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Many Valued Logics: Interpretations, Representations and Applications

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"I am entitled not to recognize the principle of bivalence, and to accept the view that besides truth and falsehood exist other truth-values, including at least more, the third truth-value. What is this third-value? I have no suitable name for it. But after the preceding explanations it should not be difficult to understand what I have in mind. I maintain that there are propositions which are neither true nor false but indeterminate. All sentences about future facts which are not yet decided belong to this category. Such sentences are neither true at present moment, for they have no real correlate. [...] If third value is introduced into logic we change its very foundations."

Jan Łukasiewicz, *On determinism* 1946

Overview

This thesis, as the research activity of the author, is devoted to establish new connections and to strengthen well-established relations between different branches of mathematics, via logic tools. Two main many valued logics, *logic of balance* and *Łukasiewicz logic*, are considered; their associated algebraic structures will be studied with different tools and these techniques will be applied in social choice theory and artificial neural networks. The thesis is structured in three parts.

Part I The logic of balance, for short $Bal(H)$, is introduced. It is showed: the relation with ℓ-Groups, i.e. lattice ordered abelian groups (Chapter 2); a functional representation (Chapter 3); the algebraic geometry of the variety of ℓ-Groups with constants (Chapter 4).

Part II A brief historical introduction of Łukasiewicz logic and its extensions is provided. It is showed: a functional representation via *generalized states* (Chapter 5); a non-linear model for MV-algebras and a detailed study of it, culminating in a categorical theorem (Chapter 6).

Part III Applications to social choice theory and artificial neural network are presented. In particular: preferences will be related to vector lattices and their cones, recalling the relation between polynomials and cones studied in Chapter 4; multilayer perceptrons will be elements of non-linear models introduced in Chapter 6 and networks will take advantages from *polynomial completeness*, which is studied in Chapter 2.

We are going to present: in Sections 1.1 and 1.2 all the considered structures, our approach to them and their (possible) applications; in Section 1.3 a focus on the representation theory for ℓ-Groups and MV-algebras.

Note that: algebraic geometry for ℓ-Groups provides a *modus operandi* which turns out to be useful not only in theoretical field, but also in applications, opening (we hope) new perspectives and intuitions, as we made in this first approach to social theory; non-linear models here presented and their relation to neural networks seem to be very promising, giving both intuitive and formal approach to many concrete problems, for instance degenerative diseases or distorted signals. All these interesting topics will be studied in future works of the author.

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Contents

Notations and Symbols

$\mathbb{N}$	set of natural numbers
$\mathbb{Z}$	set of integer numbers
$\mathbb{Q}$	set of rational numbers
$\mathbb{R}$	set of real numbers
$< a >$	ℓ-ideal generated by a
$Aff_Y(X)$	space of all affine functions from X to Y
$\mathcal{C}_{X_2}(X_1)$	set of all continuous functions from X_1 to X_2 topological spaces
$FA\ell_0(n)$	free ℓ-group over n generators
$FA\ell_H(n)$	free ℓ-group over n generators with constants in H
$Hom(A, B)$	set of homomorphisms from A to B
ℓ-Group	lattice ordered abelian group
ℓ_u-Group	lattice ordered abelian group with u strong unit
$\ell\mathcal{GR}$	variety of ℓ-groups
$\ell\mathcal{GR}_H$	variety of ℓ-groups with ell-group H of constants
M_n	piecewise linear functions with integer coefficients over $[0, 1]^n$

Chapter 1

Introduction

1.1 ℓ-Groups: Algebraic Geometry and Social Choice

We propose a systematical study of the variety of ℓ-groups via *universal algebraic geometry*. The ℓ-groups find many applications from theoretical fields, e.g. in mathematics for the study of the C^*-algebras and in physics about quantum mechanics, to applied fields, e.g. in operational research for the multiple-criteria decision analysis and in machine learning and cognitive science for the description of artificial neural networks. The study of these structures is deep and wide (e.g. see Anderson and Feil, 2012; Glass and Holland, 2012), with particular interest on geometric features and on connection with polyhedral geometry, specially in the case with strong unit (see Busaniche, Cabrer, and Mundici, 2012; Cabrer and Mundici, 2011; Cabrer and Mundici, 2012; Cabrer, 2015), thanks to the relation with Łukasiewicz logic via Mundici functor (for more details see Cignoli, d'Ottaviano, and Mundici, 2013; Mundici, 1986).

We will deal with the variety of ℓ-groups using universal algebraic geometry (see Plotkin in Plotkin, 2002), which combines the tools of classical algebraic geometry, traditionally based on the concepts of polynomial and field, with the tools of universal algebra that apply to algebraic structures of any kind (including groups, rings, etc.). The goodness of these techniques is already shown in a series of works by Sela, Kharlampovich and others; these works solve the conjectures of Tarski on finitely generated free groups, showing that these groups have the same theory (apart from the case of one generator, which gives the integers) and that this theory is decidable.

We start from very malleable objects, piecewise linear functions, a well established tool for a huge amount of applications, to get to define, in the purest way, a logic language which describes our structures. We show how to obtain and derive properties in one among algebraic, geometrical, functional analytic and logic field using information coming from the other ones. The underlying theme of this work leads us in a path through different fields; it connects algebra, geometry, functional analysis and logic through the simple ability to define objects using ℓ-equations (which are the equations between ℓ-polynomials), i.e. the ability to describe solutions of ℓ-equations from the properties of an ℓ-group and viceversa.

We use different tools and techniques to describe properties of ℓ-groups. In Section 4.1 there is a briefly overview of piecewise linear functions (generalizing many results of Baker, 1968; Beynon, 1975; Beynon, 1977). In Section 4.3 we study the connections between algebraic and geometrical properties of an ℓ-group. In Chapter 2 we extend a logic, proposed in Galli, Lewin, and Sagastume, 2004, with constants which describes our structures and we focus on the *polynomial completeness*. In particular, the main results (presented in Di Nola, Lenzi, and Vitale, sub) are:

- a completeness theorem of our logic (Theorem 2.1.2);
- a Wójcicki-type theorem (Theorem 2.2.1);
- the Nullstellensatz for ℓ-groups (Theorem 4.3.1);
- a characterization of the geometrically stable ℓ-groups (Theorem 4.4.1);
- a characterization of algebraically closed ℓ-groups (Theorem 4.6.1);
- a categorical duality between the category of algebraic sets and of coordinate algebras (Theorem 4.7.1).

Social Preferences Our choices are strictly related to our ability to compare alternatives according to different criteria, e.g. price, utility, feelings, life goals, social conventions, personal values, etc. This means that in each situation we have different *best alternatives* with respect to many criteria; usually, the context gives us the *most suitable* criteria, but no one says that there is a unique criterion. Even when we want to make a decision according to the opinions of the experts in a field we may not have a unique advice. To sum up, we have to be able to define our *balance* between different criteria and opinions, to give to each comparison a weight which describes the importance, credibility or goodness and then to include all these information in a mixed criteria. As usual, we need a formalization which gives us tools to solve these problems; properties of this formalization are well summarized by Saaty in Saaty, 1990, according to whom

> [it] must include enough relevant detail to: represent the problem as thoroughly as possible, but not so thoroughly as to lose sensitivity to change in the elements; consider the environment surrounding the problem; identify the issues or attributes that contribute to the solution; identify the participants associated with the problem.

Riesz spaces, with their double nature of both weighted and ordered spaces, seem to be the natural framework to deal with multi-criteria methods; in fact, in real problems we want to obtain an order starting from weights and to compute weights having an order.

We remark that:

- Riesz spaces are already studied and widely applied in economics, mainly supported by works of Aliprantis (see Abramovich, Aliprantis, and Zame, 1995; Aliprantis and Brown, 1983; Aliprantis and Burkinshaw, 2003);
- contrary to the main lines of research, which prefer to propose ad-hoc models for each problems, we want to analyze and propose a general framework to work with and to be able, in the future, to provide a universal translator of various approaches.

We introduce basic definitions and properties of Riesz spaces with a possible interpretation of them in the context of pairwise comparison matrices, focusing on aggregation procedures. As main results we have:

- a characterization of collective choice rules satisfying Arrow's axioms (Theorem 7.2.1);
- established an antitone Galois correspondence between total preorders and cones of a Riesz space (Theorem 7.3.1);
- a categorical duality between categories of preorders and of particular cones of a Riesz space (Theorem 7.3.2).

In Section 7 we recall some basic definitions of Riesz space and of pairwise comparison matrix (PCM). Section 7.1 is devoted to explain, also with meaningful examples, the main ideas that led us to propose Riesz spaces as suitable framework in the context of decision making; in particular it will explained how properties of Riesz spaces can be appropriate to model, and to deal with, real problems. In Sections 7.2 and 7.3 we focus on a particular method of decision making theory, i.e. PCMs; we pay special attention to:

- collective choice rules;
- classical social axioms (Arrow's axioms);
- total preorder spaces;
- duality between total preorders and geometric objects.

1.2 MV-algebras: Beyond Linearity and ANNs

Recall that *MV-algebras* are the structures corresponding to Łukasiewicz many valued logic, in the same sense in which Boolean algebras correspond to classical logic (see Blok and Pigozzi, 1989). *Riesz MV-algebras* are MV-algebras enriched with an action of the interval $[0, 1]$, which makes them appealing for applications in real analysis.

Usually free MV-algebras and Riesz MV-algebras (in particular the finitely generated ones) are represented by piecewise linear functions. But for applications it could be interesting to represent (Riesz) MV-algebras with non-linear functions. One could relax the linearity requirement and consider piecewise polynomial functions, which are important for several reasons, for instance they are the subject of the celebrated Pierce-Birkhoff conjecture, and include, in one variable, the spline functions, a kind of functions which has been deeply studied, see Schoenberg, 1946a and Schoenberg, 1946b. Other examples are Lyapunov functions used in the study of dynamical systems, see Lyapunov, 1992, and logistic functions. We will show that a possible application of non-linear MV-algebras can be found in the domain of artificial neural networks.

We stick to continuous functions, despite that for certain applications it could be reasonable to use discontinuous functions, for instance in order to model arbitrary signals in signal processing. Continuous functions are preferable for technical reasons: for instance, they preserve compact sets, and in general, they behave well with respect to topology.

So, our Riesz MV-algebras of interest will be the Riesz MV-algebras of all continuous functions from $[0,1]^n$ to $[0,1]$, which we will denote by C_n.

An important subalgebra of C_n is given by the Riesz MV-algebra of what we call *Riesz-McNaughton functions*. We call RM_n the Riesz MV-algebra of Riesz-McNaughton functions from $[0,1]^n$ to $[0,1]$. That is, $f \in RM_n$ if it is continuous, and there are affine functions $f_1, \ldots, f_m$ with *real* coefficients, such that for every $x \in [0,1]^n$ there is i with $f(x) = f_i(x)$.

In other words, RM_n is the set of all piecewise affine functions with real coefficients.

As a particular case, *McNaughton functions* are those Riesz-McNaughton functions where coefficients are *integer* rather than real. We denote by M_n the MV-algebra of McNaughton functions (it is an MV-algebra, not a Riesz MV-algebra). RM_n is a free Riesz MV-algebra in n generators. Then the free Riesz MV-algebras over n generators coincide with the isomorphic copies of RM_n. We say that a structure A is a *copy* of a structure B when A is isomorphic to B. However, we prefer *not* to identify isomorphic Riesz MV-algebras of functions, because they can consist of functions with very diverse geometric properties, which may be relevant for applications.

The main results (presented in Di Nola, Lenzi, and Vitale, 2016b) are:

- an extension of the Marra-Spada duality from MV-algebras to Riesz MV-algebras;
- a characterization of zerosets of Riesz-McNaughton functions by means of polyhedra (Theorem 6.1.3);
- a study of copies of RM_n in C_n;
- a duality between several interesting categories of Riesz MV-subalgebras of C_n and closed subsets of $[0,1]^n$ up to R-homeomorphism (Theorem 6.2.4).

Artificial Neural Network Many-valued logic has been proposed in Castro and Trillas, 1998 to model neural networks: it is shown there that, by taking as activation functions ρ the identity truncated to zero and one (i.e., $\rho(x) = (1 \wedge (x \vee 0)))$, it is possible to represent the corresponding neural network as combination of propositions of Łukasiewicz calculus.

In Di Nola, Gerla, and Leustean, 2013 the authors showed that multilayer perceptrons, whose activation functions are the identity truncated to zero and one, can be fully interpreted as logical objects, since they are equivalent to (equivalence classes of) formulas of an extension of Łukasiewicz propositional logic obtained by considering scalar multiplication with real numbers (corresponding to Riesz MV-algebras, defined in Di Nola and Leustean, 2011 and Di Nola and Leuştean, 2014).

We propose more general multilayer perceptrons which describe not necessarily linear events. We show how we can name a neural network with a formula and, vice versa, how we can associate a class of neural networks to each formula; moreover we introduce the idea of *Łukasiewicz Equivalent Neural Networks* to stress the strong connection between (very different) neural networks via Łukasiewicz logical objects. Moreover we describe the structure of these multilayer perceptrons and provide the existence of finite points (our input) which allow us to recognize goal functions, with the additional property that it is possible to use classical methods of learning process. To sum up, main results (partially presented in Di Nola, Lenzi, and Vitale, 2016a) are:

- propose Ł$\mathcal{N}$ as a privileged class of multilayer perceptrons;
- link Ł$\mathcal{N}$ with Łukasiewicz logic (one of the most important many-valued logics);
- show that we can use many properties of (Riesz) McNaughton functions for a larger class of functions;
- propose an equivalence between particular types of multilayer perceptrons, defined by Łukasiewicz logic objects;
- compute many examples of Łukasiewicz equivalent multilayer perceptrons to show the action of the free variables interpretation;
- describe our networks;
- argue on a suitable selection of input.

We think that using (in various ways) the *interpretation layer* it is possible to encode and describe many phenomena (e.g. degenerative diseases, distorted signals, etc), always using the descriptive power of the Łukasiewicz logic formal language.

1.3 Representation of ℓ-Groups and MV-algebras

Representation theorems have played a crucial role in the study of abstract structures. Representation theory provides a new and deep understanding of the properties in several fields, presents different perspectives and has various applications in many areas of mathematics. As showed in the literature (e.g. Riesz representation theorem for vector lattices (Rudin, 1987, Theorem 2.14), Di Nola representation theorem for MV-algebras (Cignoli, d'Ottaviano, and Mundici, 2013, Theorem 9.5.1)), special attention is paid to embeddings in functional spaces.

We focus on the space of particular homomorphisms between an archimedean ℓ-group (a semisimple MV-algebra, respectively) and a vector lattice (a Riesz MV-algebra, respectively), i.e. the set of the *generalized states*, introducing a quite natural generalization of the well-studied states on ℓ-groups and MV-algebras (see also Goodearl, 2010; Mundici, 2011). We provide a framework, in which it is possible to encode and decode more information than usual.

Archimedean ℓ-groups and semisimple MV-algebras are widely and deeply studied and different representation theorems are known in the literature (see for example Bigard, Keimel, and Wolfenstein, 1977; Boccuto and Sambucini, 1996; Darnel, 1994; Filter et al., 1994; Glass, 1999; Goodearl, 2010 and Cignoli, d'Ottaviano, and Mundici, 2013; Mundici, 1986; Mundici, 2011; Pulmannová, 2013, respectively). In particular, for archimedean ℓ-groups the Bernau representation theorem (see also Bernau, 1965) provides a functional description of these kinds of structures. The statement of the theorem is the following.

Theorem (Glass, 1999, Theorem 5.F) *Given an Archimedean ℓ-group G there is an ℓ-embedding $\iota : G \hookrightarrow D(X)$ of G into the vector lattice of almost finite continuous functions on a Stone space $X = S(B)$, where B is the Boolean algebra of polars in G.*

Furthermore, Pulmannová presents a representation theorem for semisimple MV-algebras via states on an effect algebra.

Theorem (Pulmannová, 2013, Theorem 4.5) *Given an Archimedean MV-algebra A there is an embedding of A into the MV-algebra of all pairwise commuting effects on a complex Hilbert space.*

One of the motivations of this work is to give a representation which is convenient to work with (we consider simple objects, i.e. affine or continuous functions), but, on the other hand, is powerful enough to express significant properties of our studied objects (the involved functions act on generalized states).

Generalized states take values in a Dedekind complete vector lattice, in which it is possible to give generalizations of Hahn-Banach, extension and sandwich-type theorems. Many of these results are presented in the literature (see Boccuto and Candeloro, 1994; Bonnice and Silverman, 1967; Chojnacki, 1986; Fuchssteiner and Lusky, 1981; Ioffe, 1981; Kusraev and Kutateladze, 1984; Kusraev and Kutateladze, 2012; Lipecki, 1979; Lipecki, 1980; Lipecki, 1982; Lipecki, 1985; Luschgy and Thomsen, 1983). This fact has led us to consider and use techniques which will allow to reproduce, in the framework of MV-algebras, these results and their implications in applications.

Indeed, these kinds of theorems have many applications (see Aliprantis and Burkinshaw, 2003; Aliprantis and Burkinshaw, 2006; Aubert and Kornprobst, 2006; Boccuto, Gerace, and Pucci, 2012; Boyd and Vandenberghe, 2004; Brezis, 2010; Fremlin, 1974; Hildenbrand, 2015; Kusraev and Kutateladze, 2012; Rockafellar and Wets, 2009), for example convex analysis and properties of conjugate convex functions, which are useful to prove duality theorems; image restoring problems; subdifferential and variational calculus; convex operators; least norm problems; interpolation; statistical optimization; minimization problems; vector programs; economy equilibria. We recall that MV-algebras are the algebraic semantic of Łukasiewicz logic (ŁL) (see also Cignoli, d'Ottaviano, and Mundici, 2013), one of the first non-classical logics, and Riesz MV-algebras (see Di Nola and Leuştean, 2014) of an extension of ŁL. This has implications also in the mathematical logic field. Moreover, these structures have also several applications (see Amato, Di Nola, and Gerla, 2002; Hussein and Barriga, 2009; Hassan and Barriga, 2006; Kroupa and Majer, 2012), among which artificial neural networks; image compression; image contrast control; game theory. Our approach could give some further developments in applications of both fields, by considering more abstract structures which contain more relevant information on the treated objects.

The main results (presented in Di Nola, Boccuto, and Vitale, sub) are:

- a representation theorem for archimedean ℓ_u-groups, using extremal states (Theorem 3.2.1);
- a representation theorem for archimedean ℓ_u-groups, simply by means of states (Theorem 3.2.2);

- a representation theorem for semisimple MV-algebras, via *generalized states* (Theorem 5.0.5).

Part I

Logic of Balance

Preliminaries

Varieties and Categories A signature τ is a set of function symbols each of which has an arity which is a natural number. We admit also symbols with arity zero which we will call *constants*. Now let τ be a signature and X a set of variables, then T(X) denotes the set of the terms (or τ-terms) in the signature τ on the set X of variables, which are inductively defined (for more details see Burris and Sankappanavar, 1981). We call *variety* of algebras the class of all algebraic structures on a specific signature satisfying a given set of identities. The variety identities are expressions in the form $p(x) = q(x)$ where $p(x)$ and $q(x)$ belong to the set T(X). Note that every variety Θ can be regarded as category whose morphisms are the homomorphisms in Θ. For more details on categories and functors see Mac Lane, 1978.

Congruences Let A be an algebra of signature τ and let θ be an equivalence relation. Then θ is a congruence on A if it satisfies the following compatibility properties: $\forall f \in \tau$, and $\forall a_i, b_i \in A$, $i = 1, \dots, n$, such that $a_i \theta b_i$, we have $f^A(a_1, \dots, a_n) \theta f^A(b_1, \dots, b_n)$. We denote by Con(A) the set of all congruences on the algebra A. If θ is a congruence on A, then the quotient algebra of A with respect to θ, denoted by A/θ is the algebra whose support is the support of A modulo θ and whose operations satisfy the identity $f^{A/\theta}(a_1/\theta, \dots, a_n/\theta) = f^A(a_1, \dots, a_n)/\theta$, where $a_1, \dots, a_n \in A$ and f is a n-ary functional symbol in τ. Obviously quotient algebras of A have the same signature of A.

Free Algebras Let K be a class of algebras with a signature τ (i.e. a τ-algebra), $A \in K$ and X be a subset of A. We say that A is free over X if X generates A and for every $B \in K$ and for every function $\alpha : X \to B$ there exists a unique homomorphism $\beta : A \longrightarrow B$ which extends α (ie, such that $\beta(x) = \alpha(x)$ for $x \in X$), in this case we say that A has the universal property of the applications for K on X. The size of the generating set determines the free algebra in the following sense.

Theorem 1.3.1. *(See Burris and Sankappanavar, 1981, Theorem 10.7) Let A_1 and A_2 two algebras in a class K free over X_1 and X_2 respectively. If $|X_1| = |X_2|$, then $A_1 \cong A_2$.*

Thanks to the previous theorem, for every cardinal λ, the free algebra on λ elements is unique up to isomorphism and will be denoted by $F(\lambda)$. We say also that $F(\lambda)$ is the free algebra of K over λ generators. In each variety there is a $F(\lambda)$ for every cardinal λ.

Equations Let us fix a variety Θ and a finite set $X = \{x_1, \dots, x_n\}$ and consider equations of the form $w = w'$, $w, w' \in F(X)$. Every such equation is considered also as a formula in the logic in the variety. In the later case we write $w \equiv w'$. A homomorphism $\mu : F(X) \to A$ is a root of the equation

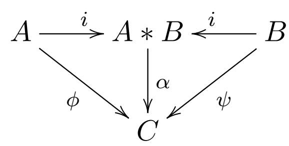
$w(x_1, \ldots, x_n) = w'(x_1, \ldots, x_n)$, if $w(\mu(x_1), \ldots, \mu(x_n)) = w'(\mu(x_1), \ldots, \mu(x_n))$. This also means that the pair (w, w') belong to $Ker\mu$. We will identify the pair (w, w') and the equation $w = w'$. In order to get a reasonable geometry in Θ we have to consider the equations with constants.

Algebras with a Fixed Algebra of Constants

Definition 1.3.1. *Let Θ be a variety , H be a fixed algebra in Θ. Consider a new variety, denoted by Θ_H. The language of Θ_H is the language of Θ plus a constant c_h for every $h \in H$. The axioms are the axioms of Θ plus all equations $c_{f(h_1,\ldots,h_n)} = f(c_{h_1}, \ldots, c_{h_N})$. Θ_H can be also viewed as a category. The objects have the form (G, g), where $g : H \to G$ is a homomorphism in Θ, not necessarily injective. We will say that (G, g) is faithful if g is injective, roughly speaking G is a faithful H-algebra if it contains a designated copy of H, which we can identify with H. Let us consider $g : H \to G$ and $g' : H \to G'$, then $\mu : G \to G'$ is a morphisms in Θ_H iff μ is a homomorphism of Θ and $g' = \mu g$.*

Let us define the free product $A * B$, where A and B are objects in a variety Θ, as follows:

1. $A * B$ is generated by $A \cup B$;
2. let $\phi : A \to C$ and $\psi : B \to C$ be morphisms, then there exists a unique morphism $\alpha : A * B \to C$ such that this is a commutative diagram:



A free algebra $F = F(X)$ in Θ_H has the form $H * F_0(X)$, where $F_0(X)$ is the free algebra in Θ over X, $*$ is the free product in Θ and the embedding $i_H : H \to F(X) = H * F_0(X)$ follows from the definition of free product. A H-algebra (G, g) is called a faithful H-algebra if $g : H \to G$ is an injection. The free algebra (F, i_H) is faithful. A H-algebra H with the identical $H \to H$ is also faithful and all other H-algebras H are isomorphic to this one. All of them are simple, i.e., they do not have faithful subalgebras and congruences. Let (G, g) be a H-algebra, and $\mu : G \to G'$ is a homomorphism in Θ, then, by $g' = \mu g$, G' becomes a H-algebra, and μ is a homomorphism of H-algebras. We say that T congruence of G is faithful if the H-algebra G/T is faithful. Let us consider (G, g), (G', g') and the homomorphism $\mu : G \to G'$; if (G', g') is a faithful H-algebra, then (G, g) is a faithful H-algebra. Moreover if $T = Ker\mu$, then T is a faithful congruence and G/T is also faithful.

The Variety of ℓ-groups An ℓ-group is a structure $(G, +, -, 0, \leq)$ such that $(G, +, -, 0)$ is an abelian group, $(G, \leq)$ is a lattice ordered set and $\forall a, b, c \in G$ we have $a \leq b \Rightarrow a + c \leq b + c$ (compatibility property). G is a *totally ordered* group when $(G, \leq)$ is a totally ordered set, i.e. a chain; and we say that G is divisible if for every $n \in \mathbb{N}$ and for every g in G there exists x such that $nx = g$. Equivalently we can consider the structure $(G, +, -, 0, \wedge, \vee)$, where $x \leq y \Leftrightarrow x \wedge y = x$. Note that ℓ-groups form a variety in the sense of universal algebra, in fact it is possible to express them via the following axioms:

1. $\forall a, b, c \in G \; a + (b + c) = (a + b) + c$;
2. $\forall a \in G \; a + 0 = a = 0 + a$;
3. $\forall a \in G \; a + (-a) = 0 = -a + a$;
4. $\forall a, b \in G \; a + b = b + a$;
5. $\forall a, b \in G \; a \wedge b = b \wedge a$;
6. $\forall a, b \in G \; a \vee b = b \vee a$;
7. $\forall a, b, c \in G \; a \vee (b \vee c) = (a \vee b) \vee c$;
8. $\forall a, b, c \in G \; a \wedge (b \wedge c) = (a \wedge b) \wedge c$;
9. $\forall a, b \in G \; a \vee (a \wedge b) = a$;
10. $\forall a, b \in G \; a \wedge (a \vee b) = a$;
11. $\forall a, b, c \in G \; c + (a \wedge b) = (c + a) \wedge (c + b)$;
12. $\forall a, b, c \in G \; c + (a \vee b) = (c + a) \vee (c + b)$.

We denote it with $\ell\mathcal{GR}$ ($\ell\mathcal{GR}_H$ if we fix an ℓ-group H of constants) and $FA\ell_0(n)$ the free ℓ-group over n generators ($FA\ell_H(n)$ if we fix an ℓ-group H of constants).

We assume also the following notation: $|a| = a \vee (-a)$ (absolute value). An ℓ-ideal of an ℓ-group is a subgroup J of G such that if $x \in J$ and $|y| \leq |x|$ then $y \in J$. We will denote by $< a >$ the ℓ-ideal generated by a. In the variety of ℓ-groups congruences are identified with ℓ-ideals. We say u strong unit of G ℓ-group if and only if $0 \leq u \in G$ and $\forall x \in G$ there is an integer n such that $x \leq nu$. We say that u_G is an *order unit* of G iff for every $x \in G$ there is a positive integer n with $|x| \leq nu_G$. We denote by (G, u_G) and (R, u_R) an abelian ℓ-group and a vector lattice (or Riesz space) with order units u_G and u_R, respectively. A partially ordered abelian group G is said to be *archimedean* iff for every $x, y \in G$ with $nx \leq y$ for every $n \in \mathbb{N}$ we have $x \leq 0$. A partially ordered abelian group G is *unperforated* iff for every $n \in \mathbb{N}$ and $x \in G$ with $nx \geq 0$ we get $x \geq 0$ (see also Goodearl, 2010, Definitions, pp. 19-20). A subgroup (resp. subspace) M of G (resp. R) is said to be *cofinal* iff for every $x \in G$ (resp. R) there is $z \in M$ with $z \geq x$. If $x_0 \in G \setminus M$, then span$(M \cup \{x_0\})$ denotes the subgroup of G generated by M and x_0, namely span$(M \cup \{x_0\}) := \{z + nx_0 : z \in M, n \in \mathbb{Z}\}$. Analogously, given $x_0 \in R \setminus M$, we denote by span$(M \cup \{x_0\})$ the subspace of R generated by M and x_0, that is span$(M \cup \{x_0\}) := \{z + \alpha x_0 : z \in M, \alpha \in \mathbb{R}\}$.

If we define the quotient group G/J, with J ℓ-ideal, the operations $a/J \vee b/J = (a \vee b)/J$ and $a/J \wedge b/J = (a \wedge b)/J$ set $a/J = a + J$, lateral of a, then G/J is an ℓ-group. Moreover, if we consider the canonical projection $\rho_J : G \to G/J$ which associates to each element its lateral, we can see that $ker(\rho_J) = J$. An ℓ-ideal J is called prime if and only if J is proper and the ℓ-group G/J is totally ordered. Let G and H be ℓ-groups, $f : G \longrightarrow H$ is a homomorphism of ℓ-groups ($f \in Hom(G, H)$) if and only if f is a homomorphism of groups and of lattices. If G and H are ℓ-groups and $f : G \to H$ is an homomorphism then $ker(f) = f^{-1}(0)$ is an ℓ-ideal of G and $G/ker(f)$ is isomorphic to an ℓ-subgroup of H. A function $\mu : (G, u_G) \to (R, u_R)$ is an *ℓ_u-homomorphism* iff it is a monotone homomorphism of groups such that $\mu(u_G) = u_R$ (here and in the sequel, we refer to the reduct abelian lattice group of R). We denote by $Hom(G, R)$ (resp. $\ell_u Hom(G, R)$) the set of all monotone group homomorphisms (resp. ℓ_u-homomorphisms) between G and R, and by $S(G, u_G)$ the space of all *states* between G and $\mathbb{R}$ (see also Goodearl, 2010). Note that, when $R = \mathbb{R}$, then $\ell_u Hom(G, \mathbb{R}) = S(G, u_G)$. If $\mathcal{K} \subset Hom(G, R)$, then we say that $\mu \in \mathcal{K}$ is *extremal* iff, whenever $\mu_1, \mu_2 \in \mathcal{K}$ and $\mu = \alpha\mu_1 + (1 - \alpha)\mu_2$ with $\alpha \in [0, 1]$, we get $\mu = \mu_1$ or $\mu = \mu_2$. The set of all extremal elements of $\mathcal{K}$ is denoted by $Ext(\mathcal{K})$.

If X is a real vector space, then a *convex combination* of elements $x_1, \ldots, x_n$ of X is a linear combination of the form $\sum_{i=1}^{n} \alpha_i x_i$, where $\sum_{i=1}^{n} \alpha_i = 1$ and $\alpha_i \geq 0$ for each $i = 1, \ldots, n$. If X_1 and X_2 are real vector spaces and C_i is a convex subset of X_i, $i = 1, 2$, then a function $f : X_1 \to X_2$ is said to be *affine* iff f preserves convex combinations. We denote by $Aff_Y(X)$ the space of all affine functions from X to Y.

Chapter 2

The Logic *Bal*(H)

We start from ***Bal***, defined in Galli, Lewin, and Sagastume, 2004; this logic, associated with ℓ-Groups, describes the balance of opposing forces, i.e. theorems could be interpreted as balanced states, and models some features of arguments in which conflicting pieces of evidence are confronted, e.g. police investigations, political influences, etc. *Equilibrium* is the only one distinguished truth value, which will be interpreted as the zero af an ℓ-group.

Then we introduce ***Bal***(H), where H is a fixed ℓ-group of constants. Let us consider a set of propositional variables and the language $\boldsymbol{L}_{\boldsymbol{Bal}(H)} = \{\rightarrow, ^+, \{c_h\}_{h\in H}\}$. As usual the terms of our logic are defined inductively as follows: propositional variables and constants are terms, if ϕ and ψ are terms then $\phi \rightarrow \psi$ and ϕ^+ are terms. Axioms and rules are the following.

Axioms

BAL1 $(\phi \rightarrow \psi) \rightarrow ((\theta \rightarrow \phi) \rightarrow (\theta \rightarrow \psi))$

BAL2 $(\phi \rightarrow (\psi \rightarrow \theta)) \rightarrow (\psi \rightarrow (\phi \rightarrow \theta))$

BAL3 $((\phi \rightarrow \psi) \rightarrow \psi) \rightarrow \phi$

BAL4 $\phi^{++} \rightarrow \phi^+$

BAL5 $((\phi \rightarrow \psi)^+ \rightarrow (\psi \rightarrow \phi)^+) \rightarrow (\psi \rightarrow \phi)$

C1 $c_{a-b} \rightarrow (c_b \rightarrow c_a)$

C2 $c_{a\vee b} \rightarrow (c_b \rightarrow c_a)^+ \oplus c_b$

where $x \oplus y := (x \rightarrow (x \rightarrow x)) \rightarrow y$.

Rules

$$\frac{\phi, \phi \rightarrow \psi}{\psi} \quad (MP)$$

$$\frac{\phi, \psi}{\phi \rightarrow \psi} \quad (G)$$

$$\frac{\phi}{\phi^+} \quad (PI)$$

$$\frac{(\phi \rightarrow \psi)^+}{(\phi^+ \rightarrow \psi^+)^+} \quad (MI)$$

Let us consider $G \in \ell\mathcal{GR}^H$, if we consider a map v' from the propositional variables of ***Bal***(H) to G, we can consider the H-valuation v recursively defined as follows:

- $v(x) = v'(x)$ for all x variable
- $v(c_h) = h$ for all $h \in H$
- $v(\phi \to \psi) = v(\psi) - v(\phi)$
- $v(\phi^+) = max\{\phi, 0\}$

We say that v satisfies ϕ iff $v(\phi) = 0$.

2.1 Polynomial Completeness and Completeness Theorem

We want to prove the completeness of our new logic (for more details on the definition of completeness in logic you can see Burris and Sankappanavar, 1981). For this reason we investigate the role of the introduced constants in the Lindenbaum algebras (which are exactly, up to isomorphism, $FA\ell_H(\aleph_0)$) and the impact on the logic side.

Until now, we did not stress the deep difference between ℓ-polynomials and the associated functions, some intuitions can came out from many observations, but in this section we focus on the formalism behind these ideas and we try to express the properties which an ℓ-group G have to have to be *polynomially complete*. An analogous definition is presented in Belluce, Di Nola, and Lenzi, 2014 in the field of MV-algebras.

Definition 2.1.1. *An ℓ-group G is polynomially complete w.r.t. H ($PC(H)$) iff for every n, if we consider $p(\bar{x}; \bar{h}) \in FA\ell_H(n)$ such that $\forall \bar{g} \in G^n$ $p(\bar{g}; \bar{h}) = 0$ then $p(\bar{x}; \bar{h})$ is the zero polynomial, where $\bar{h} = (h_1, \ldots, h_m)$ represents the constants of p in H. We will say that G is polynomially complete (for short PC) iff G is $PC(H)$ for every $H \leq G$.*

Theorem 2.1.1. *The following are equivalent:*

1. *G is $PC(H)$;*
2. *if $p, q \in FA\ell_H(n)$ induce the same function over G then $p = q$;*
3. *if $p, q \in FA\ell_H(n)$ induce the same function over G then they induce the same function in every extension of G.*

Proof. $(1 \Leftrightarrow 2)$ It follows by the fact that in the variety of ℓ-groups every equality $p = q$ can be write $p - q = 0$.

$(3 \Rightarrow 2)$ We have that $FA\ell_H(n) \leq FA\ell_G(n)$ and $FA\ell_G(n)$ is an extension of G.

$(2 \Rightarrow 3)$ Trivial.

□

To sum up we have that G is $PC(H)$ when G is *big enough* to separate polynomials in $FA\ell_H$.

Proposition 2.1.1. *Let us consider $\{G_i\}_{i \in I}$ finite family of ℓ-groups such that they are $PC(H)$. The cartesian product $G = \prod_{i \in I} G_i$ is $PC(H^{|I|})$.*

Proof. Let p be an ℓ-polynomial such that $Z_G(p) = G$. This means that for every $j \in I$ and $g = (g^{(i)})_{i \in I} \in G$ $p_j(g) = 0_{G_j}$, i.e. $p(g^{(j)}) = 0_{G_j}$; but for each j G_j is PC(H), then $p = 0$. □

Proposition 2.1.2. *If G is $PC(H_1)$ then for each $G' \geq G$ and $H_2 \leq H_1$ we have that G' is $PC(H_1)$ and G is $PC(H_2)$.*

Proof. It is straightforward by definition. □

Corollary 2.1.1. *G is $PC(G)$ iff G is PC.*

Proposition 2.1.3. *$\mathbb{Z}$ is not PC, but it is $PC(\{0\})$.*

Proof. In Section 4.2.3(the case with constants) there are presented non-zero ℓ-polynomials sf_n which induce the zero function over $\mathbb{Z}$, i.e. $\mathbb{Z}$ is not PC.

On the other hand when we consider $FA\ell_0(n)$ and the direction of the closed cones (which are zero sets) generated by $\mathbb{Z}^n$ are dense in the space of the directions in $\mathbb{R}^n$, i.e. the only ℓ-polynomial that induce the zero function is the zero polynomial. □

In the next proposition we prove that real numbers are polynomially complete, i.e the concepts of ℓ-polynomial with constants in $\mathbb{R}$ and of the induced function coincide over $\mathbb{R}$. As said already in Section 4.1, we have focused on $\mathbb{R}$ by the fact that $\mathbb{R}$ is more suitable than $\mathbb{Z}$ in the study of algebraic, geometrical and logical properties, also for non-homogeneous ℓ-polynomial, i.e. in a logic with constants.

Proposition 2.1.4. *$\mathbb{R}$ is PC.*

Proof. Let us consider an ℓ-polynomial $p \equiv p(\bar{x}, \bar{c})$ where $\bar{c} \in \mathbb{R}^m$. Let us suppose $p(\bar{x}, \bar{c}) = 0$ for each $\bar{x} \in \mathbb{R}^n$, we want to prove that p is the zero polynomial. Let us consider $\chi_i = \pi_i \circ \psi \circ \phi : \mathbb{R} \to \mathbb{R}^*$, where

- $\phi : \mathbb{R} \to FA\ell_{\mathbb{R}}(n)$ is the natural embedding associating each element of $\mathbb{R}$ to the constant polynomial;
- $\psi : FA\ell_{\mathbb{R}}(n) \to (\mathbb{R}^*)^I$ is the embedding provided in Labuschagne and Van Alten, 2007, Lemma 2.4 and $\mathbb{R}^*$ is an ultrapower of $\mathbb{R}$;
- $\pi_i : (\mathbb{R}^*)^I \to \mathbb{R}^*$ is the canonical projection for each $i \in I$.

In general χ_i are not injective. If χ_i are injective then they are a elementary embeddings, by the fact that $\mathbb{R}$ and $\mathbb{R}^*$ are divisible totally ordered ℓ-groups which are model complete.

By this we have that for each $i \in I$ $p(\bar{x}, \chi_i(\bar{c})) = 0$ for every $\bar{x} \in (\mathbb{R}^*)^n$, then $p(\bar{x}, \psi \circ \phi(\bar{c})) = 0$ in $((\mathbb{R}^*)^I)^n$. Since $FA\ell_{\mathbb{R}}(n)$ is embedded in $((\mathbb{R}^*)^I)^n$ $p(\bar{x}, \bar{c})$ is the zero polynomial.

The other possibility is that for some $i \in I$ $\chi_i(\mathbb{R}) = \{0\}$, but we have the following chain of implications:

$$p(\bar{x}, \bar{c}) = 0 \Rightarrow \frac{1}{n} p(\bar{x}, \bar{c}) = 0 \Rightarrow p(\bar{x}, \frac{1}{n}\bar{c}) = 0.$$

Considering the limit $n \to +\infty$ we have that $p(\bar{x}, 0) = 0$. So replying the construction above we have the result. □

Corollary 2.1.2. *Every divisible totally ordered archimedean ℓ-group G is PC.*

Proof. The proof is analogous to Proposition 2.1.4. □

Proposition 2.1.5. *$\mathbb{R}^*$ ultrapower of $\mathbb{R}$ is PC.*

Proof. Let $\mathbb{R}^* = \mathbb{R}^I/U$ be an ultrapower of $\mathbb{R}$, where U is an ultrafilter on the set I. Let us consider an ℓ-polynomial $p \equiv p(\bar{x}, \bar{c})$ where p is a non-zero polynomial and $\bar{c} \in (\mathbb{R}^*)^m$, i.e. $c = (\bar{c}_i)_{i\in I}/U$.. So there exists $\mathbb{R}^* \subseteq G$ such that for some $\bar{g} \in g^n$ $p(\bar{g}, \bar{c}) \neq 0$.

Let $G^* = G^I/U$, and let ϕ be the canonical embedding of G in G^*. From $p(\bar{g}, (\bar{c}_i)_{i\in I}) \neq 0$, since ϕ is an elementary embedding we have $p(\phi(\bar{g}), (\bar{c}_i)_{i\in I}) \neq 0$. By Łoś Theorem we have that $p(\bar{g}, \bar{c_i}) \neq 0$ for each i in some J where $J \subseteq I$ and $J \in U$.

By Proposition 2.1.4 for every $i \in J$ there exists $\bar{k}_i \in \mathbb{R}^n$ such that $p(\bar{k}_i/U, \bar{c}_i/U) \neq 0$. Now it is enough to note that $p((\bar{k}_i)_{i\in J}/U, (\bar{c}_i)_{i\in J}/U) \neq 0$, i.e. $(\bar{k}_i)_{i\in J}/U \in \mathbb{R}^*$ is not a root of the polynomial p, to have the result. □

Corollary 2.1.3. *Every ultrapower of PC ℓ-groups is PC.*

Proof. The proof is analogous to Proposition 2.1.5. □

Now we can state the following theorem of completeness.

Theorem 2.1.2. *[Completeness Theorem] If G is $PC(H)$ then*

$$\vdash_{\textbf{\textit{Bal}}(H)} \phi \quad \Leftrightarrow \quad \models_G \phi.$$

Proof. Let us consider the non trivial implication $\Leftarrow$. Let us suppose that $\models_G \phi$. This means, by definition, that for all $g \in G$ $v(\varphi)(g) = 0$. G is $PC(H)$ so $v(\varphi)$ is the zero polynomial, i.e. $\vdash_{\textbf{\textit{Bal}}(H)} \phi$. □

2.1.1 A Characterization of Totally Ordered PC ℓ-Groups

In analogy with Belluce, Di Nola, and Lenzi, 2014 it is also possible to characterize totally ordered PC ℓ-groups as follows.

Definition 2.1.2. *A totally ordered ℓ-group G is quasi-divisible if for every $a < b$ and for every positive integer N there is c such that $a < Nc < b$.*

Proposition 2.1.6. *For every totally ordered ℓ-group G the following are equivalent:*

1. *G is polynomially complete;*
2. *G is order dense in its divisible hull;*
3. *G is quasidivisible.*

Proof. ($1 \Rightarrow 3$) Let us suppose for absurd that G is PC but not quasidivisible. This means that there are $a < b \in G$ and N such that for every $g \in G$, $Ng \leq a$ or $b \leq Ng$. If we consider the polynomial $p(x, (a, b)) := |(Ng - a) \vee 0| \wedge |(b - Ng) \vee 0|$, then it is equal to 0 for every $x \in G$. By the fact that G', the divisible hull of G, is quasidivisible there exists $g' \in G'$ such that $p(g', (a, b)) \neq 0$, which is an absurd.

($2 \Rightarrow 1$) Let p be in $FA\ell_G(n)$ such that $p(\bar{g}') \neq 0$ for some $g' \in G'$, divisible hull of G. By the fact that G' and $\mathbb{R}$ are divisible totally ordered ℓ-groups, they enjoy the same first order properties; so there exist $I_1, \ldots, I_k$ nontrivial intervals of G' such that $p((x_1, \ldots, x_k)) \neq 0$ for each $(x_1, \ldots, x_k) \in I = I_1 \times \ldots I_k$. By order density, I contains a point $g \in G^k$ such that $p(g) \neq 0$. In this way we have that if a polynomial p induces zero in G then p induces zero in G', but by Corollary 2.1.2 we know that G' is PC then we can conclude that G is PC.

(2 ⇔ 3) See Belluce, Di Nola, and Lenzi, 2014, Proposition 6.8. □

Corollary 2.1.4. *Every totally ordered ℓ-group can be embedded in a PC totally ordered ℓ-group.*

Corollary 2.1.5. *$\mathbb{Q}$ is PC.*

2.2 A Wójcicki-type Theorem

In Cignoli, d'Ottaviano, and Mundici, 2013; Marra and Spada, 2012; Mundici, 2011 some results, known as Wójcicki's Theorem, play a crucial role in the connection between syntax and semantics. Here we propose an analogous result in our framework.

Definition 2.2.1. *We say that f ℓ-polynomial is CNB (completely not bounded) iff*

$$\forall g \in G^+ \, \exists (g_1, \ldots, g_n) \, : \, (k_1, \ldots, k_n) > (g_1, \ldots, g_n) \to f(k_1, \ldots, k_n) > g.$$

Theorem 2.2.1. *Let G be an archimedean totally ordered $PC(H)$ ℓ-group, where $H \leq G$. Let f and g in $FA\ell_H(n)$. If g is a CNB ℓ-polynomial we have that:*

$$Z_G(f) \supseteq Z_G(g) \;\Leftrightarrow < f > \subseteq < g > .$$

Proof. ⇐ Trivial.

⇒ By Anderson and Feil, 2012, Theorem 2.3 we have that $G \lesssim \mathbb{R}$, so f and g can be seen as piecewise linear functions from $\mathbb{R}^n$ to $\mathbb{R}$; then we can consider $\{P_i\}_{i \in I}$ standard simplicial subdivision of the domain such that both f and g are linear on P_i, for every $i \in I$. Let C be the hyper cube $[-M, M]^n$, such that every vertex of $\{P_i\}_{i \in I}$ is in the interior of C. Adapting Cignoli, d'Ottaviano, and Mundici, 2013, Lemma 3.4.8 we have that $|f| \leq m|g|$ on C and, through the fact that g is CNB, we have that $|f| \leq m|g|$ on G^n; but G is $PC(H)$, so $|f| \leq m|g|$ in $FA\ell_H(n)$. □

Corollary 2.2.1. *Let f and g be in $FA\ell_0(n)$. We have that:*

$$Z_{\mathbb{Z}}(f) \supseteq Z_{\mathbb{Z}}(g) \;\Leftrightarrow < f > \subseteq < g > .$$

Corollary 2.2.2. *Let f and g be in $FA\ell_{\mathbb{R}}(n)$. We have that:*

$$Z_{\mathbb{R}}(f) \supseteq Z_{\mathbb{R}}(g) \;\Leftrightarrow < f > \subseteq < g > .$$

2.2.1 Examples

Let us consider $G = H = \mathbb{Z}$ and the functions $f = sf_0 \vee (x-2) \vee (-x)$ and $g = 0 \vee (x-1) \vee (-x)$, where sf_0 is defined in Section 4.2.3. We already observed that $\mathbb{Z}$ is not $PC(\mathbb{Z})$ (see Proposition 2.1.3). It is easy to see that g is CNB, $\mathbb{Z}$ is archimedean and $Z_G(g) \subseteq Z_G(f)$, but there is no natural number m such that $|f| \leq m|g|$.

Let us consider $G = H = \mathbb{R}$ and the functions $f = 0 \vee (x-2)$ and $g = (x \vee 0) \wedge 1$. In this case $\mathbb{R}$ is PC, but g is not CNB so, as before, there is no natural number m such that $|f| \leq m|g|$.

Chapter 3

Functional Representations of ℓ-Groups

3.1 Preliminary Results

Given any two subgroups M, Z of G with $M \subset Z$ and any monotone homomorphism $\mu_0 : M \to R$, set $E(\mu_0, Z) := \{\nu : Z \to R$, ν is a monotone homomorphism, $\nu_{|M} = \mu_0\}$, and let us denote by $Ext(E(\mu_0, Z))$ the set of all extremal elements of $E(\mu_0, Z)$ (see also Lipecki, 1979).

We now prove the following

Theorem 3.1.1. *The set $Ext(\ell_u Hom(G, R))$ is nonempty.*

In order to demonstrate Theorem 3.1.1, we first prove the following two lemmas.

Lemma 3.1.1. (see also Lipecki, 1979, Lemma 1) *Let M be a cofinal subgroup of G, $\mu_0 : M \to R$ be a monotone homomorphism and $x_0 \in G \setminus M$. Then $Ext(E(\mu_0, span(M \cup \{x_0\}))) \neq \emptyset$.*

Proof. Let $T_e(x_0) := \bigwedge\Big\{\dfrac{1}{n}\,\mu_0(z) : z \in M, n \in \mathbb{N}, nx_0 \le z\Big\}$. For each $z \in M$ and $n \in \mathbb{N}$ set $\nu(z+nx_0) := \mu_0(z)+nT_e(x_0)$. In Lipecki, 1985 it is shown that $\nu \in E(\mu_0, \text{span}(M \cup \{x_0\}))$. Moreover observe that, if $\lambda \in E(\mu_0, \text{span}(M \cup \{x_0\}))$, then for every $z \in M$ and $n \in \mathbb{N}$ with $nx_0 \le z$ we get $n\lambda(x_0) = \lambda(n\,x_0) \le \lambda(z) = \mu_0(z)$, so that $\lambda(x_0) \le \dfrac{1}{n}\,\mu_0(z)$. By arbitrariness of z and n, we obtain

$$\lambda(x_0) \le T_e(x_0). \tag{3.1}$$

Now suppose that $\nu = \alpha\nu' + (1-\alpha)\nu''$, where ν', $\nu'' \in E(\mu_0, \text{span}(M \cup \{x_0\})$ and $\alpha \in [0, 1]$. Taking into account (3.1), we get

$$\begin{aligned} \nu(x_0) &= \alpha\,\nu'(x_0) + (1-\alpha)\nu''(x_0) \le \\ &\le \alpha\,T_e(x_0) + (1-\alpha)T_e(x_0) = T_e(x_0) = \nu(x_0), \end{aligned} \tag{3.2}$$

and thus all inequalities in (3.2) are equalities. Furthermore, taking $\alpha = 1$ and $\alpha = 0$ in (3.2), we get $\nu'(x_0) = \nu(x_0)$ and $\nu''(x_0) = \nu(x_0)$, respectively and hence, by construction, $\nu'(t) = \nu''(t) = \nu(t)$ for each $t \in M \cup \{x_0\}$. This concludes the proof. □

Lemma 3.1.2. (see also Lipecki, 1979, Lemma 2) *Let $\mu_0 : M \to R$ be a monotone homomorphism and Z, Z_1 be two subgroups of G with $M \subset Z \subset Z_1$. If $\nu \in Ext(E(\mu_0, Z))$ and $\nu_1 \in Ext(E(\nu, Z_1))$, then $\nu_1 \in Ext(E(\mu_0, Z_1))$.*

Proof. Let $\nu_1 = \alpha\nu' + (1-\alpha)\nu''$, with $\nu', \nu'' \in E(\mu_0, Z_1)$ and $\alpha \in [0,1]$. We get $\nu'_{|Z}, \nu''_{|Z} \in E(\mu_0, Z)$, and thus $\nu'_{|Z} = \nu''_{|Z} = \nu$, since $\nu \in Ext(E(\mu_0, Z))$. So ν', $\nu'' \in E(\nu, Z_1)$, and therefore, as $\nu_1 \in Ext(E(\nu, Z_1))$, we obtain $\nu' = \nu'' = \nu_1$. This ends the proof. □

Proof of Theorem 3.1.1. Let u_G be an order unit of G and $M = \mathbb{Z}u_G := \{nu_G : n \in \mathbb{Z}\}$. Set $\mu_0(nu_G) = nu_R$ for every $n \in \mathbb{Z}$. Let $\mathcal{M}$ be the family of all pairs (Z, ν), where Z is a subgroup of G, $M \subset Z$ and $\nu \in Ext(E(\mu_0, Z))$. We say that $(Z_1, \nu_1) \le (Z_2, \nu_2)$ if and only if $Z_1 \subset Z_2$ and $\nu_2 \in E(\nu_1, Z_2)$. By construction, $(\mathcal{M}, \le)$ is a nonempty partially ordered class. We claim that $\mathcal{M}$ is inductive. If $\{(Z_\iota, \nu_\iota) \colon \iota \in \Lambda\}$ is a chain in $\mathcal{M}$, then set $Z_0 := \bigcup_{\iota \in \Lambda} Z_\iota$ and $\nu_0(t) = \nu_\iota(t)$ if $t \in Z_\iota$. It is not difficult to check that ν_0 is well-defined, $(Z_0, \nu_0) \in \mathcal{M}$ and $(Z_\iota, \nu_\iota) \le (Z_0, \nu_0)$ for every $\iota \in \Lambda$. By virtue of the Zorn Lemma, $\mathcal{M}$ has a maximal element of the type (Z, ν). We claim that $Z = G$. Indeed, if $x_0 \in G \setminus Z$, then, arguing analogously as in Lemma 3.1.1, there should be an element of $\mathcal{M}$ defined on span$(Z \cup \{x_0\})$, getting a contradiction with maximality. This concludes the proof. □

The next result will be useful in the sequel.

Theorem 3.1.2. (see Fuchssteiner and Lusky, 1981, Theorem 1.3.3) *Let G be a partially ordered abelian group, R be a Dedekind complete vector lattice, $p : G \to R$ be a monotone and subadditive function, with $p(nx) = np(x)$ for every $x \in G$ and $n \in \mathbb{N} \cup \{0\}$. Set $\mathcal{K} := \{\mu : G \to R$: μ is a monotone homomorphism and $\mu(t) \le p(t)$ for every $t \in G\}$. Then for every $x \in G$ we get $p(x) = \max_{\mu \in \mathcal{K}} \mu(x)$.*

3.1.1 Some Properties of Extremal States

We now prove the following Krein-Mil'man-type theorem, which extends Theorem 3.1.2 to extremal vector lattice-valued homomorphisms (see also Kusraev and Kutateladze, 1984, Theorem 1.4.3, Kusraev and Kutateladze, 2012, Theorem 2.2.2, Lipecki, 1982, Theorem 5).

Theorem 3.1.3. *Let G, R, p, $\mathcal{K}$ be as in Theorem 3.1.2. Then for each $x \in G$ we get $p(x) = \max_{\mu \in Ext(\mathcal{K})} \mu(x)$.*

Proof. Fix arbitrarily $x \in G$, and let $M = \mathbb{Z}x := \{nx : n \in \mathbb{Z}\}$. For every $n \in \mathbb{Z}$ set $\mu_0(nx) := np(x)$. It is not difficult to see that μ_0 is monotone, additive and $\mu_0(t) \le p(t)$ for each $t \in M$.

Choose arbitrarily $x_0 \in G \setminus M$, and set

$$\begin{aligned} \beta_e(x_0) &:= \bigwedge\Big\{\frac{p(z+nx_0)-\mu_0(z)}{n} : z \in M,\, n \in \mathbb{N}\Big\}, \\ \beta_i(x_0) &:= \bigvee\Big\{\frac{\mu_0(z)-p(z-nx_0)}{n} : z \in M,\, n \in \mathbb{N}\Big\}. \end{aligned} \tag{3.3}$$

We claim that $\beta_i(x_0) \le \beta_e(x_0)$. Indeed, since $\mu_0 \le p$ on M and thanks to subadditivity of p, for every $n, n' \in \mathbb{N}$ and $z, z' \in Z$ we get

$$\begin{aligned} &\tfrac{\mu_0(z)}{n} + \tfrac{\mu_0(z')}{n'} = \tfrac{n\mu_0(z')+n'\mu_0(z)}{n\,n'} = \tfrac{\mu_0(nz'+n'z)}{n\,n'} \le \\ \le\; &\tfrac{p(nz'+n'z)}{n\,n'} = \tfrac{p(nz'+n\,n'x_0+n'z-n\,n'x_0)}{n\,n'} \le \\ \le\; &\tfrac{n'p(z+nx_0)+np(z'-n'x_0)}{n\,n'} = \tfrac{p(z+nx_0)}{n} + \tfrac{p(z'-n'x_0)}{n'}, \end{aligned}$$

and hence

$$\frac{\mu_0(z') - p(z' - n'x_0)}{n'} \le \frac{p(z + nx_0) - \mu_0(z)}{n}. \tag{3.4}$$

Taking in (3.4) the infimum with respect to z and n and the supremum with respect to z' and n', we get $\beta_i(x_0) \le \beta_e(x_0)$, that is the claim.

Let now $a \in R$ with $\beta_i(x_0) \le a \le \beta_e(x_0)$, and for every $z \in M$ and $n \in \mathbb{N}$ put $\nu(z + nx_0) := \mu_0(z) + n\,a$. Observe that ν is well-defined. indeed, if $z_1 + n_1x_0 = z_2 + n_2x_0$, then $z_1 - z_2 = (n_2 - n_1)x_0$, and this it possible if and only if $z_1 = z_2$ and $n_1 = n_2$. It is easy to check that ν is additive. We now prove that $\mu_0(z) + n\,a \ge 0$ (resp. ≤ 0) whenever $z \in M$, $n \in \mathbb{Z}$ and $z + nx_0 \ge 0$ (resp. ≤ 0). If $n = 0$, this is an immediate consequence of positivity of μ_0. Now consider the case $n > 0$. If $z + nx_0 \ge 0$, then, as p is monotone, we get $\dfrac{-p(-z - nx_0)}{n} \ge 0$, and hence

$$a \ge \beta_i(x_0) \ge \frac{\mu_0(-z) - p(-z - nx_0)}{n} \ge \frac{\mu_0(-z)}{n} = \frac{-\mu_0(z)}{n},$$

from which we obtain $\mu_0(z) + n\,a \ge 0$. If $z + nx_0 \le 0$, then $p(z + nx_0) \le 0$, and so

$$a \le \beta_e(x_0) \le \frac{p(z + nx_0) - \mu_0(z)}{n} \le \frac{-\mu_0(z)}{n}.$$

Thus we get $\mu_0(z) + n\,a \le 0$. If $n < 0$, then $z + nx_0 \ge 0$ if and only if $-z - nx_0 \le 0$ and thus, taking into account the previous step we get

$$0 \ge \mu_0(-z) - n\,a = -\mu_0(z) - n\,a,$$

namely $\mu_0(z) + n\,a \ge 0$. Analogously it is possible to check that, if $n < 0$ and $z + nx_0 \le 0$, then $\mu_0(z) + n\,a \le 0$. Thus, ν is positive.

Now observe that, if $\lambda \in E(\mu_0, \text{span}(M \cup \{x_0\}))$ and $\lambda(t) \le p(t)$ for every $t \in \text{span}(M \cup \{x_0\})$, then for each $z \in M$ and $n \in \mathbb{N}$ we get

$$\mu_0(z) + n\lambda(x_0) = \lambda(z + nx_0) \le p(z + nx_0),$$

whence $\lambda(x_0) \le \dfrac{p(z + nx_0) - \mu_0(z)}{n}$. Taking the infimum with respect to z and n, we get $\lambda(x_0) \le \beta_e(x_0)$. Moreover, for every $z \in M$ and $n \in \mathbb{N}$ we have

$$-\mu_0(z) + n\lambda(x_0) = -\lambda(z - nx_0) \ge -p(z - nx_0),$$

and thus $\lambda(x_0) \le \dfrac{\mu_0(z) - p(z - nx_0)}{n}$. Passing to the supremum, we obtain $\lambda(x_0) \ge \beta_i(x_0)$ (see also Boccuto and Candeloro, 1994).

Now, proceeding analogously as in Lemmas 3.1.1, 3.1.2 and Theorem 3.1.1, set

$$E'(\nu, Z) := \{\nu \in R^Z,\ \nu \text{ is a monotone homomorphism},\ \nu_{|M} = \mu_0,\ \nu \le p \text{ on } G\},$$

let $Ext(E'(\nu, Z))$ be the set of all extremal elements of $E'(\nu, Z)$, and take $\beta_e(x_0)$ instead of $T_e(x_0)$. Taking into account that $\lambda(x_0) \le \beta_e(x_0)$, let us consider the class $\mathcal{M}'$ of all pairs of the type (Z, ν), where Z is a subgroup of G, $M \subset Z$ and $\nu \in Ext(E'(\mu_0, Z))$. Arguing analogously as in the proof of Theorem 3.1.1, it is possible to check that $\mathcal{M}'$ is inductive, and so, by virtue of the Zorn Lemma, $\mathcal{M}'$ admits a maximal element, in which $Z = G$. Indeed, if $x_0 \in G \setminus Z$, proceeding similarly as in Lemma 3.1.1, it would be possible to construct an element of $\mathcal{M}'$ defined on span$(Z \cup \{x_0\})$, getting a contradiction with maximality. So, for every $x \in G$, $\mathcal{K}_x$ has at least an extremal element, where $\mathcal{K}_x$ is the set of all monotone homomorphisms $\mu : G \to R$ with $\mu(t) \le p(t)$ for each $t \in G$ and $\mu(x) = p(x)$. From this we obtain the assertion. □

We now recall the following

Proposition 3.1.1. (see Mundici, 2011, Proposition 10.3) *Let $A := \Gamma(G, u_G)$ be an MV-algebra with its associated unital ℓ-group (G, u_G). Then for every state s of (G, u_G) the restriction $s|_A$ of s to A is a state of A. The map $s \mapsto s|_A$ is an affine isomorphism of $S((G, u_G)) \subset \mathbb{R}^G$ onto $S(A) \subset [0, 1]^A$. Thus, the extremal states of (G, u_G) are in one-one correspondence with the extremal states of A.*

3.2 Vector Lattice-Valued States and ℓ-Groups

Here we prove our main theorems, extending Goodearl, 2010, Theorem 7.7 to the vector lattice setting. To this aim, we first give the following

Lemma 3.2.1. *Let G be an archimedean ℓ-group, R be a Dedekind complete vector lattice, with order units u_G and u_R, respectively. If $x \in G$ has the property that $\mu(x) = 0$ for each $\mu \in Ext(\ell_u Hom(G, R))$, then $x = 0$.*

Proof. For each $x \in G$, set $p(x) = \bigwedge\Big\{\frac{k}{l}u_R : k \in \mathbb{Z}, l \in \mathbb{N}, lx \le ku_G\Big\}$. It is not difficult to check that $p(0) = 0$, $p(u_G) = u_R$ and $p(-u_G) = -u_R$. Moreover, for each $x \in G$ and $n \in \mathbb{N}$ we have

$$\begin{aligned} p(nx) &= \bigwedge\Big\{\frac{nk}{nl}u_R : k \in \mathbb{Z}, l \in \mathbb{N}, nlx \le ku_G\Big\} = \\ &= \bigwedge\Big\{\frac{nk}{h}u_R : k \in \mathbb{Z}, h \in \mathbb{N}, hx \le ku_G\Big\} \\ &= n \cdot \bigwedge\Big\{\frac{k}{h}u_R : k \in \mathbb{Z}, h \in \mathbb{N}, hx \le ku_G\Big\} = n\, p(x). \end{aligned}$$

Furthermore, for every x_1, $x_2 \in G$ with $x_1 \le x_2$ we get

$$\begin{aligned} p(x_1) &\le \bigwedge\Big\{\frac{k}{l}u_R : k \in \mathbb{Z}, l \in \mathbb{N}, lx_1 \le ku_G\Big\} \le \\ &\le \bigwedge\Big\{\frac{k}{l}u_R : k \in \mathbb{Z}, l \in \mathbb{N}, lx_2 \le ku_G\Big\} = p(x_2), \end{aligned}$$

and hence p is monotone. Now we claim that p is subadditive. Fix arbitrarily $k_1, k_2 \in \mathbb{Z}$, $l_1, l_2 \in \mathbb{N}$, with $l_j x_j \le k_j u_G$, $j = 1, 2$. We have

$$\frac{k_1 l_2 + k_2 l_1}{l_1 l_2} u_R = \frac{k_1}{l_1} u_R + \frac{k_2}{l_2} u_R,$$

$$l_1 l_2 (x_1 + x_2) = l_1 l_2 x_1 + l_1 l_2 x_2 \le (k_1 l_2 + k_2 l_1) u_G.$$

Thus, we obtain

$$\begin{aligned} p(x_1+x_2) &= \bigwedge\Big\{\frac{k^*}{l^*}u_R : k^*\in\mathbb{Z}, l^*\in\mathbb{N}, l^*(x_1+x_2)\le k^*u_G\Big\} \le \\ &\le \frac{k_1 l_2 + k_2 l_1}{l_1 l_2}u_R = \frac{k_1}{l_1}u_R + \frac{k_2}{l_2}u_R. \end{aligned}$$

From this, by arbitrariness of k_j, l_j, $j = 1, 2$, it follows that $p(x_1 + x_2) \le p(x_1) + p(x_2)$. Thus p satisfies the hypotheses of Theorem 3.1.2. Let $\mathcal{K}$ be as in Theorem 3.1.2, then for each $x \in G$ we get

$$\begin{aligned} p(x) &= \max_{\mu\in Ext(\mathcal{K})} \mu(x), \\ p(-x) &= \max_{\mu\in Ext(\mathcal{K})} -\mu(x), \\ p(x)\vee p(-x) &= \max_{\mu\in Ext(\mathcal{K})} |\mu(x)|, \\ p(x)\vee p(-x) &= |p(x)|\vee|p(-x)|. \end{aligned} \tag{3.5}$$

Furthermore observe that, by construction, $p(x) = T_e(x)$ for all $x \in G$, where $T_e(x)$ is as in the proof of Lemma 3.1.1.

Put

$$r(x) = -p(-x) = \bigvee\Big\{\frac{h}{q}u_R : h\in\mathbb{Z}, q\in\mathbb{N}, hu_G\le qx\Big\}$$

and $v(x) = |p(x)| \vee |r(x)|$ for every $x \in G$. First of all note that, if $\mu(t) \le p(t)$ for each $t \in G$, then in particular $\mu(u_G) \le p(u_G) = u_R$, $\mu(-u_G) \le p(-u_G) = -u_R$, and hence $-\mu(u_G) = \mu(-u_G) \le u_R$. Thus, $\mu(u_G) = u_R$.

Conversely, if $\mu : G \to R$ is a monotone homomorphism with $\mu(u_G) = u_R$, then μ is an extension of the function μ_0 defined as in the proof of Theorem 3.1.1, and hence, proceeding analogously as in the proof of Theorem 3.1.3, it is possible to check that $\mu(t) \le T_e(t) = p(t)$ for every $t \in G$. Thus we get

$$\begin{aligned} 0 &= \max\{|\mu(x)| : \mu\in Ext(\ell_u Hom(G,R))\} = \\ &= \max\{|\mu(x)| : \mu\in Ext(\mathcal{K})\} = v(x)\ge 0. \end{aligned}$$

From this it follows that $v(x) = 0$, and hence $p(x) = r(x) = 0$. Set now

$$w(x) = \bigwedge\Big\{\frac{j}{n}u_R : j,n\in\mathbb{N}, -ju_G\le nx\le ju_G\Big\}.$$

Fix arbitrarily $\varepsilon > 0$. Then, by proceeding analogously as in Goodearl, 2010, Proposition 7.12, we find h, $k \in \mathbb{Z}$, l, $q \in \mathbb{N}$ with $hu_G \le qx$, $lx \le ku_G$,

$$-\varepsilon u_R = r(x) - \varepsilon u_R \le \frac{h}{q} \le r(x) = 0,$$

$$0 = p(x) \le \frac{k}{l} \le p(x) + \varepsilon u_R = \varepsilon u_R.$$

Set $j := |h|l \vee |k|q$. We get

$$0 \le w(x) \le \frac{j}{lq} = \frac{|h|}{q} \vee \frac{|k|}{l} \le \varepsilon u_R,$$

$$-ju_G \leq hlu_G \leq lqx \leq kqu_G \leq ju_G.$$

From this and arbitrariness of ε it follows that $w(x) = 0$. Finally, we prove that $x = 0$. To this aim, we first claim that the set $\mathcal{O}_1$ is infinite, where $\mathcal{O}_j = \{n \in \mathbb{N} : n|x| \leq ju_G\}$ for every $j \in \mathbb{N}$. Otherwise, let $n_0 = \max \mathcal{O}_1$. It is easy to check that, if $j \in \mathbb{N}$ and $q \in \mathcal{O}_1$, then $jq \in \mathcal{O}_j$. We now prove the converse implication. Pick $j, q \in \mathbb{N}$ with $jq \in \mathcal{O}_j$. Then $qj|x| \leq ju_G$, namely $j(u_G - q|x|) \geq 0$. Since G is an abelian ℓ-group, G is unperforated (see also Goodearl, 2010, Proposition 1.22), and hence $u_G - q|x| \geq 0$, that is $q \in \mathcal{O}_1$. Thus, $\max \mathcal{O}_j = n_0 j$, and then

$$\begin{aligned} w(x) &= \bigwedge\Big\{\frac{j}{n}u_R : j, n \in \mathbb{N}, n|x| \leq ju_G\Big\} = \\ &= \bigwedge\Big\{\frac{j}{n}u_R : j \in \mathbb{N}, n \in \mathcal{O}_j\Big\} = \frac{1}{n_0}u_R \neq 0, \end{aligned}$$

getting a contradiction. Thus $\mathcal{O}_1$ is infinite, namely there exist infinitely many positive integers t with $n_t \in \mathcal{O}_1$. We claim that $\mathcal{O}_1 = \mathbb{N}$. Indeed, for each $n \in \mathbb{N}$ there is $t_0 \in \mathbb{N}$ with $n \leq n_{t_0}$, and hence $n|x| \leq n_{t_0}|x| \leq u_G$. From this, since G is archimedean, by Goodearl, 2010, Proposition 1.23 it follows that $|x| = 0$, that is $x = 0$. This ends the proof. □

Now let R be a Dedekind complete vector lattice with order unit u_R, and set

$$\|x\|_{u_R} := \min\{\alpha \in \mathbb{R}^+ : |x| \leq \alpha u_R\}. \tag{3.6}$$

It is not difficult to see that the map $\|\cdot\|_{u_R}$ in (3.6) is well-defined and is a norm. In particular, note that the implication $[\,\|x\|_{u_R} = 0 \Longrightarrow x = 0]$ can be proved by arguing analogously as in the proof of the implication $[w(x) = 0 \Longrightarrow x = 0]$ in Lemma 3.2.1.

We consider the family

$$\mathcal{B} := \{B(\varepsilon, J) : \varepsilon > 0, J \text{ is a finite subset of } G\},$$

where

$$\begin{gathered} B(\varepsilon, J) = B(\varepsilon, \{x_1, x_2, \ldots, x_n\}) = \\ = \{f \in R^G : \|f(x_i)\|_{u_R} \leq \varepsilon,\, x_i \in J,\, i = 1, 2, \ldots, n\} = \\ = \{f \in R^G : |f(x_i)| \leq \varepsilon u_R,\, x_i \in J,\, i = 1, 2, \ldots, n\} \end{gathered}$$

for each ε and J. It is not difficult to see that $\mathcal{B}$ is a base of neighborhoods of 0. We equip R^G with the *product topology*, namely the topology τ generated by $\mathcal{B}$, and we endow $\ell_u Hom(G, R)$ with the topology induced by τ.

Let G be as in Lemma 3.2.1. The *evaluation map* is the application ψ which to every point $x \in G$ associates the function $\psi(x) : \ell_u Hom(G, R) \to R$, defined by

$$\psi(x)(\mu) = \mu(x), \quad \mu \in \ell_u Hom(G, R). \tag{3.7}$$

It is not difficult to check that ψ is affine and continuous on $\ell_u Hom(G, R)$. Thus, the evaluation map ψ in (3.7) can be viewed as a function $\psi : G \to Aff_R(\ell_u Hom(G, R))$.

For each $x \in G$ we consider the restriction of $\psi(x)$ to $\mathcal{C}_R(Ext(\ell_u Hom(G, R)))$, where the sets R and $Ext(\ell_u Hom(G, R))$ are endowed with the topology generated by the norm $\| \cdot \|_{u_R}$ and with the topology induced by τ, respectively. Hence, a function $\phi : G \to \mathcal{C}_R(Ext(\ell_u Hom(G, R)))$ is defined, by setting

$$\phi(x)(\mu) = \mu(x), \quad \mu \in Ext(\ell_u Hom(G, R)), \tag{3.8}$$

which we call again *evaluation map*.

Note that ψ and ϕ are positive homomorphisms, and that $\psi(u_G)$ ($\phi(u_G)$, respectively) is the constant function, which associates to every $\mu \in \ell_u Hom(G, R)$ ($Ext(\ell_u Hom(G, R))$, respectively) the value u_R (see also Goodearl, 2010).

Our main results here proved are the injectivity of the evaluation maps ψ and ϕ. We give the following

Theorem 3.2.1. *Let G and R be as in Lemma 3.2.1. Then the map*

$$\phi : G \to \mathcal{C}_R(Ext(\ell_u Hom(G, R))),$$

defined as in (3.8), is an injective ℓ_u-homomorphism, i.e. a faithful representation.

Proof. It is a direct consequence of Lemma 3.2.1. □

Theorem 3.2.2. *Let G and R be as in Lemma 3.2.1. Then the map*

$$\psi : G \to Aff_R(\ell_u Hom(G, R)),$$

defined by setting $\psi(x)(\mu) = \mu(x)$, $x \in G$, $\mu \in \ell_u Hom(G, R)$, is an injective ℓ_u-homomorphism, that is a faithful representation.

Proof. By construction, $\psi(x) : \ell_u Hom(G, R) \to R$ defines an affine function on the space of $\ell_u Hom(G, R)$, and $\psi \in \ell_u Hom(G, Aff_R(\ell_u Hom(G, R)))$. Using the same notations as in Lemma 3.2.1, to prove the theorem it is enough to consider $\mathcal{K}$ and $\ell_u Hom(G, R)$ instead of $Ext(\mathcal{K})$ and $Ext(\ell_u Hom(G, R))$, respectively, and proceeding analogously as in (3.5), it is sufficient to deal with $\max_{\mu \in \mathcal{K}} \mu(x)$ instead of $\max_{\mu \in Ext(\mathcal{K})} \mu(x)$, getting the injectivity of ψ. □

In general, the condition of archimedeanness of the involved ℓ-group G cannot be dropped. Indeed, we get the following two results (see also Goodearl, 2010, Theorem 7.7).

Proposition 3.2.1. *Let G and R be as in Theorem 3.2.1, and ϕ be as in (3.8). If ϕ is an injective ℓ_u-homomorphism, then G is archimedean.*

Proof. Note that R is Dedekind complete, and then R is archimedean. Thus, $\mathcal{C}_R(Ext(\ell_u Hom(G, R)))$ is archimedean too. Since there is an ℓ_u-isomorphism between G and a substructure of $\mathcal{C}_R(Ext(\ell_u Hom(G, R)))$, then we get the result. □

Proposition 3.2.2. *Let G and R be as in Theorem 3.2.1, and ψ be as in (3.7). If ψ is an injective ℓ_u-homomorphism, then G is archimedean.*

Proof. It is enough to observe that R is archimedean, since R is Dedekind complete, and then $Aff_R(\ell_u Hom(G, R))$ is archimedean. There is an ℓ_u-isomorphism between G and a substructure of $Aff_R(\ell_u Hom(G, R))$, and thus the assertion follows. □

Remarks 3.2.1. (a) In general the condition of Dedekind completeness of the involved vector lattice R cannot be dropped. Indeed, Dedekind completeness is a necessary and sufficient condition for R in order that the Hahn-Banach, extension and sandwich-type theorems hold (see also Boccuto and Candeloro, 1994; Bonnice and Silverman, 1967; Ioffe, 1981; To, 1971).

(b) In general, if μ is a monotone homomorphism defined in a cofinal subgroup of an ℓ-group and with values in another ℓ-group, then μ does not satisfy extension-type theorems. Indeed, let us define μ on the group of all even integers endowed with order unit 2, with values in $\mathbb{Z}$ equipped with order unit 1, by setting $\mu(2n) = n$, $n \in \mathbb{Z}$. Then, μ does not admit any additive monotone extension defined on the whole of $\mathbb{Z}$ (see also Lipecki, 1980). Moreover, in Boccuto, 1995, Theorem 5.3 it is shown that, if G is a rational vector lattice, R is a Dedekind complete abelian ℓ-group and $p : G \to R$ is a function with $p(nx) = np(x)$ for every $x \in G$ and $n \in \mathbb{N} \cup \{0\}$, then R contains necessarily a Dedekind complete vector lattice, containing the range of p. So, it is natural to assume that our involved functionals take values in a (Dedekind complete) vector lattice rather than in an ℓ-group.

Chapter 4

Algebraic Geometry over ℓ-Groups

4.1 Piecewise Linear Functions

In this section we generalize some well-know results presented in Baker, 1968; Beynon, 1975; Beynon, 1977. Usually ℓ-groups are studied considering the set $\mathbb{Z}$ of the integers, but $\mathbb{Z}$ can be seen as a subset of $\mathbb{R}$. In Section 2.1 we will study the *polynomial completeness property* which induces the authors to choose the ℓ-group of real numbers (equipped with the usual order) instead of the integers, moreover in this way all the propositions can be immediately extended to vector lattices.

For these reasons in this section we consider all ℓ-polynomials as piecewise linear functions from $\mathbb{R}^n$ to $\mathbb{R}$ where each variable has integer coefficient, and we will characterize the zero sets of these functions.

We show that in general the zero set of a set of functions is:

- a closed cone of $\mathbb{R}^n$, if we consider polynomial functions without constants; and in particular if we consider a finite set of functions the associated zero set is a closed integral polyhedral cone in $\mathbb{R}^n$ (defined below);
- a closed set in the topology of $\mathbb{R}^n$, if we consider polynomial functions with constants; and in particular if we consider a finite set of functions the associated zero set is a rational polyhedron in $\mathbb{R}^n$.

Let n be a positive integer. Consider the additive group of continuous functions from $\mathbb{R}^n$ to $\mathbb{R}$ with the pointwise ordering, and let $\pi_i : \mathbb{R}^n \longrightarrow \mathbb{R}$, $1 \leq i \leq n$, be the projection functions: $\pi_i(x_1, \ldots, x_n) = x_i$.

We can also consider $FA\ell_0(n)$ the lattice-ordered sublattice subgroup generated by these n projections, which precisely consists of all continuous real-valued piecewise linear homogeneous functions over $\mathbb{R}^n$, where each piece has integer coefficients. Let $hlin_{\mathbb{Z}}(\mathbb{R}^n, \mathbb{R})$ be the set of all homogeneous linear polynomials with integer coefficients, and every $g \in hlin_{\mathbb{Z}}(\mathbb{R}^n, \mathbb{R})$ is equal to $\sum_i^n z_i\pi_i$, where $z_i \in \mathbb{Z}$. It results that $FA\ell_0(n)$ can be defined as follows: $FA\ell_0(n) = \{f = \bigwedge_i \bigvee_j f_{ij} : \mathbb{R}^n \to \mathbb{R} \mid f_{ij} \in hlin_{\mathbb{Z}}(\mathbb{R}^n, \mathbb{R})\}$. On the other hand we can, as in Plotkin, 2002, consider polynomial functions with constants. In particular, let us consider $FA\ell_{\mathbb{Z}}(n)$ defined as follows:

$$FA\ell_{\mathbb{Z}}(n) = \{f = \bigwedge_i \bigvee_j (f_{ij} + h_{i,j}) \mid f_{ij} \in hlin_{\mathbb{Z}}(\mathbb{R}^n, \mathbb{R}) \quad h_{i,j} \in \mathbb{Z}\}.$$

Definition 4.1.1. *A cone in $\mathbb{R}^n$ is a subset K of $\mathbb{R}^n$ which is invariant under multiplication by positive scalars. K is a closed cone if K is also closed in the topology of $\mathbb{R}^n$. We can define the cone generated by a subset X of $\mathbb{R}^n$ as follows: $Cone(X) = \{x \in \mathbb{R}^n \mid \exists \alpha \in \mathbb{R}^{\geq 0}\ \exists \widetilde{x} \in X\ :\ x = \alpha \widetilde{x}\}$.*

Definition 4.1.2. *A subspace $\sum_{i=1}^{n} m_i x_i = 0$ ($m_i \in \mathbb{Z}$) is an integral hyperspace, the corresponding n-dimensional subsets $\sum_{i=1}^{n} m_i x_i \geq 0$ are called closed integral half-spaces. An integral polyhedral cone is convex if it is obtained by finite intersections from integral half-spaces. A closed integral polyhedral cone is a cone obtainable by finite unions of intersections from closed integral half-spaces.*

Definition 4.1.3. *For $f \in FA\ell_0(n)$, let $Z_0(f)$ be the zero set of f, i.e.*

$$Z_0(f) = \{x \in \mathbb{R}^n\ :\ f(x) = 0\}.$$

Let $S(f)$ be the support of f, i.e. $S(f) = \{x \in \mathbb{R}^n\ :\ f(x) \neq 0\}$. If K is a subset of $\mathbb{R}^n$, let $S_K(f)$ be the support of f in K, i.e.

$$S_K(f) = S(f) \cap K.$$

From Baker, 1968 we have the following proposition and its corollary.

Proposition 4.1.1. *Let $f, g \in FA\ell_0(n)$ and let K be a closed integral polyhedral cone in $\mathbb{R}^n$. Suppose that $S_K(f) \subseteq S_K(g)$. Then there is a natural number m such that $|f| \leq m|g|$ on K.*

Corollary 4.1.1. *Let J be an ℓ-ideal of $FA\ell_0(n)$. Suppose that $g \in J$ and $S(f) \subseteq S(g)$. Then $f \in J$.*

This is not true in $FA\ell_{\mathbb{Z}}(n)$. In fact, let us consider $f = (x-1) \vee 0$ and $g = (x \vee 0) \wedge 1$, where f and g are in $FA\ell_{\mathbb{Z}}(1)$. We have that $S_{\mathbb{R}^{\geq 0}}(f) \subseteq S_{\mathbb{R}^{\geq 0}}(g)$, but there is no natural number m such that $|f| \leq m|g|$ on $\mathbb{R}^{\geq 0}$. Recall the notion of CNB ℓ-polynomial from definition 2.2.1.

Theorem 4.1.1. *Let f and g be an ℓ-polynomial and a CNB ℓ-polynomial respectively and K a closed integral polyhedral cone in $\mathbb{R}^n$. Suppose that $S_K(f) \subseteq S_K(g)$. Then there is a natural number m such that $|f| \leq m|g|$ on K.*

Corollary 4.1.2. *Let J be an ℓ-ideal of $FA\ell_{\mathbb{R}}(n)$. Suppose that $g \in J$, g is CNB and $S(f) \subseteq S(g)$. Then $f \in J$.*

Corollary 4.1.3. *Let f and g be an ℓ-polynomial and a CNB ℓ-polynomial respectively. We have that:*

$$Z(f) \supseteq Z(g)\ \Leftrightarrow <f> \subseteq <g>$$

where $<f>$ and $<g>$ are the ℓ-ideals generated by f and g.

Note that the definition of zero set can be generalized as follows.

Definition 4.1.4. *Let us consider $\{f_i\}_{i \in I}$ set of continuous functions from $\mathbb{R}^n$ to $\mathbb{R}$, then we have $Z(\{f_i\}_{i \in I}) = \{x \in \mathbb{R}^n\ :\ \forall i\ f_i(x) = 0\}$.*

In the rest of the paper we will write Z_0 and Z_H when we want to stress that we are considering homogeneous piecewise linear and affine functions.

4.2 Characterization of Zero Sets

4.2.1 The Case Without Constants

In this section we prove a generalization of the following proposition, presented in Baker, 1968.

Proposition 4.2.1. *Baker, 1968, Lemma 3.2 The zero sets $Z(f)$, $f \in FA\ell_0(n)$, are precisely the closed integral polyhedral cones in $\mathbb{R}^n$.*

We now reproduce Proposition 4.2.1 considering not only finitely generated ℓ-ideals but a generic one.

Remarks 4.2.1. *Z has the following properties:*

1. *A finite union of zero sets is a zero set, it is trivial to see that*

$$Z(\{f_i\}_{i\in I}) \cup Z(\{f_j\}_{j\in J}) = Z(\{|f_i| \wedge |f_j|\}_{(i,j)\in I\times J})$$

   for every set I and J;

2. *an infinite intersection of zero sets is a zero set, i.e.*

$$\bigcap_{i\in\alpha} Z(f_i) = Z(\{f_i\}_{i\in\alpha})$$

   for every index set α (note that we can suppose α countable because $FA\ell(n)$ is countable for every $n \in \mathbb{N}$);

3. *If we consider $U \subseteq FA\ell(n)$ then $Z(U) = Z(id(U))$, where $id(U)$ is the ℓ-ideal generated by U;*

4. *if $U = \{f_1, ..., f_m\}$ then $Z(U) = Z(\{f_1, ..., f_m\}) = Z(f)$, where $f = |f_1| \vee ... \vee |f_m|$;*

5. *in particular we have $Z(g) = Z(|g|)\ \forall g \in FA\ell(n)$.*

By remarks we will say that there exists a non negative polynomial f (which is computable as in remark 4) for every finitely-generated ideal J, such that $Z(J) = Z(f)$.

Proposition 4.2.2. *The zero sets $Z(\{f_i\})$, $\{f_i\} \subseteq FA\ell_0(n)$, are precisely the closed cones in $\mathbb{R}^n$.*

Proof. Since every element of $FA\ell_0(n)$ is a continuous function the zero set of every its element is closed, then $\bigcap_i Z(f_i)$ is again closed. Moreover if $f(\overline{y}) = 0$ then $\forall\alpha \in \mathbb{R}^{\geq 0}\quad f(\alpha\overline{y}) = 0$, so $Z(f)$ is a closed cone for every f, hence $\bigcap_i Z(f_i)$ is always a closed cone.

Vice versa, let us consider a closed cone $C \neq \{\bar{0}\}$ (if $C = \{\bar{0}\}$ then we have $C = Z(|x_1| + ... + |x_n|)$), the cube $K = [-1, 1]^n$, and the cube boundary ∂K. $\mathrm{C} \cap \partial K$ is a closed subset of $\mathbb{R}^n$, so we can write

$$\mathrm{C} \cap \partial K = \bigcap_{i\in\alpha} \bigcup_{j\in J_i} r_{i,j}$$

where: α is an index set; J_i is a finite set $\forall i \in \alpha$; $r_{i,j}$ is a hypercube of the form $r_{i,j} = \bigcap_{l \in L}(s_{i,j,l}) \cap \partial K$; L is a finite set; $s_{i,j,l}$ is a closed half-space of the form $\{x_m \leq q_l\}$ or $\{x_m \geq q_l\}$, $q_l \in \mathbb{Q}$. Let us focus on $s_{i,j,l}$ in the form $x_m \leq q_l$, we note that $\forall i, j, l\ \exists h\ :\ |x_h| = 1$ and $q_l = \frac{a_l}{b_l}$, where $a_l \in \mathbb{Z}$ and $b_l \in \mathbb{N}$, so we have:

- if $x_h = 1$ then $x_m \leq q_l = q_l x_h = \frac{a_l x_h}{b_l}$. i.e. $s_{i,j,l} = Z(b_l x_m - a_l x_h \vee 0)$;
- if $x_h = -1$ then $x_m \leq q_l = q_l(-x_h) = \frac{-a_l x_h}{b_l}$. i.e. $s_{i,j,l} = Z(b_l x_m + a_l x_h \vee 0)$.

Similarly if $s_{i,j,l}$ is in the form $x_m \geq q_l$. Summing up there exists $f_{i,j,l} \in FA\ell_0(n)$ for all i, j and l such that $s_{i,j,l} = Z(f_{i,j,l})$, so we have $r_{i,j} = \bigcap_{l \in L}(Z(f_{i,j,l})) \cap \partial K$, and by remark (2) we can say $r_{i,j} = Z(\{f_{i,j,l}\}_{l \in L}) \cap \partial K$.

Then

$$C \cap \partial K = \bigcap_{i \in \alpha} \bigcup_{j \in J} r_{i,j} = \bigcap_{i \in \alpha} \bigcup_{j \in J} (Z(\{f_{i,j,l}\}_{l \in L}) \cap \partial K) = (\bigcap_{i \in \alpha} \bigcup_{j \in J} Z(\{f_{i,j,l}\}_{l \in L})) \cap \partial K,$$

but we have $\bigcap_{i \in \alpha} \bigcup_{j \in J} Z(\{f_{i,j,l}\}_{l \in L}) = Z(\{f_\nu\})$, where each f_ν can be written as in the previous remarks. For the chain of equations we can say that the cones generated by $C \cap \partial K$ and $Z(\{f_\nu\}) \cap \partial K$ are equal. It is enough to remember that C and $Z(\{f_\nu\})$ are closed cones and by this we have the following chain of equalities:

$$C = Cone(C \cap \partial K) = Cone(Z(\{f_\nu\}) \cap \partial K) = Z(\{f_\nu\}).$$

□

So we are considering subsets of $FA\ell_0(n)$ and by Z we can generate (all) the closed cone of R^n.

4.2.2 The Case With Constants

Proposition 4.2.3. *The zero sets $Z(\{f_i\})$, $\{f_i\} \subseteq FA\ell(n)$, are precisely the closed set in $\mathbb{R}^n$.*

Proof. We can always consider the (rational) rectangle which is zero set of some particular ℓ-polynomial with constants; the topology generated by (rational) rectangles is equal to the Euclidean topology. We have trivially that $Z(\{f_i\})$ is a closed set of $\mathbb{R}^n$ in the standard topology; conversely if we consider C a closed set of $\mathbb{R}^n$ we can always approximate with a family of (rational) rectangles (definable as zero sets of particular ℓ-polynomials). □

Proposition 4.2.4. *The operator ZI is exactly the standard closure in Euclidean spaces.*

Remark 4.2.1. *If we consider the ℓ-groups $\mathbb{Z}$ and $\mathbb{Q}$ we have the following facts:*

- $Z_0^{\mathbb{Z}} I_0^{\mathbb{Z}}(C) = Cone(C) \cap \mathbb{Z}$
- $Z^{\mathbb{Z}} I^{\mathbb{Z}}(C) = \bar{C} \cap \mathbb{Z}$
- $Z_0^{\mathbb{Q}} I_0^{\mathbb{Q}}(C) = Cone(C) \cap \mathbb{Q}$

- $Z^{\mathbb{Q}}I^{\mathbb{Q}}(C) = \bar{C} \cap \mathbb{Q}$

where the superscript indicates the ℓ-group in which the operator acts and the set C is considered.

4.2.3 Examples

The One-Dimensional Case Without Constants

We know that for every set $\mathcal{F}$ of functions in $FA\ell_0(n)$ the set $Z_0(\mathcal{F})$ is a closed cone of $\mathbb{R}^n$ with the vertex in the origin (Proposition 4.2.2). In one-dimensional case there are only the following closed cones:

- $\{0\}$;
- $[0, +\infty[$;
- $]-\infty, 0]$;
- $\mathbb{R}$.

So we can say that if we consider a subset C of $\mathbb{R}$ the corresponding subset $Z_0I_0(C)$ can be one of the cones presented before. To be more precise we have the following characterization.

Proposition 4.2.5. *For all C subset of $\mathbb{R}$ we have:*

$$\mathbf{Z_0I_0}(\mathbf{C}) = \begin{cases} \{0\} & \textit{if} \quad C = \{0\} \\ [0, +\infty[& \textit{if} \quad C \cap]0, +\infty[\neq \emptyset \quad \textit{and} \quad C \cap]-\infty, 0[= \emptyset \\ ]-\infty, 0] & \textit{if} \quad C \cap]0, +\infty[= \emptyset \quad \textit{and} \quad C \cap]-\infty, 0[\neq \emptyset \\ \mathbb{R} & \textit{if} \quad C \cap]0, +\infty[\neq \emptyset \quad \textit{and} \quad C \cap]-\infty, 0[\neq \emptyset \end{cases}$$

The proof is quite trivial and it descends from Proposition 4.2.2 below.

The One-Dimensional Case With Constants

We can consider the more complex case of the operator ZI. In this case we can start to study C when it is a point, an (open, closed or half-closed) interval and a (open or closed) ray.

To describe all these situations we prefer use some useful functions in the form

$$p_{\frac{m}{n}}(x) := |(nx - m)|$$

and

$$r^{+}_{\frac{m}{n}}(x) := |(nx - m) \wedge 0|$$

$$r^{-}_{\frac{m}{n}}(x) := |(-nx + m) \wedge 0|$$

where $m \in \mathbb{Z}$ and $n \in \mathbb{N}$, the first gives us a rational point as zero set and the second ones a rational closed ray. We will use also the notation p_q, r^-_q and r^+_q where $q = \frac{m}{n}$. Why do we choose these functions? Because with these functions we can determinate, in a standard way, the action of ZI on subsets of $\mathbb{R}$, and they give us a presentation of the generators of the ideal $I(C)$.

By this we can say that if C is a finite set of rational points and rational closed rays or intervals we have an explicit description of the ideal $I(C)$, which is finitely generated by a combination of these particular functions.

Let us consider $\bar{x}$ a non-rational point in $\mathbb{R}$. We cannot use $p_{\frac{m}{n}}$ because $\bar{x}$ is not in $\mathbb{Q}$ so we can think to approximate $\bar{x}$ from both sides with two series of rational rays in the following way. We know that there are an increasing series $\{q_i^l\}$ and a decreasing series $\{q_j^r\}$ in $\mathbb{Q}$ converging to $\bar{x}$ from left and from right. Now if we consider $I(\{\bar{x}\})$ which contains all the ℓ-polynomials such that $f(\bar{x}) = 0$, we have that $r^+_{q_i^l}$ and $r^-_{q_j^r}$ are in $I(\bar{x})$. In this case we have that the ideal $I(\bar{x})$ can not be finitely generated, in fact if $I(\bar{x})$ is finitely generated for the structure of our space there exists $p_{\bar{x}}(x) = |(nx - m)|$, with $m, n \in \mathbb{Z} \setminus \{0\}$, such that $p_{\bar{x}}(\bar{x}) = 0$, but it is impossible because in this way we have that $\bar{x} = \frac{m}{n}$ and $\bar{x} \in \mathbb{R} \setminus \mathbb{Q}$.

The same construction is possible with closed, open and half-closed intervals (finite or infinite).

Let us consider $C = \mathbb{Z}$ as exemplar of infinite discrete set of points and the ℓ-polynomial functions (separation functions)

$$sf_n(x) = ((x - n) \wedge (-x + n + 1)) \vee 0$$

for each $n \in \mathbb{Z}$, where trivially $sf_n \in I(\mathbb{Z})\ \forall n \in \mathbb{Z}$ so we have $\mathbb{Z} \subseteq ZI(\mathbb{Z}) \subseteq Z(\{sf_n\}_{n\in\mathbb{Z}}) = \mathbb{Z}$. More easily we can consider the function defined by the following series

$$F_{\mathbb{Z}}(x) = \sum_{n\in\mathbb{Z}} sf_n(x),$$

and observe that $Z(\{sf_n\}_{n\in\mathbb{N}}) = Z(F_{\mathbb{Z}})$; we have to note that $F_{\mathbb{Z}}$ is not a polynomial because we have an infinite sum, and so $I(\mathbb{Z})$ is not finitely generated.

Note that all these considerations and constructions can be easily extended in the multidimensional case, and they give us a useful tool to classify and recognize finitely generated ideals.

4.3 The ℓ-Operators Z and I

Definition 4.3.1. *We consider $FA\ell_0(X)$ the free abelian ℓ-group on $X = \{x_1, ..., x_n\}$ finite set of generators; we will also use $FA\ell_0(n)$, where $|X| = n$. An important result (seeBigard, Keimel, and Wolfenstein, 1977) tells us that every free ℓ-group is a subdirect product of groups isomorphic to $\mathbb{Z}$, moreover we can express it in the following way:*

$$FA\ell_0(n) = \{f = \bigwedge_i \bigvee_j f_{ij} \mid f_{ij} \in hlin_{\mathbb{Z}}\}$$

where $f \in hlin_{\mathbb{Z}}$ iff $f = \sum_{i=1}^n z_i x_i$, with $z_i \in \mathbb{Z}$.
In particular, by universal properties of free algebras, it follows that every ℓ-group is homomorphic image of a subdirect product of groups isomorphic to $\mathbb{Z}$, since each ℓ-group is homomorphic image of the free ℓ-group.

We can fix an ℓ-group H and consider the variety $\ell\mathcal{GR}_H$. In $\ell\mathcal{GR}_H$ we have the free algebra $FA\ell_H(n)$ as follows

$$FA\ell_H(n) = \{f = \bigwedge_i \bigvee_j (f_{ij} + h_{i,j}) \mid f_{ij} \in hlin_{\mathbb{Z}} \quad h_{i,j} \in H\}.$$

We will write $FA\ell(n)$ to indicate $FA\ell_0(n)$ and $FA\ell_H(n)$ when the context is clear or when the results work for both.

Definition 4.3.2. *Let G an ℓ-group, let $A \subseteq Hom(FA\ell(n), G)$, whose elements are seen as points of G^n, and let $U \subseteq FA\ell(n)$ a set of polynomials in $FA\ell(n)$, then we can define the following operators*

$$Z_G \,:\, \mathcal{P}(FA\ell(n)) \longrightarrow \mathcal{P}(Hom(FA\ell(n), G))$$

where $Z_G(U) := \{\mu : FA\ell(n) \to G \mid U \subseteq Ker\mu\}$, and

$$I_G \,:\, \mathcal{P}(Hom(FA\ell(n), G)) \longrightarrow \mathcal{P}(FA\ell(n))$$

where $I_G(A) := \bigcap_{\mu \in A} Ker\mu$.

We say that $Z_G(U)$ is the ℓ-algebraic set (or ℓ-zero set) determined by U and $I_G(A)$ is the ℓ-ideal determinated by A (we can say it also G-closed ℓ-ideal). As in classical algebraic geometry and in Plotkin, 2002 we will identify $Hom(FA\ell(X), G)$ with the Cartesian product G^n, and we have:

$$I_G(A) = \{p \in FA\ell(n) \mid \forall \bar{a} \in A\, p(\bar{a}) = 0\} = \bigcap_{\bar{a} \in A} I_G(\bar{a}),$$

where $A \subseteq G^n$.

Remark 4.3.1. *Note that if $G = \mathbb{R}$ then the definitions, given in the Section 4.1, of zerosets and ideals coincide with those of $Z_{\mathbb{R}}$ and $I_{\mathbb{R}}$. Moreover the five properties of Z_0 in Remarks 4.2.1 hold also for Z_G in a generic ℓ-group G.*

We can also consider the operators $I_G Z_G$ and $Z_G I_G$ as follows:

$$I_G Z_G(U) = I_G(Z_G(U)) = \{p \in FA\ell(n) \mid \forall \bar{a} \in Z_G(U)\, p(\bar{a}) = 0\}$$

$$Z_G I_G(A) = Z_G(I_G(A)) = \{\overline{x} \in G^n \mid f(\overline{x}) = 0\, \forall f \in I_G(A)\}$$

Some properties of I_G and Z_G are independent from the ℓ-group G, in those cases we will write I and Z.

Definition 4.3.3. *Let $(A, \leq)$ and $(B, \leq)$ be two partially ordered sets. A Galois correspondence consists of two monotone functions: $F : A \to B$ and $G : B \to A$, such that for all a in A and b in B, we have $F(a) \leq b \quad \Leftrightarrow \quad a \leq G(b)$.*

Operators I_G and Z_G form a Galois correspondence between $(\mathcal{P}(Hom(FA\ell(X), G)), \subseteq)$ and $(\mathcal{P}(FA\ell(X)), \subseteq)$. In the variety of ℓ-groups the operator Z has a more general meaning. In fact each equation of ℓ-polynomials of the form $w = w'$ can be put in the form $z = 0$ where $z = w - w'$. We can consider $w, w' \in FA\ell(n)$ and then we can ask if (and when) $w \equiv w'$. Let us fix an ℓ-group G and define $Val_G(w \equiv w')$ as follows:

$$Val_G(w \equiv w') = \{\overline{\mu} : FA\ell(n) \to G \mid w^{\overline{\mu}} = w'^{\overline{\mu}}\},$$

and in the variety of ℓ-groups, we have that $Z_G(\{w - w'\}) = Val_G(w \equiv w')$.

4.3.1 The Nullstellensatz for ℓ-groups

The Hilbert's Nullstellensatz is a theorem in algebraic geometry that relates varieties and ideals in polynomial rings over algebraically closed fields. Let K be an algebraically closed field (such as the field of complex numbers) and let consider the polynomial ring $K[x_1, x_2, ..., x_n]$ and let I be an ideal in this ring. The affine variety $V(I)$ defined by this ideal consists of all n-tuples $\bar{k} = (k_1, ..., k_n)$ in K^n such that $f(\bar{k}) = 0$ for all f in I. The theorem of zeros Hilbert states that if p is some polynomial in $K[x_1, x_2, ..., x_n]$ such that $p(\bar{k}) = 0$ for all $\bar{k}$ in $V(I)$, then there exists a natural number r such that p^r is in the I. With the usual notation in algebraic geometry, the Nullstellensatz can also be formulated as $I(Z(J)) = \sqrt{J}$ for every ideal J, where $\sqrt{I} = \{x \in A | \exists n \in \mathbb{N} : x^n \in I\}$. In this section we propose a variation of Hilbert's Nullstellensatz. Instead of an algebraically closed field K and the polynomial ring over it we will consider a generic ℓ-group G and ℓ-polynomials. Note that every ℓ-polynomial is equal to zero in the zero of every ℓ-group, property equivalent to the algebraic closure requested to the fields. We will define an ℓ-radical $_\ell\sqrt{I}$ such that we have, in the Theorem 4.3.1, $I(Z(J)) = _\ell\sqrt{J}$.

Definition 4.3.4. *Let J be an ℓ-ideal of $FA\ell(n)$. We can define the ℓ-radical $_\ell\sqrt{J}$ as follows*

$$_\ell\sqrt{J} = \bigcap_{J \subseteq I(\bar{a})} I(\bar{a})$$

Note that every ℓ-radical is the intersection of ideals and therefore it is itself an ideal.

Lemma 4.3.1. *Let be J an ℓ-ideal of $FA\ell(n)$. $I(Z(J)) = _\ell\sqrt{J}$*

Proof. By Lemma 4.3.2 $I(Z(J)) = \bigcap_{\bar{y} \in Z(J)} I(\bar{y})$, but we have also that

$$\bigcap_{\bar{y} \in Z(J)} I(\bar{y}) = \bigcap\{I(\bar{y}) \,|\, \forall f \in J \; f(\bar{y}) = 0\} = \bigcap\{I(\bar{y}) \,|\, J \subseteq I(\bar{y})\}.$$

□

Lemma 4.3.2. *Let U be a subset of $FA\ell(n)$, so $I(Z(U)) = \bigcap_{\bar{y} \in Z(U)} I(\bar{y})$*

Proof. Let f be in I(Z(U)), this means that $f(\bar{y}) = 0 \; \forall \bar{y} \in Z(U)$ or equivalently $f \in I(\bar{y}) \; \forall \bar{y} \in Z(U)$ but $f \in I(\bar{y}) \; \forall \bar{y} \in Z(U) \iff f \in \bigcap_{\bar{y} \in Z(U)} I(\bar{y})$ □

By previous lemmas we have the following theorem.

Theorem 4.3.1. *(Nullstellensatz for ℓ-groups)*
$I(Z(J)) = {}_\ell\sqrt{J}$, moreover the ideals J such that $I(Z(J)) = J$ are exactly the ℓ-radical ideals.

4.3.2 Closure Operators

Definition 4.3.5. *A closure operator is a map Γ from a powerset $\mathcal{P}(S)$ of a set S to $\mathcal{P}(S)$ such that $X \subseteq \mathcal{P}(X)$, $X \subseteq Y$ implies $\Gamma(X) \subseteq \Gamma(Y)$, and $\Gamma(\Gamma(X)) = \Gamma(X)$.*

Proposition 4.3.1. *ZI and IZ are closure operators.*

Proof. We have $X \subseteq ZI(X)$ by Galois connection properties. Let consider $X, Y \subseteq G^n$, we have that $X \subseteq Y \Rightarrow I(X) \supseteq I(Y) \Rightarrow ZI(X) \subseteq ZI(Y)$. Let us consider $\bar{a}$ in $ZI(ZI(X))$, by definition it exists an f in $I(ZI(X))$ such that $f(\bar{a}) = 0$; but we have $I(ZI(X)) = IZ(I(X)) = I(X)$. So we have f in $I(X)$ such that $f(\bar{a}) = 0$, i.e. $\bar{a} \in ZI(X)$.

Analogously can be proved that IZ is a closure operator. □

4.4 Geometrically Stable ℓ-groups

Let us consider the sets $\mathcal{K}_{(G)}$ and $\mathcal{C}_{(G)}$ of the zero sets and of the ℓ-ideals. In general we know that the intersection of zero sets is a zero set, and the intersection of ℓ-ideals is an ℓ-ideal; but the union of zero sets (or ℓ-ideals) is not necessary a zero set (or ℓ-ideal), then $(\mathcal{K}_{(G)}, \cup, \cap)$ and $(\mathcal{C}_{(G)}, \cup, \cap)$ are not structured as lattices. Let us define the operation $\overline{\cup}$ as follows:

$$\begin{aligned} Z(X)\overline{\cup}Z(Y) &= ZI(Z(X) \cup Z(Y)), \\ I(X)\overline{\cup}I(Y) &= IZ(I(X) \cup I(Y)). \end{aligned}$$

So we can consider the complete lattices $(\mathcal{K}_{(G)}, \overline{\cup}, \cap)$ and $(\mathcal{C}_{(G)}, \overline{\cup}, \cap)$.

Proposition 4.4.1. *The lattices $(\mathcal{K}_{(G)}, \overline{\cup}, \cap)$ and $(\mathcal{C}_{(G)}, \overline{\cup}, \cap)$ are dual.*

Proof. It follows from the Proposition 4.3.1. □

Definition 4.4.1. *Let G be an ℓ-group. G is geometrically stable if for all $Z_G(X)$, $Z_G(Y)$ we have $Z_G(X)\overline{\cup}Z_G(Y) = Z_G(X) \cup Z_G(Y)$.*

Recall that in Zariski topology on G^n closed sets are finite unions and arbitrary intersections of zero sets and it is the minimal topology in the space such that all zero sets are closed. Note that if G is geometrically stable then closed sets are all zero sets.

Definition 4.4.2. *A closure operator Γ on a powerset is called topological when it commutes with finite unions and $\Gamma(\emptyset) = \emptyset$. The fixpoints of a topological closure operator are closed under finite unions and arbitrary intersections, so they are the closed sets of a topology.*

Theorem 4.4.1. *Let G be an ℓ-group. The following are equivalent:*

1. *G is geometrically stable;*
2. *G is totally ordered;*
3. *$Z_G I$ is a topological operator;*
4. *IZ_G is a topological operator.*

The proof of the theorem naturally follows from the following lemmas.

Lemma 4.4.1. *If G is not totally ordered then $Z_G I$ is not a topological operator in n dimensions for all $n \geq 2$.*

Proof. G is not totally ordered so there are $w, z \in G \setminus \{0\}$ such that $w \wedge z = 0$. In fact if we have $c, d \in G$ that are non comparable we can consider $w = c-(c\wedge d)$ and $z = d-(c\wedge d)$. Let us consider the projections $x_1, x_2 \in FA\ell(n)$, with $n \geq 2$, and $\bar{a} = (a_1, a_2, ..., a_n) \in G^n$ such that $a_1 = w$, $a_2 = z$ and $a_i = 0$ for all $i = 3, ..., n$. We can define $Z_1 = Z_G(x_1)$ and $Z_2 = Z_G(x_2)$. Now it is sufficient to prove that $\bar{a} \in Z_G I(Z_1 \cup Z_2) \setminus (Z_G I(Z_1) \cup Z_G I(Z_2))$; but $\bar{a} \notin Z_i$ because a_1 and a_2 are not equal to zero and by remark $Z_G I(Z_i) = Z_i$. By the theorem of Hahn we know that $G \subseteq \bigoplus_{i\in I} \mathbb{R}_i$, where I is the set of all prime ideals of G. Let $I_w = \{i \in I \,|\, w_i = 0\}$ and $I_z = \{i \in I \,|\, z_i = 0\}$, by $w \wedge z = 0$ we have $I = I_w \cup I_z$. Now let consider $f \in I(Z_1 \cup Z_2) = I(Z_1) \cap I(Z_2)$, in particular $f \in I(Z_2)$ i.e. $f(w, 0, ..., 0) = 0$ then $f_i(w, z, 0, ..., 0) = f(w_i, z_i, ..., 0) = 0 \; \forall i \in I_z$; in a similar way we have $f_i = 0$ $\forall i \in I_w$ by $f(0, z, ..., 0) = 0$. So $f_i = 0$ for all i in I i.e. $f(w, z, 0, ..., 0) = 0$, then $\bar{a} \in Z_G I(Z_1 \cup Z_2)$.

□

Lemma 4.4.2. *For all $X, Y \subseteq G^n$ we have $I(X \cup Y) = I(X) \cap I(Y)$.*

Proof. We have $I(X \cup Y) \supseteq I(X) \cap I(Y)$ by definition. Let us consider $p \notin I(X) \cap I(Y)$, so $\exists \bar{a} \in X$ such that $p(\bar{a}) \neq 0$ or $\exists \bar{b} \in Y$ such that $p(\bar{b}) \neq 0$, then we can say that $\exists \bar{c} \in X \cup Y$ such that $p(\bar{c}) \neq 0$, i.e. $p \notin I(X \cup Y)$ □

For all I, J ℓ-ideals of $FA\ell(n)$ and for all G ℓ-group we have $Z(I \cap J) \supseteq Z(I) \cup Z(J)$ by definition.

Lemma 4.4.3. *For all I, J ℓ-ideals of $FA\ell(n)$ and for all G totally ordered ℓ-group, we have $Z_G(I \cap J) = Z_G(I) \cup Z_G(J)$.*

Proof. Let us consider $\bar{a} \in Z_G(I \cap J)$, this means that $\forall p \in I \cap J \,|p(\bar{a}) = 0$. Suppose that $\bar{a} \notin Z(I)$, i.e. $\exists q_I \in I$ such that $|q_I(\bar{a})| \neq 0$. Now let $q_J \in J$, so $|q_I(\bar{a})| \wedge |q_J(\bar{a})| = 0$, because $|q_I| \wedge |q_J| \in I \cap J$. Now we use our hypothesis of total ordering of G and we can say $|q_J(\bar{a})| = 0$, and by the arbitrariness of q_J we have $\bar{a} \in Z(J)$. □

If we consider the case in which $G = \mathbb{R}$, by the total order of $\mathbb{R}$, we have that $Z_{\mathbb{R}} I$ and $I Z_{\mathbb{R}}$ are topological operators, i.e. $\mathbb{R}$ is geometrically stable.

4.5 Geometrically Noetherian ℓ-Groups

Definition 4.5.1. *Let G and H be ℓ-groups. G is called geometrically Noetherian w.r.t. H iff for every $n \in \mathbb{N}$ and for every system of equations T in $FA\ell_H(n)$ there exists T_0 finite subset of T such that $Z(T) = Z(T_0)$.*

Remarks 4.5.1. *Trivial, but useful, remarks are the following ones:*

- *if $H_1 \leq H_2$ and G is geometrically Noetherian w.r.t. H_2 then G is geometrically Noetherian w.r.t. H_1, in particular if G is not geometrically Noetherian w.r.t. $\{0\}$ then G is not geometrically Noetherian w.r.t. any H;*

- *if $G_1 \leq G_2$ and G_2 is geometrically Noetherian w.r.t. H then G_1 is geometrically Noetherian w.r.t. H;*

- *if $G_1 \cong G_2$ and G_2 is geometrically Noetherian w.r.t. H then G_1 is geometrically Noetherian w.r.t. H.*

Lemma 4.5.1. *$\mathbb{Z}$ is not geometrically Noetherian w.r.t. $\{0\}$.*

Proof. Let us consider $n = 2$ and the closed cone $C = \{(x, y) \,|\, 0 \leq y \leq \sqrt{2}x,\ x \geq 0\}$. By the characterization of zero sets there exists $\{f_i\}_{i \in I}$, an infinite set of polynomials, such that $C = Z(\{f_i\}_{i \in I})$. If $\mathbb{Z}$ were geometrically Noetherian w.r.t. $\{0\}$ then there exists I' finite subset of I such that $Z(\{f_j\}_{j \in I'}) = Z(\{f_i\}_{i \in I}) = C$, i.e. we have that $\sqrt{2}$ is a rational, but it is an absurdum. □

Proposition 4.5.1. *An ℓ-group G is geometrically Noetherian w.r.t. H iff $G = \{0\}$.*

Proof. It is trivial that $G = \{0\}$ is geometrically Noetherian w.r.t. H, for all H ℓ-group of constants.

Let us consider $G \neq \{0\}$, then we have that there exists G' ℓ-subgroup of G such that $G' \cong \mathbb{Z}$.

By lemma and a previous remark we have that G' is not geometrically Noetherian w.r.t. $\{0\}$ so G is not geometrically Noetherian w.r.t. $\{0\}$; and by the first remark G is not geometrically Noetherian w.r.t. any H. □

4.6 Algebraically Closed ℓ-Groups

We would like to study the notion of algebraically closed ℓ-group by following Plotkin, 2002. However if we follow Plotkin literally we end up of a definition of H-algebraically closed ℓ-group (Definition 4.6.1) which is trivial except for $H = \{0\}$. For completeness we give the general definition. Moreover we give a weaker definition which we call weakly H-algebraically closed, which is not trivial also for $H \neq \{0\}$ in general.

Definition 4.6.1. *Let G be an ℓ-group and let $H \leq G$. G is H-algebraically closed iff for every J ℓ-ideal such that $J \subset FA\ell_H(n)$ we have $Z_G(J) \neq \emptyset$.*

Proposition 4.6.1. *f is a strong unit of $\mathbb{R}^{\mathbb{R}^n}$ equipped with the pointwise order iff f is CNB and $Z(f) = \emptyset$.*

Proposition 4.6.2. *Let J be an ℓ-ideal $J \subseteq FA\ell_H(n)$. $J = FA\ell_H(n)$ iff there exists u strong unit such that $u \in J$.*

Proposition 4.6.3. *Let G be an ℓ-group, then G is not H-algebraically closed for each $H \neq \{0\}$.*

Proof. Let G be an ℓ-group and let us consider $J =< h >$, where $h \in H \setminus \{0\}$. By Proposition 4.6.2 $J \neq FA\ell_H(n)$, but $Z_G(J) = \emptyset$. □

By Proposition 4.6.3, Definition 4.6.1 is trivial in our context. Less trivial definitions are the following.

Definition 4.6.2. *Let G be an ℓ-group. G is algebraically closed iff for every J ℓ-ideal such that $J \subset FA\ell_0(n)$ we have $\bar{Z}_G(J) \neq \emptyset$, where $\bar{Z}_G(J)$ is the set $Z_G(J) \setminus \{0\}$.*

Definition 4.6.3. *Let G be an ℓ-group. Let H be an ℓ-group such that $H \leq G$. G is weakly H-algebraically closed if for every polynomial $f \in FA\ell_H(n)$ such that $< f >$ is proper we have $Z_G(f) \neq \emptyset$.*

Proposition 4.6.4. *$\mathbb{Q}$ is not algebraically closed.*

Proof. Let us consider $J = \{f \in FA\ell_0(2) \mid f(1, \sqrt{2}) = 0\}$. J is a proper ℓ-ideal, but $\bar{Z}_{\mathbb{Q}}(J) = \emptyset$. □

Proposition 4.6.5. *$\mathbb{Z}$ is weakly $\{0\}$-algebraically closed.*

Proof. Let $f \in FA\ell_H(n)$ such that $< f >= J \subset FA\ell_H(n)$. By Proposition 4.6.2 and by the nature of our objects we have $Z_{\mathbb{Z}}(f) \neq \emptyset$, i.e. $Z_{\mathbb{Z}}(J) \neq \emptyset$. □

Corollary 4.6.1. *Every G ℓ-group is weakly $\{0\}$-algebraically closed.*

Definition 4.6.4. *Let X be a set with $A = \{A_i\}_{i \in I}$ a family of subsets of X. A has the finite intersection property (FIP) if for any finite subcollection $K \subseteq I$ the intersection $\bigcap_{i \in K} A_i$ is not empty.*

Proposition 4.6.6. *Let J be an ℓ-ideal of $FA\ell_0(n)$. J is proper iff $\{\bar{Z}_G(f_i)\}_{f_i \in J}$ has the FIP.*

Proof. $\Rightarrow$ Let us consider J ℓ-ideal such that $\{\bar{Z}_G(f_i)\}_{f_i \in J}$ has not the FIP. This means that there exists $f_1, \dots, f_m$ with $\bar{Z}_G(f_1, \dots, f_m) = \emptyset$; by this we have $f = \bigvee_{i=1}^m |f_i| \in J$, but by easy observation f is a strong unit of $FA\ell_0(n)$ and then J is not proper.

$\Leftarrow$ Let us consider J non-proper ℓ-ideal. By Proposition 4.6.2 there exists u strong unit of $FA\ell_0(n)$ such that $u \in J$; so $\bar{Z}_G(u) = \emptyset$, i.e. $\{\bar{Z}_G(f_i)\}_{f_i \in J}$ has not the FIP. □

Theorem 4.6.1. *Let G be an ℓ-group. We have the following equivalence:*

1. *G is algebraically closed;*
2. *the Zariski topology on $(G \setminus \{0\})^n$ is compact.*

Proof. $1 \Rightarrow 2$ Let us consider G algebraically closed ℓ-group and $\{\bar{Z}_G(f_i)\}_{i \in I}$ a family of closed sets indexed by I which has the FIP. By Proposition 4.6.6 the set $\{f_i\}_{i \in I}$ is included in some J proper ℓ-ideal. By the fact that G is algebraically closed we have

$$\bigcap_{i \in I} \bar{Z}_G(f_i) \supseteq \bar{Z}_G(J) \neq \emptyset,$$

and by a characterization of compact topology we have that the Zariski topology on $(G \setminus \{0\})^n$ is compact.

$2 \Rightarrow 1$ Let J be a proper ℓ-ideal. Let us consider $\{f_i\}_{i=1,\dots,m}$, a finite subset of J. By Proposition 4.6.2 and Corollary 4.6.1 we have that $\bar{Z}_G(\{f_i\}_{i=1,\dots,m}) \neq \emptyset$, but the Zariski topology on $(G \setminus \{0\})^n$ is compact so we can say that $\bar{Z}_G(J) \neq \emptyset$. □

Corollary 4.6.2. *The ℓ-group $\mathbb{R}$ is algebraically closed.*

4.7 Categorical Duality

In this section we propose a categorical duality between the categories of zero sets (or equivalently algebraic sets) and of the coordinate algebras.

We define now the categories $K_{\ell-Gr}$ (of the algebraic sets) and $C_{\ell-Gr}$ (of coordinate algebras). The $K_{\ell-Gr}$ objects are (X, A, H), where A is an algebraic set in $Hom(FA\ell_H(X), H)$; while the $C_{\ell-Gr}$ objects are $(FA\ell_H(X)/I, H)$ where I is an H-closed ℓ-ideal in $FA\ell_H(X)$. Let define the morphisms $(X, A, H_1) \to (Y, B, H_2)$. We consider the homomorphisms $\delta : H_1 \to H_2$, $s : FA\ell_{H_2}(Y) \to FA\ell_{H_1}(X)$, $\nu : FA\ell_{H_1}(X) \to H_1$ and the commutative diagram:

$$\begin{array}{ccc} FA\ell_{H_2}(Y) & \xrightarrow{\ s\ } & FA\ell_{H_1}(X) \\ \big\downarrow \nu' & & \big\downarrow \nu \\ H_2 & \xleftarrow[\ \delta\ ]{} & H_1 \end{array}$$

For every homomorphism $\nu : FA\ell_{H_1}(X) \to H_1$ we consider the homomorphism $\nu' = \delta\nu s$ that we can express also through the application $(s, \delta) : Hom(FA\ell_{H_1}(X), H_1) \to Hom(FA\ell_{H_2}(Y), H_2)$ such that $(s, \delta)(\nu) = \nu'$. The couple (s, δ) is admissible with respect to A and B if $\nu' \in B$ for all $\nu \in A$. Let (s, δ) be an admissible couple with respect to A and B, we fix δ and we consider the map $[s]_\delta : A \to B$, obtained by restricting (s, δ). The couple $([s]_\delta, \delta)$ will be the morphism $(X, A, H_1) \to (Y, B, H_2)$ and we define the composition of two morphism in the following way $([s']_{\delta'}, \delta')([s]_\delta, \delta) = ([ss']_{\delta'\delta}, \delta'\delta) : (X, A, H_1) \to (Z, C, H_3)$ where $([s']_{\delta'}, \delta') : (Y, B, H_2) \to (Z, C, H_3)$ and $([s]_\delta, \delta) : (X, A, H_1) \to (Y, B, H_2)$.

We can state the following duality theorem.

Theorem 4.7.1. *The category of algebraic sets and of coordinate algebras are dually isomorphic.*

The proof of the theorem follows from the lemmas below.

This duality allows us to reconstruct, as particular cases, key results presented in Baker, 1968; Beynon, 1975; Beynon, 1977; Cabrer and Mundici, 2011; Cabrer, 2015 and Belluce, Di Nola, and Lenzi, 2014; Cabrer and Mundici, 2009; Marra and Spada, 2012, in the fields of ℓ-groups and MV-algebras. In fact, recall that the Mundici functor Γ associates to an ℓ-group G with a strong unit u the MV-algebra interval $[0, u]$; the introduction of constants makes it possible to consider $[0, u]$ as an algebraic set.

Lemma 4.7.1. *The map $F : K_{\ell-Gr} \to C_{\ell-Gr}$ from the category of algebraic sets to the category of coordinate algebras defined as follows:*

(i) $F((X, A, H)) = (FA\ell_H(X)/A', H)$;

(ii) $F(([s]_\delta, \delta)) = (\sigma_s, \delta)$;

is a contravariant functor.

Proof. Let I_1 be an ℓ-ideal of $FA\ell_{H_1}(X)$ and let I_2 be an ℓ-ideal of $FA\ell_{H_2}(Y)$. Suppose $s : FA\ell_{H_2}(Y) \to FA\ell_{H_1}(X)$ is an admissible homomorphism with respect to I_1 and I_2. Define $\sigma_s : FA\ell_{H_2}(Y)/I_2 \to FA\ell_{H_1}(X)/I_1$ as the homomorphism such that $\sigma_s \circ \rho_2 = \rho_1 \circ s$ where ρ_1 and ρ_2 are the canonical epimorphisms by the ℓ-ideals I_1 and I_2 respectively; or equivalently σ_s can be defined by the commutativity of the following diagram:

$$\begin{array}{ccc} FA\ell_{H_2}(Y) & \xrightarrow{s} & FA\ell_{H_1}(X) \\ \downarrow \rho_2 & & \downarrow \rho_1 \\ FA\ell_{H_2}(Y)/I_2 & \xrightarrow{\sigma_s} & FA\ell_{H_1}(X)/I_1 \end{array}$$

σ_s is also well defined, in fact, if we consider $p(\overline{y}) + I_2 = q(\overline{y}) + I_2 \in FA\ell_{H_2}(Y)/I_2$,

$$\rho_2(p(\overline{y})) = \rho_2(q(\overline{y})) \qquad (*)$$

or equivalently

$$p(\overline{y}) - q(\overline{y}) \in I_2 \qquad (**)$$

then we can see, by definition, the following chain of equalities: $\sigma_s(\rho_2(p(\overline{y}))) = \rho_1(s(p(\overline{y}))) = s(p(\overline{y}))+I_1$ and similarly for $q(\overline{y})$ $\sigma_s(\rho_2(q(\overline{y}))) = \rho_1(s(q(\overline{y}))) = s(q(\overline{y}))+I_1$; but by (**) and the admissibility of s we have $s(p(\overline{y}))-s(q(\overline{y})) \in I_1$ and then $s(p(\overline{y})) + I_1 = s(q(\overline{y})) + I_1$. □

Likewise, suppose σ is a morphism of the category $C_{\ell-Gr}$ from $FA\ell_{H_2}(Y)/I_2$ to $FA\ell_{H_1}(X)/I_1$. We can define the admissible homomorphism $s_\sigma : FA\ell_{H_2}(Y) \to FA\ell_{H_1}(X)$ such that $\sigma \circ \rho_2 = \rho_1 \circ s_\sigma$ where again ρ_1, ρ_2 are the canonical projections; s_σ can be expressed also by the following commutative diagram:

$$\begin{array}{ccc} FA\ell_{H_2}(Y) & \xrightarrow{s_\sigma} & FA\ell_{H_1}(X) \\ \downarrow \rho_2 & & \downarrow \rho_1 \\ FA\ell_{H_2}(Y)/I_2 & \xrightarrow{\sigma} & FA\ell_{H_1}(X)/I_1 \end{array}$$

from which we can derive the morphism $[s_\sigma]$ of category $K_{\ell-Gr}$.

Lemma 4.7.2. *The map $G : C_{\ell-Gr} \to K_{\ell-Gr}$ from the category of coordinate algebras to the category of algebraic sets defined as follows*

(i) $G((FA\ell_{H_1}(X)/I, H)) = (X, I', H)$;

(ii) $G((\sigma, \delta)) = ([s_\sigma], \delta)$;

is a contravariant functor.

Lemma 4.7.3. *The composed functor $GF : K_{\ell-Gr} \to K_{\ell-Gr}$ is the identity functor of the category $K_{\ell-Gr}$.*

Proof. Let us consider an object (X, A, H) and a morphism $[s]$ of the category $K_{\ell-Gr}$. We have

$$GF(X, A, H) = F(G(X, A, H)) = F((FA\ell_{H_1}(X)/A', H)) = (X, A'', H) = (X, A, H).$$

Moreover, if we consider a morphism $[s] : (X, A_1, H_1) \to (Y, A_2, H_2)$, we have that the domain and codomain coincide with those of $GF([s])$ and $GF([s]) = F(G([s])) = F(\sigma_s) = [s_{\sigma_s}]$, but by definition $\rho_1 \circ s_{\sigma_s} = \sigma_s \circ \rho_2 = \rho_1 \circ s$. Now take any $p(\overline{y}) \in FA\ell_{H_2}(Y)$. We derive $s_{\sigma_s}(p(\overline{y})) + A_1' =$

$s(p(\overline{y})) + A_1'$. From this we get that $s_{\sigma_s}(p(\overline{y})) - s(p(\overline{y})) \in A_1'$ and then for the definition of the closure operator this is equivalent to saying that $0 = \mu(s_{\sigma_s}(p(\overline{y})) - s(p(\overline{y}))) = \mu(s_{\sigma_s}(p(\overline{y}))) - \mu(s(p(\overline{y})) \, \forall \mu \in A_1$. Then we get that $\mu(s_{\sigma_s}(p(\overline{y}))) = \mu(s(p(\overline{y}))$ and since $p(\overline{y}) \in FA\ell_{H_2}(Y)$ is arbitrary we get $s = s_{\sigma_s}$. □

In a similar way we obtain the following result.

Lemma 4.7.4. *The composed functor $FG : C_{\ell-Gr} \to C_{\ell-Gr}$ is the identity functor of the category $C_{\ell-Gr}$.*

Part II

Łukasiewicz Logic and its Extensions

Preliminaries

Łukasiewicz Logic and MV-Algebras. The system of axioms for propositional Łukasiewicz logic uses implication and negation as the primitive connectives:

$(A \to B) \to ((B \to C) \to (A \to C))$
$((A \to B) \to B) \to ((B \to A) \to A)$
$(\neg B \to \neg A) \to (A \to B)$.

MV-algebras are the algebraic structures associated to Łukasiewicz logic, in the same sense in which Boolean algebras correspond to classical logic. An *MV-algebra* is a structure $(A, \oplus, \neg, 0)$ where $(A, \oplus, 0)$ is a commutative monoid and:

- $\neg\neg x = x$;
- $x \oplus \neg 0 = \neg 0$;
- $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$ (Mangani's axiom).

Other useful notations are:

- $1 = \neg 0$;
- $n.x = x \oplus x \ldots \oplus x$ (we iterate the sum n times);
- $x \odot y = \neg(\neg x \oplus \neg y)$;
- $x \vee y = \neg(\neg x \oplus y) \oplus y$;
- $x \wedge y = \neg(\neg x \vee \neg y)$.

In every MV-algebra we have a partial order $x \leq y$ which holds if and only if there is z such that $y = x \oplus z$. This order is always a lattice order, where the supremum of two elements is $x \vee y$ and the infimum is $x \wedge y$.

An *ideal* of an MV-algebra A is a subset of A which is closed under sum and is closed downwards in the order of A. If $X \subseteq A$, we denote by $id(X)$ the ideal generated by X. An ideal J is called *principal* if there is an element $f \in A$ which generates J. In this case we write $J = id(f)$.

We denote by A/J the quotient MV-algebra given by an MV-algebra A modulo an ideal J of A.

Recall that an MV algebra is called *semisimple* if the intersection of its maximal ideals is zero. Examples of semisimple MV algebras are C_n and its subalgebras, including M_n and RM_n.

Given a subset S of C_n, and a subset $C \subseteq [0,1]^n$, we denote by $S|_C$ the set of all restrictions of functions in S to C.

Let $C \subseteq [0,1]^m$ and $D \subseteq [0,1]^n$. We call *Z-map* from C to D any n-tuple of McNaughton functions in M_m which sends C to D. We call *Z-homeomorphism* between C and D an invertible Z-map from C to D whose inverse is a Z-map from D to C.

Recall that a *convex polyhedron* is the convex hull of a tuple of real points, and that a *polyhedron* is the union of finitely many convex rational polyhedra. A *simplex of dimension* k is the convex hull of $k+1$ points which is not contained in affine subspaces of dimension less than k. Recall also that a *rational convex polyhedron* is the convex hull of a tuple of rational points, and that a *rational polyhedron* is the union of finitely many convex rational polyhedra.

Rational Łukasiewicz Logic and divisible MV-Algebras. Here we recall the definition of *rational Łukasiewicz logic*, an extension of Łukasiewicz logic, introduced in Gerla, 2001. Formulas are built via the binary connective $\oplus$ and the unary ones $\neg$ and δ_n in the standard way. An assignment is a function $v : Form \to [0,1]$ such that:

- $v(\neg\varphi))1 - v(\varphi)$
- $v(\varphi \oplus \psi) = min\{1, \varphi + \psi\}$
- $v(\delta_n \varphi) = \frac{v(\varphi)}{n}$

For each formula $\varphi(X_1, \ldots, X_n)$ it is possible to associate the truth function $TF(\varphi, \iota) : [0,1]^n \to [0,1]$, where:

- $\iota = (\iota_1, \ldots, \iota_n) : [0,1]^n \to [0,1]^n$
- $TF(X_i, \iota) = \iota_i$
- $TF(\neg\varphi, \iota) = 1 - TF(\varphi, \iota)$
- $TF(\delta_n \varphi, \iota) = \frac{TF(\varphi,\iota)}{n}$

Note that in most of the literature there is no distinction between a McNaughton function and a MV-formula, but it results that, with a different interpretation of the free variables, we can give meaning to MV-formulas by means of other, possibly non-linear, functions (e.g. we consider generators different from the canonical projections $\pi_1, \ldots, \pi_n$, such as polynomial functions, Lyapunov functions, logistic functions, sigmoidal functions and so on).

Real Łukasiewicz Logic and Riesz MV-Algebras. We follow Di Nola and Leuştean, 2014. A *Riesz MV-algebra* is a structure $(R, \cdot, \oplus, \neg, 0)$ where $(R, \oplus, \neg, 0)$ is an MV-algebra and the operation $\cdot : [0,1] \times R \to R$ satisfies the following identities, where $x, y \in R$ and $q, r \in [0,1]$:

- $r(x \odot \neg y) = (rx) \odot \neg(ry)$;
- $(r \odot \neg q)x = \neg(rx) \odot qx)$;
- $r(qx) = (rq)x$;
- $1x = x$.

A *Riesz MV polynomial* is an expression built from variables and 0 by applying the operations $\cdot, \oplus, \neg$ and multiplication by any number $c \in [0,1]$. Note that a free Riesz MV algebra on n generators is given by the set of all Riesz MV polynomials in n variables, modulo the ideal of all polynomials which are zero in every Riesz MV algebra. A free Riesz MV-algebra on n generators is concretely described by Riesz-McNaughton functions.

Possible Generalizations We can say that Riesz MV-algebras are MV-algebras equipped with a sort of module structure on $[0, 1]$, thought of as a multiplicative monoid. The situation can be generalized in many ways. For instance, $[0, 1]$ can be replaced with a product MV-algebra, so that Riesz MV-algebras generalize to MV-modules on a product MV-algebra.

Product MV-algebras arose in the attempt of understanding the interplay between the MV-algebra structure and the multiplicative structure of $[0, 1]$. They are axiomatized, for instance, in Dvurečenskij and Riečan, 1999. Examples of product MV-algebras are the sets of continuous functions from any fixed topological space to $[0, 1]$. The Mundici equivalence between MV-algebras and ℓu-groups extends to one between product MV-algebras and ℓu-rings. Actually, as explained in Di Nola and Leuştean, 2014, Riesz MV-algebras were born as a weakening of product MV-algebras.

On the Definition of Constituent of a Function In the definition of McNaughton function, it is required that the function has a finite tuple of affine constituents. The notion of constituent can be vastly generalized to nonlinear situations as those considered.

Definition 4.7.1. *Let f be a function from a set X to a set Y. A tuple of functions $(f_1, \dots, f_m)$ is called a* constituent tuple *of f if the domain of each f_i is a subset of X and for every $x \in X$ there is i such that $f(x) = f_i(x)$.*

Definition 4.7.2. *Let A be a set of functions. f is called* piecewise-A *if it admits a tuple of constituents in A.*

Definition 4.7.3. *Let f be a function from a set X to a set Y. Let A be a set of functions. Let K be a set of subsets of X. We will say that f is* piecewise-(A, K) *if there are finitely many pairs $(f_1, K_1), \dots, (f_m, K_m)$ such that each f_i is in A, f_i is defined (at least) everywhere in K_i, and for every $x \in X$ there is i such that $x \in K_i$ and $f(x) = f_i(x)$.*

Note that by definition, every McNaughton function is piecewise-A, where A is the set of all affine functions with integer coefficients from subsets of $[0, 1]^n$ to $[0, 1]$. More precisely:

Theorem 4.7.2. *(see Cignoli, d'Ottaviano, and Mundici, 2013) In the terminology above, every McNaughton function is piecewise-(A, K), where K is the set of all rational polyhedra included in $[0, 1]^n$ (as noted by a referee, every piecewise (A, K)-function is continuous, so it is a McNaughton function).*

In other words, the theorem says that the domains of the affine constituents of a McNaughton function can always be taken to be rational polyhedra.

We will see that Theorem 4.7.2 extends to Riesz McNaughton functions, see Theorem 6.1.2.

The Marra-Spada Duality In Marra and Spada, 2012 we find a careful proof of several facts on MV-algebras which were previously considered as folklore. In particular we have:

Theorem 4.7.3. *(see Marra and Spada, 2012) There is a duality between the category of finitely generated, semisimple MV-algebras and the category of closed subsets of $[0, 1]^n$ with Z-maps as morphisms.*

Actually Marra and Spada, 2012 describes a more general adjunction for arbitrary MV-algebras, including infinitely generated and non-semisimple MV-algebras, but here we stick to the semisimple, finitely generated case for simplicity.

Ideals and homomorphisms Ideals of MV-algebras correspond to congruences of MV-algebras. Moreover, as a consequence of Di Nola and Leuştean, 2014, Remark 3, every MV-algebraic congruence in a Riesz MV-algebra is also a Riesz MV-algebraic congruence. In this sense, the "ideals" of an Riesz MV-algebra can be identified with the ideals of its MV-algebraic reduct, and the same holds for maximal ideals. We denote by R/J the quotient Riesz MV algebra given by R modulo its ideal J.

A further consequence of the above considerations on congruences is the following:

Lemma 4.7.5. *Every homomorphism between the MV-algebra reducts of two Riesz MV-algebras A and B is also a homomorphism between A and B.*

Proof. A map $f : A \to B$ is a homomorphism if and only if $ker(f) = \{(x, y)|f(x) = f(y)\}$ is an congruence. □

Now recall the Di Nola embedding theorem:

Theorem 4.7.4. *(see Di Nola, 1991, Di Nola, 1993) Ever MV-algebra embeds in a power of an ultrapower of $[0, 1]$.*

By the previous lemma and theorem, in the Riesz context we have:

Corollary 4.7.1. *Every Riesz MV-algebra embeds in a power of an ultrapower of $[0, 1]$.*

The I-V connection It is useful to adopt the following notations:

- $I(C) = \{f \in RM_n : f(c) = 0 \; for \; every \; c \in C\}$ is the annihilator ideal of $C \subseteq [0, 1]^n$;
- $V(X) = \{x \in [0, 1]^n : f(x) = 0 \; for \; every \; f \in X\}$ is the vanishing locus of the set $X \subseteq RM_n$.

Note that there is an isomorphism between $RM_n|_C$ and $RM_n/I(C)$.

Lemma 4.7.6. *Let C, D be two closed subsets of $[0, 1]^n$ such that C is not included in D. Then there is a function $f \in M_n$ which is identically zero on D but not identically zero on C.*

Proposition 4.7.1. *For every set $X \subseteq RM_n$, $V(X)$ is closed. Moreover for every closed set $C \subseteq [0, 1]^n$ we have $C = V(I(C))$.*

Proof. The first point holds because Riesz-McNaughton functions are continuous.

For the second point, $C \subseteq V(I(C))$ follows by definition of I and V. Conversely, suppose $x \notin C$. By Lemma 4.7.6 there is $f \in M_n$ such that $f = 0$ in C and $f(x) \neq 0$. Since $M_n \subseteq RM_n$, we conclude $x \notin V(I(C))$. □

We can say that a Riesz MV-algebra is *semisimple* if its MV algebra reduct is semisimple. Examples of semisimple Riesz MV-algebras are C_n and its Riesz MV-subalgebras, including RM_n.

We have the following criterion for semisimplicity:

Lemma 4.7.7. *A finitely generated Riesz MV-algebra $R = RM_n/J$ is semisimple if and only if J is an intersection of maximal ideals of RM_n.*

Proof. Suppose $R = RM_n/J$ is semisimple. Let $\pi : RM_n \to RM_n/J$ the quotient map. The maximal ideals of R are the ideals M/J where $M \in Max(RM_n)$ and $M \supseteq J$. Since R is semisimple we have

$$\bigcap_{M \in Max(RM_n), M \supseteq J} M/J = 0,$$

and by applying the inverse mapping π^{-1} we infer

$$\bigcap_{M \in Max(RM_n), M \supseteq J} M = J,$$

so J is an intersection of maximal ideals.

The converse is analogous.

□

Maximal ideals of free Riesz MV-algebras are characterized as follows:

Lemma 4.7.8. *A subset J of RM_n is a maximal ideal if and only if $J = I(c)$ for some $c \in [0,1]^n$. Moreover the map sending $c \in [0,1]^n$ to $I(c) \in Max(RM_n)$ is a homeomorphism.*

Proof. Each $I(c)$ is a maximal ideal because the quotient $RM_n/I(c)$ is isomorphic to $[0,1]$ via evaluation of functions in c, and $[0,1]$ is a simple Riesz MV-algebra (the unique one, see Di Nola and Leuştean, 2014, Corollary 1).

Conversely, let M be a maximal ideal. If $M \neq I(c)$ for every c, then for every c there is $f_c \in M$ with $f_c(c) \neq 0$, and by continuity, $f_c \neq 0$ in an open neighborhood U_c of c. By compactness there are $c_1, \ldots, c_k$ such that

$$U_{c_1} \cup \ldots \cup U_{c_k} = [0,1]^n.$$

So, the function

$$f = f_{c_1} \oplus \ldots \oplus f_{c_k}$$

belongs to M, is nonzero everywhere in $[0,1]^n$, and by compactness, f has a real minimum $m > 0$. Taking an integer $N > 1/m$, we have $N.f = 1$, so $1 \in M$, contrary to the fact that M is a proper ideal.

We omit the proof that the map is a homeomorphism. □

More generally we have:

Corollary 4.7.2. *Let $C \subseteq [0,1]^n$ be closed. There is a homeomorphism between the topological spaces C and $Max(RM_n|_C)$. The homeomorphism sends $c \in C$ to*

$$\{f \in RM_n|_C : f(c) = 0\}.$$

Putting together Lemmas 4.7.7 and 4.7.8 we have:

Corollary 4.7.3. *A finitely generated Riesz MV-algebra $R = RM_n/J$ is semisimple if and only if $J = I(V(J))$.*

The following is a kind of analogue of Hilbert's Nullstellensatz on zerosets of polynomials in algebraically closed fields, see Hilbert and Sturmfels, 1993:

Corollary 4.7.4. *For every set $J \subseteq RM_n$, $I(V(J))$ is the intersection of all maximal ideals containing J.*

Chapter 5

Functional Representations and Generalized States

If G is an ℓ_u-group, then the states of G and the ℓ_u-homomorphisms from G to $\mathbb{R}$ coincide. So, when we consider ℓ_u-homomorphisms from G to a vector lattice R, actually we deal with *generalized states* on an ℓ_u-group.

On the other hand, a state of an MV-algebra is a convex combination of MV-homomorphisms. In this case, we consider convex combinations of these MV-homomorphisms.

For this reason we need to give a more general definition of state of an MV-algebra, as proposed below.

Definition 5.0.4. *Let A be an MV-algebra and S be a Riesz MV-algebra. We say that $s : A \to S$ is a* generalized state *iff $s(1_A) = 1_S$ and $s(x) \oplus_R s(y) = s(x \oplus_A y) \oplus_R s(x \odot y)$ for every $x, y \in A$. We denote by $ST(A, S)$ the set of all generalized states from A to S.*

Analogously as in the context of the states (see also Mundici, 2011, Proposition 10.2) we have the following propositions.

Proposition 5.0.2. *Every generalized state s of an MV-algebra sarisfies the following properties.*

(a) *If $x \leq y$ then $s(x) \leq s(y)$;*

(b) $s(0_A) = 0_S$;

(c) *$s(x \oplus_A y) = s(x) \oplus_S s(y)$ whenever $x, y \in A$ and $x \odot_A y = 0_A$.*

Proposition 5.0.3. *Let $A = \Gamma(G, u_G)$ be an MV-algebra with its associated unital ℓ-group (G, u_G). Let $S = \Gamma(R, u_R)$ be a Riesz MV-algebra with its associated unital vector lattice (R, u_R). Then for every $s \in \ell_u Hom(G, R)$ the restriction of s to A is an element of $ST(A, S)$. The map $\gamma : s \mapsto s|_A$ is an affine isomorphism.*

Definition 5.0.5. *Let X be a set of functions from A to S, where A is any abstract non-empty set and $(S, \oplus_S, 0_S, \cdot_S, \neg_S)$ is a Riesz MV-algebra. We denote by $(Aff^*_S(X), \oplus, 0, \cdot, \neg)$ the set of functions from A to S such that $\underline{1} \in Aff^*_S(X)$, where $\underline{1}(a) = 1_S = \neg_S 0_S$ for all $a \in A$ and the other functions are recursively defined as follows.*

(i) *$x \in Aff^*_S(X)$ for all $x \in X$;*

(ii) *if $\alpha \in [0, 1]$ and $v \in Aff^*_S(X)$, then $\alpha \cdot v \in Aff^*_S(X)$, where $\alpha \cdot v(a) = \alpha \cdot_S v(a)$ for every $a \in A$;*

(iii) *if $v \in Aff^*_S(X)$, then $\neg v \in Aff^*_S(X)$, where $(\neg v)(a) = \neg_S v(a)$;*

(iv) *if* $v, w \in Aff_S^*(X)$*, then* $v \oplus w \in Aff_S^*(X)$*, where* $(v \oplus w)(a) = v(a) \oplus_S w(a)$.

We now give the following results, in the (Riesz) MV-algebra context.

Proposition 5.0.4. *Let* X *be a set of functions from* A *to* S*, where* A *is any abstract non-empty set and* S *is a Riesz MV-algebra. Then* $Aff_S^*(X)$ *is a Riesz MV-algebra of functions from* A *to* S.

Proposition 5.0.5. $Aff^*_{\Gamma(R)}(X) = \Gamma(Aff_R(X))$.

Theorem 5.0.5. *Let* A *be an MV-algebra,* $S = \Gamma(R, u_R)$ *be a Riesz MV-algebra, where* R *is a Dedekind complete vector lattice with order unit* u_R*. Then the following are equivalent:*

(1) A *is semisimple;*

(2) *the map* $\phi_\Gamma : A \hookrightarrow Aff_S^*(ST(A, S))$ *defined by* $\phi_\Gamma(a) = \hat{a}$*, where* $\hat{a}(\nu) = \nu(a)$*,* $a \in A$ *and* $\nu \in ST(A, S)$*, is an injective MV-homomorphism;*

(3) *the map* $\psi_\Gamma : A \hookrightarrow \mathcal{C}_R(Ext(ST(A, S)))$*, defined by* $\psi_\Gamma(a) = \hat{a}$*, where* $\hat{a}(\nu) = \nu(a)$*,* $a \in A$ *and* $\nu \in Ext(ST(A, S))$*, is an injective MV-homomorphism.*

We know that $S(A) = Conv(Hom_{MV}(A, [0, 1]))$. Define $Aff^*(X) = Aff^*_{[0,1]}(X)$, where $[0, 1]$ is the standard Riesz MV-algebra. In $Aff^*(X)$, for all $y \in Y$ we get $\underline{1}(y) = 1$, $\oplus$ is the sum truncated to 0 and 1, $\cdot$ is the scalar multiplication and $\neg v = 1 - v$. So we have the following corollary, which provides a representation in the space of affine functions on the set of states of A.

Corollary 5.0.5. *Let* A *be a semisimple MV-algebra. Then the application* $\phi^* : A \hookrightarrow Aff^*(S(A))$ *defined by* $\phi(a) = \hat{a}$ *where* $\hat{a}(h) = h(a)$*,* $a \in A$*, is an injective MV-homomorphism.*

Chapter 6

Non-Linear Functional Representation and Interpretation

6.1 A Marra-Spada Duality for Semisimple Riesz MV-algebras

We wish to define a Marra-Spada-like duality between the category of finitely generated, semisimple Riesz MV-algebras and the category of closed subsets of $[0,1]^n$ with suitable morphisms. In order to define these morphisms, we have to replace Z-maps with *R-maps*, which are tuples of Riesz-McNaughton functions, rather than tuples of McNaughton functions. Likewise, Z-homeomorphisms must be replaced by *R-homeomorphisms*, which are invertible R-maps.

In analogy with Theorem 4.7.3 we have:

Theorem 6.1.1. *There is a duality RMS (for Riesz-Marra-Spada) between the category of finitely generated, semisimple Riesz MV-algebras and the category of closed subsets of $[0,1]^n$ with R-maps.*

This duality is a pair of functors, but we feel free to call RMS both functors. Rather than giving a full proof of Theorem 6.1.1, we limit ourselves to defining RMS on objects and morphisms, and we observe that the proof of Marra and Spada, 2012 for the MV algebra case goes through. On objects, the duality is as follows.

Given a semisimple Riesz MV-algebra R with n generators, we have $R = RM_n/J$ where J is an ideal of RM_n, and we associate to R the vanishing set $V(J)$, which is a closed subset of $[0,1]^n$.

Conversely, given a closed set $C \subseteq [0,1]^n$, it is natural to associate to C the Riesz MV-algebra of Riesz-McNaughton functions restricted to C, which we denote by $RM_n|_C$. Note that the latter MV-algebra is semisimple.

On morphisms, we extend the duality as follows.

Consider an MV algebra morphism h from a Riesz MV-algebra $A = RM_n/J$ to a Riesz MV-algebra $B = RM_m/K$. Choose $f_i \in h(\pi_i/J)$, for $i = 1, \dots, n$. Then $RMS(h)$ sends $c \in V(K)$ to the tuple $(f_1(c), \dots, f_n(c))$. It results that $RMS(h)$ is a well defined R-map from $V(K)$ to $V(J)$.

Conversely, given an R- map g from a closed set $C \subseteq [0,1]^n$ to a closed set $D \subseteq [0,1]^m$, we define $RMS(g)$ as the function from $RM_n|_D$ to $RM_n|_C$ given by composition with g.

Lemma 6.1.1. *Let H be a m-tuple of functions in C_n. The Riesz MV- subalgebra generated by H is isomorphic to $RM_m|_{Range(H)}$.*

Proof. The map ϕ sending $f \in RM_m$ to $f \circ H$ is a surjective homomorphism from RM_m to the Riesz MV-subalgebra generated by H, and we have $\phi(f) = \phi(g)$ if and only if $f = g$ on the range of H. So, ϕ induces a bijection between the subalgebra generated by H and the Riesz MV-algebra of Riesz-McNaughton functions in m variables restricted to the range of H. □

From the lemma and the Marra-Spada duality other similar results can be derived, for instance:

Lemma 6.1.2. *Let H be an m-tuple in C_n and let K be an m'-tuple in $C_{n'}$. The Riesz MV-subalgebras generated by H and K are isomorphic if and only if their ranges are R-homeomorphic.*

Lemma 6.1.3. *Let $C \subseteq [0,1]^m, D \subseteq [0,1]^n$ be two closed sets. Then $RM_m|_C$ embeds in $RM_n|_D$ if and only if there is a surjective R-map from D to C.*

In the next lemma, we say that an R-map $f : C \to D$ is *left invertible* if there is an R map $g : D \to C$ such that $x = g(f(x))$ for every $x \in C$.

Lemma 6.1.4. *Let $A = RM_n/J$, $B = RM_m/K$ be two finitely generated, semisimple Riesz algebras. Then there is a surjection from A to B if and only if there is a left invertible R-map from $V(K)$ to $V(J)$.*

We find it interesting to notice:

Proposition 6.1.1. *Given a semisimple MV-algebra $A = M_n/J$, let $R(A) = RM_n|_{V(J)}$.*

Then $R(A)$ is a semisimple Riesz MV-algebra, and $Max(A)$ and $Max(R(A))$ are canonically homeomorphic (hence, by Corollary 4.7.2, they are canonically homeomorphic to $V(J)$ with its usual Euclidean topology inherited from $[0,1]^n$).

The definition of $R(A)$ above gives also another simple construction of the Riesz hull of a semisimple MV-algebra A defined and constructed in Diaconescu and Leuştean, 2015.

In fact, first A is isomorphic to $M_n|_{V(J)}$. Moreover, by definition, the Riesz hull of an MV-algebra A is a Riesz MV-algebra where A embeds and which is generated by A as a Riesz MV-algebra. Now, A embeds into $R(A)$ because every McNaughton function is a Riesz-McNaughton function. Moreover, the n projections generate $R(A)$ as a Riesz MV-algebra, and the projections belong to A, hence A generates $R(A)$ as a Riesz MV-algebra.

In the Riesz setting, Theorem 4.7.2 becomes:

Theorem 6.1.2. *Every Riesz-McNaughton function is piecewise-(A, K), where A is the set of affine functions with real coefficients, and K is the set of all polyhedra included in $[0,1]^n$.*

In other words, the theorem says that the domains of the affine constituents of a McNaughton function can always be taken to be polyhedra.

Proof. Let $f \in RM_n$. The proof goes by induction on the shortest Riesz MV polynomial p which defines f.

If p is a projection x_i then p is affine on the whole cube.

If $p = \neg q$ or $p = cq$ with $c \in [0,1]$ the statement follows from the inductive hypothesis.

Consider $p = q \oplus r$. Then q and r are piecewise (A, K). So there is a finite set of polyhedra $\{\gamma_i\}_{i \in I}$ which cover the cube, where both q and r are affine. So, $q + r$ is also affine in γ_i; hence, both

$$\delta_i = \{x \in \gamma_i : q + r \leq 1\}$$

and

$$\eta_i = \{x \in \gamma_i : q + r \geq 1\}$$

are polyhedra. Moreover $q \oplus r = q + r$ in δ_i and $q \oplus r = 1$ in η_i. So, $p = q \oplus r$ is affine in δ_i and η_i, and p is affine on the finite set of polyhedra $\{\delta_i\}_{i \in I} \cup \{\eta_i\}_{i \in I}$. □

Corollary 6.1.1. *Every zeroset of a Riesz-McNaughton function is a polyhedron.*

Proof. Let f be a Riesz-McNaughton function. By the previous theorem, there are polyhedra $P_1, \ldots, P_k$ which cover the cube and where f is affine. But the zeroset of an affine function on each P_i is a polyhedron, and taking the union for $i = 1, \ldots, k$, we conclude that the zero set of f is a polyhedron. □

We have also the converse:

Lemma 6.1.5. *Every polyhedron included in $[0, 1]^n$ is the zeroset of a Riesz-McNaughton function.*

Proof. Let $P \subseteq [0, 1]^n$ be a polyhedron. We can suppose that P is a simplex of dimension n. Let us take a finite set F of simplexes of dimension at most n, such that:

- P is an element of F,
- every face of an element of F is in F,
- the union of F is $[0, 1]^n$, and
- the intersection of any two elements of F either is empty or is a face of both.

For every $\sigma \in F$, let σ_0 be the set of all vertices of σ which belong to P, and σ_1 be the other vertices of σ. There is a unique affine function f_σ from σ to $[0, 1]$ which sends σ_0 to 0 and σ_1 to 1. In fact, let $\sigma_0 = \{v_0, \ldots, v_m\}$ and $\sigma_1 = \{v_{m+1}, \ldots, v_s\}$. Let $f_\sigma(v_i) = 0$ for $i = 0, \ldots, m$ and $f_\sigma(v_i) = 1$ for $i = m + 1, \ldots, s$. Now extend f_σ to σ as follows: if

$$x = \lambda_0 v_0 + \ldots + \lambda_s v_s,$$

where $0 \leq \lambda_i \leq 1$ and $\Sigma_i \lambda_i = 1$, then we let

$$f_\sigma(x) = \lambda_0 f_\sigma(v_0) + \ldots + \lambda_s f_\sigma(v_s).$$

Moreover for every $\sigma, \tau \in F$, we have $f_\sigma(x) = f_\tau(x)$ for every $x \in \sigma \cap \tau$. So the partial functions f_σ extend to a unique, continuous, piecewise affine function $f : [0, 1]^n \to [0, 1]$ which is zero on P and nonzero on $[0, 1]^n \setminus P$. □

Summing up we have:

Theorem 6.1.3. *The zerosets of Riesz-McNaughton functions coincide with the polyhedra included in $[0, 1]^n$.*

6.1.1 Finitely Presented Case

Recall that a *finitely presented* Riesz MV-algebra is one of the form RM_n/J, where J is a finitely generated ideal (recall that in MV algebras, finitely generated ideals are principal).

First of all we give the Riesz MV-algebra analogous of Wojcicki Theorem (for the latter see Marra and Spada, 2012):

Lemma 6.1.6. *Every principal ideal of RM_n is an intersection of maximal ideals.*

Proof. Let $f \in RM_n$. It is enough to show $id(f) = I(V(f))$.

Clearly $f \in I(V(f))$ so $id(f) \subseteq I(V(f))$.

Conversely, let $g \in I(V(f))$. By definition of I and V, every zero of f is also a zero of g. Now, by Theorem 6.1.2, f and g are piecewise affine, and the pieces are polyhedra. Consider a triangulation T of $[0,1]^n$ into finitely many polyhedra such that in every element of T, both f and g are affine. Let V be the set of all vertices of the elements of T. Note that V is finite. Let N be an integer sufficiently large to ensure $g(v)/f(v) \leq N$ for every $v \in V$ such that $f(v) \neq 0$. Then $g(v) \leq Nf(v)$ for every $v \in V$. So, for every polyhedron $P \in T$, we have $g(v) \leq Nf(v)$ for every vertex v of P, and since f, g are affine in P, we conclude $g \leq Nf$ in P, and taking the union over $P \in T$, we have $g \leq Nf$ on the whole $[0,1]^n$. So, $g \leq N.f$ and $g \in id(f)$. □

Now, in analogy with Marra and Spada, 2012 we observe:

Corollary 6.1.2. *Every finitely presented Riesz MV-algebra is semisimple.*

Proof. This follows from Lemma 4.7.7 and the previous lemma. □

The previous results allow us to specialize the duality as follows:

Theorem 6.1.4. *The duality RMS specializes to a duality between polyhedra included in $[0,1]^n$ and finitely presented Riesz MV-algebras.*

Proof. If $C \subseteq [0,1]^n$ is a polyhedron, then by Theorem 6.1.3 we have $C = V(f)$ for some $f \in M_n$, hence $C = V(J)$ where J is the ideal generated by f. Then $RMS(C) = RM_n|_C$ is finitely presented because it is isomorphic to RM_n/J and J is principal.

Conversely, if $A = RM_n/J$ is finitely presented, and J is an ideal generated by a function $f \in RM_n$, then $V(J) = V(f)$ is a polyhedron again by Theorem 6.1.3. □

Likewise, in the MV-algebra case, the duality of Theorem 4.7.3 specializes to a duality between rational polyhedra and finitely presented MV-algebras, see Marra and Spada, 2012.

6.1.2 Examples of Riesz MV-algebras

Before going into further technicalities, let us consider some examples.

Consider the function $h(x) = x^2$ seen as a function from $[0,1]$ to $[0,1]$. Clearly, $h(x)$ is not an element of RM_1, because, for instance, its second derivative is nonzero everywhere. So, $h(x)$ does not generate RM_1 as a Riesz MV subalgebra of C_1. However, since $h(x)$ is a homeomorphism of $[0,1]$, $h(x)$ generates a copy of RM_1 in C_1. This copy consists exactly of all continuous piecewise Aff_h- functions, where Aff_h is the set of all compositions $l \circ h$ where l is an affine function with real coefficients. Since $h(x) = x^2$ is a quadratic polynomial, the MV algebra generated by h consists of piecewise quadratic functions.

Likewise, a continuum of examples can be obtained by taking $h(x) = x^\alpha$, where α is any positive real number. So we obtain:

Theorem 6.1.5. *C_1 contains a continuum of copies of RM_1.*

When α is an integer, $h(x)$ generates an MV-algebra of piecewise polynomial functions (isomorphic to RM_1).

Other examples are the spline functions. Usually spline functions are piecewise polynomial functions where a certain degree of regularity. If we limit ourselves to require continuity, then we have sets of continuous, piecewise polynomial functions of any fixed degree which have the structure of a Riesz MV-algebra.

By contrast, note that regular splines do not form a Riesz MV-algebra (neither an MV-algebra). For instance, the functions x^2 and $(1-x)^2$ are regular (i.e. C^∞) splines, but $x^2 \wedge (1-x)^2$ has a singularity in $x = 1/2$.

Another example is the logistic function. Usually a logistic function has the form $f(x) = L/1 + e^{-k(x-x_0)}$ and has the real line as a domain. If we insist that the function (restricted to $[0,1]$) must belong to C_1, then suitable values of L, k, x_0 must be chosen. If $f \in C_1$, then $Range(f)$ will be a subsegment of $[0,1]$, which is (in our terminology) R-homeomorphic to $[0,1]$, so f generates a copy of RM_1.

Pulmannova Pulmannová, 2013 shows that every semisimple MV-algebra embeds into the MV-algebra of multiplication operators between 0 and 1 on the space of L^2 functions on a compact set. We note that multiplication operators are closed under multiplication by any real $c \in [0,1]$, so they form a Riesz MV-algebra. Since every MV-algebra morphism between two Riesz MV-algebras is a Riesz MV-algebra morphism, every semisimple Riesz MV-algebra embeds into a Riesz MV-algebra of multiplication operators of an L^2 space.

In Di Nola, Gerla, and Leustean, 2013, Riesz MV-algebras are applied to neural networks; in fact, multilayer perceptrons can be modeled with certain functions of C_n; and conversely, every Riesz-McNaughton function can be associated to a neural network.

6.2 Riesz MV-subalgebras

In the examples we have seen that C_1 contains continuum many copies of RM_1. More generally:

Proposition 6.2.1. *Let $h \in C_1$ be any nonconstant map. Then h generates a copy of RM_1.*

Proof. This follows from Lemma 6.1.1 by taking $n = 1$ and $H = h$ since $Range(h)$ is a segment of $[0,1]$ which is R-homeomorphic to $[0,1]$. □

Of course, every constant function generates a Riesz MV-algebra isomorphic to $[0,1]$ which cannot contain any copy of RM_1 (e.g. because $[0,1]$ is totally ordered, whereas RM_1 is not totally ordered).

We have seen that C_1 contains continuously many copies of RM_1. In fact it is enough to consider the Riesz MV-algebras generated by x^α with $\alpha \in [0,1]$. Likewise in n dimensions we can consider the Riesz MV-algebras generated by the n-tuples $(x_1^\alpha, \dots, x_n^\alpha)$ and we obtain:

Corollary 6.2.1. *C_n contains continuously many copies of RM_n.*

Definition 6.2.1. *Let C be a closed subset of $[0,1]^m$.*

*We say that C is Rn-*fat *if there is a R-map F such that $F(C)$ is included in $[0,1]^n$ and contains a nonempty open subset of $[0,1]^n$.*

*We say that C is Rn-*slim *if C is not Rn-fat.*

Lemma 6.2.1. *A closed subset C of $[0,1]^m$ is Rn-fat if and only if there is a surjective R-map from C to $[0,1]^n$.*

Proof. If the R-map from C to $[0,1]^n$ exists, then clearly, C is Rn-fat. Conversely, suppose F is an R-map and $F(C)$ has nonempty interior. in $[0,1]^n$. Then $F(C)$ contains a product of n rational intervals $[a_1,b_1] \times \ldots \times [a_n,b_n]$. Let g_i be a McNaughton function such that $g_i(a_i) = 0$ and $g_i(b_i) = 1$. Let $G = (g_1, \dots, g_n)$. Then $(G \circ F)|_C$ is a surjective R-map from C to $[0,1]^n$. □

Lemma 6.2.2.

- *The union of two Rn-slim closed subsets of $[0,1]^m$ is Rn-slim;*
- *the image of an Rn-slim closed subset of $[0,1]^m$ under a R-map is Rn-slim;*
- *if $m < n$, then $[0,1]^m$ is Rn-slim.*

Proof. For the first point, let C, D be two Rn-slim closed subsets. Suppose by contradiction $C \cup D$ is Rn-fat. Then there is a R-map F such that $F(C \cup D)$ contains an open subset O of $[0,1]^n$. Note $F(C \cup D) = F(C) \cup F(D)$. Hence we have $O \subseteq F(C) \cup F(D)$. Since $F(C)$ is closed, $O \setminus F(C)$ is an open subset of $[0,1]^n$, and it is nonempty, otherwise O would be included in $F(D)$ and D would be Rn-fat; so C is Rn-fat, contrary to the Rn-slimness of C. So $C \cup D$ is Rn-slim.

For the second point, let C be closed in $[0,1]^m$ and Rn-slim. Let F be a R-map. Let $D = F(C)$. Suppose for an absurdity that D is Rn-fat. Then there is a R-map F' such that $F'(D)$ contains an open in $[0,1]^n$. So, the image of C under the R-map $F' \circ F$ contains an open in $[0,1]^n$, contrary to the slimness of C. So, D is also Rn-slim.

For the third point, suppose for an absurdity that $[0,1]^m$ is Rn-fat. Then there is a R-map F such that $F([0,1]^m)$ has nonempty interior in $[0,1]^n$. Taking affine constituents of F, we have a tuple G of affine functions such that $G([0,1]^m)$ has nonempty interior in $[0,1]^n$. Since $m < n$, this is impossible by elementary linear algebra considerations. □

For MV-algebras we have the following:

Theorem 6.2.1. *An n-tuple of functions of C_n, say $H = (h_1, \dots, h_n)$, generates a copy of M_n if and only if the function H from $[0,1]^n$ to itself is surjective.*

By Proposition 6.2.1, the analogous of this theorem for Riesz MV algebras is *false*.

However, the implication from right to left still holds:

Proposition 6.2.2. *Let $H = (h_1, \ldots, h_n)$ be a n-tuple of elements of C_n that gives a surjective map from $[0,1]^n$ to $[0,1]^n$. Then H generates a copy of RM_n.*

Proof. Suppose H is surjective. Then $Range(H) = [0,1]^n = Range(\pi_1, \ldots, \pi_n)$, where π_i are the projections from $[0,1]^n$ to $[0,1]$. By Lemma 6.1.2, $RM_n|_{Range(H)}$ is isomorphic to RM_n, so the Riesz MV-algebra generated by H is isomorphic to RM_n. □

On the other hand, consider $n = 1$ and the function $h(x) = 1/2x$ from $[0,1]$ to $[0,1]$. The range of h is $[0,1/2]$ which is R-homeomorphic to $[0,1]$ (via the pair of R-maps $(1/2x, 2.x)$). Hence, by Lemma 6.1.2, $RM_1|_{Range(h)}$ is isomorphic to RM_1, despite $h : [0,1] \to [0,1]$ is not surjective.

The same argument gives an interesting structural difference between RM_n and M_n which we describe now.

Recall that an algebraic structure is called *Hopfian* if every surjective endomorphism is an automorphism. Hopfianity is an interesting algebraic generalization of finiteness. There is a celebrated theorem by Malcev to the effect that every residually finite, finitely generated algebra in any variety is Hopfian, see Evans, 1969.

Now we continue with the following lemma of universal algebra, for which we acknowledge professor B. Steinberg:

Lemma 6.2.3. *Let V be a variety with finitary operations generated by finite algebras. Let F a free finitely generated object of V. Then F is Hopfian. Moreover, let X be a minimal cardinality generating set of F. Then X is a free basis of F.*

Proof. Since V is generated by finite algebras, the relatively free algebras in V are residually finite (the homomorphisms into the finite algebras generating V separate points). Any finitely generated, residually finite universal algebra (with finitary operations) is Hopfian by a theorem of Malcev (see Evans, 1969). So F is Hopfian.

Now suppose X is a minimal cardinality finite generating set for F. Let Y be a free basis. It must have at least as many elements as X so we can choose an onto map from Y to X. This must extend to a surjective endomorphism from F to F, which must be an automorphism since F is Hopfian. But then our onto map from Y to X is 1 to 1, so X is a free basis. □

Note that the variety of MV-algebras is generated by finite algebras, so the proof of the previous lemma implies the following theorem.

Theorem 6.2.2. *M_n is Hopfian for every integer n.*

However we prove:

Theorem 6.2.3. *RM_n is* not *Hopfian.*

Proof. Consider for simplicity $n = 1$. Since $[0,1/2]$ is R-homeomorphic to $[0,1]$, we have that $RM_1|_{[0,1/2]}$ is isomorphic to RM_1. The former Riesz MV algebra is isomorphic to $RM_1/I([0,1/2])$, so there is an isomorphism

$$\iota : RM_1/I([0,1/2]) \to RM_1.$$

Let

$$\pi : RM_1 \to RM_1/I([0,1/2])$$

be the quotient map. Consider

$$\sigma = \iota \circ \pi : RM_1 \to RM_1.$$

Then σ a surjective endomorphism σ of RM_1 whose kernel is $I([0, 1/2])$, which is not the zero ideal (for instance, it contains the function $x \odot x$). So, σ is not an automorphism. □

We have the following category theoretic theorem.

Theorem 6.2.4. *Consider the map ρ sending the Riesz MV-algebra generated by an m-tuple H of functions in C_n to the range of H.*

Then ρ is well defined up to R-homeomorphism.

Moreover, ρ can be extended to a duality between the following subcategories of finitely generated Riesz MV- subalgebras of C_n (with Riesz MV-algebra homomorphisms as morphisms) and closed subsets of $[0, 1]^n$ up to R-homeomorphism (with R-maps as morphisms), respectively:

1. *the copies of RM_k and the sets R-homeomorphic to $[0, 1]^k$;*
2. *the Riesz MV-algebras containing a copy of RM_k and the Rk-fat sets;*
3. *the Riesz MV-algebras embeddable in RM_k and the sets S such that there is a surjective R-map from $[0, 1]^k$ to S;*
4. *the homomorphic images of RM_k and the sets S such that there is a left invertible R-map from S to $[0, 1]^k$.*

Here, m, n, k are arbitrary positive integers.

Proof. The Riesz MV-algebra generated by H is isomorphic to $RM_m|_{Range(H)}$. Hence, if H and K generate the same algebra, then $RM_m|_{Range(H)}$ is isomorphic to $RM_m|_{Range(K)}$, and by Lemma 6.1.2, $Range(H)$ and $Range(K)$ are R-homeomorphic. This proves that ρ is well defined up to R-homeomorphism.

Since the maximal space of $RM_m|_{Range(H)}$ is $Range(H)$ and the maximal space of RM_k is $[0, 1]^k$, the first point follows from Lemma 6.1.2.

By Lemma 6.1.3, RM_k embeds in $RM_m|_{Range(H)}$ if and only if there is a surjective R-map from $Range(H)$ to $[0, 1]^k$, that is, $Range(H)$ is Rk-fat. This proves the second point.

The third point again follows from Lemma 6.1.3, and similarly, the fourth point follows from Lemma 6.1.4. □

Proposition 6.2.3. *If $m < n$, then* no *m-tuple of functions of C_n can generate a Riesz MV-algebra containing a copy of RM_n.*

Proof. Let A be a Riesz MV-algebra generated by m functions $f_1, \dots, f_m$. Then the range of $(f_1, \dots, f_m)$ is Rn-slim, and also the range of any tuple of elements of A is Rn-slim by Lemma 6.2.2. Suppose there is an isomorphism ϕ from RM_n to a Riesz MV-subalgebra of A. Let $l_i = \phi(\pi_i)$. Then the range of $(l_1, \dots, l_n)$ is Rn-slim whereas the range of $(\pi_1, \dots, \pi_n)$ is Rn-fat. So the range of $(\pi_1, \dots, \pi_n)$ is not contained in the range of $(l_1, \dots, l_n)$. By Lemma 4.7.6 there is a function $f \in RM_n$ such that $f \circ (l_1, \dots, l_n)$ is identically zero but $f \circ (\pi_1, \dots, \pi_n)$ is not identically zero. So ϕ cannot exist. □

Corollary 6.2.2. *For every $m < n$, RM_m does not contain any isomorphic copy of RM_n.*

Proof. This is because for $m < n$, every n-tuple in RM_m has an Rn-slim image. □

On the other hand:

Proposition 6.2.4. *C_n contains a copy of RM_m for* every *m, n.*

Proof. We know that C_n contains copies of M_m. Now the Riesz MV-algebra generated by a copy of M_m in the Riesz MV-algebra C_n is a Riesz MV-algebra isomorphic to RM_m. □

The construction above provides a *canonical* copy of RM_m in C_n for *every* m, n. For instance, consider $m = 2$ and $n = 1$. Let S be the continuous surjective function from $[0, 1]$ to $[0, 1]^2$ given in . Write $S = (S_1, S_2)$. Then S_1 and S_2 generate a copy of RM_2 in C_1.

6.3 A Categorial Theorem

Lemma 6.3.1. *(see Mundici, 2011) The image of a rational polyhedron P under a definable map F is a rational polyhedron.*

Lemma 6.3.2. *Let $C \subseteq [0, 1]^m, D \subseteq [0, 1]^n$ be two closed sets. Then $M_m|_C$ embeds in $M_n|_D$ if and only if there is a surjective definable map from D to C.*

Proof. Let F be a definable map from D onto C. Then the function from f to $f \circ F$ is an injective homomorphism from $M_m|_C$ to $M_n|_D$.

Conversely, suppose that $M_m|_C$ embeds in $M_n|_D$. Call j the embedding.

Let us consider the definable map g from D to C given simply by the counterimage map j^{-1} between the maximal spaces of the two MV-algebras. This map is surjective. In fact, let I be a maximal ideal of $M_m|_C$. Since j is injective, $j(I)$ is a proper ideal of $M_n|_D$. By Zorn Lemma there is a maximal ideal M in $M_n|_D$ such that $j(I) \subseteq M$. Then $I \subseteq j^{-1}(M)$ and, since I is maximal, $I = j^{-1}(M)$. So, g is a surjective definable map from D to C. □

Lemma 6.3.3. *Let A, B be two finitely generated, semisimple MV-algebras. Then there is a surjection from A to B if and only if there is a definable homeomorphism from $Max(B)$ to a subset of $Max(A)$.*

Proof. Suppose that A and B are semisimple, A is generated by n elements and B is generated by m elements. Then A is isomorphic to $M_n|_{Max(A)}$ and B is isomorphic to $M_m|_{Max(B)}$.

Suppose there is a surjection from A to B. Then by Mundici, 2011, Lemma 3.12 there is a definable homeomorphism from $Max(B)$ to a subset of $Max(A)$.

Conversely, suppose that j is a definable homeomorphism from $Max(B)$ to a subset of $Max(A)$. Then B is isomorphic to $M_n|_{j(Max(B))}$. Consider the map s sending $f \in M_n|_{Max(A)}$ to $f|_{j(Max(B))} \in M_n|_{j(Max(B))}$. Every function $g \in M_n|_{j(Max(B))}$ is a definable map, so it can be extended to a definable map on $Max(A)$. This means that the map s is surjective. So there is a surjection from A to B. □

Lemma 6.3.4. *(see Marra and Spada, 2012) Let C be a closed subset of $[0,1]^m$ and let D be a closed subset of $[0,1]^{m'}$. $M_m|_C$ is isomorphic to $M_{m'}|_D$ if and only if C and D are definably homeomorphic.*

Lemma 6.3.5. *Let H be an m-tuple in C_n and let K be an m'-tuple in $C_{n'}$. The subalgebras generated by $\tilde{H}$ and $\tilde{K}$ are isomorphic if and only if their ranges are definably homeomorphic.*

Proof. Let C be a closed subset of $[0,1]^m$ and let K be a closed subset of $[0,1]^{m'}$. By Lemma 6.3.4, $M_m|_C$ is isomorphic to $M_{m'}|_D$ if and only if C and D are definably homeomorphic.

Then $M_m|_{Range(H)}$ is isomorphic to $M_{m'}|_{Range(K)}$ if and only if the two ranges are definably homeomorphic. So the algebras generated by H and K are isomorphic if and only if the ranges are definably homeomorphic. □

In particular, if H, K are two m-tuples in C_n with the same range, then the subalgebras generated by $\tilde{H}$ and $\tilde{K}$ are isomorphic, so these subalgebras share every property invariant under MV-algebra isomorphism. Note however that H and K could have very different geometric properties, despite having the same range. For instance, H could be differentiable and K could not.

Theorem 6.3.1. *Consider the map ρ sending the MV-algebra generated by an m-tuple H of functions in C_n to the range of H. Then ρ is well defined up to definable homeomorphism. Moreover, ρ can be extended to a functorial duality between the following subcategories of finitely generated MV-subalgebras of C_n (with MV-algebra homomorphisms as morphisms) and closed subsets of $[0,1]^n$ up to definable homeomorphism (with definable maps as morphisms), respectively:*

1. *the copies of M_k and the sets definably homeomorphic to $[0,1]^k$;*

2. *the MV-algebras containing a copy of M_k and the k-fat sets;*

3. *the MV-algebras embeddable in M_k and the sets S such that there is a surjective definable map from $[0,1]^k$ to S;*

4. *the homomorphic images of M_k and the sets S such that there is an injective definable map from S to $[0,1]^k$;*

5. *the finitely presented MV-algebras and the rational polyhedra;*

6. *the projective MV-algebras and the Z-retracts of $[0,1]^h$ for some h (for the definition of Z-retract see Mundici, 2011).*

Proof. Since the maximal space of $M_m|_{Range(H)}$ is $Range(H)$ and the maximal space of M_k is $[0,1]^k$, the first point follows from Lemma 6.3.5.

By Lemma 6.3.2, M_k embeds in $M_m|_{Range(H)}$ if and only if there is a surjective definable map from $Range(H)$ to $[0,1]^k$, that is, $Range(H)$ is k-fat. This proves the second point.

The third point again follows from Lemma 6.3.2, and similarly, the fourth point follows from Lemma 6.3.3.

For the fifth point, if H generates a finitely presented subalgebra A of C_n then, by Mundici, 2011, A is isomorphic to the restriction of M_m to a rational polyhedron P. But A is also isomorphic to the restriction of M_m to the range of H. The range of H is definably homeomorphic to P, and by Lemma 6.3.1, the range of H is itself a rational polyhedron. The converse is analogous.

For the last point, if H generates a projective subalgebra A of C_n, by Cabrer and Mundici, 2009, A is isomorphic to the restriction of M_m to a Z-retract P of $[0,1]^k$ for some k. But A is also isomorphic to the restriction of M_m to the range of H. The range of H is definably homeomorphic to P, so the range of H is itself a Z-retract of $[0,1]^k$. The converse is analogous. □

Part III

Applications

Chapter 7

Social Preferences

Preliminaries

We will use $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ to indicate, respectively, the set of natural, integer and real numbers. We will indicate with $<$ and $\leq$ the usual (strict and non-strict) orders and $\preceq$ will be the order of the considering example and it will be defined in each context.

7.0.1 Riesz Spaces

Definition 7.0.1. *A structure* $\mathcal{R} = (R, +, \cdot, \bar{0}, \preceq)$ *is a Riesz space (or a vector lattice) if and only if:*

- $\mathcal{R} = (R, +, \cdot, \bar{0})$ *is a vector space over the field* $\mathbb{R}$*;*
- $(R, \preceq)$ *is a lattice;*
- $\forall a, b, c \in R$ *if* $a \preceq b$ *then* $a + c \preceq b + c$*;*
- $\forall \lambda \in \mathbb{R}^+$ *if* $a \preceq b$ *then* $\lambda \cdot a \preceq \lambda \cdot b$*.*

A Riesz space $(R, +, \cdot, \bar{0}, \preceq)$ is said to be *archimedean* iff for every $x, y \in R$ with $n \cdot x \preceq y$ for every $n \in \mathbb{N}$ we have $x \preceq \bar{0}$. A Riesz space $(R, +, \cdot, \bar{0}, \preceq)$ is said to be linearly ordered iff $(R, \preceq)$ is totally ordered. We will denote by R^+ the subset of positive elements of R Riesz space (the *positive cone*), i.e. $R^+ = \{a \in R \mid \bar{0} \preceq a\}$. We say that u is a *strong unit* of R iff for every $a \in R$ there is a positive integer n with $|a| \leq n \cdot u$, where $|a| = (a) \vee (-a)$.

Examples:

1. An example of non-linearly ordered Riesz space is the vector space $\mathbb{R}^n$ equipped with the order $\preceq$ such that $(a_1, \ldots, a_n) \preceq (b_1, \ldots, b_n)$ if and only if $a_i \leq b_i$ for all $i = 1, \ldots, n$; it is also possible to consider $(1, \ldots, 1)$ as strong unit.

2. A non-archimedean example is $\mathbb{R} \times_{LEX} \mathbb{R}$ with the lexicographical order, i.e. $(a_1, a_2) \preceq (b_1, b_2)$ if and only if $a_1 < b_1$ or ($a_1 = b_1$ and $a_2 \leq b_2$); in this case $(1, 0)$ is a strong unit.

3. $(\mathbb{R}, +, \cdot, 0, \leq)$, which is the only (up to isomorphism) archimedean linearly ordered Riesz space, as showed in Labuschagne and Van Alten, 2007; obviously 1 can be seen as the standard strong unit.

4. $(\mathbb{R}^C, +, \cdot, \mathbf{0}, \preceq)$ the space of (not necessarily continuous) functions from C compact subset of $\mathbb{R}$, e.g. the closed interval $[0, 1]$, to $\mathbb{R}$, such that

for every $f, g \in \mathbb{R}^C$ and $\alpha \in \mathbb{R}$ we have $(f+g)(x) = f(x) + g(x)$, $(\alpha \cdot f)(x) = \alpha f(x)$, $f \preceq g \Leftrightarrow f(x) \leq g(x)\ \forall x \in C$ and $\mathbf{0}$ is the zero-constant function; if we consider continuous functions the one-constant function $\mathbf{1}$ is a strong unit.

5. $(M_n(R), +, \cdot, 0_{n\times n}, \preceq)$ the space of $n \times n$ matrices over R Riesz space with component-wise operations and order as in example (1).

Definition 7.0.2. *A cone in R^n is a subset K of R^n which is invariant under multiplication by positive scalars. A polyhedral cone is convex if it is obtained by finite intersections of half-spaces.*

Cones play a crucial role in Riesz spaces theory, as showed in Aliprantis and Tourky, 2007 with also some applications (e.g. to linear programming Aliprantis and Tourky, 2007, Corollary 3.43). Another remarkable example of this fruitful tool is the well-known Baker-Beynon duality (see Beynon, 1975), which shows that the category of finitely presented Riesz spaces is dually equivalent to the category of (polyhedral) cones in some Euclidean space. Analogously to Euclidean spaces, in R^n (with R generic Riesz space) we can consider *orthants*, i.e. a subset of R^n defined by constraining each Cartesian coordinate to be $x_i \preceq \bar{0}$ or $x_i \succeq \bar{0}$. Here we introduce the definition of *TP-cones*, which will be useful in the sequel.

Definition 7.0.3. *Let us consider L cone. We say that L is a TP-cone if it is the empty-set, or an orthant or an intersection of them.*

7.0.2 Pairwise Comparison Matrices

Let $N = \{1, 2, ..., n\}$ be a set of alternatives. Pairwise comparison matrices (PCMs) are one of the way in which we can express preferences. A PCM has the form:

$$X = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & & x_{2n} \\ \vdots & & \ddots & \vdots \\ x_{n1} & \dots & \dots & x_{nn} \end{pmatrix}. \tag{7.1}$$

The generic element x_{ij} express a vis-à-vis comparison, the intensity of the preference of the element i compared with j. The request is that from these matrices we can deduce a vector which represents preferences; more in general we want to provide an order $\lesssim_X$. In literature there are many formalizations and definitions of PCMs, e.g. preference ratios, additive and fuzzy approaches. In Cavallo and D'Apuzzo, 2009 authors introduce PCMs over abelian linearly ordered group, showing that all these approaches use the same algebraic structure. A forthcoming paper provides a more general framework, archimedean linearly ordered Riesz spaces to deal with *aggregation* of PCMs. We want to go beyond the archimedean property and the linear order. Using different Riesz spaces with various characteristics it is possible to describe and solve a plethora of concrete issues.

PCMs are used in the Analytic Hierarchy Process (AHP) introduced by Saaty in Saaty, 1977; it is successfully applied to many Multi-Criteria Decision Making (MCDM) problems, such as facility location planning, marketing, energetic and environmental requalification and many others (see

Badri, 1999; Hua Lu et al., 1994; Racioppi, Marcarelli, and Squillante, 2015; Vaidya and Kumar, 2006).

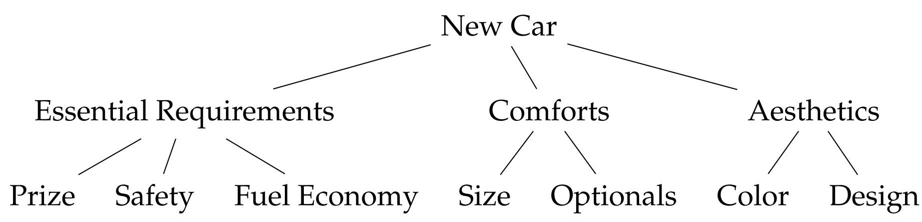
As interpretation in the context of PCMs we will say that alternative i is preferred to j if and only if $\bar{0} \preceq x_{ij}$.

7.1 Preferences via Riesz Spaces

Why should we use an element of a Riesz space to express the intensity of a preference? As showed in Cavallo and D'Apuzzo, 2009; Cavallo, Vitale, and D'Apuzzo, 2009, Riesz spaces provide a general framework to present at-once all approaches and to describe properties in the context of PCMs. Preferences via Riesz spaces are *universal*, in the sense that *(I)* they can express a ratio or a difference or a fuzzy relation, *(II)* the obtained results are true in every formalization and *(III)* Riesz spaces are a common language which can be used as a bridge between different points of view.

What does it mean non-linear intensity? In multi-criteria methods decision makers deals with many (maybe conflicting) objectives and intensity of preferences is expressed by a (real) number in each criteria. In AHP we have different PCMs, which describe different criteria; if we consider $\mathbb{R}^n$ [see example (2) above] we are just writing all these matrices as a unique matrix with vectors as elements. Actually, we can consider each component of a vector as the standard way to represent the intensity preference and the vector itself as the natural representation of multidimensional (i.e. multi-criteria) comparison. This construction has its highest expression in the subfield of MCDM called Multi-Attribute Decision Making, which has several models and applications in military system efficiency, facility location, investment decision making and many others (e.g. see Belton, 1986; Torrance et al., 1996; Xu, 2015; Zanakis et al., 1998)

Does it make sense to consider non-archimedean Riesz space in this context? Let us consider the following example. A worker with economic problems has to buy a car. We can consider the following hierarchy:



It is clear that Essential Requirements (ER), Comforts (C) and Aesthetics (A) cannot be just weighted and combined as usual. In fact, we may have the following two cases:

- we put probability different to zero on (C) and (A) and in the process can happen that the selected car is not the most economically convenient or even too expensive for him (remember that the worker has a low budget and he has to buy a car), and this is an undesired result.

- conversely, to skip the case above, we can just consider (ER) as unique criterion and neglect (C) and (A). Also in this case we have a non-realistic model, indeed our hierarchy does not take into account that if two cars have the same rank in (ER) then the worker will choose the car with more optionals or with a comfortable size for his purposes.

In a such situation it seems to be natural to consider a lexicographic order [see example (2) above] such as $(\mathbb{R} \times_{LEX} \mathbb{R}) \times_{LEX} \mathbb{R}$, where each component of a vector $(x, y, z) \in (\mathbb{R} \times_{LEX} \mathbb{R}) \times_{LEX} \mathbb{R}$ is a preference intensity in (RE), (C) and (A) respectively (we may shortly indicate the hierarchy with $(RE) \times_{LEX} (C) \times_{LEX} (A)$). We remark that *lexicographic preferences* cannot be represented by any continuous utility function (see Debreu, 1954).

Which kinds of intensity can we express with functions? This approach is one of the most popular and widely studied one, under the definition of *utility functions*. These functions provide a cardinal presentation of preferences, which allows to work with choices using a plethora of different tools, related to the model (e.g. see Harsanyi, 1953; Houthakker, 1950; Levy and Markowitz, 1979). We want to stress that in example (4) we consider functions from a compact to $\mathbb{R}$, without giving a meaning of the domain, which can be seen as a time interval, i.e. in this framework it is also possible to deal with Discounted Utility Model and intertemporal choices (e.g. see Frederick, Loewenstein, and O'donoghue, 2002). Manipulation of a particular class of these functions (i.e. piecewise-linear functions defined over $[0,1]^n$) in the context of Riesz MV-algebras is presented in Di Nola, Lenzi, and Vitale, 2016b. Furthermore, it is possible to consider more complex examples, for instance we can consider the space $\mathbb{R}^F$ of functionals, where F is a general archimedean Riesz space with strong unit (e.g. see Cerreia-Vioglio et al., 2015).

7.2 On Collective Choice Rules for PCMs and Arrow's Axioms

In this section we want to formalize and characterize Collective Choice Rules f in the context of *generalized PCMs*, i.e. PCMs with elements in a Riesz space, which satisfy classical conditions in social choice theory.

Let R be a Riesz space. Let us consider m experts/decision makers and n alternatives. A collective choice rule f is a function

$$f : GM_n^m \to GM_n$$

such that

$$f(X^{(1)}, \ldots, X^{(m)}) = X$$

where X is a *social* matrix, GM_n is the set of all matrices (PCMs) over R with n alternatives such that for every $i \in \{1, \ldots, n\}$ $x_{ii} = \bar{0}$. f can be seen also as follows:

$$f = (\tilde{f}_{ij})_{1 \leq i,j \leq n},$$

where

$$\tilde{f}_{ij} : GM_n^m \to R.$$

Note that GM_n is a subspace of $M_n(R)$ (see example (5)), i.e. it is a Riesz space. Let us introduce properties related with axioms of democratic legitimacy and informational efficiency required in Arrow's theorem.

$$\forall i, j\ (\exists f_{ij} : R^m \to R\ :\ \tilde{f}_{ij}(X^{(1)}, \ldots, X^{(m)}) = f_{ij}(x_{ij}^{(1)}, \ldots, x_{ij}^{(m)})) \qquad \text{(Property } I^*\text{)}$$

$$\forall i, j\ (f_{ij}((R^m)^+) \subseteq R^+) \qquad \text{(Property } P^*\text{)}$$

$$\nexists i \in \{1, \ldots, m\}\ : \forall X^{(j)}, \text{with } j \neq i\ \ (f(X^{(1)}, \ldots, X^{(i)}, \ldots, X^{(m)}) = X) \qquad \text{(Property } D^*\text{)}$$

Theorem 7.2.1. *Let R be a Riesz space and let f be a function $f : (R^{n^2})^m \to R^{n^2}$. f is a collective choice rule satisfying Axioms of Arrow's theorem if and only if f has properties I^*, P^* and D^*.*

Proof. Unrestricted Domain (Axiom U). The first axiom asserts that f has to be defined on all the space GM_n^m, i.e. decision makers (DMs) can provide every possible matrix as input. This is equivalent to say that f is defined on $(R^{n^2})^m$.

Independence from irrelevant alternatives (Axiom I). The second axiom says that the relation between two alternatives is influenced only by these alternatives and not by other ones, i.e. it is necessary and sufficient to know how DMs compare just these two alternative. This is equivalent to property I^*.

Pareto principle (Axiom P). The third axiom states that f has to compute a preference if it is expressed unanimously by DMs. This is equivalent to property P^*.

Non-dictatorship (Axiom D). The last axiom requires democracy, that is no one has the right to impose his preferences to the entire society. This is equivalent to property D^*. □

In Theorem 7.2.1 it is presented a characterization of collective social rules which respect Arrow's axioms; however it does not guarantee that the social matrix produce a *consistent* preference, in fact not all PCMs provide an order on the set of alternatives. We will study this feature in Section 7.3.

7.3 On Social Welfare Function Features

Social welfare functions (SWFs) are all the collective choice rules which provide a total preorder on the set of alternatives. We can decompose a SWF g as follows:

$$g = \omega \circ f,$$

where f is a collective choice rule having properties I^*, P^* and D^*, and ω is a function such that

$$\omega :\ GM_n \to \mathbf{TP},$$

where $\mathbf{TP}$ is the set of total preorders on the set of alternatives. Let us consider a social matrix $X = f(X^{(1)}, \ldots, X^{(m)})$. We want to characterize property of ω such that g is a social welfare function.

Let us recall the definition of transitive PCM.

Definition 7.3.1. *Cavallo and D'Apuzzo, 2015, Definition 3.1 A pairwise comparison matrix* X *is transitive if and only if* $(\bar{0} \preceq x_{ij} \text{ and } \bar{0} \preceq x_{jk}) \Rightarrow \bar{0} \preceq x_{ik}$

It is trivial to check that if X is *transitive*, then it is possible to directly compute an order which expresses the preferences over alternatives. In fact, let X be a GM_n, it has two properties:

(ρ) $x_{ii} = \bar{0}$, (Reflexivity)
(γ) $\forall i, j \in \{1, \ldots, n\}$ $x_{ij} \in R$. (Completeness)

If we have also that

(τ) $(\bar{0} \preceq x_{ij} \text{ and } \bar{0} \preceq x_{jk}) \Rightarrow \bar{0} \preceq x_{ik}$ (Transitivity)

We say that an order $\lesssim_X$ is *compatible with* X if and only if we have that:

$$\bar{0} \preceq x_{ij} \quad \Leftrightarrow \quad j \lesssim_X i.$$

An analogous definition is proposed in Trockel, 1998 in the context of utility functions.

Proposition 7.3.1. *Let* X *be a transitive* GM_n *(*TGM_n*) then there exists a unique total preorder* $\lesssim_X$ *compatible with* X*. Or equivalently, the correspondence*

$$\theta : TGM_n \rightarrow \mathbf{TP}$$

which associates to each $X \in TGM_n$ *a preorder* $\lesssim_X$ *compatible with* X *itself is a surjective function. Moreover* $\lesssim_X \equiv \lesssim_{\alpha \cdot X}$ *for every* $\alpha \in \mathbb{R}^+$*, and* $\lesssim_X \equiv \gtrsim_{\alpha \cdot X}$ *for every* $\alpha \in \mathbb{R}^-$.

Let $\mathcal{C}(R) = \{A \subseteq R \mid A \text{ is a cone}\}$ be the set of all closed cones of R Riesz space. By Proposition 7.3.1 we can consider the function Φ

$$\Phi : \mathbf{TP} \rightarrow \mathcal{C}(TGM_n)$$

such that

$$\Phi(\lesssim) = \{X \in TGM_n \mid \lesssim \text{ is compatible with } X\}$$

Proposition 7.3.2. *The function* Φ *is injective.*

We can define an order relation $\ll$ over $\mathbf{TP}$ as follows:

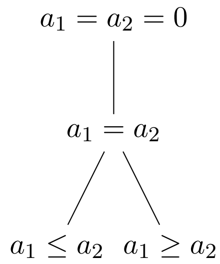
$$\lesssim_1 \ll \lesssim_2 \quad \Leftrightarrow \quad i \lesssim_2 j \rightarrow i \lesssim_1 j .$$

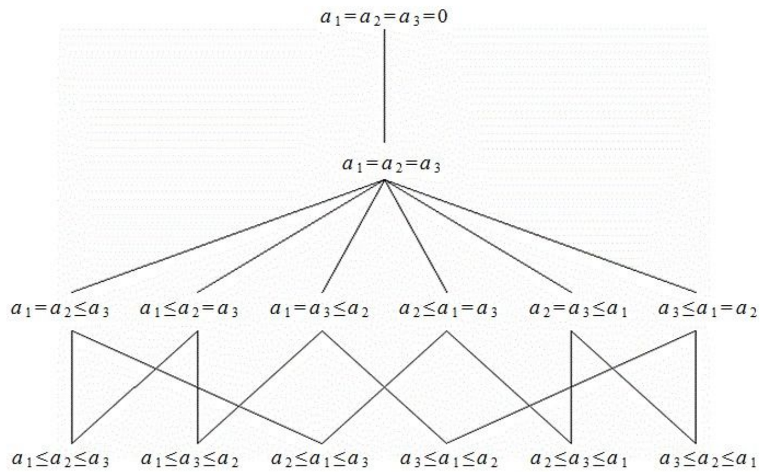
It is also possible to denote with $\lesssim = \lesssim_1 \vee \lesssim_2$ as the total preorder such that

$$i \lesssim j \quad \Leftrightarrow \quad i \lesssim_1 j \text{ and } i \lesssim_2 j.$$

Remark 7.3.1. *By easy considerations, we have that* $\Phi(\lesssim_1) \cap \Phi(\lesssim_2) = \Phi(\lesssim_1 \vee \lesssim_2)$*. Moreover, note that* $\mathbf{TP}$ *is closed with respect to* $\vee$*, i.e.* $(\mathbf{TP}, \vee)$ *is a join-semilattice.*

Examples Let us consider n alternatives. The spaces of total preorder with $n = 2$ and $n = 3$ have the following configurations:





Note that in each space we have exactly one atom which expresses indifference. We call *basic total preorder* an element which is minimal in $(\mathbf{TP}, \ll)$.

Remark 7.3.2. *In order to deal with aggregation of many TGM_n we added a root ($\top$), which can be interpreted as impossibility to make a social decision (related to Condorcet's paradox and Arrow's impossibility theorem in the context of PCMs). We put*

$$\Phi(\top) = \emptyset.$$

Proposition 7.3.3. *Every $\lesssim$ total preorder different from $\top$ can be written as $\bigvee_i \lesssim_i$, where $\lesssim_i$ are basic total preorders.*

Proof. If $\lesssim$ has no identities then it is a basic total preorders. For each identity $a_i = a_j$ in $\lesssim$ we can consider $\lesssim_h \vee \lesssim_k$, with $\lesssim_h$ and $\lesssim_k$ basic total preorders such that $a_i \lesssim_h a_j$, $a_j \lesssim_k a_i$ and preserve all the other relations of $\lesssim$. □

Proposition 7.3.4. *Let $\lesssim$ be a basic total preorder over n elements. We have that $\Phi(\lesssim)$ is an orthant in TGM_n.*

Proof. By the fact that $\lesssim$ is a basic total preorder we have that $a_i \lesssim a_j$ or $a_j \lesssim a_i$ for each alternatives a_i and a_j, i.e. $x_{ij} \succeq \bar{0}$ or $x_{ij} \preceq \bar{0}$. □

Analogously to θ we can define Θ in this way:

$$\Theta : \mathcal{C}(TGM_n) \rightarrow \mathbf{TP}$$

where $\Theta(\emptyset) = \top$ and

$$\Theta(K) = \Phi^{-1}\left(\bigcap_{\substack{C\in\Phi(\mathbf{TP}) \\ C\cap K\neq\emptyset}} C\right).$$

By Remark 7.3.1 we have that the function is well-defined.

Definition 7.3.2. *Let $(A, \leq_A)$ and $(B, \leq_B)$ be two partially ordered sets. An antitone Galois correspondence consists of two monotone functions: $F : A \rightarrow B$ and $G : B \rightarrow A$, such that for all a in A and b in B, we have $F(a) \leq_B b \quad \Leftrightarrow$ $a \geq_A G(b)$.*

Now we can state the following result.

Theorem 7.3.1. *The couple (Θ, Φ) is an antitone Galois correspondence between $(\mathcal{C}(TGM_n), \subseteq)$ and $(\mathbf{TP}, \ll)$.*

Proof. Let K be an element of $\mathcal{C}(TGM_n)$ and $\lesssim$ an element of $\mathbf{TP}$. Let $\lesssim_K$ be $\Theta(K)$.The proof follows by this chain of equivalence:

$$\Theta(K) \ll\lesssim \quad\Leftrightarrow\quad (i \lesssim j \rightarrow i \lesssim_k j) \quad\Leftrightarrow\quad (X \in \Phi(\lesssim) \rightarrow X \in K) \quad\Leftrightarrow\quad K \supseteq \Phi(\lesssim).$$

□

We denote by K_n the subset of $\mathcal{C}(TGM_n)$ of all the cones L such that $L \in \Phi(\mathbf{TP})$.

Proposition 7.3.5. *Let L be a cone of TGM_n. We have that*

$$L \in \Phi(\mathbf{TP}) \quad\Leftrightarrow\quad L\ is\ a\ TP-cone.$$

Proof. ($\Rightarrow$) Let L be in $\Phi(\mathbf{TP})$, this means that $L = \emptyset$ or $L = \Phi(\lesssim)$ for some $\lesssim$ total preorder. Using Proposition 7.3.3 and Remark 7.3.1 we have:

$$L = \Phi(\lesssim) = \Phi(\bigvee_i \lesssim_i) = \bigcap_i \Phi(\lesssim_i),$$

where $\lesssim_i$ are basic total preorders. By Proposition 7.3.4 and Definition 7.0.3 we have that L is a TP-cone.

($\Leftarrow$) Let L be a TP-cone. We have that:

- if $L = \emptyset$ then $L \in \Phi(\mathbf{TP})$;
- if L is an orthant then for each i and j $x_{ij} \succeq \bar{0}$ or $x_{ij} \preceq \bar{0}$, which is equivalent to say that there exists $\lesssim$ (basic) total preorder such that $a_i \lesssim a_j$ or $a_j \lesssim a_i$, i.e. $L \in \Phi(\mathbf{TP})$;
- if L is an intersection of O_i orthants then

$$L = \bigcap_i O_i = \bigcap_i \Phi(\lesssim_i) = \Phi(\bigvee_i \lesssim_i),$$

for some $\lesssim_i$ basic total preorders, i.e. $L \in \Phi(\mathbf{TP})$.

□

7.3.1 Categorical Duality

In this subsection we provide a categorical duality between the categories of total preorders and of TP-cones (for basic definition on categories see Mac Lane, 1978).

Let us define the categories $\mathbb{TP}_n$ (of total preorders) and $\mathbb{K}_n$ (of TP-cones in TGM_n). In $\mathbb{TP}_n$ the objects are total preorder on n elements and arrows are defined by order $\ll$, i.e.

$$\lesssim_1 \rightarrow \lesssim_2 \quad \Leftrightarrow \quad \lesssim_1 \ll \lesssim_2 .$$

In a similar way we define $\mathbb{K}_n$ whose objects are TP-cones in the space TGM_n and arrows are defined by inclusion.

Theorem 7.3.2. *Categories of preorders and of TP-cones are dually isomorphic.*

Proof of Theorem 7.3.2 descends from lemmas below.

Lemma 7.3.1. *The maps $\mathit{\Theta} : \mathbb{K}_n \rightarrow \mathbb{TP}_n$ and $\mathit{\Phi} : \mathbb{TP}_n \rightarrow \mathbb{K}_n$ defined as follows*

- $\mathit{\Theta}(C) = \Theta(C)$
- $\mathit{\Theta}(\rightarrow) = \leftarrow$
- $\mathit{\Phi}(\lesssim) = \Phi(\lesssim)$
- $\mathit{\Phi}(\rightarrow) = \leftarrow$

are contravariant functors.

Proof. Let us consider C and D TP-cones, such that $C \rightarrow D$. We have that:

$$C \rightarrow D \quad \Leftrightarrow \quad C \subseteq D \quad \Leftrightarrow \quad \Theta(C) \gg \Theta(C) \quad \Leftrightarrow \quad \Theta(C) \leftarrow \Theta(D).$$

Analogously, if we consider $\lesssim_1$ and $\lesssim_2$ total preorders over n elements, such that $\lesssim_1 \rightarrow \lesssim_2$, then:

$$\lesssim_1 \rightarrow \lesssim_2 \quad \Leftrightarrow \quad \lesssim_1 \ll \lesssim_2 \quad \Leftrightarrow \quad \Phi(\lesssim_1) \supseteq \Phi(\lesssim_2) \quad \Leftrightarrow \quad \Phi(\lesssim_1) \leftarrow \Phi(\lesssim_2).$$

□

Lemma 7.3.2. *The composed functors $\mathit{\Phi\Theta} : \mathbb{K}_n \rightarrow \mathbb{K}_n$ and $\mathit{\Theta\Phi} : \mathbb{TP}_n \rightarrow \mathbb{TP}_n$ are the identity functors of the categories $\mathbb{K}_n$ and $\mathbb{TP}_n$ respectively.*

Proof. Let us consider K TP-cone, we have that

$$\mathit{\Phi\Theta}(K) = \Phi(\Theta(K)) = \Phi\left(\Phi^{-1}\left(\bigcap_{\substack{C \in \Phi(\mathbf{TP}) \\ C \cap K \neq \emptyset}} C\right)\right) = \bigcap_{\substack{C \in \Phi(\mathbf{TP}) \\ C \cap K \neq \emptyset}} C,$$

but K is a TP-cone, i.e. $K \in \Phi(\mathbf{TP})$, hence

$$\bigcap_{\substack{C \in \Phi(\mathbf{TP}) \\ C \cap K \neq \emptyset}} C = K.$$

Vice versa, let $\lesssim$ be a total preorder, then

$$\Theta\Phi(\lesssim) = \Theta(\Phi(\lesssim)) = \Theta(\{X \in TGM_n \mid \lesssim \textit{ is compatible with } X\}).$$

Let us denote by $K_{\lesssim} = \{X \in TGM_n \mid \lesssim \textit{ is compatible with } X\}$, therefore we have:

$$\Theta(K_{\lesssim}) = \Phi^{-1}\left(\bigcap_{\substack{C \in \Phi(\mathbf{TP}) \\ C \cap K \neq \emptyset}} C\right) = \Phi^{-1}(K_{\lesssim}) = \lesssim .$$

In both cases arrows are preserved by Lemma 7.3.1. $\square$

Chapter 8

Artificial Neural Networks

8.1 Multilayer Perceptrons

Artificial neural networks are inspired by the nervous system to process information. There exist many typologies of neural networks used in specific fields. We will focus on feedforward neural networks, in particular multilayer perceptrons, which have applications in different fields, such as speech or image recognition. This class of networks consists of multiple layers of neurons, where each neuron in one layer has directed connections to the neurons of the subsequent layer. If we consider a multilayer perceptron with n inputs, l hidden layers, ω^h_{ij} as weight (from the j-th neuron of the hidden layer h to the i-th neuron of the hidden layer $h+1$), b_i real number and ρ an activation function (a monotone-nondecreasing continuous function), then each of these networks can be seen as a function $F : [0,1]^n \to [0,1]$ such that

$$F(x_1, \dots, x_n) = \rho(\sum_{k=1}^{n^{(l)}} \omega^l_{0,k} \rho(\dots(\sum_{i=1}^{n} \omega^1_{l,i} x_i + b_i)\dots))).$$

The following theorem explicits the relation between rational Łukasiewicz logic and multilayer perceptrons.

Theorem 8.1.1. *(See Amato, Di Nola, and Gerla, 2002, Theorem III.6) Let the function ρ be the identity truncated to zero and one.*

- *For every l, n, $n^{(2)}$, …, $n^{(l)} \in \mathbb{N}$, and $\omega^h_{i,j}, b_i \in \mathbb{Q}$, the function $F : [0,1]^n \to [0,1]$ defined as*

  $$F(x_1, \dots, n_n) = \rho(\sum_{k=1}^{n^{(l)}} \omega^l_{0,k} \rho(\dots(\sum_{i=1}^{n} \omega^1_{l,i} x_i + b_i)\dots)))$$

  is a truth function of an MV-formula with the standard interpretation of the free variables;

- *for any f truth function of an MV-formula with the standard interpretation of the free variables, there exist l, n, $n^{(2)}$, …, $n^{(l)} \in \mathbb{N}$, and $\omega^h_{i,j}, b_i \in \mathbb{Q}$ such that*

  $$f(x_1, \dots, n_n) = \rho(\sum_{k=1}^{n^{(l)}} \omega^l_{0,k} \rho(\dots(\sum_{i=1}^{n} \omega^1_{l,i} x_i + b_i)\dots))).$$

8.2 Łukasiewicz Equivalent Neural Networks

In this section we present a logical equivalence between different neural networks, proposed in Di Nola, Lenzi, and Vitale, 2016a.

When we consider a surjective function from $[0,1]^n$ to $[0,1]^n$ we can still describe non-linear phenomena with an MV-formula, which corresponds to a function which can be decomposed into "regular pieces", not necessarily linear (e.g. a piecewise sigmoidal function) (for more details see Di Nola, Lenzi, and Vitale, 2016b).

The idea is to apply, with a suitable choice of generators, all the well established methods of MV-algebras to piecewise non-linear functions.

Definition 8.2.1. *We call $\L\mathcal{N}$ the class of the multilayer perceptrons such that:*

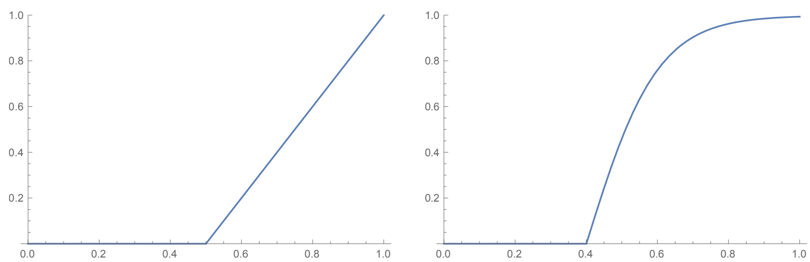
- *the activation functions of all neurons from the second hidden layer on is $\rho(x) = (1 \wedge (x \vee 0))$, i.e. the identity truncated to zero and one;*
- *the activation functions of neurons of the first hidden layer have the form $\iota_i \circ \rho(x)$ where ι_i is a continuous function from $[0,1]$ to $[0,1]$.*

8.2.1 Examples of Łukasiewicz Equivalent Neural Networks

Let us see now some examples of Łukasiewicz equivalent neural networks (seen as the functions $\psi(\varphi(\bar{x}))$). In every example we will consider a Riesz MV-formula $\psi(\bar{x})$ with many different φ interpretations of the free variables $\bar{x}$, i.e. the activation functions of the interpretation layers.

Example 1

A simple one-variable example of Riesz MV-formula could be $\psi = \bar{x} \odot \bar{x}$. Let us plot the functions associated with this formula when the activation functions of the interpretation layer is respectively the identity truncate function to 0 and 1 and the $LogSigm$.



$\varphi(\bar{x}) := id(x)$ $\varphi(\bar{x}) := LogSigm(x)$

FIGURE 8.1: $\psi(\bar{x}) = \bar{x} \odot \bar{x}$

In all the following examples we will have (a), (b) and (c) figures, which indicate respectively these variables interpretations:

(a) x and y as the canonical projections π_1 and π_2;

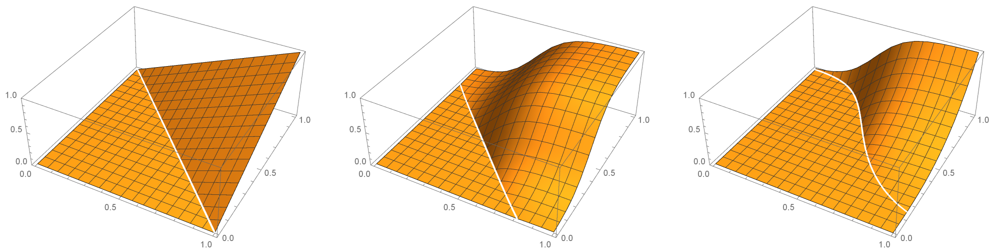
(b) both x and y as $LogSigm$ functions, applied only on the first and the second coordinate respectively, i.e. $LogSigm \circ \rho(\pi_1)$ and $LogSigm \circ \rho(\pi_2)$ (as in the example 1);

(c) x as $LogSigm$ function, applied only on the first coordinate, and y as the cubic function π_2^3.

We show how, by changing projections with arbitrary functions φ, we obtain functions (b) and (c) "similar" to the standard case (a), which, however, are no more "linear". The "shape" of the function is preserved, but distortions are introduced.

Example 2: The $\odot$ Operation

We can also consider, in a similar way, the two-variables formula $\psi(\bar{x}, \bar{y}) = \bar{x} \odot \bar{y}$ (figure 8.2).

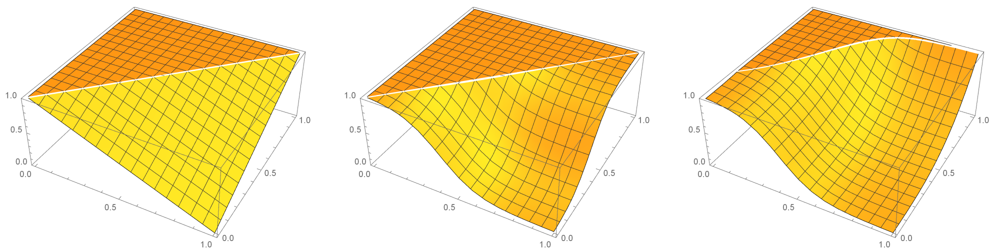


(a) (b) (c)

FIGURE 8.2: $\psi(\bar{x}, \bar{y}) = \bar{x} \odot \bar{y}$

Example 3: The Łukasiewicz Implication

As in classical logic, also in Łukasiewicz logic we have *implication* ($\rightarrow$), a propositional connective which is defined as follows: $\bar{x} \rightarrow \bar{y} = \bar{x}^* \oplus \bar{y}$ (figure 8.3).

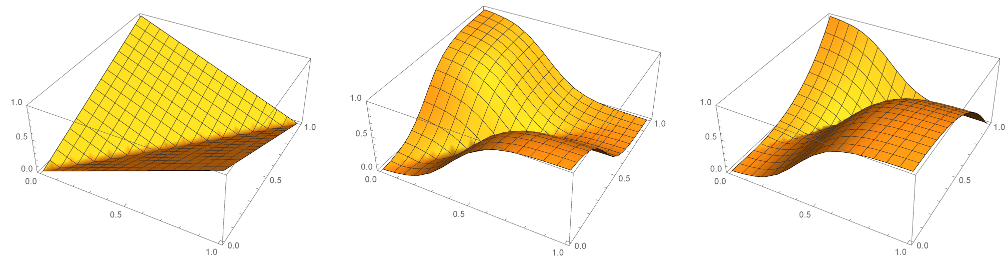


(a) (b) (c)

FIGURE 8.3: $\psi(\bar{x}, \bar{y}) = \bar{x} \rightarrow \bar{y}$

Example 4: The Chang Distance

An important MV-formula is $(\bar{x} \odot \bar{y}^*) \oplus (\bar{x}^* \odot \bar{y})$, called *Chang Distance*, which is the absolute value of the difference between x and y in the usual sense (figure 8.4).



(a) (b) (c)

FIGURE 8.4: $\psi(\bar{x}, \bar{y}) = (\bar{x} \odot \bar{y}^*) \oplus (\bar{x}^* \odot \bar{y})$

8.3 Function Approximation Problems

8.3.1 Input Selection and Polynomial Completeness

The connection between MV-formulas and truth functions (evaluated over particular algebras) is analyzed in Belluce, Di Nola, and Lenzi, 2014, via *polynomial completeness*. It is showed that in general two MV-formulas may not coincide also if their truth functions are equal. This strange situation happens when the truth functions are evaluated over a "not suitable" algebra, as explained hereinafter.

Definition 8.3.1. *An MV-algebra A is polynomially complete if for every n, the only MV-formula inducing the zero function on A is the zero.*

Proposition 8.3.1. *Belluce, Di Nola, and Lenzi, 2014, Proposition 6.2 Let A be any MV-algebra. The following are equivalent:*

- *A is polynomially complete;*
- *if two MV-formulas φ and ψ induce the same function on A, then $\varphi = \psi$;*
- *if two MV-formulas φ and ψ induce the same function on A, then they induce the same function in every extension of A;*

Proposition 8.3.2. *Belluce, Di Nola, and Lenzi, 2014, Corollary 6.14 If A is a discrete MV-chain, then A is not polynomially complete.*

Roughly speaking an MV-algebra A is polynomially complete if it is able to distinguish two different MV-formulas. This is strictly linked with back-propagation and in particular with the input we choose; in fact Proposition 8.3.2 implies that an homogeneous subdivision of the domain is not a suitable choice to compare two piecewise linear functions (remember that S_n, the MV-chain with n elements, has the form $S_n = \{\frac{i}{n-1} \mid i = 0, \ldots, n-1\}$).

So we have to deal with finite input, trying to escape the worst case in which the functions coincide only over the considered points. The next results guarantee the existence of finitely many input such that the local equality between the piecewise linear function and the truth function of an MV-formula is an identity.

Proposition 8.3.3. *Let $f : [0,1] \to [0,1]$ be a rational piecewise linear function. There exists a set of points $\{x_1, \ldots, x_m\} \subset [0,1]$, with f derivable in each x_i, such that if $f(x_i) = TF(\varphi, (\pi_1))(x_i)$ for each i and $TF(\varphi, (\pi_1))$ has the minimum number of linear pieces then $f = TF(\varphi, (\pi_1))$.*

Proof. Let f be a rational piecewise linear function and $I_1, \ldots, I_m$ be the standard subdivision of $[0,1]$ such that $f_j := f|_{I_j}$ is linear for each $j = 1, \ldots, m$. Let us consider $x_1, \ldots, x_m$ irrational numbers such that $x_j \in I_j \; \forall j$. It is a trivial observation that f is derivable in each x_i and that $\{f_j\}_{j=1,\ldots,m}$ are linear components of $TF(\varphi, (\pi_1))$ if $f(x_i) = TF(\varphi, (\pi_1))(x_i)$; by our choice to consider the minimum number of linear pieces and by the fact that $f = TF(\psi, (\pi_1))$, for some ψ, we have that $f = TF(\varphi, (\pi_1))$. □

Now we give a definition which will be useful in the sequel.

Definition 8.3.2. *Let $x_1, \ldots, x_k$ be real numbers and $z_0, z_1, \ldots, z_k$ be integers. We say that $x_1, \ldots, x_k$ are integral affine independent iff $z_0 + z_1 x_1 + \ldots + z_k x_k = 0$ imply that $z_i = 0$ for each $i = 0, \ldots, k$.*

Note that there exists integral affine independent numbers. For example $\log_2(p_1), \log_2(p_2), \ldots, \log_2(p_n)$, where $p_1, \ldots, p_n$ are distinct prime number, are integral affine independent; it follows by elementary property of logarithmic function and by the fundamental theorem of arithmetic.

Lemma 8.3.1. *Let f and g affine functions from $\mathbb{R}^n$ to $\mathbb{R}$ with rational coefficients. We have that $f = g$ iff $f(\bar{x}) = g(\bar{x})$, where $\bar{x} = (x_1, \ldots, x_n)$ and $x_1, \ldots, x_n$ are integral affine independent.*

Proof. It follows by Definition 8.3.2. □

Integral affine independence of coordinates of a point is, in some sense, a weaker counterpart of polynomial completeness. In fact it does not guarantee identity of two formulas, but just a local equality of their components.

Theorem 8.3.1. *Let $f : [0,1]^n \to [0,1]$ be a rational piecewise linear function ($\mathbb{Q}M_n$). There exists a set of points $\{\bar{x}_1, \ldots, \bar{x}_m\} \subset [0,1]^n$, with f differentiable in each $\bar{x}_i$, such that if $f(\bar{x}_i) = TF(\varphi, (\pi_1, \ldots, \pi_n))(\bar{x}_i)$ for each i and $TF(\varphi, (\pi_1, \ldots, \pi_n))$ has the minimum number of linear pieces then $f = TF(\varphi, (\pi_1, \ldots, \pi_n))$.*

Proof. It follows by Lemma 8.3.1 and the proof is analogous to Proposition 8.3.3. □

By the fact that the function is differenziable in each $\bar{x}_i$, it is possible to use gradient methods for the back-propagation.

As shown in Di Nola, Lenzi, and Vitale, 2016b and in Section 8.2 it is possible to consider more general functions than piecewise linear ones as interpretation of variables in MV-formulas. Let us denote by $M_n^{(h_1,\ldots,h_n)}$ the following MV-algebra

$$M_n^{(h_1,\ldots,h_n)} = \{f \circ (h_1, \ldots, h_n) \mid f \in M_n \text{ and } h_i : [0,1] \to [0,1] \; \forall i = 1, \ldots, n\}.$$

Likewise in the case of piecewise linear functions we say that $g \in M_n^{(h_1,\ldots,h_n)}$ is $(h_1, \ldots, h_n)$-piecewise function, $g_1, \ldots, g_m$ are the $(h_1, \ldots, h_n)$-components of g and $I_1, \ldots, I_k$, connected sets which form a subdivision of $[0,1]^n$, are $(h_1, \ldots, h_n)$-pieces of g, i.e. $g|_{I_i} = g_j$ for some $j = 1, \ldots, m$.

Now we give a generalization of Definition 8.3.2 and an analogous of Theorem 8.3.1.

Definition 8.3.3. *Let $x_1, \ldots, x_k$ be real numbers, $z_0, z_1, \ldots, z_k$ integers and $h_1, \ldots, h_k$ functions from $[0,1]$ to itself. We say that $x_1, \ldots, x_k$ are integral affine $(h_1, \ldots, h_k)$-independent iff $z_0 + z_1 h_1(x_1) + \ldots + z_k h_k(x_k) = 0$ imply that $z_i = 0$ for each $i = 0, \ldots, k$.*

For instance let us consider the two-variable case $(h_1, h_2) = (x^2, y^2)$; we trivially have that $\sqrt{\log_2(p_1)}, \sqrt{\log_2(p_2)}$ are integral affine (x^2, y^2)-independent.

Theorem 8.3.2. *Let $(h_1, \ldots, h_n) : [0,1]^n \to [0,1]^n$ be a function such that $h_i : [0,1] \to [0,1]$ is injective and continuous for each i. Let $g : [0,1]^n \to [0,1]$ be an element of $M_n^{(h_1, \ldots, h_n)}$. There exists a set of points $\{\bar{x}_1, \ldots, \bar{x}_m\} \subset [0,1]^n$ such that if $g(\bar{x}_i) = TF(\varphi, (h_1, \ldots, h_n))(\bar{x}_i)$ for each i and $TF(\varphi, (h_1, \ldots, h_n))$ has the minimum number of $(h_1, \ldots, h_n)$-pieces then $g = TF(\varphi, (h_1, \ldots, h_n))$.*

Proof. It is sufficient to note that injectivity allows us to consider the functions h_i^{-1}, in fact if $h_1, \ldots, h_n$ are injective functions then there exist integral affine $(h_1, \ldots, h_n)$-independent numbers and this bring us back to Theorem 8.3.1. □

8.3.2 On the Number of Hidden Layers

One of the important features of a multilayer perceptron is the number of hidden layers. In this section we show that, in our framework, three hidden layers are able to compute the function approximation.

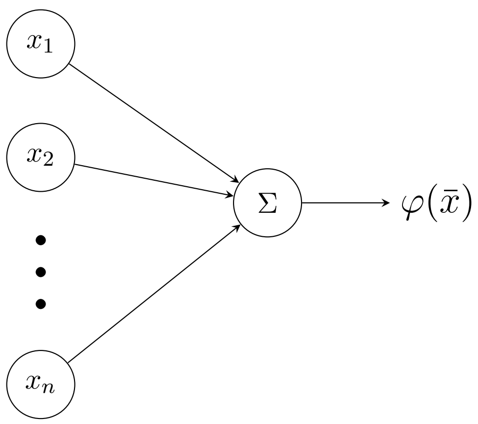
We refer to Di Nola and Lettieri, 2004 for definition of *simple McNaughton functions*. As natural extention we have the following one.

Definition 8.3.4. *We say that $f \in \mathbb{Q}M_n$ is simple iff there is a real polynomial $g(x) = ax + b$, with rational coefficients such that $f(x) = (g(x) \wedge 1) \vee 0$, for every $x \in [0,1]^n$.*

Proposition 8.3.4. *Let us consider $f \in \mathbb{Q}M_n$ and $\bar{x} = (x_1, \ldots, x_n)$ a point of $[0,1]^n$ such that $x_1, \ldots, x_n$ are integral affine independent. If $f(\bar{x}) \notin \{0,1\}$ then there exists a unique simple rational McNaughton function g such that $f(\bar{x}) = g(\bar{x})$.*

Proof. It is straightforword by definition. □

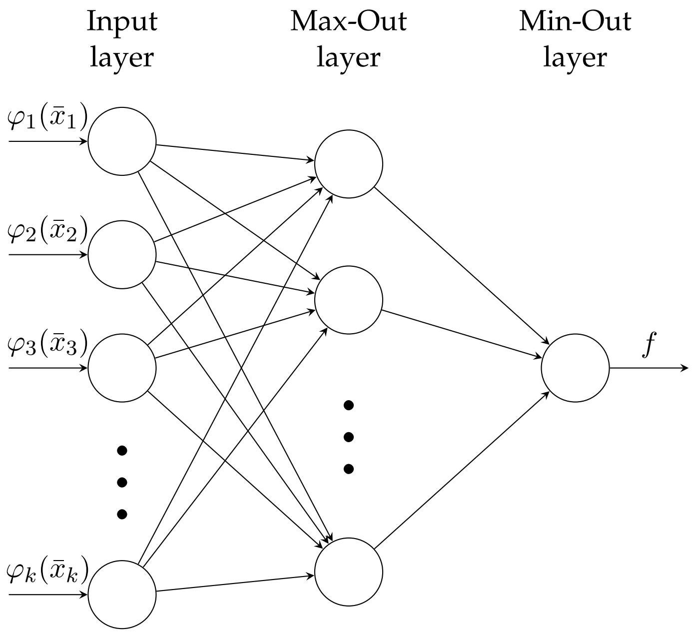
Via Proposition 8.3.4, it is possible to consider the following perceptron.



Every rational McNaughton function can be written in the following way:

$$f(\bar{x}) = \bigwedge_i \bigvee_j \varphi_{ij}(\bar{x})$$

where φ_{ij} are simple $\mathbb{Q}M_n$. By this well-known representation it is suitable to consider the following multilayer perceptron:



where φ_i are the linear components of f and $\bar{x}_i$ are points as described before. Note that these networks are universal approximators (see Kreinovich, Nguyen, and Sriboonchitta, 2016).

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