.\, \top]\!]_{\mathcal{M}} \cong [\![\vec{y} \,.\, \psi]\!]_{\mathcal{M}} \subseteq M^k \cong M = \{0\}\ .$$

By axiom NT of $\mathbb{L}oc_V^2$, this is not possible. So we have the following result.

Proposition 5.5.9. *Let $V = V(\{S_i\}_{i\in I}, \{S_j^\omega\}_{j\in J})$ be a Komori variety. Then the theories $\mathbb{L}oc_V^2$ and $\mathbb{G}_{(I,J)}$ are not bi-interpretable.*

5.5.2 When is $\mathbb{L}oc_V^2$-mod(Set) algebraic?

By Theorem 5.2.19, the theory $\mathbb{L}oc_V^2$ is of presheaf type, whence its category $\mathbb{L}oc_V^2$-mod(**Set**) of set-based models is finitely accessible, i.e., it is the ind-completion of its full subcategory on the finitely presentable objects. It is natural to wonder under which conditions this category is also *algebraic* (i.e., equivalent to the category of finite-product-preserving functors from a small category with finite products to **Set**, cf. Chapter 1 of [43]). Indeed, in Section 4.4 we proved that the theory of perfect MV-algebras is Morita-equivalent to an algebraic theory, namely the theory of ℓ-groups, whence its category of set-based models is algebraic.

As shown by the following proposition, the category $\mathbb{L}oc_V^2$-mod(**Set**) cannot be algebraic for an arbitrary proper subvariety V.

Proposition 5.5.10. *Let (I, J) be a reduced pair such that $I \neq \emptyset$ and $J \neq \emptyset$ and $V = V(\{S_i\}_{i\in I}, \{S_j^\omega\}_{j\in J})$ the corresponding variety. Then $\mathbb{L}oc_V^2$-mod*(**Set**) *is not algebraic.*

Proof. Given $n \in I$ and $m \in J$, we have that $S_n, S_m^\omega \in \mathbb{L}oc_V^2$-mod(**Set**). If $\mathbb{L}oc_V^2$-mod(**Set**) is algebraic then there exists the coproduct $\mathcal{A}$ of S_n and S_m^ω in $\mathbb{L}oc_V^2$-mod(**Set**). Since $\mathcal{A}$ belongs to $\mathbb{L}oc_V^2$-mod(**Set**), it has finite rank. By Theorem 5.2.3, either $\mathcal{A}$ is simple or there exists $j \in J$ such that rank$(\mathcal{A})/j$. Since there is an MV-algebra homomorphism $S_m^\omega \to \mathcal{A}$, $\mathcal{A}$ cannot be simple. So rank$(\mathcal{A})/j$ for some $j \in J$. But $m/$rank$(\mathcal{A})$ and (I, J) is a reduced pair, so $m = j$ and hence rank$(\mathcal{A}) = m$. On the other hand, n divides rank$(\mathcal{A})$ since there is an MV-algebra homomorphism $S_n \to \mathcal{A}$, so n/m. Since (I, J) is a reduced pair, this implies that $n = m$; but this is absurd since in a reduced pair (I, J), $I \cap J = \emptyset$. □

On the other hand, as we shall prove below, for varieties V generated by a single chain (which can be either a finite simple algebra or a Komori chain), the corresponding category $\mathbb{L}oc_V^2$-mod(**Set**) is algebraic.

If V is generated by one simple MV-algebra S_n, the models of the theory $\mathbb{G}_{(\{n\},\emptyset)}$ in **Set** are the triples of the form $(\{0\}, 0, \downarrow k)$, where $k \in \delta(n)$. Thus, we have that:

$$\mathbb{G}_{(\{n\},\emptyset)}\text{-mod}(\mathbf{Set}) \simeq (\delta(n), /) .$$

Hence, the category $\mathbb{L}oc_V^2$-mod(**Set**) is algebraic if and only if the poset of divisors of n is an algebraic category.

If instead V is generated by one Komori chain S_n^ω, we have that:

$$\mathbb{G}_{(\emptyset,\{n\})}\text{-mod}(\mathbf{Set}) \simeq \mathbb{L}'\text{-mod}(\mathbf{Set}) \times (\delta(n), /),$$

where $\mathbb{L}'$ is the theory of ℓ-groups with an arbitrary constant. Since the theory $\mathbb{L}'$ is algebraic, the category $\mathbb{L}oc_V^2$-mod(**Set**) is algebraic if the poset category $(\delta(n), /)$ is (cf. Proposition 5.5.14 below).

In Chapter 4 of [43], the authors characterized the algebraic categories as the free cocompletions under sifted colimits (i.e., those colimits which commute with all finite products in **Set**) of a small category (which can be recovered from it as the full subcategory on the *perfectly presentable objects*, i.e., those objects whose corresponding covariant representable functor preserves sifted colimits). In Chapter 6 of *op. cit.*, they gave an alternative characterization in terms of strong generators of perfectly presentable objects:

Definition 5.5.11 (Definition 6.1 [43])**.** A set of objects $\mathcal{G}$ in a category $\mathcal{A}$ is called a *generator* if two morphisms $x, y : A \to B$ are equal whenever $x \circ g = y \circ g$ for every morphism $g : G \to A$ with domain G in $\mathcal{G}$. A generator $\mathcal{G}$ is called *strong* if a monomorphism $m : A \to B$ is an isomorphism whenever every morphism $g : G \to B$ with domain G in $\mathcal{G}$ factors through m.

Theorem 5.5.12 (Theorem 6.9 [43])**.** *The following conditions on a category $\mathcal{A}$ are equivalent:*

(i) $\mathcal{A}$ is algebraic;

(ii) $\mathcal{A}$ is cocomplete and has a strong generator of perfectly presentable objects.

Remark 5.5.13. By Corollary 6.5 [43], if $\mathcal{A}$ has coproducts and every object of $\mathcal{A}$ is a colimit of objects from $\mathcal{G}$, then $\mathcal{G}$ is a strong generator.

The following result is probably well-known but we were not able to find it in the literature.

Proposition 5.5.14. *Let $\mathcal{A}$ and $\mathcal{B}$ be algebraic categories. Then the category $\mathcal{A} \times \mathcal{B}$ is algebraic.*

Proof. As colimits in $\mathcal{A} \times \mathcal{B}$ are computed componentwise, the category $\mathcal{A} \times \mathcal{B}$ has coproducts if $\mathcal{A}$ and $\mathcal{B}$ do. Let us now prove that for any objects a of $\mathcal{A}$ and b of $\mathcal{B}$ that are perfectly presentable respectively in $\mathcal{A}$ and in $\mathcal{B}$, the object (a, b) is perfectly presentable in $\mathcal{A} \times \mathcal{B}$. If $\pi_{\mathcal{A}} : \mathcal{A} \times \mathcal{B} \to \mathcal{A}$ and $\pi_{\mathcal{B}} : \mathcal{A} \times \mathcal{B} \to \mathcal{B}$ are the canonical projection functors then, for any diagram $D : \mathcal{I} \to \mathcal{A} \times \mathcal{B}$ defined on a sifted category $\mathcal{I}$, $\mathrm{colim}(D) = (\mathrm{colim}(\pi_{\mathcal{A}} \circ D), \mathrm{colim}(\pi_{\mathcal{B}} \circ D))$. So, since colimits in $\mathcal{A} \times \mathcal{B}$ are computed componentwise and sifted colimits commute with finite products in the category **Set**, we have that $\mathrm{Hom}_{\mathcal{A}\times\mathcal{B}}((a, b), \mathrm{colim}(D)) \cong \mathrm{Hom}_{\mathcal{A}}(a, \mathrm{colim}(\pi_{\mathcal{A}} \circ D)) \times \mathrm{Hom}_{\mathcal{B}}(b, \mathrm{colim}(\pi_{\mathcal{B}} \circ D)) \cong \mathrm{colim}(\mathrm{Hom}_{\mathcal{A}}(a, -) \circ \pi_{\mathcal{A}} \circ D) \times \mathrm{colim}(\mathrm{Hom}_{\mathcal{B}}(b, -) \circ \pi_{\mathcal{B}} \circ D) \cong \mathrm{colim}((\mathrm{Hom}_{\mathcal{A}}(a, -) \circ \pi_{\mathcal{A}} \circ D) \times (\mathrm{Hom}_{\mathcal{B}}(b, -) \circ \pi_{\mathcal{B}} \circ D)) \cong \mathrm{colim}(\mathrm{Hom}_{\mathcal{A}\times\mathcal{B}}((a, b), -) \circ D)$. We can thus conclude from Remark 5.5.13 that the category $\mathcal{A} \times \mathcal{B}$ is algebraic, as required. □

Proposition 5.5.15. *The category $(\delta(n), /)$ is algebraic.*

Proof. By Theorem 5.5.12 and Remark 5.5.13, it suffices to verify that the category $(\delta(n), /)$ is cocomplete and that every element is a join of perfectly presentable objects. Now, by Example 5.6(3) [43], the perfectly presentable objects of a poset are exactly the compact elements, i.e., the elements x such that for any directed join $\bigvee_{i \in I} y_i$, from $x \leq \bigvee_{i \in I} y_i$ it follows that $x \leq y_i$ for some i. The poset $(\delta(n), /)$ is cocomplete since it is finite and has finite coproducts (given by the l.c.m. and by the initial object 1). Since every element of $(\delta(n), /)$ is compact, our thesis follows. □

We can thus conclude that

Corollary 5.5.16. *The category $\mathbb{L}oc^2_V$-mod(**Set**) is algebraic if and only if V can be generated by a single chain (either of the form S_n or of the form S_n^ω).* □

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