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On neutrino mixing in Quantum Field Theory

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Abstract

Neutrino physics is one of the most important areas in research on the fundamental interactions, both theoretical and experimental. A central issue in the study of these particles is related to the properties of mixing and flavor oscillations. A necessary condition for the mixing is that neutrinos have masses, a feature which is not considered by the Standard Model, and which poses the problem of the origin and nature of these masses.

The theoretical basis of neutrino mixing has been studied in many details and in the late '90 a quantum field theory (QFT) formalism for mixed fields has been developed thanks to which it was found that mixing transformations induce a condensed structure in the flavour vacum with possible phenomenological consequences. In particular, the usual formulas of Pontecorvo oscillation are arising as the relativistic limit of exact formulas in the context of field theory. Within this framework it is inserted our work, finalized to the study of algebraic properties of the mixing transformation generator in QFT and its components.

In quantum mechanics (QM) the mixing transformation looks like a rotation operating on massive neutrino states. We show explicitly that such a rotation is not sufficient for implementing the mixing transformation at level of fields. It is necessary, in fact, also the action of a Bogoliubov transformation which induces a suitable mass shift. Such a property of Bogoliubov transformations has been already known and used since long time, *e.g.* in renormalization theory or in the dynamical generation of mass. We then analyze the condensate nature of the flavor vacuum and the rôle played by the non–commutativity between the rotation and the Bogoliubov transformation. This structure of the vacuum also suggests a thermodynamical interpretation

which we investigate, showing peculiarities in the thermal behavior due to the character of the particle-antiparticle condensate involved in the flavor vacuum. The key point in our analysis is the non–commutativity between rotation and Bogoliubov transformations, a feature which turns out to be at the origin of the inequivalence among mass and flavor vacua. From another point of view, the Bogoliubov transformations are shown to naturally arise when studying the neutrino mixing in the contest of the non–commutative Spectral Geometry in Alain Connes' construction and the algebra doubling he introduces. Given the algebraic nature of our arguments, we have good reasons to believe that the results we have obtained are general and therefore can also be extend to the mixing phenomenon of any particle, even if our analysis is limited to the case of two Dirac neutrinos.

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Contents

1 Standard Model (SM) **15**
1.1 Gauge symmetry 18
1.2 Spontaneous symmetry breaking: $U(1)$ model 19
1.3 Standard Model Lagrangian 22
1.3.1 Gauge term 22
1.3.2 Higgs and Yukawa terms 24
1.4 Mixing within the SM 25
1.4.1 Flavor and Cabibbo-Kobayashi-Maskawa matrix 26
1.4.2 Symmetry breaking; boson and fermion masses 28
1.5 Discrete Symmetries 30
1.6 Neutrino mixing within and beyond the SM 32

2 Neutrino mixing **35**
2.1 Neutrino Mixing in Quantum Mechanics: B. Pontecorvo's idea 35
2.1.1 Majorana neutrinos 36
2.1.2 Dirac neutrinos 38
2.1.3 Neutrino oscillations 40
2.2 Neutrino mixing in Quantum Field Theory formalism 42
2.2.1 Neutrino oscillations in QFT 46
2.3 Pontecorvo vs QFT formalism 52

3 Mixing generator and vacuum structure **57**
3.1 Decomposition of the mixing generator 57
3.2 Vacuum structure and non–commutativity 60
3.2.1 Vacuum energies 61

3.2.2 Small θ approximation 63
3.2.3 Condensate structure 68
3.3 Thermodynamical properties 69
3.4 Extension to the case of three flavors. 73

4 Non–commutative geometry 81
4.1 A little bit of history 81
4.2 Non–commutative examples 83
4.2.1 Landau levels 83
4.2.2 Quantum dissipation induced non–commutative geometry 88
4.3 Non–commutative Geometry 93
4.4 Non–commutative Spectral Geometry 96

5 Neutrino mixing and non–commutative spectral geometry in Connes' construction 99
5.1 Introduction to Alain Connes' Model 99
5.1.1 Standard Model and General Relativity 101
5.1.2 Spectral Action Principle 102
5.1.3 Neutrinos within the NCSG Model 105
5.1.4 Summarizing 107
5.2 Hopf Algebra, Bogoliubov transformation and neutrino mixing 110
5.2.1 Hopf Algebra 110
5.2.2 Co-product and Bogoliubov transformation 114
5.2.3 Bogoliubov trasformation, doubling and neutrino mixing 116

6 Phase space picture of neutrino mixing and oscillations 119
6.1 Flavor mixing and oscillations 119
6.1.1 Neutrino oscillations 120
6.1.2 Coupled oscillators 120
6.2 Classical phase–space picture 122
6.3 Oscillations in phase space 124

7 Conclusions and Outlook 129

Appendix - The $SU(2)$ Group **134**

Appendix - The Lorentz group **136**

Appendix - GIM mechanism **138**

Appendix - Seesaw mechanism **141**

Appendix - Work in progress : Extension to boson mixing **144**

Introduction

Neutrino was postulated for the first time by Pauli in 1930 [1] to save the energy, momentum and spin conservation in the β-decays; in contrast with Bohr, who proposed a statistical version of the conservation laws to explain the phenomenon, Pauli assumed the existence of a new particle and give it the name of "neutrone". In 1932 Chadwick [2] discovered a nuclear particle, with neutral charge and a definitely bigger mass, which he called "neutrone" too. Later on, it was Fermi [3], while formalizing the theory of β-decay, who coined the term *neutrino* for the particle postulated by Pauli, to solve the ambiguity. In 1942 Ganchang [4] proposed to use the β capture to find neutrinos experimentally; almost ten years later Cowan and Reines published [5] the result of their experiment on neutrino detection.
From that moment on, neutrinos have continuously captured the curiosity of many physicists for their peculiar nature.
One of the most fascinating characteristic of neutrinos is the phenomenon of flavor oscillations, postulated in 1957 by Pontecorvo [6, 7], on which many experiments have been done and are still taking place [8]. Many years of studies and efforts have, in fact, brought to the scientific community independent evidences of neutrino oscillations which have been obtained in a series of experiments, such as: the Super Kamiokande experiment [9] and other atmospheric neutrino experiments [10]; the SNO experiment and solar neutrino experiments [11, 12, 13, 14, 15]; the KamLAND experiment [16] and other reactor experiments [17]; the OPERA experiment [18] and long baseline experiments [19, 20, 21].
Since Pontecorvo's pioneering work, the theoretical basis of neutrino mixing

has been studied in great detail but there are still many open questions on such a puzzling particle. Within the Standard Model (SM) neutrinos are described as three $(\nu_e, \mu_\mu, \nu_\tau)$ massless left-handed fermions and the leptonic number is strictly conserved. The phenomenon of neutrino mixing, and the existence of non–zero neutrino masses, has captured much attention because it opens interesting perspectives on the physics beyond Standard Model.
Actually, in presence of mixing talking about the masses of the flavor neutrinos $(\nu_e, \mu_\mu, \nu_\tau)$ is not correct. In fact, neutrino fields entering in charge weak currents have definite flavor but not a definite mass. On the other hand, the fields (ν_1, ν_2, ν_3) with definite masses, which propagate as free fields, do not have a definite flavor. Nonetheless, the latter can be obtained as a mixture of flavor fields and viceversa, depending one which fields one considers as fundamental. It is exactly this characteristic that leads to the observed neutrino oscillations.
Nonetheless, neutrino absolute mass values are still to be found. In fact, experiments on neutrino oscillations can only measure the squared mass difference of different neutrino families, *i.e.* $(\Delta m_{12})^2, (\Delta m_{23})^2$ and it is not known whether the hierarchy is normal or inverted, *i.e.* $(\Delta m_{12})^2 \leq (\Delta m_{23})^2$ or viceversa. An upper bound for the absolute mass values comes from the recents results of the Planck mission and other cosmological measurements [22]. In fact, they revealed that the sum of the masses is constrained $\sum_i m_i < 0.23eV$. Neutrino masses are, thus, much smaller than the masses of the other fermions (*i.e.* leptons and quarks). It seems unlikely [23] that the origin of neutrino masses is the same of quarks and lepton, *i.e.* the Standard Model Higgs. Another open question about neutrinos raises again from cosmology: the asymmetry between baryons and anti-baryons in the Universe cannot have been developed without any CP violation during the first phases of the Universe. The only known source of CP violation, the ones in the quark mixing sector, is not sufficient to explain the observations. A CP violation in the lepton sector could be responsible for such asymmetry, making the study of neutrino oscillation even more interesting.
Another question mark concerning neutrinos is about their actual nature: whether they are Dirac or Majorana particlesis still to be determined.

Recently [25, 26] a rich non–perturbative vacuum structure has been discovered to be associated with the mixing of fermion fields in the context of Quantum Field Theory. The careful study of such a structure [27] has led to the determination of the exact QFT formulas for neutrino oscillations [28], exhibiting new features with respect to the usual quantum mechanical Pontecorvo formula [23]. Actually, it turns out that the non–trivial nature of the mixing transformations is a general feature and manifests itself also in the case of bosons [28, 29]. Other non–trivial features, such as the occurrence of a geometric (Berry-Anandan) phase in field mixing, has been also pointed out [30]. The QFT formalism has shown the unitary inequivalence of the vacuum for neutrino fields with definite flavour (flavour vacuum) and the ones with definite mass (mass vacuum). The unitary inequivalence between representations of the canonical (anti-)commutation relations is, in fact, a characteristic feature of QFT, where many inequivalent representations (many different Hilbert space) are allowed for a given dynamics. This aspect is absent in quantum mechanics due to the von Neumann theorem, which states that only one Hilbert space is admitted due to the finiteness of the number of the degrees of freedom of the system under consideration. Many physically relevant aspects in the mixing and oscillation phenomenon are consequences of such a QFT characteristic feature. For example, flavor neutrinos considered as fundamental objects break Lorentz invariance, in that they do not satisfy the standard dispersion relation $E^2 - k^2 = m^2$[1] [31, 32]. Also, the presence of a condensate structure associated ti mixing has suggested the possibility of dynamical generation of mixing [33] - [39].

It is in such a framework that our work is set. We focus on the algebraic structure of the field mixing generator in QFT and on its components, investigating the compatibility of the mixing transformation at level of states and fields. We find that at a very basic algebraic level, the origin of such incompatibility resides in the non–commutativity between rotation and Bogoliubov transformations. A new type of transformations - *non–diagonal Bogoliubov transformations* - arises as a consequence of such non–commutativity. Thus,

[1] In the text we shall always use natural units.

a non–commutative structure led us to investigate how the neutrino mixing is introduced in the context of the "non–commutative spectral geometry" (NCSG) formulated by Alain Connes [40] within which one can get the Lagrangian of the Standard Model minimally coupled with gravity.
The term *non–commutative geometry* was used for the first time by von Neumann to denote in general a geometry in which the algebra of functions forms a non–commutative algebra. Exactly like in the quantization of classical phase-space, coordinates are substituted by generators of the algebra [41] but, since these do not commute they cannot be simultaneously diagonalized and the space disappears. An intuitive idea to *think* a point in this case could be a Planck cell of dimension given by the Planck area [42], analogously to how Bohr cells replace classical phase-space points. NCSG combines non–commutative geometry and spectral triples. In the Connes' construction the coupling with gravity is obtained thanks to the fact that this construction uses a group of symmetry which encodes both the diffeomorphism, which control general relativity, and the local gauge invariance, on which the SM is based. The key point in Connes' construction is the doubling of the space, which induce the doubling of the algebra. Such doubling can be seen as the seed of neutrino mixing, naturally encoding it in NCSG, *i.e.* there is an internal, self-consistent reason to introduce neutrino mixing in the calculation. It is important to stress that our analysis in this thesis is limited to the case of two Dirac neutrinos. However, we have good reasons to believe that the present results are general, since our arguments are of algebraic nature. The results also extend to the mixing phenomenon of any particle, and are not limited to the case of Dirac neutrinos. Extension to three neutrinos is in our plans. In Ref. [43] it has been discussed the non–commutative algebraic structure in quantum cosmology. In this connection, with special reference to the role played by the extension of the Wheeler-DeWitt (WDW) equation in minisuperspace cosmology, we have been also studying general perturbation terms to be added to the Hamiltonian, applied to known semiclassical solutions [44], *e.g.* a stochastic interaction term, for which first order perturbative corrections are computed. Such an interaction can be used to describe the interaction of the cosmological background with the microscopic d.o.f. of

the gravitational field (see Refs. [45], [46, 47]). In this thesis, however, we will not discuss such an issues of non–commutative quantum Cosmology which would lead us too far from the central object of study.
Finally, it has been shown that the phenomenon of neutrino mixing and oscillations can be equivalently understood and described in quantum information language: time dependent (single–particle) entanglement among neutrino flavor states is generated by the time evolution of the system and could be in principle used for quantum information tasks [48, 49].

In view of these developments, it is definitely interesting any attempt to reformulate such a phenomenon in a new language and from an alternative point of view. We thus, have considered the correspondence of neutrino mixing transformations with transformations in (classical) phase space, which could be at some point implemented with non-commutativity in order to fully reproduce the quantum mechanical framework. Here we make a first step in this direction, defining a consistent classical phase space picture of neutrino mixing and oscillations leading to the same oscillation formulas obtained in the quantum formulation.

This thesis is structured as follows: in *Chapter 1* we briefly summarize the main aspects of the Standard Model and of neutrino mixing. *Chapter 2* is dedicated to neutrino mixing and oscillations, considering both the quantum mechanics approach and the quantum field theory formalism. In *Chapter 3* we analyse the mixing generator, decomposing it into components, and the flavor vacuum structure, studying its thermodynamical properties. A non–commutative structure will arise. *Chapter 4* is focused on non–commutativity, giving some known examples of non–commutative systems, and briefly presenting non–commutative geometry and NCSG elements. In *Chapter 5* we introduce further notions on Alain Connes' construction; we summarize how neutrinos appear within this construction and we relate the algebra doubling, which is a crucial element of the NCSG model, to the Hopf non–commutative algebra and Bogoliubov transformations, which play a key role in the neutrino mixing. In *Chapter 6*, in order to better understand the mixing phenomenon, we study a classical system analogue for it. We then close with our *Conclusions and Outlook*.

Chapter 1

Standard Model (SM)

In general, in physics a standard model is a model describing a certain category of physical phenomenons which is compatible with a considerable quantity of non–trivial experimental proofs. The word “model”, in contrast with “theory”, refers to the fact that a satisfactory degree of internal consistency it is not reached, thus the “model” is considered as an approximation of a more complete physical theory which is yet to be discovered.

Last century has been a crucial one in the science progress. Since the 1930s discoveries and theories have resulted in significant understanding of the fundamental structure of the matter. There are four fundamental interactions: electromagnetic, weak, strong nuclear and gravitational. These interactions govern the building blocks of everything can be found in the Universe: the fundamental particles. The Standard Model of particle physics (SM) is precisely that of the elementary constituents of matter and the forces, or interactions, between these elementary constituents. It concerns the electromagnetic, weak, and strong nuclear interactions, as well as classifying all the known subatomic particles.The development of the Standard Model was driven by theoretical and experimental particle physicists. For theorists, the SM is a paradigm of a quantum field theory, which exhibits a wide range of physics including spontaneous symmetry breaking, anomalies and non–perturbative behavior. It is used as a basis for building more exotic mod-

els that incorporate hypothetical particles, extra dimensions, and elaborate symmetries (such as supersymmetry) in an attempt to explain experimental results at variance with the SM, such as the existence of dark matter and neutrino oscillations. From the point of view of the field theory, which is the language in which the SM is formulated, both constituents of matter and interactions are described by fields, *i.e.* operators defined at every point of space-time.
Let us briefly recall the key point of the SM, we remark that a detailed description of the SM falls outside the purpose of this thesis. For a more comprehensive and exhaustive presentation we refer to [50, 51].
All the particles we know can be divided in two big groups: *bosons* and *fermions*; summarized in the following Fig.1.1

- Bosons are responsible of interaction. All interactions are, indeed, produced by the exchange of virtual quanta: the *eight gluons* for the strong interaction, responsible for the nuclear interactions, the *photon* for the electromagnetic ones, responsible for the structure of atoms and molecules, the *three bosons* $W^{\pm}$ and Z for the weak interaction, responsible for β-decay, as well as the decays of many unstable particles. All the radiation quanta for the three interaction are vector (spin-one) fields. Similarly, it is possible that the gravitational interactions result from the exchange of virtual *gravitons*, which is assumed to be a tensor, spin-two field. *Higgs boson* has spin 0.

- Fermions are the constituents of matter. They appear to be all spin half-integer particles and are divided into quarks, which are subject to strong and weak interactions, and leptons which are involved in weak interactions but not in strong ones.

Let us outline quark and lepton features in a few additional points:

1. Quarks and gluons do not appear as free particles. They form a large number of bound states, the hadrons.

2. At present we know the existence of six quark species, called “flavours”.

Each one appears under three forms, called "colours" (no relation with the ordinary sense of these words), *i.e.* Red, Green and Blue.

3. The u , c and t quarks have electric charge equal to $\frac{2}{3}$ times the electric charge of the proton, while the other three d , s and b have charge equal to $\frac{-1}{3}$.

4. Quarks and leptons seem to fall into three distinct groups, or *families.* This family structure is one of the great puzzles in elementary particle physics. The first family is composed by the electron and its associated neutrino as well as the up and down quarks. These quarks are the constituents of protons and neutrons. The role of each member of this family in the structure of matter is obvious. The role of the other two families remains a bit obscure. The muon and the tau leptons seem to be heavier versions of the electron but they cannot be viewed as excited states of it because they seem to carry their own quantum numbers. The associated quarks with exotic names such as charm, strange, top and bottom, form new, unstable hadrons which are not present in ordinary matter. They decay by weak interactions.

5. The sum of all electric charges inside any family (particles and anti-particles) is equal to zero.

One can intuitively represent the force transmitted between two bodies as the result of the exchange of one of the bosons mediators. Moreover, since the theory is relativistic, it is possible the conversion of energy in the field and vice versa, which means that, not only the interactions of exchange processes, but also processes of creation and destruction of particles of matter are possible. Essential components of the Standard Model are the gauge symmetry and the spontaneous breaking of this symmetry which produces the masses. These components are examined in the following sections in the simple case of the symmetry $U(1)$.

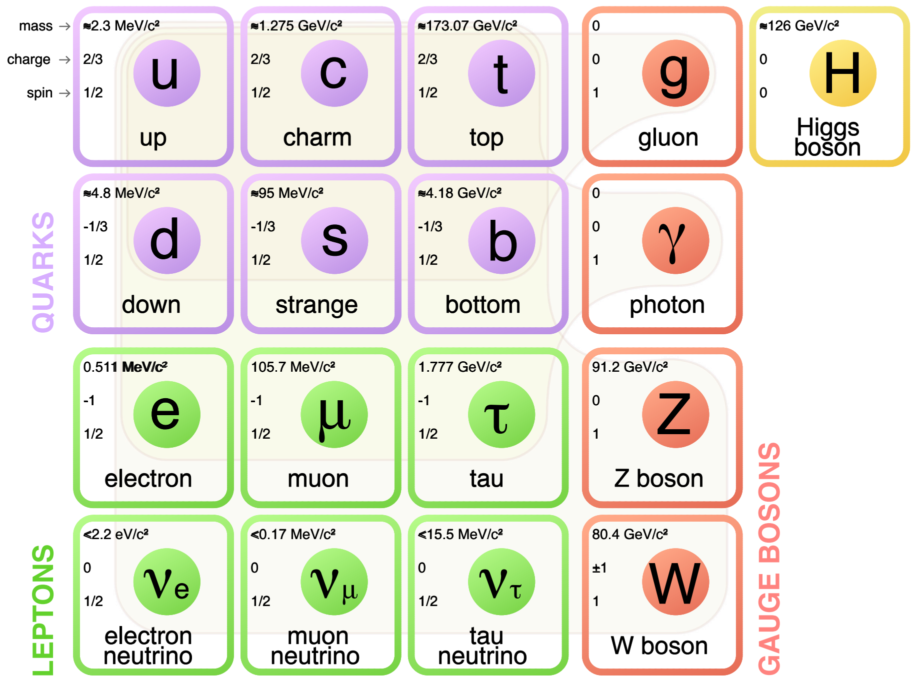


Figure 1.1: Scheme of the particles included in the Standard Model.

1.1 Gauge symmetry

The gauge symmetry is a local transformation (that depends on the point of space-time), which leaves invariant the Lagrangian $\mathcal{L}$, or more precisely, the action

$$S = \int d^4x \mathcal{L}(x).$$

In its simplest form, the symmetry acts on a scalar field $\varphi(x)$ and on a vector one $(A_0(x), A_1(x), A_2(x), A_3(x))$ in the following manner:

$$\varphi(x) \to e^{i\theta(x)}\phi(x)\,, \qquad A_\mu \to A_\mu + \frac{i}{e}\partial_\mu \theta(x), \tag{1.1}$$

to leave invariant the Lagrangian:

$$\mathcal{L} = -F_{\mu\nu}F^{\mu\nu} + D_\mu \phi^* D^\mu \phi \tag{1.2}$$

with

$$F_{\mu\nu} = D_\mu A_\nu - D_\nu A_\mu \,, \qquad\qquad D_\mu = \partial_\mu + ieA_\mu. \tag{1.3}$$

The physical meaning of gauge invariance is that all observable which can be measured experimentally are independent from the transformation in Eq.(1.1): for example, a phase change in the scalar field leaves unchanged all the physical quantities as long as one also transforms the A_μ field under a gauge transformation. As in quantum mechanics, in fields theory each symmetry corresponds to a conserved quantity. Using the Noether theorem one can see that the conserved quantity in the case of the gauge symmetry is the electric charge. At this point it is important to notice that the gauge symmetry is not compatible with a field with non–zero mass. In fact, a mass term $m^2 A_\mu A_\mu$ is not invariant under the transformations in Eq.(1.1). So this symmetry may be valid for the description of a massless photon, but not, for example, for the mediator of weak interactions, which have masses of the order of 100 GeV.

1.2 Spontaneous symmetry breaking: $U(1)$ model

The breaking of the symmetry in a field theory can occur in two different ways. One talks about an *explicit symmetry breaking* when the Lagrangian is not invariant under a given symmetry. For example, let one define a transformation $U(1)$ for a charged scalar field $\phi(x) : \phi(x) \to e^{i\theta(x)}\phi(x)$. If one term of the Lagrangian is proportional to $\phi^2(x)$ the gauge symmetry is explicitly broken; for example let us consider

$$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi + V(\phi^2).$$

One term of the Lagrangian will be invariant under such transformation : $\phi^*\phi \to \phi^*\phi$ but another one won't: $\phi^2(x) \to e^{2i\theta(x)}\phi^2(x)$. On the other hand, one has a *spontaneous breaking of symmetry* when the Lagrangian is invariant under a given symmetry but the vacuum state, *i.e.*, the minimum energy state, it is not. The simplest example of spontaneous breaking is given by the scalar model with *global* invariance $U(1)$:

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - V(\phi); \tag{1.4}$$

$$V(\phi) = \lambda(\phi^*\phi)^2 - \mu^2(\phi^*\phi) \;\; \lambda,\; \mu^2 > 0\,; \tag{1.5}$$

$$U(1) : \phi(x) \to e^{i\theta}\phi(x). \tag{1.6}$$

The potential $V(\phi)$ has a minimum whenever $(\phi^*\phi) = \frac{\mu}{2\lambda} \equiv v^2$, considering the $U(1)$ symmetry. In other words, there is a degeneracy; in particular the vacuum is infinite times degenerate. The symmetry is broken in the moment one chooses one particular vacuum to build the field theory among the degenerate ones. Because of the invariance given by the symmetry, one can choose as vacuum the field configuration in which $\langle\phi(x)\rangle = v\,\forall x$. (In nature there exist many examples of spontaneous symmetry breaking; one is given by the ferromagnetic materials which are described by a rotation invariant Hamiltonian. In the fundamental state a non–zero spin alignment exists, *i.e.* a magnetization $M \neq 0$.) The field can, thus, be rewritten as $\phi(x) = v + \sigma(x) + i\chi(x)$, $\langle 0|\phi(x)|0\rangle = v$, where $\sigma(x)$ and $\chi(x)$ are the "small oscillations" fields, with average value equal to zero, which one can quantize following the canonical rules. Re-writing the Lagrangian in Eq. (1.4) with the condition in Eq. (1.5) in terms of these fields, one obtains:

$$\mathcal{L} = \partial_\mu\sigma\partial^\mu\sigma + \partial_\mu\chi\partial^\mu\chi - 4\lambda v^2\sigma^2 - \lambda(\sigma^2+\chi^2)^2 - 2v\sigma(\sigma^2+\chi^2). \tag{1.7}$$

One, thus, obtain a (*e.g.* Higgs) boson σ with mass $M_H^2 \sim \lambda v^2$ and a (Goldstone) boson with zero mass which interact with the trilinear an quartic interaction determined by the potential. In general the number of Goldstone bosons with zero mass, which exist only in the case of spontaneous breaking of a continuous symmetry, is given by the number of broken generators, *i.e.* the number of generator which do not leave the vacuum invariant. In fact, because of the potential invariance, every point obtained from the vacuum through one of these transformation has the same value of the potential.

The small oscillation in the direction of these broken generators correspond to "flat" directions, *i.e.* zero mass particle. Qualitatively different phenomena are obtained when there is a local symmetry breaking. Considering the Lagrangian in Eq.(1.4) with a gauge symmetry, *i.e.* an interaction with a vector field A_μ :

$$\mathcal{L} = -F_{\mu\nu}F^{\mu\nu} + D_\mu\phi^* D^\mu\phi - V(\phi); \tag{1.8}$$

$$V(\phi) = \lambda(\phi^*\phi)^2 - \mu^2(\phi^*\phi) \;\; \lambda, \mu^2 > 0\,; \tag{1.9}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu; \tag{1.10}$$

$$D_\mu = \partial_\mu + ieA_\mu. \tag{1.11}$$

This Lagrangian is invariant under the gauge symmetry

$$U(1) : \phi(x) \to e^{i\theta}\phi(x)\,,\; A_\mu(x) \to A_\mu(x) + \frac{i}{e}\partial_\mu\theta(x).$$

Similar considerations to the ones we have done above lead to the conclusion that there is a spontaneous breaking with $\langle A_\mu\rangle = 0$, $\langle\phi\rangle = v \neq 0$. It is convenient to choose the following parametrization $\phi(x) = H(x)\exp[i\frac{\theta(x)}{v}]$, with $H(x)$ and $\theta(x)$ real fields and $\langle H\rangle = v$. Again it will be possible to describe the small oscillations writing $H(x) = v + \sigma(x)$. The Lagrangian, thus, becomes:

$$\mathcal{L} = -F_{\mu\nu}F^{\mu\nu} + \partial_\mu\sigma\partial^\mu\sigma + \partial_\mu\theta\partial^\mu\theta + e^2v^2A_\mu A^\mu - 4\lambda v^2\sigma^2 + \ldots \tag{1.12}$$

where the dots indicate the interaction terms between A_μ, σ, θ. As in the case of the $U(1)$ global symmetry breaking, one obtains a Goldstone boson θ and a Higgs boson σ. Nonetheless, the photon A_μ, has gained mass equal to ev. The θ field can be eliminated with a particular choice of the gauge (unitary gauge). Indeed, choosing a transformation with the parameter $-\frac{\theta}{v}$ such as $\phi(x) = H(x)\exp[i\frac{\theta(x)}{v}] \to \exp[-i\frac{\theta(x)}{v}]H(x)\exp[i\frac{\theta(x)}{v}] = H(x)$, the θ field disappears from the theory. On the other hand, the field A_μ gain a longitudinal component, *i.e.* proportional (in the Fourier transform) to k_μ where k is the photon momentum: $A_\mu(x) \to A_\mu(x) - \frac{i}{e}\partial_\mu\frac{\theta(x)}{v}$. The counting of the degrees of freedom is consistent: before the breaking one has one massless photon with 2 transverse polarization and one charged scalar, *i.e.* 4 degrees of freedom; after the symmetry breaking one has one massive photon whit another longitudinal polarization and a non–charged scalar (Higgs), again 4 degrees of freedom. This mechanism by which, when a gauge symmetry is spontaneously broken, the boson vector gains mass "eating" a Goldstone boson which gives it its longitudinal polarization, is called Higgs mechanism.

1.3 Standard Model Lagrangian

1.3.1 Gauge term

The Standard Model Lagrangian is totally determined by the renormalization requirement, by the gauge symmetry and by the particle it describes. The group which describes the gauge symmetry is the non–abelian group $SU(3)_{color} \otimes SU(2)_{weak} \otimes U(1)_{hypercharge}$. The subgroup $SU(3)_{color}$ describes the color, *i.e.* the strong interaction charge. The subgroup $SU(2)_{weak} \otimes U(1)_{hypercharge}$ describes the electroweak interaction, in this sector one also have the spontaneous symmetry breaking and the generation of the masses, and is the sector we are going to describe in more detail. Given the $SU(2)_{weak} \otimes U(1)_{hypercharge}$ symmetry and the particle representation in such symmetry, the Lagrangian contains all and only the renormalizable interaction, *i.e.* with operator of dimension 4 or smaller. Regarding the particles, or matter field, let us focus on the first family: the fermionic fields u, d, ν, e. A Dirac spinor is a 4 component field formed by two Weyl spinors:

$$\Psi = \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix}. \tag{1.13}$$

The two Weyl spinors Ψ_L, Ψ_R are the usual two components spinors of the quantum mechanics, but belong to two different representation of the Lorentz group (Appendix - Lorenz group). Physically, they substantially correspond to electrons with spin parallel and antiparallel to the direction of motion. One of the main features of the SM is that weak interaction treat in a different way left handed and right handed fermions, putting them in two different $SU(2)$ representations. In particular, left fields are described with a doublet of weak isospin, while right handed fields are singlet:

$$Q_L \equiv \begin{pmatrix} u_L \\ d_L \end{pmatrix} \to U(x) \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \qquad d_R \to d_R,\ u_R \to u_R \tag{1.14}$$

where $U(x)$ is an $SU(2)$ matrix depending on the point x. One also introduces three gauge fields A_1, A_2, A_3, which are an isospin triplet, and a field B_μ, which is a singlet. The gauge transformations for these fields are:

$$A^a_\mu \to A'^a_\mu \ :\ A'^a_\mu \tau^a = U A^a_\mu \tau^a U^{-1} + (\partial_\mu U)U^{-1}, \tag{1.15}$$

$$B_\mu \to B'_\mu = B_\mu + i\partial_\mu \theta, \tag{1.16}$$

with τ^a, $a = 1, 2, 3$ Dirac matrices and $U(x) = \exp[i\alpha^a(x)\tau^a + i\theta(x)Y]$. It is now possible to write the gauge terms of the SM Lagrangian, invariant under the transformations in Eqs. (1.14), (1.15):

$$\begin{aligned} \mathcal{L}_{gauge} \quad = \quad & -\frac{1}{4}Tr\{\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}\} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} + \sum_k i\bar{\Psi}_k \mathbf{D}_\mu \gamma^\mu \Psi_k\,; \qquad (1.17) \\ & \mathbf{A}_\mu = A^a_\mu T^a\,, \\ & \mathbf{D}_\mu = \partial_\mu - ig\mathbf{A}_\mu - ig' B_\mu Y\,, \\ & \mathbf{F}_{\mu\nu} = i[\partial_\mu - ig\mathbf{A}_\mu, \partial_\nu - ig\mathbf{A}_\nu] = F^a_{\mu\nu}T^a\,, \\ & B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu\,. \end{aligned}$$

Note that the generator of the $SU(2)$ group T^a satisfies the relations $[T^a, T^b] = i\varepsilon_{abc}T^c$ and act on the fields depending on the representations; *i.e.* $T^a e_R = 0$, so that e_R does not interact with the non–abelian gauge fields A_i. The k index, thus, vary on the 5 representation of the first family : $L = (\nu_L, e_L)$[1],

[1] Since we are considering only the first family in order to have a lighter notation here we indicate $\nu_e \equiv \nu$.

$Q = (u_L, d_L)$, e_R, u_R, d_R, having considered the neutrino as *massless*.

1.3.2 Higgs and Yukawa terms

The gauge symmetry is not compatible with a non–zero mass of the gauge boson, this fact contradicts the experimental observations of gauge boson masses of the order of $100\,GeV$. In contrast with what happens in QED, in which a fermionic mass term $m\bar{\Psi}\Psi$ is allowed by the abelian symmetry $\Psi \to e^{i\theta}\Psi$, in the SM this term is not allowed because of the gauge symmetry and the left-right fermions asymmetry. In fact, one can write a mass term as:

$$m\bar{\Psi}\Psi = m(\bar{\Psi}_R\Psi_L + \bar{\Psi}_L\Psi_R)\,, \qquad \Psi_L = \frac{1-\gamma_5}{2}\Psi\ ,\ \Psi_R = \frac{1+\gamma_5}{2}\Psi, \tag{1.18}$$

while a mass term for quark u, d, for example, is written as:

$$m_u(\bar{u}_L u_R + \bar{u}_R u_L) + m_d(\bar{d}_L d_R + \bar{d}_R d_L), \tag{1.19}$$

and this term is clearly non–invariant under the gauge transformation defined in Eq.(1.14) which rotates only the left components. Fermion masses, like the boson masses, have to be generated via the spontaneous symmetry breaking mechanism. To this purpose, the Higgs sector has to be included in the Lagrangian. This term is made up by a $SU(2)$ scalar: ϕ.

$$\phi = \begin{pmatrix} \varphi_+ \\ \varphi_0 \end{pmatrix}\ ;\ \bar{\phi} \equiv i\sigma_2\phi^* = \begin{pmatrix} \varphi_0^* \\ -\varphi_- \end{pmatrix}, \tag{1.20}$$

which transforms under gauge as:

$$\phi \to U\phi \qquad \bar{\phi} \equiv i\sigma_2\phi^* \to i\sigma_2 U^*\phi^* = Ui\sigma_2\phi^* = U\bar{\phi} \tag{1.21}$$

i.e. ϕ and $\bar{\phi}$ transform in the same way. With the Higgs field one can build the following invariant renormalizable terms of dim 4[2]:

$$(\phi^\dagger_\alpha \phi_\alpha)^2 = (\varphi_- \varphi_+ + \varphi_0 \varphi_0^*)^2 \,, \tag{1.22}$$
$$\bar{Q}^\alpha_L d_R \phi^\alpha + h.c. = \bar{u}_L \phi_+ u_R + \bar{d}_L \phi_0 d_R + h.c. \,.$$

The term in the second line of Eq.(1.22) is called "Yukawa" term and it is responsible for the fermion masses, together with analogue terms which involves left $Q_L = (U_L, D_L)$ and right leptons. Regarding the $U(1)$ invariance, the following relations hold:

$$y(D_R) - y(Q_L) = q_D - (q_D - t^3_D) = t^3_D = -\frac{1}{2} \,,$$

$$y(U_R) - y(Q_L) = q_U - (q_U - t^3_U) = t^3_U = \frac{1}{2} \,,$$

therefore, $y(\phi) = \frac{1}{2}$. Finally, the part of the SM Lagrangian which involves the Higgs doublet and its interaction with the fermions is:

$$Y_d \bar{Q}_L \phi d_R + Y_u \bar{Q}_L \bar{\phi} u_R + h.c. - V(\phi) \,, \qquad V(\phi) = \lambda(\phi^\dagger \phi - v^2)^2 \,, \tag{1.23}$$

with Y_d, Y_u, λ arbitrary dimensionless constant, while v is an arbitrary constant with mass dimension. It is important to observe that the form of this interaction is the most general possible: these are all and only the interactions compatible with the gauge symmetry and with the request of renormalizability for the theory.

1.4 Mixing within the SM

Experimental evidences show that the some of the particles of different families summarized in Figure 1.1, mix between each other, so that a particle of a given family transforms into one of another family. This happens, for

[2] α are the isospin $SU(2)$ index; h.c. means hamiltonian conjugate, $\varphi_{\equiv}\varphi^*_+$.

instance, for quarks (in brackets the quark content of the mesons):

$$K^+(u\bar{s}) \to \pi^0(u\bar{u})e^+\nu_e\,,$$

$$B^0(d\bar{b}) \to \pi^-(d\bar{u})l^+\nu_l\,.$$

The quantitative analysis of such processes, and thus the confrontation between theory and experiment, is difficult because of the fact that the quarks do not exist as free particles and lies outside the purpose of this thesis.

1.4.1 Flavor and Cabibbo-Kobayashi-Maskawa matrix

In order to better understand this phenomenon, one should go back to the Higgs sector and Yukawa terms in Eq.(1.23) and consider the existence of 3 families (i, j are family indecs):

$$-\lambda(\phi^\dagger\phi - v^2)^2 + \bar{Q}^i_L M^D_{ij} D^j_R \phi + h.c. + \bar{Q}^i_L M^U_{ij} U^j_R \bar{\phi}^\alpha + h.c. \tag{1.24}$$

Because the Yukawa couplings generally involve fermions belonging to different families, flavour-space fermion mass matrices are not diagonal. This means that mass eigenstates are different from weak eigenstates which have definite gauge transformation properties. Since a generic matrix M can be diagonalized via a unitary matrix the expression in Eq.(1.24) can be simplified with a basis change: $Q_L \to V^\dagger_{Q_L} Q_L$, $U_L \to V_{U_R} U_R$, $D_R \to V_{D_R} D_R$ with V unitary matrices.[3] One can choose a basis in which M^U is diagonal, $V^\dagger_{Q_L} M^U V_{D_R} = Diag(m^u_i)$ where m^u_i are the eigenvalues (real). Moreover, since $M^D = HV$ one can choose $V_{D_R} = V^\dagger V^\dagger_{Q_L}$. Eq.(1.24) can thus be rewritten as

$$-\lambda(\phi^\dagger\phi - v^2)^2 + \bar{Q}^i_L \Lambda^D_{ij} D^j_R \phi + h.c. + \lambda^u_i \bar{Q}^i_L U^j_R \bar{\phi}^\alpha + h.c.\,, \tag{1.25}$$

with $H_D = H^\dagger_D$, $\lambda^u_i > 0$ and Λ 3x3 hermitian matrix. Quark masses can be

[3]Such a basis change is allowed because kinetic terms are not affected by it, and thus propagating states are invariant.

obtained from Eq.(1.25) via $\phi \to \langle\phi\rangle$:

$$
\begin{aligned}
&Z_\mu[\bar{d}_L^i(t_3 - s_w^2 q)\gamma^\mu d_L^i + \bar{d}_R^i(t_3 - s_w^2 q)\gamma^\mu d_R^i + d \leftrightarrow u] + \\
&+W_\mu^+ \bar{u}_L^i \gamma^\mu d_L^i + v\bar{d}_L^i \Lambda_d^{ij} d_R^j + \lambda_i^u v \bar{u}_L^i u_R^i + h.c.\,. \qquad (1.26)
\end{aligned}
$$

Eq.(1.26) shows that it is not possible to apply an unitary transformation on the fields, *i.e.* a base change, which diagonalizes at the same time the mass terms and the gauge interactions. Actually, it is possible to diagonalize Λ_d^{ij} with a rotation of the fields d_L, d_R but this inevitably leads to non–diagonal gauge interaction terms in the charge sector. On the other hand, whichever rotation of the fields leaves invariate the non–charged interaction. It is common use to choose the basis in which the mass terms, *i.e.* the propagators, are diagonal in the flavor. Thus one has to diagonalize Λ so that $d_{L,R} \to V^{CKM} d_{L,R}$ with $(V^{CKM})^\dagger V^{CKM} = 1$. One obtains

$$
m_i^d d_L^i d_R^i + h.c. m_i^u \bar{u}_L^i u_R^i + h.c. \qquad (1.27)
$$

and

$$
(W_\mu^- \bar{u}_L^i V_{CKM}^{ij} \gamma^\mu d_L^j + h.c.)\,. \qquad (1.28)
$$

The unitary matrix V^{CKM} is called Cabibbo-Kobayashi-Maskawa 3x3 matrix

$$
\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}\,.
$$

Considering all the symmetries of the system one has that V is a matrix with four parameter: three rotation angles θ_{ij} and a complex phase $e^{i\delta}$.

$$
V^{CKM} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{-i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{-i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{-i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{-i\delta_{13}} & c_{23}c_{13} \end{pmatrix},
$$

where $c_{ij} = \cos\theta_{ij}$, $s_{ij} = \sin\theta_{ij}$.

1.4.2 Symmetry breaking; boson and fermion masses

The form of the potential in Eq.(1.25) causes the spontaneous symmetry breaking of the $SU(2) \otimes U(1)$ which leads the Higgs scalar to gain a vacuum expectation value (vev) $\langle 0|\phi|0\rangle$. The direction of such symmetry breaking it is not arbitrary because it has to respect $U(1)_{em}$. This means that it cannot be the charged field φ^+ to gain a non–zero vev since it is not invariant under $U(1)_{em}\,\varphi^+ \to e^{i\theta}\varphi^+$. It is indeed the non–charged part that gains a vev v:

$$\phi_0 = \begin{pmatrix} 0 \\ v \end{pmatrix} \; ; \qquad \phi = \begin{pmatrix} \phi_1 + i\varphi_2 \\ v + \sigma + i - \varphi_3 \end{pmatrix} . \tag{1.29}$$

The symmetry breaking does not occur in a simple way: ϕ_0 is invariant under the generator $Q \equiv T_3 + Y$ while it is not invariant under $T_3 - Y$, T_2, T_1. The structure of the symmetry breaking is thus $SU(2)_L \otimes U(1)_Y \to U(1)_{em}$. Accordingly, one have 3 Goldstone bosons φ_1, φ_2, φ_3, which correspond to the directions in which the symmetry has been broken. Those bosons are the ones which give mass to the W^+, W^-, Z bosons via Higgs mechanism. In fact, the Higgs sector of the Lagrangian reads:

$$\mathcal{L}_{Higgs} = (D_\mu\phi)^\dagger D_\mu\phi - \lambda(\phi^\dagger\phi v^2)^2 . \tag{1.30}$$

The part which gives the masses to the bosons can be rewritten as:

$$\begin{aligned} &\langle\phi\rangle(gA^a_\mu T^a + g'B_\mu Y)(gA^b_\mu T^b + g'B_\mu Y)\langle\phi\rangle = \\ &g^2v^2(A^1_\mu A^1_\mu + A^2_\mu A^2_\mu) + v^2(gA^3_\mu - g'B_\mu)^2 . \end{aligned} \tag{1.31}$$

The spectrum is thus made by the following bosons with the corresponding masses[4]:

$$W^{\pm} \equiv \frac{A_1 \mp i A_2}{\sqrt{2}} \, ; \; M_W = g v \, , \tag{1.32}$$

$$Z \equiv c_W A_3 - s_W B \, ; \; M_Z = \sqrt{g^2 + g'^2} v \, , \tag{1.33}$$

$$A \equiv s_W A_3 + c_W B \, ; \; M_\gamma = 0 \, , \tag{1.34}$$

with

$$g A^3_\mu T^3 + g' B_\mu Y = \frac{g}{c_W} Z_\mu (T_3 - s^2_W Q) + e Q A_\mu \, , \qquad e \equiv g s_W = g' c_W \, . \tag{1.35}$$

This last expression clarifies the reason why the photon does not gain mass: $Q\langle\phi\rangle = 0$. What gains mass is, on the other hand, $g A^3_\mu T^3 + g' B_\mu Y$, which is the Z boson. The $U(1)_{em}$ invariance ensures that the photon does not acquires mass at any order of the perturbation. The interaction terms between fermion and bosons can be rewritten as:

$$\frac{g}{c_W} Z_\mu \sum_k i \bar{\Psi}_k \gamma_\mu (T^3 - s^2 Q) \Psi_k + \bar{\Psi}_k \gamma_\mu Q \Psi_k + g (W^+_\mu \bar{\Psi}_k \gamma_\mu T^+ \bar{\Psi}_k + h.c.) \tag{1.36}$$

Quarks masses are obtained by Eq.(1.23) substituting $\phi \to \langle\phi\rangle$:

$$Y_d v \bar{d}_L d_R + h.c. Y_d v \bar{u}_L u^i_R + h.c. - V(\phi) \, . \tag{1.37}$$

Fermions, thus, acquire a mass proportional to the vev $m \sim Y v$, which means that the Y coupling of the fermions with the Higgs boson are proportional to their masses.

[4] Here $c_w = \cos\theta_w$, $s_w = \sin\theta_w$ and θ_w is the weak mixing angle, which characterizes the embedding of $U(1)_{em}$ into the full gauge group $SU(2) \otimes U(1)$

1.5 Discrete Symmetries

Parity (P), Charge Conjugation (C)

Discrete symmetries are very important in physics; among the most important one, there are Parity (P), which inverts the sign of the spatial coordinates of a four vector and leaves unchanged the time component, and Charge Conjugation (C). As an example on how these symmetries work one can show how they act on the electromagnetic current $J_\mu = e\bar{u}\gamma_\mu u$. Under the action of C the charge changes its sign and therefore the whole current switches sign. Conversely, under the action of P, only the three velocities change their signs. Therefore, $J_\mu \xrightarrow{C} -J_\mu$ while $J_\mu \xrightarrow{P} J_\mu$. Moreover, A_μ transformation can be derived from the knowledge of the fact that the $J_\mu A^\mu$ Lagrangian term is invariant in QED, thus $A_\mu \xrightarrow{C} -A_\mu$ while $A_\mu \xrightarrow{P} A_\mu$. Let us now analyze the more general case of current (charged or not) which involves either left handed or right handed fields, starting with the parity P. A left-handed fermion is characterized by parallel spin and momentum in the high energy limit. Parity leaves invariant the spin, switching the sign of the momentum, therefore changing the left-handed fermion into a right-handed one, with antiparallel spin and momentum.

$$\bar{u}_L\gamma_\mu d_L \xrightarrow{P} \bar{u}_R\gamma_\mu d_R\,; \quad \bar{u}_R\gamma_\mu d_R \xrightarrow{P} \bar{u}_L\gamma_\mu d_L\,. \tag{1.38}$$

The Standard Model is, therefore, not invariant under P. For example the charged currents only involve left fermions. The term of charged current is transformed by P in a term with only right fields which do not exist in the Lagrangian:

$$W^-_\mu\bar{u}_L\gamma_\mu d_L + h.c. \xrightarrow{P} W^-_\mu\bar{u}_R\gamma_\mu d_R + h.c.\,. \tag{1.39}$$

Intuitively charged current distinguish between right-handed and left-handed fermions. Charge Conjugation changes the sign of the charge of the particle. Since a fermion with parallel spin and momentum is left-handed and a fermion with anti parallel spin and momentum is right-handed, C, besides conjugating the fields, switches left and right, therefore :

$$\bar{u}_L\gamma_\mu d_L \xrightarrow{P} -\bar{d}_R\gamma_\mu u_R\,; \quad \bar{u}_R\gamma_\mu d_R \xrightarrow{P} -\bar{d}_L\gamma_\mu u_L\,. \tag{1.40}$$

Hence, the SM is not invariant under C either:

$$W_\mu^-\bar{u}_L\gamma_\mu d_L + W_\mu^+\bar{d}_L\gamma_\mu u_L \xrightarrow{C} W_\mu^-\bar{u}_R\gamma_\mu d_R + W_\mu^+\bar{d}_R\gamma_\mu u_R\,. \tag{1.41}$$

It seems natural to wonder whether the SM is invariant under the simultaneous application of C ad P (CP) since the first would switch left to right and the second would switch back right to left:

$$\begin{aligned} W_\mu^-\bar{u}_L\gamma_\mu d_L + W_\mu^+\bar{d}_L\gamma_\mu u_L &\xrightarrow{C} W_\mu^-\bar{u}_R\gamma_\mu d_R + W_\mu^+\bar{d}_R\gamma_\mu u_R \\ &\xrightarrow{P} W_\mu^-\bar{u}_L\gamma_\mu d_L + W_\mu^+\bar{d}_L\gamma_\mu u_L\,. \end{aligned} \tag{1.42}$$

Nevertheless, one should consider the CKM matrix terms since C conjugates the Lagrangian terms. The global effect is:

$$W_\mu^-\bar{u}_L^i V_{ij}\gamma_\mu d_L^j + h.c. \xrightarrow{CP} W_\mu^-\bar{u}_R\gamma_\mu d_R + W_\mu^-\bar{u}_L^i V_{ij}^*\gamma_\mu d_L^j + h.c.\,. \tag{1.43}$$

which means that in order for the Lagrangian to be CP invariant CKM matrix terms have to be real. Such a thing is not possible with three families: a complex phase factor remains. This phase is the reason why in the SM one has violations of CP, for example in the $K-\bar{K}$ case.[5]

Leptonic number and baryonic number

Other important discrete symmetries in the SM are the baryon and the lepton number. The baryon number, of course, counts the baryons (for which

[5]Such a thinking would not happen with only one or two families.

its value is 1), *i.e.* the heavy hadrons like neutron and proton; while it is 0 for the light hadrons like meson π, K etc. $B = \frac{1}{3}$ for any quark, and 1 for any baryon composed by three quarks. The symmetry is a global $U(1)_B$ so that $\psi \rightarrow e^{i\theta B}\psi$, where B is the baryon number. The baryon number is conserved. Such a symmetry is an accidental one, *i.e.* it is not direct consequence of the gauge symmetry, it rather depends on the gauge symmetry, the renormalizability and chosen representation for the elementary particles. Finally, the lepton number (1 for leptons and -1 for anti-leptons) it is a good symmetry, in fact, in the SM not only the total leptonic number $N_e + N_\mu + N_\tau$ is conserved but also the single family ones N_e, N_μ, N_τ separately, since there are no current which link different families, as happens with the quarks.

1.6 Neutrino mixing within and beyond the SM

As we already said in the Introduction, it is an experimental fact that neutrinos oscillate in flavor, *i.e.* change from a family to another, for instance an electronic neutrino ν_e can transform into a muonic ν_μ or tauonic ν_τ one, or viceversa. The importance of neutrino oscillations is in the fact that they are signature of small neutrino masses and neutrino mixing. The discovery of this phenomenon took more than forty years. Let us now consider briefly the evolution of original ideas of neutrino masses, mixing and oscillations. In 1930, introducing neutrino, Pauli assumed [1] that this particle is a weakly interacting particle with no charge, spin 1/2 and a really small mass, much smaller than electron mass. In 1933 by Fermi [3] and Perrin [52] proposed the first method to measure neutrino mass: search for effects of neutrino mass via detailed investigation of the high-energy part of β-spectra which correspond to the emission of neutrino with a small energy. For the first period no effects of neutrino masses were found in these experiments until the first limit for the upper bound of the neutrino mass was obtained $m_\nu \leq 500eV$ [54]. In 1957 after the violation of the parity in the β-decay was discovered [55], the two-component neutrino theory was proposed by Landau [53], Lee and

Yang [56] and Salam [57]. Let us consider the Dirac equation for the field of neutrino with mass, in order to demonstrate the idea of the two-component neutrino

$$i\gamma^{\alpha}\partial_{\alpha}\nu(x) - m_{\nu}\nu(x) = 0\,. \tag{1.44}$$

For left-handed and right-handed components $\nu_L(x)$ and $\nu_R(x)$ from Eq.(1.44) we have two coupled equations

$$i\gamma^{\alpha}\partial_{\alpha}\nu_L(x) - m_{\nu}\nu_R(x) = 0 \tag{1.45}$$

and

$$i\gamma^{\alpha}\partial_{\alpha}\nu_R(x) - m_{\nu}\nu_L(x) = 0\,. \tag{1.46}$$

Landau, Lee and Yang and Salam assumed that the neutrino mass is equal to zero, taking into account the bound $m_{\nu} \leq (100-200)eV$. Choosing $m = 0$ from Eq.(1.45) and Eq.(1.46) they obtain two decoupled Weyl equations

$$i\gamma^{\alpha}\partial_{\alpha}\nu_{L,R}(x) = 0 \tag{1.47}$$

and the neutrino field can be in this case $\nu_L(x)$ or $\nu_R(x)$. If the neutrino field is $\nu_L(x)$ $(\nu_R(x))$

1. the general Hamiltonian of the β-decay has the form

$$H_I^{\beta} = \sum_i G_i(\bar{p}O_i n)(\bar{e}O_i\frac{1}{2}(1 \pm \gamma_5)\nu) + h.c. \tag{1.48}$$

   where the index i runs over S, V, T, A, P (scalar, vector, etc). Thus, the two-component neutrino theory allows for a large violation of parity, observed in the β-decay, thanks to the vectorial and the axial contributes.

2. Neutrino helicity is equal to $-1(+1)$ and antineutrino helicity is equal to $+1(-1)$ in the case of $\nu_L(x)$ $(\nu_R(x))$.

the neutrino helicity was indeed measured in 1958 in the famous M. Goldhaber et al. [58] experiment. In this experiment it was possible to determine the longitudinal polarization of neutrinos. It was found that the neutrino is a left-handed particle. Thus, the neutrino field is $\nu_L(x)$. It is interesting to notice that equations (1.47) for a massless particle were discussed by Pauli in his encyclopedia article "General Principles of Quantum Mechanics" (1933). Pauli wrote that because equation $\nu_L(x)$ $(\nu_R(x))$ is not invariant under space reflection it is "not applicable to the physical reality". From the point of view of the two-component theory large violation of parity in the β-decay and other leptonic processes is ultimately connected with zero-mass neutrinos. This point of view changed after Feynman and Gell-Mann [59], Marshak and Sudarshan [60] proposed in 1958 $V - A$ theory. This theory was based on the assumption that in the Hamiltonian of the weak interaction enter left-handed components of all fields. This means that the violation of parity in the weak interaction is not connected with exceptional properties of neutrinos. Moreover, after the $V - A$ theory it was natural to turn up arguments and consider neutrino as a particle with a mass different from zero.

Nevertheless, the two-component neutrino theory was a nice and the simplest theoretical possibility. It was in a perfect agreement with numerous experiments on the investigations of weak processes. Probably this was the main reason why during many years there was a common opinion that neutrinos are massless particles. The Glashow-Weinberg-Salam Standard Model was build under the assumption of massless two-component neutrinos, which means that neutrino oscillations are an experimental proof of the necessity to implement and modify the SM into a more complete physical theory.

Chapter 2

Neutrino mixing

2.1 Neutrino Mixing in Quantum Mechanics: B. Pontecorvo's idea

The first idea of neutrino masses, mixing and oscillations was suggested by B.Pontecorvo [6] in 1957. He thought that there is an analogy between leptons and hadrons and he believed that in the lepton world exists a phenomenon analogous to the famous $K_0 \leftrightarrow \bar{K}_0$ oscillations [50]. The only possible candidate were neutrinos. At that time only one neutrino type was known. Possible oscillations in this case are $\nu_L \leftrightarrow \bar{\nu}_L$ and $\bar{\nu}_R \leftrightarrow \nu_R$. According to the two-component neutrino theory the states $\bar{\nu}_L$ and ν_R do not exist. Such states were a problem for B. Pontecorvo [61]. The discovery of the muonic neutrino ν_μ solved this problem, in the paper of the 1978 B. Pontecorvo and S.M.Bilenky [62] analyzed the $\nu_e \leftrightarrow \nu_\mu$ oscillation, using both Majorana and Dirac neutrinos.

2.1.1 Majorana neutrinos

The first theory of neutrino oscillations was based on the two component neutrino theory, whose Hamiltonian is made up only by left-handed components of the neutrino fields and right-handed components of antineutrino fields:

$$\begin{aligned} \nu_{eL} &= \frac{1+\gamma_5}{2}\nu_e, \quad \nu_{\mu L} = \frac{1+\gamma_5}{2}\nu_\mu\,, \\ \nu^c_{eR} &= \frac{1-\gamma_5}{2}\nu^c_e = (\nu_{eL})^c, \quad \nu^c_{\mu R} = \frac{1-\gamma_5}{2}\nu^c_\mu = (\nu_{\mu L})^c\,. \end{aligned} \tag{2.1}$$

Where $\nu^c_{e,\mu} = C\bar{\nu}_{e,\mu}$ is the charge conjugate spinor, and the matrix C satisfies the following relations:

$$\begin{aligned} &C^+C = 1\,, \\ &C\gamma^T_\alpha C^{-1} = -\gamma_\alpha\,, \\ &C^T = -C\,. \end{aligned} \tag{2.2}$$

The Hamiltonian, quadratic in the neutrino field will, then, be:

$$H = m_{\bar{e}e}\bar{\nu}^c_{eR}\nu_{eL} + m_{\bar{\mu}\mu}\bar{\nu}^c_{\mu R}\nu_{\mu L} + m_{\bar{\mu}e}(\bar{\nu}^c_{\mu R}\nu_{eL} + \bar{\nu}^c_{eR}\nu_{\mu L}) + h.c.\,, \tag{2.3}$$

where $m_{\bar{e}e}$, $m_{\bar{\mu}\mu}$ and $m_{\bar{\mu}e}$ have the dimension of a mass and are real if the Eq. (2.3) is invariant under CP-transformation. This Hamiltonian does not conserve lepton number and can be written more compactly as

$$H = \bar{\nu}^c_R M \nu_L + \bar{\nu}_L M^+ \nu^c_R\,, \tag{2.4}$$

where

$$\nu_L = \begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \end{pmatrix} \qquad \nu^c_R = \begin{pmatrix} \nu^c_{eR} \\ \nu^c_{\mu R} \end{pmatrix} \tag{2.5}$$

and

$$M = \begin{pmatrix} m_{\bar{e}e} & m_{\bar{\mu}e} \\ m_{\bar{\mu}e} & m_{\bar{\mu}\mu} \end{pmatrix}\,. \tag{2.6}$$

If the matrix M in Eq. (2.6) is real, it is possible to rewrite the Eq. (2.4) as follows:

$$H = \bar{\nu}_R^c M(\nu_L + \nu_R^c) + \bar{\nu}_L M(\nu_R^c + \nu_L) = \bar{\chi} M \chi \,, \tag{2.7}$$

where

$$\chi = \nu_L + \nu_R^c = \begin{pmatrix} \nu_{eL} + \nu_{eR}^c \\ \nu_{\mu L} + \nu_{\mu R}^c \end{pmatrix} = \begin{pmatrix} \chi_e \\ \chi_\mu \end{pmatrix} \tag{2.8}$$

and

$$\chi^c = C\bar{\chi} = \chi \,. \tag{2.9}$$

χ_e and χ_μ are fields of Majorana neutrinos [24], and their appearance is due to the non–conservation of the lepton numbers. Diagonalizing the matrix M by using the orthogonal matrix U ($U^T U = 1$)

$$M = U M_0 U^T \,, \tag{2.10}$$

it is possible to find the matrix M_0:

$$M_0 = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \,. \tag{2.11}$$

The Hamiltonian in Eq. (2.7) becomes

$$H = \bar{\phi} M_0 \phi = \sum_{\sigma=1,2} m_\sigma \bar{\phi}_\sigma \phi_\sigma \,, \tag{2.12}$$

with $\phi = U^T \chi$ or $\chi = U\phi$, so that

$$\nu_{eL} = \sum_{\sigma=1,2} U_{1\sigma} \phi_{\sigma L}, \quad \nu_{\mu L} = \sum_{\sigma=1,2} U_{2\sigma} \phi_{\sigma L} \tag{2.13}$$

are the relations between the fields present in the Hamiltonian of the ordinary weak interaction (ν_{eL} and $\nu_{\mu L}$) and the fields of Majorana neutrinos with masses m_1 and m_2 (ϕ_1, ϕ_2). With those assumptions fields which appears in the usual weak interaction hamiltonian are linear superposition of fields of Majorana neutrinos, with non–vanishing masses. In fact, having the orthogonal matrix U the general form:

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} . \tag{2.14}$$

It follows:

$$\nu_{eL} = \cos\theta\phi_{1L} + \sin\theta\phi_{2L}, \quad \nu_{\mu L} = -\sin\theta\phi_{1L} + \cos\theta\phi_{2L} . \tag{2.15}$$

It is clear that the angle θ characterizes the degrees of mixing of ϕ_1, ϕ_2. Moreover it is possible to demonstrate that:

$$\begin{aligned} m_{\bar{e}e} &= \cos^2\theta m_1 + \sin^2\theta m_2 , \\ m_{\bar{\mu}\mu} &= \sin^2\theta m_1 + \cos^2\theta m_2 , \\ m_{\bar{\mu}e} &= \sin\theta\cos\theta(-m_1 + m_2) . \end{aligned} \tag{2.16}$$

So that

$$\tan 2\theta = \frac{2m_{\bar{\mu}e}}{m_{\bar{\mu}\mu} - m_{\bar{e}e}} , \tag{2.17}$$

$$m_{1,2} = \frac{1}{2}(m_{\bar{e}e} + m_{\bar{\mu}\mu} \pm \sqrt{(m_{\bar{e}e} - m_{\bar{\mu}\mu})^2 + 4m_{\bar{\mu}e}^2} . \tag{2.18}$$

The oscillation is possible only if $\theta \neq 0$ and $m_1 \neq m_2$, *i.e.* $m_{\bar{\mu}e} \neq 0$ and at least one between $m_{\bar{\mu}\mu}$ and $m_{\bar{e}e}$ non–vanishing, while the maximum mixing happens when $\theta = \frac{\pi}{4}$, $m_{\bar{\mu}\mu} = m_{\bar{e}e}$ and $m_{\bar{\mu}e} \neq 0$. In such case follows:

$$\nu_{eL} = \frac{1}{\sqrt{2}}(\phi_{1L} + \phi_{2L}), \quad \nu_{\mu L} = \frac{1}{\sqrt{2}}(-\phi_{1L} + \phi_{2L}). \tag{2.19}$$

2.1.2 Dirac neutrinos

Comparing the hadron current of the standard theory $j^h_\alpha = (j^h_\alpha)_C + (j^h_\alpha)_{GIM}$ [1] with the ordinary lepton current $j^l_\alpha = (\bar{\nu}_e\gamma_\alpha e_L) + (\bar{\nu}_\mu\gamma_\alpha\mu_L)$ it is clear that whereas linear superpositions of the d and s quark are present in the hadron current, in the lepton current ν_e and ν_μ are unmixed (*i.e.* the

[1]For more details on GIM mechanism go to Appendix - GIM mechanism

charged hadron current does not conserve strangeness, while the lepton current is conserving lepton number). In order to remove this difference it is possible to assume that there exist two neutrinos, *i.e.* Dirac neutrinos ν_1 and ν_2 with finite masses (m_1, m_2) such that, considering a general mixing angle θ :

$$\begin{aligned} \nu_e &= \nu_1 \cos\theta + \nu_2 \sin\theta\,, \\ \nu_\mu &= -\nu_1 \sin\theta + \nu_2 \cos\theta\,. \end{aligned} \tag{2.20}$$

If $\theta = 0$ there is no mixing, while $\theta = \frac{\pi}{4}$ is the angle corresponding to the maximum mixing.

In both schemes, *i.e.* considering Majorana neutrinos and Dirac neutrinos, the neutrino masses are non–vanishing and orthogonal combinations of the operators of neutrino fields are present in the hamiltonian but, of course, Majorana neutrinos are Majorana particles while Dirac Neutrinos are Dirac particles. Moreover in Majorana theory lepton number, *i.e.* L_e and L_μ are not conserved, while in Dirac theory the sum $L_e + L_\mu$ is conserved. In fact, the mass term of the Hamiltonian in the Dirac theory is:

$$H_1 = m_1 \bar{\nu}_1 \nu_1 + m_2 \bar{\nu}_2 \nu_2\,, \tag{2.21}$$

which, in terms of ν_e and ν_μ becomes:

$$H_1 = m_{ee} \bar{\nu}_e \nu_e + m_{\mu\mu} \bar{\nu}_\mu \nu_\mu + m_{\mu e} (\bar{\nu}_\mu \nu_e + \bar{\nu}_e \nu_\mu)\,, \tag{2.22}$$

where

$$\begin{aligned} m_{ee} &= \cos^2\theta m_1 + \sin^2\theta m_2\,, \\ m_{\mu\mu} &= \sin^2\theta m_1 + \cos^2\theta m_2\,, \\ m_{\mu e} &= \sin\theta\cos\theta(-m_1 + m_2)\,. \end{aligned} \tag{2.23}$$

Clearly m_{ee} and $m_{\mu\mu}$ are the bare masses of the electron and muon neutrinos, and the Hamiltonian does not conserve separately L_e and L_μ, but their sum. With this said, Dirac neutrinos are treated on the same foot as all the other particles, whereas in the Majorana scheme neutrinos occupy a special place among the other fundamental particles. In fact, there is no analogy between leptons and quark in Majorana theory due to the fact that every types of neutrino is associated with two states, while in Dirac scheme the association is with four states.

2.1.3 Neutrino oscillations

Considering the state vectors of electron and muon neutrinos $|\nu_e\rangle$, $|\nu_\mu\rangle$ with momentum $\mathbf{p}$ and helicity -1, follows:

$$|\nu_l\rangle = \sum_{\sigma=1,2} U_{l\sigma} |\nu_\sigma\rangle , \qquad (l = e, \mu) \tag{2.24}$$

where $|\nu_\sigma\rangle \quad (\sigma = 1, 2)$ is the state vector of the neutrino with mass m_σ, momentum $\mathbf{p}$ and helicity -1 and, thus, describes both Majorana and Dirac neutrinos, while the matrix U has the same form of the one in Eq. (2.14). So:

$$H |\nu_\sigma\rangle = E_\sigma |\nu_\sigma\rangle \tag{2.25}$$

where H is the full Hamiltonian and

$$E_\sigma = \sqrt{m_\sigma^2 + \mathbf{p}^2}. \tag{2.26}$$

Moreover

$$|\nu_\sigma\rangle = \sum_{l=e,\mu} U_{l\sigma} |\nu_l\rangle . \tag{2.27}$$

Let us now consider the behavior of a beam of neutrinos, produced in some weak process. At the initial time ($t = 0$) such a beam is described by vector $|\nu_l\rangle$. At the time t the state vector of the beam is given by:

$$|\nu_l\rangle_t = e^{-iHt} |\nu_l\rangle = \sum_{\sigma=1,2} U_{l\sigma} e^{-iE_\sigma t} |\nu_\sigma\rangle . \tag{2.28}$$

Since $|\nu_l\rangle$ is not an eigenstate of the Hamiltonian H, such a neutrino beam is not described by a stationary state, as it should be in usual theories, but by a superposition of stationary states. Expanding the state vector in term of vectors $|\nu_{l'}\rangle$ follows:

$$|\nu_l(t)\rangle = \sum_{l'_e,\mu} a_{\nu_{l'}\nu_l}(t)\,|\nu_{l'}\rangle \tag{2.29}$$

where

$$a_{\nu_{l'}\nu_l}(t) = \sum_{\sigma=1,2} U_{l\sigma} e^{-iE_\sigma t} U_{l'\sigma} \,. \tag{2.30}$$

is the probability amplitude of finding $|\nu_{l'}\rangle$ at a time t after the generation of $|\nu_l\rangle$. Of course:

$$a_{\nu_{l'}\nu_l}(0) = \sum_{\sigma=1,2} U_{l\sigma} U_{l'\sigma} = \delta_{l'l} \,. \tag{2.31}$$

Clearly in the case $m_1 \neq m_2$ and $U_{l\sigma} \neq \delta_{l\sigma}$, $a_{\nu_e\nu_\mu}(t) = a_{\nu_\mu\nu_e}(t) \neq 0$, *i.e.* there is an oscillation $\nu_e \leftrightarrow \nu_\mu$, the probability of transition is given by:

$$w_{\nu_{l'}\nu_l}(t) = w_{\nu_{l'}\nu_l} = \sum_{\sigma,\sigma'} U_{l\sigma} U_{l'\sigma} U_{l\sigma'} U_{l'\sigma'} \cos{(E_\sigma - E_{\sigma'})t} \,, \tag{2.32}$$

which satisfies:

$$\sum_{l'=e,\mu} w_{\nu_{l'}\nu_l}(t) = 1. \tag{2.33}$$

In particular if $p \gg m_1, m_2$, *i.e.* $E_1 - E_2 = \frac{m_1^2 - m_2^2}{2p}$, defying $w_{\nu_{l'}\nu_l}(R)$ as the probability to find $\nu_{l'}$ at a distance R from the source ν_l, follows:

$$\begin{aligned} w_{\nu_e\nu_e}(R) &= w_{\nu_\mu\nu_\mu}(R) = 1 - \frac{1}{2}\sin^2 2\theta(1 - \cos 2\pi\frac{R}{L}) \,, \\ w_{\nu_e\nu_\mu}(R) &= w_{\nu_\mu\nu_e}(R) = \frac{1}{2}\sin^2 2\theta(1 - \cos 2\pi\frac{R}{L}) \,, \end{aligned} \tag{2.34}$$

where $L = \frac{4\pi p}{|m_1^2 - m_2^2|}$ is the oscillation length and together with θ can be connected with the parameter of both schemes.

2.2 Neutrino mixing in Quantum Field Theory formalism

Let us start by introducing the Lagrangian:

$$\mathcal{L}(x) = \bar{\Psi}_f(x)(i\not{\partial} - M)\Psi_f(x) \tag{2.35}$$

$$= \bar{\Psi}_m(x)(i\not{\partial} - M_d)\Psi_m(x), \tag{2.36}$$

with $\Psi_f^T = (\nu_e, \nu_\mu)$ being the flavor fields and

$$M = \begin{pmatrix} m_e & m_{e\mu} \\ m_{e\mu} & m_\mu \end{pmatrix}.$$

The flavor fields are connected to the free fields $\Psi_m^T = (\nu_1, \nu_2)$, with

$$M_d = diag(m_1, m_2)\,,$$

by the mixing relations of two (Dirac) neutrino fields with definite flavors ν_e, ν_μ:

$$\nu_e(x) = \cos\theta\ \nu_1(x)\ +\ \sin\theta\ \nu_2(x)\,,$$
$$\nu_\mu(x) = -\sin\theta\ \nu_1(x)\ +\ \cos\theta\ \nu_2(x)\,. \tag{2.37}$$

where ν_1, ν_2 are the (free) neutrino fields with definite masses m_1, m_2, respectively, and θ is the mixing angle. The fields quantization setting is the standard one; the ψ_m are free fields and their explicit expansion in terms of the ladder operators is

$$\nu_i(x) = \frac{1}{\sqrt{V}}\sum_{\mathbf{k},r}\left[u^r_{\mathbf{k},i}(t)\alpha^r_{\mathbf{k},i}\ + v^r_{-\mathbf{k},i}(t)\beta^{r\dagger}_{-\mathbf{k},i}\right]e^{i\mathbf{k}\cdot\mathbf{x}}\,,\qquad i = 1, 2\,, \tag{2.38}$$

where $r = 1, 2$ is the helicity index, $u^r_{\mathbf{k},i}(t) = e^{-i\omega_{k,i}t}u^r_{\mathbf{k},i}$ and $v^r_{\mathbf{k},i}(t) = e^{i\omega_{k,i}t}v^r_{\mathbf{k},i}$, with $\omega_{k,i} = \sqrt{\mathbf{k}^2 + m_i^2}$. The $\alpha^r_{\mathbf{k},i}$ and the $\beta^r_{\mathbf{k},i}$, are the annihilation operators for the vacuum state $|0\rangle_{1,2} \equiv |0\rangle_1 \otimes |0\rangle_2$: $\alpha^r_{\mathbf{k},i}|0\rangle_{1,2} = \beta^r_{\mathbf{k},i}|0\rangle_{1,2} = 0$. The anticommutation relations are, indeed, the standard ones:

$$\{\nu^\alpha_i(x), \nu^{\beta\dagger}_j(y)\}_{t=t'} = \delta^3(\mathbf{x}-\mathbf{y})\delta_{\alpha\beta}\delta_{ij}\ , \qquad \alpha,\beta = 1,..,4\ , \tag{2.39}$$

$$\{\alpha^r_{\mathbf{k},i}, \alpha^{s\dagger}_{\mathbf{q},j}\} = \delta_{\mathbf{kq}}\delta_{rs}\delta_{ij}; \qquad \{\beta^r_{\mathbf{k},i}, \beta^{s\dagger}_{\mathbf{q},j}\} = \delta_{\mathbf{kq}}\delta_{rs}\delta_{ij}, \quad i,j = 1,2\ . \tag{2.40}$$

While, the orthonormality and completeness relations are :

$$\begin{aligned}
&\sum_\alpha u^{r\alpha *}_{\mathbf{k},i} u^{s\alpha}_{\mathbf{k},i} = \sum_\alpha v^{r\alpha *}_{\mathbf{k},i} v^{s\alpha}_{\mathbf{k},i} = \delta_{rs}, \\
&\sum_\alpha u^{r\alpha *}_{\mathbf{k},i} v^{s\alpha}_{-k,i} = \sum_\alpha v^{r\alpha *}_{-k,i} u^{s\alpha}_{\mathbf{k},i} = 0, \\
&\sum_r \left(u^{r\alpha *}_{\mathbf{k},i} u^{r\beta}_{\mathbf{k},i} + v^{r\alpha *}_{-k,i} v^{r\beta}_{-k,i} \right) = \delta_{\alpha\beta}.
\end{aligned} \tag{2.41}$$

With this in mind, Eqs.(2.37) can be recast as [25, 26, 63]:

$$\begin{aligned}
\nu^\alpha_e(x) &= G^{-1}_\theta(t)\ \nu^\alpha_1(x)\ G_\theta(t)\,, \\
\nu^\alpha_\mu(x) &= G^{-1}_\theta(t)\ \nu^\alpha_2(x)\ G_\theta(t)\,,
\end{aligned} \tag{2.42}$$

where the generator $G_\theta(t)$ is given by

$$G_\theta(t) = \exp\left[\theta \int d^3\mathbf{x} \left(\nu^\dagger_1(x)\nu_2(x) - \nu^\dagger_2(x)\nu_1(x)\right)\right]\ , \tag{2.43}$$

is an element of $SU(2)$ [25], which can be written as

$$G_\theta(t) = \exp[\theta(S_+(t) - S_-(t))] \tag{2.44}$$

with

$$S_+(t) = S^\dagger_-(t) \equiv \int d^3\mathbf{x}\ \nu^\dagger_1(x)\nu_2(x)\,. \tag{2.45}$$

Indeed, by introducing

$$S_3 \equiv \frac{1}{2}\int d^3\mathbf{x}\left(\nu_1^\dagger(x)\nu_1(x) - \nu_2^\dagger(x)\nu_2(x)\right) \ , \tag{2.46}$$

the $SU(2)$ algebra is closed (at fixed t):

$$[S_+(t), S_-(t)] = 2S_3 \quad , \quad [S_3, S_\pm(t)] = \pm S_\pm(t) \ , \tag{2.47}$$

The explicit form of $S_+(t)$ in terms of the ladder operators is

$$S_+^{\mathbf{k},r} = \left(U_\mathbf{k}^* \ \alpha_{\mathbf{k},1}^{r\dagger}\alpha_{\mathbf{k},2}^r - \epsilon^r \ V_\mathbf{k}^* \ \beta_{-\mathbf{k},1}^r\alpha_{\mathbf{k},2}^r + \epsilon^r \ V_\mathbf{k} \ \alpha_{\mathbf{k},1}^{r\dagger}\beta_{-\mathbf{k},2}^{r\dagger} + U_\mathbf{k} \ \beta_{-\mathbf{k},1}^r\beta_{-\mathbf{k},2}^{r\dagger}\right) \tag{2.48}$$

where we defined $S_\pm = \sum_\mathbf{k} S_\pm^\mathbf{k}$, with $S_+^\mathbf{k} = S_-^{\mathbf{k}\dagger} \equiv \sum_r S_+^{\mathbf{k},r}$, $\epsilon^r = (-1)^r$ and

$$U_\mathbf{k}(t) \equiv u_{\mathbf{k},2}^{r\dagger}(t)u_{\mathbf{k},1}^r(t) = v_{-\mathbf{k},1}^{r\dagger}(t)v_{-\mathbf{k},2}^r(t) = \ |U_\mathbf{k}| \, e^{i(\omega_{k,2}-\omega_{k,1})t} \, , \tag{2.49}$$

$$V_\mathbf{k}(t) \equiv \epsilon^r \ u_{\mathbf{k},1}^{r\dagger}(t)v_{-\mathbf{k},2}^r(t) = -\epsilon^r \ u_{\mathbf{k},2}^{r\dagger}(t)v_{-\mathbf{k},1}^r(t) = \ |V_\mathbf{k}| \, e^{i(\omega_{k,2}+\omega_{k,1})t} \, , \tag{2.50}$$

$$\begin{aligned} |U_k| \equiv & \ \left(\tfrac{\omega_{k,1}+m_1}{2\omega_{k,1}}\right)^{\frac{1}{2}}\left(\tfrac{\omega_{k,2}+m_2}{2\omega_{k,2}}\right)^{\frac{1}{2}}\left(1 + \tfrac{k^2}{(\omega_{k,1}+m_1)(\omega_{k,2}+m_2)}\right) , \\ |V_k| \equiv & \ \left(\tfrac{\omega_{k,1}+m_1}{2\omega_{k,1}}\right)^{\frac{1}{2}}\left(\tfrac{\omega_{k,2}+m_2}{2\omega_{k,2}}\right)^{\frac{1}{2}}\left(\tfrac{k}{(\omega_{k,2}+m_2)} - \tfrac{k}{(\omega_{k,1}+m_1)}\right) , \end{aligned} \tag{2.51}$$

which translates to

$$|U_\mathbf{k}| = \frac{|\mathbf{k}|^2 + (\omega_{k,1}+m_1)(\omega_{k,2}+m_2)}{2\sqrt{\omega_{k,1}\omega_{k,2}(\omega_{k,1}+m_1)(\omega_{k,2}+m_2)}}, \qquad |U_\mathbf{k}|^2 + |V_\mathbf{k}|^2 = 1. \tag{2.52}$$

Thus, by use of $G_\theta(t)$, the flavor fields, in Eq. (2.42), can be expanded as:

$$\nu_\sigma(x) \ = \sum_{r=1,2}\int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}}\left[u_{\mathbf{k},i}^r(t)\alpha_{\mathbf{k},\sigma}^r(t) + v_{-\mathbf{k},i}^r(t)\beta_{-\mathbf{k},\sigma}^{r\dagger}(t)\right] e^{i\mathbf{k}\cdot\mathbf{x}} \, , \tag{2.53}$$

with $(\sigma, i) = (e, 1), (\mu, 2)$ and the flavor annihilation operators are defined as $\alpha_{\mathbf{k},\sigma}^r(t) \equiv G_\theta^{-1}(t)\alpha_{\mathbf{k},i}^r G_\theta(t)$ and $\beta_{-\mathbf{k},\sigma}^{r\dagger}(t) \equiv G_\theta^{-1}(t)\beta_{-\mathbf{k},i}^{r\dagger}G_\theta(t)$. Using Eqs. (2.43),(2.44),(3.7) for $\mathbf{k} = (0, 0, |\mathbf{k}|)$, we can write down the explicit form of the ladder operator :

$$\alpha^r_{\mathbf{k},e}(t) = \cos\theta\; \alpha^r_{\mathbf{k},1} \;+\; \sin\theta\; \left(U^*_{\mathbf{k}}(t)\; \alpha^r_{\mathbf{k},2} \;+\; \epsilon^r\; V_{\mathbf{k}}(t)\; \beta^{r\dagger}_{-\mathbf{k},2}\right) \,, \tag{2.54}$$

$$\alpha^r_{\mathbf{k},\mu}(t) = \cos\theta\; \alpha^r_{\mathbf{k},2} \;-\; \sin\theta\; \left(U_{\mathbf{k}}(t)\; \alpha^r_{\mathbf{k},1} \;-\; \epsilon^r\; V_{\mathbf{k}}(t)\; \beta^{r\dagger}_{-\mathbf{k},1}\right) \,, \tag{2.55}$$

$$\beta^r_{-\mathbf{k},e}(t) = \cos\theta\; \beta^r_{-\mathbf{k},1} \;+\; \sin\theta\; \left(U^*_{\mathbf{k}}(t)\; \beta^r_{-\mathbf{k},2} \;-\; \epsilon^r\; V_{\mathbf{k}}(t)\; \alpha^{r\dagger}_{\mathbf{k},2}\right) \,, \tag{2.56}$$

$$\beta^r_{-\mathbf{k},\mu}(t) = \cos\theta\; \beta^r_{-\mathbf{k},2} \;-\; \sin\theta\; \left(U_{\mathbf{k}}(t)\; \beta^r_{-\mathbf{k},1} \;+\; \epsilon^r\; V_{\mathbf{k}}(t)\; \alpha^{r\dagger}_{\mathbf{k},1}\right) \,. \tag{2.57}$$

An inspection of the Eqs. (2.54),(2.55),(2.56),(2.57) shows that the mixing transformations for the ladder operator produce a "nested" operator with a rotation and a time dependent Bogoliubov transformation with coefficient $U_{\mathbf{k}}(t)$, $V_{\mathbf{k}}(t)$.
Finally, the action of the mixing generator on the vacuum $|0\rangle_{1,2}$ is non–trivial; at finite volume V we have:

$$|0(t)\rangle_{e,\mu} \equiv G^{-1}_\theta(t)\; |0\rangle_{1,2} \,. \tag{2.58}$$

In the infinite volume limit one obtains [25]

$$\lim_{V\to\infty}\; {}_{1,2}\langle 0|0(t)\rangle_{e,\mu} = \lim_{V\to\infty}\; e^{\frac{V}{(2\pi)^3}\int d^3\mathbf{k}\; ln\; (1-\sin^2\theta\; |V_{\mathbf{k}}|^2)} = 0\,. \tag{2.59}$$

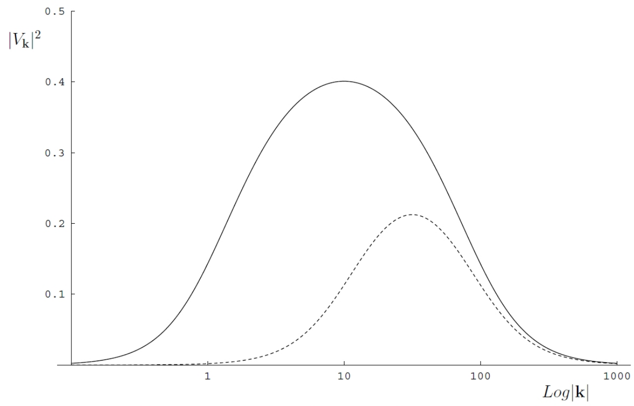
Eq.(2.59) expresses the unitary inequivalence between the flavor and the mass representations and shows the non–trivial nature of the mixing transformations in Eq.(2.37) resulting in the condensate structure of the flavor vacuum [25, 64]. The form of the flavor vacuum (at $t = 0$) is the following one:

$$\begin{aligned}|0\rangle_{e,\mu} = \prod_{\mathbf{k},r} \Big[&(1-\sin^2\theta|V_{\mathbf{k}}|^2) - \epsilon^r \sin\theta\cos\theta|V_{\mathbf{k}}|(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2} + \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}) + \\ &+\epsilon^r \sin^2\theta|V_{\mathbf{k}}||U_{\mathbf{k}}|(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},1} - \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},2}) \\ &+\sin^2\theta|V_{\mathbf{k}}|^2\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2}\alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}\Big]\; |0\rangle_{1,2}\,.\end{aligned} \tag{2.60}$$

From Eq.(2.60) it is evident that the condensate nature made of particle-antiparticle pairs with same or different masses. The condensation density of the flavor vacuum is given by

$$ {}_{e,\mu}\langle 0(t)|\alpha^{r\dagger}_{\mathbf{k},i}\alpha^{r}_{\mathbf{k},i}|0(t)\rangle_{e,\mu} = {}_{e,\mu}\langle 0(t)|\beta^{r\dagger}_{\mathbf{k},i}\beta^{r}_{\mathbf{k},i}|0(t)\rangle_{e,\mu} = \sin^2\theta\;|V_{\mathbf{k}}|^2, \quad (2.61) $$

with $i = 1, 2$, and the same result for antiparticles. Note that the $|V_{\mathbf{k}}|^2$ has a maximum at $\sqrt{m_1 m_2}$ and $|V_{\mathbf{k}}|^2 \simeq \frac{(m_2-m_1)^2}{4|\mathbf{k}|^2}$ for $|\mathbf{k}| \gg \sqrt{m_1 m_2}$.



Solid line: $m_1 = 1$, $m_2 = 100$; Dashed line: $m_1 = 10$, $m_2 = 100$.

Eq. (2.59) can only hold in the QFT framework; since there unitarily inequivalent representations exist, contrarily to what happens in Quantum Mechanics (QM) where the von Neumann theorem states the unitary equivalence of the representations of the canonical anticommutation relations. Eq. (2.59) also expresses the non–perturbative nature of the field mixing mechanism.

2.2.1 Neutrino oscillations in QFT

The single (mixed) particle flavor state is given by

$$ |\alpha^{r}_{\mathbf{k},\sigma}(t)\rangle \equiv \alpha^{r\dagger}_{\mathbf{k},\sigma}(t)|0(t)\rangle_{e,\mu} = G^{-1}_{\theta}(t)\alpha^{r\dagger}_{\mathbf{k},i}|0\rangle_{1,2}\;, \quad (2.62) $$

where $\sigma, i = e, 1$ or $\mu, 2$. States with particle number higher than one are obtained similarly by operating repeatedly with the creation operator $\alpha^{r\dagger}_{\mathbf{k},\sigma}$.

Majorana neutrinos

The momentum operator for the free fields is

$$\mathbf{P}_i = \sum_{r=1,2} \int d^3\mathbf{k}\, \mathbf{k} \Big(\alpha^{r\dagger}_{\mathbf{k},i} \alpha^r_{\mathbf{k},i} - \alpha^{r\dagger}_{-\mathbf{k},i} \alpha^r_{-\mathbf{k},i} \Big) \ , \tag{2.63}$$

with $i = 1, 2$. For mixed fields, one has $\mathbf{P}_\sigma(t) = G^{-1}_\theta(t) \mathbf{P}_i G_\theta(t)$, namely

$$\mathbf{P}_\sigma(t) = \sum_{r=1,2} \int d^3\mathbf{k}\, \mathbf{k} \Big(\alpha^{r\dagger}_{\mathbf{k},\sigma}(t) \alpha^r_{\mathbf{k},\sigma}(t) - \alpha^{r\dagger}_{-\mathbf{k},\sigma}(t) \alpha^r_{-\mathbf{k},\sigma}(t) \Big), \tag{2.64}$$

for $\sigma = e, \mu$ with $\mathbf{P}_e(t) + \mathbf{P}_\mu(t) = \mathbf{P}_1 + \mathbf{P}_2 \equiv \mathbf{P}$ and $[\mathbf{P}, G_\theta(t)] = 0$. The total momentum is of course conserved, $[\mathbf{P}, H] = 0$, with H denoting the Hamiltonian. The expectation value on the flavor vacuum of the momentum operator $\mathbf{P}_\sigma(t)$ vanishes at all times:

$${}_{e,\mu}\langle 0(t) | \mathbf{P}_\sigma(\mathbf{t}) | 0(t) \rangle_{e,\mu} = 0, \quad \sigma = e, \mu \ . \tag{2.65}$$

The state $|\alpha^r_{\mathbf{k},e}\rangle \equiv |\alpha^r_{\mathbf{k},e}(0)\rangle$ is an eigenstate of the momentum operator $\mathbf{P}_e(0)$ at time $t = 0$, $\mathbf{P}_e(0)|\alpha^r_{\mathbf{k},e}\rangle \equiv \mathbf{k}|\alpha^r_{\mathbf{k},e}\rangle$. At time $t \neq 0$ the normalized expectation value for the momentum in such a state is [25]

$$\begin{aligned} \mathcal{P}^e_{\mathbf{k},\sigma}(t) \ &\equiv \ \frac{\langle \alpha^r_{\mathbf{k},e} | \mathbf{P}_\sigma(t) | \alpha^r_{\mathbf{k},e} \rangle}{\langle \alpha^r_{\mathbf{k},e} | \mathbf{P}_\sigma(0) | \alpha^r_{\mathbf{k},e} \rangle} \\ &= \ |\{\alpha^r_{\mathbf{k},e}(t), \alpha^{r\dagger}_{\mathbf{k},e}(t')\}|^2 + |\{\alpha^{r\dagger}_{-\mathbf{k},e}(t), \alpha^{r\dagger}_{\mathbf{k},e}(t')\}|^2 \ , \end{aligned}$$

for $\sigma = e, \mu$. Note that $\mathcal{P}^e_{\mathbf{k},\sigma}(t)$ behaves actually as a "charge operator". Indeed, the operator $\alpha^{r\dagger}_{\mathbf{k},i} \alpha^r_{\mathbf{k},i} - \alpha^{r\dagger}_{-\mathbf{k},i} \alpha^r_{-\mathbf{k},i}$ is the fermion number operator. Therefore, the explicit calculation of $\mathcal{P}^e_{\mathbf{k},\sigma}(t)$ provides the flavor charge oscillation. We obtain

$$\begin{aligned}\mathcal{P}^{e}_{\mathbf{k},e}(t) &= 1-\sin^2 2\theta \\ &\times\left[|U_{\mathbf{k}}|^2 \sin^2 \frac{\omega_{k,2}-\omega_{k,1}}{2}t + |V_{\mathbf{k}}|^2 \sin^2 \frac{\omega_{k,2}+\omega_{k,1}}{2}t\right], \\ \mathcal{P}^{e}_{\mathbf{k},\mu}(t) &= \sin^2 2\theta \\ &\times\left[|U_{\mathbf{k}}|^2 \sin^2 \frac{\omega_{k,2}-\omega_{k,1}}{2}t + |V_{\mathbf{k}}|^2 \sin^2 \frac{\omega_{k,2}+\omega_{k,1}}{2}t\right] .\end{aligned}$$

Dirac neutrinos

The Lagrangian for two free Dirac fields, with masses m_1 and m_2 - Eq.(2.35):

$$\mathcal{L}(x) \;=\; \bar{\Psi}_m(x)\,(i\,\partial\!\!\!/ - M_d)\,\Psi_m(x)\,,$$

where[2] $\Psi_m^T = (\nu_1, \nu_2)$ and $M_d = diag(m_1, m_2)$, is invariant under global $U(1)$ phase transformations of the type $\Psi'_m = e^{i\alpha}\,\Psi_m$: as a result, one has the conservation of the Noether charge $Q = \int d^3\mathbf{x}\, I^0(x)$ (with $I^\mu(x) = \bar{\Psi}_m(x)\,\gamma^\mu\,\Psi_m(x)$) which is indeed the total charge of the system (*i.e.* the total lepton number). One may onsider then the global $SU(2)$ transformation [65]:

$$\Psi'_m(x) \;=\; e^{i\alpha_j\tau_j}\,\Psi_m(x), \qquad\qquad j \;=\; 1,2,3\,, \tag{2.66}$$

with $\tau_j = \sigma_j/2$ and σ_j being the Pauli matrices. For $m_1 \neq m_2$, the Lagrangian is not generally invariant under the above transformations. One has [66] indeed:

$$\delta\mathcal{L}(x) \;=\; i\alpha_j\,\bar{\Psi}_m(x)\,[\tau_j, M_d]\,\Psi_m(x) \;=\; -\alpha_j\,\partial_\mu J^\mu_{m,j}(x)\,, \tag{2.67}$$

$$J^\mu_{m,j}(x) \;=\; \bar{\Psi}_m(x)\,\gamma^\mu\,\tau_j\,\Psi_m(x), \qquad\qquad j \;=\; 1,2,3\,. \tag{2.68}$$

[2]The subscript m denotes quantities which are in terms of fields with definite masses.

Explicitly:

$$J^{\mu}_{m,1}(x) = \frac{1}{2}\left[\bar{\nu}_1(x)\,\gamma^{\mu}\,\nu_2(x) + \bar{\nu}_2(x)\,\gamma^{\mu}\,\nu_1(x)\right], \tag{2.69}$$

$$J^{\mu}_{m,2}(x) = \frac{i}{2}\left[\bar{\nu}_1(x)\,\gamma^{\mu}\,\nu_2(x) - \bar{\nu}_2(x)\,\gamma^{\mu}\,\nu_1(x)\right], \tag{2.70}$$

$$J^{\mu}_{m,3}(x) = \frac{1}{2}\left[\bar{\nu}_1(x)\,\gamma^{\mu}\,\nu_1(x) - \bar{\nu}_2(x)\,\gamma^{\mu}\,\nu_2(x)\right]. \tag{2.71}$$

The charges $Q_{m,j}(t) \equiv \int d^3\mathbf{x}\, J^0_{m,j}(x)$ satisfy the $su(2)$ algebra (at equal times): $[Q_{m,j}(t), Q_{m,k}(t)] = i\,\epsilon_{jkl}\,Q_{m,l}(t)$. The Casimir operator is proportional to the total (conserved) charge: $C_m = \frac{1}{2}Q$ and that, since $Q_{m,3}$ is conserved in time, one has

$$Q_1 \equiv \frac{1}{2}Q + Q_{m,3} \quad , \quad Q_2 \equiv \frac{1}{2}Q - Q_{m,3}\,, \tag{2.72}$$

$$Q_i = \sum_r \int d^3\mathbf{k}\left(\alpha^{r\dagger}_{\mathbf{k},i}\alpha^{r}_{\mathbf{k},i} - \beta^{r\dagger}_{-\mathbf{k},i}\beta^{r}_{-\mathbf{k},i}\right) \quad , \quad i = 1,2. \tag{2.73}$$

These are nothing but the Noether charges associated with the non–interacting fields ν_1 and ν_2: in the absence of mixing, they are the flavor charges, separately conserved for each generation. On the other hand, one may rewrite the Lagrangian in the flavor basis[3] (cf. Eq.(2.36)):

$$\mathcal{L}(x) = \bar{\Psi}_f(x)\,(i\,\partial\!\!\!/ - M)\,\Psi_f(x)$$

where $\Psi_f^T = (\nu_e, \nu_\mu)$ and $M = \begin{pmatrix} m_e & m_{e\mu} \\ m_{e\mu} & m_\mu \end{pmatrix}$. Obviously, $\mathcal{L}$ is still invariant under $U(1)$. When considering the $SU(2)$ transformation [65]:

$$\Psi'_f(x) = e^{i\alpha_j\tau_j}\,\Psi_f(x), \tag{2.74}$$

$$\delta\mathcal{L}(x) = i\alpha_j\,\bar{\Psi}_f(x)\,[\tau_j, M]\,\Psi_f(x) = -\alpha_j\,\partial_\mu J^{\mu}_{f,j}(x)\,, \tag{2.75}$$

$$J^{\mu}_{f,j}(x) = \bar{\Psi}_f(x)\,\gamma^{\mu}\,\tau_j\,\Psi_f(x), \qquad j = 1,2,3. \tag{2.76}$$

[3]Subscript f denotes here flavor

the charges $Q_{f,j}(t) \equiv \int d^3\mathbf{x}\, J^0_{f,j}(x)$ satisfy the $su(2)$ algebra. Note that, because of the off–diagonal (mixing) terms in the mass matrix M, $Q_{f,3}$ is not anymore conserved. This implies an exchange of charge between ν_e and ν_μ, resulting in the phenomenon of flavor oscillations. One may indeed define the *flavor charges* for mixed fields as

$$Q_e(t) \;\equiv\; \int d^3\mathbf{x}\, \nu_e^\dagger(x)\nu_e(x) \;=\; \frac{1}{2}Q \,+\, Q_{f,3}(t)\,, \tag{2.77}$$

$$Q_\mu(t) \;\equiv\; \int d^3\mathbf{x}\, \nu_\mu^\dagger(x)\nu_\mu(x) \;=\; \frac{1}{2}Q \,-\, Q_{f,3}(t)\,, \tag{2.78}$$

where $Q_e(t) \,+\, Q_\mu(t) \;=\; Q$. They are related to the Noether charges as

$$Q_\sigma(t) \;=\; G^{-1}_\theta(t)\, Q_i\, G_\theta(t)\,, \tag{2.79}$$

with $(\sigma, i) = (e, 1), (\mu, 2)$. From Eq.(2.79), it follows that the flavor charges are diagonal in the flavor ladder operators:

$$Q_\sigma(t) \;=\; \sum_r \int d^3\mathbf{k} \left(\alpha^{r\dagger}_{\mathbf{k},\sigma}(t)\alpha^{r}_{\mathbf{k},\sigma}(t) \,-\, \beta^{r\dagger}_{-\mathbf{k},\sigma}(t)\beta^{r}_{-\mathbf{k},\sigma}(t) \right)\,, \tag{2.80}$$

with $\sigma = e, \mu$. In the Heisenberg picture the state for a particle with definite (electron) flavor, spin and momentum is defined as[4]:

$$|\alpha^r_{\mathbf{k},e}\rangle \equiv \alpha^{r\dagger}_{\mathbf{k},e}(0)|0\rangle_{e,\mu} = G^{-1}_\theta(0)\alpha^{r\dagger}_{\mathbf{k},1}|0\rangle_{1,2}\,, \tag{2.81}$$

where $|0\rangle_{e,\mu} \equiv |0(0)\rangle_{e,\mu}$. Note that the $|\alpha^r_{\mathbf{k},e}\rangle$ is an eigenstate of $Q_e(t)$, at $t = 0$: $Q_e(0)|\alpha^r_{\mathbf{k},e}\rangle = |\alpha^r_{\mathbf{k},e}\rangle$. Thus ${}_{e,\mu}\langle 0|Q_\sigma(t)|0\rangle_{e,\mu} \;=\; 0$ and

$$\begin{aligned} \mathcal{Q}_{\mathbf{k},\sigma}(t) \;&\equiv\; \langle\alpha^r_{\mathbf{k},e}|Q_\sigma(t)|\alpha^r_{\mathbf{k},e}\rangle \\ &=\; \left|\left\{\alpha^r_{\mathbf{k},\sigma}(t), \alpha^{r\dagger}_{\mathbf{k},\rho}(0)\right\}\right|^2 \,+\, \left|\left\{\beta^{r\dagger}_{-\mathbf{k},\sigma}(t), \alpha^{r\dagger}_{\mathbf{k},\rho}(0)\right\}\right|^2 . \end{aligned} \tag{2.82}$$

[4]Similar results are obtained for a muon neutrino state: $|\alpha^r_{\mathbf{k},\mu}\rangle \equiv \alpha^{r\dagger}_{\mathbf{k},\mu}(0)|0\rangle_{e,\mu}$.

Charge conservation is ensured at any time: $\mathcal{Q}_{\mathbf{k},e}(t) + \mathcal{Q}_{\mathbf{k},\mu}(t) = 1$. The oscillation formulas for the flavor charges are then [67]

$$\begin{aligned}\mathcal{Q}_{\mathbf{k},e}(t) &= 1 - \sin^2(2\theta)\,|U_{\mathbf{k}}|^2\, \sin^2\left(\frac{\omega_{k,2}-\omega_{k,1}}{2}t\right) \\ &\quad + \sin^2(2\theta)\,|V_{\mathbf{k}}|^2\, \sin^2\left(\frac{\omega_{k,2}+\omega_{k,1}}{2}t\right), \end{aligned} \tag{2.83}$$

$$\begin{aligned}\mathcal{Q}_{\mathbf{k},\mu}(t) &= \sin^2(2\theta)\,|U_{\mathbf{k}}|^2\, \sin^2\left(\frac{\omega_{k,2}-\omega_{k,1}}{2}t\right) \\ &\quad + \sin^2(2\theta)\,|V_{\mathbf{k}}|^2\, \sin^2\left(\frac{\omega_{k,2}+\omega_{k,1}}{2}t\right). \end{aligned} \tag{2.84}$$

This result is exact. There are two differences with respect to the usual formula for neutrino oscillations: the amplitudes are energy dependent, and there is an additional oscillating term. In the relativistic limit ($|\mathbf{k}| \gg \sqrt{m_1 m_2}$) one obtains ($\theta = \pi/4$):

$$\begin{aligned}\mathcal{Q}_{\mathbf{k},\mu}(t) \simeq &\left(1 - \frac{(\Delta m)^2}{4|\mathbf{k}|^2}\right)\sin^2\left[\frac{\Delta m^2}{4|\mathbf{k}|}\,t\right] \\ &+ \frac{(\Delta m)^2}{4k^2}\sin^2\left[\left(|\mathbf{k}| + \frac{m_1^2+m_2^2}{4|\mathbf{k}|}\right)\,t\right]. \end{aligned} \tag{2.85}$$

The usual QM formulas [62], are thus approximately recovered. Observe that for small times:

$$\mathcal{Q}_{\mathbf{k},\mu}(t) \simeq \frac{(m_2-m_1)^2}{4}\left(1 + \frac{m_1^2+m_2^2}{2|\mathbf{k}|^2} + \frac{(m_1+m_2)^2}{4|\mathbf{k}|^2}\right)t^2. \tag{2.86}$$

Thus, even for the case of relativistic neutrinos, QFT corrections are in principle observable (for sufficiently small time arguments). Note that the above quantities are not interpreted as probabilities, rather they have a sense as *statistical averages*, *i.e.* as mean values. This is because the structure of the theory for mixed field is that of a many–body theory, where does not make sense to talk about single–particle states. This situation has a formal analogy with QFT at finite temperature, where only statistical averages are well defined. It can be indeed explicitly checked that

$$\langle \widetilde{\alpha}^r_{\mathbf{k},e} | \widetilde{Q}_\sigma(t) | \widetilde{\alpha}^r_{\mathbf{k},e} \rangle \; = \; \langle \alpha^r_{\mathbf{k},e} | Q_\sigma(t) | \alpha^r_{\mathbf{k},e} \rangle \tag{2.87}$$

which ensure the cancellation of the arbitrary mass parameters.

2.3 Pontecorvo vs QFT formalism

We have seen how Pontecorvo mixing transformations are written as a rotation of the states with definite masses $|\nu_1\rangle$, $|\nu_2\rangle$, into those with definite flavor $|\nu_e\rangle$ and $|\nu_\mu\rangle$ as [62] (cf. Eq.(2.20)):

$$|\nu_e\rangle \;=\; \cos\theta\,|\nu_1\rangle + \sin\theta\,|\nu_2\rangle, \tag{2.88}$$

$$|\nu_\mu\rangle \;=\; \cos\theta\,|\nu_2\rangle - \sin\theta\,|\nu_1\rangle. \tag{2.89}$$

On the other hand, Standard Model is formulated in terms of fields and there neutrino mixing appears in the following form [68] (cf. Eq.(2.37)):

$$\begin{aligned} \nu_e(x) &= \cos\theta\;\nu_1(x) \,+\, \sin\theta\;\nu_2(x)\,, \\ \nu_\mu(x) &= \cos\theta\;\nu_2(x) \,-\, \sin\theta\;\nu_1(x)\,, \end{aligned}$$

where $x \equiv (\mathbf{x}, t)$, which is a rotation-like transformation. However its generator

$$G(t;\theta,m_1,m_2) = \exp\left\{\theta \int d^3\mathbf{x} \left(\nu_1^\dagger(x)\nu_2(x) - \nu_2^\dagger(x)\nu_1(x)\right)\right\}\;, \tag{2.90}$$

is not the one of a rotation. The question then arise to what extent the two above transformations are equivalent[5]. It has been shown [25] that this is not the case and indeed a deep conceptual difference is present between mixing of states and mixing of fields. Let us, thus, consider the expansion for the

[5]Our analysis is limited to the case of two Dirac neutrinos. Extension to three neutrinos is in our plans. However, we have good reasons to believe that the present results are general, since our arguments are of algebraic nature.

Dirac fields ν_1 and ν_2 with definite masses appearing in Eqs.(2.90),(2.90), as presented in Eq.(2.38)[6] :

$$\nu_i(x) = \sum_r \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \left[u^r_{\mathbf{k},i}(t)\alpha^r_{\mathbf{k},i} \; + v^r_{-\mathbf{k},i}(t)\beta^{r\dagger}_{-\mathbf{k},i}\right] e^{i\mathbf{k}\cdot\mathbf{x}}, \quad i = 1,2. \quad (2.91)$$

Then, Eqs.(2.88),(2.89) can be seen as arising by the application to the vacuum state $|0\rangle_{1,2}$ of the following operators:

$$\alpha^{r\dagger}_{\mathbf{k},e} \;=\; \cos\theta\, \alpha^{r\dagger}_{\mathbf{k},1} + \sin\theta\, \alpha^{r\dagger}_{\mathbf{k},2}, \quad (2.92)$$

$$\alpha^{r\dagger}_{\mathbf{k},\mu} \;=\; \cos\theta\, \alpha^{r\dagger}_{\mathbf{k},2} - \sin\theta\, \alpha^{r\dagger}_{\mathbf{k},1}. \quad (2.93)$$

The generator of such transformation is indeed a rotation $R(\theta)$:

$$\begin{aligned} R(\theta) \;&=\; \exp\left\{\theta \sum_r \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \left[\left(\alpha^{r\dagger}_{\mathbf{k},1}\alpha^r_{\mathbf{k},2} + \beta^{r\dagger}_{-\mathbf{k},1}\beta^r_{-\mathbf{k},2}\right)e^{i\psi_k}\right.\right. \\ &- \left.\left.\left(\alpha^{r\dagger}_{\mathbf{k},2}\alpha^r_{\mathbf{k},1} + \beta^{r\dagger}_{-\mathbf{k},2}\beta^r_{-\mathbf{k},1}\right)e^{-i\psi_k}\right]\right\}, \end{aligned} \quad (2.94)$$

where in full generality a phase ψ_k has been included. The action of the rotation generator on the annihilation operators gives:

$$R(\theta)^{-1}\alpha^r_{\mathbf{k},1}R(\theta) \;=\; \cos\theta\, \alpha^r_{\mathbf{k},1} \;+\; e^{i\psi_k}\, \sin\theta\, \alpha^r_{\mathbf{k},2}, \quad (2.95)$$

$$R(\theta)^{-1}\alpha^r_{\mathbf{k},2}R(\theta) \;=\; \cos\theta\, \alpha^r_{\mathbf{k},2} \;-\; e^{-i\psi_k}\, \sin\theta\, \alpha^r_{\mathbf{k},1}, \quad (2.96)$$

$$R(\theta)^{-1}\beta^{r\dagger}_{\mathbf{k},1}R(\theta) \;=\; \cos\theta\, \beta^{r\dagger}_{\mathbf{k},1} \;+\; e^{-i\psi_k}\, \sin\theta\, \beta^{r\dagger}_{\mathbf{k},2}, \quad (2.97)$$

$$R(\theta)^{-1}\beta^{r\dagger}_{\mathbf{k},2}R(\theta) \;=\; \cos\theta\, \beta^{r\dagger}_{\mathbf{k},2} \;-\; e^{i\psi_k}\, \sin\theta\, \beta^{r\dagger}_{\mathbf{k},1}. \quad (2.98)$$

Notice that the unitary operator $R^{-1} = R^\dagger$ leaves the vacuum invariant:

$$R^{-1}(\theta)|0\rangle_{1,2} = |0\rangle_{1,2}\;. \quad (2.99)$$

[6] where $u^r_{\mathbf{k},i}(t) = e^{-i\omega_{k,i}t}u^r_{\mathbf{k},i}$ and $v^r_{-\mathbf{k},i}(t) = e^{i\omega_{k,i}t}v^r_{-\mathbf{k},i}$, with $\omega_{k,i} = \sqrt{\mathbf{k}^2 + m_i^2}$. The $\alpha^r_{\mathbf{k},i}$ and the $\beta^r_{-\mathbf{k},i}$ $(r = 1,2)$, are the annihilation operators for the vacuum state $|0\rangle_{1,2} \equiv |0\rangle_1 \otimes |0\rangle_2$: $\alpha^r_{\mathbf{k},i}|0\rangle_{1,2} = \beta^r_{\mathbf{k},i}|0\rangle_{1,2} = 0$.

Before applying the same operator R on the fields Ψ_m, let us introduce another canonical transformation which will be useful in the following, the Bogoliubov transformation:

$$\begin{aligned}
\tilde{\alpha}^r_{\mathbf{k},i} &\equiv B_i^{-1}(\Theta_i)\,\alpha^r_{\mathbf{k},i}\,B_i(\Theta_i) \\
&= \cos\Theta_{\mathbf{k},i}\,\alpha^r_{\mathbf{k},i} - \epsilon^r\,e^{-i\phi_{k,i}}\,\sin\Theta_{\mathbf{k},i}\,\beta^{r\dagger}_{-\mathbf{k},i}, \qquad (2.100)\\
\tilde{\beta}^r_{-\mathbf{k},i} &\equiv B_i^{-1}(\Theta_i)\,\beta^r_{-\mathbf{k},i}\,B_i(\Theta_i) \\
&= \cos\Theta_{\mathbf{k},i}\,\beta^r_{-\mathbf{k},i} + \epsilon^r\,e^{i\phi_{k,i}}\,\sin\Theta_{\mathbf{k},i}\,\alpha^{r\dagger}_{\mathbf{k},i}, \qquad (2.101)
\end{aligned}$$

with $i = 1, 2$ and the generator(s)

$$B_i(\Theta_i) = \exp\left\{\sum_r \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}}\Theta_{\mathbf{k},i}\,\epsilon^r\left[\alpha^r_{\mathbf{k},i}\beta^r_{-\mathbf{k},i}e^{-i\phi_{k,i}} - \beta^{r\dagger}_{-\mathbf{k},i}\alpha^{r\dagger}_{\mathbf{k},i}e^{i\phi_{k,i}}\right]\right\}, \qquad (2.102)$$

with $i = 1, 2$. Since $[B_1(\Theta_1), B_2(\Theta_2)] = 0$, we may also define

$$B(\Theta_1, \Theta_2) \equiv B_2(\Theta_2)B_1(\Theta_1). \qquad (2.103)$$

Note that, at odd with the case of the rotation, the Bogoliubov transformation does not leave invariant the vacuum $|0\rangle_{1,2}$

$$\begin{aligned}
|\widetilde{0}\rangle_{1,2} &\equiv B^{-1}(\Theta_1, \Theta_2)|0\rangle_{1,2} \\
&= \prod_{i=1,2}\prod_{\mathbf{k},r}\left[\cos\Theta_{\mathbf{k},i} + \epsilon^r e^{i\phi_{k,i}}\,\sin\Theta_{\mathbf{k},i}\,\alpha^{r\dagger}_{\mathbf{k},i}\beta^{r\dagger}_{-\mathbf{k},i}\right]|0\rangle_{1,2}. \qquad (2.104)
\end{aligned}$$

Indeed, in the infinite volume limit, the states $|\widetilde{0}\rangle_{1,2}$ and $|0\rangle_{1,2}$ become orthogonal, thus giving rise to inequivalent representations [64]. This is a well-known feature of QFT [69] reflecting into the non–unitary nature (in the infinite volume limit) of the generator of Bogoliubov transformations. We are now ready to explore the action of the above rotation - Eq.(2.94) -

on the fields ν_1 and ν_2:

$$R^{-1}(\theta)\nu_1(x)R(\theta) = \cos\theta\,\nu_1(x) + \sin\theta\sum_r\int\frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}}\,e^{i\mathbf{k}\cdot\mathbf{x}}\Big(e^{i\psi_k}\alpha^r_{\mathbf{k},2}\,u^r_{\mathbf{k},1}(t) + e^{-i\psi_k}\beta^{r\dagger}_{\mathbf{k},2}\,v^r_{-\mathbf{k},1}(t)\Big), \quad (2.105)$$

$$R^{-1}(\theta)\nu_2(x)R(\theta) = \cos\theta\,\nu_2(x) - \sin\theta\sum_r\int\frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}}\,e^{i\mathbf{k}\cdot\mathbf{x}}\Big(e^{-i\psi_k}\alpha^r_{\mathbf{k},1}\,u^r_{\mathbf{k},2}(t) + e^{i\psi_k}\beta^{r\dagger}_{\mathbf{k},1}\,v^r_{-\mathbf{k},2}(t)\Big). \quad (2.106)$$

We see that the above expressions do not fully reproduce the mixing at level of fields, cf.Eqs.(2.90),(2.90): the problem is that the last term in the r.h.s. of these equations appears as the expansion of the field in the "wrong" basis. However, it is possible to recover the wanted expression by means of a suitable Bogoliubov transformation, which implements a mass shift. Let us see this for the field ν_1:

$$\begin{aligned} &B_2^{-1}(\Theta_2)\,R^{-1}(\theta)\,\nu_1(x)\,R(\theta)\,B_2(\Theta_2) = \\ &= \cos\theta\,\nu_1(x)+\sin\theta\sum_r\int\frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}}e^{i\mathbf{k}\cdot\mathbf{x}}\Big(e^{i\psi_k}\tilde{\alpha}^r_{\mathbf{k},2}u^r_{\mathbf{k},1}(t)+e^{-i\psi_k}\tilde{\beta}^{r\dagger}_{\mathbf{k},2}v^r_{-\mathbf{k},1}(t)\Big) \\ &= \cos\theta\,\nu_1(x)+\sin\theta\sum_r\int\frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}}e^{i\mathbf{k}\cdot\mathbf{x}}\Big(e^{i\psi_k}\alpha^r_{\mathbf{k},2}\hat{u}^r_{\mathbf{k},1}(t)+e^{-i\psi_k}\beta^{r\dagger}_{\mathbf{k},2}\hat{v}^r_{-\mathbf{k},1}(t)\Big), \end{aligned} \quad (2.107)$$

where

$$\hat{u}^r_{\mathbf{k},1}(t) = u^r_{\mathbf{k},1}e^{-i\omega_{k,1}t}e^{i\psi_k}\cos\Theta_{\mathbf{k},2} + \epsilon^r v^r_{-\mathbf{k},1}e^{i\omega_{k,1}t}e^{-i\phi_{k,2}}e^{-i\psi_k}\sin\Theta_{\mathbf{k},2}\,, \quad (2.108)$$

$$\hat{v}^r_{-\mathbf{k},1}(t) = v^r_{-\mathbf{k},1}e^{i\omega_{k,1}t}e^{-i\psi_k}\cos\Theta_{\mathbf{k},2} - \epsilon^r u^r_{\mathbf{k},1}e^{-i\omega_{k,1}t}e^{-i\phi_{k,2}}e^{-i\psi_k}\sin\Theta_{\mathbf{k},2}\,. \quad (2.109)$$

Thus for $\hat{\Theta}_2 \equiv \Theta_{\mathbf{k},2} = \arccos\left(e^{-i\psi_k}U_{\mathbf{k}}(t)\right)$, with $U_{\mathbf{k}}(t) \equiv u^{r\dagger}_{\mathbf{k},2}(t)u^r_{\mathbf{k},1}(t)$, the Bogoliubov transformation $B_2(\hat{\Theta}_2)$ produces [70] a mass shift by $m_2 - m_1$ such that[7]

$$\hat{u}^r_{\mathbf{k},1}(t) = u^r_{\mathbf{k},2}(t)\,, \qquad \hat{v}^r_{-\mathbf{k},1}(t) = v^r_{-\mathbf{k},2}(t). \quad (2.110)$$

[7] An equivalent choice is $\hat{\Theta}_{\mathbf{k},2} = \arcsin\left(e^{i\phi_{k,2}}e^{i\psi_k}V_{\mathbf{k}}(t)\right)$ with $V_{\mathbf{k}}(t) \equiv \epsilon^r\, u^{r\dagger}_{\mathbf{k},1}(t)v^r_{-\mathbf{k},2}(t)$.

In definitive, the action of $B_2^{-1}(\hat{\Theta}_2)\, R^{-1}(\theta)$ produces the desired rotation-like transformation of the field ν_1 - cf. Eq.(2.90). A similar reasoning can be done for ν_2, using $B_1^{-1}(\hat{\Theta}_1)\, R^{-1}(\theta)$, with $\hat{\Theta}_1 = \arccos\left(e^{i\psi_k} U_{\mathbf{k}}(t)\right)$. We can conclude that

$$\nu_e = G_2^{-1} \nu_1 G_2$$

$$\nu_\mu = G_1^{-1} \nu_2 G_1$$

where

$$G_1 = B_1^{-1}(\hat{\Theta}_1)\, R^{-1}(\theta),$$

$$G_2 = B_2^{-1}(\hat{\Theta}_2)\, R^{-1}(\theta).$$

In closing this Section, we note that the rôle of the Bogoliubov transformation in the process of (dynamical) mass generation is well known, see for example Refs.[34, 36].

Chapter 3

Mixing generator and vacuum structure

In the previous Chapter, we have shown the incompatibility of the mixing transformation as mere rotations both for states and fields, and the necessity of implementing a mass shift in order to reproduce the correct relations for fields: such an operation is highly non–trivial and indeed requires infinite energy (in the infinite volume limit). On the other hand, these results are incomplete in that two different generators are needed for ν_1 and ν_2, whereas we know the algebraic generator for fields to be that of Eq.(2.90). It thus arises the problem of the decomposition of such generator in terms of rotation and Bogoliubov transformations.In order to achieve such decomposition we notice that the mixing generator $G_\theta(t)$, Eq(2.43), is a function of m_1, m_2 and θ. In this Chapter we shall see how one can disentangle the mass dependence of $G_\theta(t)$ from its dependence on the mixing angle.

3.1 Decomposition of the mixing generator

Let us, thus, recall [71] from Eqs. (2.94), (2.102) the definition of $B_i(\Theta_i)$, $i = 1, 2$, the generators of Bogoliubov transformations, and $R(\theta)$, the rotation generator:

$$R(\theta) = \exp\Big\{\theta \sum_{\mathbf{k},r} \Big[\Big(\alpha^{r\dagger}_{\mathbf{k},1}\alpha^{r}_{\mathbf{k},2} + \beta^{r\dagger}_{-\mathbf{k},1}\beta^{r}_{-\mathbf{k},2}\Big)e^{i\psi_k} - \Big(\alpha^{r\dagger}_{\mathbf{k},2}\alpha^{r}_{\mathbf{k},1} + \beta^{r\dagger}_{-\mathbf{k},2}\beta^{r}_{-\mathbf{k},1}\Big)e^{-i\psi_k}\Big]\Big\},$$

$$B_i(\Theta_i) = \exp\Big\{\sum_{\mathbf{k},r} \Theta_{\mathbf{k},i}\, \epsilon^r \Big[\alpha^{r}_{\mathbf{k},i}\beta^{r}_{-\mathbf{k},i}e^{-i\phi_{k,i}} - \beta^{r\dagger}_{-\mathbf{k},i}\alpha^{r\dagger}_{\mathbf{k},i}e^{i\phi_{k,i}}\Big]\Big\}.$$

Since $[B_1(\Theta_1), B_2(\Theta_2)] = 0$, we may also define $B(\Theta_1,\Theta_2) \equiv B_2(\Theta_2)B_1(\Theta_1)$ and then calculate how ladder operators transform under $B(\Theta_1,\Theta_2)$ - cf. Eqs. (2.100),(2.101):

$$\begin{aligned}\tilde{\alpha}^{r}_{\mathbf{k},i} &\equiv B^{-1}(\Theta_1,\Theta_2)\,\alpha^{r}_{\mathbf{k},i}\,B(\Theta_1,\Theta_2)\\ &= \cos\Theta_{\mathbf{k},i}\,\alpha^{r}_{\mathbf{k},i} - \epsilon^r\, e^{-i\phi_{k,i}}\,\sin\Theta_{\mathbf{k},i}\,\beta^{r\dagger}_{-\mathbf{k},i}\,,\end{aligned}$$

$$\begin{aligned}\tilde{\beta}^{r}_{-\mathbf{k},i} &\equiv B^{-1}(\Theta_1,\Theta_2)\,\beta^{r}_{-\mathbf{k},i}\,B(\Theta_1,\Theta_2)\\ &= \cos\Theta_{\mathbf{k},i}\,\beta^{r}_{-\mathbf{k},i} + \epsilon^r\, e^{i\phi_{k,i}}\,\sin\Theta_{\mathbf{k},i}\,\alpha^{r\dagger}_{\mathbf{k},i}\,.\end{aligned}$$

On the other hand, we know the action of the rotation generator on the annihilation operators to give - cf. Eqs. (2.95-2.98):

$$\begin{aligned}R(\theta)^{-1}\alpha^{r}_{\mathbf{k},1}R(\theta) &= \cos\theta\,\alpha^{r}_{\mathbf{k},1} + e^{i\psi_k}\,\sin\theta\,\alpha^{r}_{\mathbf{k},2}\,,\\ R(\theta)^{-1}\alpha^{r}_{\mathbf{k},2}R(\theta) &= \cos\theta\,\alpha^{r}_{\mathbf{k},2} - e^{-i\psi_k}\,\sin\theta\,\alpha^{r}_{\mathbf{k},1}\end{aligned}$$

and similar ones for the $\beta^{r}_{\mathbf{k},i}$. The action of the above transformations on the vacuum for the free fields ν_1, ν_2 is given by

$$\begin{aligned}|\widetilde{0}\rangle_{1,2} &\equiv B^{-1}(\Theta_1,\Theta_2)|0\rangle_{1,2}\\ &= \prod_{i=1,2}\prod_{\mathbf{k},r}\Big[\cos\Theta_{\mathbf{k},i} + \epsilon^r e^{i\phi_{k,i}}\,\sin\Theta_{\mathbf{k},i}\,\alpha^{r\dagger}_{\mathbf{k},i}\beta^{r\dagger}_{-\mathbf{k},i}\Big]|0\rangle_{1,2}\,, \end{aligned}\tag{3.1}$$

$$R^{-1}(\theta)|0\rangle_{1,2} = |0\rangle_{1,2}\,. \tag{3.2}$$

Define now $\tilde{R} \equiv \tilde{R}(\theta, \Theta_1, \Theta_2) = B^{-1}(\Theta_1, \Theta_2) R^{-1}(\theta) B(\Theta_1, \Theta_2)$, which can be written as

$$\tilde{R} = \exp\Big\{\theta \sum_{\mathbf{k},r} \Big[\Big(\tilde{\alpha}^{r\dagger}_{\mathbf{k},1}\tilde{\alpha}^{r}_{\mathbf{k},2} + \tilde{\beta}^{r\dagger}_{-\mathbf{k},1}\tilde{\beta}^{r}_{-\mathbf{k},2}\Big) e^{i\psi_k} - \Big(\tilde{\alpha}^{r\dagger}_{\mathbf{k},2}\tilde{\alpha}^{r}_{\mathbf{k},1} + \tilde{\beta}^{r\dagger}_{-\mathbf{k},2}\tilde{\beta}^{r}_{-\mathbf{k},1}\Big) e^{-i\psi_k}\Big]\Big\}, \qquad (3.3)$$

Thereby, using Eqs.(2.100), (2.101) and imposing the following constraint

$$U_{\mathbf{k}}(t) = e^{-i\psi_k}\ \cos(\Theta_{\mathbf{k},1} - \Theta_{\mathbf{k},2})\,, \qquad (3.4)$$

$$V_{\mathbf{k}}(t) = e^{\frac{(\phi_{k,1}+\phi_{k,2})}{2}}\ \sin(\Theta_{\mathbf{k},1} - \Theta_{\mathbf{k},2})\,, \qquad (3.5)$$

we obtain

$$\tilde{R} = \exp\Big\{\sum_r \Big(U^*_{\mathbf{k}}\ \alpha^{r\dagger}_{\mathbf{k},1}\alpha^{r}_{\mathbf{k},2} - \epsilon^r\ V^*_{\mathbf{k}}\ \beta^{r}_{-\mathbf{k},1}\alpha^{r}_{\mathbf{k},2} + \epsilon^r\ V_{\mathbf{k}}\ \alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2} + U_{\mathbf{k}}\ \beta^{r}_{-\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2}\Big) - \sum_r \Big(U_{\mathbf{k}}\ \alpha^{r\dagger}_{\mathbf{k},2}\alpha^{r}_{\mathbf{k},1} + \epsilon^r\ V^*_{\mathbf{k}}\ \beta^{r}_{-\mathbf{k},2}\alpha^{r}_{\mathbf{k},1} - \epsilon^r\ V_{\mathbf{k}}\ \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1} + U^*_{\mathbf{k}}\ \beta^{r}_{-\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}\Big)\Big\}, \qquad (3.6)$$

which indeed coincides with the expression of the mixing generator $G_\theta(t)$

$$G_\theta(t) = \exp\Big\{\sum_r \Big(U^*_{\mathbf{k}}\ \alpha^{r\dagger}_{\mathbf{k},1}\alpha^{r}_{\mathbf{k},2} - \epsilon^r\ V^*_{\mathbf{k}}\ \beta^{r}_{-\mathbf{k},1}\alpha^{r}_{\mathbf{k},2} + \epsilon^r\ V_{\mathbf{k}}\ \alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2} + U_{\mathbf{k}}\ \beta^{r}_{-\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2}\Big) - \sum_r \Big(U_{\mathbf{k}}\ \alpha^{r\dagger}_{\mathbf{k},2}\alpha^{r}_{\mathbf{k},1} + \epsilon^r\ V^*_{\mathbf{k}}\ \beta^{r}_{-\mathbf{k},2}\alpha^{r}_{\mathbf{k},1} - \epsilon^r\ V_{\mathbf{k}}\ \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1} + U^*_{\mathbf{k}}\ \beta^{r}_{-\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}\Big)\Big\} \qquad (3.7)$$

obtained using Eqs. (2.44),(2.48), provided we make the following identifications:

$$\psi_k = (\omega_{k,1} - \omega_{k,2})\,t\,, \qquad (3.8)$$

$$\phi_{k,i} = 2\,\omega_{k,i}\,t\,, \qquad (3.9)$$

$$\Theta_{\mathbf{k},i} = \frac{1}{2}\cot^{-1}\left(\frac{|\mathbf{k}|}{m_i}\right)\,. \qquad (3.10)$$

In definitive, we have shown [70] that it is possible to decompose the mixing generator G_θ in the following way (for $t = 0$)

$$G(\theta, m_1, m_2) = B^{-1}(\Theta_1, \Theta_2)\, R(\theta)\, B(\Theta_1, \Theta_2)\,, \tag{3.11}$$

i.e., we have been able to write G as a product of operators depending only on the masses or on the mixing angle. From the above decomposition, it is quite evident that the non–trivial nature of the mixing generator arises as a consequence of the non–commutativity of the rotation generator with the generator of Bogoliubov transformation(s). From Eq.(3.8), we see that $\Theta_{\mathbf{k},i}$ are functions of the masses and the momentum only. Thus we can regard the generator $B(\Theta_1, \Theta_2)$, where the momentum has been integrated out, as dependent on the mass parameters, *i.e.* as $B(m_1, m_2)$. Then we rewrite Eq.(3.11) as

$$G_t(\theta, m_1, m_2) = B_t^{-1}(m_1, m_2)\, R_t(\theta)\, B_t(m_1, m_2)\,, \tag{3.12}$$

where we have included now the dependence on time as an index t and the notation is $f(\Theta_i(m_i)) \equiv f(m_i)$. Finally, let us consider the flavor vacuum (for simplicity at $t = 0$), which can be written as

$$|0\rangle_{e,\mu} \equiv G^{-1}|0\rangle_{1,2} \;=\; |0\rangle_{1,2} \;+\; \Big[B(m_1, m_2)\;,\; R^{-1}(\theta)\Big]\;|\widetilde{0}\rangle_{1,2}\,, \tag{3.13}$$

where we see as the condensate structure arises as a consequence of the non–vanishing commutator $[B, R^{-1}]$.

3.2 Vacuum structure and non–commutativity

We use now the decomposition found in the previous section to investigate the nature of the flavor vacuum.

3.2.1 Vacuum energies

The first investigative approach we undertake is a step-by-step one, starting from $|0\rangle_{1,2}$ and acting on it with the above Bogoliubov transformations and rotation generators. In the following table we report the vacuum expectation values of the (unordered) Hamiltonian on the various *vacua* obtained by acting step-by-step with the above generators.

State	$\langle H_{\mathbf{k},1} + H_{\mathbf{k},2}\rangle$
$\|0\rangle_1 \otimes \|0\rangle_2 \equiv \|0\rangle_{1,2}$	$-(\omega_{k,1} + \omega_{k,2})$
$B^{-1}(m_1, m_2)\|0\rangle_{1,2} \equiv \|\widetilde{0}\rangle_{1,2}$	$-(k + k)$
$R^{-1}(\theta)B^{-1}(m_1, m_2)\|0\rangle_{1,2} = R^{-1}(\theta)\|\widetilde{0}\rangle_{1,2}$	$-k(2\cos^2\theta + (\frac{\omega_{k,1}}{\omega_{k,2}} + \frac{\omega_{k,2}}{\omega_{k,1}})\sin^2\theta)$
$B(m_1, m_2)R^{-1}(\theta)B^{-1}(m_1, m_2)\|0\rangle_{1,2} \equiv \|0\rangle_{e,\mu}$	$-(\omega_{k,1} + \omega_{k,2})(1 - 2\sin^2\theta\sin^2(\Theta_{\mathbf{k},1} - \Theta_{\mathbf{k},2}))$

From the results shown in the table we can understand the steps that lead to the non–equivalence between the two vacua. Starting from the vacuum for the free fields $|0\rangle_{1,2}$, the first Bogoliubov transformation(s) acts as a mass shift that brings the two species of neutrinos to have a massless vacuum: the tilde vacuum. The second transformation that acts is a rotation. The tilde vacuum, though, is no more invariant under such rotation because the latter is written in terms of the mass ladder operators and not of the tilde ones. The last transformation to act is the inverse Bogoliubov transformation which however cannot put back masses at their original position, since in the meanwhile they have been "rotated" by R. This mechanism is essentially due to the non–vanishing commutator $[B, R^{-1}]$. The situation is clarified by Fig.3.1, where we plot, for sample value of the parameters, the (absolute values of) expectation values of $H_{\mathbf{k},1}$ and $H_{\mathbf{k},}$ for the above vacua. The arrows indicate the "way" from $|0\rangle_{1,2}$ to $|0\rangle_{e,\mu}$, making clear the origin of the energy gap between these two vacua. In fact, we can see how the first Bogoliubov transformation(s) - A $\rightarrow$ B - moves our point to one with the same horizontal and vertical axes value equal to the sample values chosen for the momentum. The tilde vacuum is clearly non–invariant under the action of the rotation - B $\rightarrow$ C - whose effect is an energy gap between the two vacua. Finally the inverse Bogoliubov transformation - C $\rightarrow$ D - fails to reproduce the starting energy values. In the following figures, we consider

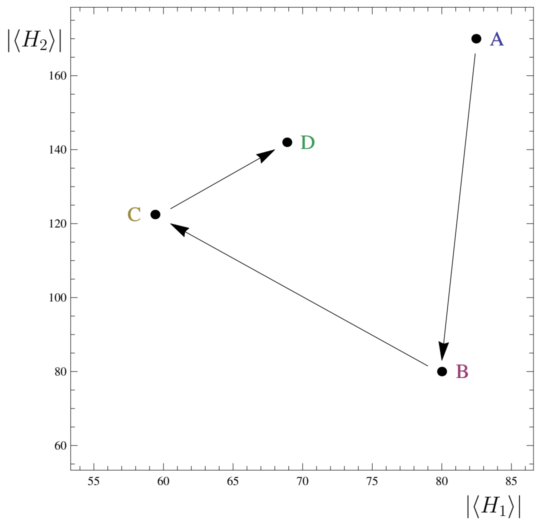


Figure 3.1: Plot of vacuum expectation values of H_1 and H_2 for the states of Table 1. Sample values of parameters are chosen for $\theta = \pi/4$, $k = 80$, $m_1 = 20$, $m_2 = 150$. We put $A = |0\rangle_{1,2}$, $B = |\widetilde{0}\rangle_{1,2}$, $C = R^{-1}(\theta)|\widetilde{0}\rangle_{1,2}$, and $D = |0\rangle_{e,\mu}$.

other values of the parameters, showing various limits in which the effects of the condensate tends to be less important and eventually to disappear.

In Fig.3.2 we see that in the limit of the high momentum the points - A, B, C, D - tends to overlap. Fig.3.3 and Fig.3.4 show how the smaller the mixing angle is the smaller the energy gap between $|0\rangle_{1,2}$ and $|0\rangle_{e,\mu}$ becomes. Finally, in Fig.3.5 we can see how even for the maximum mixing angle one has no mixing when the difference between the masses is vanishing - D and A coincide.

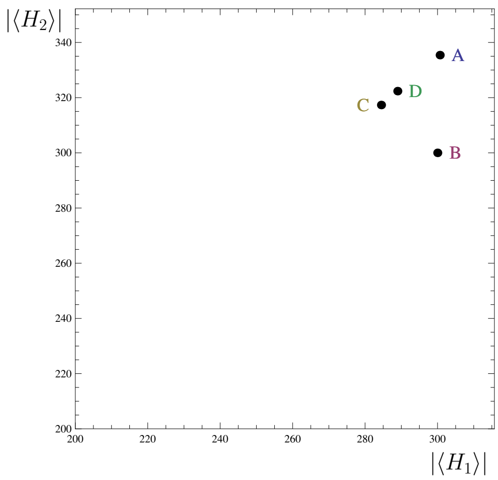


Figure 3.2: Plot of vacuum expectation values of H_1 and H_2 for the states of Table 1. Sample values of parameters are chosen for $\theta = \pi/4$, $k = 300$, $m_1 = 20$, $m_2 = 150$.

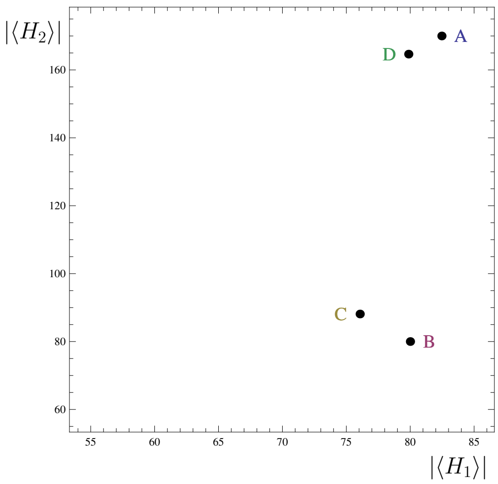


Figure 3.3: Plot of vacuum expectation values of H_1 and H_2 for the states of Table 1. Sample values of parameters are chosen for $\theta = \pi/10$, $k = 80$, $m_1 = 20$, $m_2 = 150$.

3.2.2 Small θ approximation

Another useful approach consists in analyzing what happens to the generator G for small mixing angle θ. From Eq.(3.11) it appears evident that the difference between G (generator of a rotation in the fields) and R (gener-

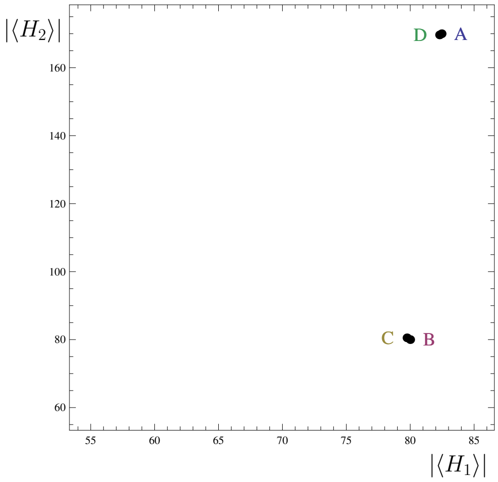


Figure 3.4: Plot of vacuum expectation values of H_1 and H_2 for the states of Table 1. Sample values of parameters are chosen for $\theta = \pi/40$, $k = 80$, $m_1 = 20$, $m_2 = 150$.

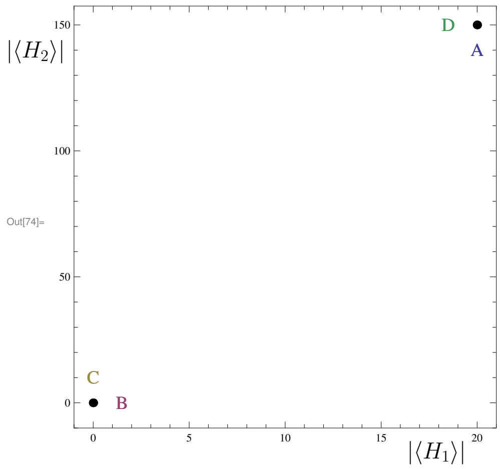


Figure 3.5: Plot of vacuum expectation values of H_1 and H_2 for the states of Table 1. Sample values of parameters are chosen for $\theta = \pi/4$, $k = 0$, $m_1 = 20$, $m_2 = 150$.

ator of a rotation in the ladder operators) relies in the non–zero value of the commutator $[R, B]$. In order to better understand how this decomposition works we shall recall the expansion of $G(\theta)$ at t=0

$$G(\theta) = \exp\left\{\theta \sum_r \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \Big[U_\mathbf{k} \left(\alpha^{r\dagger}_{\mathbf{k},1}\alpha^r_{\mathbf{k},2} + \beta^r_{-\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2} - \alpha^{r\dagger}_{\mathbf{k},2}\alpha^r_{\mathbf{k},1} - \beta^r_{-\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}\right) \right.$$
$$\left. +\epsilon^r V_\mathbf{k} \left(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2} - \beta^r_{-\mathbf{k},1}\alpha^r_{\mathbf{k},2} + \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1} - \beta^r_{-\mathbf{k},2}\alpha^r_{\mathbf{k},1}\right)\Big]\right\}, \quad (3.14)$$

where we know

$$U_\mathbf{k} \;=\; \cos(\Theta_{\mathbf{k},2} - \Theta_{\mathbf{k},1})\,, \qquad V_\mathbf{k} \;=\; \sin(\Theta_{\mathbf{k},2} - \Theta_{\mathbf{k},1})\,.$$

For $(\Theta_{\mathbf{k},2} - \Theta_{\mathbf{k},1})$ small, we have

$$U_\mathbf{k} \cong 1\,, \qquad V_\mathbf{k} \cong (\Theta_{\mathbf{k},2} - \Theta_{\mathbf{k},1})\,. \quad (3.15)$$

In such an approximation, by expanding $G(\theta)$ up to the first order in θ we obtain:

$$G(\theta) = \mathbb{1} + \theta \sum_r \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \left\{ \left[\alpha^{r\dagger}_{\mathbf{k},1}\alpha^r_{\mathbf{k},2} + \beta^{r\dagger}_{-\mathbf{k},1}\beta^r_{-\mathbf{k},2} - \alpha^{r\dagger}_{\mathbf{k},2}\alpha^r_{\mathbf{k},1} - \beta^{r\dagger}_{-\mathbf{k},2}\beta^r_{-\mathbf{k},1}\right] \right.$$
$$\left. +(\Theta_{\mathbf{k},2} - \Theta_{\mathbf{k},1})\epsilon^r \left[\alpha^r_{\mathbf{k},1}\beta^r_{-\mathbf{k},2} - \beta^{r\dagger}_{-\mathbf{k},2}\alpha^{r\dagger}_{\mathbf{k},1} + \alpha^r_{\mathbf{k},2}\beta^r_{-\mathbf{k},1} - \beta^{r\dagger}_{-\mathbf{k},1}\alpha^{r\dagger}_{\mathbf{k},2}\right]\right\}. \quad (3.16)$$

We recognize in Eq.(3.16) the following operators[1]:

$$J^r_{\mathbf{k},1} \equiv \frac{1}{2}\Big[(\alpha^r_{\mathbf{k},1}\beta^r_{-\mathbf{k},1} - \beta^{r\dagger}_{-\mathbf{k},1}\alpha^{r\dagger}_{\mathbf{k},1}) - (\alpha^r_{\mathbf{k},2}\beta^r_{-\mathbf{k},2} - \beta^{r\dagger}_{-\mathbf{k},2}\alpha^{r\dagger}_{\mathbf{k},2})\Big], \quad (3.17)$$
$$J^r_{\mathbf{k},2} \equiv -\frac{1}{2}\Big[(\alpha^r_{\mathbf{k},1}\beta^r_{-\mathbf{k},2} - \beta^{r\dagger}_{-\mathbf{k},2}\alpha^{r\dagger}_{\mathbf{k},1}) + (\alpha^r_{\mathbf{k},2}\beta^r_{-\mathbf{k},1} - \beta^{r\dagger}_{-\mathbf{k},1}\alpha^{r\dagger}_{\mathbf{k},2})\Big], \quad (3.18)$$
$$J^r_{\mathbf{k},3} \equiv \frac{1}{2}\Big[(\alpha^{r\dagger}_{\mathbf{k},1}\alpha^r_{\mathbf{k},2} + \beta^{r\dagger}_{-\mathbf{k},1}\beta^r_{-\mathbf{k},2}) - (\alpha^{r\dagger}_{\mathbf{k},2}\alpha^r_{\mathbf{k},1} + \beta^{r\dagger}_{-\mathbf{k},2}\beta^r_{-\mathbf{k},1})\Big], \quad (3.19)$$

which close the $su(2)$ algebra: $[J^r_{\mathbf{k},i}, J^r_{\mathbf{k},j}] = \varepsilon_{ijk} J_{\mathbf{k},k}$ with $i,j,k = 1,2,3$. Moreover, in the above limit, it is also possible to expand $V_\mathbf{k}$ in terms of the

[1] $J^r_{\mathbf{k},1} \equiv \frac{1}{2}(K^r_{\mathbf{k},1} - K^r_{\mathbf{k},2})$ with $K^r_{\mathbf{k},i} \equiv \alpha^r_{\mathbf{k},i}\beta^r_{-\mathbf{k},i} - \beta^{r\dagger}_{-\mathbf{k},i}\alpha^{r\dagger}_{\mathbf{k},i}$ and $\ln B_i(\Theta_{\mathbf{k},i}) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\Theta_{\mathbf{k},i}K_{\mathbf{k},i}$; $\ln R(\theta) = \theta \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\; 2\; J_{\mathbf{k},3}$.

adimensional parameter $a \equiv \frac{(m_2-m_1)^2}{m_1 m_2}$ so that

$$U_{\mathbf{k}} \cong 1\,, \quad V_{\mathbf{k}} \;\cong\; a\tilde{V}_{\mathbf{k}}, \tag{3.20}$$

up to $o[(a)^2]$ where $\tilde{V}_{\mathbf{k}} \equiv \frac{|\mathbf{k}|\sqrt{m_1 m_2}}{2(|\mathbf{k}|^2+m_1 m_2)}$ and thus,

$$G(\theta) \cong \mathbb{I} + \theta \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}}\, 2 \sum_r J^r_{\mathbf{k},3} - \theta\, a \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}}\, 2\,\tilde{V}_{\mathbf{k}} \sum_r \epsilon^r J^r_{\mathbf{k},2}\,. \tag{3.21}$$

It is easy to see as this generator becomes the identity when $\theta = 0$ and is equivalent to a mere rotation when $a = 0$, *i.e.* $m_2 = m_1$. Moreover, the last term shows the explicit dependance on the true physical parameters of the mixing transformation, *i.e.* θ and a. Notice that the adimensional parameter a appears at second order in the expansion, being linked with the commutator $J_{\mathbf{k},2} = [J_{\mathbf{k},3}, J_{\mathbf{k},1}]$ which can be interpreted [71] as a *non–diagonal Bogoliubov transformation*, and is the first non–trivial term which contributes to the flavor vacuum structure. This feature can be further understood by looking at the tilde vacuum, defined as (cf. Eq.(3.1)):

$$\begin{aligned} |\tilde{0}\rangle_{1,2} &\equiv B^{-1}(\Theta_1,\Theta_2)|0\rangle_{1,2} \\ &\cong \left\{\mathbb{I} + \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \sum_r \left[\Theta_{\mathbf{k},1}\, \alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},1} + \Theta_{\mathbf{k},2}\, \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},2}\right]\right\}|0\rangle_{1,2}\,, \end{aligned} \tag{3.22}$$

for $\Theta_{\mathbf{k},i}$ small, and comparing it with the flavor vacuum obtained in our approximation:

$$\begin{aligned} |0\rangle_{e,\mu} &\equiv G^{-1}|0\rangle_{1,2} \\ &\cong \left\{\mathbb{I} + \theta\, a \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}}\, \tilde{V}_{\mathbf{k}} \sum_r \epsilon^r \left[\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2} + \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}\right]\right\}|0\rangle_{1,2}\,. \end{aligned} \tag{3.23}$$

Notice that, although the operatorial structure of the two above equations is similar, Eq.(3.23) exhibits non–diagonal operatorial terms. From Eq.(3.23) we see that $|0\rangle_{e,\mu}$ cannot be reduced as a tensor product of vectors built on $|0\rangle_{1,2}$: this indeed confirms that the phenomenon of flavor mixing is related to the entanglement of mass eigenstates (see [72] for the discussion of entanglement in the context of particle mixing and oscillations). Another interesting feature of this phenomenon appears as one analyses more closely the parameter a, which in order to exist needs at least two fermion families to be present. In fact, with just one family the only adimensional parameter one can form is $\frac{|k|}{m}$, which however depends on k and thus cannot be extracted from the integrals. The same considerations can be done also looking at the explicit form of the mixing generator $G(\theta)$ at second order approximation in θ:

$$
\begin{aligned}
G(\theta) \cong \quad & \mathbb{I} + 2\,\theta \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \sum_r J^r_{\mathbf{k},3} \,-\, 2\,\theta\, a \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \tilde{V}_{\mathbf{k}} \sum_r \epsilon^r J^r_{\mathbf{k},2} \\
& - \theta\, a^2 \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \tilde{V}^2_{\mathbf{k}} \sum_r J^r_{\mathbf{k},3} \\
& + 2\,\theta^2\, a \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \tilde{V}_{\mathbf{k}} \sum_r \epsilon^r (J^r_{\mathbf{k},1} - 2 J^r_{\mathbf{k},2} J^r_{\mathbf{k},3}) \\
& + \theta^2\, a^2 \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \tilde{V}^2_{\mathbf{k}} \sum_r (J^2_{\mathbf{k},2} - J^2_{\mathbf{k},3}).
\end{aligned} \tag{3.24}
$$

It is easy to see that the complete operatorial structure of the flavor vacuum is obtained to the same approximation by application of this operator to the vacuum $|0\rangle_{1,2}$, see Eq.(3.25).

$$
\begin{aligned}
|0\rangle_{e,\mu} \;=\; & \prod_{\mathbf{k},r} \Big[(1 - \sin^2\theta\, |V_{\mathbf{k}}|^2) \,+ \\
& \quad - \epsilon^r \sin\theta\, \cos\theta\, |V_{\mathbf{k}}|\, (\alpha^{r\dagger}_{\mathbf{k},1} \beta^{r\dagger}_{-\mathbf{k},2} + \alpha^{r\dagger}_{\mathbf{k},2} \beta^{r\dagger}_{-\mathbf{k},1}) \,+ \\
& \quad + \epsilon^r \sin^2\theta\, |V_{\mathbf{k}}||U_{\mathbf{k}}|\, (\alpha^{r\dagger}_{\mathbf{k},1} \beta^{r\dagger}_{-\mathbf{k},1} - \alpha^{r\dagger}_{\mathbf{k},2} \beta^{r\dagger}_{-\mathbf{k},2}) \,+ \\
& \quad + \sin^2\theta\, |V_{\mathbf{k}}|^2\, \alpha^{r\dagger}_{\mathbf{k},1} \beta^{r\dagger}_{-\mathbf{k},2} \alpha^{r\dagger}_{\mathbf{k},2} \beta^{r\dagger}_{-\mathbf{k},1} \Big]\, |0\rangle_{1,2}\,.
\end{aligned} \tag{3.25}
$$

To this end, notice that $J_{\mathbf{k},3}|0\rangle_{1,2} = 0$ and

$$
J^2_{\mathbf{k},2}|0\rangle_{1,2} = \sum_r (-1 + \alpha^{r\dagger}_{\mathbf{k},1} \beta^{r\dagger}_{-\mathbf{k},2} \alpha^{r\dagger}_{\mathbf{k},2} \beta^{r\dagger}_{-\mathbf{k},1}) |0\rangle_{1,2}.
$$

3.2.3 Condensate structure

Finally, let us express the flavor vacuum by means of the full finite decomposition in Eq.(3.11):

$$|0\rangle_{e,\mu} \;=\; |0\rangle_{1,2} \;+\; \Big[B(m_1,m_2)\;,\;R^{-1}(\theta)\Big]\;|\widetilde{0}\rangle_{1,2}\,, \tag{3.26}$$

where $|\widetilde{0}\rangle_{1,2}$ is defined in Eq.(3.1). We, thus, see how a condensate nature, made of particle-antiparticle pairs with same or different masses [25], arises as a consequence of the non–vanishing commutator $[B, R^{-1}]$. Indeed, a condensate is already present in the Bogoliubov vacuum $|\widetilde{0}\rangle_{1,2}$, for which it is possible to compute a condensation density:

$${}_{1,2}\langle\widetilde{0}|\alpha^{r\dagger}_{\mathbf{k},i}\alpha^{r}_{\mathbf{k},i}|\widetilde{0}\rangle_{1,2} = {}_{1,2}\langle\widetilde{0}|\beta^{r\dagger}_{-\mathbf{k},i}\beta^{r}_{-\mathbf{k},i}|\widetilde{0}\rangle_{1,2} = \sin^2\Theta_{\mathbf{k},i}, \tag{3.27}$$

with $i = 1,2$. The condensation density of the flavor vacuum differs from the one of the Bogoliubov vacuum and is given by

$$\begin{aligned} {}_{e,\mu}\langle 0(t)|\alpha^{r\dagger}_{\mathbf{k},i}\alpha^{r}_{\mathbf{k},i}|0(t)\rangle_{e,\mu} &= {}_{e,\mu}\langle 0(t)|\beta^{r\dagger}_{-\mathbf{k},i}\beta^{r}_{-\mathbf{k},i}|0(t)\rangle_{e,\mu} \\ &= \sin^2\theta\;\sin^2(\Theta_{\mathbf{k},1}-\Theta_{\mathbf{k},2}), \end{aligned} \tag{3.28}$$

with $i = 1,2$. We stress that, such condensation density, vanishes when either $\theta = 0$ and/or $m_1 = m_2$, which are the cases in which there is no mixing. As a result of the non–vanishing commutator in Eq.(3.26) , one finds a gap in the vev of the energy on the two vacua:

$$\begin{aligned} \Delta E_{\mathbf{k}} &\equiv {}_{e,\mu}\langle 0|H_{\mathbf{k}}|0\rangle_{e,\mu} - {}_{1,2}\langle 0|H_{\mathbf{k}}|0\rangle_{1,2} \\ &= 2(\omega_{k,1}+\omega_{k,2})\sin^2\theta\sin^2(\Theta_{\mathbf{k},1}-\Theta_{\mathbf{k},2})\,, \end{aligned} \tag{3.29}$$

where $H_{\mathbf{k}} \equiv H_{\mathbf{k},1} + H_{\mathbf{k},2}$ - cf. Tab(3.14).

3.3 Thermodynamical properties

Having found a condensate structure in the flavor vacuum it make sense to investigate about the possibility of a thermodynamical interpretation for such vacuum. We proceed in analogy with Thermo Field Dynamics (TFD) for fermions, where a thermal vacuum is generated by means of a suitable Bogoliubov transformation: $|0(\vartheta)\rangle \equiv B(\vartheta)|0\rangle$, where $\vartheta \equiv \vartheta(\beta)$. In doing so, a "fictitious" system (the *tilde* system), with the same structure of the physical system, is introduced and is interpreted as a thermal bath. According to [73], such a state can be written as

$$|0(\vartheta)\rangle = \prod_{\mathbf{k},r} \left[\cos\vartheta_{\mathbf{k}} + \sin\vartheta_{\mathbf{k}}\, \alpha_{\mathbf{k}}^{r\dagger} \tilde{\alpha}_{\mathbf{k}}^{r\dagger} \right] |0\rangle_{1,2}, \tag{3.30}$$

or in the form

$$|0(\vartheta)\rangle = \exp\left(-\frac{S_a}{2}\right)|I\rangle = \exp\left(-\frac{S_{\tilde{a}}}{2}\right)|I\rangle\,, \tag{3.31}$$

with

$$|I\rangle \equiv \exp\left(\sum_{r,\mathbf{k}} \tilde{a}_{-\mathbf{k}}^{r\dagger} a_{\mathbf{k}}^{r\dagger}\right)|0\rangle, \tag{3.32}$$

$$S_a = -\sum_{r,\mathbf{k}} \left(a_{\mathbf{k}}^{r\dagger} a_{\mathbf{k}}^{r} \ln\sin^2\vartheta_{\mathbf{k}} + a_{\mathbf{k}}^{r} a_{\mathbf{k}}^{r\dagger} \ln\cos^2\vartheta_{\mathbf{k}} \right)\,, \tag{3.33}$$

and $a_{\mathbf{k}}^{r}$ and $a_{\mathbf{k}}^{r\dagger}$ are fermion operators. In the derivation of the above expressions one makes use of the following relations

$$e^{-\frac{S_a}{2}} a_{\mathbf{k}}^{\dagger} e^{\frac{S_a}{2}} = \tan\vartheta_{\mathbf{k}} a_{\mathbf{k}}^{\dagger}\,, \qquad e^{-\frac{S_a}{2}} \tilde{a}_{\mathbf{k}}^{\dagger} e^{\frac{S_a}{2}} = \tilde{a}_{\mathbf{k}}^{\dagger}\,. \tag{3.34}$$

A similar expression holds for $S_{\tilde{a}}$. S_a (or $S_{\tilde{a}}$) can, thus, be interpreted as the entropy function associated to the vacuum condensate. We also have

$$n_k \equiv \langle 0(\vartheta)| a_{\mathbf{k}}^{r\dagger} a_{\mathbf{k}}^{r} |0(\vartheta)\rangle = \sin^2(\vartheta_{\mathbf{k}})\,. \tag{3.35}$$

The expectation value of the Hamiltonian $H = \sum_k \epsilon_k a^\dagger_{\mathbf{k}} a_{\mathbf{k}}$ is

$$\langle 0(\vartheta)|H|0(\vartheta)\rangle = \sum_k \epsilon_k n_k .$$

We will use $\omega_k = \epsilon_k - \mu$, with μ being the chemical potential. The vev on the thermal vacuum of the entropy is:

$$\langle 0(\vartheta)|S_a|0(\vartheta)\rangle = -2\sum_{\mathbf{k}} \big(n_{\mathbf{k}} \ln(n_{\mathbf{k}}) + (1-n_{\mathbf{k}})\ln(1-n_{\mathbf{k}})\big)\,. \tag{3.36}$$

We also considers the following quantity:

$$\Omega = \langle 0(\vartheta)|H - \frac{1}{\beta}S_a - \mu N|0(\vartheta)\rangle\,, \tag{3.37}$$

which can be identified as a thermodynamical potential [73]. Extremization of Ω with respect to $\vartheta_{\mathbf{k}}$ leads to the Fermi-Dirac distribution.

$$n_k = \frac{1}{e^{\beta\omega_k}+1}. \tag{3.38}$$

We now apply a similar reasoning also for the case of the flavor vacuum generated by $G_t(\theta, m_1, m_2)$ as in Eq.(2.90) and assume that it is possible to rewrite it as:

$$|0\rangle_{e,\mu} = e^{-\frac{S^f_i}{2}}|I_f\rangle\,, \tag{3.39}$$

where $i = 1, 2$, f denotes "flavor", and[2]

$$\begin{aligned} S^f_i \equiv \sum_{\mathbf{k}} S^f_{k,i} \;&=\; -\sum_{\mathbf{k}} \Big\{(\alpha^\dagger_{\mathbf{k},i}\alpha_{\mathbf{k},i} + \beta^\dagger_{-\mathbf{k},i}\beta_{-\mathbf{k},i})\ln\sin^2\Gamma_k \\ &+\; (\alpha_{\mathbf{k},i}\alpha^\dagger_{\mathbf{k},i} + \beta_{-\mathbf{k},i}\beta^\dagger_{-\mathbf{k},i})\ln\cos^2\Gamma_k\Big\}\,, \end{aligned} \tag{3.40}$$

with $\sin\Gamma_k \equiv |V_{\mathbf{k}}|\sin\theta$. We have the following relations (to be compared

[2]Here we have chosen to separate the physical and tilde systems (in analogy with TFD) according to the mass index.

with Eq.(3.34))

$$e^{-\frac{S_i^f}{2}}\alpha^\dagger_{\mathbf{k},j}e^{\frac{S_i^f}{2}} = e^{\delta_{ij}\ln\tan\Gamma_k}\ \alpha^\dagger_{\mathbf{k},j}\ , \tag{3.41}$$

$$e^{-\frac{S_i^f}{2}}\beta^\dagger_{-\mathbf{k},j}e^{\frac{S_i^f}{2}} = e^{\delta_{ij}\ln\tan\Gamma_k}\ \beta^\dagger_{-\mathbf{k},j}. \tag{3.42}$$

In order to check whether or not the ansatz in Eq.(3.39) is consistent, we evaluate it at the first order approximation in θ for small $\Theta_{\mathbf{k},2}-\Theta_{\mathbf{k},1}$:

$$S_i^f \simeq -\sum_{\mathbf{k}}\Big\{(\alpha^\dagger_{\mathbf{k},i}\alpha_{\mathbf{k},i}+\beta^\dagger_{-\mathbf{k},i}\beta_{-\mathbf{k},i})\ln\theta(\Theta_{\mathbf{k},2}-\Theta_{\mathbf{k},1})\Big\}, \tag{3.43}$$

$$e^{-\frac{S_i^f}{2}}\alpha^\dagger_{\mathbf{k},j}e^{\frac{S_i^f}{2}} \simeq e^{\delta_{ij}\ln\theta(\Theta_{\mathbf{k},2}-\Theta_{\mathbf{k},1})}\alpha^\dagger_{\mathbf{k},j}\ , \tag{3.44}$$

$$e^{-\frac{S_i^f}{2}}\beta^\dagger_{-\mathbf{k},j}e^{\frac{S_i^f}{2}} \simeq e^{\delta_{ij}\ln\theta(\Theta_{\mathbf{k},2}-\Theta_{\mathbf{k},1})}\ \beta^\dagger_{-\mathbf{k},j}. \tag{3.45}$$

We have

$$|I_f\rangle \simeq \prod_{\mathbf{k},r}\exp\Big\{\epsilon^r\big(\alpha^\dagger_{\mathbf{k},1}\beta^\dagger_{-\mathbf{k},2}+\alpha^\dagger_{\mathbf{k},2}\beta^\dagger_{-\mathbf{k},1}\big)\Big\}|0\rangle_{1,2}\ , \tag{3.46}$$

thus the identity in Eq.(3.39) is satisfied in this approximation - cf Eq.(3.23). This is indeed sufficient for the following considerations. Further discussion on the thermodynamical structure of $|0\rangle_{e,\mu}$ will be presented elsewhere.

We recall [25] that it is possible to rewrite $|U_{\mathbf{k}}|^2$ in terms of two adimensional parameters

$$|U_{\mathbf{k}}|^2 = \frac{1}{2}\Big(1+\frac{1}{\sqrt{1+a(p/(p^2+1))^2}}\Big)\ , \tag{3.47}$$

with

$$p \equiv \frac{|\mathbf{k}|}{\sqrt{m_1 m_2}}\ ,\quad a \equiv \frac{(m_2-m_1)^2}{m_1 m_2}\ . \tag{3.48}$$

In Fig.3.2, Γ_k is plotted as a function of a for various values of p. Finally, we define the difference $\Delta S^f_{k,i}$ between the vev of the entropy operator Eq.(3.40)

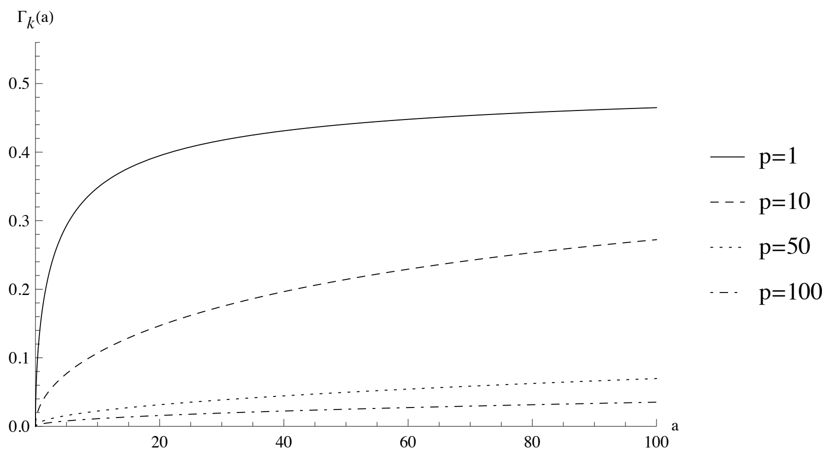


Figure 3.6: Plot of $\Gamma_k(a)$. Sample values of $\theta = \frac{\pi}{4}$.

computed on the two different vacua

$$\begin{aligned} \Delta S^f_{k,i} &= {}_{e,\mu}\langle 0|S^f_{k,i}|0\rangle_{e,\mu} - {}_{1,2}\langle 0|S^f_{k,i}|0\rangle_{1,2} \\ &= -2\sin^2\Gamma_k \, \ln\tan^2\Gamma_k\,. \end{aligned} \tag{3.49}$$

We can now consider the ratio $\frac{\Delta S^f_{k,i}}{\Delta E_{k,i}}$, where the latter is the gap energy defined in the previous Section Eq.(3.29) for the field i, obtaining

$$\beta_{k,i} = \frac{\Delta S^f_{k,i}}{\Delta E_{k,i}} = -\frac{\ln\tan^2\Gamma_k}{\omega_{k,i}}\,, \tag{3.50}$$

which, however, depends on the momentum. In fact, unlike the standard TFD case, in which the parameter $\vartheta_{\mathbf{k}}$ is determined only by the relation in Eq.(3.38), in the present case the Bogoliubov angle is already set with the condition $\Theta_{\mathbf{k},i} = \frac{1}{2}\cot^{-1}(\frac{|\mathbf{k}|}{m_i})$.This results in an impossibility to introduce a well defined temperature or equivalently in a deviation from the Fermi distribution, due to the non–diagonal pairs in the condensate structure of the flavor vacuum. On the other hand, starting from a different viewpoint, one can investigate the relation between the flavor vacuum and a thermal vacuum state of the form $|0(\beta_1, \beta_2)\rangle \equiv |0(\beta_1)\rangle \otimes |0(\beta_2)\rangle$ with

$$|0(\beta_i)\rangle \equiv \prod_{\mathbf{k},r} \Big[\cos\gamma_{k,i}(\beta_i) + \sin\gamma_{k,i}(\beta_i)\, \alpha^{r\dagger}_{\mathbf{k},i}\beta^{r\dagger}_{-\mathbf{k},i}\Big]|0\rangle_i\,, \tag{3.51}$$

where $i = 1, 2$, $\gamma_{k,i}(\beta_i)$ are the parameters of the Bogoliubov transformations depending on the temperature. We consider the total number operator on the flavor vacuum

$$N_f(k) \equiv {}_{e,\mu}\langle 0|N_{\mathbf{k},1} + N_{\mathbf{k},2}|0\rangle_{e,\mu} = 2\,\sin^2\theta\,|V_{\mathbf{k}}|^2$$

while the vev on the thermal vacuum gives[3]

$$N_F(k) \equiv \langle 0(\beta_1, \beta_2)|N_{\mathbf{k},1} + N_{\mathbf{k},2}|0(\beta_1, \beta_2)\rangle = \frac{1}{e^{\beta_1\omega_{k,1}} + 1} + \frac{1}{e^{\beta_2\omega_{k,2}} + 1}.$$

One may wonder to what extent, $N_F(k)$ can fit $N_f(k)$ for given values of the parameters m_1, m_2 and θ, by adjusting the free parameters β_1 and β_2. From Fig.3.7 we see that this is somehow possible only for the right tail of the distribution $N_f(k)$; on the other hand, for low momenta, the behavior of the two distributions is quite different. This fact boils down to a structural difference between the two states $|0\rangle_{e,\mu}$ and $|0(\beta_1, \beta_2)\rangle$. These states differ because in the condensate structure of the "thermal" state $|0(\beta_1, \beta_2)\rangle$ are missing terms of the form $(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2} + \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1})|0\rangle_{1,2}$ - cf. Eq.(3.25) - due to the non–diagonal Bogoliubov transformation discussed previously.

3.4 Extension to the case of three flavors.

We start by considering the following Lagrangian density describing three Dirac fields with a mixed mass term:

$$\mathcal{L}(x) \,=\, \bar{\Psi}_f(x)\,(i\,\partial\!\!\!/ - \mathsf{M})\,\Psi_f(x)\,, \tag{3.52}$$

where $\Psi_f^T = (\nu_e, \nu_\mu, \nu_\tau)$ and $\mathsf{M} = \mathsf{M}^\dagger$ is the mixed mass matrix.

[3]F stands for Fermi

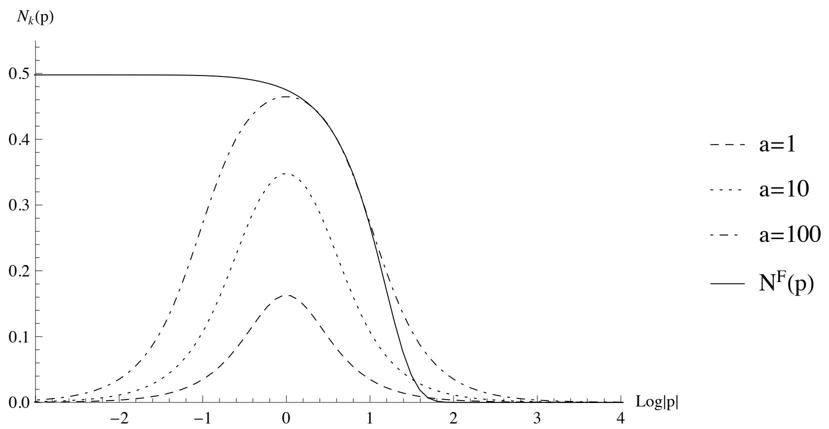


Figure 3.7: Plot of $N_f(p)$ and $N_F(p)$ against $Log|p|$ for different values of a . Sample values of parameters for $N_f(p)$ are $\theta = \pi/4$ and $m_1 = 20$. Sample values of parameters for $N_F(p)$ are $a = 100$, $T_1 = 10^4$ and $T_2 = 7.8\,10^4$.

Among the various possible parameterizations of the mixing matrix for three fields, we choose to work with the following one since it is the familiar parameterization of the CKM matrix [68, 74]:

$$
\begin{aligned}
\Psi_f &= \mathcal{U}\Psi_m \qquad (3.53)\\
&= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}\Psi_m,
\end{aligned}
$$

with[4] $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$, being θ_{ij} the mixing angle between ν_i, ν_j and $\Psi_m^T = (\nu_1, \nu_2, \nu_3)$.

Using Eq.(3.53), we diagonalize the quadratic form of Eq.(3.52), which then reduces to the Lagrangian for three Dirac fields, with masses m_1, m_2 and m_3:

$$
\mathcal{L}(x) \;=\; \bar{\Psi}_m(x)\left(i\;\partial\!\!\!/ - \mathsf{M}_d\right)\Psi_m(x)\,, \qquad (3.54)
$$

where $\mathsf{M}_d = diag(m_1, m_2, m_3)$.

Following Ref.[25, 26], it is possible to construct the generator for the

[4] $\Psi = \Psi(x)$

mixing transformation (3.53) and define[5]

$$\nu_\sigma^\alpha(x) \equiv G_\theta^{-1}(t)\, \nu_i^\alpha(x)\, G_\theta(t), \tag{3.55}$$

where $(\sigma, i) = (e,1), (\mu,2), (\tau,3)$, and

$$G_\theta(t) = G_{23}(t) G_{13}(t) G_{12}(t)\,, \tag{3.56}$$

where

$$G_{12}(t) \equiv \exp\Big[\theta_{12} L_{12}(t)\Big]; \tag{3.57}$$

$$L_{12}(t) = \int d^3x \left[\nu_1^\dagger(x)\nu_2(x) - \nu_2^\dagger(x)\nu_1(x)\right],$$

$$G_{23}(t) \equiv \exp\Big[\theta_{23} L_{23}(t)\Big]; \tag{3.58}$$

$$L_{23}(t) = \int d^3x \left[\nu_2^\dagger(x)\nu_3(x) - \nu_3^\dagger(x)\nu_2(x)\right],$$

$$G_{13}(t) \equiv \exp\Big[\theta_{13} L_{13}(\delta, t)\Big]; \tag{3.59}$$

$$L_{13}(\delta, t) = \int d^3x \left[\nu_1^\dagger(x)\nu_3(x) e^{-i\delta} - \nu_3^\dagger(x)\nu_1(x) e^{i\delta}\right],$$

It is evident from the above form of the generators, that the phase δ is unavoidable for three field mixing, while it can be incorporated in the definition of the fields in the two flavor case. The free fields ν_i (i=1,2,3) can be

[5] Let us consider for example the generation of the first row of the mixing matrix $\mathcal{U}$. One has $\partial\nu_e/\partial\theta_{23} = 0$; and $\partial\nu_e/\partial\theta_{13} = G_{12}^{-1}G_{13}^{-1}[\nu_1, L_{13}]G_{13}G_{12} = G_{12}^{-1}G_{13}^{-1}e^{-i\delta}\nu_3 G_{13}G_{12}$, thus:

$$\partial^2\nu_e/\partial\theta_{13}^2 \;=\; -\nu_e \quad \Rightarrow \quad \nu_e \;=\; f(\theta_{12})\cos\theta_{13} + g(\theta_{12})\sin\theta_{13};$$

with the initial conditions (from Eq.(3.55)): $f(\theta_{12}) = \nu_e|_{\theta_{13}=0}$ and $g(\theta_{12}) = \partial\nu_e/\partial\theta_{13}|_{\theta_{13}=0} = e^{-i\delta}\nu_3$. We also have

$$\partial^2 f(\theta_{12})/\partial\theta_{13}^2 \;=\; -f(\theta_{12}) \quad \Rightarrow \quad f(\theta_{12}) \;=\; A\cos\theta_{12} + B\sin\theta_{12}$$

with the initial conditions $A = \nu_e|_{\theta=0} = \nu_1$ and $B = \partial f(\theta_{12})/\partial\theta_{12}|_{\theta=0} = \nu_2$, and $\theta = (\theta_{12}, \theta_{13}, \theta_{23})$.

quantized in the usual way [75] (we use $t \equiv x_0$):

$$\nu_i(x) = \sum_r \int d^3k \left[u^r_{\mathbf{k},i}(t) \alpha^r_{\mathbf{k},i} + v^r_{-\mathbf{k},i}(t) \beta^{r\dagger}_{-\mathbf{k},i} \right] e^{i\mathbf{k}\cdot\mathbf{x}}, \qquad i = 1,2,3\,, \tag{3.60}$$

with $u^r_{\mathbf{k},i}(t) = e^{-i\omega_{k,i}t} u^r_{\mathbf{k},i}$, $v^r_{\mathbf{k},i}(t) = e^{i\omega_{k,i}t} v^r_{\mathbf{k},i}$ and $\omega_{k,i} = \sqrt{\mathbf{k}^2 + m_i^2}$. The vacuum for the mass eigenstates is denoted by $|0\rangle_m$: $\alpha^r_{\mathbf{k},i}|0\rangle_m = \beta^r_{\mathbf{k},i}|0\rangle_m = 0$. The anticommutation relations are the usual ones; the wave function orthonormality and completeness relations are those of Ref.[25]. By use of $G_\theta(t)$, the flavor fields can be expanded as:

$$\nu_\sigma(x) = \sum_r \int d^3k \left[u^r_{\mathbf{k},i}(t) \alpha^r_{\mathbf{k},\sigma}(t) + v^r_{-\mathbf{k},i}(t) \beta^{r\dagger}_{-\mathbf{k},\sigma}(t) \right] e^{i\mathbf{k}\cdot\mathbf{x}} \tag{3.61}$$

with $(\sigma, i) = (e,1), (\mu,2), (\tau,3)$. The flavor annihilation operators are defined as $\alpha^r_{\mathbf{k},\sigma}(t) \equiv G^{-1}_\theta(t) \alpha^r_{\mathbf{k},i} G_\theta(t)$ and $\beta^{r\dagger}_{-\mathbf{k},\sigma}(t) \equiv G^{-1}_\theta(t) \beta^{r\dagger}_{-\mathbf{k},i} G_\theta(t)$. For further reference, it is useful to list explicitly the flavor annihilation/creation operators (see also Ref.[25]). In the reference frame $\mathbf{k} = (0,0,|\mathbf{k}|)$ the spins decouple and their form is particularly simple:

$$\begin{aligned}
\alpha^r_{\mathbf{k},e}(t) &= c_{12}c_{13}\; \alpha^r_{\mathbf{k},1} + s_{12}c_{13} \left(U^{\mathbf{k}*}_{12}(t)\; \alpha^r_{\mathbf{k},2} + \epsilon^r V^{\mathbf{k}}_{12}(t)\; \beta^{r\dagger}_{-\mathbf{k},2} \right) \\
&+ e^{-i\delta}\; s_{13} \left(U^{\mathbf{k}*}_{13}(t)\; \alpha^r_{\mathbf{k},3} + \epsilon^r V^{\mathbf{k}}_{13}(t)\; \beta^{r\dagger}_{-\mathbf{k},3} \right)\,, \\
\alpha^r_{\mathbf{k},\mu}(t) &= \left(c_{12}c_{23} - e^{i\delta}\; s_{12}s_{23}s_{13} \right)\; \alpha^r_{\mathbf{k},2} \\
&- \left(s_{12}c_{23} + e^{i\delta}\; c_{12}s_{23}s_{13} \right) \left(U^{\mathbf{k}}_{12}(t)\; \alpha^r_{\mathbf{k},1} - \epsilon^r V^{\mathbf{k}}_{12}(t)\; \beta^{r\dagger}_{-\mathbf{k},1} \right) \\
&+ s_{23}c_{13} \left(U^{\mathbf{k}*}_{23}(t)\; \alpha^r_{\mathbf{k},3} + \epsilon^r V^{\mathbf{k}}_{23}(t)\; \beta^{r\dagger}_{-\mathbf{k},3} \right)\,, \\
\alpha^r_{\mathbf{k},\tau}(t) &= c_{23}c_{13}\; \alpha^r_{\mathbf{k},3} \\
&- \left(c_{12}s_{23} + e^{i\delta}\; s_{12}c_{23}s_{13} \right) \left(U^{\mathbf{k}}_{23}(t)\; \alpha^r_{\mathbf{k},2} - \epsilon^r V^{\mathbf{k}}_{23}(t)\; \beta^{r\dagger}_{-\mathbf{k},2} \right) \\
&+ \left(s_{12}s_{23} - e^{i\delta}\; c_{12}c_{23}s_{13} \right) \left(U^{\mathbf{k}}_{13}(t)\; \alpha^r_{\mathbf{k},1} - \epsilon^r V^{\mathbf{k}}_{13}(t)\; \beta^{r\dagger}_{-\mathbf{k},1} \right)\,,
\end{aligned} \tag{3.62}$$

$$
\begin{aligned}
\beta^r_{-\mathbf{k},e}(t) &= c_{12}c_{13}\,\beta^r_{-\mathbf{k},1} + s_{12}c_{13}\left(U^{\mathbf{k}*}_{12}(t)\,\beta^r_{-\mathbf{k},2} - \epsilon^r V^{\mathbf{k}}_{12}(t)\,\alpha^{r\dagger}_{\mathbf{k},2}\right) \\
&+ e^{i\delta}\,s_{13}\left(U^{\mathbf{k}*}_{13}(t)\,\beta^r_{-\mathbf{k},3} - \epsilon^r V^{\mathbf{k}}_{13}(t)\,\alpha^{r\dagger}_{\mathbf{k},3}\right)\,, \\
\beta^r_{-\mathbf{k},\mu}(t) &= \left(c_{12}c_{23} - e^{-i\delta}\,s_{12}s_{23}s_{13}\right)\,\beta^r_{-\mathbf{k},2} \\
&- \left(s_{12}c_{23} + e^{-i\delta}\,c_{12}s_{23}s_{13}\right)\left(U^{\mathbf{k}}_{12}(t)\,\beta^r_{-\mathbf{k},1} + \epsilon^r\,V^{\mathbf{k}}_{12}(t)\,\alpha^{r\dagger}_{\mathbf{k},1}\right) \\
&+ s_{23}c_{13}\left(U^{\mathbf{k}*}_{23}(t)\,\beta^r_{-\mathbf{k},3} - \epsilon^r\,V^{\mathbf{k}}_{23}(t)\,\alpha^{r\dagger}_{\mathbf{k},3}\right)\,, \\
\beta^r_{-\mathbf{k},\tau}(t) &= c_{23}c_{13}\,\beta^r_{-\mathbf{k},3} \\
&- \left(c_{12}s_{23} + e^{-i\delta}\,s_{12}c_{23}s_{13}\right)\left(U^{\mathbf{k}}_{23}(t)\,\beta^r_{-\mathbf{k},2} + \epsilon^r V^{\mathbf{k}}_{23}(t)\,\alpha^{r\dagger}_{\mathbf{k},2}\right) \\
&+ \left(s_{12}s_{23} - e^{-i\delta}\,c_{12}c_{23}s_{13}\right)\left(U^{\mathbf{k}}_{13}(t)\,\beta^r_{-\mathbf{k},1} + \epsilon^r V^{\mathbf{k}}_{13}(t)\,\alpha^{r\dagger}_{\mathbf{k},1}\right)\,.
\end{aligned}
\tag{3.63}
$$

These operators satisfy canonical (anti)commutation relations at equal times. The main difference with respect to their "naive" quantum-mechanical counterparts is in the anomalous terms proportional to the V_{ij} factors. In fact, $U^{\mathbf{k}}_{ij}$ and $V^{\mathbf{k}}_{ij}$ are Bogoliubov coefficients defined as:

$$
V^{\mathbf{k}}_{ij}(t) = |V^{\mathbf{k}}_{ij}|\; e^{i(\omega_{k,j}+\omega_{k,i})t} \quad , \quad U^{\mathbf{k}}_{ij}(t) = |U^{\mathbf{k}}_{ij}|\; e^{i(\omega_{k,j}-\omega_{k,i})t} \tag{3.64}
$$

$$
|U^{\mathbf{k}}_{ij}| = \left(\frac{\omega_{k,i}+m_i}{2\omega_{k,i}}\right)^{\frac{1}{2}}\left(\frac{\omega_{k,j}+m_j}{2\omega_{k,j}}\right)^{\frac{1}{2}}\left(1+\frac{|\mathbf{k}|^2}{(\omega_{k,i}+m_i)(\omega_{k,j}+m_j)}\right) = \cos(\xi^{\mathbf{k}}_{ij}) \tag{3.65}
$$

$$
|V^{\mathbf{k}}_{ij}| = \left(\frac{\omega_{k,i}+m_i}{2\omega_{k,i}}\right)^{\frac{1}{2}}\left(\frac{\omega_{k,j}+m_j}{2\omega_{k,j}}\right)^{\frac{1}{2}}\left(\frac{|\mathbf{k}|}{(\omega_{k,j}+m_j)} - \frac{|\mathbf{k}|}{(\omega_{k,i}+m_i)}\right) = \sin(\xi^{\mathbf{k}}_{ij}) \tag{3.66}
$$

$$
|U^{\mathbf{k}}_{ij}|^2 + |V^{\mathbf{k}}_{ij}|^2 = 1 \tag{3.67}
$$

where $i, j = 1, 2, 3$ and $j > i$. The following identities hold:

$$V_{23}^{\mathbf{k}}(t)V_{13}^{\mathbf{k}*}(t) + U_{23}^{\mathbf{k}*}(t)U_{13}^{\mathbf{k}}(t) = U_{12}^{\mathbf{k}}(t), \tag{3.68}$$

$$V_{23}^{\mathbf{k}}(t)U_{13}^{\mathbf{k}*}(t) - U_{23}^{\mathbf{k}*}(t)V_{13}^{\mathbf{k}}(t) = -V_{12}^{\mathbf{k}}(t)$$

$$U_{12}^{\mathbf{k}}(t)U_{23}^{\mathbf{k}}(t) - V_{12}^{\mathbf{k}*}(t)V_{23}^{\mathbf{k}}(t) = U_{13}^{\mathbf{k}}(t), \tag{3.69}$$

$$U_{23}^{\mathbf{k}}(t)V_{12}^{\mathbf{k}}(t) + U_{12}^{\mathbf{k}*}(t)V_{23}^{\mathbf{k}}(t) = V_{13}^{\mathbf{k}}(t)$$

$$V_{12}^{\mathbf{k}*}(t)V_{13}^{\mathbf{k}}(t) + U_{12}^{\mathbf{k}*}(t)U_{13}^{\mathbf{k}}(t) = U_{23}^{\mathbf{k}}(t), \tag{3.70}$$

$$V_{12}^{\mathbf{k}}(t)U_{13}^{\mathbf{k}}(t) - U_{12}^{\mathbf{k}}(t)V_{13}^{\mathbf{k}}(t) = -V_{23}^{\mathbf{k}}(t) \ ,$$

$$\xi_{13}^{\mathbf{k}} = \xi_{12}^{\mathbf{k}} + \xi_{23}^{\mathbf{k}}, \tag{3.71}$$

$$\xi_{ij}^{\mathbf{k}} = \arctan\left(|V_{ij}^{\mathbf{k}}| \,/\, |U_{ij}^{\mathbf{k}}|\right) \ .$$

As already observed in Ref.[25] we remark that, in contrast with the case of two flavor mixing, the condensation densities are now different for particles of different masses:

$$\begin{aligned}
\mathcal{N}_1^{\mathbf{k}} &= {}_f\langle 0(t)|N_{\alpha_1}^{\mathbf{k},r}|0(t)\rangle_f = {}_f\langle 0(t)|N_{\beta_1}^{\mathbf{k},r}|0(t)\rangle_f \qquad (3.72)\\
&= s_{12}^2 c_{13}^2\, |V_{12}^{\mathbf{k}}|^2 + s_{13}^2\, |V_{13}^{\mathbf{k}}|^2 ,\\
\mathcal{N}_2^{\mathbf{k}} &= {}_f\langle 0(t)|N_{\alpha_2}^{\mathbf{k},r}|0(t)\rangle_f = {}_f\langle 0(t)|N_{\beta_2}^{\mathbf{k},r}|0(t)\rangle_f \qquad (3.73)\\
&= \left|-s_{12}c_{23} + e^{i\delta}\, c_{12}s_{23}s_{13}\right|^2\, |V_{12}^{\mathbf{k}}|^2 + s_{23}^2 c_{13}^2\, |V_{23}^{\mathbf{k}}|^2 ,\\
\mathcal{N}_3^{\mathbf{k}} &= {}_f\langle 0(t)|N_{\alpha_3}^{\mathbf{k},r}|0(t)\rangle_f = {}_f\langle 0(t)|N_{\beta_3}^{\mathbf{k},r}|0(t)\rangle_f \qquad (3.74)\\
&= \left|-c_{12}s_{23} + e^{i\delta}\, s_{12}c_{23}s_{13}\right|^2 |V_{23}^{\mathbf{k}}|^2 + \left|s_{12}s_{23} + e^{i\delta}\, c_{12}c_{23}s_{13}\right|^2 |V_{13}^{\mathbf{k}}|^2 .
\end{aligned}$$

On the other hand, the generator of the mixing matrix $\mathcal{U}$ of Eq.(3.53) is only one of the various forms in which a 3×3 unitary matrix can be parameterized. Indeed, the generator Eq.(3.56) can be used for generating such alternative parameterizations. To see this, let us first define in a more general way the generators, G_{ij} including phases for all of them:

$$G_{12}(t) \equiv \exp\Big[\theta_{12} L_{12}(\delta_{12}, t)\Big]; \tag{3.75}$$

$$L_{12}(\delta_{12}, t) = \int d^3x \left[\nu_1^\dagger(x)\nu_2(x) e^{-i\delta_{12}} - \nu_2^\dagger(x)\nu_1(x) e^{i\delta_{12}}\right],$$

$$G_{23}(t) \equiv \exp\Big[\theta_{23} L_{23}(\delta_{23}, t)\Big]; \tag{3.76}$$

$$L_{23}(\delta_{23}, t) = \int d^3x \left[\nu_2^\dagger(x)\nu_3(x) e^{-i\delta_{23}} - \nu_3^\dagger(x)\nu_2(x) e^{i\delta_{23}}\right],$$

$$G_{13}(t) \equiv \exp\Big[\theta_{13} L_{13}(\delta_{13}, t)\Big]; \tag{3.77}$$

$$L_{13}(\delta_{13}, t) = \int d^3x \left[\nu_1^\dagger(x)\nu_3(x) e^{-i\delta_{13}} - \nu_3^\dagger(x)\nu_1(x) e^{i\delta_{13}}\right].$$

Using such form we can extend the results we have obtained so far to the case of three flavor mixing case, thanks to their algebraic nature. We see that, following the reasoning of the previous sections, it is possible to rewrite $G_{ij}(t)$ as:

$$G_{ij}(t) \equiv G_{ij}(t, \theta, \Theta_i, \Theta_j, \delta) = B^{-1}(\Theta_i, \Theta_j)\, R(\theta, \delta_{ij})\, B(\Theta_i, \Theta_j), \tag{3.78}$$

where $B(\Theta_i, \Theta_j)$ is the standard Bogoliubov transformation we have used till now and

$$R(\theta, \delta_{ij}) = \exp\Big\{\theta \sum_{\mathbf{k},r} \Big[\Big(\alpha^{r\dagger}_{\mathbf{k},1}\alpha^r_{\mathbf{k},2} e^{-i\delta_{ij}} + \beta^{r\dagger}_{-\mathbf{k},1}\beta^r_{-\mathbf{k},2} e^{i\delta_{ij}}\Big) e^{i\psi_k} - \Big(\alpha^{r\dagger}_{\mathbf{k},2}\alpha^r_{\mathbf{k},1} e^{i\delta_{ij}} + \beta^{r\dagger}_{-\mathbf{k},2}\beta^r_{-\mathbf{k},1} e^{-i\delta_{ij}}\Big) e^{-i\psi_k}\Big]\Big\}.$$

In order to better analyze the role and nature of the phase $e^{i\delta_{ij}}$ we also expand the generator G for small mixing angles θ_{ij} and for $(\Theta_{\mathbf{k},j} - \Theta_{\mathbf{k},i})$ small, as in the previous sections, up to the first order in θ_{ij}. In order to do that we define the following operators[6]:

$$J^{i,j\,r}_{\mathbf{k},1} \equiv \frac{1}{2}\Big[(\alpha^r_{\mathbf{k},i}\beta^r_{-\mathbf{k},i} - \beta^{r\dagger}_{-\mathbf{k},i}\alpha^{r\dagger}_{\mathbf{k},i}) - (\alpha^r_{\mathbf{k},j}\beta^r_{-\mathbf{k},j} - \beta^{r\dagger}_{-\mathbf{k},j}\alpha^{r\dagger}_{\mathbf{k},j})\Big], \tag{3.79}$$

$$J^{i,j\,r}_{\mathbf{k},2} \equiv -\frac{1}{2}\Big[(\alpha^r_{\mathbf{k},i}\beta^r_{-\mathbf{k},j}e^{-i\delta_{ij}} - \beta^{r\dagger}_{-\mathbf{k},j}\alpha^{r\dagger}_{\mathbf{k},i}e^{-i\delta_{ij}}) + (\alpha^r_{\mathbf{k},j}\beta^r_{-\mathbf{k},i}e^{i\delta_{ij}} - \beta^{r\dagger}_{-\mathbf{k},i}\alpha^{r\dagger}_{\mathbf{k},j}e^{i\delta_{ij}})\Big], \tag{3.80}$$

$$J^{i,j\,r}_{\mathbf{k},3} \equiv \frac{1}{2}\Big[(\alpha^{r\dagger}_{\mathbf{k},i}\alpha^r_{\mathbf{k},j}e^{-i\delta_{ij}} + \beta^{r\dagger}_{-\mathbf{k},i}\beta^r_{-\mathbf{k},j}e^{i\delta_{ij}}) - (\alpha^{r\dagger}_{\mathbf{k},j}\alpha^r_{\mathbf{k},i}e^{i\delta_{ij}} + \beta^{r\dagger}_{-\mathbf{k},j}\beta^r_{-\mathbf{k},i}e^{-i\delta_{ij}})\Big], \tag{3.81}$$

which do not close the $su(2)$ algebra as in the two flavor case, because of the phase $e^{i\delta_{ij}}$. The mixing generator in the above limit, thus reads:

$$\begin{aligned} G(\theta) \cong 1\!\!\mathrm{I} \;&+\; \sum_{i,j=1,2,3}^{i\neq j} \theta_{ij} \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}}\, 2 \sum_r J^{i,j\,r}_{\mathbf{k},3} \\ &-\; \sum_{i,j=1,2,3}^{i\neq j} \theta_{ij}\, a_{ij} \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}}\, 2\, \tilde{V}^{ij}_{\mathbf{k}} \sum_r \epsilon^r J^{i,j\,r}_{\mathbf{k},2} \\ &+\; o[\theta_{ij}\theta_{jk}]. \end{aligned} \tag{3.82}$$

where $a_{ij} \equiv \frac{(m_j-m_i)^2}{m_i m_j}$. The same structure we have seen in the case of two flavor neutrinos arises.

[6] $J^{i,j\,r}_{\mathbf{k},1} \equiv \frac{1}{2}(K^{i\,r}_{\mathbf{k}} - K^{j\,r}_{\mathbf{k}})$ with $K^{i\,r}_{\mathbf{k}} \equiv \alpha^r_{\mathbf{k},i}\beta^r_{-\mathbf{k},i} - \beta^{r\dagger}_{-\mathbf{k},i}\alpha^{r\dagger}_{\mathbf{k},i}$ and $\ln B_i(\Theta_{\mathbf{k},i}) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\Theta_{\mathbf{k},i}K_{\mathbf{k},i}$; $\ln R(\theta_{ij}) = \theta_{ij}\int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\, 2\, J^{ij\,r}_{\mathbf{k},3}$.

Chapter 4

Non–commutative geometry

In the previous Chapter a non–commutative structure has arisen. In that the non–trivial nature of the mixing generator arises as a consequence of the non–commutativity of the rotation generator with the generator of Bogoliubov transformation(s), as if the action of the Bogoliubov transformation(s) results in an "anisotropy" in the ladder operator space. This leads us to study neutrino mixing in a non–commutative framework. For this reason we shall introduce non–commutativity and non–commutative spectral geometry.

4.1 A little bit of history

Already in 1930's Heisenberg proposed to supplant the space-time continuum by a lattice structure, so that one could control the divergences which had troubled quantum electrodynamics from the very beginning, even though a lattice structure breaks Lorentz invariance and thus can barely be considered as fundamental. In 1947 Snyder had the idea of using a non–commutative structure at small length scales, so that the result would have been to introduce an effective cut-off in field theory similar to a lattice, and maintaining Lorentz invariance at the same time [76]. Almost at the same time the renormalization program eventually became a successful prescription for predicting numbers from the theory of quantum electrodynamics and Snyder idea was, to a great extent, unheeded. It was von Neumann who, later on, used for the first time the term *non–commutative geometry* to denote in general a geometry in which the algebra of functions is a non–commutative algebra. Exactly like in the quantization of classical phase-space, coordinates are substituted by generators of the algebra [41] but, since these do not commute they cannot be simultaneously diagonalized and the space disappears. An intuitive idea to *think* a point in this case could be a Planck cell of dimension given by the Planck area [42] analogously to how Bohr cells replace classical phase-space points. In fact, the ultraviolet divergences of quantum field theory could be eliminated in case one can find a coherent description for the structure of space-time which were pointless on small length scales. Indeed, the elimination of these divergences is equivalent to coarsegraining the structure of space-time over small length scales: if one sets an ultraviolet cut-off, the theory does not see length scales smaller than Λ^{-1}. When calculating a Feynman diagram one has to put a cut-off on the momentum variables in the integrands, neglecting any interest in regions of space-time of volume less than Λ^{-4}. The more Λ grows, the smaller the forbidden region becomes but it can never be made to vanish: there is a fundamental length scale, much larger than the Planck length, below which the notion of a point is of no practical meaning. A very simple and elegant way of introducing such a scale in a Lorentz-invariant way is through the introduction of non–commuting space-time *coordinates*.

One usually replaces the four Minkowski coordinates x_μ by four generators q_μ of a non–commutative algebra which satisfy commutation relations of the form

$$[q^\mu, q^\nu] = i\tilde{k}q^{\mu\nu} \tag{4.1}$$

The parameter $\tilde{k}$ is a fundamental area scale of the order of the Planck area: $\tilde{k} \approx \mu_P^{-2} = G\hbar$. Eq.(4.1) contains information about the algebra: first, if the right-hand side does not vanish it states that at least some of the q^μ do not commute; moreover, it is possible to identify the original coordinates with the generators q^μ in the limit $\tilde{k} \to 0$:

$$\lim_{\tilde{k}\to 0} q^\mu = x^\mu .$$

It is reasonable to think that once one has made space-time 'non–commutative' he has to do the same with the Poincarè group. This reasoning leads naturally to the notion of a q-deformed Poincarè (or Lorentz) group which act on a very particular non–commutative version of Minkowski space called q-Minkowski space [77]. The idea of a q–deformation goes back to Sylvester [78]. It was taken up later by Weyl [79] and Schwinger [80] to produce a finite version of quantum mechanics. In the second part of the 80s mathematicians, particularly Connes [81] and Woronowicz [82], succeeded in generalizing the notion of differential structure to non–commutative geometry. This lead to a revived interest in Snyder's idea: just as it is possible to give many differential structures to a given topological space it is possible to define many differential calculi over a given algebra.

4.2 Non–commutative examples

4.2.1 Landau levels

Let us consider the mote of an electron in the (x, y) plane immersed in a magnetic filed $\mathbf{B} = B\hat{z}$ orthogonal to the plane, with m_e the electron mass. The Lagrangian of the electron, considering the vector potential $\mathbf{A} : \nabla \wedge \mathbf{A} = \mathbf{B}$, is:

$$L = \frac{1}{2} m_e v^2 + e\mathbf{A}\mathbf{v} \,. \tag{4.2}$$

Therefore, the equation of motions are:

$$m_e \frac{d^2}{dt}\mathbf{r} = e\mathbf{v} \wedge \mathbf{B}\,, \tag{4.3}$$

and $\mathbf{p} = m_e\mathbf{v} + e\mathbf{A}$ is the canonical momentum, which leads to the Hamiltonian being:

$$H = \frac{1}{2m_e}(\mathbf{p} - e\mathbf{A})^2\,. \tag{4.4}$$

Of course, the electron describes a circular motion around the center of coordinate $\mathbf{R} = (X, Y, 0)$ and radius r_0. The cyclotron frequency is $\omega_c = \frac{|e|B}{m_e}$ and the solution of the Eq.(4.3)is:

$$\mathbf{r} = \mathbf{R} + r_0(\cos(\omega_c t + \delta), \sin(\omega_c t + \delta), 0) \tag{4.5}$$

with velocity

$$\mathbf{v} = \omega_c r_0(-\sin(\omega_c t + \delta), \cos(\omega_c t + \delta), 0), \tag{4.6}$$

while $\mathbf{R}$ and δ are defined by the initial conditions of the problem. It is possible to introduce new variables $(\xi, \eta, 0)$ which describe the position of the electron with respect to the center $\mathbf{R}$. In such a notation Eq.(4.5), (4.6) become:

$$\mathbf{r} = (X + \xi, Y + \eta, 0),$$

$$\mathbf{v} = \omega_c(-\eta, \xi, 0).$$

Choosing now the symmetric gauge $\mathbf{A} = (-B\frac{y}{2}, B\frac{x}{2}, 0)$ as vector potential, the canonical momentum becomes $\mathbf{p} = \frac{1}{2}eB(-Y + \eta, X - \xi, 0)$ and the third component of the angular momentum

$$L_z = \frac{1}{2}eB(R^2 - r_0^2)$$

is an integral of motion. In Quantum Mechanics the Hamiltonian for an electron in a magnetic field is:

$$H = \frac{1}{2m_e}[\mathbf{p} - e\mathbf{A}(\mathbf{r})]^2 + g\mu_B m_s B \tag{4.7}$$

where $\mathbf{p}$ is the canonical momentum which satisfies the usual commutation relation $[p_\alpha, r_\beta] = -i\hbar\delta_{\alpha\beta}$, $\alpha, \beta = x, y, z$; $g \approx 2$ is the Landé factor, μ_B is the Bohr magneton and $m_s = \pm 1/2$ is the electron spin. But let us focus on the first part of the Hamiltonian, and neglect the spin term. We know that, if $\mathbf{A}(x, y, z) = 0$ the electrons will move freely in the plane (x, y) and are described by plane waves with energy $\epsilon_k = \frac{\hbar^2 k^2}{2m_e} = \frac{\hbar^2}{2m_e}(k_x^2 + k_y^2)$. Once one has fixed the magnetic field, the choice of the gauge is not univocal: the Hamiltonian does not depend explicitly from the magnetic field, it rather depends only on the potential vector, which means that a different choice of gauge would lead to a different wave function. For this reason, in order to describe an electron in a magnetic field, one has to replace the momentum $\mathbf{p}$ with an operator invariant under gauge transformations [83]: the dynamic momentum

$$\mathbf{p} \to \boldsymbol{\pi} = m_e\mathbf{v} = (\mathbf{p} - e\mathbf{A}).$$

There is a caveat: the components of the dynamic momentum *do not commute* between themselves. In fact,

$$\begin{aligned}
[\pi_x, \pi_y] &= [p_x + eA_x(\mathbf{r}), p_y + eA_y(\mathbf{r})] = e\{[p_x, A_y] - [p_y, A_z]\} = \\
&= -ie\hbar\Big(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\Big) = -i\hbar(\nabla \wedge \mathbf{B})_z = -ie\hbar B = \\
&= -i\frac{\hbar^2}{l_B^2} = \frac{i}{L}^2,
\end{aligned} \tag{4.8}$$

where $l_B = \sqrt{\frac{\hbar}{|e|B}}$ is the magnetic length. Such a result is very interesting in that this commutator is gauge invariant, since the two component are invariant themselves. One can write the Hamiltonian in terms of the dynamic momentum:

$$H = \frac{1}{2m_e}\pi^2 = \frac{1}{2m_e}(\pi_x^2 + \pi_y^2).$$

Since π_x and π_y appear in a quadratic form, such Hamiltonian has the same structure of a one-dimension harmonic oscillator. It can be useful, thus, to define the ladder operators with the usual commutation rule $[a, a^\dagger] = 1$:

$$a = \frac{l_B}{\sqrt{2}\hbar}(\pi_x - i\pi_y)\,, \qquad a^\dagger = \frac{l_B}{\sqrt{2}\hbar}(\pi_x + i\pi_y).$$

Consequently, the Hamiltonian can be rewritten as:

$$H = \hbar\omega\Big(a^\dagger a + \frac{1}{2}\Big). \tag{4.9}$$

The eigenstate of the Hamiltonian in Eq.(4.9) are the same of the number operator $a^\dagger a|n\rangle = n|n\rangle$, and the ladder operators act in the usual way:

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle\,, \qquad a|n\rangle = \sqrt{n}|n-1\rangle\,,$$

with $n > 0$ and $|0\rangle : a|0\rangle = 0$. The eigenstate associated to the n^{th} energy level is determined by

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle\,.$$

Such energy levels are discretized as for the harmonic oscillator

$$E_n = \hbar\omega(n + \frac{1}{2})\,, \tag{4.10}$$

and the discretized levels are called *Landau levels*. Notice that this is not a small change: the spectrum looks very very different from that of a free particle in the absence of a magnetic field. Following Ref.[84], let us now define the cyclotron radius vector $\rho = (\rho_x, \rho_y)$ as

$$\rho_x = -L^2\pi_y,\ \rho_y = L^2\pi_x. \tag{4.11}$$

If $m\mathbf{v} = \hbar\pi$ were classical, then ρ would be the radius vector from the center of the circular cyclotron orbit to the position of the charge. When quantum mechanics is employed, the notion of a cyclotron orbit becomes blurred because the vector cyclotron radius has components which are non–

commutative,

$$[\rho_x, \rho_y] = iL^2. \tag{4.12}$$

The energy of the charged particle may still be written in terms of the cyclotron radius ρ and the cyclotron frequency ω_c as

$$H = \frac{1}{2} m\omega_c^2 \rho^2 = \frac{1}{2} m\omega_c^2 (\rho_x^2 + \rho_y^2), \tag{4.13}$$

Let us stop a moment to introduce perhaps the clearest example of a non–commutative geometry is the "plane". Suppose that (X, Y) represents the coordinates of a "point" in such a plane and further suppose that the coordinates do not commute; *i.e.*, $[X, Y] = iL^2$, where L is the geometric length scale in the non–commutative plane. The physical meaning of L becomes evident upon placing

$$Z = \frac{X + iY}{L\sqrt{2}}, \; Z^* = \frac{X - iY}{L\sqrt{2}} \; ,$$

$$[Z, Z^*] = 1 \tag{4.14}$$

into the non–commutative Pythagoras' definition of distance S; it is

$$S^2 = X^2 + Y^2 = L^2(2Z^*Z + 1) \,. \tag{4.15}$$

From the known properties of the oscillator destruction Z and creation Z^* operators in Eqs. (4.14),(4.15), it follows that the Pythagorean distance is quantized in units of the length scale L according to

$$S_n^2 = L^2(2n + 1) \,, \tag{4.16}$$

where $n = 0, 1, 2, 3, \ldots$. The non–commutative geometrical Pythagorean theorem yields the quantized radius vector values

$$\rho_n^2 = L^2(2n + 1), \; n = 0, 1, 2, 3, \ldots \,. \tag{4.17}$$

Eqs. (4.13)(4.17) imply the Landau magnetic energy spectrum in Eq. (4.10). Note that the position of the charge $\mathbf{r} = (x, y)$ has components which commute $[x, y] = 0$, but these do not commute with the cyclotron radius components; *i.e.*, we have from Eqs. (18) and (20) that

$$[\rho_x, x] = [\rho_y, y] = 0, \; [\rho_x, y] = [x, \rho_y] = iL^2.$$

We then introduce the coordinate $\mathbf{R} = (X, Y)$ as the center of the cyclotron orbit via

$$\mathbf{r} = \mathbf{R} + \boldsymbol{\rho}$$

and find that

$$[X, Y] = -iL^2.$$

Thus $\mathbf{R} = (X, Y)$ and $\boldsymbol{\rho} = (\rho_x, \rho_y)$ represent two independent pairs of geometric canonical conjugate variables; *i.e.*, $[R_i, \rho_j] = 0$.

4.2.2 Quantum dissipation induced non–commutative geometry

Another example of a non–commutative system is the quantum dissipation induced non–commutative geometry, see Ref. [84]. The quantum properties of a "position coordinate" x of a particle are best described by making "two copies" of the coordinate $x \to (x_+, x_-)$. For computing averages of any possible associated operator (say Q) employing a reduced density matrix (say ρ) one must integrate over both coordinates (x_+ and x_-) in the copies; *i.e.*, the averaged value of Q is of the form

$$\begin{aligned} \langle Q \rangle &= Tr(\rho Q) \\ &= \int\int \langle x_+|\rho|x_-\rangle\langle x_-|Q|x_+\rangle dx_+ dx_- \, . \end{aligned} \tag{4.18}$$

For a particle moving in one dimension with an Hamiltonian

$$H = \frac{p^2}{2M} + U(x) = -\frac{\hbar^2}{2M}\Big(\frac{\partial}{\partial x}\Big)^2 + U(x) \tag{4.19}$$

the time dependence,

$$\rho(t) = e^{-iHt/\hbar}\rho\, e^{iHt/\hbar} \tag{4.20}$$

reads (in the coordinate representation)

$$\langle x_+|\rho(t)|x_-\rangle = e^{-i(H_+-H_-)t/\hbar}\langle x_+|\rho(t)|x_-\rangle \tag{4.21}$$

where the two copies of the Hamiltonian

$$H_\pm = \frac{p_\pm^2}{2M} + U(x_\pm) = -\frac{\hbar^2}{2M}\Big(\frac{\partial}{\partial x_\pm}\Big)^2 + U(x_\pm) \tag{4.22}$$

drive x_+ forward in time and drive x_- backward in time. The equation of motion for the density matrix is then:

$$i\hbar\frac{\partial}{\partial t}\langle x_+|\rho(t)|x_-\rangle = \mathcal{H}_0\langle x_+|\rho(t)|x_-\rangle\,, \tag{4.23}$$

where

$$\mathcal{H}_0 = H_+ - H_- = \frac{p_+^2}{2M} + U(x_+) - \frac{p_-^2}{2M} - U(x_-). \tag{4.24}$$

The notion of quantum dissipation enters into our considerations if there is a coupling to a thermal reservoir yielding a mechanical resistance R. The full equation of motion has the form

$$i\hbar\frac{\partial}{\partial t}\langle x_+|\rho(t)|x_-\rangle = \mathcal{H}\langle x_+|\rho(t)|x_-\rangle - \langle x_+|N[\rho(t)]|x_-\rangle\,, \tag{4.25}$$

where $N[\rho] \approx \frac{ik_BTR}{\hbar}[x,[x,\rho]]$ describes the effects of the reservoir random thermal noise and the new "Hamiltonian" $\mathcal{H}_0 \to \mathcal{H}$ for motion in the (x_+, x_-)

plane has been discussed in the previous section

$$\mathcal{H} = \frac{1}{2M}\{\Big(p_+ - \frac{Rx_-}{2}\Big)^2 - \Big(p_- + \frac{Rx_+}{2}\Big)^2\} + U(x_+) - U(x_-)\,. \qquad (4.26)$$

The velocity components (v_+, v_-) in the (x_+, x_-) plane may be found from the Hamiltonian equation

$$v_\pm = \dot{x}_\pm = \frac{\partial \mathcal{H}}{\partial p_\pm} = \pm\frac{1}{M}\Big(p_\pm \mp \frac{Rx_\mp}{2}\Big). \qquad (4.27)$$

Similarly,

$$\dot{p}_\pm = -\frac{\partial \mathcal{H}}{\partial x_\pm} = \mp U'(x_\pm) \mp \frac{Rv_\mp}{2}. \qquad (4.28)$$

From Eqs.(4.27),(4.28) it follows that

$$M\dot{v_\pm} + Rv_\mp + U'(x_\pm) = 0\,. \qquad (4.29)$$

The classical equation of motion including dissipation thereby holds true if $x_+(t) \approx x_-(t) \approx x(t)$. Dissipation induced quantum interference takes place if and only if the forward in time paths differ appreciably from the backward in time paths The commutation relations in dissipative (x_+, x_-) plane may now be derived. If we define

$$Mv_\pm = \hbar K_\pm\,, \qquad (4.30)$$

then one finds from Eq.(4.27) that

$$[K_+, K_-] = \frac{iR}{\hbar} = \frac{i}{L^2}\,. \qquad (4.31)$$

A canonical set of conjugate position coordinates (ξ_+, ξ_-) may be defined by

$$\xi_\pm = \mp L^2 K, \quad [\xi_+, \xi_-] = iL^2\,. \qquad (4.32)$$

Another canonical set of conjugate position coordinates (X_+, X_-) may be defined by

$$x_+ = X_+ + \xi_+,\ x_- = X_- + \xi_-,\ \ [X_+, X_-] = iL^2 \tag{4.33}$$

Note that $[X_a, \xi_b] = 0$, where $a, b = \pm$. For the case of pure friction in which the potential $U = 0$, Eqs.(4.26), imply

$$\mathcal{H}_{friction} = \frac{\hbar}{2M}(K_+^2 - K_-^2) = -\frac{\hbar^2}{2ML^4}(\xi_+^2 - \xi_-^2)\,. \tag{4.34}$$

The equations of motion read

$$\dot{\xi}_\pm = \frac{i}{\hbar}[\mathcal{H}_{friction}, \xi\pm] = \frac{\hbar}{ML^2}(\xi_\mp) = -\frac{R}{M}\xi_\mp = -\Gamma\xi_\mp\,, \tag{4.35}$$

with the solution

$$\begin{pmatrix} \xi_+(t) \\ \xi_-(t) \end{pmatrix} = \begin{pmatrix} \cosh(\Gamma t) & -\sinh(\Gamma t) \\ -\sinh(\Gamma t) & \cosh(\Gamma t) \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}. \tag{4.36}$$

Eq.(4.36) describes the hyperbolic orbit

$$\xi_-^2(t) - \xi_+^2(t) = \frac{2L^2\mathcal{H}_{friction}}{\hbar\Gamma}\,. \tag{4.37}$$

A comparison can be made between the non–commutative dissipative plane and the non–commutative Landau magnetic plane as shown in Fig.4.1.The circular orbit in Fig.4.2 for the magnetic problem is here replaced by the hyperbolic orbit. In light of the minus sign in the "kinetic" energy,

$$\mathcal{H}_{friction} = \frac{M}{2}(v_+^2 - v_-^2) \tag{4.38}$$

it is best to view the metric as pseudo-Euclidean or equivalently we can use the Minkowski metric $uw = u_+w_+ - u_-w_-$. In fact, the quantum dissipative eigenvalue problem $\mathcal{H}_{friction}\tilde{\rho}\omega = \hbar\omega\tilde{\rho}$ is formally identical to the relativistic charged scalar field equation in $(1+1)$- dimensional quantum

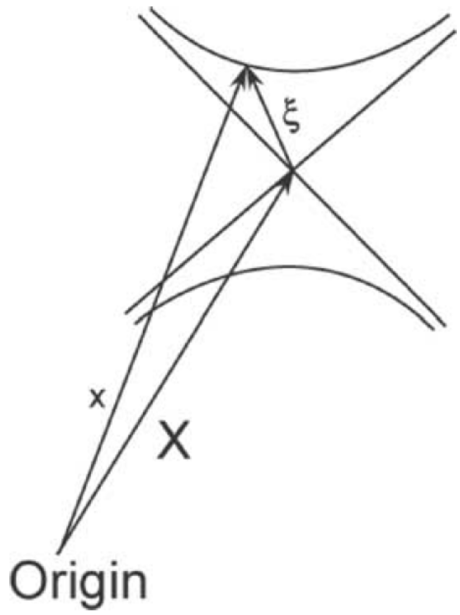


Figure 4.1: The figure shows the hyperbolic path of a particle moving in the $x = (x_+, x_-)$ plane. A non–commuting coordinate pair is $X = (X_+, X_-)$ which points from the origin to hyperbolic center. Another non–commuting coordinate pair is $\xi = (\xi_+, \xi_-)$ which points from the center of the orbit to the position on the hyperbola $x = X + \xi$.

electrodynamics; *i.e.*,

$$\{-d_\mu d^\mu + \left(\frac{mc}{\hbar}\right)^2 \psi(x)\} = 0\,,$$

$$K_\mu = id_\mu = -i\partial_\mu + \frac{eA_\mu}{\hbar c}\,,$$

$$[K_\mu, K_\nu] = \frac{i\hbar e F_{\mu\nu}}{c}\,. \tag{4.39}$$

Since in $(1+1)$-dimensional electrodynamics, the only non–zero tensor components describe the electric field $F_{10} = F_{01} = E$, it follows by comparing Eqs.(4.31),(4.39) that the analogy is exact if $L^2 = \frac{\hbar}{R} = \frac{\hbar c}{eE}$.

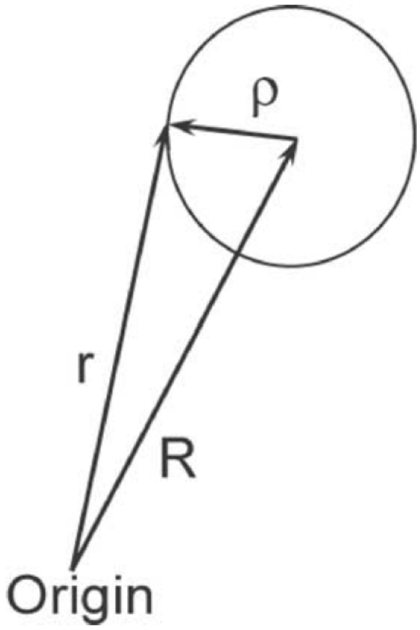


Figure 4.2: The figure shows a charge e moving in a circular cyclotron orbit. A non–commuting coordinate pair is $R = (X, Y)$ which points fromthe origin to the orbit center. Another non–commuting coordinate pair is $\rho = (\rho_x, \rho_y)$ which points from the center of the orbit to the charge position $r = R + \rho$.

4.3 Non–commutative Geometry

It is really important to understand how to extend common tools and notions from spaces in which the geometry is commutative to non–commutative geometry ones. The first thing to be adapted will, then, be the metric. This is a crucial point because, given the line element ds:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu , \tag{4.40}$$

the distance between two points $d(x, y)$ is given by:

$$d(x, y) = \inf \int_\gamma ds \, , \tag{4.41}$$

where γ varies on all the possible path between x and y. It is possible to introduce non–commutative geometry substituting the common mathematical tools, *i.e.* real variable, differential and derivatives, integral calculation etc, with new ones. In 1994 Connes introduced [40] a "translator" thanks to which it is possible to pass from a commutative context to a non–commutative one, as follows:

Commutative Geometry	**Non–commutative Geometry**
Real variable	Self-adjoint operator in $\mathcal{H}$
Complex variable	Operator in $\mathcal{H}$
Infinitesimal dx^μ	Compact operator in $\mathcal{H}$
Infinitesimal of α-order	Compact operator in $\mathcal{H}$ such as its characteristic values $\mu_n = o(n^{-\alpha})$ if $n \to \infty$
Differential of a real or complex variable	$df = [F, f] = Ff - fF$, with F self-adjoint operator $\in L^1(H)$
Integral	Diximier trace $Tr_w(T)$
Line element $ds = \sqrt{g_{\mu\nu} dx^\mu dx^\nu}$	$ds = \frac{1}{D}$ with D fermion propagator

The range of a complex variable is replaced by the spectrum $Spec(T)$, of the correspondent operator T. If the variable is real the operator must be self-adjoint, to have a real spectrum. Infinitesimal variables are replaced by compact operators. An operator T in $\mathcal{H}$ is called compact if $\forall \epsilon > 0$, expectation values of T are superiorly bounded with ϵ in a subspace of $\mathcal{H}$. In particular, $\forall \alpha > 0$, $\alpha \in \mathbb{R}$ $\mu_n(T) = o(n^{-\alpha})$ for $n \to \infty$ defines α. Note that if T_1 is α_1-order and T_2 is α_2-order, then $T_1 T_2$ is of order $(\alpha_1 + \alpha_2)$. The differential df of a real of complex variable expressed by:

$$df = \sum \frac{\partial f}{\partial x^i} dx^i$$

is replaced by the commutator $df = [F, f]$. This is the same passage that is done from the classical mechanic to the quantum mechanic. In fact, in this case there is the substitution of the Poisson brackets $\{f, g\}$ with the commutators $[f, g]$, in such way the Leinbitz rule is still valid, *i.e.* $d(fg) = d(f)g + fd(g)$. Another fundamental tool of the non–commutative geometry is the Diximier trace, which replace the integral calculation. Diximier trace satisfies the following property:

1. $Tr_w(T)$ is linear in T;

2. If $T \geq 0$ then $Tr_w(T) \geq 0$;

3. If S is a bounded operator then $Tr_w(ST) = Tr_w(TS)$;

4. $Tr_w(T)$ does not depend on the choice of the inner product in $\mathcal{H}$;

5. $Tr_w(T) = 0$ if $\mu_n(T) = o(n^{-1})$.

Moreover for $T \geq 0$ we have

$$Tr_w(T) = \lim_w \frac{\sum_{n=0}^{N} \mu_n(T)}{LogN} .$$

The most important consequence of being in a non–commutative space is that it is no more possible to measure precisely x^μ since it does not commute with ds, *i.e.* $[x_\mu, x_\nu] = iL^2\varepsilon_{\mu\nu}$. For that reason, in non–commutative geometry distance between two point (x, y) is calculated as:

$$d(x, y) = \sup\{|f(x) - f(y)| : f \in \mathcal{A}, ||[\mathcal{D}, f]|| \leq 1\}. \tag{4.42}$$

where the norm is the norm of operators in Hilbert space. Points x and y are used to convert $\mathcal{A}$ element, *i.e.* f, in scalar quantity, while $ds = \frac{1}{\mathcal{D}} \neq \sqrt{g_{\mu\nu}dx^\mu dx^\nu}$, because $\mathcal{D}$ is the fermionic propagator. Interesting studies of

Connes' construction with special reference to fermion degrees of freedom and NCG in gauge theories have been pursued in large number of publication. For brevity we quote here only two of them [85] [86].

4.4 Non–commutative Spectral Geometry

Non–commutative Spectral Geometry is given by a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{A}$ is an involution of operators on the finite-dimensional Hilbert space $\mathcal{H}$, and $\mathcal{D}$ is a self-adjoint unbounded operator in $\mathcal{H}$. This construction is analogous to the Fourier transform in commutative spaces, and, being of spectral nature, creates a link with the experimental data. Within Connes' theory there is the concept of *spectral dimension*, which is a subset Π of the complex surface C, in which spectral functions (*i.e.* $\zeta(z)$) associated to the algebra elements and Dirac operator) have singularities. Another main aspect of NCSG is the *real structure* which is fundamental within Connes' model. A real structure on a spectral triple $\mathcal{F} = (\mathcal{A}, \mathcal{H}, D)$ is an anti-linear isometry $J : \mathcal{H} \to \mathcal{H}$, such as:

$$J^2 = \epsilon, \quad JD = \epsilon^{'} DJ, \quad J\gamma = \epsilon^{''} \gamma J,$$

where $\epsilon, \epsilon^{'}, \epsilon^{''} \in \{-1, 1\}$ depend on n, as follows:

$$
\begin{aligned}
(n, \epsilon, \epsilon^{'}, \epsilon^{''}) &\equiv (0,1,1,1) \\
&\equiv (1,1,-1,0) \\
&\equiv (2,-1,1,-1) \\
&\equiv (3,-1,1,0) \\
&\equiv (4,-1,1,1) \\
&\equiv (5,-1,-1,0) \\
&\equiv (6,1,1,-1) \\
&\equiv (7,1,1,0)\,.
\end{aligned}
\tag{4.43}
$$

A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ with a real structure J is called *real spectral triple.* Let $\mathcal{H} = \mathcal{L}^2(\mathcal{M}, S)$ be the Hilbert space of square integrable sections of the spinor bundle, $\mathcal{A} = C^\infty(\mathcal{M})$ be the algebra of smooth functions on $\mathcal{M}$ acting on $\mathcal{H}$ as simple multiplication operators

$$
(f\xi)(x) = f(x)\xi(x), \quad \forall f \in C^\infty(\mathcal{M}) \quad \text{and} \quad \forall \xi \in \mathcal{L}^2(\mathcal{M}, S), \tag{4.44}
$$

and $D = \partial\!\!\!/_{\mathcal{M}} = i\gamma^\mu \nabla^s_\mu$ (where ∇^s_μ is the spin connection $\nabla^s_\mu = \partial_\mu + \frac{1}{2}\omega^{ab}_\mu \gamma_{ab}$) The algebra $\mathcal{A}$, related to the gauge group of local gauge transformations, is the algebra of coordinates; all information about space are encoded in $\mathcal{A}$. As already said, in order to account for the SM effect on a 4-dimensional manyfold, Connes introduces the doubling of the space, obtained as the product between a manyfold $\mathcal{M}$ and a non–commutative discrete space $\mathcal{F}$, *i.e.* $\mathcal{M} \times \mathcal{F}$, because in this way it is possible to obtain SM action staring from Maxwell-Dirac one. The product geometry is specified by the rules:

$$
\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2, \qquad \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2. \tag{4.45}
$$

Hence for $\mathcal{M} \times \mathcal{F}$ the rules read:

$$\begin{aligned}
&\mathcal{A} = C^\infty(\mathcal{M}) \otimes \mathcal{A}_\mathcal{F} = C^\infty(\mathcal{M}, \mathcal{A}_\mathcal{F}), \\
&\mathcal{H} = \mathcal{L}^2(\mathcal{M}, S) \otimes \mathcal{H}_\mathcal{F} = \mathcal{L}^2(\mathcal{M}, S \otimes \mathcal{H}_\mathcal{F}) \\
&D = \partial\!\!\!/_\mathcal{M} \otimes 1 + \gamma_5 \otimes D_\mathcal{F}\,,
\end{aligned} \tag{4.46}$$

where γ_5 is the chirality operator in 4-dimension.

Chapter 5

Neutrino mixing and non–commutative spectral geometry in Connes' construction

5.1 Introduction to Alain Connes' Model

For reader's convenience we briefly summarize some of the basic features and ingredients of the Connes construction for the Standard Model with neutrino mixing, minimally coupled to gravity. At Planck scale, space-time should show his quantum nature and all the four fundamental interaction should be unified. Considering the action functional at low energy scale $S = S_{E-H} + S_{SM}$, which is the sum of the Einstein-Hilbert action (S_{E-H}) and SM action (S_{SM}), it can be noticed that the two parts do not share the same symmetries. The former is ruled by outer automorphism invariance (diffeomorphism) the latter by inner automorphism (local gauge transformation). Near the Planck scale, this sum fails to capture the correct description of physics, and one may argue that the distinct feature between the underlying symmetries of the two parts of S may be at the origin of the unsuccessful search for a unified theory of all interactions including gravity. The full group of invariance of the total (including gravity and matter) action functional S is the semi-direct product $U = \mathcal{G} \rtimes \mathrm{Diff}(\mathcal{M})$ of the group $\mathcal{G}$ of gauge transformations of the matter sector (the standard model) and the group $\mathrm{Diff}(\mathcal{M})$ of diffeomorphisms of the manifold $\mathcal{M}$. For those reasons a real theory of quantum gravity is still missing, but there exist theories which aim to explain what happen at high energy scale. The very difference between Alain Connes' model and the other theories is that in Connes' one space-time geometry is embedded in the action of the theory, while other theories need to postulate a posteriori the geometry at Planck scale. The aim of this model is to deal with the correct space-time symmetry U. Moreover, non–commutative Spectral Geometry (NCSG) offers a variety of phenomenological consequences, so it is possible to use this model to investigate problematics such as first phases of the Universe or high energy physics. Treating the Standard Model as a phenomenological one Connes tries to unify interactions through the unification of the symmetry group U. In General Relativity space-time is seen as a "deformable veil" in which each point is described with four coordinates. Connes describes the Universe as two identical sheets separated by a non–commutative discrete dimension, in that way he doubles the Einstein space-time and defines a discrete dimension, *i.e.* to solve the "symmetry problem" Connes considered a model of a two-sheeted space, made from the product of a four dimensional smooth compact Reimannian manifold $\mathcal{M}$ with a fixed spin structure, by a discrete non–commutative space $\mathcal{F}$ composed by only two points. In this approach the SM of the electroweak and strong interactions, even if is seen as a phenomenological model, specifies the geometry of space-time in such a way so that the Maxwell-Dirac action functional leads to the SM action. Following this proposal, the geometric space is defined as the tensor product of continuous geometry $\mathcal{M}$ for space-time by an internal geometry $\mathcal{F}$ for the SM. Connes' NCSG is then a non–euclidean geometry, in which coordinates do not commutates, *i.e.* $[x_i, x_j] = iL^2\varepsilon_{ij}$, where εij is anti-symmetric and x_i are the spacial coordinates. As we shall see, the non–commutative nature of the discrete space $\mathcal{F}$ is given by the spectral triple $(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{A}$ is an involution of operators on the finite-dimensional

Hilbert space $\mathcal{H}$ of Euclidean fermions, and D is, in general, a self-adjoint unbounded operator in $\mathcal{H}$. Following this procedure Alain Connes obtain a geometrical explanation for the Standard Model. In particular, one of the crucial aspect of Connes' construction is the choice of the operator D, because, as we shall see, it is the inverse of the line element. In reference [87] Connes and Chamseddine define a class of Dirac operator which is represented by D. Note that, in this way, the Dirac operator plays simultaneously two roles: it defines the dynamics of matter and the kinematics of gravity. [88]

5.1.1 Standard Model and General Relativity

The main aim of Alain Connes' model is to demonstrate that the full Lagrangian of SM minimally coupled with the gravity can be easily obtained using non–commutative geometry. The Standard Model of elementary particles, already briefly summarized in Chapter 1, is a quantum field theory, which is consistent with both quantum mechanic and special relativity, and describes all the known elementary particles and three of the four fundamentals forces, i.e strong interaction, weak interaction and electromagnetic interaction. For that reason SM can not be considered as the ultimate model of fundamental interactions, not involving one of them (*i.e.* Gravitational interaction is not described in the SM) , even if SM predictions are quite fully confirmed by experimental data. Particles, in SM, are divided in two big categories:

- *fermions*, *i.e.* quarks and leptons, which constitute matter. Fermions are divided in three family or generation: (u, d, ν_e, e), (c, s, ν_μ, μ) and (t, b, ν_τ, τ);
- *bosons*, which mediates interactions: photon γ for the electromagnetic interaction, two charged bosons $W^\pm$ and the neutral one Z^0 for the weak interaction and the eight gluon $\}$ for the strong interaction. Moreover, the model includes the Higgs boson H^0 which is responsible,

through a process of spontaneous symmetry breaking, of the particles' masses.

At classical level, gravity is well described by the theory of General Relativity; dynamic is obtain from the Einstein field equation:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda_c g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{5.1}$$

where $c = 1$, $g_{\mu\nu}$ is the metric tensor $diag(1, -1, -1, -1)$, $T_{\mu\nu}$ is the energy-momentum tensor, Λ_c is the cosmological constant and G is the gravitational constant. Equation (5.1) can be written considering the Einstein-Hilbert action:

$$S_{EH} = \frac{1}{16\pi G}\int_{\mathcal{M}} \sqrt{-g}(R - 2\Lambda_c)dx^4\,, \tag{5.2}$$

hence the density of Lagrangian is

$$\mathcal{L}_{EH} = \frac{1}{16\pi G}(R - 2\Lambda_c)\,. \tag{5.3}$$

Having said that, it is possible to define the Lagrangian of SM minimally coupled with the gravity:

$$\mathcal{L} = \mathcal{L}_{SM} + \mathcal{L}_{EH}\,, \tag{5.4}$$

where $\mathcal{L}_{SM}$ is the Lagrangian of the SM. [100]

5.1.2 Spectral Action Principle

The main result of Connes' model are obtained using the spectral action principle, called in that way because it depends only on the spectrum of the Dirac operator and affirms that, within the NCSG, the bosonic euclidean action depends only on the spectrum of the Dirac operator as follows:

$$Tr(f(\frac{D}{\Lambda})), \tag{5.5}$$

where Λ and D have dimension of a mass and in particular Λ fixes the energy scale and f is a cut-off function. Thanks to the fact that, given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, the spectral action can be expressed in terms o a Λ power series:

$$Tr(f(\frac{D}{\Lambda})) \sim \sum_{k\in\Pi} f_k \Lambda^k \fint |D|^{-k} + f(0)\zeta_{D(0)} + \dots, \tag{5.6}$$

where the sum is done on the spectral dimension Π. The non–commutative integral is defined in terms of the zeta function $\zeta_{D(s)} = Tr(|D|^{-s})$ and momenta f_k are defined by

$$f_k \;=\; \int_0^\infty f(u)u^{k-1}\,du \quad k>0 \quad f_0 \equiv f(0)\,.$$

Trace $Tr(f(\frac{D}{\Lambda}))$ can be expressed perturbatively in terms of the Seeley-deWitt coefficient and its asymptotic expansion is:

$$\begin{aligned} Tr(f(\frac{D}{\Lambda})) \sim & 2\Lambda^4 f_4 a_0 + 2\Lambda^2 f_2 a_2 + f_0 a_4 + \dots \\ & + \Lambda^{-2k} f_{-2k} a_{4+2k} + \dots\;, \end{aligned} \tag{5.7}$$

where cut-off function f stretch to zero at infinity, and appears only in the momenta f_k:

$$\begin{aligned} f_0 &\equiv f(0), \\ f_k &\equiv \int_0^\infty f(u)u^{k-1}\,du \quad k>0, \\ f_{2k} &= (-1)^k \frac{k!}{(2k)!} f(2k)(0). \end{aligned}$$

In truth, since the expansion in zero vanishes [87], the asymptotic expansion, neglecting the vanishing terms is:

$$Tr(f(\frac{D}{\Lambda})) \sim 2\Lambda^4 f_4 a_0 + 2\Lambda^2 f_2 a_2 + f_0 a_4. \tag{5.8}$$

where f_0, f_2 and f_4 are three parameters of the model and are related to the coupling constant at unification, the gravitational constant and the cosmological constant. In particular, the term in Λ^4 gives the cosmological term, the term in Λ^2 gives the Einstein-Hilbert action functional, and the Λ-independent term yields the Yang-Mills action for the gauge fields corresponding to the internal degrees of freedom of the metric. The fermionic term can be obtained adding $\frac{1}{2}\langle J\psi, D\psi\rangle$, where J is a real structure on the spectral triple and ψ is a spinor in the Hilbert space $\mathcal{H}$ of the quarks and leptons. The computation of the all spectral action functional gives the full Lagrangian for the Standard Model minimally coupled with gravity, with neutrino mixing and Majorana mass terms.

$$\begin{aligned} S = \int d^4x\sqrt{g}\Big[&\frac{1}{2k_0^2}R + \alpha_0 C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \gamma_0 + \tau_0 R^* R^* \\ &+ \frac{1}{4}G^i_{\mu\nu}G^{\mu\nu i} + \frac{1}{4}F^\alpha_{\mu\nu}F^{\mu\nu\alpha} + \frac{1}{4}B_{\mu\nu}B^{\mu\nu} \\ &+ \frac{1}{2}|D_\mu\mathbf{H}|^2 - \mu_0^2|\mathbf{H}|^2 - \xi_0 R|\mathbf{H}|^2 + \lambda_0|\mathbf{H}|^4\Big]\,, \end{aligned} \tag{5.9}$$

where

$$\begin{aligned} \frac{1}{k_0^2} &= \frac{96 f_2\Lambda^2 - f_0 c}{12\pi^2}\,, \\ \mu_0^2 &= 2\frac{f_2\Lambda^2}{f_0} - \frac{e}{a}\,, \\ \alpha_0 &= -\frac{3f_0}{10\pi^2}\,, \\ \tau_0 &= \frac{11 f_0}{60\pi^2}\,, \\ \gamma_0 &= \frac{1}{\pi^2}(48 f_4\Lambda^4 - f_2\Lambda^2 c + \frac{f_0}{4}d)\,, \\ \lambda_0 &= \frac{\pi^2}{2f_0}\frac{b}{a^2}\,, \\ \xi_0 &= \frac{1}{12}\,, \end{aligned}$$

$\mathbf{H}$ is the rescaled Higgs field ϕ, *i.e.* $\mathbf{H} = (\sqrt{\frac{af_0}{\pi}})\phi$. In Eq.(5.9) we find a quadratic term in the Weyl curvature $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$ and in the topologic term R^*R^*, but not in the Ricci tensor R. Moreover, it is important to notice that the penultimate term represent the coupling between gravity and SM, while the other terms are linked to the gauge fields. Thus this approach leads to a geometric explanation of the SM; in particular, the vacuum expectation value of the Higgs field is related to the non–commutative distance between the two sheets. We want to underline that in Connes' model neutrino mixing is obtained for analogy with the quarks case, in fact the mixing comes from a mixing matrix, namely U^{lep}, as we show in the next section.

5.1.3 Neutrinos within the NCSG Model

In the context on NCSG, neutrinos appear naturally as Majorana spinors (so that neutrinos are their own antiparticles), for which the mass terms in the Lagrangian can be written as

$$\frac{1}{2}\sum_{\lambda\kappa}\bar{\psi}_{\lambda L}\mathcal{S}_{\lambda\kappa}\hat{\psi}_{\kappa R}+\frac{1}{2}\sum_{\lambda\kappa}\overline{\bar{\psi}_{\lambda L}\mathcal{S}_{\lambda\kappa}\hat{\psi}_{\kappa R}}\ ,$$

where the subscript $_{L,R}$ stand for left-handed, right-handed states, respectively. The off-diagonal parts of the symmetric matrix $\mathcal{S}_{\lambda\kappa}$ are the Dirac mass terms, while the diagonal ones are the Majorana mass terms. Within NCSG, one can show [89] the existence of a Dirac operator $\mathcal{D}_{\mathcal{F}}$ for the algebra

$$\begin{aligned}\mathcal{A}_{\mathcal{F}} &= \{(\lambda, q_L, \lambda, m)|\lambda\in\mathbb{C}, q_L\in\mathbb{H}, m\in M_3(\mathbb{C})\}\\ &\sim \mathbb{C}\oplus\mathbb{H}\oplus M_3(\mathbb{C})\ ,\end{aligned}$$

with off-diagonal terms. In particular, one can show [89] that there exist 3×3 matrices (3 for the number of generations) $\Upsilon_e, \Upsilon_\nu, \Upsilon_d, \Upsilon_u$ and a symmetric 3×3 matrix (3 for the number of generations) Υ_R, such that $\mathcal{D}_{\mathcal{F}}$ is of the form

$$\mathcal{D}_{\mathcal{F}}(\Upsilon)=\begin{pmatrix} S & T^\star \\ T & \overline{S}\end{pmatrix}\ . \tag{5.10}$$

S is a linear map

$$S = S_l\oplus(S_q\otimes 1_3)\ ,$$

with 1_3 the identity 3×3 matrix and

$$S_l = \begin{pmatrix} 0 & 0 & \Upsilon_\nu^\star & 0 \\ 0 & 0 & 0 & \Upsilon_e^\star \\ \Upsilon_\nu & 0 & 0 & 0 \\ 0 & \Upsilon_e & 0 & 0 \end{pmatrix} , \; S_q = \begin{pmatrix} 0 & 0 & \Upsilon_u^\star & 0 \\ 0 & 0 & 0 & \Upsilon_d^\star \\ \Upsilon_u & 0 & 0 & 0 \\ 0 & \Upsilon_d & 0 & 0 \end{pmatrix} ,$$

with the subsripts $_q$ and $_l$ denoting quarks and leptons, respectively. The * denotes adjoints, while $\bar{S} = \bar{S}_l \oplus (1_3 \otimes \bar{S}_q)$ act on $\mathcal{H}_{\bar{f}}$ by the complex conjugate matrices, where we have splitted $\mathcal{H}_\mathcal{F}$ according to $\mathcal{H}_\mathcal{F} = \mathcal{H}_f \oplus \mathcal{H}_{\bar{f}}$. Finally, T a linear map so that $T(\nu_R) = \Upsilon_R \bar{\nu}_R$. The presence of the symmetric matrix Υ_R in the Dirac operator of the finite geometry $\mathcal{F}$ accounts for the Majorana mass terms, while Υ_ν is the neutrino Dirac mass matrix. Hence, the restriction of $\mathcal{D}_\mathcal{F}(\Upsilon)$ to the subspace of $\mathcal{H}_\mathcal{F}$ with the $(\nu_R, \nu_L, \bar{\nu}_R, \bar{\nu}_L)$ basis can be written as a matrix [89]

$$\begin{pmatrix} 0 & M_\nu^\star & M_R^\star & 0 \\ M_\nu & 0 & 0 & 0 \\ M_R & 0 & 0 & \bar{M}_\nu^\star \\ 0 & 0 & \bar{M}_\nu & 0 \end{pmatrix} , \tag{5.11}$$

where $M_\nu = (2M/g)K_\nu$ with

$$2M = \left[\frac{\mathrm{Tr}(\Upsilon_\nu^\star \Upsilon_\nu + \Upsilon_e^\star \Upsilon_e + 3(\Upsilon_u^\star \Upsilon_u + \Upsilon_d^\star \Upsilon_d)}{2} \right]^{1/2} , \tag{5.12}$$

K_ν the neutrino Dirac mass matrix and M_R the Majorana mass matrix. The equations of motion of the spectral action imply that the largest eigenvalue of M_R is of the order of the unification scale. The Dirac mass M_ν turns out to be of the order of the Fermi energy, thus much smaller. In conclusion, the way the NCSG model has been built, it can account for neutrino mixing and the seesaw mechanism. [1]

[1] More details on seesaw mechanism can be found in Appendix - Seesaw mechanism.

5.1.4 Summarizing

Non–commutative spectral geometry is based on three *ansatz*:

• At some energy level, close but below the Planck scale, geometry is described by the product of a four-dimensional smooth compact Riemannian manifold $\mathcal{M}$ with a fixed spin structure by a discrete non–commutative space $\mathcal{F}$ composed by only two points. The non–commutativity of $\mathcal{F}$ can be expressed by a real spectral triple $\mathcal{F} = (\mathcal{A}_\mathcal{F}, \mathcal{H}_\mathcal{F}, D_\mathcal{F})$, where $\mathcal{A}_\mathcal{F}$ is an involution of operators on the finite-dimensional Hilbert space $\mathcal{H}_\mathcal{F}$ of Euclidean fermions, and $\mathcal{D}_\mathcal{F}$ is a self-adjoint unbounded operator in $\mathcal{H}_\mathcal{F}$. The algebra $\mathcal{A}_\mathcal{F}$ contains all information usually carried by the metric. The axioms of the spectral triples imply that the Dirac operator of the internal space, $D_\mathcal{F}$, is the fermionic mass matrix. The Dirac operator is the inverse of the Euclidean propagator of fermions. The spectral geometry for $\mathcal{M} \times \mathcal{F}$ is thus given by

$$\begin{aligned} \mathcal{A} &= C^\infty(\mathcal{M}) \otimes \mathcal{A}_\mathcal{F} = C^\infty(\mathcal{M}, \mathcal{A}_\mathcal{F}) \ , \\ \mathcal{H} &= \mathcal{L}^2(\mathcal{M}, S) \otimes \mathcal{H}_\mathcal{F} = \mathcal{L}^2(\mathcal{M}, S \otimes \mathcal{H}_\mathcal{F}) \\ D &= D_\mathcal{M} \otimes 1 + \gamma_5 \otimes D_\mathcal{F} \ , \end{aligned}$$

where $C^\infty(\mathcal{M}, \mathcal{C})$ is the algebra of smooth complex valued functions on $\mathcal{M}$; $\mathcal{L}^2(\mathcal{M}, S)$ is the space of square integrable Dirac spinors over $\mathcal{M}$; $D_\mathcal{M}$ is the Dirac operator $\partial\!\!\!/_\mathcal{M} = \sqrt{-1}\gamma^\mu \nabla^s_\mu$ on $\mathcal{M}$; and γ_5 is the chirality operator in the four-dimensional case.

• The finite dimensional algebra $\mathcal{A}_\mathcal{F}$, which is the main input, is chosen to be [87]

$$\mathcal{A}_\mathcal{F} = M_a(\mathbb{H}) \oplus M_k(\mathbb{C}) \ , \tag{5.13}$$

with $k = 2a$ and $\mathbb{H}$ being the algebra of quaternions. This choice was made due to the three following reasons: (i) the model should account for massive neutrinos and neutrino oscillations so it cannot be a left-right symmetric model, like for instance $\mathbb{C} \oplus \mathbb{H}_\mathrm{L} \oplus \mathbb{H}_\mathrm{R} \oplus M_3(\mathbb{H})$; (ii) non–commutative geometry imposes constraints on algebras of operators in the Hilbert space; and (iii) one should avoid fermion doubling. The first possible value for the even number k is 2, corresponding to a Hilbert space of four fermions, but this choice is ruled out from the existence of quarks. The next possible value is $k = 4$ leading to the correct number of $k^2 = 16$ fermions, four leptons and twelve quark, in each of the three generations. This is the most economical choice that can account for the SM.

• The action functional is dictated by the spectral action principle, which affirms that the bosonic part of the action functional depends only on the spectrum of the Dirac operator $\mathcal{D}$ and is of the form

$$\mathrm{Tr}\left(f\left(\frac{\mathcal{D}}{\Lambda}\right)\right) , \tag{5.14}$$

where f is a positive even function of the real variable and it falls to zero for large values of its argument, while the parameter Λ fixes the energy scale. Thus, the action functional sums up eigenvalues of the Dirac operator which are smaller than the cut-off scale Λ. Since the bosonic action only depends on the spectrum of the line element, *i.e.* the inverse of the Dirac operator, one concludes that $\mathcal{D}$ contains all information about the bosonic part of the action. The trace, Eq. (5.14), is then evaluated with heat kernel techniques and is given in terms of geometrical Seeley-deWitt coefficients a_n. Since f is a cut-off function, its Taylor expansion at zero vanishes. Therefore, its asymptotic expansion depends only on the three momenta f_0, f_2 and f_4, which are related to the coupling constant at unification, the gravitational constant and the cosmological constant, respectively. In this sense, the choice of the test function f plays only a limited rôle. Hence,

$$\mathrm{Tr}\left(f\left(\frac{\mathcal{D}}{\Lambda}\right)\right) \sim 2\Lambda^4 f_4 a_0 + 2\Lambda^2 f_2 a_2 + f_0 a_4 , \tag{5.15}$$

where

$$f_k = \int_0^\infty f(u) u^{k-1} du .$$

The gravitational Einstein action is thus obtained by the expansion of the action functional. The coupling with fermions is obtained by adding to the trace, Eq. (5.14), the term

$$\mathrm{Tr}\frac{1}{2}\langle J\psi, \mathcal{D}\psi\rangle \ , \tag{5.16}$$

where J is the real structure on the spectral triple and ψ is an element in the space $\mathcal{H}_{\mathcal{F}}$. In the presence of gauge fields A, there is a modification in the metric (within non–commutative geometry, one does not focus on $g_{\mu\nu}$ but on the Dirac operator instead), leading to the inner fluctuations of the metric (we now drop the subscript $_{\mathcal{F}}$ for simplicity)

$$\mathcal{D} \to \mathcal{D}_A = \mathcal{D} + A + \epsilon' JAJ^{-1} \ , \tag{5.17}$$

where A is a self-adjoint operator of the form

$$A = \sum_j a_j[D, b_j] \ , \quad a_j, b_j \in \mathcal{A} \ ,$$

J is an antilinear isometry and $\epsilon' \in \{-1, 1\}$. Applying the action principle to $\mathcal{D}_A$ one obtains the combined Eistein-Yang-Mills action. Thus, the fermions of the SM provide the Hilbert space of a spectral triple for a suitable algebra, while the bosons arise as inner fluctuations of the corresponding Dirac operator.

In conclusion, the full Lagrangian of the SM minimally coupled to gravity, is obtained as the asymptotic expansion (in inverse powers of Λ) of the spectral action for the product geometry $\mathcal{M}\times\mathcal{F}$. This geometric model can explain the SM phenomenology [91, 92]. Moreover, since this model lives by construction at very high energies, it can provide a natural framework to address early universe cosmological issues [93]-[102].

5.2 Hopf Algebra, Bogoliubov transformation and neutrino mixing

The doubling of the space introduced by Connes, as we already said, leads to the doubling of the algebra which is a central ingredient of Hopf algebras with the operator doubling implied by the coalgebra. The coproduct operation is indeed a map $\mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ which duplicates the algebra. In particular, as we shall see, the deformed Hopf algebra is relevant within NCSG. In fact, in order to be non–commutative, needs to be "deformed". As we shall see, the coproduct of the deformed algebra is strictly related with the Bogoliubov transformations, which are the same transformation, as we have seen in Chapter 2 and 3, that, combined with a rotation, appear in the neutrino mixing transformation. This link is crucial for the purpose of this thesis, since is the chain ring between the doubling of the algebra of Connes' construction and the neutrino mixing. For that reason we will summarize the main features of Hopf algebra, coproduct operator and Bogoliubov transformations.

5.2.1 Hopf Algebra

An algebra μ over a field $\mathcal{F}$ is a vectorial space $\mathcal{A}$ with a bilinear mapping:

$$\begin{aligned} &\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \qquad && \text{product}\,, \\ &\eta : \mathcal{F} \to \mathcal{A} && \text{identity}\,, \end{aligned} \tag{5.18}$$

such that:

$$\begin{aligned} &\mu \circ (id \times \mu) = \mu \circ (\mu \circ id) \qquad && \text{associativity}\,, \\ &\eta \circ (id \times \eta) = id = \eta \circ (\eta \circ id) && \text{existence} \quad \text{of} \quad \text{identity}\,. \end{aligned} \tag{5.19}$$

A co-algebra Δ over a field $\mathcal{F}$ is a vectorial space $\mathcal{A}$ with a bilinear mapping:

$$\begin{aligned} &\Delta : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} && \mathrm{co-product}\,, \\ &\epsilon : \mathcal{F} \to \mathcal{A} && \mathrm{co-identity}\,, \end{aligned} \tag{5.20}$$

such that:

$$\begin{aligned} &\Delta \circ (id \times \Delta) = \Delta \circ (\Delta \circ id) && \mathrm{co-associativity}\,, \\ &\epsilon \circ (id \times \epsilon) = id = \epsilon \circ (\epsilon \circ id) && \mathrm{existence \quad of \quad co-identity}\,. \end{aligned} \tag{5.21}$$

A bialgebra Δ over a field $\mathcal{F}$ is a vectorial space $\mathcal{A}$ which is simultaneously an algebra and a co-algebra. An *Hopf algebra* is a bialgebra equipped with a linear mapping $\gamma : \mathcal{A} \to \mathcal{A}$

$$\mu \circ (id \times \gamma) \circ \Delta = \eta \circ \epsilon\,. \tag{5.22}$$

Summarizing an Hopf algebra is an algebra in which holds:

$$\begin{aligned} &\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} && \mathrm{product}\,, \\ &\eta : \mathcal{F} \to \mathcal{A} && \mathrm{identity}\,, \\ &\Delta : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} && \mathrm{co-product}\,, \\ &\epsilon : \mathcal{F} \to \mathcal{A} && \mathrm{co-identity}\,, \\ &\gamma : \mathcal{A} \to \mathcal{A} && \mathrm{antipode}\,. \end{aligned} \tag{5.23}$$

Clearly, every algebra equipped with multiplication, can be promoted to be a Hopf algebra defying $\forall x \in \mathcal{A}$:

$$\begin{aligned} \Delta &: x \to x \otimes x\,, \\ \epsilon &: x \to 1\,, \\ \gamma &: x \to x^{-1}\,. \end{aligned} \tag{5.24}$$

The co-product operator, *i.e.* the operator which doubles the considered algebra, has a central role in Physics. In fact coproduct structure is used in the familiar addition of energy, momentum and angular momentum. In general, coproduct of a generic operator ΔO is an homomorphism defined as:

$$\Delta O = O \otimes \mathbb{1} + \mathbb{1} \otimes O \equiv O_1 + O_2 \,, \tag{5.25}$$

therefore it is commutative. From the above, it is evident the close link between the Hopf algebra, just defined, and the structure of the mathematical model of Alain Connes, see Eqs (4.45). In particular, as we shall see, it is the algebra of Hops deformed that is relevant to the construction of the NCSG. Let us consider, then, at first the algebra of fermions $h(1|1)$ and then the corresponding deformed algebra $h_q(1|1)$. The fermionic algebra is generated by the set of operators $\{a, a^\dagger, H, N\}$ with commutation relation:

$$\begin{aligned} \{a, a^\dagger\} =& 2H \,, \\ [N, a] =& -a \,, \\ [N, a^\dagger] =& a^\dagger \,, \\ [H, \bullet] =& 0 \,. \end{aligned} \tag{5.26}$$

The Casimir operator is $C = 2HN - a^\dagger a$ and the co-product is defined as follows:

$$\begin{aligned} \Delta a &= a \otimes \mathbf{1} + \mathbf{1} \otimes a \equiv a_1 + a_2 \ , \\ \Delta a^\dagger &= a^\dagger \otimes \mathbf{1} + \mathbf{1} \otimes a^\dagger \equiv a_1^\dagger + a_2^\dagger \ , \end{aligned} \tag{5.27}$$

$$\begin{aligned} \Delta H &= H \otimes \mathbb{1} + \mathbb{1} \otimes H \equiv H_1 + H_2 \ , \\ \Delta N &= N \otimes \mathbb{1} + \mathbb{1} \otimes N \equiv N_1 + N_2 \ . \end{aligned} \tag{5.28}$$

Following a well known construction (see *e.g.*, Ref. [103] and references there quoted), the deformed algebra $h_q(1 \mid 1)$ is defined by

$$\{a_q, a_q^\dagger\} = [2H]_q \ , \quad [N, a_q] = -a_q \ , \quad [N, a_q^\dagger] = a_q^\dagger \ , \tag{5.29}$$

where $[H, \bullet] = 0$, with $N_q \equiv N$ and $H_q \equiv H$, while $[x]_q$ is defined by

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}} \ . \tag{5.30}$$

The Casimir operator $\mathcal{C}_q$ is given by $\mathcal{C}_q = N[2H]_q - a_q^\dagger a_q$. Thus, the deformed Hopf coproduct is given by

$$\begin{aligned}
\Delta a_q &= a_q \otimes q^H + q^{-H} \otimes a_q \ , \\
\Delta a_q^\dagger &= a_q^\dagger \otimes q^H + q^{-H} \otimes a_q^\dagger \ , \\
\Delta H &= H \otimes \mathbb{1} + \mathbb{1} \otimes H \ , \\
\Delta N &= N \otimes \mathbb{1} + \mathbb{1} \otimes N \ ,
\end{aligned} \tag{5.31}$$

In the fundamental representation we have $H = 1/2$ and the Casimir operator is thus zero, $\mathcal{C}_q = 0$. Note that the q-deformed coproduct definition is such that $[\Delta a_q, \Delta a_q^\dagger] = [2\Delta H]_q$, etc., namely the q-coproduct algebra is isomorphic with the one defined by Eq. (5.29). Requiring $a, a^\dagger$ and $a_q, a_q^\dagger$ to be adjoint operators implies that q can only be of modulus one, hence $q \sim e^{i\theta}$. Note that in the fundamental representation $h(1 \mid 1)$ and $h_q(1 \mid 1)$ coincide, as it happens in the spin 1/2 representation; the differences appearing only at the level of the corresponding coproducts (and in the higher spin representations). In conclusion, we have now the prescription to work in the two-mode space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ with the non–commutative q-deformed Hopf algebra. Note that (in standard notation) $a \otimes 1 \equiv a_1$, $1 \otimes a \equiv a_2$, with $\{a_i, a_j\} = 0 = \{a_i, a_j^\dagger\}$, $i \neq j$, $i, j = 1, 2$. Also note that for consistency with the coproduct isomorphism, the Hermitian conjugation of the coproduct must be supplemented by the inversion of the two spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ in the two-mode space $\mathcal{H}$. Following Ref. [90], we recall that the four-dimensional smooth compact Riemannian manifold $\mathcal{M}$ (for spacetime) with a fixed spin structure S is fully encoded by its Dirac spectral triple $(\mathcal{A}_1, \mathcal{H}_1, \mathcal{D}_1) = (C^\infty(\mathcal{M})\mathcal{M}, \mathcal{L}^2(\mathcal{M}, S), \not{\partial}_\mathcal{M})$. Considering its product with the finite geometry $(\mathcal{A}_2, \mathcal{H}_2, \mathcal{D}_2) = (\mathcal{A}_\mathcal{F}, \mathcal{H}_\mathcal{F}, \mathcal{D}_\mathcal{F})$, the product geometry $\mathcal{M} \times \mathcal{F}$

is given by

$$
\begin{aligned}
\mathcal{A} &= \mathcal{A}_1 \otimes \mathcal{A}_2 \ , \ \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \ , \\
\mathcal{D} &= \mathcal{D}_1 \otimes 1 + \gamma_1 \otimes \mathcal{D}_2 \ , \\
\gamma &= \gamma_1 \otimes \gamma_2 \ , \quad J = J_1 \otimes J_2 \ ,
\end{aligned}
\tag{5.32}
$$

with

$$
J^2 = -1, \quad [J, \mathcal{D}] = 0, \quad [J_1, \gamma_1] = 0, \quad \{J, \gamma\} = 0, \tag{5.33}
$$

where, as customary, square and curl brackets denote commutators and anti-commutators, respectively.

5.2.2 Co-product and Bogoliubov transformation

The co-product turn out to be strictly related to the Bogoliubov transformation. It is, in fact, possible to identify a_1 and a_2 such as $a_2 \equiv \tilde{a}_1$ and $a_2^\dagger \equiv \tilde{a}_1^\dagger$, where $\tilde{\bullet}$ denotes the so called "tilde-coniugation" defined in TFD. Let us recall the "tilde-coniugation" rules:

$$
\begin{aligned}
&\widetilde{(OO')} = \tilde{O}\tilde{O}', \\
&(\alpha\widetilde{O + \beta} O') = \alpha^* \tilde{O} + \beta^* \tilde{O}', \\
&\widetilde{(O^\dagger)} = \tilde{O}^\dagger, \\
&\widetilde{(\tilde{O})} = O, \\
&\{O, \tilde{O}'\} = \{\tilde{O}'^\dagger\} = 0, \\
&|\widetilde{O(\beta)}\rangle = |O(\beta)\rangle \, .
\end{aligned}
\tag{5.34}
$$

By resorting now to the result of Ref. [103], we show that the coproduct turns out to be related to the Bogoliubov transformations. Let us define the operators A_q and B_q, as

$$\begin{aligned} A_q &\equiv \frac{\Delta a_q}{\sqrt{[2]_q}} = \frac{1}{\sqrt{[2]_q}}(e^{i\theta}a_1 + e^{-i\theta}a_2) \ , \\ B_q &\equiv \frac{1}{i\sqrt{[2]_q}}\frac{\delta}{\delta\theta}\Delta a_q = \frac{1}{\sqrt{[2]_q}}(e^{i\theta}a_1 - e^{-i\theta}a_2) \ , \end{aligned} \tag{5.35}$$

obtained from Eqs. (5.31) with $q = q(\theta) \equiv e^{i2\theta}$. The anticommutation relations read

$$\begin{aligned} &\{A_q, A_q^\dagger\} = 1 \quad , \quad \{B_q, B_q^\dagger\} = 1 \ , \\ &\{A_q, B_q\} = 0 \quad , \quad \{A_q, B_q^\dagger\} = \tan 2\theta \ . \end{aligned} \tag{5.36}$$

Let us then construct the operators

$$\begin{aligned} a(\theta) &= \frac{1}{\sqrt{2}}\big(A(\theta) + B(\theta)\big) \ , \\ \tilde{a}(\theta) &= \frac{1}{\sqrt{2}}\big(A(\theta) - B(\theta)\big) \ , \end{aligned} \tag{5.37}$$

where

$$\begin{aligned} A(\theta) &\equiv \tfrac{\sqrt{[2]_q}}{2\sqrt{2}}\big[A_{q(\theta)} + A_{q(-\theta)} + A^\dagger_{q(\theta)} - A^\dagger_{q(-\theta)}\big] \ , \\ B(\theta) &\equiv \tfrac{\sqrt{[2]_q}}{2\sqrt{2}}\big[B_{q(\theta)} + B_{q(-\theta)} - B^\dagger_{q(\theta)} + B^\dagger_{q(-\theta)}\big]. \end{aligned} \tag{5.38}$$

So that

$$\begin{aligned} &\{A(\theta), A^\dagger(\theta)\} = 1, \\ &\{B(\theta), B^\dagger(\theta)\} = 1, \\ &\{A(\theta), B^\dagger(\theta)\} = 0. \end{aligned} \tag{5.39}$$

and all other anti-commutators are equal zero. It's also possible to write :

$$\begin{aligned} A(\theta) &= \frac{1}{\sqrt{2}}\big(a(\theta) + \tilde{a}(\theta)\big), \\ B(\theta) &= \frac{1}{\sqrt{2}}\big(a(\theta) - \tilde{a}(\theta)\big). \end{aligned} \tag{5.40}$$

Hence,

$$
\begin{aligned}
a(\theta) &= a_1 \cos\theta - i\, a_2^\dagger \sin\theta \ , \\
\tilde{a}(\theta) &= a_2 \cos\theta + i\, a_1^\dagger \sin\theta \ ,
\end{aligned} \tag{5.41}
$$

with $\{a(\theta), \tilde{a}(\theta)\} = 0$. The only non–zero anticommutation relations are

$$
\{a(\theta), a^\dagger(\theta)\} = 1 \ , \ \{\tilde{a}(\theta), \tilde{a}^\dagger(\theta)\} = 1 \ . \tag{5.42}
$$

Equation (5.41) is the Bogoliubov transformation of the pair of creation and annihilation operators (a_1, a_2) into $(a(\theta), \tilde{a}(\theta))$. Equations (5.37)-(5.41) show that the Bogoliubov-transformed operators $a(\theta)$ and $\tilde{a}(\theta)$ are linear combinations of the coproduct operators defined in terms of the deformation parameter $q(\theta)$ and their θ-derivatives; namely the Bogoliubov transformation is implemented in differential form (in θ) as

$$
\begin{aligned}
a(\theta) &= \frac{1}{4}\Big(1 - i\frac{\delta}{\delta\theta}\Big) \times \Delta\big[a_q + a_{q^{-1}} - (a_q^\dagger - a_{q^{-1}}^\dagger)\big] \\
&= \frac{1}{\sqrt{2}}\Big[e^{\alpha(1 - i(\frac{\delta}{\delta\theta})} - e^{-\alpha(1 - i(\frac{\delta}{\delta\theta})}\Big] \times \Delta\big[a_q + a_{q^{-1}} - (a_q^\dagger - a_{q^{-1}}^\dagger)\big]
\end{aligned} \tag{5.43}
$$

$$
\begin{aligned}
\tilde{a}(\theta) &= \frac{1}{4}\Big(1 + i\frac{\delta}{\delta\theta}\Big) \times \Delta\big[a_q + a_{q^{-1}} - (a_q^\dagger - a_{q^{-1}}^\dagger)\big] \\
&= \frac{1}{\sqrt{2}}\Big[e^{\alpha(1 + i(\frac{\delta}{\delta\theta})} - e^{-\alpha(1 + i(\frac{\delta}{\delta\theta})}\Big] \times \Delta\big[a_q + a_{q^{-1}} - (a_q^\dagger - a_{q^{-1}}^\dagger)\big]
\end{aligned} \tag{5.44}
$$

where $\alpha = \frac{1}{4}\log 2$. Notice in Eq. (5.41) the antilinearity of the *tilde conjugation* $c\mathcal{O} \to c^*\tilde{\mathcal{O}}$ which reminds of the antilinearity of the J isometry introduced in Chapter 2. [2]

5.2.3 Bogoliubov trasformation, doubling and neutrino mixing

[2]For more details on this and other features of the q-deformed Hopf algebra and the Bogoliubov transformation, we refer the reader to Refs. [103] and [104].

We have already widely seen how crucial the role of Bogoliubov transformation is in neutrino mixing generator [71, 70]. Since deformed coproducts are a basis of Bogoliubov transformations, we have shown that the field mixing ultimately rests on the algebraic structure of the deformed coproduct in the non–commutative Hopf algebra embedded in the algebra doubling of NCSG. Moreover, if we consider again the explicit calculation of $\mathcal{P}^e_{\mathbf{k},\sigma}(t)$, as in Chapter 2, which provides the flavor charge oscillation, we obtain

$$\begin{aligned}
\mathcal{P}^e_{\mathbf{k},e}(t) &= 1 - \sin^2 2\theta \\
&\quad \times \left[|U_{\mathbf{k}}|^2 \sin^2 \frac{\omega_{k,2} - \omega_{k,1}}{2} t + |V_{\mathbf{k}}|^2 \sin^2 \frac{\omega_{k,2} + \omega_{k,1}}{2} t \right], \\
\mathcal{P}^e_{\mathbf{k},\mu}(t) &= \sin^2 2\theta \\
&\quad \times \left[|U_{\mathbf{k}}|^2 \sin^2 \frac{\omega_{k,2} - \omega_{k,1}}{2} t + |V_{\mathbf{k}}|^2 \sin^2 \frac{\omega_{k,2} + \omega_{k,1}}{2} t \right] .
\end{aligned}$$

Notice that in the absence of the condensate contribution, *i.e.* in the $|V_{\mathbf{k}}| \to 0$ limit ($|U_{\mathbf{k}}| \to 1$), the usual QM Pontecorvo approximation of the oscillation formulae is obtained. In the same limit, the non–commutative structure of the Hopf coproduct algebra (and the related Bogoliubov transformation) is lost. The quantum field non–perturbative structure is thus essential for the NCSG construction. It is worth noting that besides our discussion on neutrino mixing, Bogoliubov transformations are also relevant for quantum aspects of the theory. Indeed, as we already noticed, they are known to describe the transition among unitarily inequivalent representations of the canonical (anti)commutation relations in quantum field theory (QFT) at finite temperature and are therefore a key tool in the description of the non–equilibrium dynamics of symmetry breaking phase transitions [104, 105, 106, 107]. Here we have shown that Bogoliubov transformations are encoded in the very same structure of the algebra doubling of Connes construction. This links the NCSG construction with the non–equilibrium dynamics of the early universe, as well as with elementary particle physics, but this is out of the scope of this thesis. In this Chapter we have seen how there is a chain, which naturally encodes neutrino mixing in the Connes' construction. As we already said, one of crucial aspects of Connes' model is the doubling of the space, with the

$\mathcal{M} \otimes \mathcal{F}$, which leads to the introduction of the doubled algebra $\mathcal{A} \equiv \mathcal{A}_1 \otimes \mathcal{A}_2$, which is, indeed, equipped with a coproduct $\Delta a = a \otimes 1 + 1 \otimes a_q$. In truth, since the algebra is a non–commutative one, the coproduct must be deformed, *i.e.* $\Delta a_q = a_q \otimes q^H + q^{-H} \otimes a_q$, in order to be non–commutative. We have seen [110, 111] that the deformed coproduct is strictly linked to Bogoliubov transformation, in fact Bogoliubov transformed annihilation and creation operators are linear combination of the coproduct operators defined in terms of the deformation parameter $q(\theta)$ and their θ-derivatives. Those Bogoliubov transformation, combined with a rotation, are the same present in the mixing transformation which links the mass field annihilation and creation operators with the flavor ones, leading to the unitary inequivalence between the two vacuum, *i.e.* mass vacuum and flavor vacuum.

Chapter 6

Phase space picture of neutrino mixing and oscillations

In this chapter, we consider the correspondence of neutrino mixing transformations with transformations in (classical) phase space as an attempt to reformulate such a phenomenon in a new language and from alternative points of view. This can be interesting since it can help to better understand a phenomenon which has a specific quantum nature by means of some classical system analogue for it. Also, the (classical) phase space picture could be at some point implemented with non-commutativity in order to fully reproduce the quantum mechanical framework. Here we make a first step in this direction, defining a consistent classical phase space picture of neutrino mixing and oscillations leading to the same oscillation formulas obtained in the quantum formulation.

6.1 Flavor mixing and oscillations

Let us briefly review the simplest description for neutrino oscillations and we rephrase this in terms of a system of coupled oscillators.

6.1.1 Neutrino oscillations

Pontecorvo mixing transformations [62] are written as a rotation of the states with definite masses m_1, m_2, $|\nu_1\rangle$, $|\nu_2\rangle$, into those with definite flavors $|\nu_e\rangle$ and $|\nu_\mu\rangle$ as:

$$|\nu_e\rangle = \cos\theta\,|\nu_1\rangle + \sin\theta\,|\nu_2\rangle, \tag{6.1}$$

$$|\nu_\mu\rangle = \cos\theta\,|\nu_2\rangle - \sin\theta\,|\nu_1\rangle. \tag{6.2}$$

Neutrino oscillations arise from time evolution of the state $|\nu_e\rangle$ in Eq.(6.1) which gives:

$$|\nu_e(t)\rangle = \cos\theta\, e^{-i\omega_1 t}\,|\nu_{_1}\rangle \,+\, \sin\theta\, e^{-i\omega_2 t}\,|\nu_{_2}\rangle. \tag{6.3}$$

When computing the probability amplitude of finding a given neutrino, *e.g.* ν_e, at a time t from its generation, one obtains flavor oscillations:

$$P_{\nu_e\to\nu_e}(t) = |\langle\nu_e|\nu_e(t)\rangle|^2 = 1 - \sin^2 2\theta\, \sin^2\left(\frac{\Delta\omega}{2}t\right) = 1 - P_{\nu_e\to\nu_\mu}(t). \tag{6.4}$$

One can also verify that flavor conservation holds:

$$|\langle\nu_e|\nu_e(t)\rangle|^2 \,+\, |\langle\nu_\mu|\nu_e(t)\rangle|^2 \,=\, 1. \tag{6.5}$$

6.1.2 Coupled oscillators

Since we are interested in the mixing transformation properties and in the above treatment no reference appears to the fermionic or bosonic nature of the particles, we treat such particles as bosons and consider the simplest case of two harmonic oscillators with different frequencies. The (normal ordered) Hamiltonian for such system is

$$: H(\hat{a}_{_{1,2}};\hat{a}^\dagger_{_{1,2}}) := \omega_1\,\hat{a}_1^\dagger\hat{a}_1 + \omega_2\,\hat{a}_2^\dagger\hat{a}_2\,, \tag{6.6}$$

with $|a_1\rangle = \hat{a}_1^\dagger|0\rangle_1$, $|a_2\rangle = \hat{a}_2^\dagger|0\rangle_2$, $[\hat{a}_i,\hat{a}_j^\dagger] = \delta_{ij}$. We use $\hbar = 1$. Eqs.(6.1),(6.2) can be seen as arising by the application to the vacuum state $|0\rangle_{1,2} = |0\rangle_1 \otimes$

$|0\rangle_2$ of the following flavor operators (adopting the conventional neutrino terminology):

$$\hat{a}_e^\dagger = \cos\theta\, \hat{a}_1^\dagger + \sin\theta\, \hat{a}_2^\dagger, \qquad \hat{a}_\mu^\dagger = \cos\theta\, \hat{a}_2^\dagger - \sin\theta\, \hat{a}_1^\dagger. \tag{6.7}$$

The transformation in Eq.(6.7) is a canonical one. The Hamiltonian in Eq.(6.6) is written in terms of the flavor ladder operators as:

$$: H(\hat{a}_{e,\mu}; \hat{a}_{e,\mu}^\dagger) := \omega_e\, \hat{a}_e^\dagger \hat{a}_e + \omega_\mu\, \hat{a}_\mu^\dagger \hat{a}_\mu + \omega_{e\mu}\, (\hat{a}_e^\dagger \hat{a}_\mu + \hat{a}_\mu^\dagger \hat{a}_e), \tag{6.8}$$

with

$$\begin{aligned} \omega_e &= \omega_1 \cos^2\theta + \omega_2 \sin^2\theta, \\ \omega_\mu &= \omega_2 \cos^2\theta + \omega_1 \sin^2\theta, \\ \omega_{e\mu} &= (\omega_2 - \omega_1) \sin\theta \cos\theta. \end{aligned} \tag{6.9}$$

Thus we also have $\omega_1 + \omega_2 = \omega_e + \omega_\mu$. We remark that the hermiticity of $H(\hat{a}_{e,\mu}; \hat{a}_{e,\mu}^\dagger)$ is preserved, it is not spoiled by the non–diagonal mixing terms. In order to discuss flavor oscillations within the classical phase–space picture we shall give in a moment, we consider here the Heisenberg picture. Thus we introduce the flavor number operator $\hat{N}_e(t) = \hat{a}_e^\dagger(t)\hat{a}_e(t)$ on the one–particle state $|a_e\rangle$ using - cf. Eq.(6.7):

$$\hat{a}_e(t) = \cos\theta\, \hat{a}_1\, e^{-i\omega_1 t} + \sin\theta\, \hat{a}_2\, e^{-i\omega_2 t}, \quad \hat{a}_\mu(t) = \cos\theta\, \hat{a}_2\, e^{-i\omega_2 t} - \sin\theta\, \hat{a}_1\, e^{-i\omega_1 t} \tag{6.10}$$

and h.c. We obtain the expectation value:

$$\begin{aligned} \langle a_e|\hat{N}_e(t)|a_e\rangle &= \langle a_e|(\cos\theta\, \hat{a}_1^\dagger\, e^{i\omega_1 t} + \sin\theta\, \hat{a}_2^\dagger\, e^{i\omega_2 t})(\cos\theta\, \hat{a}_1\, e^{-i\omega_1 t} + \sin\theta\, \hat{a}_2\, e^{-i\omega_2 t})|a_e\rangle \\ &= 1 - 2\sin^2 2\theta\, \sin^2\left(\frac{\omega_2 - \omega_1}{2} t\right). \end{aligned} \tag{6.11}$$

In the same way we obtain

$$\langle a_e|\hat{N}_\mu(t)|a_e\rangle = 2\sin^2 2\theta\, \sin^2\left(\frac{\omega_2 - \omega_1}{2} t\right) \tag{6.12}$$

where $\hat{N}_\mu(t) = \hat{a}^\dagger_\mu(t)\hat{a}_\mu(t)$.

6.2 Classical phase–space picture

We now consider a classical analogue of the above system, *i.e.* two harmonic oscillators, with the same mass m but different frequencies $\omega_1 \neq \omega_2$. The total Hamiltonian is:

$$H(Q_{_{1,2}}; P_{_{1,2}}) = \frac{P_1^2}{2m} + \frac{\omega_1^2 m}{2} Q_1^2 + \frac{P_2^2}{2m} + \frac{\omega_2^2 m}{2} Q_2^2. \tag{6.13}$$

Defining

$$\alpha_i = \sqrt{\frac{m\omega_i}{2}} \left(Q_i + i\frac{P_i}{m\omega_i} \right), \qquad \alpha_i^* = \sqrt{\frac{m\omega_i}{2}} \left(Q_i - i\frac{P_i}{m\omega_i} \right), \tag{6.14}$$

one can rewrite the Hamiltonian in Eq.(6.13) as

$$H(\alpha_{_{1,2}}; \alpha^*_{_{1,2}}) = \omega_1 \, |\alpha_1|^2 + \omega_2 \, |\alpha_2|^2. \tag{6.15}$$

where α_i are c–number counterparts of the above ladder operators and may be thought as eigenvalues of a coherent state corrisponding to the annihilation operator $\hat{a}_i$. In analogy with Eq.(6.7) we write the following transformations:

$$\alpha_e = \cos\theta \, \alpha_1 + \sin\theta \, \alpha_2, \qquad \alpha_\mu = \cos\theta \, \alpha_2 - \sin\theta \, \alpha_1. \tag{6.16}$$

and c.c., and substitute them in Eq.(6.15), obtaining the Hamiltonian in the "flavor" variables:

$$H(\alpha_{e,\mu}; \alpha^*_{e,\mu}) = \omega_e |\alpha_e|^2 + \omega_\mu |\alpha_\mu|^2 + \omega_{e\mu}(\alpha_e^* \alpha_\mu + \alpha_\mu^* \alpha_e). \tag{6.17}$$

where ω_e, ω_μ and $\omega_{e\mu}$ are given in Eq.(6.8). Eq.(6.15) and Eq.(6.17) represent alternative forms of the Hamiltonian Eq.(6.13) in terms of $(\alpha_1, \alpha_2, \alpha_1^*, \alpha_2^*)$, $(\alpha_e, \alpha_\mu, \alpha_e^*, \alpha_\mu^*)$, respectively. It is now interesting to ask whether the same Hamiltonian can be written in terms of other canonical variables (Q_e, Q_μ, P_e, P_μ) in such a way to close the chain depicted in the following graph:

$$\begin{bmatrix} H(Q_{1,2};P_{1,2}) & \stackrel{4}{\longleftrightarrow} & H(\alpha_{1,2};\alpha^*_{1,2}) \\ & & \\ \updownarrow 1 & & \updownarrow 3 \\ & & \\ H(Q_{e,\mu};P_{e,\mu}) & \stackrel{2}{\longleftrightarrow} & H(\alpha_{e,\mu};\alpha^*_{e,\mu}) \end{bmatrix} \tag{6.18}$$

In analogy with Eq.(6.14) we define ($\sigma = e, \mu$):

$$\alpha_\sigma = \sqrt{\frac{m\omega_\sigma}{2}}\left(Q_\sigma + i\frac{P_\sigma}{m\omega_\sigma}\right), \quad \alpha^*_\sigma = \sqrt{\frac{m\omega_\sigma}{2}}\left(Q_\sigma - i\frac{P_\sigma}{m\omega_\sigma}\right). \tag{6.19}$$

Inspired by Eq.(6.7) and Eq.(6.16), we introduce the following canonical transformations:

$$Q_e = \cos\theta\sqrt{\tfrac{\omega_1}{\omega_e}}Q_1 + \sin\theta\sqrt{\tfrac{\omega_2}{\omega_e}}Q_2, \quad P_e = \cos\theta\sqrt{\tfrac{\omega_e}{\omega_1}}P_1 + \sin\theta\sqrt{\tfrac{\omega_e}{\omega_2}}P_2, \tag{6.20}$$

$$Q_\mu = \cos\theta\sqrt{\tfrac{\omega_2}{\omega_\mu}}Q_2 - \sin\theta\sqrt{\tfrac{\omega_1}{\omega_\mu}}Q_1, \quad P_\mu = \cos\theta\sqrt{\tfrac{\omega_\mu}{\omega_2}}P_2 - \sin\theta\sqrt{\tfrac{\omega_\mu}{\omega_1}}P_1. \tag{6.21}$$

They indeed guarantee the conservation of the Poisson brackets. For example:

$$\begin{aligned} \{Q_e, P_e\} &= \sum_{i=1}^{2}\left(\frac{\partial Q_e}{\partial Q_i}\frac{\partial P_e}{\partial P_i} - \frac{\partial Q_e}{\partial P_i}\frac{\partial P_e}{\partial Q_i}\right) \\ &= \cos^2\theta\sqrt{\frac{\omega_1}{\omega_e}}\sqrt{\frac{\omega_e}{\omega_1}} + \sin^2\theta\sqrt{\frac{\omega_2}{\omega_e}}\sqrt{\frac{\omega_e}{\omega_2}} = 1. \end{aligned} \tag{6.22}$$

The generator of the canonical (point) transformation in Eqs.(6.20),(6.21) is of the F_2–type [109]:

$$F_2(Q_{1,2}, P_{e,\mu}) = \left(\cos\theta\sqrt{\frac{\omega_1}{\omega_e}}Q_1 + \sin\theta\sqrt{\frac{\omega_2}{\omega_e}}Q_2\right)P_e + \left(\cos\theta\sqrt{\frac{\omega_2}{\omega_\mu}}Q_2 - \sin\theta\sqrt{\frac{\omega_1}{\omega_\mu}}Q_1\right)P_\mu;$$

$$P_{1,2} = \frac{\partial F_2}{\partial Q_{1,2}}, \qquad Q_{e,\mu} = \frac{\partial F_2}{\partial P_{e,\mu}}.$$

In order for the chain in Eq.(6.18) to close, we invert the transformations in Eqs.(6.20),(6.21) and substitute them in Eq.(6.13). We obtain

$$\begin{aligned} H(Q_{e,\mu};P_{e,\mu}) &= \frac{P_e^2}{2m}+\frac{P_\mu^2}{2m}+\frac{m\omega_e^2}{2}Q_e^2+\frac{m\omega_\mu^2}{2}Q_\mu^2 \\ &+ \omega_{e\mu}\Big(\frac{P_e}{\sqrt{m\omega_e}}\frac{P_\mu}{\sqrt{m\omega_\mu}}+(\sqrt{m\omega_e}Q_e)(\sqrt{m\omega_\mu}Q_\mu)\Big). \end{aligned} \tag{6.23}$$

We also verify that, in turn, substituting Eq.(6.19) in Eq.(6.17), we obtain again Eq.(6.23). Consistently with the canonicity of the used transformations, $H(Q_{e,\mu};P_{e,\mu})$ turns out to be a real quantity; the imaginary terms proportional to Q_eP_μ and $Q_\mu P_e$ cancel. From Eq.(6.23) one can derive the equations of motion for these variables:

$$\dot{Q}_e=\frac{P_e}{m}+\omega_{e\mu}\frac{P_\mu}{m\sqrt{\omega_e\omega_\mu}},\quad \dot{P}_e=-m\omega_e^2-\omega_{e\mu}m\sqrt{\omega_e\omega_\mu}Q_\mu, \tag{6.24}$$

$$\dot{Q}_\mu=\frac{P_\mu}{m}+\omega_{e\mu}\frac{P_e}{m\sqrt{\omega_e\omega_\mu}},\quad \dot{P}_\mu=-m\omega_\mu^2-\omega_{e\mu}m\sqrt{\omega_e\omega_\mu}Q_e, \tag{6.25}$$

which can be rewritten in the following form:

$$\begin{cases} \ddot{Q}_e=-(\omega_e^2+\omega_{e\mu}^2)Q_e-\omega_{e\mu}Q_\mu\left(\frac{\omega_e\omega_\mu+\omega_\mu^2}{\sqrt{\omega_e\omega_\mu}}\right) \\ \ddot{Q}_\mu=-(\omega_\mu^2+\omega_{e\mu}^2)Q_\mu-\omega_{e\mu}Q_e\left(\frac{\omega_e\omega_\mu+\omega_e^2}{\sqrt{\omega_e\omega_\mu}}\right) \end{cases} \tag{6.26}$$

Summarizing, we have found the analogue, at classical level, of Pontecorvo mixing transformations. These are given by the the canonical (classical) transformations Eqs.(6.20),(6.21).

6.3 Oscillations in phase space

We wonder whether it is possible to obtain the oscillation formulas Eqs.(6.11),(6.12) within the classical model in terms of the Q_σ, P_σ variables. We know that the (real) solutions of Hamilton's equations for the Hamiltonian Eq.(6.13) are:

$$Q_1(t) = A_1\, e^{i\omega_1 t} + A_1^*\, e^{-i\omega_1 t}, \qquad Q_2(t) = A_2\, e^{i\omega_2 t} + A_2^*\, e^{-i\omega_2 t}. \tag{6.27}$$

and

$$P_1(t) = i\omega_1 A_1 e^{i\omega_1 t} - i\omega_1 A_1^* e^{-i\omega_1 t}, \qquad P_2(t) = i\omega_2 A_2 e^{i\omega_2 t} - i\omega_2 A_2^* e^{-i\omega_2 t}. \tag{6.28}$$

We now set $m = 1$ for simplicity and consider the following quantities:

$$\alpha_1(t) = \frac{1}{2}\left(\sqrt{\omega_1} Q_1(t) + i\frac{1}{\sqrt{\omega_1}} P_1(t)\right) \;=\; A_1^* \sqrt{\omega_1} e^{-i\omega_1 t} \tag{6.29}$$

$$\alpha_2(t) = \frac{1}{2}\left(\sqrt{\omega_2} Q_2(t) + i\frac{1}{\sqrt{\omega_2}} P_2(t)\right) \;=\; A_2^* \sqrt{\omega_2} e^{-i\omega_2 t} \tag{6.30}$$

and c.c.. Using Eqs.(6.20),(6.21) the quantity $\alpha_e(t)$ and $\alpha_\mu(t)$ take the following form:

$$\begin{aligned} \alpha_e(t) &= \frac{1}{2}\left(\sqrt{\omega_e} Q_e(t) + i\frac{1}{\sqrt{\omega_e}} P_e(t)\right) \qquad (6.31) \\ &= \cos\theta A_1^* \sqrt{\omega_1} e^{-i\omega_1 t} + \sin\theta A_2^* \sqrt{\omega_2} e^{-i\omega_2 t}, \\ \alpha_\mu(t) &= \frac{1}{2}\left(\sqrt{\omega_\mu} Q_\mu(t) + i\frac{1}{\sqrt{\omega_\mu}} P_\mu(t)\right) \qquad (6.32) \\ &= \cos\theta A_2^* \sqrt{\omega_2} e^{-i\omega_2 t} - \sin\theta A_1^* \sqrt{\omega_1} e^{-i\omega_1 t}. \end{aligned}$$

Setting as initial conditions

$$e: \qquad A_1 = e^{i\gamma}\frac{\cos\theta}{\sqrt{\omega_1}}, \quad A_2 = e^{i\gamma}\frac{\sin\theta}{\sqrt{\omega_2}} \tag{6.33}$$

with γ arbitrary real, we obtain

$$|\alpha_e(t)|^2\Big|_e = 1 - 2\sin^2 2\theta\, \sin^2\left(\frac{\omega_2 - \omega_1}{2} t\right), \tag{6.34}$$

which shows that already at a classical level we can reproduce flavor oscillation formulas. By the same reasoning, we have:

$$|\alpha_\mu(t)|^2\Big|_e = 2\sin^2 2\theta\, \sin^2\Big(\frac{\omega_2-\omega_1}{2}t\Big). \tag{6.35}$$

Choosing as initial conditions:

$$\mu: \qquad A_1 = e^{i\gamma}\frac{-\sin\theta}{\sqrt{\omega_1}}, \quad A_2 = e^{i\gamma}\frac{\cos\theta}{\sqrt{\omega_2}}\,, \tag{6.36}$$

we obtain similar results as above, cf. Eqs.(6.34),(6.35), by exchanging $e \leftrightarrow \mu$.

$$|\alpha_\mu(t)|^2\Big|_\mu = 1 - 2\sin^2 2\theta\, \sin^2\Big(\frac{\omega_2-\omega_1}{2}t\Big), \tag{6.37}$$

and

$$|\alpha_e(t)|^2\Big|_\mu = 2\sin^2 2\theta\, \sin^2\Big(\frac{\omega_2-\omega_1}{2}t\Big). \tag{6.38}$$

We note that, in the above derivation, the quantities $|\alpha_\sigma(t)|^2\Big|_\rho$ behave exactly as the expectation values of the operators $\hat{N}_\sigma(t)$ on the state $|a_\rho\rangle$.

On the same line of reasoning, we can consider the "expectation value" of the Hamiltonian Eq.(6.17) on the "flavor state", corresponding to the initial conditions in Eq.(6.33):

$$H(\alpha_{e,\mu};\alpha^*_{e,\mu})\Big|_e = \omega_e + (\omega_\mu-\omega_e)|\alpha_\mu(t)|^2\Big|_e + \omega_{e\mu}\Big(\alpha^*_e(t)\alpha_\mu(t)+\alpha^*_\mu(t)\alpha_e(t)\Big)\Big|_e, \tag{6.39}$$

where we used $|\alpha_e(t)|^2\Big|_e = 1 - |\alpha_\mu(t)|^2\Big|_e$. Following Ref.[?], we can regard the "free" Hamiltonians $\tilde{H}_e = \omega_e\,|\alpha_e(t)|^2$ and $\tilde{H}_\mu = \omega_\mu\,|\alpha_\mu(t)|^2$ as free energies F_σ, and write:

$$H = \sum_{\sigma=e,\mu}\Big(F_\sigma(t) + TS_\sigma(t)\Big), \tag{6.40}$$

where we make the identification $T = \tan 2\theta$ and

$$S_\sigma = \frac{1}{4}\delta\omega\Big(\alpha_e^*(t)\alpha_\mu(t) + \alpha_\mu^*(t)\alpha_e(t)\Big), \tag{6.41}$$

with $\delta\omega = \frac{2}{\tan 2\theta}\,\omega_{e\mu}$. We have:

$$S_e(t)\Big|_e = -\frac{1}{4}\delta\omega \sin 4\theta\, \sin^2\Big(\frac{\omega_2 - \omega_1}{2}t\Big). \tag{6.42}$$

The other results are summarized in Table 6.1, from which we see how the energetic balance is recovered.

Table 6.1: Energetic balance.

init.cond.	H	F_e	F_μ	$TS_e = TS_\mu$
e	ω_e	$\omega_e\,\lvert\alpha_e(t)\rvert^2\Big\rvert_e$	$\omega_\mu\,\lvert\alpha_\mu(t)\rvert^2\Big\rvert_e$	$\frac{1}{2}\delta\omega\lvert\alpha_\mu(t)\rvert^2\Big\rvert_e$
μ	ω_μ	$\omega_e\,\lvert\alpha_e(t)\rvert^2\Big\rvert_\mu$	$\omega_\mu\,\lvert\alpha_\mu(t)\rvert^2\Big\rvert_\mu$	$-\frac{1}{2}\delta\omega\lvert\alpha_e(t)\rvert^2\Big\rvert_\mu$

Note finally, that the integral of the entropy expectation value over an oscillation cycle, is only dependent on the mixing angle:

$$\int_0^\tau S_e(t)\Big|_e dt = \pi\cos^2 2\theta \sin 2\theta, \tag{6.43}$$

where the period is $\tau = \frac{2\pi}{\omega_2 - \omega_1}$. Such a quantity, being independent of dynamical parameters, can be related to other geometric invariants as geometric phase for oscillating neutrinos [30].

In conclusion we have considered a simple classical model in phase-space resembling the quantum one used for the description of neutrino oscillations. We have found that it is indeed possible to obtain flavor oscillation formulas by considering classical analogues of the (time–dependent) flavor number operators and (initial) neutrino states. By resorting to previous results [?], we have given a thermodynamical interpretation of the phenomenon of flavor oscillations in terms of energy fluxes between two open subsystems. One interesting output of the what we just presented is a form for the classical hamiltonian describing flavor oscillations. This can be useful for identifying classical (hamiltonian) systems analogue for this phenomenon. Another possible direction of investigation which is suggested by the above hamiltonian form Eq.(6.23) is the mapping of such system into an equivalent non–commutative one. On such aspects, work is in progress.

Chapter 7

Conclusions and Outlook

The mixing phenomenon appears today to be a truly quantum field phenomenon, whose quantum mechanical approximation reproduces the Pontecorvo formalism and its results. Far from being a simple rotation among massive neutrinos, the field mixing appears to be related to the complex mathematical structure of QFT. The discovery of the unitary inequivalence between the massive neutrino vacuum and the flavor neutrino vacuum has, in fact, displayed the condensate nature of the later, while the origin of the non–vanishing neutrino masses remains a still open problem.
In order for the mixing to occur, neutrino masses need to be non–zero and different among themselves, on the one side; on the other side, the mixing angle must be non–zero. The question then arises if any relation exists between the masses and the mixing angle, and which one is such a relation, if any. In this thesis we have analyzed such apparently puzzling dependence of the mixing mechanism both, on the neutrino mass values and on the mixing angle. The result we have obtained is that a possibility exists to disentangle in the mixing transformation the dependence of the mixing generator on the angle from the one on the masses. The decomposition we obtained shows the generator components, which depends on the physical parameters of the mixing transformation, *i.e.* θ and a. It is, in fact, possible to rewrite the generator of flavor mixing transformations as a rotation, depending only by the

mixing angle θ, transformed under a Bogoliubov transformation, depending only on the masses m_1 and m_2. Such a decomposition explicitly shows that the rotation at the level of the state, *i.e.* Pontecorvo mixing transformation, is not sufficient for implementing the mixing transformation at level of fields. It is necessary, in fact, also the action of a Bogoliubov transformation which generates a suitable mass shift. These two transformations do not commute among themselves and this fact produces important effects on the vacuum structure which has the structure of a $SU(N)$ generalized coherent state (condensate of particle-antiparticle pairs). The way this occurs shows the crucial role played by the condensate of the flavor vacuum, thus reinforcing the QFT nature of the mixing phenomenon. Once the vacuum has acquired its condensate structure, it is no more invariant under rotation by the mixing angle θ.

We should stress that the Bogoliubov transformations appearing in the above decomposition are responsible for the mass shift in the fermion fields. Such a property of Bogoliubov transformations has been already known and used since long time [73, 69, 34], *e.g.* in renormalization theory or in the dynamical generation of mass [33, 34, 35, 36]. In fact, the same type of transformations was used in Ref.[38, 39], where the generation of masses and mixing was studied in the context of dynamical symmetry breaking. Bogoliubov transformations are also used in recent studies of neutrino mixing in astrophysics [112, 113] and in a curved spacetime [114] . Another property which is worthwhile investigating in such framework is the Lorentz invariance; in fact, as we said before, flavor states behave as real physical entities [63], rather then the mass eigenstates, even if they do not satisfy the standard dispersion relation $E^2 - k^2 = m^2$.
Focusing on the algebraic structure of the mixing generator, we have introduced the concept of *non–diagonal* Bogoliubov transformation, as the first non–trivial term which contributes to the flavor vacuum structure. Moreover, the condensate structure of the vacuum suggests a thermodynamical interpretation which we investigated, showing peculiarities in the thermal behavior due to the character of the particle-antiparticle condensate involved in the flavor vacuum. Such an issue will be further investigated in a future

work.

In our analysis a non–commutative structure appeared which lead us to investigate on how the neutrino mixing is introduced in the context of the "non–commutative spectral geometry" (NCSG) formulated by Alain Connes within which one can get the Lagrangian of the Standard Model minimally coupled with gravity. We have found that the doubling of the algebra $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$ acting on the space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, used by Connes, naturally leads to the Bogoluibov transformation, which once again play a key role in the mixing. In fact, the Bogoliubov transformed operators $a(\theta)$ and $\tilde{a}(\theta)$ are linear combinations of the coproduct operators defined in terms of the deformation parameter $q(\theta)$ and its θ-derivatives, obtained from the doubled algebra $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$. Neutrino mixing is thus intimately related to the algebra doubling and, as such, it is intrinsically present in the NCSG model. We remark that Bogoliubov transformations act on operators, so our discussion is framed in the quantum operator formalism. Thus, the doubling of the algebra in Connes' construction appears to be grounded in the QFT Hopf deformed algebra, and in turn this has been shown to involve field mixing. Working with fields introduces crucial features in the formalism. From the one side, it means that we have an infinite number of degrees of freedom (therefore we have to consider the continuum or the infinite volume limit). On the other side, as it emerges from the discussion presented above, the algebra doubling, through the Bogoliubov transformations, combines the field operator positive frequency part with the negative frequency one, leading to the non–commutative features. It has, in fact, been shown in Ref. [115] that the gauge structure of the Standard Model is implicit in the algebra doubling, a key ingredient of the NCSG construction.

Let us stress once again that our arguments are of algebraic nature and therefore general. Our results can, indeed, be extended to the mixing phenomenon of any particle. We showed the extension to the three flavor neutrinos to be of difficult computation due to the phase term. On the other hand, our approach could lead to further insight on the nature of such a term which leads to the CP violation and we will focus on such analysis in a future work.

Nonetheless, we already obtained some results on boson mixing, reported in *Appendix - Extension to boson mixing.* Finally, we have considered a phase space description of a classical system, analogue to the two mixing neutrino system, defying corresponding transformations. Such a reformulation could be, in a future work, implemented with non–commutativity, in order to fully reproduce the quantum mechanical framework.

Appendix - The $SU(2)$ Group

The $SU(2)$ of the unitary matrices with determinant one is generated by the hermitian matrices $\tau^a = \frac{\sigma^a}{2}$ where the Pauli matrices σ^a

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ ; \tag{1}$$

satisfy the fundamental properties:

$$\sigma_i \sigma_j = \delta_{ij} + i\varepsilon_{ijk}\sigma_k, \tag{2}$$

$$[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k, \tag{3}$$

$$\sigma_i \sigma_j \sigma_k = -\sigma_i^* = -\sigma^t. \tag{4}$$

$$Tr[\sigma_a] = 0, \tag{5}$$

$$Tr[\sigma_a \sigma_b] = 2\delta_{ab}, \tag{6}$$

$$Tr[\sigma_a \sigma_b \sigma_c] = 2i\varepsilon_{abc}, \tag{7}$$

$$Tr[\sigma_a \sigma_b \sigma_c \sigma_d] = 2(\delta_{ab}\delta_{cd} + \delta_{bc}\delta_{da} - \delta_{ac}\delta_{bd}). \tag{8}$$

The generators τ^a are normalized so that $Tr[\tau^a \tau^b] = \frac{1}{2}\delta_{ab}$ and satisfy $[\tau_i, \tau_j] = i\varepsilon_{ijk}\tau_k$ where ε_{ijk} is the Levi-Civita symbol, which defines the group structure constants. Every matrix $U \in SU(2)$ can be written as $U = \exp[i\tau^a \alpha^a]$, $a = 1, 2, 3$; α^a real.

The fundamental representation

The fundamental representation is defined by a complex spinor s of spin $\frac{1}{2}$ which that transforms like $s_i \to s'_i = U_{ij} s_j$. Note that $s^* \to U^* s^*$ while, given the Pauli matrices properties, $s^c \equiv i\sigma^2 s^* \to U s^c$, $s^{*c} \to U^* s^{*c}$. It is said that the s representation is in $\frac{1}{2}$ while $s^* \in \frac{1}{2}^*$.

The Adjoint representation

The Adjoint representation is defined by a real vector of spin 1: A_i, $i = 1, 2, 3$ which transforms as

$$A'_i \sigma^i = U A_i \sigma^i U^\dagger \,,\; U \in SU(2) \to A'_i = R_{ij} A_j \,,\; R \in SO(3). \tag{9}$$

The $SO(3)$ is thus locally isomorphous to to the $SU(2)$ group, *i.e.* has the same structure constants.

Two spin composition

Given two spinors $s, w \in \frac{1}{2}$ there exist two possible decompositions of the tensor product in irreducible representations:

$$\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1 \,, \qquad \frac{1}{2} \otimes \frac{1}{2}^* = 0 \oplus 1. \tag{10}$$

The irreducible representations can be classified accordingly to their symmetry property to the exchange $s \rightleftharpoons w$.

Appendix - The Lorentz group

Lorentz transformations are described by the operator $\Lambda = \exp[i(\mathbf{J}_+\boldsymbol{\theta}_+ + \mathbf{J}_-\boldsymbol{\theta}_-)]$ where the operators $\mathbf{J}_\pm$ satisfies the $SU(2)$ algebra commutation relations and $\boldsymbol{\theta}_+ = \boldsymbol{\alpha} - i\boldsymbol{\beta}$, $\boldsymbol{\theta}_+^* = \boldsymbol{\alpha} + i\boldsymbol{\beta}$; the parameters $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ are real and describe respectively rotations and boosts. The group is, thus, isomorphous to $SU(2) \otimes SU(2)$ and the fundamental representation is described by the two components Weyl spinors $\psi_L = (\frac{1}{2}, 0)$ and $\psi_R = (0, \frac{1}{2})$ which transform as follows:

$$\psi_L \to e^{i\frac{\boldsymbol{\sigma}}{2}\boldsymbol{\alpha} + \frac{\boldsymbol{\sigma}}{2}\boldsymbol{\beta}}\psi_L\,, \quad \psi_R \to e^{i\frac{\boldsymbol{\sigma}}{2}\boldsymbol{\alpha} - \frac{\boldsymbol{\sigma}}{2}\boldsymbol{\beta}}\psi_R. \tag{11}$$

One can switch from a representation to another using the real antisymmetric metric tensor $\varepsilon \equiv i\sigma_2$, $\varepsilon^2 = -1$. In fact, using $\sigma_2\sigma_i^*\sigma_2 = -\sigma_i$:

$$\psi_L \to e^{i\frac{\boldsymbol{\sigma}}{2}\boldsymbol{\alpha} + \frac{\boldsymbol{\sigma}}{2}\boldsymbol{\beta}}\psi_L \Rightarrow \psi_L^c \equiv i\sigma_2\psi_L^* \to e^{-i\frac{\boldsymbol{\sigma}^*}{2}\boldsymbol{\alpha} + \frac{\boldsymbol{\sigma}^*}{2}\boldsymbol{\beta}}(-i\sigma_2)(i\sigma_2)\psi_L^* = e^{i\frac{\boldsymbol{\sigma}}{2}\boldsymbol{\alpha} - \frac{\boldsymbol{\sigma}}{2}\boldsymbol{\beta}}\psi_L^c. \tag{12}$$

The Lorentz scalar can be written using only two left spinors with the decomposition $(\frac{1}{2}, 0) \otimes (\frac{1}{2}, 0) = (0, 0) + (1, 0)$ on in a similar way with two right spinors. The right correct combination is $\psi_L\varepsilon\psi_L$, $\psi_R\varepsilon\psi_R$. In fact, the relation among the Pauli matrices can be rewritten as $\sigma_i^t\sigma_2 = -\sigma_2\sigma_i$, from which:

$$\begin{aligned} \psi_L^t\sigma_2\psi_L \to \psi_L^t e^{i\frac{\boldsymbol{\sigma}^t}{2}\boldsymbol{\alpha} + \frac{\boldsymbol{\sigma}^t}{2}\boldsymbol{\beta}}\sigma_2 e^{i\frac{\boldsymbol{\sigma}}{2}\boldsymbol{\alpha} + \frac{\boldsymbol{\sigma}}{2}\boldsymbol{\beta}}\psi_L &= \psi_L^t\sigma_2 e^{i\frac{-\boldsymbol{\sigma}}{2}\boldsymbol{\alpha} + \frac{-\boldsymbol{\sigma}}{2}\boldsymbol{\beta}} e^{i\frac{\boldsymbol{\sigma}}{2}\boldsymbol{\alpha} + \frac{\boldsymbol{\sigma}}{2}\boldsymbol{\beta}}\psi_L && (13) \\ &= \psi_L^t\sigma_2\psi_L. && (14) \end{aligned}$$

In order to see how the four-vectors transform, one should consider that the spatial three-vector transform, under rotation, as:

$$A'_a\sigma_a = UA_a\sigma_aU^\dagger \quad U = e^{i\frac{\boldsymbol{\sigma}}{2}\boldsymbol{\alpha}}, \tag{15}$$

$$A'_aA'_a = Tr\{A'_a\sigma_aA'_b\sigma_b\} = Tr\{UA_a\sigma_aU^\dagger UA_b\sigma_bU^\dagger\} = A_aA_a. \tag{16}$$

One can directly verify with the Pauli matrix algebra that, indeed, $A'_a = (\mathcal{R}A)_a$ with $\mathcal{R}$ orthogonal matrix which describes the rotation of parameter α. Since $\psi^\dagger\sigma_aA_a\psi$ is invariant if $\psi \to U\psi$, $\psi^\dagger\sigma_a\psi$ has to transform as a vector. On the other hand, this is implicit by Eq.(15) which can be written as:

$$UA_a\sigma_aU^\dagger = (\mathcal{R}A)_a\sigma = \mathcal{R}_{ab}A_b\sigma_a = A_a(\mathcal{R}^{-1}\sigma)_a \Rightarrow U\sigma_aU^\dagger = (\mathcal{R}^{-1}\sigma)_a, \tag{17}$$

where $\mathcal{R}^t = \mathcal{R}^{-1}$. In the four-vectors case one can extend Eq.(15) as[1] :

$$A'^\mu\sigma_\mu = MA^\mu\sigma_\mu M^\dagger \quad \sigma_\mu = (1, \sigma_i) \quad M = \exp\,[i\boldsymbol{\sigma\alpha} + \boldsymbol{\sigma\beta}]. \tag{18}$$

Attention is needed because $M^{-1} \neq M^\dagger$ and $Tr\{\sigma_\mu\sigma_\nu\} \neq g_{\mu\nu}$; one has to define $\bar{\sigma} = (1, -\sigma_i)$ so that $Tr\{\sigma_\mu\sigma_\nu\} = g_{\mu\nu}$. Moreover $A'^\mu\bar{\sigma}_\mu = M^{-1\dagger}A^\mu\sigma_\mu M^{-1} = MA^\mu\sigma_\mu M^\dagger = A'^\mu\sigma_\mu$, thus:

$$\begin{aligned} A'_\mu A'^\mu = Tr\{(A'\sigma)(A'\bar{\sigma})\} &= Tr\{M(A\sigma)M^\dagger M^{-1\dagger}(A\bar{\sigma})M^{-1}\} \\ &= Tr\{(A\sigma)(A\bar{\sigma})\} = A_\mu A^\mu. \end{aligned} \tag{19}$$

Finally, $\psi_L^\dagger(A^\mu\sigma_\mu)\psi_L$ where $\psi_L \to M\psi_L$ and $\psi_R^\dagger(A^\mu\bar{\sigma}_\mu)\psi_R$ where $\psi_L \to M^{-1\dagger}\psi_R$ are invariant, therefore $\psi_L^\dagger\sigma_\mu\psi_L$, $\psi_R^\dagger\bar{\sigma}_\mu\psi_R$, transforms as four-vectors.

[1]Note that $\sigma_\mu = \sigma_\mu^\dagger$, therefore $A'^\mu\sigma_\mu = MA^\mu\sigma_\mu M^{-1}$ cannot be used because $M^{-1} \neq M^\dagger$

Appendix - GIM mechanism

The Glashow-Iliopoulos-Maiani (GIM) mechanism was discovered in 1970 by Sheldon Lee Glashow, John Iliopoulos, and Luciano Maiani [116]. This mechanism describes a way to naturally suppress Flavour Changing Neutral Currents (FCNC), as well as $\Delta S = 2$ transitions, in weak interactions. In order to formulate the theory of this mechanism, a fourth quark flavour, "charm", which was not yet known, was introduced.
Considering Flavor Changing Weak Processes, we recall that they obey certain selection rules, according to experimental data:

1. The $\Delta S = 1$ rule, which states that the flavour number, in this case strangeness S, changes by at most one unit.

2. The allowed $\Delta Flavour = 1$ processes involve only charged currents

It follows that $\Delta S = 2$ transitions, as well as FCNC processes, must occur only at second order in the weak interactions. The best experimental evidence for the first is the measured $K_L - K_S$ mass difference which equals $3.4810^{-12} MeV$ and, for the second, the branching ratio $B_{\mu^+\mu^-} = \frac{\Gamma(K_L \to \mu^+\mu^-)}{\Gamma(KL \to all)}$ which equals 6.8710^{-9} (PDG). The GIM mechanism offers a natural explanation for both. It is based on two ingredients:

- The first is a generalisation of the Cabibbo universality principle for the charged weak current. With only three quark flavours,u , d and s, Cabibbo postulated that the charged weak current is given by

$$J_\mu(x) = \bar{u}(x)\gamma_\mu(1+\gamma_5)[\cos\theta d(x) + \sin\theta s(x)], \tag{20}$$

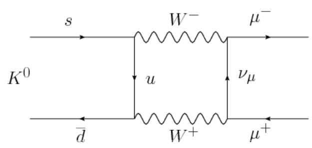


Figure 1: The one-loop contribution to $K_0 \to \mu^+\mu^-$ in a three quark theory.

where θ denotes the Cabibbo angle, *i.e.* mismatch between the flavour symmetry breaking directions chosen by the strong and weak interactions. Eq. (20) can be interpreted as saying that the u quark is coupled to a certain linear combination of the d and s quarks, $d_C = \cos\theta d + \sin\theta s$. The orthogonal combination, namely $s_C = -\sin\theta d + \cos\theta s$ remains uncoupled. With the addition of a fourth quark c with electric charge $\frac{2}{3}$, GIM conjectured that the full charged weak current is given by

$$J_\mu(x) = \bar{u}(x)\gamma_\mu(1+\gamma_5)d_C(x) + \bar{c}\gamma_\mu(1+\gamma_5)s_C(x) \tag{21}$$

or , in another notation,

$$J_\mu(x) = \bar{U}(x)\gamma_\mu(1+\gamma_5)CD(x), \tag{22}$$

with

$$U = \begin{pmatrix} u \\ c \end{pmatrix}; \quad D = \begin{pmatrix} d \\ s \end{pmatrix}; \quad C = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta. \end{pmatrix} \tag{23}$$

The important point is that, now, the current J_3 , given by the commutator of J and $J^\dagger$, is diagonal in flavour space. As a result in a gauge theory the neutral current, which is a linear superposition of J_3 and the electromagnetic current, will also be diagonal. This solves the first part of the problem, namely it ensures that FCNC processes will not be generated in the tree approximation. However, this is not enough to explain the observed rates. For example, the $K_L \to \mu^+\mu^-$ decay can be generated by the box diagram of Figure 1 which, in a renormalisable gauge theory, is expected to give a branching ratio of order $g^4 \sim \alpha^2 \sim 10^{-4}$, with α the fine structure constant.

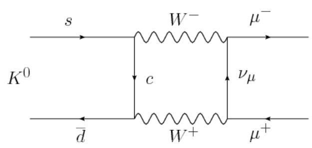


Figure 2: The charm quark contribution.

- Secondly GIM observed that, with a fourth quark, there is a second diagram, with c replacing u , Figure 2. In the limit of exact flavour symmetry the two diagrams cancel. The breaking of flavour symmetry induces a mass difference between the quarks, so the sum of the two diagrams is of order $g^4 \frac{(m_c^2 m_u^2)}{m_W^2} \sim \alpha^2 \frac{m_c^2}{m_W^2}$. With the measured charm quark mass $m_c \sim 1.27 GeV$ (PDG), the predicted rates are in agreement with observation. Before the experimental discovery of the charm particles, this mechanism was used to put upper limits on their masses. The same mechanism applies to the present theory with six quark flavours with the Cabibbo-Kobayashi-Maskawa matrix replacing the C matrix of Eq. (23).

Appendix - Seesaw mechanism

Quarks and leptons seem to fall into three distinct groups, or *families*. This family structure is one of the great puzzles in elementary particle physics. One of the problematic point of this division is the huge mismatch between the neutrino masses (of the order of eV) and the correspondent lepton masses (which are millions of times heavier). In fact, in order to be in the same multiplet, the lepton and the neutrino should have a similar mass, which they have not. The seesaw mechanism is used to explain this divergence [117]. There are several types of seesaw mechanism, each extending the Standard Model. We will analyze just the simplest version, Type 1, which extends the Standard Model by assuming two or more additional right-handed neutrino fields inert under the electroweak interactions, and the existence of a very large mass scale. This allows the mass scale to be identifiable with the postulated scale of grand unification. This model produces a light neutrino, for each of the three known neutrino flavors, and a corresponding very heavy neutrino for each flavor, which has yet to be observed. The simple mathematical principle behind the seesaw mechanism is the following property of any 2×2 matrix

$$A = \begin{pmatrix} 0 & M \\ M & B \end{pmatrix}, \tag{24}$$

where B is taken to be much larger than M. It has two very disproportioned eigenvalues:

$$\lambda_{\pm} = \frac{B \pm \sqrt{B^2 + 4M^2}}{2}. \tag{25}$$

The larger eigenvalue, λ_+, is approximately equal to B, while the smaller eigenvalue is approximately equal to

$$\lambda_- \approx -\frac{M^2}{B}\,. \tag{26}$$

Thus, $|M|$ is the geometric mean of λ_+ and $-\lambda_-$, since the determinant equals $\lambda_+\lambda_- = -M^2$. As we can see, if one of the eigenvalues goes up, the other goes down, and vice versa. For this reason the mechanism took his nice name, *"seesaw"*. As we already said, this mechanism serves to explain why the neutrino masses are so small [118] [119]. The matrix A is essentially the mass matrix for the neutrinos. The Majorana mass B component is comparable to the GUT scale and violates lepton number; while the components M, the Dirac mass, is of order of the much smaller electroweak scale, the vacuum expectation value (*vev*) below. The smaller eigenvalue λ_- then leads to a very small neutrino mass comparable to $1eV$, which is in qualitative accord with experiments, sometimes regarded as supportive evidence for the framework of Grand Unified Theories. The 2×2 matrix A arises in a natural manner within the standard model by considering the most general mass matrix allowed by gauge invariance of the standard model action, and the corresponding charges of the lepton and neutrino fields. Let the Weyl spinor χ be the neutrino part of a left-handed lepton isospin doublet (the other part being the left-handed charged lepton),

$$L = \begin{pmatrix} \chi \\ \chi' \end{pmatrix}, \tag{27}$$

as it is present in the minimal standard model without neutrino masses, and let η be a postulated right-handed neutrino Weyl spinor which is a singlet under weak isospin (*i.e.* does not interact weakly, such as a sterile neutrino). There are now three ways to form Lorentz covariant mass terms, giving either

$$\frac{1}{2}B'\chi^\alpha\chi_\alpha, \quad \frac{1}{2}B\eta^\alpha\eta_\alpha, \quad \text{or} \quad M\eta^\alpha\chi_\alpha, \tag{28}$$

and their complex conjugates, which can be written as a quadratic form,

$$\frac{1}{2}\begin{pmatrix}\chi\\ \eta\end{pmatrix}\begin{pmatrix}B' & M\\ M & B\end{pmatrix}\begin{pmatrix}\chi & \eta\end{pmatrix}. \tag{29}$$

Since the right-handed neutrino spinor is uncharged under all standard model gauge symmetries, B is a free parameter which can in principle take any arbitrary value. The parameter M is forbidden by electroweak gauge symmetry, and can only appear after its spontaneous breakdown through a Higgs mechanism, like the Dirac masses of the charged leptons. This means that M is naturally of the order of the vacuum expectation value of the standard model Higgs field,

$$vev \quad v \approx 246GeV, \qquad |\langle H\rangle| = \frac{v}{\sqrt{2}}M_t = O(\frac{v}{\sqrt{2}}) \approx 174GeV\,. \tag{30}$$

If the dimensionless Yukawa coupling is of order $y \approx 1$. It can be chosen smaller consistently, but extreme values $y \gg 1$ can make the model non–perturbative. The parameter B' on the other hand, is completely forbidden, since no renormalizable singlet under weak hypercharge and isospin can be formed using these doublet components. This is the origin of the pattern and hierarchy of scales of the mass matrix A within the *Type 1* seesaw mechanism. The large size of B can be motivated in the context of grand unification. In such models, enlarged gauge symmetries may be present, which initially force $B = 0$ in the unbroken phase, but generate a non–vanishing large value $B \approx M_{GUT} \approx 10^{15}GeV$, around the scale of their spontaneous symmetry breaking, so, given an $M \approx 100GeV$, one has $\lambda_- \approx 0.01eV$. A huge scale has thus induced a dramatically small neutrino mass for the eigenvector $\nu \approx \chi^{-\frac{M}{B}\eta}$. We remark that seesaw mechanism is a really interesting feature of the Physics beyond the SM. A deep investigation of the mechanism is out of the scope of this thesis. but there is a large number of publication on this matter, here we cite a few interesting articles [120] [121] [122] [123].

Appendix - Work in progress : Extension to boson mixing

The results we have obtained in this thesis are of algebraic nature and can be extended also to the boson mixing. Already in Refs. [25, 26] an extension of the QFT formalism of the mixing to the boson case and to the three flavor case is presented. Nonetheless, the "nested" structure we have treated and disentangled arises also in these other cases.

Following Ref.[28] we used the same reasoning we applied and showed in this thesis for neutrino mixing in the case of boson mixing.

The mixing relations are:

$$\begin{aligned} \phi_A(x) &= \phi_1(x)\ \cos\theta + \phi_2(x)\ \sin\theta \\ \phi_B(x) &= -\phi_1(x)\ \sin\theta + \phi_2(x)\ \cos\theta \end{aligned} \tag{31}$$

where generically we denote the mixed fields with suffixes A and B. Let the fields $\phi_i(x)$, $i = 1, 2$, be free complex fields with definite masses. Their conjugate momenta are $\pi_i(x) = \partial_0\phi_i^\dagger(x)$ and the commutation relations are the usual ones:

$$[\phi_i(x), \pi_j(y)]_{t=t'} = \left[\phi_i^\dagger(x), \pi_j^\dagger(y)\right]_{t=t'} = i\delta^3(\mathbf{x}-\mathbf{y})\,\delta_{ij} \quad i,j = 1,2 \tag{32}$$

with the other equal–time commutators vanishing. The Fourier expansions of fields and momenta are:

$$\phi_i(x) = \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{2\omega_{k,i}}} \left(a_{\mathbf{k},i}\, e^{-i\omega_{k,i}t} + b^\dagger_{\mathbf{k},i}\, e^{i\omega_{k,i}t} \right) e^{i\mathbf{k}\cdot\mathbf{x}} \tag{33}$$

$$\pi_i(x) = i \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} \sqrt{\frac{\omega_{k,i}}{2}} \left(a^\dagger_{\mathbf{k},i}\, e^{i\omega_{k,i}t} - b_{\mathbf{k},i}\, e^{-i\omega_{k,i}t} \right) e^{i\mathbf{k}\cdot\mathbf{x}}\,, \tag{34}$$

where $\omega_{k,i} = \sqrt{\mathbf{k}^2 + m_i^2}$ and $\left[a_{\mathbf{k},i}, a^\dagger_{\mathbf{p},j}\right] = \left[b_{\mathbf{k},i}, b^\dagger_{\mathbf{p},j}\right] = \delta^3(\mathbf{k}-\mathbf{p})\delta_{ij}$, with $i,j = 1,2$ and the other commutators vanishing. We will consider stable particles, which will not affect the general validity of our results.

We now proceed in a similar way to what has been done for fermions and recast Eqs.(31) into the form:

$$\phi_A(x) = G^{-1}_\theta(t)\; \phi_1(x)\; G_\theta(t) \tag{35}$$

$$\phi_B(x) = G^{-1}_\theta(t)\; \phi_2(x)\; G_\theta(t) \tag{36}$$

and similar ones for $\pi_A(x)$, $\pi_B(x)$. $G_\theta(t)$ denotes the operator which implements the mixing transformations (31):

$$G_\theta(t) = exp\left[-i\theta \int d^3x \left(\pi_1(x)\phi_2(x) - \phi_1^\dagger(x)\pi_2^\dagger(x) - \pi_2(x)\phi_1(x) + \phi_2^\dagger(x)\pi_1^\dagger(x)\right)\right]\,,$$

which is (at finite volume) a unitary operator: $G^{-1}_\theta(t) = G_{-\theta}(t) = G^\dagger_\theta(t)$. The generator of the mixing transformation in the exponent of $G_\theta(t)$ can also be written as

$$G_\theta(t) = \exp[\theta(S_+(t) - S_-(t))]\;. \tag{37}$$

The operators

$$S_+(t) = S_-^\dagger(t) \equiv -i \int d^3x (\pi_1(x)\phi_2(x) - \phi_1^\dagger(x)\pi_2^\dagger(x))\,,$$

together with

$$S_3 \equiv \frac{-i}{2}\int d^3x \Big(\pi_1(x)\phi_1(x) - \phi_1^\dagger(x)\pi_1^\dagger(x) - \pi_2(x)\phi_2(x) + \phi_2^\dagger(x)\pi_2^\dagger(x)\Big) \qquad (38)$$

$$S_0 = \frac{Q}{2} \equiv \frac{-i}{2}\int d^3x \Big(\pi_1(x)\phi_1(x) - \phi_1^\dagger(x)\pi_1^\dagger(x) + \pi_2(x)\phi_2(x) - \phi_2^\dagger(x)\pi_2^\dagger(x)\Big)\ , \qquad (39)$$

close the $su(2)$ algebra (at each time t): $[S_+(t), S_-(t)] = 2S_3$, $[S_3, S_\pm(t)] = \pm S_\pm(t)$, $[S_0, S_3] = [S_0, S_\pm(t)] = 0$. Note that S_3 and S_0 are time independent. It is useful to write down explicitly the expansions of the above generators in terms of annihilation and creation operators:

$$S_+(t) = \int d^3k \Big(U^*_{\mathbf{k}}(t)\, a^\dagger_{\mathbf{k},1} a_{\mathbf{k},2} - V^*_{\mathbf{k}}(t)\, b_{-\mathbf{k},1} a_{\mathbf{k},2} + V_{\mathbf{k}}(t)\, a^\dagger_{\mathbf{k},1} b^\dagger_{-\mathbf{k},2} - U_{\mathbf{k}}(t)\, b_{-\mathbf{k},1} b^\dagger_{-\mathbf{k},2}\Big)$$

$$S_-(t) = \int d^3k \Big(U_{\mathbf{k}}(t)\, a^\dagger_{\mathbf{k},2} a_{\mathbf{k},1} - V_{\mathbf{k}}(t)\, a^\dagger_{\mathbf{k},2} b^\dagger_{-\mathbf{k},1} + V^*_{\mathbf{k}}(t)\, b_{-\mathbf{k},2} a_{\mathbf{k},1} - U^*_{\mathbf{k}}(t)\, b_{-\mathbf{k},2} b^\dagger_{-\mathbf{k},1}\Big)$$

$$S_3 = \frac{1}{2}\int d^3k \Big(a^\dagger_{\mathbf{k},1} a_{\mathbf{k},1} - b^\dagger_{-\mathbf{k},1} b_{-\mathbf{k},1} - a^\dagger_{\mathbf{k},2} a_{\mathbf{k},2} + b^\dagger_{-\mathbf{k},2} b_{-\mathbf{k},2}\Big)$$

$$S_0 = \frac{1}{2}\int d^3k \Big(a^\dagger_{\mathbf{k},1} a_{\mathbf{k},1} - b^\dagger_{-\mathbf{k},1} b_{-\mathbf{k},1} + a^\dagger_{\mathbf{k},2} a_{\mathbf{k},2} - b^\dagger_{-\mathbf{k},2} b_{-\mathbf{k},2}\Big)\ .$$

The transformations for the ladder operators are:

$$\alpha^r_{\mathbf{k},A}(t) = \cos\theta\, \alpha^r_{\mathbf{k},1} + \sin\theta\left(U^*_{\mathbf{k}}(t)\, \alpha^r_{\mathbf{k},2} + \epsilon^r V_{\mathbf{k}}(t)\, \beta^{r\dagger}_{-\mathbf{k},2}\right)\ , \qquad (40)$$

$$\alpha^r_{\mathbf{k},B}(t) = \cos\theta\, \alpha^r_{\mathbf{k},2} - \sin\theta\left(U_{\mathbf{k}}(t)\, \alpha^r_{\mathbf{k},1} - \epsilon^r V_{\mathbf{k}}(t)\, \beta^{r\dagger}_{-\mathbf{k},1}\right)\ , \qquad (41)$$

and c.c., with

$$V_{\mathbf{k}}(t) \equiv |V_{\mathbf{k}}(t)| e^{i(\omega_{k,1}+\omega_{k,2})t}, \qquad U_{\mathbf{k}}(t) \equiv |U_{\mathbf{k}}(t)| e^{i(\omega_{k,2}-\omega_{k,1})t}$$

$$|V_{\mathbf{k}}(t)| = \frac{1}{2}\Big(\sqrt{\frac{\omega_{k,1}}{\omega_{k,2}}} - \sqrt{\frac{\omega_{k,2}}{\omega_{k,1}}}\Big), \qquad |U_{\mathbf{k}}(t)|^2 - |V_{\mathbf{k}}(t)|^2 = 1$$

. Identifying

$$U_{\mathbf{k}}(t) \;=\; e^{-i\psi_k}\;\cosh(\Theta_{\mathbf{k},1} - \Theta_{\mathbf{k},2})\,, \tag{42}$$

$$V_{\mathbf{k}}(t) \;=\; e^{\frac{(\phi_{k,1}+\phi_{k,2})}{2}}\;\sinh(\Theta_{\mathbf{k},1} - \Theta_{\mathbf{k},2})\,, \tag{43}$$

we obtain, as we expected:

$$G(\theta, m_1, m_2) = B^{-1}(\Theta_1, \Theta_2)\, R(\theta)\, B(\Theta_1, \Theta_2)\,, \tag{44}$$

provided we make the following identifications:

$$\psi_k \;=\; (\omega_{k,1} - \omega_{k,2})\, t\,, \tag{45}$$

$$\phi_{k,i} \;=\; 2\,\omega_{k,i}\, t\,, \tag{46}$$

$$\Theta_{\mathbf{k},i} \;=\; \ln\sqrt{m_i}\,. \tag{47}$$

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