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Elliptic Operators with Unbounded Coefficients

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Alla mia famiglia

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Abstract

Aim of this manuscript is to give generation results and some Hardy inequalities for elliptic operators with unbounded coefficients of the form

$$\mathcal{A}u = \operatorname{div}(aDu) + F \cdot Du + Vu,$$

where V is a real valued function, $a(x) = (a_{kl}(x))$ is symmetric and satisfies the ellipticity condition and a and F grow to infinity. In particular, we mainly deal with Schrödinger type operators, i.e., operators of the form $\mathcal{A} = a\Delta + V$. The case of the whole operator is also considered in the sense that a weighted Hardy inequality for these operators is provided. Finally we will consider the higher order elliptic operator perturbed by a singular potential $A = \Delta^2 - c|x|^{-4}$.

Due to their importance for the strong relation with Schrödinger operators, in Chapter 1, we provide a survey on the most significant proofs of Hardy's inequalities appeared in literature. Furthermore, we generalise Hardy inequality proving a weighted inequality with respect to a measure $d\mu = \mu(x)\,dx$ satisfying suitable local integrability assumptions in the weighted spaces $L^2_\mu(\mathbb{R}^N) = L^2(\mathbb{R}^N, d\mu)$. We claim that for all $u \in H^1_\mu(\mathbb{R}^N)$, $c \le c_{0,\mu}$

$$c\int_{\mathbb{R}^N} \frac{u^2}{|x|^2}\,d\mu \le \int_{\mathbb{R}^N} |\nabla u|^2\,d\mu + C_\mu \int_{\mathbb{R}^N} u^2\,d\mu$$

holds with $c_{0,\mu}$ optimal constant.

In Chapter 2 we recall the result of Baras-Goldstein concerning the existence and non-existence of positive solutions to the Schrödinger equation

$$\partial_t u = \Delta u + \frac{c}{|x|^2}u.$$

We present in details the Cabré-Martel approach for such problems.

Chapter 3 deals with the study of generation properties in L^p-spaces of the Schrödinger type operator L_0 with unbounded diffusion

$$L_0 u = Lu + Vu = (1 + |x|^\alpha)\Delta u + \frac{c}{|x|^2}u,$$

where $\alpha \geq 0$ and $c \in \mathbb{R}$. The proofs are based on some L^p-weighted Hardy inequality and perturbation techniques.

Finally, in Chapter 4, we study the biharmonic operator perturbed by an inverse fourth-order potential

$$A = A_0 - V = \Delta^2 - \frac{c}{|x|^4},$$

where c is any constant such that $c < C^* := \left(\frac{N(N-4)}{4}\right)^2$. Making use of the Rellich inequality, multiplication operators and off-diagonal estimates, we prove that the semigroup generated by $-A$ in $L^2(\mathbb{R}^N)$, $N \geq 5$, extrapolates to a bounded holomorphic C_0-semigroup on $L^p(\mathbb{R}^N)$ for all $p \in [p_0', p_0]$, where $p_0 = \frac{2N}{N-4}$ and p_0' is its dual exponent. Furthermore, we study the boundedness of the Riesz transform

$$\Delta A^{-1/2} := \frac{1}{\Gamma(1/2)} \int_0^\infty t^{-1/2} \Delta e^{-tA}\, dt$$

on $L^p(\mathbb{R}^N)$ for all $p \in (p_0', 2]$. Thus, we obtain $W^{2,p}$-regularity of the solution to the evolution equation with initial datum in $L^p(\mathbb{R}^N)$ for $p \in (p_0', 2]$.

Introduction

Elliptic operators with bounded coefficients have been widely studied in the literature both in $\mathbb{R}^N$ and in open subsets of $\mathbb{R}^N$ and nowadays they are well understood. Recently, the interest in operators with unbounded coefficients has grown considerably due to their numerous applications in many fields of science, such as quantum mechanics, fluid dynamics, e.g., in the study of Navier-Stokes equations with a rotating obstacle, see [22, 33] and the references therein. Moreover, in biology, when studying the motion of a particle acting under a force perturbed by noise, see for example [26], or stochastic analysis and mathematical finance, where stochastic models lead to equations with unbounded coefficients, e.g., the well known Black-Scholes equation introduced in [9] and some structure models of interest rate derivatives, see for example [11, 20]. This class of operator is a generalisation of the operators with bounded coefficients and historically, in the mathematical literature, the subject is studied using several approaches, with ideas and methods from partial differential equations, Dirichlet forms, stochastic processes, stochastic differential equations. One can wonder about many aspects such as existence, uniqueness, regularity or integral representation of solutions to the abstract Cauchy problem associated to an elliptic operator with unbounded coefficients of the form

$$\mathcal{A}u = \mathrm{div}(aDu) + F \cdot Du + Vu,$$

where V is a real-valued function, $a(x) = (a_{kl}(x))$ satisfies the ellipticity condition and a and F grow to infinity. If one assumes that $\mathcal{A}$ has coefficients belonging to $C^{\alpha}_{loc}(\mathbb{R}^N)$ it is then possible to prove that there exists a positive semigroup $(T(t))_{t\geq 0}$ such that the parabolic problem associated to $\mathcal{A}$

$$\begin{cases} u_t(t,x) = \mathcal{A}u(t,x) & t > 0, x \in \mathbb{R}^N, \\ u(0,x) = f(x) & x \in \mathbb{R}^N, \end{cases}$$

admits a classical solution for every $f \in C_b(\mathbb{R}^N)$ given by $u(t,x) = T(t)f(x)$, see [8]. The solution is constructed through an approximation procedure as the limit of solutions to suitable Dirichlet problems on bounded domains and it is given by a semigroup $T(t)$ applied to the initial datum f. Moreover, this solution admits an integral representation by

$$T(t)f(x) = \int_{\mathbb{R}^N} p(t,x,y)f(y)\,dy$$

with p, the so called integral kernel, a positive function. The semigroup $(T(t))_{t\geq 0}$ is generated by $\mathcal{A}$ in a weak sense and it is often called Markov semigroup in the case where $V \equiv 0$.

The operator $\mathcal{A}$ can also be defined via the sesquilinear form

$$Q(u,v) = \int_{\mathbb{R}^N} a\nabla u \cdot \nabla \overline{v}\,dx - \int_{\mathbb{R}^N} F \cdot \nabla u \overline{v}\,dx - \int_{\mathbb{R}^N} V u \overline{v}\,dx,$$

u, $v \in C_c^\infty(\mathbb{R}^N)$. It is possible to derive properties for the operator $\mathcal{A}$ and generation results through properties of the form Q.

In this manuscript we mainly deal with Schrödinger type operators, i.e., operators with vanishing drift term, $\nabla a + F = 0$. The case of the whole operator is also considered in the sense that a weighted Hardy's inequality for this operators is provided. Finally we will consider the higher order elliptic operator perturbed by a singular potential $A = \Delta^2 - c|x|^{-4}$.

Due to its importance for the strong relation with the Schrödinger operator, in Chapter 1, a survey on the most significant proofs of Hardy inequality appeared in literature is presented. We start with the proof of the Hardy original inequality for the one dimensional case appeared for the first time in 1920 in [34]. He states the following: Given a square-integrable function f on $(0,\infty)$, then f is integrable over the interval $(0,x)$ for each positive x and

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\,dt\right)^2 dx \leq 4\int_0^\infty f^2(x)\,dx.$$

Moreover, the inequality is sharp. Since then, alternative proofs and various generalizations and variants of the inequality occurred. The well known one in $L^2(\mathbb{R}^N)$, for $N \geq 2$, states that

$$\left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{u^2(x)}{|x|^2}\,dx \leq \int_{\mathbb{R}^N} |\nabla u(x)|^2\,dx \tag{1}$$

holds for every $u \in H^1(\mathbb{R}^N)$. We give different proofs of both the inequality itself and the sharpness of the constant through different strategies. There are also several generalisations to the L^p-setting. In particular, we will prove that for every $u \in W^{1,p}(\mathbb{R}^N)$ with compact support, one has

$$\gamma_\alpha \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^2} |x|^\alpha dx \leq \int_{\mathbb{R}^N} |\nabla u|^2 \, |u|^{p-2} |x|^\alpha dx \tag{2}$$

with optimal constant $\gamma_\alpha = \left(\frac{N+\alpha-2}{p}\right)^2$. Furthermore, we generalise Hardy's inequality proving a weighted inequality with respect to a measure $d\mu = \mu(x)\,dx$ satisfying suitable local integrability assumptions in the weighted spaces $L^2_\mu(\mathbb{R}^N) = L^2(\mathbb{R}^N, d\mu)$. We claim that for all $u \in H^1_\mu(\mathbb{R}^N)$ and $c \leq c_{0,\mu}$

$$c \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} \, d\mu \leq \int_{\mathbb{R}^N} |\nabla u|^2 \, d\mu + C_\mu \int_{\mathbb{R}^N} u^2 \, d\mu$$

holds with optimal constant. The interest in studying such an inequality is its relation with the parabolic problem associated to the Kolmogorov operator perturbed by a singular potential

$$Lu = \Delta u + \frac{\nabla \mu}{\mu} \cdot \nabla u + c|x|^{-2} u.$$

Finally we provide a proof of the Rellich inequality for all $u \in H^2(\mathbb{R}^N)$

$$\left(\frac{N(N-4)}{4}\right)^2 \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^4} \, dx \leq \int_{\mathbb{R}^N} |\Delta u(x)|^2 \, dx.$$

In Chapter 2 we focus our attention on the well known Schrödinger operator

$$H = -\Delta - V = -\Delta - c|x|^{-2}$$

$c \in \mathbb{R}$. Schrödinger type operators have widely been studied in literature. Several authors characterize different classes of potentials, domain, kernel estimates, spectrum and eigenvalues and the associated evolution equation, i.e., $u_t = \Delta u + V(x)u$ in $(0, T] \times \mathbb{R}^N$ has been studied, see for example [16, 8, 60, 61]. We limit ourselves in considering the critical potential $V(x) = c|x|^{-2}$ and studying the generation of a semigroup for a suitable realisation of $-H$ in $L^2(\mathbb{R}^N)$. Then, we will state some upper bounds for the heat kernel of the associated semigroup, and finally, we will provide a non-existence result in $L^2(\mathbb{R}^N)$, Theorem 0.1 below. It is known that the potential V is highly singular in the sense that it belongs to a border line case where the strong maximum principle and Gaussian bounds may fail, cf. [3]. Moreover, it is not in the Kato class potentials. If $V \leq \frac{c}{|x|^{2-\varepsilon}}$, then the initial value problem associated to $-H$ is well-posed. But for $\varepsilon = 0$ the problem may not have positive solutions. The characterization of the existence of positive weak solutions to the parabolic problem associated with the operator $-H$, i.e.,

$$\begin{cases} u_t - \Delta u = c|x|^{-2}u & (0,T) \times \mathbb{R}^N, \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{3}$$

was first discovered by Baras and Goldstein in [7]. They prove that a positive weak solution to (3) exists if and only if $c \le c_0(N) := \left(\frac{N-2}{2}\right)^2$. Moreover, Cabré and Martel in [12] provide a simpler proof of the non-existence result. In particular, they define the bottom of the spectrum of the operator H as

$$\lambda_1 = \inf_{\varphi \in H^1(\mathbb{R}^N)\setminus\{0\}} \left(\frac{\int_{\mathbb{R}^N} \left(|\nabla \varphi|^2 - V\varphi^2\right)\, dx}{\int_{\mathbb{R}^N} \varphi^2\, dx} \right).$$

Then, they prove the following.

Theorem 0.1. *For $N \ge 3$, $c \ge 0$, $f \ge 0$, consider the problem*

$$\begin{cases} u_t = \Delta u + \frac{c}{|x|^2}u & t > 0, x \in \mathbb{R}^N, \\ u(0,\cdot) = f \ \in L^2(\mathbb{R}^N). \end{cases} \tag{4}$$

(i) If $\lambda_1 > -\infty$ (that is $c \le c_0(N)$), then there exists a function $u \in C([0,\infty), L^2(\mathbb{R}^N))$, weak solution of (4)*, exponentially bounded, i.e.,*

$$\|u(t)\| \le Me^{\omega t}\|f\|. \tag{5}$$

(ii) If $\lambda_1 = -\infty$ (that is $c > c_0(N)$), then for all $0 \le f \in L^2(\mathbb{R}^N) \setminus \{0\}$ there is no positive weak solution of (4) *satisfying* (5)*.*

It is then clear that the existence of positive solutions to (4) is related to Hardy's inequality (1) on the space $L^2(\mathbb{R}^N)$. The nonexistence of solutions is due to the optimality of the constant in the Hardy inequality. Therefore, studying the bottom of the spectrum is equivalent to studying the Hardy inequality and the sharpness of the best possible constant.

We also give generation results in $L^2(\mathbb{R}^N)$ via form methods. The associated form to H is

$$Q(u,v) = \int_{\mathbb{R}^N} \nabla u \cdot \nabla \overline{v}\, dx - \int \frac{c}{|x|^2} u\overline{v}\, dx$$

with $D(Q) = \{u \in H^1(\mathbb{R}^N) : \||V|^{\frac{1}{2}}u\|_2 < \infty\}$. Studying the properties of the form one is able to prove that the associated operator $-H_2$ in $L^2(\mathbb{R}^N)$

is the generator of a holomorphic strongly continuous contraction semigroup on $L^2(\mathbb{R}^N)$.

Chapter 3 is devoted to the presentation of the work in [23]. We consider the Schrödinger type operator L_0 with unbounded diffusion

$$L_0u = Lu + Vu = (1+|x|^\alpha)\Delta u + \frac{c}{|x|^2}u$$

with $\alpha \geq 0$ and $c \in \mathbb{R}$. The aim is to obtain sufficient conditions on the parameters ensuring that L_0 with a suitable domain generates a quasi-contractive and positivity preserving C_0-semigroup in $L^p(\mathbb{R}^N)$, $1 < p < \infty$. The tools we have at our disposal are the Hardy inequality (2) and generation results for the unperturbed elliptic operator

$$L = (1+|x|^\alpha)\Delta$$

which have been proved in [24, 39, 46]. As said before, the inverse square potential V is highly singular, however, Okazawa in [53] obtains generation results for the operator $H = -\Delta - V$ through perturbation techniques. In particular, he proves that the realisation $-H_p$ of $-H$ in $L^p(\mathbb{R}^N)$, $1 < p < \infty$, with domain $W^{2,p}(\mathbb{R}^N)$, generates a contractive and positive C_0-semigroup in $L^p(\mathbb{R}^N)$ and $C_c^\infty(\mathbb{R}^N)$ is a core for $-H_p$, if $N > 2p$ and

$$c < \frac{(p-1)(N-2p)N}{p^2} := \beta$$

see [53, Theorem 3.11]. In the case where $N \leq 2p$, it is proved that $-H_p$ with domain $D(H_p) = W^{2,p}(\mathbb{R}^N) \cap \{u \in L^p(\mathbb{R}^N) : |x|^{-2}u \in L^p(\mathbb{R}^N)\}$ is m-sectorial if $c < \beta$, see [53, Theorem 3.6]. If one replaces the Laplacian by the operator L similar results can be obtained. Therefore, we treat the operator L_0 as a perturbation of the operator L and we are able to prove the following theorems, where we distinguish the two cases $\alpha \leq 2$ and $\alpha > 2$ because the hypotheses on the unperturbed operator L are different.

Theorem 0.2. *Assume* $0 \leq \alpha \leq 2$. *Set* $k = \min\{\frac{N(p-1)(N-2p)}{p^2}, (p-1)\gamma_0 = \frac{(p-1)(N-2)^2}{p^2}\}$. *If* $2p < N$ *and* $\alpha \leq (N-2)(p-1)$ *then, for every* $c < k$ *the operator* $L + \frac{c}{|x|^2}$ *endowed with the domain* D_p *defined as follows*

$$D_p = \{u \in W^{2,p}(\mathbb{R}^N) : |x|^\alpha |D^2u|, |x|^{\alpha/2}|\nabla u| \in L^p(\mathbb{R}^N)\}$$

generates a contractive positive C_0*-semigroup in* $L^p(\mathbb{R}^N)$. *Moreover,* $C_c^\infty(\mathbb{R}^N)$ *is a core for such an operator. Finally, the closure of* $\left(L + \frac{k}{|x|^2}, D_p\right)$ *generates a contractive positive* C_0*-semigroup in* $L^p(\mathbb{R}^N)$.

Theorem 0.3. *Assume $\alpha > 2$. Set $k = \min\{\frac{N(p-1)(N-2p)}{p^2}, (p-1)\gamma_0\}$. If $\frac{N}{N-2} < p < \frac{N}{2}$ and $\alpha < \frac{N(p-1)}{p}$, then for every $c < k$ the operator $L + \frac{c}{|x|^2}$ endowed with the domain $\widehat{D_p}$*

$$\widehat{D_p} = \{u \in W^{2,p}(\mathbb{R}^N) : |x|^{\alpha-2}u, |x|^{\alpha-1}|\nabla u|, |x|^{\alpha}|D^2 u| \in L^p(\mathbb{R}^N)\}$$

generates a contractive positive C_0-semigroup in $L^p(\mathbb{R}^N)$. Moreover, $C_c^\infty(\mathbb{R}^N)$ is a core for such an operator. Finally, the closure of $\left(L + \frac{k}{|x|^2}, \widehat{D_p}\right)$ generates a contractive positive C_0-semigroup in $L^p(\mathbb{R}^N)$.

When $2p \geq N$, if at least $2p - N \leq \alpha$, then, still relying on an abstact theorem by Okazawa [53, Theorem 1.6], similar results are obtained but with domain $D_p \cap D(|x|^{-2})$ and $\widehat{D_p} \cap D(|x|^{-2})$ respectively. Furthermore, we also discuss the generation of a C_0-semigroup for the operator

$$\widetilde{L}_0 u := (1 + |x|^\alpha)\Delta u - \eta |x|^\beta u + \frac{c}{|x|^2}u,$$

where η is a positive constant, $\alpha > 2$ and $\beta > \alpha - 2$. In this case less conditions on the parameters occur in order to obtain similar theorems as before.

Chapter 4 deals with the results in paper [31]. In particular, we study the biharmonic operator perturbed by an inverse fourth-order potential

$$A = A_0 - V = \Delta^2 - \frac{c}{|x|^4},$$

where c is any constant such that $c < C^* := \left(\frac{N(N-4)}{4}\right)^2$. We prove that the semigroup generated by $-A$ in $L^2(\mathbb{R}^N)$, $N \geq 5$, extrapolates to a bounded holomorphic C_0-semigroup on $L^p(\mathbb{R}^N)$ for all $p \in [p_0', p_0]$ where $p_0 = \frac{2N}{N-4}$ and p_0' is its dual exponent. Observe that the unperturbed operator, $A_0 = \Delta^2$, is studied by Davies in [18]. In particular, he proves that for $N < 4$, $-A_0$ generates a bounded C_0-semigroup on $L^p(\mathbb{R}^N)$ for all $1 \leq p < \infty$ and that Gaussian-type estimates for the heat kernel hold. The result is different when the dimension is greater than the order of the operator, $N > 4$. In this case he proves that the semigroup $(e^{-tA_0})_{t\geq 0}$ on $L^2(\mathbb{R}^N)$ extends to a bounded holomorphic C_0-semigroup on $L^p(\mathbb{R}^N)$ for all $p \in [p_0', p_0]$. An analogous situation holds when one replaces A_0 by A. However, much more recently, Quesada and Rodríguez-Bernal in [56], using abstract parabolic arguments, prove that the biharmonic operator $-A_0$ generates a holomorphic semigroup in some suitable scale spaces $W^{4\alpha,p}(\mathbb{R}^N)$ for every $1 < p < \infty$ without restriction to the dimension N.

We study the operator $A = \Delta^2 - V$ in $L^2(\mathbb{R}^N)$ via quadratic form methods. It is associated with the form

$$\mathfrak{a}(u,v) = \int_{\mathbb{R}^N} \Delta u \Delta \overline{v}\, dx - \int_{\mathbb{R}^N} V u \overline{v}\, dx$$

with $D(\mathfrak{a}) = \{u \in H^2(\mathbb{R}^N) : \||V|^{1/2}u\|_2 < \infty\}$. Studying the properties of the form $\mathfrak{a}$, one obtains generation of a contractive holomorphic C_0-semigroup on $L^2(\mathbb{R}^N)$. Then, making use of multiplication operators and off-diagonal estimates, we prove that, for $N \geq 5$, the semigroup $(e^{-tA})_{t\geq 0}$ extrapolates to a bounded holomorphic C_0-semigroup on $L^p(\mathbb{R}^N)$ for all $p \in [p_0', p_0]$.

Furthermore, we study the boundedness of the Riesz transform

$$\Delta A^{-1/2} := \frac{1}{\Gamma(1/2)} \int_0^\infty t^{-1/2} \Delta e^{-tA}\, dt$$

on $L^p(\mathbb{R}^N)$ for all $p \in (p_0', 2]$. The boundedness of $\Delta A^{-1/2}$ on $L^p(\mathbb{R}^N)$ implies that the domain of $A^{1/2}$ is included in the Sobolev space $W^{2,p}(\mathbb{R}^N)$. Thus, we obtain $W^{2,p}$-regularity of the solution to the evolution equation with initial datum in $L^p(\mathbb{R}^N)$. In the setting of higher order operators, Blunck and Kunstmann in [10] apply the Calderón-Zygmund theory for a non-integral operator L to obtain estimates on $\Delta L^{-1/2}$ since, in general, operators of order $2m$ do not satisfy Gaussian bounds if $2m < N$. More precisely, they prove an abstract criterion for estimates of the type

$$\|BL^{-\alpha} f\|_{L^p(\Omega)} \leq C_p \|f\|_{L^p(\Omega)}, \qquad p \in (q_0, 2],$$

where B, L are linear operators, $\alpha \in [0,1)$, $q_0 \in [1,2)$ and Ω is a measure space. We apply this criterion to our situation $(B, L, \Omega) = (\Delta, A, \mathbb{R}^N)$ with $q_0 = p_0'$ that together with the obtained off-diagonal estimates, leads to the following result.

Theorem 0.4. *The parabolic problem associated to $-A = -\Delta^2 + \frac{c}{|x|^4}$, $c < C^*$,*

$$\begin{cases} \partial_t u(t) = -Au(t) & \text{for } t \geq 0, \\ u(0) = f, \end{cases}$$

admits a unique solution for each initial datum $f \in L^p(\mathbb{R}^N)$, $p \in [p_0', p_0]$. Moreover, if $f \in L^p(\mathbb{R}^N)$ for $p \in (p_0', 2]$, then $u(t) \in W^{2,p}(\mathbb{R}^N)$ for every $t > 0$.

Chapter 1

Hardy's inequalities

Hardy's inequalities play an important role in analysis with several applications in many fields such as partial differential equations, harmonic analysis, spectral theory or quantum mechanics [7, 15, 61]. The aim of this chapter is to survey the most significant proofs of the classical Hardy inequality appeared in literature since 1920, when Hardy first shows the inequality in the one dimensional case. We are also interested in a generalization of the inequality in $L^p(\mathbb{R}^N)$ for $1 < p < \infty$. Furthermore, we will prove some weighted Hardy inequalities with respect to a measure $d\mu$. Finally, we present the Rellich inequality, the analogous of Hardy's for higher order operators.

The main motivation for Hardy in establishing the inequality is the attempt to simplify the proof of Hilbert's double series theorem, [35, Theorem 315]. In fact, in [34], the Hardy inequality is stated for the first time in the one dimensional case and the optimality of the constant is claimed in the sense that it cannot be replaced by any smaller one. The original inequality says the following. Given a square-integrable function f on $(0, \infty)$, then f is integrable over the interval $(0, x)$ for each positive x and

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f(t)\, dt \right)^2 dx \le 4 \int_0^\infty f^2(x)\, dx.$$

Since then, alternative proofs and various generalizations and variants of the inequality occurred. The well known one in $L^2(\mathbb{R}^N)$, for $N \ge 2$, on which we will focus our attention, states that for every $u \in H^1(\mathbb{R}^N)$, $N \ge 2$,

$$\left(\frac{N-2}{2} \right)^2 \int_{\mathbb{R}^N} \frac{u^2(x)}{|x|^2}\, dx \le \int_{\mathbb{R}^N} |\nabla u(x)|^2\, dx. \tag{1.1}$$

When $N = 2$ there is nothing to prove. Therefore, we will concentrate on the case $N \ge 3$. As stated before the constant is sharp, however, it is never achieved. Indeed, the equality in (1.1) is attained considering the radial functions $u(x) = |x|^{-\frac{N-2}{2}}$, which do not belong to $H^1(\mathbb{R}^N)$.

There are several generalizations to the L^p-setting. In particular, we will focus our attention on the inequality

$$\left(\frac{N+\alpha-2}{p}\right)^2 \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^2}|x|^\alpha dx \leq \int_{\mathbb{R}^N} |\nabla u|^2 \, |u|^{p-2}|x|^\alpha dx \tag{1.2}$$

for every $u \in W^{1,p}(\mathbb{R}^N)$ with compact support, $N \geq 3$ and $\alpha \geq 0$.

Hardy's inequality has a strong relation with the Schrödinger operator $H = -\Delta - c|x|^{-2}$, i.e., the Laplacian perturbed by the inverse-square potential. In fact, one of the most important consequences of the inequality concerning evolution equations was showed by Baras and Goldstein. In the pioneering work [7], they establish a relation between (1.1) and the parabolic problem associated to $-H$. In particular, they consider the problem

$$\begin{cases} u_t - \Delta u = c|x|^{-2}u & (0,T) \times \mathbb{R}^N, \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{1.3}$$

with $0 < T \leq \infty$, $u_0 \in L^1_{loc}(\mathbb{R}^N)$ and $u_0 \geq 0$ a.e. In order to understand the criticality of the potential, the best constant in Hardy's inequality is required. They show that, if $0 \leq c \leq \left(\frac{N-2}{2}\right)^2$, then there exists a positive weak solution to (1.3) global in time. If instead, $c > \left(\frac{N-2}{2}\right)^2$, then there does not exist a positive weak solution to (1.3).

1.1 Proofs

We provide different proofs of both the inequality itself and the optimality of the constant. To start with, we consider Hardy's proof for the one dimensional case as it appears in the book by Hardy, Littlewood and Pólya, [35, Theorem 327].

Proposition 1.1. *If $f \in L^2(0,\infty)$ and $F(x) = \int_0^x f(t)\,dt$, then*

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^2 dx < 4\int_0^\infty f^2(x)\,dx, \tag{1.4}$$

unless $f \equiv 0$. The constant 4 is the best possible.

Proof. If $0 < \xi < X$, integrating by parts, we have

$$\begin{aligned}\int_{\xi}^{X} \left(\frac{F(x)}{x}\right)^2 dx &= -\int_{\xi}^{X} F^2(x) \frac{d}{dx} x^{-1}\, dx \\ &= \xi^{-1}F^2(\xi) - X^{-1}F^2(X) + 2\int_{\xi}^{X} x^{-1}F(x)f(x)\, dx.\end{aligned}$$

Since F is a primitive of f, when f^2 is integrable, one has that $F^2(x) = o(x)$. Hence, letting $\xi \to 0$ and applying Hölder's inequality

$$\begin{aligned}\int_0^X \left(\frac{F(x)}{x}\right)^2 dx &< 2\int_0^X \frac{F(x)}{x} f(x)\, dx \\ &\le 2\left(\int_0^X \left(\frac{F(x)}{x}\right)^2 dx\right)^{1/2} \left(\int_0^X f^2(x)\, dx\right)^{1/2}. \qquad (1.5)\end{aligned}$$

If f is different from 0 in $(0, X)$, the left-hand side of (1.5) is positive. Hence, it gives

$$\int_0^X \left(\frac{F(x)}{x}\right)^2 dx \le 4\int_0^X f^2(x)\, dx. \qquad (1.6)$$

When we make $X \to \infty$ we obtain (1.4), except that '$<$' is replaced by '$\le$'. In particular, the integral on the left-hand side of (1.4) is finite. It follows that all the integrals in (1.5) remain finite when X is replaced by ∞, and that

$$\begin{aligned}\int_0^\infty \left(\frac{F(x)}{x}\right)^2 dx &\le 2\int_0^\infty \frac{F(x)}{x} f(x)\, dx \\ &\le 2\left(\int_0^\infty \left(\frac{F(x)}{x}\right)^2 dx\right)^{1/2} \left(\int_0^\infty f^2(x)\, dx\right)^{1/2}. \qquad (1.7)\end{aligned}$$

The last sign of the inequality may be replaced by '$<$' unless $x^{-1}F$ and f are effectively proportional. This would make f a power of x, and then $\int f^2\, dx$ would be divergent. Hence,

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^2 dx < 2\left(\int_0^\infty \left(\frac{F(x)}{x}\right)^2 dx\right)^{1/2} \left(\int_0^\infty f^2(x)\, dx\right)^{1/2} \qquad (1.8)$$

unless f is nul. Since the integral on the left-hand side is positive and finite, (1.4) now follows from (1.8) as (1.6) followed by (1.5).

In order to prove that the constant is the best possible (inequality is sharp), we have to prove that for each $a \leq 4$ the following inequality holds

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^2 dx \geq a \int_0^\infty f^2(x)\, dx \tag{1.9}$$

for some function f satisfying the assumptions in Proposition 1.1. To this purpose, given $\varepsilon > 0$, consider the function $f \in L^2(0,\infty)$

$$f(x) = \begin{cases} 0 & \text{if } x < 1, \\ x^{-\frac{1}{2}-\varepsilon} & \text{if } x \geq 1. \end{cases}$$

Hence, we will have

$$F(x) = \begin{cases} 0 & \text{if } x \leq 1, \\ \frac{2}{1-2\varepsilon}\left(x^{\frac{1}{2}-\varepsilon} - 1\right) & \text{if } x \geq 1. \end{cases}$$

By direct computation we get

$$\int_0^\infty f^2(x)\, dx = \int_1^\infty x^{-1-2\varepsilon}\, dx = \frac{1}{2\varepsilon},$$

and

$$\begin{aligned} \int_0^\infty \frac{F^2(x)}{x^2}\, dx &= \left(\frac{2}{1-2\varepsilon}\right)^2 \int_1^\infty x^{-2}(x^{1-2\varepsilon} - 2x^{\frac{1}{2}-\varepsilon} + 1)\, dx \\ &= \left(\frac{2}{1-2\varepsilon}\right)^2 \left(\frac{1}{2\varepsilon} - \frac{4}{1+2\varepsilon} + 1\right). \end{aligned}$$

Therefore, inequality (1.9) is

$$\left(\frac{2}{1-2\varepsilon}\right)^2 \left(1 - \frac{8\varepsilon}{1+2\varepsilon} + 2\varepsilon\right) \geq a$$

which, letting $\varepsilon \to 0$, holds for every $a \leq 4$. $\square$

We now provide a proof of the classical Hardy inequality (1.1) in $L^2(\mathbb{R}^N)$ for $N \geq 3$. It has been published by several authors, see for example [2, 6]. This proof essentially follows the lines of the one of Hardy for the one dimensional case.

Proposition 1.2. *For each $u \in H^1(\mathbb{R}^N)$, $N \geq 3$, the following inequality holds, with optimal constant,*

$$\left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{u^2(x)}{|x|^2}\, dx \leq \int_{\mathbb{R}^N} |\nabla u(x)|^2\, dx.$$

Proof. By a density argument we can consider $u \in C_c^\infty(\mathbb{R}^N)$. Then the following identity holds,

$$u^2(x) = -\int_1^\infty \frac{d}{dt} u^2(tx)\,dt = -2\int_1^\infty u(tx)x \cdot \nabla u(tx)\,dt.$$

Hence,

$$\int_{\mathbb{R}^N} \frac{u^2(x)}{|x|^2}\,dx = -2\int_{\mathbb{R}^N}\int_1^\infty \frac{u(tx)}{|x|}\frac{x}{|x|}\cdot \nabla u(tx)\,dt\,dx.$$

By a change of variables, Fubini's theorem and Hölder's inequality, it follows that

$$\begin{aligned}
\int_{\mathbb{R}^N} \frac{u^2(x)}{|x|^2}\,dx &= -2\int_1^\infty \frac{dt}{t^{N-1}} \int_{\mathbb{R}^N} \frac{u(y)}{|y|}\nabla u(y)\cdot \frac{y}{|y|}\,dy \\
&= -\frac{2}{N-2}\int_{\mathbb{R}^N} \frac{u(y)}{|y|}\nabla u(y)\cdot \frac{y}{|y|}\,dy \\
&\le \frac{2}{N-2}\left(\int_{\mathbb{R}^N} \frac{u^2(y)}{|y|^2}\,dy\right)^{1/2}\left(\int_{\mathbb{R}^N} |\nabla u(y)|^2\,dy\right)^{1/2}.
\end{aligned}$$

We can then conclude the inequality.

As regards the optimality of the constant, given $\varepsilon > 0$, define the radial function $U \in C(\mathbb{R}^N)$ by $U(x) = \phi(r)$ where $|x| = r$ and ϕ is given by

$$\phi(r) = \begin{cases} A & \text{if } r \in [0,1], \\ Ar^{\frac{2-N}{2}-\varepsilon} & \text{if } r \ge 1, \end{cases}$$

where $A = \frac{2}{N-2+2\varepsilon}$. Now, since the function U does not belong in general to $L^2(\mathbb{R}^N)$, we consider the truncated functions $U_n = \vartheta_n U$ with $\vartheta_n(x) = \vartheta\left(\frac{x}{n}\right)$, ϑ being a smooth function supported in $B(2)$ and $\vartheta = 1$ in $B(1)$. Now $(U_n)_{n\in\mathbb{N}} \subset H^1(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N} \frac{U_n^2}{|x|^2}\,dx \to \int_{\mathbb{R}^N} \frac{U^2}{|x|^2}\,dx$ and $\int_{\mathbb{R}^N} |\nabla U_n|^2\,dx \to \int_{\mathbb{R}^N} |\nabla U|^2\,dx$ as $n \to \infty$.

Indeed, by direct computation, we get

$$\begin{aligned}
\int_{\mathbb{R}^N} |\nabla U_n(x)|^2\,dx = \omega_N \int_1^{2n} & r^{-(1+2\varepsilon)} \vartheta_n^2(r)\,dr \\
& + \frac{1}{n^2} \int_{B(2n)\setminus B(n)} U^2(x) \left|\nabla \vartheta\left(\frac{x}{n}\right)\right|^2 dx \\
& + \frac{2}{n} \int_{B(2n)\setminus B(n)} U(x)\vartheta_n(x)\nabla U(x)\cdot\nabla\vartheta\left(\frac{x}{n}\right) dx \\
& < \frac{1}{A^2} \int_{B(2n)\setminus B(1)} \frac{U_n^2(x)}{|x|^2}\,dx \\
& + \frac{1}{n^2} \int_{B(2n)\setminus B(n)} U^2(x) \left|\nabla \vartheta\left(\frac{x}{n}\right)\right|^2 dx \\
& + \frac{2}{n} \int_{B(2n)\setminus B(n)} U(x)\vartheta_n(x)\nabla U(x)\cdot\nabla\vartheta\left(\frac{x}{n}\right) dx,
\end{aligned}$$

where ω_N is the measure of the $(N-1)-$dimensional unit sphere. Now, letting $n \to \infty$ one obtains for each $\varepsilon > 0$

$$\int_{\mathbb{R}^N} \frac{U^2(x)}{|x|^2}\,dx > \left(\frac{2}{N-2+2\varepsilon}\right)^2 \int_{\mathbb{R}^N} |\nabla U(x)|^2\,dx.$$

This ends the proof. □

The argument of the following proof can be found in [17, p. 166]. Thanks to its simplicity, it can be easily generalized to a large class of operators, for instance to the Ornstein-Uhlenbeck operator or to the sub-Laplacian on the Heisenberg group (see for example [30]).

Proposition 1.3. *For each $\varphi \in C_c^\infty(\mathbb{R}^N)$, $N \geq 3$, the following inequality holds*

$$\left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{\varphi^2(x)}{|x|^2}\,dx \leq \int_{\mathbb{R}^N} |\nabla\varphi(x)|^2\,dx.$$

Proof. Let us consider the functions

$$f_\varepsilon(x) = (\varepsilon + |x|^2)^k$$

with $\varepsilon > 0$ and $k < 0$ to be chosen later. We can compute

$$\frac{\Delta f_\varepsilon(x)}{f_\varepsilon(x)} = \frac{(4k^2 + 2k(N-2))\,|x|^2 + 2Nk\varepsilon}{(\varepsilon + |x|^2)^2}.$$

Minimising the function $s(k) = 4k^2 + 2k(N-2)$ one obtains $k = \frac{2-N}{4}$ and then

$$\frac{\Delta f_\varepsilon(x)}{f_\varepsilon(x)} = \frac{-\left(\frac{N-2}{2}\right)^2 |x|^2 - \frac{N(N-2)}{2}\varepsilon}{(\varepsilon + |x|^2)^2}.$$

Now, take $\varphi \in C_c^\infty(\mathbb{R}^N)$ and set $\psi = \frac{\varphi}{f_\varepsilon}$. Computing and integrating by parts one obtains

$$\int_{\mathbb{R}^N} |\nabla\varphi|^2\,dx + \int_{\mathbb{R}^N} \frac{\Delta f_\varepsilon}{f_\varepsilon}\varphi^2\,dx = \int_{\mathbb{R}^N} |f_\varepsilon \nabla\psi|^2\,dx \geq 0.$$

Hence,

$$-\int_{\mathbb{R}^N} \frac{\Delta f_\varepsilon}{f_\varepsilon}\varphi^2\,dx \leq \int_{\mathbb{R}^N} |\nabla\varphi|^2\,dx.$$

Letting $\varepsilon \to 0$, by Fatou's lemma, inequality (1.1) is proved,

$$\left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2}\,dx \leq \int_{\mathbb{R}^N} |\nabla\varphi|^2\,dx.$$

□

Cabré and Martel in [12] also study the problem (1.3). We will discuss their results later. At this point, we are interested in the fact that they establish a relation between the existence of solutions to the problem (1.3) and the validity of a Hardy-type inequality. In particular, they introduce the *generalised bottom of the spectrum* of the operator $H = -\Delta - V$

$$\lambda_1 = \inf_{\varphi \in H^1(\mathbb{R}^N)\setminus\{0\}} \left(\frac{\int_{\mathbb{R}^N} \left(|\nabla\varphi|^2 - V\varphi^2\right)dx}{\int_{\mathbb{R}^N} \varphi^2 dx} \right)$$

which eventually can be $-\infty$. In fact, condition $\lambda_1 > -\infty$ implies a Hardy-type inequality of the form

$$\int_{\mathbb{R}^N} V\varphi^2\,dx \leq \int_{\mathbb{R}^N} |\nabla\varphi|^2\,dx - \lambda_1 \int_{\mathbb{R}^N} \varphi^2\,dx, \qquad \forall\, \varphi \in H^1(\mathbb{R}^N).$$

Therefore in our case, roughly speaking, inequality (1.1) says that the first eigenvalue of H is nonnegative for all $0 < c < \left(\frac{N-2}{2}\right)^2$. In conclusion, in order to prove the optimality of the constant $\left(\frac{N-2}{2}\right)^2$ in the Hardy inequality (1.1), one has to prove that $\lambda_1 = -\infty$ whenever c is greater than the Hardy constant. We provide the proof of the optimality following this strategy.

Proposition 1.4. *There exists some function $\varphi \in H^1(\mathbb{R}^N)$ such that inequality* (1.1)

$$c\int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2}\,dx \le \int_{\mathbb{R}^N} |\nabla\varphi|^2\,dx$$

does not hold if $c > \left(\frac{N-2}{2}\right)^2$.

Proof. Let γ be such that $\max\{-\sqrt{c}, -\frac{N}{2}\} < \gamma \le \frac{-N+2}{2}$, so that $|x|^{2\gamma} \in L^1_{loc}(\mathbb{R}^N)$ and $|x|^{2\gamma-2} \notin L^1_{loc}(\mathbb{R}^N)$ and $\gamma^2 < c$.

Let $n \in \mathbb{N}$ and $\vartheta \in C_c^\infty(\mathbb{R}^N)$, $0 \le \vartheta \le 1$, $\vartheta = 1$ in $B(1)$ and $\vartheta = 0$ in $B^c(2)$. Set $\varphi_n(x) = \min\{|x|^\gamma\vartheta(x), n^{-\gamma}\}$. We observe that

$$\varphi_n(x) = \begin{cases} \left(\frac{1}{n}\right)^\gamma & \text{if } |x| < \frac{1}{n}, \\ |x|^\gamma & \text{if } \frac{1}{n} \le |x| < 1, \\ |x|^\gamma\vartheta(x) & \text{if } 1 \le |x| < 2, \\ 0 & \text{if } |x| \ge 2. \end{cases}$$

The functions φ_n are in $H^1(\mathbb{R}^N)$.

Let us consider $c > \left(\frac{N-2}{2}\right)^2$. As stated before we have to prove that

$$\lambda_1 = \inf_{\varphi \in H^1(\mathbb{R}^N)\setminus\{0\}} \left(\frac{\int_{\mathbb{R}^N} \left(|\nabla\varphi|^2 - \frac{c}{|x|^2}\varphi^2\right) dx}{\int_{\mathbb{R}^N} \varphi^2 dx} \right)$$

is $-\infty$.

We get

$$\begin{aligned}
\int_{\mathbb{R}^N} \left(|\nabla \varphi_n|^2 - \frac{c}{|x|^2}\varphi_n^2 \right) dx &= \int_{B(1)\setminus B(\frac{1}{n})} \left(|\nabla |x|^\gamma|^2 - \frac{c}{|x|^2}|x|^{2\gamma} \right) dx \\
&\quad + \int_{B(2)\setminus B(1)} |\nabla |x|^\gamma \vartheta(x)|^2 \, dx \\
&\quad - \int_{B(2)\setminus B(1)} \frac{c}{|x|^2} \left(|x|^\gamma \vartheta(x) \right)^2 \, dx \\
&\quad - c \int_{B(\frac{1}{n})} \left(\frac{1}{n} \right)^{2\gamma} \frac{1}{|x|^2} \, dx \\
&\le (\gamma^2 - c) \int_{B(1)\setminus B(\frac{1}{n})} |x|^{2\gamma-2} \, dx \\
&\quad + 2 \int_{B(2)\setminus B(1)} \left(|x|^{2\gamma} |\nabla \vartheta|^2 + \gamma^2 \vartheta^2 |x|^{2\gamma-2} \right) dx \\
&\le (\gamma^2 - c) \int_{B(1)\setminus B(\frac{1}{n})} |x|^{2\gamma-2} \, dx \\
&\quad + 2(\|\nabla \vartheta\|_\infty^2 + \gamma^2) \int_{B(2)\setminus B(1)} dx \\
&= (\gamma^2 - c) \int_{B(1)\setminus B(\frac{1}{n})} |x|^{2\gamma-2} \, dx + C_1 \, . \qquad (1.10)
\end{aligned}$$

On the other hand,

$$\int_{\mathbb{R}^N} \varphi_n^2 \, dx \ge \int_{B(2)\setminus B(1)} |x|^{2\gamma} \vartheta^2(x) \, dx = C_2. \qquad (1.11)$$

Taking into account (1.10) and (1.11) one obtains

$$\lambda_1 \le \frac{\int_{\mathbb{R}^N} \left(|\nabla \varphi_n|^2 - \frac{c}{|x|^2}\varphi_n^2 \right) dx}{\int_{\mathbb{R}^N} \varphi_n^2 \, dx} \le \frac{(\gamma^2 - c) \int_{B(1)\setminus B(\frac{1}{n})} |x|^{2\gamma-2} \, dx + C_1}{C_2}.$$

We observe that $\gamma^2 - c < 0$. Now, letting n go to ∞, we get

$$\lim_{n\to\infty} \int_{B(1)\setminus B(\frac{1}{n})} |x|^{2\gamma-2} \, dx = +\infty$$

and hence $\lambda_1 = -\infty$. □

The last method of proving Hardy's inequality we would like to describe, is the one used by Mitidieri in [49], which essentially consists in applying the divergence theorem to specially chosen vector fields.

Proposition 1.5. *For each $\varphi \in C_c^\infty(\mathbb{R}^N)$, $N \geq 3$, the following inequality holds*

$$\left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2}\, dx \leq \int_{\mathbb{R}^N} |\nabla \varphi|^2\, dx.$$

Proof. For every $\lambda \geq 0$, let us consider the vector field $F(x) = \lambda \frac{x}{|x|^2}$, $x \neq 0$. Note that $F \in W^{1,1}_{loc}(\mathbb{R}^N)$. Integrating by parts and applying Hölder and Young inequalities one obtains

$$\begin{aligned}
\int_{\mathbb{R}^N} \varphi^2 \mathrm{div} F\, dx &= \lambda(N-2) \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2}\, dx \\
&= -2\lambda \int_{\mathbb{R}^N} \varphi \frac{x}{|x|^2} \cdot \nabla \varphi\, dx \\
&\leq 2\lambda \left(\int_{\mathbb{R}^N} |\nabla \varphi|^2\, dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2}\, dx \right)^{\frac{1}{2}} \\
&\leq \int_{\mathbb{R}^N} |\nabla \varphi|^2\, dx + \lambda^2 \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2}\, dx.
\end{aligned}$$

Hence,

$$(\lambda(N-2) - \lambda^2) \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2}\, dx \leq \int_{\mathbb{R}^N} |\nabla \varphi|^2\, dx.$$

By taking the maximum over λ of the function $\psi(\lambda) = \lambda(N-2) - \lambda^2$, we get the result.

□

From now on we assume $N \geq 3$ and $\alpha \geq 0$. We want to prove inequality (1.2) using the vector fields method. We set

$$\gamma_\alpha = \left(\frac{N+\alpha-2}{p}\right)^2.$$

For similar proofs we refer to [53, Lemma 2.2 & Lemma 2.3] for the case $\alpha = 0$ and [46, Appendix] for $\alpha \geq 2$. Here we give a simple proof which holds for any $\alpha \geq 0$, see [23].

Proposition 1.6. *For every $u \in W^{1,p}(\mathbb{R}^N)$ with compact support, one has*

$$\gamma_\alpha \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^2} |x|^\alpha dx \leq \int_{\mathbb{R}^N} |\nabla u|^2 \, |u|^{p-2} |x|^\alpha dx$$

with optimal constant. Moreover, the inequality holds true even if u is replaced by $|u|$.

Proof. By density, it suffices to prove the inequality for $u \in C_c^1(\mathbb{R}^N)$. So, for every $\lambda \geq 0$, let us consider the vector field $F(x) = \lambda \frac{x}{|x|^2}|x|^\alpha$, $x \neq 0$, and set $d\mu(x) = |x|^\alpha dx$. As before, integrating by parts and applying Hölder's and Young's inequalities we get

$$\begin{aligned}\int_{\mathbb{R}^N} |u|^p \mathrm{div} F\, dx &= \lambda(N-2+\alpha)\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^2}\, d\mu \\ &= -p\lambda \int_{\mathbb{R}^N} |u|^{p-2} u \nabla u \cdot \frac{x}{|x|^2}\, d\mu \\ &\leq p\lambda \left(\int_{\mathbb{R}^N} |\nabla u|^2 \, |u|^{p-2}\, d\mu\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^2}\, d\mu\right)^{\frac{1}{2}} \\ &\leq \int_{\mathbb{R}^N} |\nabla u|^2 \, |u|^{p-2}\, d\mu + \frac{\lambda^2 p^2}{4} \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^2}\, d\mu.\end{aligned}$$

Hence,

$$\left[\lambda(N-2+\alpha) - \frac{\lambda^2 p^2}{4}\right] \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^2}|x|^\alpha \, dx \leq \int_{\mathbb{R}^N} |\nabla u|^2 \, |u|^{p-2} |x|^\alpha \, dx.$$

By taking the maximum over λ of the function $\psi(\lambda) = \lambda(N-2+\alpha) - \lambda^2 p^2/4$, we get the result.
We note here that the integration by parts is straightforward when $p \geq 2$. For $1 < p < 2$, $|u|^{p-2}$ becomes singular near the zeros of u. Also in this case the integration by parts is allowed, see [45].

By using the identity $\nabla |u|^p = p|u|^{p-1}\nabla |u|$ in the computations above, the statement continues to hold true with u replaced by $|u|$.

As regards the optimality of the constant γ_α, given $\varepsilon > 0$, define the radial function $U \in C(\mathbb{R}^N)$ by $U(x) = \phi(r)$ where $|x| = r$ and ϕ is given by

$$\phi(r) = \begin{cases} A & \text{if } r \in [0,1], \\ A r^{-\frac{1}{A}} & \text{if } r \geq 1, \end{cases}$$

where $A = \frac{p}{N+\alpha-2+\varepsilon}$. As before, since the function U does not belong in general to $W^{1,p}(\mathbb{R}^N)$ and does not have compact support, we consider the truncated functions $U_n = \vartheta_n U$ with $\vartheta_n(x) = \vartheta\left(\frac{x}{n}\right)$, ϑ being a smooth function supported in $B(2)$ and $\vartheta = 1$ in $B(1)$. Now one has $(U_n)_{n\in\mathbb{N}} \subset W^{1,p}(\mathbb{R}^N)$ with compact support and $\int_{\mathbb{R}^N} |U_n|^p |x|^{\alpha-2}\, dx \to \int_{\mathbb{R}^N} |U|^p |x|^{\alpha-2}\, dx$ and $\int_{\mathbb{R}^N} |\nabla U_n|^2 |U_n|^{p-2} |x|^\alpha \, dx \to \int_{\mathbb{R}^N} |\nabla U|^2 |U|^{p-2}|x|^\alpha \, dx$ as $n \to \infty$. Indeed, by direct computation, we get

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla U_n|^2 |U_n|^{p-2}|x|^\alpha\,dx &= \omega_N A^{p-2}\int_1^{2n} r^{-(1+\varepsilon)}|\vartheta_n(r)|^p\,dr\\
&\quad+\frac{1}{n^2}\int_{B(2n)\setminus B(n)} U^2|U_n|^{p-2}\left|\nabla\vartheta\left(\frac{x}{n}\right)\right|^2|x|^\alpha\,dx\\
&\quad+\frac{2}{n}\int_{B(2n)\setminus B(n)} U\vartheta_n|U_n|^{p-2}\nabla U\cdot\nabla\vartheta\left(\frac{x}{n}\right)|x|^\alpha\,dx\\
&< \frac{1}{A^2}\int_{B(2n)\setminus B(1)}|U_n|^p|x|^{\alpha-2}\,dx\\
&\quad+\frac{1}{n^2}\int_{B(2n)\setminus B(n)}|U|^p|\vartheta_n|^{p-2}\left|\nabla\vartheta\left(\frac{x}{n}\right)\right|^2|x|^\alpha\,dx\\
&\quad+\frac{2}{n}\int_{B(2n)\setminus B(n)} U\vartheta_n|U_n|^{p-2}\nabla U\cdot\nabla\vartheta\left(\frac{x}{n}\right)|x|^\alpha\,dx,\end{aligned}$$

where ω_N is the measure of the $(N-1)-$dimensional unit sphere. Now, letting $n\to\infty$ one obtains for each $\varepsilon>0$

$$\int_{\mathbb{R}^N}|U|^p|x|^{\alpha-2}\,dx > \left(\frac{p}{N+\alpha-2+\varepsilon}\right)^2\int_{\mathbb{R}^N}|\nabla U|^2|U|^{p-2}|x|^\alpha\,dx.$$

This ends the proof. □

1.2 Weighted Hardy's inequalities

In [13] we provide a generalization of the Hardy inequality. Under suitable assumptions on μ, we prove a weighted inequality with respect to a measure $d\mu=\mu(x)dx$ in the weighted space $L^2_\mu(\mathbb{R}^N):=L^2(\mathbb{R}^N,d\mu)$. We claim that, for all $u\in H^1_\mu(\mathbb{R}^N)$, $c\le c_{0,\mu}$, the following inequality

$$c\int_{\mathbb{R}^N}\frac{u^2}{|x|^2}\,d\mu \le \int_{\mathbb{R}^N}|\nabla u|^2\,d\mu + C_\mu\int_{\mathbb{R}^N}u^2\,d\mu \qquad (1.12)$$

holds with optimal constant. The inequality is related to the following Kolmogorov equation perturbed by a singular potential

$$Lu+Vu=\left(\Delta u+\frac{\nabla\mu}{\mu}\cdot\nabla u\right)+\frac{c}{|x|^2}u.$$

The reason for studying such an inequality is, as before, the willingness of establishing the existence of positive weak solutions to the parabolic problem associated to the operator $L+V$. Recently, in [28] and [29], the operator L perturbed by the inverse square potential V is considered and the associated evolution equation

$$\begin{cases} \partial_t u(t,x) = Lu(t,x) + V(x)u(t,x) & t>0,\ x \in \mathbb{R}^N, \\ u(0,\cdot) = u_0 \in L^2_\mu(\mathbb{R}^N), \end{cases} \tag{1.13}$$

is studied. Following the Cabré-Martel idea it is proved that there still is a relation between the weak solution of (1.13) and the bottom of the spectrum of the operator $-(L+V)$

$$\lambda_1 := \inf_{\varphi \in H^1_\mu(\mathbb{R}^N)\setminus\{0\}} \left(\frac{\int_{\mathbb{R}^N} |\nabla \varphi|^2\, d\mu - \int_{\mathbb{R}^N} V\varphi^2\, d\mu}{\int_{\mathbb{R}^N} \varphi^2\, d\mu} \right).$$

Therefore, studying the bottom of the spectrum is equivalent to studying the Hardy inequality (1.12) in the weighted space $L^2_\mu(\mathbb{R}^N)$ and the sharpness of the best constant possible.

In particular, in [29], a weighted Hardy inequality for the density measure $\mu(x)$ satisfying $\mu(x) = Ke^{-\sigma(x)}$ with $c_1|x|^2 \le \sigma(x) \le c_2|x|^2$ is obtained. Under more general hypotheses on $d\mu$ the argument is extended to a larger class of Kolmogorov operators, including for example the operator $Au = \Delta u - |x|^r x\cdot\nabla u + \frac{c}{|x|^2}u$ with $r>0$ and $c \le c_\sigma$, however, the optimality of the constant c_σ is not obtained.

With the purpose of generalizing these results for a larger range of Kolmogorov type operators, we look for conditions on the measure $d\mu$ in order that (1.12) holds with optimal constant.

Observe that a special case is given when $\mu_A(x) = e^{-\frac{1}{2}\langle Ax,x\rangle}$, where A is a positive real Hermitian $N\times N$ matrix. The operator L becomes the well-known symmetric Ornstein-Uhlenbeck operator $Lu = \Delta u - Ax\cdot\nabla u$, which has been extensively studied in literature.

From now on we denote by $c_0(N) = \left(\frac{N-2}{2}\right)^2$ the best constant in Hardy inequality (1.1). Observe that, assuming some regularity assumptions on μ, the operator $L+V$ in $L^2_\mu(\mathbb{R}^N)$ is equivalent to the Schrödinger operator $H = \Delta + (U_\mu + V)$ in $L^2(\mathbb{R}^N)$ where

$$U_\mu := \frac{1}{4}\left|\frac{\nabla\mu}{\mu}\right|^2 - \frac{1}{2}\frac{\Delta\mu}{\mu}.$$

Indeed, taking the transformation $T\varphi = \frac{1}{\sqrt{\mu}}\varphi$ we have $L+V = THT^{-1}$. Now, roughly speaking, for $V = \frac{c}{|x|^2}$, we expect Hardy's inequality to hold if $U_\mu + \frac{c}{|x|^2} \le \frac{c_0(N)}{|x|^2}$ in a neighbourhood of the origin, that is $c \le c_0(N) - |x|^2 U_\mu$. Thus, in order to obtain a weighted Hardy's inequality, we consider the following hypotheses on $\mu(x)$:

$(H1)$ $\mu \ge 0$, $\mu^{\frac{1}{2}} \in H^1_{loc}(\mathbb{R}^N)$, $\Delta\mu \in L^1_{loc}(\mathbb{R}^N)$ and

$$c_{0,\mu} := \liminf_{x\to 0} \left(c_0(N) - |x|^2 U_\mu\right) \qquad \text{is finite;}$$

$(H2)$ for every $R > 0$ the function

$$U := U_\mu - \frac{c_0(N) - c_{0,\mu}}{|x|^2} \tag{1.14}$$

is bounded from above in $\mathbb{R}^N \setminus B(R)$ and, there exists a $R_0 > 0$ such that

$$|x|^2 U \le \frac{1}{4}\frac{1}{|\log|x||^2}$$

holds for $x \in B(R_0)$.

Under the assumptions $(H1)$ and $(H2)$ we are able to obtain (1.12) for all $c \le c_{0,\mu}$. When condition $(H2)$ is not fulfilled we still obtain the weighted Hardy inequality if we only assume

$(H2)'$ U bounded from above in $\mathbb{R}^N \setminus B(R)$ for every $R > 0$,

however, in this case, the constant $c_{0,\mu}$ is not achieved.

As regards the optimality of the constant, if

$(H3)$ there exists $\sup_{\delta\in\mathbb{R}} \left\{\frac{1}{|x|^\delta} \in L^1_{loc}(\mathbb{R}^N, d\mu)\right\} := N_0$,

then the weighted Hardy inequality (1.12) does not hold for $c > c_0(N_0) = \left(\frac{N_0-2}{2}\right)^2$. If, instead, we have

$(H3)'$ there exists $\sup_{\delta\in\mathbb{R}} \left\{\frac{1}{|x|^\delta} \in L^1_{loc}(\mathbb{R}^N, d\mu)\right\} := N_0$ and some $r > 0$ such that

$$\limsup_{\lambda\to 0^+} \lambda \int_{B(r)} |x|^{\lambda - N_0}\, d\mu = +\infty,$$

then the inequality does not hold for $c \geq c_0(N_0)$.

These hypotheses on μ allow us to treat the case $\mu(x) = ke^{-\sigma(x)}$ with $\sigma(x) = b|x|^m$ for $m, b > 0$ obtaining the operator

$$Au = \Delta u - bm|x|^{m-2}x \cdot \nabla u + \frac{c}{|x|^2}u.$$

For $m > 0$ the best constant $c_0(N_0) = c_{0,\mu} = c_0(N)$ is achieved. Moreover, under the same assumptions, one can also consider the density measure $\mu(x) = \frac{1}{|x|^\beta}e^{-\sigma(x)}$. In this case the best constant depends upon the parameter β and one needs $\beta < N - 2$. We have explicitly $N_0 = N - \beta$ and then $c_{0,\mu} = c_0(N_0) = \left(\frac{N-\beta-2}{2}\right)^2$. For this measure we obtain the operator

$$Au = \Delta u - (\beta + bm|x|^m)\frac{x}{|x|^2} \cdot \nabla u + \frac{c}{|x|^2}u.$$

If one considers the measure $\mu(x) = \frac{1}{|x|^\beta}$, then the well known Caffarelli-Nirenberg inequality is obtained. For $\varphi \in H^1(\mathbb{R}^N)$, $\beta < N - 2$,

$$\left(\frac{N-\beta-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2}|x|^{-\beta}\,dx \leq \int_{\mathbb{R}^N} |\nabla\varphi|^2|x|^{-\beta}\,dx.$$

Finally, we can multiply the measure by a logarithmic term near the origin $\mu(x) \sim \left(\log\frac{1}{|x|}\right)^\alpha$. In this case one obtains the weighted Hardy inequality with constant $c_{0,\mu} = c_0(N_0) = c_0(N)$. The constant is the best one, however, in the case $\alpha > 0$ this measure satisfies Hypotheses $(H2)'$ and $(H3)'$, then $c_0(N)$ is not achieved.

1.2.1 The inequality

Let $d\mu$ be a measure (not necessary a probability measure) with density $\mu(x)$. We begin with the proof of the following improved Hardy inequality.

Proposition 1.7. *Assume $(H1)$. Then, for any $\varphi \in C_c^\infty(\mathbb{R}^N)$, the following inequality holds*

$$c_{0,\mu}\int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2}\,d\mu \leq \int_{\mathbb{R}^N} |\nabla\varphi|^2\,d\mu + \int_{\mathbb{R}^N} U\varphi^2\,d\mu, \tag{1.15}$$

where U is given by (1.14).

Proof. One has

$$\begin{aligned}\varphi(x)\sqrt{\mu(x)} &= -\int_1^\infty \frac{d}{dt}\left(\varphi(tx)\sqrt{\mu(tx)}\right)\, dt\\ &= -\int_1^\infty x\cdot\left(\nabla\varphi(tx)+\frac{1}{2}\varphi(tx)\frac{\nabla\mu(tx)}{\mu(tx)}\right)\sqrt{\mu(tx)}\, dt.\end{aligned}$$

By Minkowski inequality for integrals and a change of variables we have

$$\begin{aligned}\left\|\frac{\varphi}{|x|}\right\|_{L^2_\mu} &= \left\|\frac{\varphi\sqrt{\mu(x)}}{|x|}\right\|_{L^2}\\ &\le \left\|\int_1^\infty \left|\frac{x}{|x|}\cdot\left(\nabla\varphi(tx)+\frac{1}{2}\varphi(tx)\frac{\nabla\mu(tx)}{\mu(tx)}\right)\sqrt{\mu(tx)}\right|\, dt\right\|_{L^2}\\ &\le \int_1^\infty \left\|\frac{x}{|x|}\cdot\left(\nabla\varphi(tx)+\frac{1}{2}\varphi(tx)\frac{\nabla\mu(tx)}{\mu(tx)}\right)\sqrt{\mu(tx)}\right\|_{L^2} dt\\ &= \int_1^\infty \left(\int_{\mathbb{R}^N}\left|\nabla\varphi(tx)+\frac{1}{2}\varphi(tx)\frac{\nabla\mu(tx)}{\mu(tx)}\right|^2\mu(tx)\, dx\right)^{\frac{1}{2}} dt\\ &= \int_1^\infty t^{-\frac{N}{2}}\, dt\left\|\nabla\varphi+\frac{1}{2}\varphi\frac{\nabla\mu}{\mu}\right\|_{L^2_\mu}\\ &= \frac{1}{\sqrt{c_0(N)}}\left\|\nabla\varphi+\frac{1}{2}\varphi\frac{\nabla\mu}{\mu}\right\|_{L^2_\mu}.\end{aligned}$$

Hence,

$$\begin{aligned}c_0(N)\int_{\mathbb{R}^N}\frac{\varphi^2}{|x|^2}\, d\mu &\le \int_{\mathbb{R}^N}|\nabla\varphi|^2\, d\mu+\frac{1}{4}\int_{\mathbb{R}^N}\varphi^2\left|\frac{\nabla\mu}{\mu}\right|^2 d\mu+\int_{\mathbb{R}^N}\nabla\varphi\cdot\nabla\mu\frac{\varphi}{\mu}\, d\mu\\ &= \int_{\mathbb{R}^N}|\nabla\varphi|^2\, d\mu+\frac{1}{4}\int_{\mathbb{R}^N}\varphi^2\left|\frac{\nabla\mu}{\mu}\right|^2 d\mu+\frac{1}{2}\int_{\mathbb{R}^N}\nabla\varphi^2\cdot\nabla\mu\, dx\\ &= \int_{\mathbb{R}^N}|\nabla\varphi|^2\, d\mu+\frac{1}{4}\int_{\mathbb{R}^N}\varphi^2\left|\frac{\nabla\mu}{\mu}\right|^2 d\mu-\frac{1}{2}\int_{\mathbb{R}^N}\varphi^2\Delta\mu\, dx\\ &= \int_{\mathbb{R}^N}|\nabla\varphi|^2\, d\mu+\frac{1}{4}\int_{\mathbb{R}^N}\varphi^2\left|\frac{\nabla\mu}{\mu}\right|^2 d\mu-\frac{1}{2}\int_{\mathbb{R}^N}\varphi^2\frac{\Delta\mu}{\mu}\, d\mu\\ &= \int_{\mathbb{R}^N}|\nabla\varphi|^2\, d\mu+\int_{\mathbb{R}^N}U_\mu\varphi^2\, d\mu.\end{aligned}$$

Then, (1.15) follows by the relation

$$U_\mu = U+\frac{c_0(N)-c_{0,\mu}}{|x|^2}.$$

□

We observe that, under assumption $(H1)$, $C_c^\infty(\mathbb{R}^N)$ is dense in $H^1_\mu(\mathbb{R}^N)$, [62, Theorem 1.1]. If moreover U is bounded from above in the whole space, the result below is a direct consequence of Proposition 1.7.

Corollary 1.8. *Assume $(H1)$ and that there exists $C_\mu \in \mathbb{R}$ such that $U \leq C_\mu$. Then, for any $u \in H^1_\mu(\mathbb{R}^N)$,*

$$c_{0,\mu}\int_{\mathbb{R}^N}\frac{u^2}{|x|^2}d\mu \leq \int_{\mathbb{R}^N}|\nabla u|^2 d\mu + C_\mu\int_{\mathbb{R}^N}u^2 d\mu.$$

If we assume the weaker condition of boundedness from above of U only for $|x|$ large enough, we obtain the following result.

Proposition 1.9. *Let us assume that hypotheses $(H1)$ and $(H2)'$ hold. Then for every $c < c_{0,\mu}$ there exists C_μ such that for any $u \in H^1_\mu(\mathbb{R}^N)$ the following inequality holds,*

$$c\int_{\mathbb{R}^N}\frac{u^2}{|x|^2}d\mu \leq \int_{\mathbb{R}^N}|\nabla u|^2 d\mu + C_\mu\int_{\mathbb{R}^N}u^2 d\mu.$$

Proof. Since $\limsup_{x\to 0}|x|^2 U = 0$ we have that for every $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that $U \leq \frac{\varepsilon}{|x|^2}$ for every $x \in B(R_\varepsilon)$ and, moreover, there exists C_μ depending on R_ε such that $U \leq C_\mu$ for every $x \in B^c(R_\varepsilon)$. Then, by Proposition 1.7, we have

$$\begin{aligned}
c_{0,\mu}\int_{\mathbb{R}^N}\frac{u^2}{|x|^2}\,d\mu &\leq \int_{\mathbb{R}^N}|\nabla u|^2 d\mu + \int_{\mathbb{R}^N}U u^2\,d\mu \\
&\leq \int_{\mathbb{R}^N}|\nabla u|^2 d\mu + \varepsilon\int_{B(R_\varepsilon)}\frac{u^2}{|x|^2}\,d\mu + C_\mu\int_{B^c(R_\varepsilon)}u^2\,d\mu \\
&\leq \int_{\mathbb{R}^N}|\nabla u|^2\,d\mu + \varepsilon\int_{\mathbb{R}^N}\frac{u^2}{|x|^2}d\mu + C_\mu\int_{\mathbb{R}^N}u^2\,d\mu.
\end{aligned}$$

The result is obtained taking $c = c_{0,\mu} - \varepsilon$. □

We look for weaker conditions with respect to the boundedness from above for U in $\mathbb{R}^N$ in order to get (1.12). To this purpose, we have to consider improved Hardy's inequalities.

The first step is to state a relation between the weighted Hardy inequality and a special improved Hardy's inequality.

Lemma 1.10. *Assume Hypothesis (H1), and the improved Hardy inequality*

$$c_0(N)\int \frac{u^2}{|x|^2}dx - \int |\nabla u|^2 dx + \int U(x)u^2 dx \le C_\mu \int_{\mathbb{R}^N} u^2 dx,\ \ u \in C_c^\infty(\mathbb{R}^N). \tag{1.16}$$

Then, the weighted Hardy inequality holds

$$c_{0,\mu}\int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2}d\mu \le \int_{\mathbb{R}^N} |\nabla \varphi|^2 d\mu + C_\mu \int_{\mathbb{R}^N} \varphi^2 d\mu,\ \ \varphi \in C_c^\infty(\mathbb{R}^N). \tag{1.17}$$

Proof. Let $\varphi \in C_c^\infty(\mathbb{R}^N)$ and set $u := \varphi\sqrt{\mu} \in H^1(\mathbb{R}^N)$ with compact support. By (1.16), which holds by density for such a function u, integrating by parts and recalling the expression (1.14) for U, one obtains

$$\begin{aligned}
c_0(N)\int_{\mathbb{R}^N} \frac{u^2}{|x|^2}\,dx &= c_0(N)\int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2}\,d\mu \\
&\le \int_{\mathbb{R}^N} |\nabla \varphi\sqrt{\mu}|^2\,dx - \int_{\mathbb{R}^N} U(x)\varphi^2\,d\mu + C_\mu\int_{\mathbb{R}^N}\varphi^2\,d\mu \\
&= \int_{\mathbb{R}^N} |\nabla\varphi|^2\,d\mu + \frac{1}{2}\int_{\mathbb{R}^N}\nabla\varphi^2\nabla\mu\,dx + \frac{1}{4}\int_{\mathbb{R}^N}\varphi^2\left|\frac{\nabla\mu}{\mu}\right|^2\,d\mu \\
&\quad - \int_{\mathbb{R}^N} U_\mu\varphi^2\,d\mu + c_0(N)\int_{\mathbb{R}^N}\frac{\varphi^2}{|x|^2}\,d\mu \\
&\quad - c_{0,\mu}\int_{\mathbb{R}^N}\frac{\varphi^2}{|x|^2}\,d\mu + C_\mu\int_{\mathbb{R}^N}\varphi^2\,d\mu \\
&= \int_{\mathbb{R}^N} |\nabla\varphi|^2\,d\mu + C_\mu\int_{\mathbb{R}^N}\varphi^2\,d\mu \\
&\quad + c_0(N)\int_{\mathbb{R}^N}\frac{\varphi^2}{|x|^2}\,d\mu - c_{0,\mu}\int_{\mathbb{R}^N}\frac{\varphi^2}{|x|^2}\,d\mu.
\end{aligned}$$

Then, inequality (1.17) follows. □

Therefore, in order to prove (1.16) we will use the following theorem, see [51].

Theorem 1.11. *For any $\varphi \in C_c^\infty(B(1))$ the following inequality holds*

$$\int_{B(1)} |\nabla\varphi|^2\,dx \ge c_0(N)\int_{B(1)}\frac{\varphi^2}{|x|^2}\,dx + \frac{1}{4}\int_{B(1)}\frac{\varphi^2}{|x|^2|\log|x||^2}\,dx. \tag{1.18}$$

Now, we suppose that U satisfies the condition $(H2)$. We finally obtain the weighted Hardy inequality (1.12).

Theorem 1.12. *Assume that hypotheses* $(H1)$ *and* $(H2)$ *hold. Then for any* $u \in H^1_\mu(\mathbb{R}^N)$*, the following inequality holds*

$$c_{0,\mu} \int_{\mathbb{R}^N} \frac{u^2}{|x|^2}\, d\mu \le \int_{\mathbb{R}^N} |\nabla u|^2\, d\mu + C_\mu \int u^2\, d\mu.$$

Proof. By Lemma 1.10 we need to prove that

$$c_0(N) \int \frac{u^2}{|x|^2} dx - \int |\nabla u|^2\, dx + \int U(x) u^2\, dx \le C_\mu \int u^2\, dx, \ \forall\ u \in C_c^\infty(\mathbb{R}^N).$$

By Hypothesis $(H2)$ on U, there exists a $R \le 1$ (otherwise one takes $R = 1$) such that $U \le \frac{1}{4} \frac{1}{|x|^2 |\log |x||^2}$ in $B(R)$. Then, if $u \in C_c^\infty(B(R))$, by a changing of variables and (1.18), one has

$$\begin{aligned}
\int_{B(R)} U u^2\, dx &\le \frac{1}{4} \int_{B(R)} \frac{u^2}{|x|^2 |\log |x||^2}\, dx \\
&\le \frac{R^{N-2}}{4} \int_{B(1)} \frac{u^2(Ry)}{|y|^2 |\log |y||^2}\, dy \\
&\le R^N \int_{B(1)} |\nabla u(Ry)|^2\, dy - R^{N-2} c_0(N) \int_{B(1)} \frac{u^2(Ry)}{|y|^2}\, dy \\
&= \int_{B(R)} |\nabla u|^2\, dx - c_0(N) \int_{B(R)} \frac{u^2}{|x|^2}\, dx. \qquad (1.19)
\end{aligned}$$

Let $u \in C_c^\infty(\mathbb{R}^N)$ and $\theta \in C_c^\infty(\mathbb{R}^N)$, $0 \le \theta \le 1$, such that $\theta = 1$ in $B(R/2)$ e $\theta = 0$ in $B^c(R)$, and $\frac{1}{2} \frac{|\nabla \theta|^2}{\theta} - \Delta \theta \le M$. Note that such a function exists. For instance, one can consider the function $\theta_0(s) = c e^{-\frac{1}{1-s^2}}$ for $|s| \le 1$ and equals to 0 for $|s| \ge 1$ and then a translation and a dilatation of this function

$$\theta(x) = \begin{cases} 1 & |x| \le \frac{R}{2}, \\ \theta_0\left(\left(|x| - \frac{R}{2}\right) \frac{2}{R}\right) & \frac{R}{2} \le |x| \le R, \\ 0 & |x| \ge R. \end{cases}$$

Therefore, by (1.19) and Hypothesis ($H2$), one obtains

$$
\begin{aligned}
c_0(N)\int_{\mathbb{R}^N}\frac{u^2}{|x|^2}\,dx+\int_{\mathbb{R}^N}Uu^2\,dx \\
&= c_0(N)\int_{B(R)}\frac{u^2}{|x|^2}\theta\,dx+\int_{B(R)}Uu^2\theta\,dx \\
&\quad + c_0(N)\int_{B^c(R/2)}\frac{u^2}{|x|^2}(1-\theta)\,dx+\int_{B^c(R/2)}Uu^2(1-\theta)\,dx \\
&\le \int_{B(R)}\left|\nabla\left(u\sqrt{\theta}\right)\right|^2\,dx+\left(\frac{c_0(N)}{(R/2)^2}+K\right)\int_{B^c(R/2)}u^2\,dx \\
&\le \int_{\mathbb{R}^N}|\nabla u|^2\theta\,dx+\int_{\mathbb{R}^N}u^2\left(\frac{1}{4}\frac{|\nabla\theta|^2}{\theta^2}-\frac{1}{2}\frac{\Delta\theta}{\theta}\right)\theta\,dx \\
&\quad + C\int_{\mathbb{R}^N}u^2\,dx \\
&\le \int_{\mathbb{R}^N}|\nabla u|^2\,dx+\int_{\mathbb{R}^N}u^2\left(\frac{1}{4}\frac{|\nabla\theta|^2}{\theta}-\frac{1}{2}\Delta\theta\right)\,dx \\
&\quad + C\int_{\mathbb{R}^N}u^2\,dx \\
&\le \int_{\mathbb{R}^N}|\nabla u|^2dx+(M+C)\int_{\mathbb{R}^N}u^2\,dx.
\end{aligned}
$$

□

1.2.2 Optimality of the constant

In this subsection we prove the sharpness of the constant. Firstly, observe that the bottom of the spectrum of the operator $-L-V=-\Delta-\frac{\nabla\mu}{\mu}\cdot\nabla-\frac{c}{|x|^2}$ is

$$
\lambda_1=\inf_{\varphi\in H^1_\mu(\mathbb{R}^N)\setminus\{0\}}\left(\frac{\int_{\mathbb{R}^N}\left(|\nabla\varphi|^2-\frac{c}{|x|^2}\varphi^2\right)\,d\mu}{\int_{\mathbb{R}^N}\varphi^2\,d\mu}\right).
$$

If Hypothesis ($H3$) holds, arguing as in Proposition 1.4, we obtain the following result.

Proposition 1.13. *Let us assume Hypothesis* ($H3$)*. Then, there exists a function in* $H^1_\mu(\mathbb{R}^N)$ *for which the weighted Hardy inequality* (1.12) *does not hold if* $c>c_0(N_0)$.

Proof. Let γ be such that $\max\{-\sqrt{c}, -\frac{N_0}{2}\} < \gamma < \min\{\frac{-N_0+2}{2}, 0\}$ so that $|x|^{2\gamma} \in L^1_{loc}(\mathbb{R}^N, d\mu)$ and $|x|^{2\gamma-2} \notin L^1_{loc}(\mathbb{R}^N, d\mu)$ and $\gamma^2 < c$.

Let $n \in \mathbb{N}$ and $\vartheta \in C_c^\infty(\mathbb{R}^N)$, $0 \le \vartheta \le 1$, $\vartheta = 1$ in $B(1)$ and $\vartheta = 0$ in $B^c(2)$. Set $\varphi_n(x) = \min\{|x|^\gamma \vartheta(x), n^{-\gamma}\}$. We observe that

$$\varphi_n(x) = \begin{cases} \left(\frac{1}{n}\right)^\gamma & \text{if } |x| < \frac{1}{n}, \\ |x|^\gamma & \text{if } \frac{1}{n} \le |x| < 1, \\ |x|^\gamma \vartheta(x) & \text{if } 1 \le |x| < 2, \\ 0 & \text{if } |x| \ge 2. \end{cases}$$

The functions φ_n are in $H^1_\mu(\mathbb{R}^N)$.

Let us consider $c > c_0(N_0)$, then, we have to prove that

$$\lambda_1 = \inf_{\varphi \in H^1(\mathbb{R}^N, d\mu) \setminus \{0\}} \left(\frac{\int_{\mathbb{R}^N} \left(|\nabla \varphi|^2 - \frac{c}{|x|^2} \varphi^2 \right) d\mu}{\int \varphi^2 \, d\mu} \right)$$

is $-\infty$. We get

$$\begin{aligned} \int_{\mathbb{R}^N} \left(|\nabla \varphi_n|^2 - \frac{c}{|x|^2} \varphi_n^2 \right) d\mu &= \int_{B(1) \setminus B\left(\frac{1}{n}\right)} \left(|\nabla |x|^\gamma|^2 - \frac{c}{|x|^2} |x|^{2\gamma} \right) d\mu \\ &\quad + \int_{B(2) \setminus B(1)} |\nabla |x|^\gamma \vartheta(x)|^2 \, d\mu \\ &\quad - \int_{B(2) \setminus B(1)} \frac{c}{|x|^2} \left(|x|^\gamma \vartheta(x) \right)^2 \, d\mu \\ &\quad - c \int_{B\left(\frac{1}{n}\right)} \left(\frac{1}{n} \right)^{2\gamma} \frac{1}{|x|^2} \, d\mu \\ &\le (\gamma^2 - c) \int_{B(1) \setminus B\left(\frac{1}{n}\right)} |x|^{2\gamma-2} \, d\mu \\ &\quad + 2 \int_{B(2) \setminus B(1)} |x|^{2\gamma} |\nabla \vartheta|^2 \, d\mu \\ &\quad + 2\gamma^2 \int_{B(2) \setminus B(1)} \vartheta^2 |x|^{2\gamma-2} \, d\mu \\ &\le (\gamma^2 - c) \int_{B(1) \setminus B\left(\frac{1}{n}\right)} |x|^{2\gamma-2} \, d\mu \\ &\quad + 2(\|\nabla \vartheta\|_\infty^2 + \gamma^2) \int_{B(2) \setminus B(1)} d\mu \\ &= (\gamma^2 - c) \int_{B(1) \setminus B\left(\frac{1}{n}\right)} |x|^{2\gamma-2} \, d\mu + C_1. \end{aligned} \tag{1.20}$$

On the other hand,

$$\int_{\mathbb{R}^N} \varphi_n^2\, d\mu \geq \int_{B(2)\setminus B(1)} |x|^{2\gamma}\vartheta^2(x)\, d\mu = C_2\ . \tag{1.21}$$

Taking into account (1.20) and (1.21) we have

$$\lambda_1 \leq \frac{\int_{\mathbb{R}^N}\left(|\nabla\varphi_n|^2 - \frac{c}{|x|^2}\varphi_n^2\right) d\mu}{\int_{\mathbb{R}^N}\varphi_n^2\, d\mu} \leq \frac{(\gamma^2 - c)\int_{B(1)\setminus B\left(\frac{1}{n}\right)} |x|^{2\gamma-2}\, d\mu + C_1}{C_2}\ .$$

We observe that $\gamma^2 - c < 0$. Taking the limit $n \to \infty$ we get

$$\lim_{n\to\infty}\int_{B(1)\setminus B\left(\frac{1}{n}\right)} |x|^{2\gamma-2}\, d\mu = +\infty$$

hence $\lambda_1 = -\infty$. □

Therefore, we obtained the following result.

Theorem 1.14. *Assume hypotheses* $(H1)$, $(H2)$ *and* $(H3)$ *with* $c_{0,\mu} = c_0(N_0)$. *Then for any* $u \in H^1_\mu(\mathbb{R}^N)$ *the following inequality holds*

$$c_0(N_0)\int_{\mathbb{R}^N}\frac{u^2}{|x|^2}d\mu \leq \int_{\mathbb{R}^N}|\nabla u|^2 d\mu + C_\mu\int u^2 d\mu$$

and $c_0(N_0)$ *is the best constant.*

If $(H3)'$ with $N_0 > 2$ holds, one obtains the following theorem.

Theorem 1.15. *Let us assume Hypothesis* $(H3)'$ *with* $N_0 > 2$. *Then, there exists a function in* $H^1_\mu(\mathbb{R}^N)$ *for which Hardy inequality* (1.12) *does not hold if* $c = c_0(N_0) = \left(\frac{N_0-2}{2}\right)^2$.

Proof. Let γ be such that $0 > \gamma > \frac{-N_0+2}{2}$ so that $|x|^{2\gamma} \in L^1_{loc}(\mathbb{R}^N, d\mu)$ and $|x|^{2\gamma-2} \in L^1_{loc}(\mathbb{R}^N, d\mu)$ and $\gamma^2 < c_0(N_0)$. Let $\vartheta \in C_c^\infty(\mathbb{R}^N)$, $0 \leq \vartheta \leq 1$, $\vartheta = 1$ in $B(1)$ and $\vartheta = 0$ in $B^c(2)$. Set $\varphi_\gamma(x) = |x|^\gamma\vartheta(x)$. We observe that $\varphi_\gamma \in H^1_\mu(\mathbb{R}^N)$.

Let us set $c = c_0(N_0)$. We have to prove that

$$\lambda_1 = \inf_{\varphi\in H^1_\mu(\mathbb{R}^N)\setminus\{0\}}\left(\frac{\int_{\mathbb{R}^N}\left(|\nabla\varphi|^2 - \frac{c}{|x|^2}\varphi^2\right) d\mu}{\int_{\mathbb{R}^N}\varphi^2\, d\mu}\right)$$

is $-\infty$.

We get

$$
\begin{aligned}
\int_{\mathbb{R}^N} \left(|\nabla \varphi_\gamma|^2 - \frac{c}{|x|^2}\varphi_\gamma^2 \right) d\mu = & \int_{B(1)} \left(|\nabla |x|^\gamma|^2 - \frac{c}{|x|^2}|x|^{2\gamma} \right) d\mu \\
& + \int_{B(2)\setminus B(1)} \left(|\nabla |x|^\gamma \vartheta(x)|^2 - \frac{c}{|x|^2}\left(|x|^\gamma \vartheta(x)\right)^2 \right) d\mu \\
& \le (\gamma^2 - c) \int_{B(1)} |x|^{2\gamma-2}\, d\mu + 2\int_{B(2)\setminus B(1)} |x|^{2\gamma}|\nabla\vartheta|^2\, d\mu \\
& + 2\gamma^2 \int_{B(2)\setminus B(1)} \vartheta^2 |x|^{2\gamma-2}\, d\mu \\
& \le (\gamma^2 - c) \int_{B(1)} |x|^{2\gamma-2}\, d\mu \\
& + 2(\|\nabla\vartheta\|_\infty^2 + \gamma^2) \int_{B(2)\setminus B(1)} d\mu \\
& = (\gamma^2 - c) \int_{B(1)} |x|^{2\gamma-2}\, d\mu + C_1 \,. \qquad (1.22)
\end{aligned}
$$

On the other hand,

$$
\int_{\mathbb{R}^N} \varphi_\gamma^2\, d\mu \ge \int_{B(1)} |x|^{2\gamma}\, d\mu \ge \int_{B(1)} d\mu = C_2 \,. \qquad (1.23)
$$

Taking into account (1.22) and (1.23), we have

$$
\lambda_1 \le \frac{\int_{\mathbb{R}^N} \left(|\nabla \varphi_\gamma|^2 - \frac{c}{|x|^2}\varphi_\gamma^2 \right) d\mu}{\int_{\mathbb{R}^N} \varphi_\gamma^2\, d\mu} \le \frac{(\gamma^2 - c)\int_{B(1)} |x|^{2\gamma-2}\, d\mu + C_1}{C_2} \,.
$$

Now, taking the limit $\gamma \to \left(\frac{-N_0+2}{2}\right)^+$, by Hypotesis $(H3)'$ one obtains

$$
\begin{aligned}
& \lim_{\gamma \to \left(\frac{-N_0+2}{2}\right)^+} (\gamma^2 - c) \int_{B(1)} |x|^{2\gamma-2}\, d\mu \\
& = \lim_{\gamma \to \left(\frac{-N_0+2}{2}\right)^+} \left(\gamma + \frac{-N_0+2}{2}\right)\left(\gamma - \frac{-N_0+2}{2}\right) \int_{B(1)} |x|^{2\gamma-2}\, d\mu \\
& = \lim_{\lambda \to 0^+} (\lambda + 2 - N_0)\, \lambda \int_{B(1)} |x|^{2\lambda - N_0}\, d\mu = -\infty.
\end{aligned}
$$

Then $\lambda_1 = -\infty$. $\square$

Therefore, we obtain the following result.

Theorem 1.16. *Assume hypotheses* $(H1)$, $(H2)'$ *and* $(H3)'$ *with* $N_0 > 2$ *and* $c_{0,\mu} = c_0(N_0)$. *Then for every* $u \in H^1_\mu(\mathbb{R}^N)$ *the following inequality holds*

$$c \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} d\mu \leq \int_{\mathbb{R}^N} |\nabla u|^2 d\mu + C_\mu \int u^2 d\mu$$

for every $c < c_0(N_0)$ *and* $c_0(N_0)$ *is the best constant.*

1.2.3 Examples

We finally give some examples of measures one can take into consideration and for which the weighted Hardy inequality holds.

Proposition 1.17. *Let* $d\mu = \rho(x)dx$ *with* $\rho(x) = e^{-b|x|^m}$, $m > 0$, $b \geq 0$, *and* $N \geq 3$. *Then there exists a positive constant* C *such that for all* $u \in H^1_\mu(\mathbb{R}^N)$ *the following inequality*

$$c_0(N) \int_{\mathbb{R}^N} \frac{u^2}{|x|^2}\, d\mu \leq \int_{\mathbb{R}^N} |\nabla u|^2\, d\mu + C \int_{\mathbb{R}^N} u^2\, d\mu$$

holds with best constant.

Proof. This measure satisfies assumption $(H1)$. Then, by a simple computation one obtains

$$U_\mu = -\frac{1}{4} b^2 m^2 |x|^{2m-2} + \frac{1}{2} bm(N + m - 2)|x|^{m-2},$$

and $\lim_{x\to 0} |x|^2 U_\mu(x) = 0$. Therefore, $c_{0,\mu} = c_0(N)$ and $U_\mu = U$ is a bounded function far from 0 and assumptions $(H2)$ and $(H3)$ are satisfied with $N_0 = N$. Then the assertion follows from Theorem 1.14. $\square$

Proposition 1.18. *Let us consider* $d\mu = \frac{1}{|x|^\beta}\rho(x)dx$ *with* $\rho(x)$ *as in Proposition 1.17,* $N \geq 3$ *and* $\beta < N - 2$. *Then there exists a constant* $C \geq 0$ *such that for all* $u \in H^1_\mu(\mathbb{R}^N)$

$$c_0(N - \beta) \int_{\mathbb{R}^N} \frac{u^2}{|x|^2}\, d\mu \leq \int_{\mathbb{R}^N} |\nabla u|^2\, d\mu + C \int_{\mathbb{R}^N} u^2\, d\mu$$

holds with best constant.

Proof. This measure satisfies assumption $(H1)$ if $\beta < N-2$. Moreover, by a simple computation, one has $c_{0,\mu} = c_0(N-\beta)$ and

$$U = -\frac{1}{4}b^2m^2|x|^{2m-2} + \frac{1}{2}bm(N+m-2-\beta)|x|^{m-2}.$$

Then $(H2)$ and $(H3)$ hold with $c_{0,\mu} = c_0(N-\beta)$, $N_0 = N-\beta$. The result follows by Theorem 1.14. □

Remark 1.19. Let us observe that if in the Proposition above β is grater or equal to $N+m-2$, we have $U \leq 0$ and the inequality holds with $C=0$. For the other values of β we are able to state the same result without the assumption $(H2)$ but only for $m \geq 2$.

By taking $b = 0$ in Proposition 1.18 we can also state the Caffarelli-Nirenberg inequality.

Corollary 1.20. *If $\beta < N-2$ and $N \geq 3$ then the following inequality holds*

$$\left(\frac{N-2-\beta}{2}\right)^2 \int_{\mathbb{R}^N} \frac{u^2}{|x|^2}|x|^{-\beta}\,dx \leq \int_{\mathbb{R}^N} |\nabla u|^2|x|^{-\beta}\,dx, \qquad \forall u \in H^1(\mathbb{R}^N).$$

We end by an example for a weight which behaves like logarithm near 0.

Proposition 1.21. *Let $\theta \in C_c^\infty(\mathbb{R}^N)$ with $0 \leq \theta \leq 1$ such that $\theta = 1$ on $B(1)$ and $\theta = 0$ on $B(2)^c$. Assume that*

$$\mu(x) = \theta(x)\left(\log\frac{1}{|x|}\right)^\alpha, \quad x \in \mathbb{R}^N.$$

- *If $\alpha \leq 0$, then there exists a constant $C \geq 0$ such that for all $u \in H^1_\mu(\mathbb{R}^N)$*

$$c_0(N)\int_{\mathbb{R}^N} \frac{u^2}{|x|^2}\,d\mu \leq \int_{\mathbb{R}^N} |\nabla u|^2\,d\mu + C\int_{\mathbb{R}^N} u^2\,d\mu$$

  holds with best constant.

- *If $\alpha > 0$, then there exist $c, C \geq 0$ such that for all $u \in H^1_\mu(\mathbb{R}^N)$*

$$c\int_{\mathbb{R}^N} \frac{u^2}{|x|^2}\,d\mu \leq \int_{\mathbb{R}^N} |\nabla u|^2\,d\mu + C\int_{\mathbb{R}^N} u^2\,d\mu$$

  holds for every $c < c_0(N)$ and $c_0(N)$ is the best constant.

Proof. One has that μ satisfies assumptions $(H1)$, $(H2)'$ and $(H3)$ with $c_{0,\mu} = c_0(N)$ and $N_0 = N$ for all $\alpha \in \mathbb{R}$. Moreover, μ satisfies $(H2)$ if and only if $\alpha \le 0$ and μ satisfies $(H3)'$ if and only if $\alpha > 0$. Indeed, by a simple computation, one obtains for $x \in B(1)$

$$|x|^2 U_\mu = \left(\frac{1}{4} - \frac{(\alpha-1)^2}{4}\right)\left(\log\frac{1}{|x|}\right)^{-2} + \frac{\alpha}{2}(N-2)\left(\log\frac{1}{|x|}\right)^{-1}. \quad (1.24)$$

We have $\limsup_{x\to 0} |x|^2 U_\mu = 0$, then the constant in Hardy's inequality is $c_0(N)$. By (1.24) it is easy to check that assumption $(H2)$ is satisfied if and only if $\alpha \le 0$. Then, for $\alpha \le 0$ Theorem 1.14 applies.

Now $(H3)'$ is

$$\lim_{\gamma\to 0^+} \gamma \int_{B(r)} |x|^{\gamma-N}\, d\mu = +\infty$$

for some positive $r < 1$. One has

$$\begin{aligned}\gamma \int_{B(r)} \frac{|x|^\gamma}{|x|^N}\, d\mu &= \omega_N \int_0^r \gamma s^{\gamma-1}\left(\log\frac{1}{s}\right)^{\alpha} ds \\ &= \omega_N \int_0^r s^{\gamma-1}\alpha\left(\log\frac{1}{s}\right)^{\alpha-1} ds + \omega_N\left[s^\gamma\left(\log\frac{1}{s}\right)^{\alpha}\right]_0^r.\end{aligned}$$

The second term is uniformly bounded for every $\gamma > 0$. As regards the first term, it grows to infinity for $\gamma \to 0^+$ if and only if $\alpha - 1 > -1$. Therefore, for $\alpha > 0$, the assertion follows from Theorem 1.16. □

1.3 Rellich's inequality

Among the several generalization of the Hardy inequality that have been provided, we will need in the following the analogous for higher order operators, which is called Rellich's inequality, and reads as follows, for all $u \in H^2(\mathbb{R}^N)$,

$$\left(\frac{N(N-4)}{4}\right)^2 \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^4}\, dx \le \int_{\mathbb{R}^N} |\Delta u(x)|^2\, dx. \quad (1.25)$$

We learnt that Hardy's inequality is involved in the study of the operator $H = -\Delta - c|x|^{-2}$. Analogously, the Rellich inequality is involved in the study of the fourth order operator perturbed by the singular potential

$$A = \Delta^2 - c|x|^{-4}.$$

The Rellich inequality first appeares in 1954 in [58], but had already been proved in 1953 as the lectures [59] published posthumously show. The proof we provide here can be found in [19, Appendix].

Theorem 1.22. *For each $u \in H^2(\mathbb{R}^N)$, $N \geq 5$, the following inequality holds*

$$\int_{\mathbb{R}^N} \frac{u^2}{|x|^4}\,dx \leq k^2 \int_{\mathbb{R}^N} |\Delta u|^2\,dx$$

with optimal constant $k = \frac{4}{N(N-4)}$.

Proof. By a density argument we can consider $u \in C_c^\infty(\mathbb{R}^N)$. Then, applying (1.1) where we denote by $C = \frac{4}{(N-2)^2}$ and integrating by parts, one obtains

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{u^2}{|x|^4}\,dx &\leq C \int_{\mathbb{R}^N} \left|\nabla\left(\frac{u}{|x|}\right)\right|^2\,dx \\ &= -C \int_{\mathbb{R}^N} \frac{u}{|x|^2}\Delta u\,dx + C \int_{\mathbb{R}^N} \frac{u^2}{|x|^4}\,dx.\end{aligned}$$

Hence, by applying Hölder's inequality

$$\begin{aligned}(1-C)\int_{\mathbb{R}^N} \frac{u^2}{|x|^4}\,dx &\leq -C \int_{\mathbb{R}^N} \frac{u}{|x|^2}\Delta u\,dx \\ &\leq C\left(\int_{\mathbb{R}^N} |\Delta u|^2\,dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} \frac{u^2}{|x|^4}\,dx\right)^{\frac{1}{2}}.\end{aligned}$$

Therefore, one obtains

$$\int_{\mathbb{R}^N} \frac{u^2}{|x|^4}\,dx \leq \left(\frac{C}{1-C}\right)^2 \int_{\mathbb{R}^N} |\Delta u|^2\,dx.$$

Now observe that $\frac{C}{1-C} = k$ and, since C is the optimal constant in Hardy's inequality (1.1), k^2 is the best constant in Rellich inequality. □

Remark 1.23. Observe that in Theorem 1.22 there is the constraint on the dimension, $N \geq 5$, due to the integrability of the left-hand side of the Rellich inequality. If one considers functions $u \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$, inequality (1.25) hols true for every $N \neq 2$ (see [58]).

Chapter 2

Schrödinger operator

The Schrödinger operator is a partial differential operator on $\mathbb{R}^N$ of the form

$$H = -\Delta - V,$$

where Δ is the N-dimensional Laplace operator $\Delta = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$, and $V = V(x)$ is a real-valued function which is not necessarily supposed to be smooth, continuous or bounded. The interest in studying this operator mostly comes from quantum mechanics. In fact, the instantaneous configuration of a nonrelativistic particle is represented by a function $\psi(t,x) = e^{-iHt}\psi(0,x) \in L^2(\mathbb{R}^N)$ which is called wave function. It is the solution of the well-known Schrödinger equation which controls the evolution of a quantum system, and, in units with $\hbar = m = 1$, reads as follows

$$\frac{\partial \psi}{\partial t} = -iH\psi.$$

The operator H is also called the Hamiltonian of the nonrelativistic quantum system. The total energy of the system $(H\psi, \psi)$, is divided between the kinetic energy $(-\Delta\psi, \psi)$, and the potential energy $(V\psi, \psi)$ (that's the reason why one can discuss general potentials). The statistical interpretation of the wave function is that $|\psi(t,x)|^2$ is the probability density for finding the particle at point x at time t, for more details on this topic see [32]. What one is especially interested in, are the stationary states, the functions $\psi \in L^2$ such that $H\psi = E\psi$, i.e., the eigenfunctions of the operator H, which correspond to discrete (quantized) excitations of the system. Moreover, particular attention is devoted to eigenfunctions at the bottom of the spectrum. The quantity $\inf \sigma(H)$ is called the ground state energy and if it is an eigenvalue, the corresponding eigenfunction is called the ground state and it represents the configuration of the system with the smallest total energy.

For these reasons one looks at the real operator $H = -\Delta - V$ as defined at the beginning. The operator H arises from the following sesquilinear form. For a simple account on sesquilinear form theory we refer to Appendix C.

$$Q(u,v) = (\nabla u, \nabla v) - (Vu, v), \qquad u,v \in C_c^\infty(\mathbb{R}^N).$$

The Schrödinger operator H has largely been studied in literature by several authors. Only few among the several works are [16, 8, 60, 61]. Different classes of potentials have been characterized, generation results, characterization of the domain, kernel estimates, spectrum and eigenvalues have been computed and the associated evolution equation, i.e., $u_t = \Delta u + Vu$ in $(0,T] \times \mathbb{R}^N$ has been studied.

Aim of this chapter is to consider the critical potential $V(x) = c|x|^{-2}$ a.e. with $c \in \mathbb{R}$. We will study the generation of a semigroup for a suitable realisation of $-H$ in $L^2(\mathbb{R}^N)$. Then, we will state some upper bounds for the heat kernel of the associated semigroup and finally, we will provide a non-existence result in $L^2(\mathbb{R}^N)$.

2.1 Generation results

From now on we will consider $N \geq 3$ and $c \in \mathbb{R}$, $c \leq c_0(N) = \left(\frac{N-2}{2}\right)^2$, and we will focus our attention on the operator

$$Hu = -\Delta u - Vu = -\Delta u - \frac{c}{|x|^2}u.$$

The operator H has widely been studied in literature, we refer e.g. to [16, 53, 57, 61]. We want to investigate on the solution of the parabolic problem associated to $-H$ in $L^2(\mathbb{R}^N)$, i.e.,

$$\begin{cases} u_t = \Delta u + \frac{c}{|x|^2}u & t > 0, x \in \mathbb{R}^N, \\ u(0,\cdot) = f \in L^2(\mathbb{R}^N). \end{cases} \tag{2.1}$$

To this purpose, we construct a semigroup in $L^2(\mathbb{R}^N)$ generated by a suitable realisation of $-H$. We treat the operator via form methods. The associated form Q to H is

$$Q(u,v) = (\nabla u, \nabla v) - \left(\frac{c}{|x|^2}u, v\right)$$

with $u,\, v \in D(Q) = \{u \in H^1(\mathbb{R}^N) : \||V|^{\frac{1}{2}}u\|_2 < \infty\}$. The Hardy inequality (1.1) implies that for $c \le c_0(N)$ the quadratic form associated to Q is accretive, i.e.,

$$Q(u) := \int_{\mathbb{R}^N} |\nabla u(x)|^2\, dx - c \int_{\mathbb{R}^N} \frac{u^2(x)}{|x|^2}\, dx \ge 0.$$

Moreover, Hardy inequality also says that $D(Q) = H^1(\mathbb{R}^N)$. Observe that, since Q is a symmetric, positive semi-definite sesquilinear form, thanks to Cauchy-Schwarz inequality, one obtains that Q is continuous. We investigate now the closeness of the form Q. If we choose $c < c_0(N)$, there exists a $0 < \alpha < 1$ such that $\frac{c}{\alpha} < c_0(N)$ and then

$$\int_{\mathbb{R}^N} |\nabla u|^2\, dx \le \frac{1}{1-\alpha} Q(u).$$

This, together with the Hardy inequality, ensures us that the norm $\|\cdot\|_Q$, i.e., the norm associated with the form Q and the norm $\|\cdot\|_{H^1}$ are equivalent. Hence, $(D(Q), \|\cdot\|_Q)$ is complete, i.e., Q is closed. Therefore, Q satisfies conditions (C.1)-(C.4), and by Theorem C.8, the associated operator $-H_2$ in $L^2(\mathbb{R}^N)$ is the generator of a holomorphic strongly continuous contraction semigroup on $L^2(\mathbb{R}^N)$ that we will denote by $(e^{-tH})_{t\ge 0}$.

Therefore, the parabolic problem (2.1) with $c < c_0(N)$, $N \ge 3$, admits a unique solution $u(t,x) = e^{-tH} f(x)$ for every initial datum f in $L^2(\mathbb{R}^N)$.

If instead, $c = c_0(N)$, by [54, Lemma 1.29], one obtains Q is closable and a suitable extension of $-H$ associated to the closure of Q is the generator of a holomorphic contractive C_0-semigroup on $L^2(\mathbb{R}^N)$.

Remark 2.1. If $c \le 0$, it is easy to verify that the following two conditions of Beurling-Deny are satisfied by Q:

(i) $u \in D(Q)$ implies $|u| \in D(Q)$ and $Q(|u|) \le Q(u)$,

(ii) $0 \le u \in D(Q)$ implies $u \wedge 1 \in D(Q)$ and $Q(u \wedge 1) \le Q(u)$.

Then, for $c \le 0$, by [54, Corollary 2.18], the semigroup $(e^{-tH})_{t\ge 0}$ in $L^2(\mathbb{R}^N)$ is a symmetric sub-Markov semigroup, i.e., each operator e^{-tH} is symmetric, $e^{-tH} \ge 0$ for all $t \ge 0$, and $\|e^{-tH} f\|_\infty \le \|f\|_\infty$ for all $f \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and all $t \ge 0$.

2.1.1 Kernel estimates

Various estimates for the heat kernel of the semigroup associated with the operator $-Hu = \Delta u + \frac{c}{|x|^2}u$ have been proved in literature, either for short-time or long time, see for example [43, 48, 50, 63] and the references therein. Here we want to state some upper bounds for the case $c \geq 0$. We learnt that the potential $V = c|x|^{-2}$ represents a critical case for positive c. Nevertheless, it is possible to prove that the solution to the parabolic problem (2.1) associated to $-H = \Delta + c|x|^{-2}$ is given by a holomorphic contractive C_0-semigroup $(e^{-tH})_{t\geq 0}$ when $c \leq c_0(N)$. Moreover, optimal estimates for the heat kernel of the semigroup holds.

In order to get a bound on the kernel of $(e^{-tH})_{t\geq 0}$, one shall use the standard technique of transference to a weighted L^2-space and ultracontractivity methods by Davies, see [16]. He proves that Sobolev inequalities are related to ultracontractive properties of semigroups, and hence to uniform bounds on their heat kernels by the following standard result. Let $-L$ be the generator of a semigroup on $L^2(\mathbb{R}^N)$ and Q the associated form to L. Under suitable assumptions on the form Q, the Sobolev inequality

$$\|f\|_{L^{2^*}}^2 \leq c_2 Q(f), \qquad f \in D(Q), \tag{2.2}$$

is equivalent to the ultracontractivity of the semigroup, i.e., the $L^2 - L^\infty$ bound

$$\|e^{-tL}\|_{2\to\infty} \leq ct^{-\frac{N}{4}} \qquad \forall t > 0.$$

The lattest inequality, by a duality argument, yields the same estimate for the $L^1 - L^2$ norm. This is equivalent to the fact that $(e^{-tL})_{t\geq 0}$ admits an integral representation

$$e^{-tL}f(x) = \int_{\mathbb{R}^N} p(t,x,y)f(y)\,dy, \quad t > 0, \text{ a.e. } x \in \mathbb{R}^N, f \in L^2(\mathbb{R}^N),$$

where $p(t,x,y) > 0$ a.e. $y \in \mathbb{R}^N$, for all $t > 0$ and for all $x \in \mathbb{R}^N$. If (2.2) holds, p is such that

$$0 \leq p(t,x,y) \leq c_1 t^{-\frac{N}{2}}, \quad t > 0, \text{ a.e. in } \mathbb{R}^N.$$

In particular, since it may happen that $(e^{-tH})_{t\geq 0}$ is not ultracontractive, one shall first look for a suitable transformation of H to an ultracontractive operator. Therefore, one should show that there exists a suitable weight function φ such that, considering the unitary operator

$$U : u \in L^2(\mathbb{R}^N) \to \varphi^{-1}u \in L^2_\varphi := L^2(\mathbb{R}^N, \varphi^2\,dx),$$

then the operator H is unitarily equivalent to a self-adjoint Schrödinger type operator H_φ on L^2_φ for which (2.2) holds. Indeed, one could consider the transformation $H_\varphi = UHU^{-1}$ with $H_\varphi v = \varphi^{-1}H(\varphi v)$, and then, since $e^{-tH_\varphi}v = \varphi^{-1}e^{-tH}(\varphi v)$, the heat kernels of $(e^{-tH})_{t\geq 0}$ and $(e^{-tH_\varphi})_{t\geq 0}$ are related by

$$p(t,x,y) = p_\varphi(t,x,y)\varphi(x)\varphi(y) \qquad t>0, x,y\in\mathbb{R}^N.$$

Hence, one can obtain estimates on $p(t,x,y)$ via estimates on $p_\varphi(t,x,y)$.

Using this thechnique Liskevich and Sobol in [37] and Milman and Semenov in [47] prove estimates from above for the kernel of the semigroup generated by $-H$ in $L^2(\mathbb{R}^N)$. Their results can be resumed with the following theorem [55, Theorem 3.1]. We state the result according to the notations of this manuscript.

Theorem 2.2. *Let $H = -\Delta - c|x|^{-2}$. Assume $0 \leq c \leq c_0(N)$. The semigroup $(e^{-tH})_{t\geq 0}$ admits an integral representation with a kernel p, namely*

$$e^{-tH}f(x) = \int_{\mathbb{R}^N} p(t,x,y)f(y)\,dy.$$

Moreover, there exist positive constants $C>0$ and $b>1$ such that for all $t>0$ and all $x,y\in\mathbb{R}^N\setminus\{0\}$

$$0 \leq p(t,x,y) \leq Ct^{-\frac{N}{2}}e^{-\frac{|x-y|^2}{bt}}\varphi(t,x)\varphi(t,y)$$

where

$$\varphi(t,x) = \begin{cases} \left(\frac{|x|}{\sqrt{t}}\right)^\alpha & |x|\leq\sqrt{t},\\ 1 & |x|\geq\sqrt{t}, \end{cases}$$

with $2\alpha = 2-N+2\sqrt{c_0(N)-c}$.

2.2 Non-existence result

One can also discuss the non-existence of positive solutions on $L^2(\mathbb{R}^N)$ when $c>c_0(N)$. As already mentioned, Baras and Goldstein in [7] characterize the existence of positive weak solutions to the parabolic problem associated with the Schrödinger operator $-H$ in $L^1(\mathbb{R}^N)$. In particular, they study the problem

$$\begin{cases} u_t - \Delta u = c|x|^{-2}u & (0,T)\times\mathbb{R}^N,\\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{2.3}$$

with $0 < T \leq \infty$, $u_0 \in L^1_{loc}(\mathbb{R}^N)$ and $u_0 \geq 0$ a.e. They show that, if $0 \leq c \leq \left(\frac{N-2}{2}\right)^2$, then for each initial datum $u_0 \geq 0$ such that $|x|^{-\alpha}u_0 \in L^1(\mathbb{R}^N)$, where α is the smallest root of $(N-2-\alpha)\alpha = c$, there exists a positive weak solution to (2.3) global in time. If instead, $c > \left(\frac{N-2}{2}\right)^2$, then, for each $T > 0$ and each $u_0 \in L^1_{loc}(\mathbb{R}^N)$ with $u_0 \geq 0$, $u_0 \not\equiv 0$, there does not exist a positive weak solution of (2.3). Moreover, there is the so called instantaneous blow-up phenomena. Cabré and Martel in [12] provide a new and simpler proof of the non-existence result. We want to present here their technique in $L^2(\mathbb{R}^N)$. Thus, we will consider the problem

$$\begin{cases} u_t = \Delta u + \frac{c}{|x|^2}u & t > 0, x \in \mathbb{R}^N, \\ u(0,\cdot) = f \ \in L^2(\mathbb{R}^N), \end{cases} \tag{2.4}$$

for $N \geq 3$, $c \geq 0$, $f \geq 0$. First of all, we specify what we mean by weak solution of (2.4). A function $u \geq 0$ is a weak solution of (2.4) if, for each $T, R > 0$, we have $u \in C([0,T], L^2(\mathbb{R}^N))$, $Vu \in L^1((0,T) \times B(R), dt\,dx)$ and

$$\int_0^T \int_{\mathbb{R}^N} u(-\phi_t - \Delta\phi)\,dx\,dt - \int_{\mathbb{R}^N} f\phi(0,\cdot)\,dx = \int_0^T \int_{\mathbb{R}^N} \frac{c}{|x|^2}u\phi\,dx\,dt,$$

for all $\phi \in W^{2,1}_{loc}(Q_T)$ having compact support such that $\phi(T,\cdot) = 0$ and $Q_T = [0,T] \times \mathbb{R}^N$. If $T = \infty$, we say that u is a global weak solution of (2.4). Let us recall the generalised bottom of the spectrum of the operator H

$$\lambda_1 = \inf_{\varphi \in H^1(\mathbb{R}^N)\setminus\{0\}} \left(\frac{\int_{\mathbb{R}^N} \left(|\nabla\varphi|^2 - V\varphi^2\right)\,dx}{\int_{\mathbb{R}^N} \varphi^2\,dx} \right).$$

They prove the following.

Theorem 2.3. *Consider the problem* (2.4)*.*

(i) If $\lambda_1 > -\infty$ (that is $c \leq c_0(N)$), then there exists a function $u \in C([0,\infty), L^2(\mathbb{R}^N))$, weak solution of (2.4)*, exponentially bounded, i.e.,*

$$\|u(t)\| \leq Me^{\omega t}\|f\|. \tag{2.5}$$

(ii) If $\lambda_1 = -\infty$ (that is $c > c_0(N)$), then for all $0 \leq f \in L^2(\mathbb{R}^N) \setminus \{0\}$ there does not exist a positive weak solution of (2.4) *satisfying* (2.5)*.*

Proof. (i) Let us set

$$V_n(x) = \min\{c|x|^{-2}, n\}.$$

The functions V_n are bounded and nonnegative, $0 \leq V_n \leq n$. Therefore, the following problems

$$\begin{cases} \partial_t u_n = \Delta u_n + V_n u_n & x \in \mathbb{R}^N, t > 0, \\ u_n(0,x) = f \in L^2(\mathbb{R}^N), \end{cases} \tag{2.6}$$

admit a holomorphic solution that we will denote by $u_n(t,x) = T_n(t)f(x)$. Moreover, by Theorem B.5, the semigroups $(T_n(t))_{t\geq 0}$ are irreducible, i.e., for nonnegative and non-identically zero initial datum f, $T_n(t)f(x) > 0$ holds for all $x \in \mathbb{R}^N$ and $t > 0$. Hence, if $f \geq 0$, $f \neq 0$, we have $u_n > 0$. Let us consider the first equation in (2.6) and multiply by u_n and integrate. We obtain

$$\begin{aligned} \int_{\mathbb{R}^N} \partial_t u_n u_n \, dx &= \int_{\mathbb{R}^N} \Delta u_n u_n \, dx + \int_{\mathbb{R}^N} V_n u_n^2 \, dx \\ &= -\int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx + \int_{\mathbb{R}^N} V_n u_n^2 \, dx. \end{aligned}$$

Therefore,

$$\frac{1}{2}\int_{\mathbb{R}^N} \partial_t u_n^2 \, dx = -\left(\int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx - \int_{\mathbb{R}^N} V_n u_n^2 \, dx\right) \leq -\lambda_1 \int_{\mathbb{R}^N} u_n^2 \, dx$$

and

$$\frac{\partial_t \int_{\mathbb{R}^N} u_n^2 \, dx}{\int_{\mathbb{R}^N} u_n^2 \, dx} \leq -2\lambda_1.$$

Integrating with respect to t we obtain

$$\|u_n\|_2^2 \leq e^{-2\lambda_1 t}$$

and then

$$\|T_n(t)f\|_2 \leq e^{-\lambda_1 t}\|f\|_2, \quad \forall t \geq 0. \tag{2.7}$$

Therefore, we have uniformly bounded C_0-semigroups on $L^2(\mathbb{R}^N)$. What's more, since $V_{n+1} \geq V_n$ one can easily show that the functions u_n are increasing. Thus, by [2, Proposition 3,6], the sequence $(T_n(t)f)$ converges in $L^2(\mathbb{R}^N)$ to a C_0-semigroup $T(t)f$ which satisfies (2.5) thanks to (2.7). Since u_n is a weak solution of (2.6), it follows that $u(t,x) = T(t)f(x)$ is a weak solution of (2.4). Therefore, there exists a positive weak solution to (2.4) exponentially bounded and $T(t)$ is the semigroup generated by $-H$.

(ii) We proceed by contradiction. We assume that there exists a positive weak solution u to (2.4) satisfying (2.5). Then, considering the problems (2.6) we have, $u_n \geq 0$ and $V \geq V_n$. By applying a weak maximum principle, arguing as in [29, Appendix], one obtains $0 < u_n \le u$ for all $n \in \mathbb{N}$. Hence, (u_n) is increasing and bounded, it converges to some function $\overline{u}$ which also satisfies $0 \le \overline{u} \le u$. Let φ be a test function such that $\|\varphi\|_2 = 1$ and multiply (2.6) by $\frac{\varphi^2}{u_n}$. Computing we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{\partial_t u_n}{u_n}\varphi^2\,dx &= \int_{\mathbb{R}^N} \Delta u_n \frac{\varphi^2}{u_n}\,dx + \int_{\mathbb{R}^N} V_n\varphi^2\,dx \\ &= -\int_{\mathbb{R}^N} \nabla u_n \cdot \nabla\left(\frac{\varphi^2}{u_n}\right)dx + \int_{\mathbb{R}^N} V_n\varphi^2\,dx \\ &= -2\int_{\mathbb{R}^N} \nabla u_n \cdot \nabla\varphi\frac{\varphi}{u_n}\,dx + \int_{\mathbb{R}^N} |\nabla u_n|^2\frac{\varphi^2}{u_n^2}\,dx \\ &\quad + \int_{\mathbb{R}^N} V_n\varphi^2\,dx.\end{aligned}$$

Thus,

$$\begin{aligned}\partial_t \int_{\mathbb{R}^N} (\log u_n)\varphi^2\,dx = -2\int \nabla u_n \cdot \nabla\varphi\frac{\varphi}{u_n}\,dx \\ + \int_{\mathbb{R}^N} |\nabla u_n|^2\frac{\varphi^2}{u_n^2}\,dx + \int_{\mathbb{R}^N} V_n\varphi^2\,dx.\end{aligned} \tag{2.8}$$

By applying Hölder's and Young's inequality we get

$$2\int \nabla u_n \cdot \nabla\varphi\frac{\varphi}{u_n}\,dx \le \int_{\mathbb{R}^N} |\nabla u_n|^2\frac{\varphi^2}{u_n^2}\,dx + \int_{\mathbb{R}^N} |\nabla\varphi|^2\,dx$$

which, substituting in (2.8), gives

$$\int_{\mathbb{R}^N} V_n\varphi^2\,dx - \int_{\mathbb{R}^N} |\nabla\varphi|^2\,dx \le \partial_t \int_{\mathbb{R}^N} (\log u_n)\varphi^2\,dx.$$

Let $t > 1$ and integrate between 1 and t, the following inequality holds

$$\begin{aligned}(t-1)\left[\int_{\mathbb{R}^N} V_n\varphi^2\,dx - \int_{\mathbb{R}^N} |\nabla\varphi|^2\,dx\right] \\ \le \int_{\mathbb{R}^N} (\log u_n)\varphi^2\,dx - \int_{\mathbb{R}^N} (\log u_n(1))\varphi^2\,dx.\end{aligned}$$

Now, letting $n \to \infty$, we get

$$\begin{aligned}(t-1)\left[\int_{\mathbb{R}^N} V\varphi^2\,dx - \int_{\mathbb{R}^N} |\nabla\varphi|^2\,dx\right] \\ \le \int_{\mathbb{R}^N} (\log \overline{u})\varphi^2\,dx - \int_{\mathbb{R}^N} (\log \overline{u}(1))\varphi^2\,dx.\end{aligned}$$

Recall that $\overline{u} \leq u$ and $\|\varphi\|_2 = 1$, applying Jensen's inequality one has

$$(t-1)\Big[\int_{\mathbb{R}^N} V\varphi^2\,dx - \int_{\mathbb{R}^N} |\nabla\varphi|^2\,dx\Big] \leq \log\left(\int_{\mathbb{R}^N} u\varphi^2\,dx\right) - \int_{\mathbb{R}^N} (\log\overline{u}(1))\varphi^2\,dx. \tag{2.9}$$

Now, since $\varphi \in C_c^\infty(\mathbb{R}^N)$ and (2.5) holds, we estimate as follows

$$\int_{\mathbb{R}^N} u\varphi^2\,dx \leq \left(\int_{\mathbb{R}^N} u^2(\varphi^2\,dx)\right)^{\frac{1}{2}} \leq \|\varphi\|_\infty \left(\int_{\mathbb{R}^N} u^2\,dx\right)^{\frac{1}{2}} \leq M\|\varphi\|_\infty e^{\omega t}.$$

Then,

$$\log \int_{\mathbb{R}^N} u\varphi^2\,dx \leq \log\|\varphi\|_\infty + \log M + \omega t.$$

Thus, in (2.9), one has

$$(t-1)\Big[\int_{\mathbb{R}^N} V\varphi^2\,dx - \int_{\mathbb{R}^N} |\nabla\varphi|^2\,dx\Big] \leq \log\|\varphi\|_\infty + \log M + \omega t - \int_{\mathbb{R}^N} (\log\overline{u}(1))\varphi^2\,dx$$

and then,

$$\int_{\mathbb{R}^N} V\varphi^2\,dx - \int_{\mathbb{R}^N} |\nabla\varphi|^2\,dx \leq \frac{\log\|\varphi\|_\infty + \log M}{t-1} + \frac{\omega t}{t-1} - \frac{1}{t-1}\int_{\mathbb{R}^N} (\log\overline{u}(1))\varphi^2\,dx.$$

Letting $t \to \infty$, one obtains

$$\int_{\mathbb{R}^N} V\varphi^2\,dx - \int_{\mathbb{R}^N} |\nabla\varphi|^2\,dx \leq \omega,$$

and since $\int_{\mathbb{R}^N} \varphi^2\,dx = 1$, we have

$$\lambda_1 \geq -\omega$$

which is a contradiction. $\square$

Chapter 3

Schrödinger type operator with unbounded diffusion

In this chapter we want to study the Schrödinger type operator L_0 with unbounded diffusion perturbed by the singular potential V,

$$L_0 u = (1 + |x|^\alpha)\Delta u + \frac{c}{|x|^2} u =: Lu + Vu.$$

This operator has been studied in the paper [23]. In particular, we look for sufficient conditions on $\alpha \geq 0$ and $c \in \mathbb{R}$ ensuring that the space of test functions $C_c^\infty(\mathbb{R}^N)$ is a core for the operator and L_0 with a suitable domain generates a quasi-contractive and positivity preserving C_0-semigroup in $L^p(\mathbb{R}^N)$, $1 < p < \infty$. Furthermore, we also discuss the generation of a C_0-semigroup for the operator

$$\widetilde{L}_0 u := (1 + |x|^\alpha)\Delta u - \eta |x|^\beta u + \frac{c}{|x|^2} u,$$

where η is a positive constant, $\alpha > 2$ and $\beta > \alpha - 2$. In this case less conditions on the parameters occur.

In order to study generation results for L_0, we treat the operator as a perturbation of the elliptic operator

$$L = (1 + |x|^\alpha)\Delta,$$

with $\alpha \geq 0$. In fact, our approach relies on the following perturbation result due to N. Okazawa, see [53, Theorem 1.7].

Theorem 3.1. *Let A and B be linear m–accretive operators in $L^p(\mathbb{R}^N)$, with $p \in (1, +\infty)$. Let D be a core of A. Assume that*

(i) *there are constants* $\tilde{c}, a \geq 0$ *and* $k_1 > 0$ *such that for all* $u \in D$ *and* $\varepsilon > 0$

$$\mathcal{R}e(Au, \|B_\varepsilon u\|_p^{2-p}|B_\varepsilon u|^{p-2}B_\varepsilon u) \geq k_1\|B_\varepsilon u\|_p^2 - \tilde{c}\|u\|_p^2 - a\|B_\varepsilon u\|_p\|u\|_p, \tag{3.1}$$

where B_ε *denote the Yosida approximation of* B*;*

(ii) $\mathcal{R}e(u, \|B_\varepsilon u\|_p^{2-p}|B_\varepsilon u|^{p-2}B_\varepsilon u) \geq 0$*, for all* $u \in L^p(\mathbb{R}^N)$ *and* $\varepsilon > 0$*;*

(iii) *there is* $k_2 > 0$ *such that* $A - k_2B$ *is accretive.*

Set $k = \min\{k_1, k_2\}$*. If* $c > -k$ *then* $A + cB$ *with domain* $D(A+cB) = D(A)$ *is* m*–accretive and any core of* A *is also a core for* $A + cB$*. Furthermore,* $A - kB$ *is essentially* m*–accretive on* $D(A)$*.*

In order to apply the above theorem, we need some preliminary results on the operator L and the p-weighted Hardy inequality (1.2) which we recall for convenience of the reader. For all $u \in W^{1,p}(\mathbb{R}^N)$ with compact support, one has

$$\gamma_\alpha \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^2}|x|^\alpha dx \leq \int_{\mathbb{R}^N} |\nabla u|^2\, |u|^{p-2}|x|^\alpha dx$$

where $N \geq 3, \alpha \geq 0$ and $\gamma_\alpha = \left(\frac{N+\alpha-2}{p}\right)^2$.

Remark 3.2. Hardy's inequality (1.2) holds even if u is replaced by $u_+ := \sup(u, 0)$, since $u_+ \in W^{1,p}(\mathbb{R}^N)$, whenever $u \in W^{1,p}(\mathbb{R}^N)$ (cf. [27, Lemma 7.6]).

3.1 Preliminary results

Let us begin with the generation results for suitable realizations L_p of the operator L in $L^p(\mathbb{R}^N)$, $1 < p < \infty$. Such results have been proved in [24, 39, 46]. More specifically, the case $\alpha \leq 2$ has been investigated in [24] for $1 < \alpha \leq 2$ and in [39] for $\alpha \leq 1$, where the authors proved the following theorem.

Theorem 3.3. *If* $\alpha \in [0, 2]$ *then, for any* $p \in (1, +\infty)$*, the realization* L_p *of* L *with domain*

$$D_p = \{u \in W^{2,p}(\mathbb{R}^N) : |x|^\alpha|D^2u|, |x|^{\alpha/2}|\nabla u| \in L^p(\mathbb{R}^N)\}$$

generates a positive and strongly continuous holomophic semigroup. Moreover $C_c^\infty(\mathbb{R}^N)$ *is a core for* L_p*.*

The case $\alpha > 2$ is more involved and is studied in [46], where the following facts are established.

Theorem 3.4. *Assume that $\alpha > 2$.*

1. *If $N = 1, 2$, no realization of L in $L^p(\mathbb{R}^N)$ generates a strongly continuous (resp. holomorphic) semigroup.*

2. *The same happens if $N \geq 3$ and $p \leq N/(N-2)$.*

3. *If $N \geq 3$, $p > N/(N-2)$ and $2 < \alpha \leq (p-1)(N-2)$, then the maximal realization L_p of the operator L in $L^p(\mathbb{R}^N)$ with the maximal domain*

$$D_{\max} = \{u \in W^{2,p}(\mathbb{R}^N) : (1 + |x|^\alpha)\Delta u \in L^p(\mathbb{R}^N)\}$$

   generates a positive C_0-semigroup of contractions, which is also holomorphic if $\alpha < (p-1)(N-2)$.

4. *If $N \geq 3$, $p > N/(N-2)$ and $2 < \alpha < \frac{N(p-1)}{p}$ the domain $D_{\max}$ coincides with the space*

$$\widehat{D_p} = \{u \in W^{2,p}(\mathbb{R}^N) : |x|^{\alpha-2}u, |x|^{\alpha-1}|\nabla u|, |x|^\alpha|D^2 u| \in L^p(\mathbb{R}^N)\}.$$

   Moreover, $C_c^\infty(\mathbb{R}^N)$ is a core for L.

If we consider the operator $\widetilde{L} := L - \eta|x|^\beta$ with $\eta > 0$ and $\beta > \alpha - 2$ then we can drop the above conditions on p, α and N, as the following result shows, see [14], where the quasi-contractivity can be deduced from the proof of Theorem 4.5 in [14].

Theorem 3.5. *Assume $N \geq 3$. If $\alpha > 2$ and $\beta > \alpha - 2$ then, for any $p \in (1, \infty)$, the realization $\widetilde{L}_p$ of $\widetilde{L}$ with domain*

$$\widetilde{D_p} = \{u \in W^{2,p}(\mathbb{R}^N) : |x|^\beta u, |x|^{\alpha-1}|\nabla u|, |x|^\alpha|D^2 u| \in L^p(\mathbb{R}^N)\}$$

generates a positive and strongly continuous quasi-contractive holomorphic semigroup. Moreover, $C_c^\infty(\mathbb{R}^N)$ is a core for $\widetilde{L}_p$.

As a consequence of Lemma 1.6 we have the following results.

Proposition 3.6. *Assume $\alpha \leq (N-2)(p-1)$. Let $V \in L^p_{\text{loc}}(\mathbb{R}^N \setminus \{0\})$. If $V(x) \leq \frac{c}{|x|^2}$, $x \neq 0$, with $c \leq (p-1)\gamma_0$, then $L + V$ with domain $C_c^\infty(\mathbb{R}^N)$ is dissipative in $L^p(\mathbb{R}^N)$.*

Proof. Let $u \in C_c^\infty(\mathbb{R}^N)$. Take $\delta > 0$ if $1 < p < 2$ and $\delta = 0$ if $p \geq 2$. Then we have

$$\begin{aligned}
(Lu, u(|u|^2+\delta)^{\frac{p-2}{2}}) &= -\int_{\mathbb{R}^N} \nabla u \cdot \nabla \left(\overline{u}(|u|^2+\delta)^{\frac{p-2}{2}}\right)(1+|x|^\alpha)dx \\
&\quad - \alpha \int_{\mathbb{R}^N} \overline{u}(|u|^2+\delta)^{\frac{p-2}{2}} \nabla u \cdot x|x|^{\alpha-2}dx \\
&= -\int_{\mathbb{R}^N} (|u|^2+\delta)^{\frac{p-2}{2}} |\nabla u|^2 (1+|x|^\alpha)dx \\
&\quad - (p-2)\int_{\mathbb{R}^N} (|u|^2+\delta)^{\frac{p-4}{2}} (|u|\nabla|u|)\cdot(\overline{u}\nabla u)(1+|x|^\alpha)dx \\
&\quad - \alpha \int_{\mathbb{R}^N} \overline{u}(|u|^2+\delta)^{\frac{p-2}{2}} \nabla u \cdot x|x|^{\alpha-2}dx.
\end{aligned}$$

So, using the identities $|\nabla|u||^2 \leq |\nabla u|^2$ and $|u|\nabla|u| = \mathcal{R}e(\overline{u}\nabla u)$, we obtain

$$\begin{aligned}
\mathcal{R}e(Lu, u|u|^{p-2}) \leq -(p-1)&\int_{\mathbb{R}^N} |\nabla|u||^2|u|^{p-2}(1+|x|^\alpha)dx \\
-\alpha &\int_{\mathbb{R}^N} |u|^{p-1}\nabla|u| \cdot x|x|^{\alpha-2}dx
\end{aligned}$$

if $p \geq 2$. The case $1 < p < 2$ can be handled similarly. Thus, by Hölder's inequality one has

$$\begin{aligned}
&\mathcal{R}e((L+V)u, u|u|^{p-2}) \\
&\quad \leq -(p-1)\int_{\mathbb{R}^N} |\nabla|u||^2|u|^{p-2}(1+|x|^\alpha)dx + \int_{\mathbb{R}^N} V|u|^p dx \\
&\quad\quad + \alpha\left(\int_{\mathbb{R}^N} |\nabla|u||^2|u|^{p-2}|x|^\alpha dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^2}|x|^\alpha dx\right)^{\frac{1}{2}}. \qquad (3.2)
\end{aligned}$$

Set

$$\begin{aligned}
I_\alpha^2 &= \int_{\mathbb{R}^N} |\nabla|u||^2|u|^{p-2}|x|^\alpha dx, &\qquad J_\alpha^2 &= \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^2}|x|^\alpha dx, \\
I_0^2 &= \int_{\mathbb{R}^N} |\nabla|u||^2|u|^{p-2} dx, &\qquad J_0^2 &= \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^2} dx.
\end{aligned}$$

Taking the assumption on V into account we obtain

$$\mathcal{R}e((L+V)u, u|u|^{p-2}) \leq -(p-1)I_0^2 - (p-1)I_\alpha^2 + c\,J_0^2 + \alpha I_\alpha J_\alpha.$$

Since $c \leq (p-1)\gamma_0$ and Lemma 1.6 holds for $\alpha = 0$, we have that $-(p-1)I_0^2 + c\,J_0^2 \leq 0$. Now, the inequality

$$-(p-1)I_\alpha^2 + \alpha I_\alpha J_\alpha \leq 0$$

holds true if

$$-(p-1)+\alpha\gamma_\alpha^{-1/2}\le 0,$$

thanks again to Lemma 1.6. The latter inequality is equivalent to $\alpha \le (N-2)(p-1)$, which is the assumption. This ends the proof. $\square$

Remark 3.7. The assumption $\alpha \le (N-2)(p-1)$ is optimal for the dissipativity of L, as proved in [46, Proposition 8.2]. Indeed, if an operator of the form $L=(s^\alpha+|x|^\alpha)\Delta$, $s>0$ is dissipative, then $\alpha \le (N-2)(p-1)$. In fact, if we suppose L dissipative, then, for every $u\in D_{max}(L)$, u real-valued,

$$\int_{\mathbb{R}^N}(s^\alpha+|x|^\alpha)u|u|^{p-2}\Delta u\,dx\le 0.$$

If $u\in C_c^\infty(\mathbb{R}^N)$ we can integrate by parts twice and, using the identity $\nabla|u|^p=pu|u|^{p-2}\nabla u$, we get

$$\int_{\mathbb{R}^N}|u|^p|x|^{\alpha-2}\,dx\le\frac{p(p-1)}{\alpha(N+\alpha-2)}\int_{\mathbb{R}^N}(s^\alpha+|x|^\alpha)|u|^{p-2}|\nabla u|^2\,dx.$$

By applying the above inequality to $u(\lambda\cdot)$, for $\lambda>0$, we obtain

$$\int_{\mathbb{R}^N}|u(x)|^p|x|^{\alpha-2}\,dx\le\frac{p(p-1)}{\alpha(N+\alpha-2)}\int_{\mathbb{R}^N}(s^\alpha\lambda^\alpha+|x|^\alpha)|u|^{p-2}|\nabla u(x)|^2\,dx.$$

Letting $\lambda\to 0$ we get

$$\int_{\mathbb{R}^N}|u(x)|^p|x|^{\alpha-2}\,dx\le\frac{p(p-1)}{\alpha(N+\alpha-2)}\int_{\mathbb{R}^N}|x|^\alpha|u|^{p-2}|\nabla u(x)|^2\,dx$$

for every $u\in C_c^\infty(\mathbb{R}^N)$ and, by density, for every $u\in W^{1,p}(\mathbb{R}^N)$ with compact support. Since $\left(\frac{p}{N+\alpha-2}\right)^2$ is the best constant in Hardy inequality (1.2), we obtain

$$\frac{p(p-1)}{\alpha(N+\alpha-2)}\ge\left(\frac{p}{N+\alpha-2}\right)^2,$$

which implies $\alpha\le(p-1)(N-2)$.

Hence, in order to apply Theorem 3.1, we have established the following corollary.

Corollary 3.8. *Assume $\alpha\le(N-2)(p-1)$. Then, the operator $L+\dfrac{c}{|x|^2}$ with $c\le(p-1)\gamma_0$ and domain $C_c^\infty(\mathbb{R}^N)$ is dissipative in $L^p(\mathbb{R}^N)$.*

Let us recall the definition of dispersivity of an operator. A (real) linear operator A with domain $D(A)$ in $L^p(\mathbb{R}^N)$ is called dispersive if

$$(Au, u_+^{p-1}) \le 0 \quad \text{for all } u \in D(A).$$

For more details on dispersive operators we refer to [52, C-II.1].

Proposition 3.9. *Assume $\alpha \le (N-2)(p-1)$. Then, the operator $L + \frac{c}{|x|^2}$ with $c \le (p-1)\gamma_0$ and domain $C_c^\infty(\mathbb{R}^N)$ is dispersive in $L^p(\mathbb{R}^N)$.*

Proof. Let $u \in C_c^\infty(\mathbb{R}^N)$ be real-valued and fix $\delta > 0$. Replacing u by u_+ in the proof of Proposition 3.6 and since $u_+ \in W^{1,p}(\mathbb{R}^N)$, we deduce that

$$\begin{aligned}(Lu, u_+(u_+^2+\delta)^{\frac{p-2}{2}}) = &-\int_{\mathbb{R}^N} (u_+^2+\delta)^{\frac{p-2}{2}} |\nabla u_+|^2 (1+|x|^\alpha) dx \\ &- (p-2)\int_{\mathbb{R}^N} (u_+^2+\delta)^{\frac{p-4}{2}} u_+^2 |\nabla u_+|^2 (1+|x|^\alpha) dx \\ &- \alpha \int_{\mathbb{R}^N} u_+(u_+^2+\delta)^{\frac{p-2}{2}} \nabla u_+ \cdot x|x|^{\alpha-2} dx.\end{aligned}$$

Then,

$$\begin{aligned}(Lu, u_+(u_+^2+\delta)^{\frac{p-2}{2}}) \le (1-p) &\int_{\mathbb{R}^N} (u_+^2+\delta)^{\frac{p-4}{2}} u_+^2 |\nabla u_+|^2 (1+|x|^\alpha) dx \\ &- \alpha \int_{\mathbb{R}^N} u_+(u_+^2+\delta)^{\frac{p-2}{2}} \nabla u_+ \cdot x|x|^{\alpha-2} dx,\end{aligned}$$

where here we take $\delta = 0$ if $p \ge 2$ and $\delta > 0$ if $1 < p < 2$. Thus, letting $\delta \to 0$ if $1 < p < 2$, and applying Hölder's inequality we obtain

$$((L + \frac{c}{|x|^2})u, u_+^{p-1}) \le (1-p)I_{0,+}^2 + (1-p)I_{\alpha,+}^2 + cJ_{0,+}^2 + \alpha I_{\alpha,+} J_{\alpha,+},$$

where

$$\begin{aligned} I_{\alpha,+}^2 &= \int_{\mathbb{R}^N} |\nabla u_+|^2 u_+^{p-2} |x|^\alpha dx, \qquad & J_{\alpha,+}^2 &= \int_{\mathbb{R}^N} \frac{u_+^p}{|x|^2} |x|^\alpha dx, \\ I_{0,+}^2 &= \int_{\mathbb{R}^N} |\nabla u_+|^2 u_+^{p-2} dx, \qquad & J_{0,+}^2 &= \int_{\mathbb{R}^N} \frac{u_+^p}{|x|^2} dx. \end{aligned}$$

As in the proof of Proposition 3.6, the assertion follows now by Lemma 1.6 and Remark 3.2. □

The next proposition deals with the operator $\widetilde{L} = L - \eta|x|^\beta$.

Proposition 3.10. *Let $V \in L^p_{loc}(\mathbb{R}^N \setminus \{0\})$ with $V \le \frac{c}{|x|^2}$, $x \neq 0$ and $c \le (p-1)\gamma_0$.*

(i) *If $\alpha \ge 2$, $\beta > \alpha - 2$ and $\eta > 0$ then the operator $\widetilde{L} + V$ with domain $C_c^\infty(\mathbb{R}^N)$ is quasi-dissipative in $L^p(\mathbb{R}^N)$.*

(ii) *If $0 \le \alpha \le (N-2)(p-1)$, $\beta = \alpha - 2$ then $\widetilde{L}+V$ with domain $C_c^\infty(\mathbb{R}^N)$ is dissipative in $L^p(\mathbb{R}^N)$ if*

$$\eta + \frac{(N+\alpha-2)^2}{pp'} - \frac{\alpha(N+\alpha-2)}{p} \ge 0. \tag{3.3}$$

Proof. *(i)* If $\beta > \alpha - 2$, applying (3.2) and Young's inequality we obtain

$$\begin{aligned} \mathcal{R}e((\widetilde{L}+V)u, u|u|^{p-2}) &\le -(p-1)I_0^2 - (p-1)I_\alpha^2 + cJ_0^2 + \varepsilon I_\alpha^2 \\ &\quad + \int_{\mathbb{R}^N} \left(\frac{\alpha^2}{4\varepsilon}|x|^{\alpha-2} - \eta |x|^\beta \right) |u|^p \, dx \\ &\le -(p-1)I_0^2 - (p-1-\varepsilon)I_\alpha^2 + cJ_0^2 + M\|u\|_p^p \end{aligned}$$

for $u \in C_c^\infty(\mathbb{R}^N)$ and any $\varepsilon > 0$, where M is a positive constant such that $\frac{\alpha^2}{4\varepsilon}|x|^{\alpha-2} - \eta|x|^\beta \le M$ for all $x \in \mathbb{R}^N$, which holds since $\beta > \alpha - 2 \ge 0$. Choosing now $\varepsilon \le p-1$ and applying (1.2) we obtain

$$\mathcal{R}e((\widetilde{L}+V)u, u|u|^{p-2}) \le M\|u\|_p^p$$

which means that $\widetilde{L}+V$ with domain $C_c^\infty(\mathbb{R}^N)$ is quasi-dissipative in $L^p(\mathbb{R}^N)$.
(ii) If $\beta = \alpha - 2$ then (3.2) gives

$$\mathcal{R}e((\widetilde{L}+V)u, u|u|^{p-2}) \le -(p-1)I_0^2 - (p-1)I_\alpha^2 + cJ_0^2 + \alpha I_\alpha J_\alpha - \eta J_\alpha^2.$$

If $\eta \ge 0$, then the conclusion easily follows as in the end of the proof of Proposition 3.6, under the assumption $c \le \gamma_0(p-1)$ and $\alpha \le (N-2)(p-1)$. If $\eta < 0$ then by Lemma 1.6 we have

$$-(p-1)I_\alpha^2 + \alpha I_\alpha J_\alpha - \eta J_\alpha^2 \le \big(-(p-1) + \alpha\gamma_\alpha^{-1/2} - \eta\gamma_\alpha^{-1} \big) I_\alpha^2.$$

The right-hand side is nonpositive if

$$\eta + \frac{(N+\alpha-2)^2}{pp'} - \frac{\alpha(N+\alpha-2)}{p} \ge 0.$$

□

Remark 3.11. Condition (3.3) is sharp. For a proof we refer to [42, Proposition 4.2].

3.2 Main results

In this section we state and prove the main results of this chapter which deal with generation of semigroups for the operators L_0 and $\widetilde{L}_0$.
In order to apply Theorem 3.1 to our situation, an inequality of the type of (3.1) is needed. We obtain the following lemma whose proof follows the same lines of [53, Lemma 3.4].

Lemma 3.12. *Set* $V_\varepsilon = \frac{1}{|x|^2+\varepsilon}$, $\varepsilon > 0$. *Assume* $\alpha \leq (N-2)(p-1)$. *Then for every* $u \in C_c^\infty(\mathbb{R}^N)$

$$\mathcal{R}e(-Lu, |V_\varepsilon u|^{p-2}V_\varepsilon u) \geq \beta_0 \int_{\mathbb{R}^N} V_\varepsilon^p |u|^p\,dx + \beta_\alpha \int_{\mathbb{R}^N} V_\varepsilon^p |u|^p |x|^\alpha\,dx, \tag{3.4}$$

where

$$\beta_0 = \frac{N(p-1)(N-2p)}{p^2}, \qquad \beta_\alpha = \frac{(Np-N-\alpha)(N+\alpha-2p)}{p^2}.$$

Moreover, if $N > 2p$ *then both* β_0 *and* β_α *are positive.*

Proof. Let $u \in C_c^\infty(\mathbb{R}^N)$ and set $u_\delta = \big((R|u|)^2+\delta\big)^{\frac{1}{2}}$, where $R^p := V_\varepsilon^{p-1}$. In the computations below, we have to take $\delta > 0$ in the case $1 < p < 2$, whereas we only take $\delta = 0$ to deal with the case $p \geq 2$. We have

$$(-Lu, |V_\varepsilon u|^{p-2}V_\varepsilon u) = -\lim_{\delta\to 0}\int_{\mathbb{R}^N} u_\delta^{p-2}R^2\bar{u}\,Lu\,dx.$$

Integrating by parts we have

$$\begin{aligned}
-\int_{\mathbb{R}^N} u_\delta^{p-2}R^2\bar{u}\,Lu\,dx &= \int_{\mathbb{R}^N} R^2\bar{u}\,\nabla u\cdot\nabla(u_\delta^{p-2})(1+|x|^\alpha)\,dx \\
&\quad + \int_{\mathbb{R}^N} u_\delta^{p-2}\nabla u\cdot\nabla(R^2\bar{u})(1+|x|^\alpha)\,dx \\
&\quad + \alpha\int_{\mathbb{R}^N} u_\delta^{p-2}R^2\overline{u}\,|x|^{\alpha-2}x\cdot\nabla u\,dx.
\end{aligned}\tag{3.5}$$

Now, computing $\nabla(u_\delta^{p-2})$ and writing $R^2\bar{u}\,\nabla u = R\bar{u}\big(\nabla(Ru) - u\nabla R\big)$ we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} R^2\bar{u}\,\nabla u\cdot\nabla(u_\delta^{p-2})(1+|x|^\alpha)dx \\
&\qquad = \frac{p-2}{2}\int_{\mathbb{R}^N} u_\delta^{p-4}R\bar{u}\,\nabla(R^2|u|^2)\cdot\nabla(R\,u)(1+|x|^\alpha)dx \\
&\qquad\quad - \frac{p-2}{2}\int_{\mathbb{R}^N} u_\delta^{p-4}R|u|^2\nabla(R^2|u|^2)\cdot\nabla R\,(1+|x|^\alpha)dx.
\end{aligned}$$

Using also the identity

$$\nabla(R^2\bar{u})\cdot\nabla u = |\nabla(R\,u)|^2 - u\nabla(R\,\bar{u})\cdot\nabla R + R\,\bar{u}\nabla R\cdot\nabla u$$

Equation (3.5) yields

$$\begin{aligned}
-\int_{\mathbb{R}^N} u_\delta^{p-2}R^2\bar{u}\,Lu\,dx = {} & \frac{p-2}{2}\int_{\mathbb{R}^N} u_\delta^{p-4}R\bar{u}\,\nabla(R^2|u|^2)\cdot\nabla(R\,u)(1+|x|^\alpha)dx \\
& + \int_{\mathbb{R}^N} u_\delta^{p-2}|\nabla(R\,u)|^2(1+|x|^\alpha)\,dx \\
& \underbrace{-\frac{p-2}{2}\int_{\mathbb{R}^N} u_\delta^{p-4}R|u|^2\nabla(R^2|u|^2)\cdot\nabla R\,(1+|x|^\alpha)dx}_{=I} \\
& + \underbrace{\int_{\mathbb{R}^N} u_\delta^{p-2}R\,\bar{u}\nabla R\cdot\nabla u(1+|x|^\alpha)\,dx}_{=J} \\
& \underbrace{-\int_{\mathbb{R}^N} u_\delta^{p-2}u\nabla(R\,\bar{u})\cdot\nabla R(1+|x|^\alpha)\,dx}_{=K} \\
& + \alpha\int_{\mathbb{R}^N} u_\delta^{p-2}R^2\overline{u}\,|x|^{\alpha-2}x\cdot\nabla u\,dx.
\end{aligned}$$

Now, introduce the function $Q = R^p$. Writing $\nabla(R^2|u|^2) = 2R|u|^2\nabla R + 2|u|R^2\nabla|u|$ we have

$$\begin{aligned}
I &= -(p-2)\int_{\mathbb{R}^N} u_\delta^{p-4}R^2\,|u|^4\,|\nabla R|^2(1+|x|^\alpha)dx \\
&\quad -(p-2)\int_{\mathbb{R}^N} u_\delta^{p-4}|u|^3\,R^3\nabla R\cdot\nabla|u|\,(1+|x|^\alpha)dx \\
&= -\frac{p-2}{p^2}\int_{\mathbb{R}^N} u_\delta^{p-4}R^{4-2p}\,|u|^4\,|\nabla Q|^2(1+|x|^\alpha)dx \\
&\quad -\frac{p-2}{p}\int_{\mathbb{R}^N} u_\delta^{p-4}|u|^3\,R^{4-p}\nabla Q\cdot\nabla|u|\,(1+|x|^\alpha)dx.
\end{aligned}$$

Moreover,

$$\begin{aligned}
J + K &= \int_{\mathbb{R}^N} u_\delta^{p-2} \Big(R\,\bar{u} \nabla R \cdot \nabla u - u \nabla(R\,\bar{u}) \cdot \nabla R \Big)(1+|x|^\alpha)\,dx \\
&= - \int_{\mathbb{R}^N} u_\delta^{p-2} |u|^2 |\nabla R|^2 (1+|x|^\alpha)\,dx \\
&\quad + 2i \int_{\mathbb{R}^N} u_\delta^{p-2}\, \mathcal{I}m(\bar{u}\,\nabla u) \cdot R \nabla R (1+|x|^\alpha)\,dx \\
&= -\frac{1}{p^2} \int_{\mathbb{R}^N} u_\delta^{p-2} |u|^2 R^{2-2p} |\nabla Q|^2 (1+|x|^\alpha)\,dx \\
&\quad + \frac{2i}{p} \int_{\mathbb{R}^N} u_\delta^{p-2} R^{2-p}\, \mathcal{I}m(\bar{u}\,\nabla u) \cdot \nabla Q (1+|x|^\alpha)\,dx.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&- \int_{\mathbb{R}^N} u_\delta^{p-2} R^2 \bar{u}\, Lu\,dx \\
&\qquad = (p-2) \int_{\mathbb{R}^N} u_\delta^{p-4} R\,|u|\, \nabla(R\,|u|) \cdot (R\,\bar{u}) \nabla(R\,u)(1+|x|^\alpha) dx \\
&\qquad\quad + \int_{\mathbb{R}^N} u_\delta^{p-2} |\nabla(R\,u)|^2 (1+|x|^\alpha)\,dx + J_\delta \\
&\qquad\quad + \frac{2i}{p} \int_{\mathbb{R}^N} u_\delta^{p-2} R^{2-p}\, \mathcal{I}m(\bar{u}\,\nabla u) \cdot \nabla Q (1+|x|^\alpha)\,dx \\
&\qquad\quad + \alpha \int_{\mathbb{R}^N} u_\delta^{p-2} R^2 \overline{u}\, |x|^{\alpha-2} x \cdot \nabla u\,dx,
\end{aligned}$$

where we have set

$$\begin{aligned}
J_\delta = &- \frac{p-2}{p^2} \int_{\mathbb{R}^N} u_\delta^{p-4} R^{4-2p}\, |u|^4\, |\nabla Q|^2 (1+|x|^\alpha) dx \\
&- \frac{p-2}{p} \int_{\mathbb{R}^N} u_\delta^{p-4} |u|^3\, R^{4-p} \nabla Q \cdot \nabla |u|\, (1+|x|^\alpha) dx \\
&- \frac{1}{p^2} \int_{\mathbb{R}^N} u_\delta^{p-2} |u|^2 R^{2-2p} |\nabla Q|^2 (1+|x|^\alpha)\,dx.
\end{aligned}$$

Now, we take the real parts of both sides and apply the identity $\mathcal{R}e(\bar{\phi}\nabla\phi) =$

$|\phi|\nabla|\phi|$ to obtain

$$\begin{aligned}
-\mathcal{R}e\int_{\mathbb{R}^N} u_\delta^{p-2}R^2\bar{u}\,Lu\,dx &= (p-2)\int_{\mathbb{R}^N} u_\delta^{p-4}R^2|u|^2|\nabla(R|u|)|^2(1+|x|^\alpha)dx\\
&\quad+\int_{\mathbb{R}^N} u_\delta^{p-2}|\nabla(R\,u)|^2(1+|x|^\alpha)\,dx+J_\delta\\
&\quad+\alpha\int_{\mathbb{R}^N} u_\delta^{p-2}R^2|u|\,|x|^{\alpha-2}x\cdot\nabla|u|\,dx\\
&=(p-2)\int_{\mathbb{R}^N} u_\delta^{p-2}|\nabla(R|u|)|^2(1+|x|^\alpha)dx\\
&\quad-(p-2)\delta\int_{\mathbb{R}^N} u_\delta^{p-4}|\nabla(R|u|)|^2(1+|x|^\alpha)dx\\
&\quad+\int_{\mathbb{R}^N} u_\delta^{p-2}|\nabla(R\,u)|^2(1+|x|^\alpha)\,dx+J_\delta\\
&\quad+\alpha\int_{\mathbb{R}^N} u_\delta^{p-2}R^2|u|\,|x|^{\alpha-2}x\cdot\nabla|u|\,dx.
\end{aligned}$$

Since the inequality $|\nabla\phi|\geq|\nabla|\phi||$ holds and $\delta=0$ if $p\geq 2$, $\delta>0$ if $1<p<2$ we can estimate as follows

$$\begin{aligned}
-\mathcal{R}e\int_{\mathbb{R}^N} u_\delta^{p-2}R^2\bar{u}\,Lu\,dx &\geq (p-1)\int_{\mathbb{R}^N} u_\delta^{p-2}|\nabla(R|u|)|^2(1+|x|^\alpha)dx+J_\delta\\
&\quad+\alpha\int_{\mathbb{R}^N} u_\delta^{p-2}R^2|u|\,|x|^{\alpha-2}x\cdot\nabla|u|\,dx
\end{aligned}$$

if $p\geq 2$ and

$$\begin{aligned}
-\mathcal{R}e\int_{\mathbb{R}^N} u_\delta^{p-2}R^2\bar{u}\,Lu\,dx &\geq (p-1)\int_{\mathbb{R}^N} u_\delta^{p-2}|\nabla(Ru)|^2(1+|x|^\alpha)dx+J_\delta\\
&\quad+\alpha\int_{\mathbb{R}^N} u_\delta^{p-2}R^2|u|\,|x|^{\alpha-2}x\cdot\nabla|u|\,dx
\end{aligned}$$

if $1<p<2$. Letting $\delta\to 0^+$, we are lead to

$$\begin{aligned}
\mathcal{R}e(-Lu,|V_\varepsilon u|^{p-2}V_\varepsilon u) &\geq (p-1)\int_{\mathbb{R}^N}(R|u|)^{p-2}|\nabla(R|u|)|^2(1+|x|^\alpha)dx\\
&\quad-\frac{p-1}{p^2}\int_{\mathbb{R}^N} R^{-p}\,|u|^p\,|\nabla Q|^2(1+|x|^\alpha)dx\\
&\quad-\frac{p-2}{p}\int_{\mathbb{R}^N}|u|^{p-1}\,\nabla Q\cdot\nabla|u|\,(1+|x|^\alpha)dx\\
&\quad+\alpha\int_{\mathbb{R}^N}|u|^{p-1}R^p|x|^{\alpha-2}x\cdot\nabla|u|\,dx,
\end{aligned}\tag{3.6}$$

where we have used again the inequality $|\nabla\phi| \geq |\nabla|\phi||$ in the first integral of the right-hand side of (3.6), since for $1<p<2$ we had $|\nabla R\,u|^2$ instead of $|\nabla(R|u|)|^2$. Now, by the identity $p|u|^{p-1}\nabla|u| = \nabla|u|^p$, integrating by parts and recalling the definition of R we infer

$$\begin{aligned}
-\frac{p-2}{p}&\int_{\mathbb{R}^N}|u|^{p-1}\,\nabla Q\cdot\nabla|u|\,(1+|x|^\alpha)dx\\
&= \frac{p-2}{p^2}\int_{\mathbb{R}^N}|u|^p\,\Delta R^p(1+|x|^\alpha)dx\\
&\quad+\frac{\alpha(p-2)}{p^2}\int_{\mathbb{R}^N}|u|^p\nabla R^p\cdot x|x|^{\alpha-2}dx\\
&= -\frac{2N(p-1)(p-2)}{p^2}\int_{\mathbb{R}^N}V_\varepsilon^p|u|^p(1+|x|^\alpha)dx\\
&\quad+\frac{4p(p-1)(p-2)}{p^2}\int_{\mathbb{R}^N}|x|^2V_\varepsilon^{p+1}|u|^p(1+|x|^\alpha)dx\\
&\quad-\frac{2\alpha(p-1)(p-2)}{p^2}\int_{\mathbb{R}^N}V_\varepsilon^p|u|^p|x|^\alpha dx
\end{aligned}$$

and

$$\begin{aligned}
\alpha&\int_{\mathbb{R}^N}|u|^{p-1}R^p|x|^{\alpha-2}x\cdot\nabla|u|\,dx\\
&= -\frac{\alpha}{p}\int_{\mathbb{R}^N}|u|^p|x|^{\alpha-2}x\cdot\nabla R^p dx-\frac{\alpha(N+\alpha-2)}{p}\int_{\mathbb{R}^N}R^p|x|^{\alpha-2}|u|^p dx\\
&= \frac{2(p-1)\alpha}{p}\int_{\mathbb{R}^N}V_\varepsilon^p|u|^p|x|^\alpha dx-\frac{\alpha(N+\alpha-2)}{p}\int_{\mathbb{R}^N}V_\varepsilon^{p-1}|u|^p|x|^{\alpha-2}dx.
\end{aligned}$$

Finally,

$$\int_{\mathbb{R}^N}|u|^pR^{-p}|\nabla R^p|^2(1+|x|^\alpha)\,dx = 4(p-1)^2\int_{\mathbb{R}^N}|x|^2V_\varepsilon^{p+1}|u|^p(1+|x|^\alpha)dx.$$

By using such formulas in (3.6) we obtain

$$\begin{aligned}
\mathcal{R}e(-Lu,|V_\varepsilon u|^{p-2}V_\varepsilon u) \geq (p-1)&\int_{\mathbb{R}^N}(R|u|)^{p-2}|\nabla(R|u|)|^2(1+|x|^\alpha)dx\\
&-\frac{4(p-1)}{p^2}\int_{\mathbb{R}^N}|x|^2V_\varepsilon^{p+1}|u|^p(1+|x|^\alpha)dx\\
&-\frac{2N(p-1)(p-2)}{p^2}\int_{\mathbb{R}^N}V_\varepsilon^p|u|^p(1+|x|^\alpha)dx\\
&+\frac{4\alpha(p-1)}{p^2}\int_{\mathbb{R}^N}V_\varepsilon^p|u|^p|x|^\alpha dx\\
&-\frac{\alpha(N+\alpha-2)}{p}\int_{\mathbb{R}^N}V_\varepsilon^{p-1}\frac{|u|^p}{|x|^2}|x|^\alpha dx.
\end{aligned}$$

Applying Lemma 1.6, and using that $|x|^2V_\varepsilon \le 1$ we are lead to

$$\begin{aligned}\mathcal{R}e(-Lu,|V_\varepsilon u|^{p-2}V_\varepsilon u) \ge{}& (p-1)\gamma_0\int_{\mathbb{R}^N}\frac{V_\varepsilon^{p-1}|u|^p}{|x|^2}dx\\ &+\frac{N+\alpha-2}{p}\left(\frac{p-1}{p}(N+\alpha-2)-\alpha\right)\int_{\mathbb{R}^N}\frac{V_\varepsilon^{p-1}|u|^p}{|x|^2}\,|x|^\alpha dx\\ &-\frac{p-1}{p^2}(4+2Np-4N)\int_{\mathbb{R}^N}V_\varepsilon^p|u|^p(1+|x|^\alpha)dx\\ &+\frac{4\alpha(p-1)}{p^2}\int_{\mathbb{R}^N}V_\varepsilon^p|u|^p|x|^\alpha dx.\end{aligned} \tag{3.7}$$

Since $\alpha \le (N-2)(p-1)$ we have $\frac{p-1}{p}(N+\alpha-2)-\alpha \ge 0$ and then from the estimate $|x|^2V_\varepsilon \le 1$ it follows that

$$\begin{aligned}\mathcal{R}e(-Lu,|V_\varepsilon u|^{p-2}V_\varepsilon u) \ge{}& (p-1)\gamma_0\int_{\mathbb{R}^N}V_\varepsilon^p|u|^p dx\\ &+\frac{N+\alpha-2}{p}\left(\frac{p-1}{p}(N+\alpha-2)-\alpha\right)\int_{\mathbb{R}^N}V_\varepsilon^p|u|^p\,|x|^\alpha dx\\ &-\frac{p-1}{p^2}(4+2Np-4N)\int_{\mathbb{R}^N}V_\varepsilon^p|u|^p(1+|x|^\alpha)dx\\ &+\frac{4\alpha(p-1)}{p^2}\int_{\mathbb{R}^N}V_\varepsilon^p|u|^p|x|^\alpha dx.\end{aligned}$$

Thus we have

$$\mathcal{R}e(-Lu,|V_\varepsilon u|^{p-2}V_\varepsilon u) \ge \beta_0\int_{\mathbb{R}^N}V_\varepsilon^p|u|^p dx+\beta_\alpha\int_{\mathbb{R}^N}V_\varepsilon^p|u|^p\,|x|^\alpha dx,$$

where

$$\beta_0=(p-1)\gamma_0-\frac{p-1}{p^2}(4+2Np-4N)=\frac{N(p-1)(N-2p)}{p^2}$$

$$\begin{aligned}\beta_\alpha ={}&\frac{N+\alpha-2}{p}\left(\frac{p-1}{p}(N+\alpha-2)-\alpha\right)\\ &-\frac{p-1}{p^2}(4+2Np-4N)+\frac{4\alpha(p-1)}{p^2}\\ ={}&\frac{(Np-N-\alpha)(N+\alpha-2p)}{p^2}.\end{aligned}$$

So, if $N>2p$ then $\beta_0>0$ and since $0\le\alpha\le(N-2)(p-1)<N(p-1)$ we deduce that $\beta_\alpha>0$. □

Remark 3.13. We rewrite estimate (3.7) as follows

$$\mathcal{R}e(-Lu,|V_\varepsilon u|^{p-2}V_\varepsilon u)\geq \beta_0\int_{\mathbb{R}^N}V_\varepsilon^p|u|^pdx+\int_{\mathbb{R}^N}(k_0+k_1V_\varepsilon|x|^2)V_\varepsilon^{p-1}|u|^p|x|^{\alpha-2}dx$$

where

$$k_0=\frac{N+\alpha-2}{p}\left(\frac{p-1}{p}(N+\alpha-2)-\alpha\right)$$
$$k_1=\frac{4\alpha(p-1)}{p^2}-\frac{p-1}{p^2}(4+2Np-4N).$$

Notice that $k_0\geq 0$ if $\alpha\leq (N-2)(p-1)$ and that $k_0+k_1=\beta_\alpha$. Now, $k_0+k_1V_\varepsilon|x|^2=f(|x|^2)$, where $f(r)=\frac{\varepsilon k_0+(k_0+k_1)r}{\varepsilon+r}$. Since $\inf\limits_{[0,\infty)} f=\min\{k_0,k_0+k_1\}=:\mu$ we find

$$\mathcal{R}e(-Lu-\mu|x|^{\alpha-2}u,|V_\varepsilon u|^{p-2}V_\varepsilon u)\geq \beta_0\int_{\mathbb{R}^N}V_\varepsilon^p|u|^pdx.$$

The easiest case (see Lemma 3.12) is when $\mu\geq 0$.

Now, we prove a similar estimate for the operator $\widetilde{L}=L-\eta|x|^\beta$.

Lemma 3.14. *Set $V_\varepsilon=\frac{1}{|x|^2+\varepsilon}$, $\varepsilon>0$. If $\beta>\alpha-2\geq 0$ and $\eta>0$, then for every $u\in C_c^\infty(\mathbb{R}^N)$*

$$\mathcal{R}e(-\widetilde{L}u-mu,|V_\varepsilon u|^{p-2}V_\varepsilon u)\geq \beta_0\int_{\mathbb{R}^N}V_\varepsilon^p|u|^pdx+\delta_\alpha\int_{\mathbb{R}^N}V_\varepsilon^p|u|^p|x|^\alpha dx,$$

where $m=\min_{x\in\mathbb{R}^N}\left(\frac{N+\alpha-2}{p}\cdot\frac{(p-1)(N-2)-\alpha}{p}|x|^{\alpha-2}+\eta|x|^\beta\right)$, β_0 is given in Lemma 3.12 and

$$\delta_\alpha=\frac{p-1}{p^2}(4\alpha-4-2Np+4N).$$

Proof. We proceed as in the proof of Lemma 3.12. From Remark 3.13 and the inequality $|x|^2V_\varepsilon\leq 1$ it follows that

$$\begin{aligned}
&\mathcal{R}e(-\widetilde{L}u,|V_\varepsilon u|^{p-2}V_\varepsilon u)\\
&\geq \beta_0\int_{\mathbb{R}^N}V_\varepsilon^p|u|^pdx\\
&\quad+\int_{\mathbb{R}^N}V_\varepsilon^{p-1}|u|^p\left(\frac{N+\alpha-2}{p}\cdot\frac{(p-1)(N-2)-\alpha}{p}|x|^{\alpha-2}+\eta|x|^\beta\right)dx\\
&\quad+\left(\frac{4\alpha(p-1)}{p^2}-\frac{p-1}{p^2}(4+2Np-4N)\right)\int_{\mathbb{R}^N}V_\varepsilon^p|u|^p|x|^\alpha dx\\
&\geq \beta_0\int_{\mathbb{R}^N}V_\varepsilon^p|u|^pdx+m\int_{\mathbb{R}^N}V_\varepsilon^{p-1}|u|^p\,dx+\delta_\alpha\int_{\mathbb{R}^N}V_\varepsilon^p|u|^p|x|^\alpha dx\,.
\end{aligned}$$

Thus the proof of the lemma is concluded. $\square$

Applying Corollary 3.8, Lemma 3.12 and Theorem 3.1 we obtain the following generation results. We distinguish the two cases $\alpha \leq 2$ and $\alpha > 2$ since the hypotheses on the unperturbed operator L are different.

Theorem 3.15. *Assume $0 \leq \alpha \leq 2$. Set $k = \min\{\beta_0, (p-1)\gamma_0\}$. If $2p < N$ and $\alpha \leq (N-2)(p-1)$ then, for every $c < k$ the operator $L + \frac{c}{|x|^2}$ endowed with the domain D_p defined in Theorem 3.3 generates a contractive positive C_0-semigroup in $L^p(\mathbb{R}^N)$. Moreover, $C_c^\infty(\mathbb{R}^N)$ is a core for such an operator. Finally, the closure of $\left(L + \frac{k}{|x|^2}, D_p\right)$ generates a contractive positive C_0-semigroup in $L^p(\mathbb{R}^N)$.*

Theorem 3.16. *Assume $\alpha > 2$. Set $k = \min\{\beta_0, (p-1)\gamma_0\}$. If $\frac{N}{N-2} < p < \frac{N}{2}$ and $\alpha < \frac{N(p-1)}{p}$, then for every $c < k$ the operator $L + \frac{c}{|x|^2}$ endowed with the domain $\widehat{D_p}$ given in Theorem 3.4 generates a contractive positive C_0-semigroup in $L^p(\mathbb{R}^N)$. Moreover, $C_c^\infty(\mathbb{R}^N)$ is a core for such an operator. Finally, the closure of $\left(L + \frac{k}{|x|^2}, \widehat{D_p}\right)$ generates a contractive positive C_0-semigroup in $L^p(\mathbb{R}^N)$.*

The proofs of the two above theorems are identical. We limit ourselves in proving the latter.

Proof of Theorem 3.16. In order to apply Theorem 3.1, set $A = -L$, $D(A) = \widehat{D_p}$, $D = C_c^\infty(\mathbb{R}^N)$ and let B be the multiplicative operator by $\frac{1}{|x|^2}$ endowed with the maximal domain $D(|x|^{-2}) = \{u \in L^p(\mathbb{R}^N);\, |x|^{-2}u \in L^p(\mathbb{R}^N)\}$ in $L^p(\mathbb{R}^N)$. We observe that the Yosida approximation B_ε of B is the multiplicative operator by $V_\varepsilon = \frac{1}{|x|^2+\varepsilon}$. Both A and B are m–accretive in $L^p(\mathbb{R}^N)$. Then, Lemma 3.12 yields (i) in Theorem 3.1 with $k_1 = \beta_0$, $\tilde{c} = 0$ and $a = 0$. The second assumption (ii) in Theorem 3.1 is obviously satisfied. The last one, (iii), holds with $k_2 = (p-1)\gamma_0$ thanks to Corollary 3.8. Then, we infer that for every $c < k$, $-L - \frac{c}{|x|^2}$ with domain $\widehat{D_p}$ is m–accretive in $L^p(\mathbb{R}^N)$ and $C_c^\infty(\mathbb{R}^N)$ is a core for $-L - \frac{c}{|x|^2}$ by Theorem 3.4. Moreover, $-L - \frac{k}{|x|^2}$ is essentially m–accretive. By the Lumer Phillips Theorem (cf. Theorem A.8) we obtain the generation result. Finally, the positivity of the semigroup is a consequence of Proposition 3.9. The dispersivity is equivalent to the positivity of the resolvent, which is equivalent to the positivity of the semigroup. □

If $2p \geq N$, then $\beta_0 \leq 0$ and we cannot apply Theorem 3.1. However, if at least $\beta_\alpha \geq 0$, that is $2p - N \leq \alpha$, then we still have a generation result, relying on the following abstract theorem by Okazawa (see [53, Theorem 1.6]).

Theorem 3.17. *Let A and B be linear m–accretive operators in $L^p(\mathbb{R}^N)$, $1 < p < +\infty$. Let D be a core of A. Assume that there are constants $\tilde{c}, a, b \geq 0$ such that for all $u \in D$ and $\varepsilon > 0$,*

$$\mathcal{R}e(Au, \|B_\varepsilon u\|_p^{2-p}|B_\varepsilon u|^{p-2}B_\varepsilon u) \geq -b\|B_\varepsilon u\|_p^2 - \tilde{c}\|u\|_p^2 - a\|B_\varepsilon u\|_p\|u\|_p,$$

where $B_\varepsilon := B(I + \varepsilon B)^{-1}$ denotes the Yosida approximation of B. If $\nu > b$ then $A + \nu B$ with domain $D(A) \cap D(B)$ is m–accretive and $D(A) \cap D(B)$ is core for A. Moreover, $A + bB$ is essentially m–accretive on $D(A) \cap D(B)$.

In our framework the above result leads to the following theorems. We recall that $D(|x|^{-2}) = \{u \in L^p(\mathbb{R}^N);\, |x|^{-2}u \in L^p(\mathbb{R}^N)\}$.

Theorem 3.18. *Assume $0 \leq \alpha \leq 2$. If $2p \geq N$ and $2p - N \leq \alpha \leq (N-2)(p-1)$ then, for every $c < \beta_0$ the operator $L + \frac{c}{|x|^2}$ endowed with the domain $D_p \cap D(|x|^{-2})$, where D_p is defined in Theorem 3.3, generates a contractive holomorphic C_0-semigroup in $L^p(\mathbb{R}^N)$. Moreover, the closure of $\left(L + \frac{\beta_0}{|x|^2}, D_p \cap D(|x|^{-2})\right)$ generates a contractive holomorphic C_0-semigroup in $L^p(\mathbb{R}^N)$.*

Theorem 3.19. *Assume $\alpha > 2$. If $2p \geq N$ and $2p - N \leq \alpha < \frac{N(p-1)}{p}$, then for every $c < \beta_0$ the operator $L + \frac{c}{|x|^2}$ endowed with the domain $\widehat{D_p} \cap D(|x|^{-2})$, where $\widehat{D_p}$ is given in Theorem 3.4, generates a contractive holomorphic C_0-semigroup in $L^p(\mathbb{R}^N)$. Moreover, the closure of $\left(L + \frac{\beta_0}{|x|^2}, \widehat{D_p} \cap D(|x|^{-2})\right)$ generates a contractive holomorphic C_0-semigroup in $L^p(\mathbb{R}^N)$.*

As before, we limit ourselves in proving the latter.

Proof of Theorem 3.19. In order to apply Theorem 3.17, set $A = -L$, $D(A) = \widehat{D_p}$, $D = C_c^\infty(\mathbb{R}^N)$ and let B be the multiplicative operator by $\frac{1}{|x|^2}$ endowed with the maximal domain $D(|x|^{-2})$ in $L^p(\mathbb{R}^N)$. Both A and B are m–accretive in $L^p(\mathbb{R}^N)$. Then, Lemma 3.12 and Theorem 3.17 (with $b = -\beta_0$, $\tilde{c} = 0$ and $a = 0$) imply that $\left(L + \frac{c}{|x|^2}, \widehat{D_p} \cap D(|x|^{-2})\right)$ is m–accretive in $L^p(\mathbb{R}^N)$ for any $c < \beta_0$ and is essentially m–accretive if $c = \beta_0$. From the assumptions $2 < \alpha < \frac{N(p-1)}{p}$ it follows that $p > N/(N-2)$ and this yields $\alpha < (N-2)(p-1)$. Therefore, by Theorem 3.4, L generates a positive C_0-semigroup of contractions, which is also holomorphic. By inspecting the proof of [46, Theorem 8.1] it turns out that there exists $\ell_\alpha > 0$ such that

$$|\mathcal{I}m(Lu, |u|^{p-2}u)| \leq \ell_\alpha \left(-\mathcal{R}e(Lu, |u|^{p-2}u)\right)$$

for every $u \in \widehat{D_p}$ (the computations can be performed for $u \in C_c^\infty(\mathbb{R}^N)$ and then one get the estimate for $u \in \widehat{D_p}$ using the fact that $C_c^\infty(\mathbb{R}^N)$ is a core for L). Now, the previous estimate continues to hold for all $u \in \widehat{D_p} \cap D(|x|^{-2})$ replacing L with $L + \frac{c}{|x|^2}$, $c \le \beta_0$. This implies that $e^{\pm i\theta}\left(L + \frac{c}{|x|^2}\right)$ is dissipative, where $\cot\theta = \ell_\alpha$. By Theorem A.10, it follows that $L + \frac{c}{|x|^2}$ is sectorial and hence generates a holomorphic semigroup in $L^p(\mathbb{R}^N)$. This ends the proof. $\square$

If we consider the operator $\widetilde{L}$ instead of L the above conditions on p can be simplified. So, by Theorem 3.5, Proposition 3.10 and Lemma 3.14, we can apply Theorem 3.1 (Theorem 3.17, respectively) since $\delta_\alpha \ge 0$ if and only if $\alpha \ge 1 + \frac{N}{2}(p-2)$.

Theorem 3.20. *Assume $\beta > \alpha - 2 > 0$ and $\eta > 0$. Set $k = \min\{\beta_0, (p-1)\gamma_0\}$. If $\alpha \ge 1 + \frac{N}{2}(p-2)$ and $N > 2p$ then for every $c < k$, the operator $\widetilde{L} + \frac{c}{|x|^2}$ endowed with the domain $\widetilde{D_p}$ given in Theorem 3.5 generates a positive and quasi-contractive C_0-semigroup in $L^p(\mathbb{R}^N)$. Moreover, $C_c^\infty(\mathbb{R}^N)$ is a core for such an operator. Finally, the closure of $\left(\widetilde{L} + \frac{k}{|x|^2}, \widetilde{D_p}\right)$ generates a positive and quasi-contractive C_0-semigroup in $L^p(\mathbb{R}^N)$.*

Theorem 3.21. *Assume $\beta > \alpha - 2 > 0$ and $\eta > 0$. If $\alpha \ge 1 + \frac{N}{2}(p-2)$ and $N \le 2p$ then for every $c < \beta_0$, the operator $\widetilde{L} + \frac{c}{|x|^2}$ endowed with the domain $\widetilde{D_p} \cap D(|x|^{-2})$ generates a quasi-contractive C_0-semigroup in $L^p(\mathbb{R}^N)$. Moreover, the closure of $\left(\widetilde{L} + \frac{\beta_0}{|x|^2}, \widetilde{D_p} \cap D(|x|^{-2})\right)$ generates a quasi-contractive C_0-semigroup in $L^p(\mathbb{R}^N)$.*

Let us end with the study of the optimality of the constant β_0 in (3.4).

Proposition 3.22. *Assume that*

$$\mathcal{R}e(-Lu, |Vu|^{p-2}Vu) \ge C\|Vu\|_p^p, \tag{3.8}$$

for some $C > 0$, where $V = \frac{1}{|x|^2}$ and $\alpha \in \mathbb{N}$. Then, $C \le \beta_0$.

Proof. Take $u(x) = v(r) \ge 0$, $r = |x|$. Then

$$\begin{aligned}\mathcal{R}e(-Lu, &|Vu|^{p-2}Vu) \\ &= -\omega_N \int_0^{+\infty} (1 + r^\alpha)\left(v'' + \frac{N-1}{r}v'\right) r^{-2(p-1)} v^{p-1} r^{N-1} dr \\ &= J,\end{aligned}$$

where ω_N denotes the measure of the unit ball in $\mathbb{R}^N$. Choose $v(r) = r^\beta e^{-r/p}$, with $\beta > \frac{2p-N}{p}$. Then

$$J = -\omega_N \int_0^{+\infty} (1+r^\alpha)\left(\beta(\beta+N-2)r^{\delta-1} + \frac{1-N-2\beta}{p} r^\delta + \frac{1}{p^2} r^{\delta+1}\right) e^{-r} dr,$$

where we have set $\delta = \beta p + N - 2p$. Notice that $\delta > 0$ thanks to the choice of β. Using the properties of the Euler Gamma function, we have

$$\begin{aligned} J = &-\omega_N \left(\beta(\beta+N-2) + \frac{1-N-2\beta}{p}\delta + \frac{1}{p^2}\delta(\delta+1)\right)\Gamma(\delta) \\ &- \omega_N \left(\beta(\beta+N-2) + \frac{1-N-2\beta}{p}(\delta+\alpha) + \frac{1}{p^2}(\delta+\alpha)(\delta+\alpha+1)\right)\Gamma(\delta+\alpha). \end{aligned}$$

Now, observe that $\|Vu\|_p^p = \omega_N \Gamma(\delta)$. Hence from (3.8) it follows that

$$\begin{aligned} C\,\Gamma(\delta) \leq &-\left(\beta(\beta+N-2) + \frac{1-N-2\beta}{p}\delta + \frac{1}{p^2}\delta(\delta+1)\right)\Gamma(\delta) \\ &- \left(\beta(\beta+N-2) + \frac{1-N-2\beta}{p}(\delta+\alpha) + \frac{1}{p^2}(\delta+\alpha)(\delta+\alpha+1)\right)\Gamma(\delta+\alpha). \end{aligned}$$

If $\alpha = n \in \mathbb{N}$ then $\Gamma(\delta+n) = (\delta+n-1)\cdots\delta\Gamma(\delta)$ and the previous estimate yields

$$\begin{aligned} C \leq &-\left(\beta(\beta+N-2) + \frac{1-N-2\beta}{p}\delta + \frac{1}{p^2}\delta(\delta+1)\right) \\ &- \left(\beta(\beta+N-2) + \frac{1-N-2\beta}{p}(\delta+n) + \frac{1}{p^2}(\delta+n)(\delta+n+1)\right)(\delta+n-1)\cdots\delta. \end{aligned}$$

Letting $\delta \to 0^+$ which corresponds to $\beta \to \frac{2p-N}{p}$ eventually implies

$$C \leq \frac{N(p-1)(N-2p)}{p^2}.$$

Hence β_0 is the best constant for (3.8) to hold in the case $\alpha \in \mathbb{N}$. $\square$

Chapter 4

The biharmonic operator

In Chapter 2 we have studied generation results in $L^2(\mathbb{R}^N)$ for the Schrödinger operator $-H = \Delta + c|x|^{-2}$, i.e., the harmonic operator perturbed by an inverse second-order potential. In this chapter we want to present the biharmonic operator perturbed by an inverse fourth-order potential studied in [31]. In particular, we consider the operator

$$A = A_0 - V = \Delta^2 - \frac{c}{|x|^4}$$

where c is any constant such that $c < C^* := \left(\frac{N(N-4)}{4}\right)^2$. We will prove that the semigroup generated by $-A$ in $L^2(\mathbb{R}^N)$, $N \geq 5$, extrapolates to a bounded holomorphic C_0-semigroup on $L^p(\mathbb{R}^N)$ for $p \in [p_0', p_0]$ where $p_0 = \frac{2N}{N-4}$ and p_0' is its dual exponent.

To this purpose, let us consider the biharmonic operator

$$A_0 = \Delta^2.$$

It is included in a class of higher order elliptic operators studied by Davies in 1995, [18]. In particular, he proves that for $N < 4$, $-A_0$ generates a bounded C_0-semigroup on $L^p(\mathbb{R}^N)$ for all $1 \leq p < \infty$ and that Gaussian-type estimates for the heat kernel hold. In fact, denoting by K the heat kernel associated to the operator A_0, he proves that there exist $c_1, c_2, k > 0$ such that

$$|K(t,x,y)| \leq c_1 t^{-N/4} e^{-c_2 \frac{|x-y|^{4/3}}{t^{1/3}} + kt}$$

for all $t > 0$ and $x, y \in \mathbb{R}^N$.

The result is different when the dimension is greater than the order of the operator, $N > 4$. In this case he proves that the semigroup $(e^{-tA_0})_{t \geq 0}$

on $L^2(\mathbb{R}^N)$ extends to a bounded holomorphic C_0-semigroup on $L^p(\mathbb{R}^N)$ for all $p \in [p_0', p_0]$, where $p_0 = \frac{2N}{N-4}$ and p_0' is its dual exponent. An analogous situation holds when one replaces A_0 by A, which was remarked for example in [38, Section 6] by Liskevich, Sobol and Vogt (see also Proposition 4.9 below).

However, much more recently, in 2014, Quesada and Rodríguez-Bernal, using abstract parabolic arguments, prove that the biharmonic operator $-A_0$ generates a holomorphic semigroup in some suitable scale spaces $W^{4\alpha,p}(\mathbb{R}^N)$ for every $1 < p < \infty$ without restriction to the dimension N. They also obtain estimates of the semigroup. The result is the following.

Theorem 4.1. *[56, Lemma 5.2] Consider the problem*

$$\begin{cases} u_t + \Delta^2 u = 0 & t > 0, x \in \mathbb{R}^N, \\ u(0) = u_0 & \text{in } \mathbb{R}^N. \end{cases}$$

(i) Then, for each $1 < p < \infty$, the above problem defines a holomorphic semigroup, $S(t)$, in the space $W^{4\alpha,p}(\mathbb{R}^N)$, $\alpha \in \mathbb{R}$, such that for any $\mu_0 > 0$ there exists C such that

$$\|S(t)\|_{\mathcal{L}(W^{4\beta,p}(\mathbb{R}^N),W^{4\alpha,p}(\mathbb{R}^N))} \leq \frac{C(\alpha-\beta)}{t^{\alpha-\beta}} e^{\mu_0 t} \qquad t > 0, \alpha, \beta \in \mathbb{R}, \alpha \geq \beta.$$

(ii) The holomorphic semigroup $S(t)$ in $L^p(\mathbb{R}^N)$, $1 < p < \infty$, satisfies

$$\|S(t)\|_{\mathcal{L}(L^p(\mathbb{R}^N),L^r(\mathbb{R}^N))} \leq \frac{M_{p,r}}{t^{\frac{N}{4}\left(\frac{1}{p}-\frac{1}{r}\right)}} e^{\mu_0 t} \qquad t > 0$$

for any $\mu_0 > 0$ and $1 < p \leq r \leq \infty$ and some $M_{p,r} > 0$.

In this chapter, we will also study the boundedness of the Riesz transform $\Delta A^{-1/2}$ on $L^p(\mathbb{R}^N)$ for all $p \in (p_0', 2]$. We can define the Riesz transform associated to A by

$$\Delta A^{-1/2} := \frac{1}{\Gamma(1/2)} \int_0^\infty t^{-1/2} \Delta e^{-tA}\, dt.$$

The boundedness of the Riesz transform on $L^p(\mathbb{R}^N)$ implies that the domain of $A^{1/2}$ is included in the Sobolev space $W^{2,p}(\mathbb{R}^N)$. Thus, we obtain $W^{2,p}$-regularity of the solution to the evolution equation with initial datum in $L^p(\mathbb{R}^N)$. The boundedness of the Riesz transforms for Schrödinger operators has widely been studied in harmonic analysis. Several authors have generalized the results for elliptic operators L of order $2m$ or for Riemannian manifolds, see for example [4, 5, 10] and the references therein. Blunck and Kunstmann in [10] apply the Calderón-Zygmund theory for non-integral operators to obtain estimates on $\Delta L^{-1/2}$ since, in general, operators of order $2m$ do not satisfy Gaussian bounds if $2m < N$. More precisely, they prove an abstract criterion for estimates of the type

$$\|BL^{-\alpha}f\|_{L^p(\Omega)} \leq C_p\|f\|_{L^p(\Omega)}, \qquad p \in (q_0, 2],$$

where B, L are linear operators, $\alpha \in [0,1)$, $q_0 \in [1,2)$ and Ω is a measure space. We apply this criterion (Theorem 4.10 below) to our situation $(B, L, \Omega) = (\Delta, A, \mathbb{R}^N)$ with $q_0 = p_0'$.

We treat the operator $A = \Delta^2 - V$ in $L^2(\mathbb{R}^N)$ as the operator associated with the form

$$\mathfrak{a}(u,v) = (\Delta u, \Delta v) - (Vu, v) = \int_{\mathbb{R}^N} \Delta u \Delta \overline{v}\, dx - \int_{\mathbb{R}^N} V u \overline{v}\, dx$$

with $D(\mathfrak{a}) = \{u \in H^2(\mathbb{R}^N) : \||V|^{1/2}u\|_2 < \infty\}$. As a consequence of the Rellich inequality (1.25)

$$\left(\frac{N(N-4)}{4}\right)^2 \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^4}\, dx \leq \int_{\mathbb{R}^N} |\Delta u(x)|^2\, dx$$

for all $u \in H^2(\mathbb{R}^N)$ with $N \geq 5$, one obtains $D(\mathfrak{a}) = H^2(\mathbb{R}^N)$ and

$$\mathfrak{a}(u) := \int_{\mathbb{R}^N} |\Delta u(x)|^2\, dx - \int_{\mathbb{R}^N} V(x)|u(x)|^2\, dx \geq \eta \int_{\mathbb{R}^N} |\Delta u(x)|^2\, dx \tag{4.1}$$

for some $\eta \in (0,1)$, i.e., $\mathfrak{a}$ is densely defined and positive semi-definite. Indeed, since $c < C^*$, there exists a $0 < \alpha < 1$ such that $\frac{c}{\alpha} < C^*$ and

$$c \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^4}\, dx \leq \alpha \int_{\mathbb{R}^N} |\Delta u(x)|^2\, dx.$$

Therefore, we have that the form $\mathfrak{a}$ with domain $D(\mathfrak{a}) = H^2(\mathbb{R}^N)$ is densely defined, accretive, closed and continuous. In order to show the closeness, which is $(D(\mathfrak{a}), \|\cdot\|_{\mathfrak{a}})$ is complete, we only need to notice that the norms $\|\cdot\|_{\mathfrak{a}}$ and $\|\cdot\|_{H^2}$ are equivalent thanks to (1.25) and (4.1). Moreover, since $\mathfrak{a}$ is a symmetric, positive semi-definite sesquilinear form, by a simple appication of the Cauchy-Schwarz inequality, $\mathfrak{a}$ is continuous. Consequently, see for example Theorem C.9, $-A$ is the generator of a C_0-semigroup $(e^{-tA})_{t\geq 0}$ on $L^2(\mathbb{R}^N)$ that is contractive and holomorphic on the sector $\Sigma(\pi/2)$.

In the following, making use of multiplication operators and off-diagonal estimates, we prove that, for $N \geq 5$, the semigroup $(e^{-tA})_{t\geq 0}$ extrapolates to a bounded holomorphic C_0-semigroup on $L^p(\mathbb{R}^N)$ for all $p \in [p_0', p_0]$ and that the Riesz transform associated to A is bounded on $L^p(\mathbb{R}^N)$ for all $p \in (p_0', 2]$.

4.1 The twisted semigroup

In order to show the boundedness of the Riesz transform and obtain off-diagonal estimates for the semigroup generated by $-A$ we use the classical Davies perturbation technique, and estimate the twisted semigroup. Therefore, denoting by α a multi-index with $|\alpha| = \alpha_1 + \cdots + \alpha_N$ and D^α the corresponding partial differential operator on $C^\infty(\mathbb{R}^N)$, we define $\mathcal{E} := \{\phi \in C^\infty(\mathbb{R}^N;\mathbb{R})$ bounded : $|D^\alpha\phi| \le 1$ for all $1 \le |\alpha| \le 2\}$ and the twisted forms

$$\mathfrak{a}_{\lambda\phi}(u,v) := \mathfrak{a}(e^{-\lambda\phi}u, e^{\lambda\phi}v)$$

with $D(\mathfrak{a}_{\lambda\phi}) = H^2(\mathbb{R}^N)$, $\lambda \in \mathbb{R}$ and $\phi \in \mathcal{E}$. A simple computation shows that

$$A_{\lambda\phi} := e^{\lambda\phi}Ae^{-\lambda\phi}$$

with $D(A_{\lambda\phi}) = \{u \in L^2(\mathbb{R}^N) : e^{-\lambda\phi}u \in D(A)\}$ is the operator associated with the form $\mathfrak{a}_{\lambda\phi}$. Moreover, there exist $0 < \gamma < 1$ and $k > 1$ such that the inequality

$$|\mathfrak{a}_{\lambda\phi}(u) - \mathfrak{a}(u)| \le \gamma\mathfrak{a}(u) + k(1+\lambda^4)\|u\|_2^2 \tag{4.2}$$

holds for all $u \in H^2(\mathbb{R}^N)$, $\lambda \in \mathbb{R}$ and $\phi \in \mathcal{E}$. Indeed, we have

$$\begin{aligned}\mathfrak{a}_{\lambda\phi}(u) = \mathfrak{a}(u) &+ \lambda^4 \int_{\mathbb{R}^N} |\nabla\phi|^4|u|^2\,dx - \lambda^2 \int_{\mathbb{R}^N} |\Delta\phi|^2|u|^2\,dx \\ &+ 4\lambda^3 i \operatorname{Im} \int_{\mathbb{R}^N} |\nabla\phi|^2\nabla\phi\cdot\nabla\overline{u}\,u\,dx + 2\lambda^2 \operatorname{Re} \int_{\mathbb{R}^N} |\nabla\phi|^2 u\Delta\overline{u}\,dx \\ &- 4\lambda^2 \operatorname{Re} \int_{\mathbb{R}^N} \Delta\phi\,\nabla\phi\cdot\nabla\overline{u}\,u\,dx + 2\lambda i \operatorname{Im} \int_{\mathbb{R}^N} \Delta\phi\,\overline{u}\Delta u\,dx \\ &- 4\lambda^2 \int_{\mathbb{R}^N} |\nabla\phi\cdot\nabla u|^2\,dx + 4\lambda i \operatorname{Im} \int_{\mathbb{R}^N} \nabla\phi\cdot\nabla\overline{u}\Delta u\,dx.\end{aligned}$$

Now, the application of (4.1), the Landau inequality

$$\|\nabla u\|_2^2 \le \|u\|_2\|\Delta u\|_2, \qquad u \in H^2(\mathbb{R}^N)$$

and Young's inequality yields for $0 < \varepsilon < 1$

$$\begin{aligned}
|\mathfrak{a}_{\lambda\phi}(u) - \mathfrak{a}(u)| &\le N^2(\lambda^4 + \lambda^2)\|u\|_2^2 \\
&\quad + 4(N\lambda^2\|u\|_2)(N^{1/2}|\lambda|\|\nabla u\|_2) + 2(N\lambda^2\varepsilon^{-1}\|u\|_2)(\varepsilon\|\Delta u\|_2) \\
&\quad + 4(N|\lambda|\|u\|_2)(N^{1/2}|\lambda|\|\nabla u\|_2) + 2(N|\lambda|\varepsilon^{-1}\|u\|_2)(\varepsilon\|\Delta u\|_2) \\
&\quad + 4N\lambda^2\|\nabla u\|_2^2 + 4(N^{1/2}|\lambda|\varepsilon^{-1}\|\nabla u\|_2)(\varepsilon\|\Delta u\|_2) \\
&\le 4\varepsilon^2\|\Delta u\|_2^2 + 4N^2(\lambda^2 + \lambda^4)\varepsilon^{-2}\|u\|_2^2 + 10N\lambda^2\varepsilon^{-2}\|\nabla u\|_2^2 \\
&\le 4\varepsilon^2\|\Delta u\|_2^2 + 4N^2(\lambda^2 + \lambda^4)\varepsilon^{-2}\|u\|_2^2 \\
&\quad + 10(N\lambda^2\varepsilon^{-3}\|u\|_2)(\varepsilon\|\Delta u\|_2) \\
&\le 9\varepsilon^2\|\Delta u\|_2^2 + 9N^2(\lambda^2 + \lambda^4)\varepsilon^{-6}\|u\|_2^2 \\
&\le (9\varepsilon^2/\eta)\mathfrak{a}(u) + 18N^2(1 + \lambda^4)\varepsilon^{-6}\|u\|_2^2.
\end{aligned}$$

For the rest of this chapter, we fix γ and k such that inequality (4.2) holds. Then the forms $\mathfrak{a}_{\lambda\phi} + 2k(1 + \lambda^4)$ are closed and uniformly sectorial (see for example Theorem C.4). Thus the operators $-A_{\lambda\phi} - 2k(1 + \lambda^4)$ generate contractive holomorphic C_0-semigroups on $L^2(\mathbb{R}^N)$ with a common sector of holomorphy $\Sigma(\Theta)$. Therewith, we can show the following lemma.

Lemma 4.2. (a) *For all $z \in \Sigma(\Theta)$, $\lambda \in \mathbb{R}$ and $\phi \in \mathcal{E}$ the following inequality holds*

$$\|e^{-zA_{\lambda\phi}}\|_{2\to 2} \le e^{2k(1+\lambda^4)\operatorname{Re} z}. \tag{4.3}$$

(b) *There exists $M_\Theta > 0$ such that*

$$\|\Delta e^{-zA_{\lambda\phi}}\|_{2\to 2} \le M_\Theta |z|^{-1/2} e^{2k(1+\lambda^4)\operatorname{Re} z} \tag{4.4}$$

holds for all $z \in \Sigma(\Theta/2)$, $\lambda \in \mathbb{R}$ and $\phi \in \mathcal{E}$.

Proof. Let $\lambda \in \mathbb{R}$ and $\phi \in \mathcal{E}$. As mentioned before, we have

$$\|e^{-z(A_{\lambda\phi}+2k(1+\lambda^4))}\|_{2\to 2} \le 1, \tag{4.5}$$

for all $z \in \Sigma(\Theta)$, which implies (4.3). Moreover, by the Cauchy formula,

$$\|(A_{\lambda\phi} + 2k(1 + \lambda^4))e^{-z(A_{\lambda\phi}+2k(1+\lambda^4))}\|_{2\to 2} \le (|z|\sin(\Theta/4))^{-1} \tag{4.6}$$

holds for all $z \in \Sigma(\Theta/2)$. Further, (4.1) and (4.2) yield

$$\begin{aligned}
(1-\gamma)\eta\|\Delta v\|_2^2 \le (1-\gamma)\mathfrak{a}(v) &\le \operatorname{Re}(\mathfrak{a}_{\lambda\phi}(v) + 2k(1 + \lambda^4)\|v\|_2^2) \\
&\le \|(A_{\lambda\phi} + 2k(1 + \lambda^4))v\|_2\|v\|_2
\end{aligned}$$

for all $v \in D(A_{\lambda\phi})$. Taking $v = e^{-z(A_{\lambda\phi}+2k(1+\lambda^4))}u$ and applying the estimates (4.5) and (4.6), we conclude (4.4) with $M_\Theta = 1/\sqrt{(1-\gamma)\eta\sin(\Theta/4)}$. □

Finally, we prove $L^p - L^q$ estimates for the twisted semigroups.

Lemma 4.3. *Let $p_0' \le p \le 2 \le q \le p_0$. Then there exists $M_{pq} > 0$ such that*

$$\|e^{-zA_{\lambda\phi}}u\|_q \le M_{pq}|z|^{-\frac{N}{4}(\frac{1}{p}-\frac{1}{q})}e^{2k(1+\lambda^4)\operatorname{Re} z}\|u\|_p$$

holds for all $z \in \Sigma(\Theta/2)$, $\lambda \in \mathbb{R}$, $\phi \in \mathcal{E}$ and $u \in L^2(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$.

Proof. Let $z \in \Sigma(\Theta/2)$, $\lambda \in \mathbb{R}$ and $\phi \in \mathcal{E}$. Then, by Sobolev's embedding theorem (cf. [1, Theorem 4.31]) and Lemma 4.2, one obtains

$$\|e^{-zA_{\lambda\phi}}u\|_{\frac{2N}{N-4}} \le C_{\mathrm{S}}\|\Delta e^{-zA_{\lambda\phi}}u\|_2 \le C_{\mathrm{S}}M_\Theta|z|^{-1/2}e^{2k(1+\lambda^4)\operatorname{Re} z}\|u\|_2 \tag{4.7}$$

for all $u \in L^2(\mathbb{R}^N)$. Applying the Riesz-Thorin interpolation theorem to $e^{-zA_{\lambda\phi}}$ with respect to the bounds (4.3) and (4.7), we achieve the $L^2 - L^q$ estimate

$$\|e^{-zA_{\lambda\phi}}\|_{2\to q} \le M_{2q}|z|^{-\frac{N}{4}(\frac{1}{2}-\frac{1}{q})}e^{2k(1+\lambda^4)\operatorname{Re} z}$$

with $M_{2q} = (C_{\mathrm{S}}M_\Theta)^{\frac{N}{2}(\frac{1}{2}-\frac{1}{q})}$. Then a duality argument yields the $L^p - L^2$ estimate. Finally, we only have to combine these two and use the semigroup property to conclude the $L^p - L^q$ estimate with $M_{pq} = (2C_{\mathrm{S}}M_\Theta)^{\frac{N}{2}(\frac{1}{p}-\frac{1}{q})}$. □

4.2 Off-diagonal estimates

In this section, we study off-diagonal estimates, which enable us to obtain the extrapolation of the semigroup $(e^{-tA})_{t\ge 0}$ and the boundedness of the Riesz transform $\Delta A^{-1/2}$.

We say that a family $(T(z))_{z\in\Sigma(\theta)}$, $\theta \in (0, \pi/2]$, of bounded linear operators on $L^2(\mathbb{R}^N)$ *satisfies $L^p - L^q$ off-diagonal estimates* for $1 \le p \le q \le \infty$ if there exist $c_1, c_2 > 0$ such that for each convex, compact subsets E, F of $\mathbb{R}^N$, for each $u \in L^2(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ supported in E and for all $z \in \Sigma(\theta)$, one has

$$\|T(z)u\|_{L^q(F)} \le c_1|z|^{-\gamma_{pq}}\exp\Big(-c_2\frac{d(E,F)^{4/3}}{|z|^{1/3}}\Big)\|u\|_p,$$

where $\gamma_{pq} = \frac{N}{4}\big(\frac{1}{p} - \frac{1}{q}\big)$ and

$$d(E,F) = \sup_{\phi\in\mathcal{E}}[\inf\{\phi(x) - \phi(y) : x \in E, y \in F\}].$$

Davies proved that this distance is equivalent to the Euclidean one if the sets E and F are disjoint, [18, Lemma 4]. We recall this result.

Lemma 4.4. *If E and F are disjoint, convex, compact subsets of $\mathbb{R}^N$, then*

$$d_e(E,F) \leq d(E,F) \leq N^{1/2} d_e(E,F),$$

where $d_e(E,F)$ is the Euclidean distance between E and F.

Remark 4.5. (a) Since the distance d between non-disjoint sets is zero, we can drop the assumption of disjointedness in the previous lemma without changing the statement.

(b) For $E, F \subset \mathbb{R}^N$ compact, convex, $x, y \in \mathbb{R}^N$ and $r > 0$ such that $E \subset B(x,r)$ and $F \subset B(y,r)$ we obtain

$$d(E,F)^{4/3} \geq 2^{-1/3}|x-y|^{4/3} - (2r)^{4/3}.$$

Indeed, we can estimate as follows

$$|x-y| \leq 2r + d_e(E,F) \leq 2r + d(E,F) \leq 2^{1/4}((2r)^{4/3} + d(E,F)^{4/3})^{3/4}.$$

The following proposition relates the results of the previous section with the notion of off-diagonal estimates.

Proposition 4.6. *Let $\theta \in (0, \pi/2]$ and $(T(z))_{z\in\Sigma(\theta)}$ be a family in $\mathcal{L}(L^2(\mathbb{R}^N))$ that satisfies*

$$T(z) = D_s T(s^4 z) D_{1/s}, \qquad s \in (0,1),\ z \in \Sigma(\theta), \tag{4.8}$$

where D_s is the dilation operator, i.e., $D_s v(x) = v(sx)$ a.e. for all $v \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$. Further let $1 \leq p \leq q < \infty$ and $M, \omega > 0$ such that

$$\|e^{\lambda\phi} T(z) e^{-\lambda\phi} u\|_q \leq M|z|^{-\gamma_{pq}} e^{\omega(1+\lambda^4)|z|} \|u\|_p$$

holds for all $z \in \Sigma(\theta)$, $\lambda > 0$, $\phi \in \mathcal{E}$ and $u \in L^2(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$. Then $(T(z))_{z\in\Sigma(\theta)}$ satisfies $L^p - L^q$ off-diagonal estimates.

Proof. Let $z \in \Sigma(\theta)$, E, F be convex, compact subsets of $\mathbb{R}^N$ and $u \in L^2(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ supported in E. Then the assumption yields

$$\begin{aligned}
\|T(z)u\|_{L^q(F)} &\leq \|e^{-\lambda\phi}\chi_F\|_\infty \|e^{\lambda\phi} T(z) e^{-\lambda\phi} e^{\lambda\phi}\chi_E u\|_q \\
&\leq e^{-\lambda \inf_F \phi} M|z|^{-\gamma_{pq}} e^{\omega(1+\lambda^4)|z|} \|\chi_E e^{\lambda\phi}\|_\infty \|u\|_p \\
&\leq e^{-\lambda(\inf_F \phi - \sup_E \phi)} M|z|^{-\gamma_{pq}} e^{\omega(1+\lambda^4)|z|} \|u\|_p
\end{aligned}$$

for all $\lambda > 0$ and $\phi \in \mathcal{E}$. Minimising the right-hand side with respect to $\phi \in \mathcal{E}$ and choosing λ as $\left(\frac{d(E,F)}{4\omega|z|}\right)^{1/3}$ we obtain

$$\|T(z)u\|_{L^q(F)} \leq e^{2\omega|z|} M|z|^{-\gamma_{pq}} \exp\Big(-c_\omega \frac{d(E,F)^{4/3}}{|z|^{1/3}}\Big)\|u\|_p$$

with $c_\omega = \frac{3}{4(4\omega)^{1/3}}$. Now, we use the scaling property to get rid of the factor $e^{2\omega|z|}$. For $s \in (0,1)$ we estimate

$$\begin{aligned}
\|T(z)u\|_{L^q(F)} &= \|D_s \chi_{sF} T(s^4 z)\chi_{sE} D_{1/s}u\|_q \\
&= s^{-\frac{N}{q}} \|\chi_{sF} T(s^4 z)\chi_{sE} D_{1/s}u\|_q \\
&\leq e^{2\omega s^4|z|} M|z|^{-\gamma_{pq}} \exp\Big(-c_\omega \frac{(d(sE,sF)/s)^{4/3}}{|z|^{1/3}}\Big) s^{-\frac{N}{p}} \|D_{1/s}u\|_p \\
&\leq e^{2\omega s^4|z|} M|z|^{-\gamma_{pq}} \exp\Big(-\frac{c_\omega}{N^{2/3}} \frac{d(E,F)^{4/3}}{|z|^{1/3}}\Big)\|u\|_p.
\end{aligned}$$

Taking $s \to 0$, we get $L^p - L^q$ off-diagonal estimates for $(T(z))_{z\in\Sigma(\theta)}$. □

Now, since $(e^{-zA})_{z\in\Sigma(\Theta/2)}$ satisfies the scaling property (4.8) thanks to the invariance of the Laplacian, we can infer from Lemma 4.3 the following statement.

Corollary 4.7. *The semigroup $(e^{-zA})_{z\in\Sigma(\Theta/2)}$ satisfies $L^p - L^q$ off-diagonal estimates for all $p \in [p_0', 2]$ and $q \in [2, p_0]$.*

Finally, we are able to state the following theorem.

Theorem 4.8. *The semigroup $(e^{-tA})_{t\geq 0}$ on $L^2(\mathbb{R}^N)$ extrapolates to a bounded holomorphic C_0-semigroup on $L^p(\mathbb{R}^N)$ for all $p \in [p_0', p_0]$. Moreover, one can choose a common sector of holomorphy for these semigroups.*

Proof. It suffices to show that the family $(e^{-zA})_{z\in\Sigma(\Theta/2)}$ is uniformly bounded on $L^p(\mathbb{R}^N)$ to infer the extrapolation to a bounded holomorphic C_0-semigroup on $L^p(\mathbb{R}^N)$. Moreover, we only have to treat the case $p \in (2, p_0]$.

Let $p \in (2, p_0]$, $z \in \Sigma(\Theta/2)$ and C_n be the cube with center $n|z|^{1/4}$ and edge length $|z|^{1/4}$ for all $n \in \mathbb{Z}^N$. Then, using the $L^2 - L^p$ off-diagonal estimates for $(e^{-zA})_{z\in\Sigma(\Theta/2)}$, Remark 4.5(b) and Hölder's inequality, we obtain

$$\begin{aligned}
\|\chi_{C_n} e^{-zA}\chi_{C_m}u\|_p &\leq c_1 e^{-c_2|m-n|^{4/3}} |z|^{-\gamma_{2p}} |C_m|^{\frac{1}{2}-\frac{1}{p}} \|\chi_{C_m}u\|_p \\
&= c_1 e^{-c_2|m-n|^{4/3}} \|\chi_{C_m}u\|_p
\end{aligned}$$

for all $m, n \in \mathbb{Z}^N$ and $u \in L^2(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ with $c_1, c_2 > 0$ independent of z, u, m and n. Since the operator $B\colon \ell^1(\mathbb{Z}^N) \to \ell^1(\mathbb{Z}^N)$ with

$$(Bx)_m = c_1 \sum_{n\in\mathbb{Z}^N} e^{-c_2|m-n|^{4/3}} x_n, \qquad m \in \mathbb{Z}^N,\ x \in \ell^1(\mathbb{Z}^N)$$

is bounded on $\ell^1(\mathbb{Z}^N)$ as well as on $\ell^\infty(\mathbb{Z}^N)$, the Riesz-Thorin interpolation theorem yields that B is also bounded on $\ell^p(\mathbb{Z}^N)$. Setting $\hat{u} = (\|\chi_{C_n} u\|_p)_{n\in\mathbb{Z}^N}$ we conclude

$$\|e^{-zA}u\|_p \leq \|B\hat{u}\|_{\ell^p} \leq \|B\|_{\ell^p\to\ell^p} \|\hat{u}\|_{\ell^p} \leq \|B\|_{\ell^p\to\ell^p} \|u\|_p$$

for all $u \in L^2(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$. □

We have provided this proof as an application of the previous results, which we also need in the next section to prove the boundedness of the Riesz transform. Actually, we could have also applied [38, Proposition 6.1], which holds in a general setting of higher order operators defined by closed, sectorial sesquilinear forms. We recall this statement according to the notations of our situation.

Proposition 4.9. *Let $\mathfrak{a}$ be a closed, sectorial sesquilinear form in $L^2(\mathbb{R}^N)$ with $D(\mathfrak{a}) = H^2(\mathbb{R}^N)$ such that for some $C, k > 0$*

$$\tfrac{1}{2}\|\Delta u\|_2^2 \leq \operatorname{Re}\mathfrak{a}(u) \leq C(\|\Delta u\|_2^2 + \|u\|_2^2)$$

and

$$|\mathfrak{a}_{\lambda\phi}(u) - \operatorname{Re}\mathfrak{a}(u)| \leq \tfrac{1}{4}\operatorname{Re}\mathfrak{a}(u) + k(1+\lambda^4)\|u\|_2^2$$

hold for all $u \in H^2(\mathbb{R}^N)$, $\lambda \geq 0$ and $\phi \in \mathcal{E}$. Then the holomorphic C_0-semigroup $(e^{-tA})_{t\geq 0}$ on $L^2(\mathbb{R}^N)$, associated with $\mathfrak{a}$, extrapolates to a holomorphic C_0-semigroup $T_p = (e^{-tA_p})_{t\geq 0}$ on $L^p(\mathbb{R}^N)$ for all $p \in [p_0', p_0]$. The sector of holomorphy of T_p and the spectrum $\sigma(A_p)$ are p-independent.

4.3 Riesz transform

We show that $\Delta A^{-1/2} \in \mathcal{L}(L^p(\mathbb{R}^N))$ for all $p \in (\frac{2N}{N+4}, 2]$. We already know that the Riesz transform of the operator A is bounded on $L^2(\mathbb{R}^N)$ thanks to the inequality

$$\eta\|\Delta u\|_2^2 \leq a(u) = \|A^{1/2}u\|_2^2, \qquad u \in H^2(\mathbb{R}^N)$$

and the selfadjointness of $A^{1/2}$. Then, provided $\Delta A^{-1/2}$ is of weak type (p_0', p_0'), we can use the Marcinkiewicz interpolation theorem to obtain the boundedness on $L^p(\mathbb{R}^N)$ for $p_0' < p \le 2$.

Let us recall the definition of weak type operators. Let (X, Σ, μ) be a measure space. An operator $L\colon L^1(\mu) \cap L^\infty(\mu) \to L^1(\mu) + L^\infty(\mu)$ is of weak type (p,p) for $1 \le p < \infty$, if there exists a constant C such that for any $f \in L^1(\mu) \cap L^\infty(\mu)$ and $\lambda > 0$, one has

$$\mu\{x : |Lf(x)| \ge \lambda\} \le C\lambda^{-p}\|f\|_p^p.$$

In order to prove that $\Delta A^{-1/2}$ is of weak type (p_0', p_0') we make use of [10, Theorem 1.1] in the following adapted form.

Theorem 4.10. *Let $1 \le p < 2 < q \le \infty$, $q_0 \in (p, \infty]$ and $(e^{-tA})_{t \ge 0}$ be a bounded holomorphic semigroup on $L^2(\mathbb{R}^N)$ such that A is injective and has dense range. Further, let $\alpha \in [0,1)$ and B a linear operator satisfying $D(A^\alpha) \subset D(B)$ and the weighted norm estimates*

$$\|\chi_{B(x,t^{1/4})} e^{-tA} \chi_{B(y,t^{1/4})}\|_{p\to q} \le c_1 t^{-\gamma_{pq}} \exp\Big(-c_2 \frac{|x-y|^{4/3}}{t^{1/3}}\Big) \tag{4.9}$$

$$\|\chi_{B(x,t^{1/4})} t^\alpha B e^{-e^{i\sigma}tA} \chi_{B(y,t^{1/4})}\|_{p\to q_0} \le c_1 t^{-\gamma_{pq_0}} \exp\Big(-c_2 \frac{|x-y|^{4/3}}{t^{1/3}}\Big) \tag{4.10}$$

hold for all $x, y \in \mathbb{R}^N$, $t > 0$, $|\sigma| < \frac{\pi}{2} - \theta$, for some $\theta > 0$. Then $BA^{-\alpha}$ is of weak type (p,p) provided $BA^{-\alpha}$ is of weak type $(2,2)$.

We can now state the main result of this section.

Theorem 4.11. *The Riesz transform $\Delta A^{-1/2}$ of the operator A is bounded on $L^p(\mathbb{R}^N)$ for all $p \in (p_0', 2]$.*

Proof. We show that the assumptions of Theorem 4.10 with $(B, \alpha, p, q, q_0) = (\Delta, 1/2, p_0', p_0, 2)$ are satisfied to infer that $\Delta A^{-1/2}$ is of weak type (p_0', p_0').

First, we observe that A is injective and selfadjoint and has therefore dense range. Moreover, we have $D(A^{1/2}) = D(\Delta)$ and $\Delta A^{-1/2}$ is bounded on $L^2(\mathbb{R}^N)$, hence of weak type $(2,2)$, as was pointed out above. Now, it remains to show that estimates of the form (4.9) and (4.10) are satisfied. Due to Remark 4.5(b), such estimates are direct consequences of $L^{p_0'} - L^{p_0}$ off-diagonal estimates for $(e^{-zA})_{z\in\Sigma(\Theta/2)}$, which we have already obtained, and $L^{p_0'} - L^2$ off-diagonal estimates for the family $(|z|^{1/2}\Delta e^{-zA})_{z\in\Sigma(\Theta/2)}$. To achieve the latter ones, we show that

$$\|e^{\lambda\phi}\Delta e^{-zA} e^{-\lambda\phi}\|_{2\to 2} \le M|z|^{-1/2} e^{\omega(1+\lambda^4)|z|} \tag{4.11}$$

holds for all $z \in \Sigma(\Theta/2)$, $\lambda \in \mathbb{R}$ and $\phi \in \mathcal{E}$ with some $M, \omega > 0$. Indeed, we compute

$$\begin{aligned} e^{\lambda\phi}\Delta e^{-zA}e^{-\lambda\phi}u &= e^{\lambda\phi}\Delta e^{-\lambda\phi}e^{-zA_{\lambda\phi}}u \\ &= (\lambda^2|\nabla\phi|^2 - \lambda\Delta\phi)e^{-zA_{\lambda\phi}}u - 2\lambda\nabla\phi\cdot\nabla e^{-zA_{\lambda\phi}}u \\ &\quad + \Delta e^{-zA_{\lambda\phi}}u, \end{aligned}$$

which can be estimated, thanks to (4.3) and (4.4), in the following way

$$\begin{aligned} \|e^{\lambda\phi}\Delta e^{-zA}e^{-\lambda\phi}u\|_2^2 &\leq 8N^2(1+\lambda^4)\|e^{-zA_{\lambda\phi}}u\|_2^2 \\ &\quad + 16N\lambda^2\|\nabla e^{-zA_{\lambda\phi}}u\|_2^2 + 4\|\Delta e^{-zA_{\lambda\phi}}u\|_2^2 \\ &\leq 16N^2(1+\lambda^4)\|e^{-zA_{\lambda\phi}}u\|_2^2 + 12\|\Delta e^{-zA_{\lambda\phi}}u\|_2^2 \\ &\leq 16N^2(1+\lambda^4)e^{4k(1+\lambda^4)|z|}\|u\|_2^2 \\ &\quad + 12M_\Theta^2|z|^{-1}e^{4k(1+\lambda^4)|z|}\|u\|_2^2 \\ &\leq 16(M_\Theta^2+N^2)|z|^{-1}e^{5k(1+\lambda^4)|z|}\|u\|_2^2. \end{aligned}$$

Combining inequality (4.11) with the $L^{p_0'} - L^2$ estimate of Lemma 4.3, we get

$$\|e^{\lambda\phi}\Delta e^{-zA}e^{-\lambda\phi}\|_{p_0'\to 2} \leq 2M_{p_0'2}M|z|^{-1}e^{\omega(1+\lambda^4)|z|}$$

for all $z \in \Sigma(\Theta/2)$, $\lambda > 0$ and $\phi \in \mathcal{E}$. Since the family $(|z|^{1/2}\Delta e^{-zA})_{z\in\Sigma(\Theta/2)}$ satisfies the scaling property (4.8), it also satisfies $L^{p_0'} - L^2$ off-diagonal estimates by Propostion 4.6.

Thus, $\Delta A^{-1/2}$ is of weak type (p_0', p_0'). Now, by the boundedness of $\Delta A^{-1/2}$ on $L^2(\mathbb{R}^N)$ and the Marcinkiewicz interpolation theorem, we conclude that $\Delta A^{-1/2} \in \mathcal{L}(L^p(\mathbb{R}^N))$ for all $p \in (p_0', 2]$. □

Finally, we obtain the following corollary.

Corollary 4.12. *The parabolic problem associated to* $-A = -\Delta^2 + \frac{c}{|x|^4}$, $c < C^*$

$$\begin{cases} \partial_t u(t) = -Au(t) & \text{for } t \geq 0, \\ u(0) = f, \end{cases}$$

admits a unique solution for each initial datum $f \in L^p(\mathbb{R}^N)$, $p \in [p_0', p_0]$. *Moreover, if* $f \in L^p(\mathbb{R}^N)$ *for* $p \in (p_0', 2]$, *then* $u(t) \in W^{2,p}(\mathbb{R}^N)$ *for every* $t > 0$.

Appendix A

Introduction to semigroup theory

Semigroup theory is a wide and well documented subject. We give some definitions and recall some important results and properties, for details and proofs we refer to [21, 36, 41].

Definition A.1. Let X be a Banach space. A family $(T(t))_{t\geq 0}$ of bounded linear operators on X is called a *semigroup* if

(i) $T(t+s) = T(t)T(s), \quad \forall\, t, s \geq 0,$

(ii) $T(0) = I$.

If, moreover,

(iii) $\lim_{t\to 0} \parallel T(t)f - f \parallel = 0, \quad \forall\, f \in X,$

we call $(T(t))_{t\geq 0}$ a C_0*-semigroup* (or *strongly continuous semigroup*).

Strongly continuous semigroups are generated by linear operators A, generally unbounded, defined on the dense subspace $D(A)$ of X

$$D(A) = \{f \in X : \lim_{t\to 0} \frac{T(t)f - f}{t} \quad \text{exists}\} \quad \text{and}$$

$$Af := \lim_{t\to 0} \frac{T(t)f - f}{t}, \quad f \in D(A).$$

The domain $D(A)$ satisfies

$$T(t)D(A) \subseteq D(A) \quad \text{and} \quad AT(t)f = T(t)Af, \quad \forall\, t \geq 0,\ f \in D(A).$$

Moreover, if $f \in D(A)$, $T(\cdot)f$ is differentiable for every $t \geq 0$ and

$$\frac{d}{dt}T(t)f = AT(t)f, \quad t \geq 0.$$

Let $A : D(A) \to X$ be a given linear operator, it is interesting to establish if A is the generator of a C_0-semigroup, i.e., if there exists a semigroup $(T(t))_{t\geq 0}$ whose generator is A. In fact, if it is the case, for each $f \in D(A)$, the abstract Cauchy problem

$$\begin{cases} \frac{du}{dt}(t) = Au(t) & t \geq 0, \\ u(0) = f, \end{cases}$$

admits a unique solution given by $u(\cdot) := T(\cdot)f$. We will also denote the semigroup generated by A as $(e^{tA})_{t\geq 0}$.

In order to verify if an operator A generates a C_0-semigroup we give the Hille-Yoshida theorem, which is a central theorem in semigroup theory. Let us first recall some definitions for an operator in a Banach space.

Definition A.2. Let $(A, D(A))$ be an operator on X, the *resolvent set* $\rho(A)$ is the following

$$\rho(A) = \{\lambda \in \mathbb{C}\,\text{s.t.}\,(\lambda I - A) : D(A) \to X\,\text{bijective with bounded inverse}\},$$

and the *resolvent operator* is defined for $\lambda \in \rho(A)$ as

$$R(\lambda, A) = (\lambda I - A)^{-1} = \int_0^\infty e^{-\lambda t} e^{tA}\, dt.$$

An important property for an operator A is the closeness.

Definition A.3. Let $(A, D(A))$ be an operator on a Banach space X. A is *closed* if $f_n \in D(A)$, $f_n \to f$ and $Af_n \to g$, then $f \in D(A)$ and $Af = g$.

If an operator is not closed we can ask if it admits a closed extension. We define the smallest closed extension by $\overline{A}$, the closure of A.

Definition A.4. An operator A on a Banach space X is closable if there exists a closed operator $C : D(C) \subseteq X \to X$ such that $D(A) \subseteq D(C)$ and $Af = Cf$ for all $f \in D(A)$.
If A is a closable operator we define the smallest closed extension $\overline{A}$ as follows

$$D(\overline{A}) = \{f \in X \text{ s.t. } \exists f_n \in D(A) : \lim_{n\to\infty} f_n = f, \lim_{n,m\to\infty} (Af_n - Af_m) = 0\},$$

and if f and $(f_n)_{n\in\mathbb{N}}$ are as above we set

$$\overline{A}f := \lim_{n\to\infty} Af_n,$$

where the limits are taken with respect to the norm of X.

One can easily show that $\overline{A}$ is a closed operator and every closed extension of A is also an extension of $\overline{A}$.

Definition A.5. A subspace D of the domain $D(A)$ is called a *core* for A if D is dense in $D(A)$ for the graph norm

$$\|f\|_A := \|f\| + \|Af\|.$$

We now give the generation theorem.

Theorem A.6. (Hille-Yoshida) *Let $A : D(A) \to X$ be a closed and densely defined operator ($\overline{D(A)} = X$). Then A is the generator of a C_0-semigroup on X if and only if there exist $\omega \geq 0$ and $M > 0$ such that, for each $\lambda > \omega$, the operator $\lambda I - A$ is invertible and its inverse $R(\lambda, A) = (\lambda I - A)^{-1}$ satisfies the following inequality*

$$\| R(\lambda, A)^n \| \leq \frac{M}{(\lambda - \omega)^n}$$

for each $n \geq 1$.
Moreover, in this setting, the semigroup $(T(t))_{t\geq 0}$ satisfies the condition

$$\| T(t) \| \leq Me^{\omega t}, \ t \geq 0. \tag{A.1}$$

In particular, the semigroup $(T(t))_{t\geq 0}$ is called *contractive* if (A.1) holds with $\omega = 0$ and $M = 1$, i.e., $\|T(t)\| \leq 1$. It is called *quasi-contractive* if (A.1) holds with $M = 1$, i.e., $\|T(t)\| \leq e^{\omega t}$.

We now define further properties of an operator in a Banach space in order to obtain a different generation theorem named after the mathematicians Lumer and Phillips.

Definition A.7. Let $(A, D(A))$ be an operator on a Banach space X.

- A is called *accretive* if $-A$ is dissipative., i.e.,

  $$\|(\lambda + A)f\| \geq \lambda\|f\|$$

  for all $\lambda > 0$ and $f \in D(A)$.

- It is *m–accretive* if A is accretive and the range $R(\lambda+A) := (\lambda+A)D(A)$ coincides with X.

- If A is accretive, it is called *essentially m-accretive* if its closure $\overline{A}$ is m-accretive.

Theorem A.8. (Lumer-Phillips) *[21, Chap.II, Theorem 3.15] For a densely defined, dissipative operator $(A, D(A))$ on a Banach space X the following statements are equivalent.*

(a) The closure $\overline{A}$ of A generates a contraction semigroup.

(b) $R(\lambda - A)$ is dense in X for some (hence all) $\lambda > 0$.

Now, we define the class of holomorphic semigroups. They play an important role in the theory of evolution equations. We denote by

$$\Sigma(\delta) = \{\lambda \in \mathbb{C} : |\mathrm{arg}\lambda| < \delta\}\backslash\{0\}$$

the sector in $\mathbb{C}$ of angle δ. A closed linear operator $(A, D(A))$ in a Banach space X is called *sectorial (of angle δ)* if there exists $0 < \delta \leq \frac{\pi}{2}$ such that the sector $\Sigma\left(\frac{\pi}{2} + \delta\right)$ is contained in the resolvent set $\rho(A)$, and for each $\varepsilon \in (0, \delta)$ there exists $M_\varepsilon \geq 1$ such that

$$\|R(\lambda, A)\| \leq \frac{M_\varepsilon}{|\lambda|} \text{ for all } 0 \neq \lambda \in \overline{\Sigma}(\pi/2 + \delta - \varepsilon).$$

For densely defined sectorial operators we can define the following family of operators via the Cauchy integral formula.

Definition A.9. Let $(A, D(A))$ be a densely defined sectorial operator of angle δ. Define $T(0) := I$ and operators $T(z)$, for $z \in \Sigma(\delta)$, by

$$T(z) := \frac{1}{2\pi i} \int_\gamma e^{\mu z} R(\mu, A)\, d\mu,$$

where γ is any piecewise smooth curve in $\Sigma(\pi/2 + \delta)$ going from $\infty e^{-i(\pi/2+\delta')}$ to $\infty e^{i(\pi/2+\delta')}$ for some $\delta' \in (|\mathrm{arg} z|, \delta)$.

One can prove that the family so defined is a strongly continuous semigroup whose generator is the sectorial operator $(A, D(A))$, see [21, Propositions 4.3, 4.4]. We then define holomorphic semigroups.

A family $(T(z))_{z\in\Sigma(\delta)\bigcup\{0\}} \subset \mathcal{L}(X)$ is a *holomorphic semigroup* (*of angle* $\delta \in (0, \frac{\pi}{2}]$) if

(i) $T(0) = I$ and $T(z_1 + z_2) = T(z_1)T(z_2), \quad \forall\, z_1, z_2 \in \Sigma(\delta)$,

(ii) The map $z \mapsto T(z)$ is holomorphic in $\Sigma(\delta)$,

(iii) $\lim_{\Sigma(\delta')\ni z\to 0} T(z)x = x \quad \forall\, x \in X \quad$ e $\quad 0 < \delta' < \delta$.

If, in addition,

(iv) $\|T(z)\|$ is bounded in $\Sigma(\delta')$ for every $0 < \delta' < \delta$,

we call $(T(z))_{z\in\Sigma(\delta)\bigcup\{0\}}$ a *bounded holomorphic semigroup.*

The following theorem deals with generation of holomorphic semigroups.

Theorem A.10. *For an operator $(A, D(A))$ on a Banach space X, the following statements are equivalent.*

(a) A generates a bounded holomorphic semigroup $(T(z))_{z\in\Sigma(\delta\cup\{0\})}$ on X.

(b) There exists $\vartheta \in (0, \pi/2)$ such that the operators $e^{\pm i\vartheta}A$ generate bounded strongly continuous semigroups on X.

(c) A generates a bounded strongly continuous semigroup $(T(t))_{t\geq 0}$ on X such that $\mathrm{rg}(T(t)) \subset D(A)$ for all $t > 0$, and

$$M := \sup_{t>0} \|tAT(t)\| < \infty.$$

(d) A is densely defined and sectorial.

We also want to recall the integral rapresentation for the fractional power of a sectorial operator A through the generated semigroup $(T(t))_{t\geq 0}$.

Proposition A.11. *Let A be the generator of a bounded C_0-semigroup $(T(t))_{t\geq 0}$, in a Banach space X. If $0 \in \rho(A)$, then*

$$(-A)^{-z} = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} T(t)dt$$

for $0 < Re(z) < 1$.

Finally, we introduce the concept of weak generator. If one deals with non strongly continuous semigroups it is not possible to define the generator in a classical way. However, one can define a *weak generator* $\widetilde{A}$ in the space of bounded continuous functions as follows

$$D(\widetilde{A}) = \{f \in C_b(\mathbb{R}^N) : \sup_{t>0} \frac{\| T(t)f - f \|_\infty}{t} < \infty \quad \text{and} \quad \exists\, g \in C_b(\mathbb{R}^N)$$
$$\text{s.t.} \quad \lim_{t\to 0} \frac{T(t)f(x) - f(x)}{t} = g(x), \quad x \in \mathbb{R}^N\}$$

$$\widetilde{A}f(x) = \lim_{t\to 0} \frac{T(t)f(x) - f(x)}{t}, \quad f \in D(\widetilde{A}), \quad x \in \mathbb{R}^N.$$

For a weak generator similar properties to a generator of a C_0- semigroup hold.

(i) $T(t)f \in D(\widetilde{A})$, for all $t \geq 0$ e $\widetilde{A}T(t)f = T(t)\widetilde{A}f$,

(ii) for all $x \in \mathbb{R}^N$, the function $t \in [0, +\infty] \to T(t)f(x)$ is C^1 and $\frac{d}{dt}T(t)f(x) = T(t)\widetilde{A}f(x)$.

Moreover, the following result holds.

Proposition A.12. *(i) $D(\widetilde{A})$ is dense in $C_b(\mathbb{R}^N)$ with respect to the bounded pointwise convergence, i.e., for all $f \in C_b(\mathbb{R}^N)$, there exists a sequence $(f_n) \subseteq D(\widetilde{A})$, uniformly bounded and pointwise convergent to f;*

(ii) $(\widetilde{A}, D(\widetilde{A}))$ is closed in $C_b(\mathbb{R}^N)$ with respect to the bounded pointwise convergence, i.e., for all $(f_n) \subset D(\widetilde{A})$ such that $f_n \to f$ and $Af_n \to g$, with $g \in X$, then $f \in D(\widetilde{A})$ and $Af = g$.

There exists a relation between the operators

$$(\widetilde{A}, D(\widetilde{A})) \quad \text{and} \quad (A, D_{max}(A))$$

where

$$D_{max}(A) = \{u \in C_b(\mathbb{R}^N) \cap X : Au \in X\}.$$

To this purpose, one can study the spectral properties of $\widetilde{A}$ and obtain the following results.

Proposition A.13. *(i) A is an extension of $\widetilde{A}$, i.e., $D(\widetilde{A}) \subseteq D_{max}(A)$ e $\widetilde{A}f = Af$, for all $f \in D(\widetilde{A})$,* *(ii)* $D(\widetilde{A}) = D_{max}(A)$ *iff* $\lambda - A$ *is injective for one (and hence all)* $\lambda > 0$.

Proposition A.14. *The following statements are equivalent.*

(i) $\lambda - A$ *is injevtive in* $D_{max}(A)$ *(for one or all* $\lambda > 0$*);*

(ii) $T(t)\mathbb{1} = \mathbb{1}$, *for all* $t \geq 0$.

Remark A.15. Property (ii) is equivalent to the conservativity of the semigroup $(T(t))$. Thus, for conservative semigroups, the weak generator $\widetilde{A}$ coincides with the operator A.

Appendix B

Markov semigroups

Markov semigroups enjoy very nice properties because of the underlying probabilistic interpretation related to Markov processes on $\mathbb{R}^N$. They provide a solution to the parabolic equation associated to a second order elliptic operator with unbounded coefficients, A. We briefly give some definitions and describe the method to associate a Markov semigroup to A. For an extensive account on Markov semigroups we refer to [8, 16, 25].

We start with the definition of transition function.

Definition B.1. Let $\mathcal{B}(\mathbb{R}^N)$ be the set of Borelians of $\mathbb{R}^N$. We call *transition function* on $\mathbb{R}^N$ a function $p : [0, +\infty) \times \mathbb{R}^N \times \mathcal{B}(\mathbb{R}^N) \mapsto [0, +\infty)$ such that

(i) $p(t, x, \cdot)$ is a probability measure on $(\mathbb{R}^N, \mathcal{B}(\mathbb{R}^N))$, for all $t \geq 0$, for all $x \in \mathbb{R}^N$;

(ii) $p(0, x, \Gamma) = \chi_\Gamma(x)$ for all $x \in \mathbb{R}^N$ and all $\Gamma \in \mathcal{B}(\mathbb{R}^N)$;

(iii) $p(t, \cdot, \Gamma)$ is Borel measurable for all $t \geq 0$ and all $\Gamma \in \mathcal{B}(\mathbb{R}^N)$;

(iv) the semigroup property is satisfied, i.e,

$$p(t + s, x, \Gamma) = \int_{\mathbb{R}^N} p(s, x, dy) p(t, y, \Gamma)$$

for all $s, t \geq 0$, $x \in \mathbb{R}^N$ and $\Gamma \in \mathcal{B}(\mathbb{R}^N)$.

Remark B.2. If the measure $p(t, x, \cdot)$ is absolutely continuous with respect to the Lebesgue measure, one can write $p(t, x, dy) = p(t, x, y)dy$, where $p(t, x, y)$ is the density.

Therefore, we can define a Markov semigroup.

Definition B.3. For all transition functions $p(t,x,dy)$, we define the *Markov semigroup* as

$$T(t)f(x) = \int_{\mathbb{R}^N} p(t,x,dy)f(y), \qquad f \in L^\infty(\mathbb{R}^N),\ t \geq 0,\ x \in \mathbb{R}^N.$$

Thanks to the definition of transition function one obtains that the Markov semigroup satisfies the following properties:

- $T(0)f(x) = f(x), \quad \forall\, f \in L^\infty(\mathbb{R}^N),\ \forall\, x \in \mathbb{R}^N$;
- The law of semigroup is satisfied;
- $T(t) \in \mathcal{L}(L^\infty(\mathbb{R}^N))$ for all $t > 0$, (i.e., $T(t)$ is a bounded linear operator on $L^\infty(\mathbb{R}^N)$);
- $T(t)$ is contractive in $L^\infty(\mathbb{R}^N)$;
- $T(t)$ is conservative, i.e., $T(t)\mathbb{1}(x) = \mathbb{1}(x)$;
- $T(t)$ is positive, i.e., $f \geq 0 \Rightarrow T(t)f \geq 0$.

Let us consider a second order elliptic differential operator with unbounded coefficients. It is possible to construct the Markov semigroup associated to the operator.

Definition B.4. Let

$$Au(x) = \sum_{i,j=1}^{N} a_{ij}(x)D_{ij}u(x) + \sum_{i=1}^{N} b_i(x)D_iu(x)$$

with $\{a_{ij}\}_{i,j=1,\ldots,N}$ e $\{b_i\}_{i=1,\ldots,N}$ real valued functions on $\mathbb{R}^N$. We define the *maximal domain of A in $C_b(\mathbb{R}^N)$* as

$$D_{max}(A) = \{u \in C_b(\mathbb{R}^N) \cap W^{2,p}_{loc}(\mathbb{R}^N),\, \forall\, 1 < p < +\infty : Au \in C_b(\mathbb{R}^N)\}.$$

Since the coefficients of A are unbounded, in order to solve the abstract Cauchy problem associated to A, and hence, construct the Markov semigroup associated to A, one can proceed with a localising argument. We briefly describe the method. One considers the problem on the balls $B(R)$, to let

then $R \to \infty$.
Thus, we consider

$$\begin{cases} \partial_t u_R(t,x) = Au_R(t,x) & t>0, \quad x \in B(R), \\ u_R(t,x) = 0 & t>0, \quad x \in \partial B(R), \\ u_R(0,x) = f(x) & \qquad\quad x \in B(R), \end{cases} \tag{B.1}$$

with $f \in C_b(\mathbb{R}^N)$. In order to guarantee existence and regolarity of the solution, one has to assume that, for some $\alpha \in (0,1)$,

(1) $a_{ij} = a_{ji} \in C^\alpha_{loc}(\mathbb{R}^N)$, $b_i \in C^\alpha_{loc}(\mathbb{R}^N)$, for all $i,j = 1,...,N$;

(2) let $a(x) = (a_{ij}(x))^N_{i,j=1}$,

$$\langle a(x)\xi, \xi\rangle = \sum_{i,j} a_{ij}(x)\xi_i\xi_j \geq \nu(x)|\xi|^2$$

for all $x, \xi \in \mathbb{R}^N$ with $\inf_{x\in K} \nu(x) > 0$, for all K compact set of $\mathbb{R}^N$.

Then, A is uniformly elliptic on each compact set of $\mathbb{R}^N$. Then, the problem (B.1) admits a unique classical solution given by a holomorphic semigroup not strongly continuous in $C(\overline{B}(R))$

$$u_R(t,x) = T_R(t)f(x), \quad t \geq 0,\, x \in \overline{B}(R).$$

The infinitesimal generator of $(T_R(t))$ is the operator $(A, D_R(A))$, ([40, Chapter 3]), where

$$D_R(A) = \{u \in C_0(\overline{B}(R)) \cap W^{2,p}(B(R)), \forall\, p \in (1,+\infty) : Au \in C(\overline{B}(R))\}.$$

The following theorem provides useful properties for $(T_R(t))$.

Theorem B.5. *(i) $(T_R(t))$ admits the following integral representation*

$$T_R(t)f(x) = \int_{B(R)} p_R(t,x,y)f(y)dy, \quad f \in C(\overline{B}(R)),\, t>0,\, x \in \overline{B}(R)$$

with strictly positive kernel $p_R \in C((0,+\infty) \times B(R) \times B(R))$. In particular, $T_R(t) \geq 0$;

(ii) $T_R(t) \in \mathcal{L}(L^p(B_R))$ for all $t \geq 0$ and for all $1 < p < +\infty$;

(iii) $T_R(t)$ is contractive in $C(\overline{B}(R))$;

(iv) *given a bounded sequence in* $(f_n)_n \subset C(\overline{B}(R))$ *such that* $f_n \to f$ *pointwise in* $\overline{B}(R)$, *with* $f \in C(\overline{B}(R))$, *then* $T_R(t)f_n \to T_R(t)f$ *pointwise for all* $t \geq 0$;

(v) *for each* $y \in \overline{B}(R)$ *fixed,* $p_R(\cdot,\cdot,y) \in C^{1+\frac{\alpha}{2},2+\alpha}([s,t_0] \times \overline{B}(R))$ *for all* $0 < s < t_0$ *and one has*

$$\partial_t p_R(t,x,y) = A p_R(t,x,y), \quad \forall\,(t,x) \in (0,+\infty) \times \overline{B}(R).$$

Remark B.6. As a consequence, since $f \in C(\overline{B}(R))$, thanks to (v), one gets

$$u_R \in C^{1+\frac{\alpha}{2},2+\alpha}([s,t_0] \times \overline{B}(R))$$

where $u_R(t,x) = T_R(t)f(x)$.

Now, in order to let $R \to +\infty$, one studies the convergence of u_R.

Proposition B.7. *Let* $f \in C_b(\mathbb{R}^N)$ *and* $t \geq 0$; *then there exists*

$$T(t)f(x) = \lim_{R\to+\infty} T_R(t)f(x), \quad \forall\, x \in \mathbb{R}^N \tag{B.2}$$

and $(T(t))$ *is a positive semigroup in* $C_b(\mathbb{R}^N)$.

Therefore, as $R \to +\infty$, one obtains a semigroup $(T(t))$ in $C_b(\mathbb{R}^N)$. The following result holds.

Proposition B.8. *For the semigroup defined by (B.2) the following integral representation holds*

$$T(t)f(x) = \int_{\mathbb{R}^N} p(t,x,y)f(y)dy, \qquad f \in C_b(\mathbb{R}^N)$$

with $p(t,x,y) > 0$ *a.e.* $y \in \mathbb{R}^N$ *for all* $t > 0$ *and for all* $x \in \mathbb{R}^N$, $p(\cdot,\cdot,y) \in C_{loc}^{1+\frac{\alpha}{2},2+\alpha}((0,+\infty) \times \mathbb{R}^N)$ *and* $\partial_t p = Ap$ *in the couple* (t,x).

Therefore, it is possible to associate to A a Markov semigroup $(T(t))$, which is positive but not strongly continuous. One can prove that $(T(t))$ is the solution to the parabolic problem

$$\begin{cases} \partial_t u(t,x) = Au(t,x) & t > 0, \quad x \in \mathbb{R}^N, \\ u(0,x) = f(x) & \qquad\qquad x \in \mathbb{R}^N. \end{cases} \tag{B.3}$$

As a consequence of the classical Schauder estimates (see [44]) one obtains the following.

Theorem B.9. *Let $f \in C_b(\mathbb{R}^N)$, then the function*

$$u(t,x) = T(t)f(x)$$

is in $C_{loc}^{1+\frac{\alpha}{2},2+\alpha}((0,+\infty)\times\mathbb{R}^N)$ and solves (B.3). What's more, if there exists a $\lambda > 0$ such that $\lambda - A$ is injective in $D_{max}(A)$ then the semigroup is the unique solution to the problem.

Remark B.10. In general, the solution to (B.3) is not unique. In order to otain uniqueness we need that there exista a $\lambda > 0$ such that $\lambda - A$ is injective in $D_{max}(A)$. Thanks to Proposition A.14, this condition is equivalent to $T(t)\mathbb{1} = \mathbb{1}$, for all $t \geq 0$.

Appendix C

Sesquilinear forms and associated operators

We give a brief introduction to sesquilinear form theory and associated operators and semigroups. Much more comprehensive account of the subject may be found in the book by Ouhabaz, [54].

Let X be a Hilbert space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $D(\mathfrak{a})$ a linear subspace of X. We denote by $(\cdot, \cdot)$ the inner product of X and by $\|\cdot\|$ the corresponding norm. An application

$$\mathfrak{a} : D(\mathfrak{a}) \times D(\mathfrak{a}) \to \mathbb{K}$$

such that for every $\alpha \in \mathbb{K}$ and $u, v, w \in D(\mathfrak{a})$ satisfies

$$\mathfrak{a}(\alpha u + v, w) = \alpha\mathfrak{a}(u, w) + \mathfrak{a}(v, w) \quad \text{and} \quad \mathfrak{a}(u, \alpha v + w) = \overline{\alpha}\mathfrak{a}(u, v) + \mathfrak{a}(u, w)$$

is called *unbounded sesquilinear form*. The space $D(\mathfrak{a})$ is called the domain of $\mathfrak{a}$ and $\mathfrak{a}(u) := \mathfrak{a}(u, u)$ the associated quadratic form.

Definition C.1. Let $\mathfrak{a} : D(\mathfrak{a}) \times D(\mathfrak{a}) \to \mathbb{K}$ be a sesquilinear form. We say that

(i) $\mathfrak{a}$ is *densely defined* if

$$D(\mathfrak{a}) \text{ is dense in } X. \tag{C.1}$$

(ii) $\mathfrak{a}$ is *accretive* if

$$\operatorname{Re}\mathfrak{a}(u) \geq 0 \text{ for all } u \in D(\mathfrak{a}). \tag{C.2}$$

(iii) $\mathfrak{a}$ is *continuous* if there exists a non-negative constant M such that

$$|\mathfrak{a}(u,v)| \leq M\|u\|_{\mathfrak{a}}\|v\|_{\mathfrak{a}} \text{ for all } u,v \in D(\mathfrak{a}) \tag{C.3}$$

where $\|u\|_{\mathfrak{a}} := \sqrt{\operatorname{Re}\mathfrak{a}(u) + \|u\|^2}$.

(iv) $\mathfrak{a}$ is *closed* if

$$(D(\mathfrak{a}), \|\cdot\|_{\mathfrak{a}}) \text{ is a complete space.} \tag{C.4}$$

If the form $\mathfrak{a}$ satisfies conditions (C.1)-(C.4) one can easily check that $\|\cdot\|_{\mathfrak{a}}$ is a norm on $D(\mathfrak{a})$, the norm associated with the form $\mathfrak{a}$, and $D(\mathfrak{a})$ is a Hilbert space.

A stronger assumption than continuity of a form is sectoriality. It is defined as follows.

Definition C.2. A sesquilinear form $\mathfrak{a}$ acting on a complex Hilbert space X is called *sectorial* if there exists a non-negative constant C such that

$$|\operatorname{Im}\mathfrak{a}(u)| \leq C\operatorname{Re}\mathfrak{a}(u) \text{ for all } u \in D(\mathfrak{a}).$$

A relation between continuity and sectoriality is given by the following lemma.

Lemma C.3. *If $\mathfrak{a}$ is an accretive and continuous sesquilinear form on a complex Hilbert space X, then $1+\mathfrak{a}$ is sectorial. More precisely, if $\mathfrak{a}$ satisfies* (C.3) *with some constant M, then*

$$|\operatorname{Im}[(u,u)+\mathfrak{a}(u)]| \leq M\operatorname{Re}[(u,u)+\mathfrak{a}(u)] \text{ for all } u \in D(\mathfrak{a}).$$

The sum $\mathfrak{a}+\mathfrak{b}$ of two sesquilinear forms $\mathfrak{a}$ and $\mathfrak{b}$ on X is defined by

$$[\mathfrak{a}+\mathfrak{b}](u,v) := \mathfrak{a}(u,v) + \mathfrak{b}(u,v),\ D(\mathfrak{a}+\mathfrak{b}) = D(\mathfrak{a}) \cap D(\mathfrak{b}).$$

The natural question that arises is that if the properties of the forms carry over in the sum. In particular, if one of the two forms, say $\mathfrak{a}$, satisfies (C.2)-(C.4), the following theorem shows that under some additional assumptions these properties are preserved.

Theorem C.4. *Let $\mathfrak{a}$ be an accretive and continuous sesquilinear form on a complex Hilbert space X. Assume that $\mathfrak{a}'$ is a sesquilinear form such that $D(\mathfrak{a}) \subseteq D(\mathfrak{a}')$ and, for some α, β non-negative constant with $\alpha < 1$, the following inequality holds*

$$|\mathfrak{a}'(u)| \leq \alpha\operatorname{Re}\mathfrak{a}(u) + \beta\|u\|^2 \text{ for all } u \in D(\mathfrak{a}).$$

Then, the form sum $\mathfrak{t} := \mathfrak{a}+\mathfrak{a}'+\beta$ with domain $D(\mathfrak{t}) = D(\mathfrak{a})$ is accretive and continuous. Moreover, $\mathfrak{t}$ is closed if and only if $\mathfrak{a}$ is closed.

As for operators, if a form is not closed we can ask if it admits a closed extension. We define the smallest closed extension by $\overline{\mathfrak{a}}$, the closure of $\mathfrak{a}$.

Definition C.5. A densely defined accretive sesquilinear form $\mathfrak{a}$ is closable if there exists a closed accretive form $\mathfrak{c} : D(\mathfrak{c}) \subseteq X \to X$ such that $D(\mathfrak{a}) \subseteq D(\mathfrak{c})$ and $\mathfrak{a}(u,v) = \mathfrak{c}(u,v)$ for all $(u,v) \in D(\mathfrak{a})$.
If $\mathfrak{a}$ is a closable form we define the smallest closed extension $\overline{\mathfrak{a}}$ as follows

$$D(\overline{\mathfrak{a}}) = \{u \in X \text{ s.t. } \exists u_n \in D(\mathfrak{a}) : \lim_{n\to\infty} u_n = u, \lim_{n,m\to\infty} \mathfrak{a}(u_n - u_m) = 0\},$$

and

$$\overline{\mathfrak{a}}(u,v) := \lim_{n\to\infty} \mathfrak{a}(u_n, v_n),$$

for $u, v \in D(\overline{\mathfrak{a}})$, where $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are any sequences of elements of $D(\mathfrak{a})$ which converge respectively to u and v and satisfy $\mathfrak{a}(u_n - u_m) \to 0$ and $\mathfrak{a}(v_n - v_m) \to 0$ as $n, m \to \infty$. The limits are taken with respect to the norm of X.

One can show the following.

Proposition C.6. *Let $\mathfrak{a}$ be a densely defined, accretive, and continuous sesquilinear form. If $\mathfrak{a}$ is closable, then $\overline{\mathfrak{a}}$ is well defined and satisfies* (C.1)-(C.4)*. In addition, every closed extension of $\mathfrak{a}$ is also an extention of $\overline{\mathfrak{a}}$.*

Now we want to define the operator associated to a form. Let $\mathfrak{a}$ be a densely defined, accretive, continuous and closed sesquilinear form on X. We can define an unbounded operator A on a linear subspace $D(A)$ of X, which is called the *operator associated to the form* $\mathfrak{a}$, as follows

$$D(A) = \{u \in X \text{ s.t. } \exists v \in X : \mathfrak{a}(u,\phi) = (v,\phi)\ \forall \phi \in D(\mathfrak{a})\},\ Au := v.$$

Therefore, we can study the properties of A as an operator on X through the form $\mathfrak{a}$ and viceversa. For example, the operator associated to a sectorial form is a sectorial operator and the converse is also true.

Proposition C.7. *Let $\mathfrak{a}$ be a densely defined, accretive, continuous and closed sesquilinear form acting on a complex Hilbert space X. Denote by A the associated operator. The following assertions are equivalent:*

- *$\mathfrak{a}$ is a sectorial form;*
- *A is a sectorial operator.*

What we are interested in are generation results for the operator A in terms of the form $\mathfrak{a}$ as the following.

Theorem C.8. *Let $\mathfrak{a}$ be a densely defined, accretive, continuous and closed sesquilinear form on a Hilbert space X. Denote by A the operator associated with $\mathfrak{a}$. Then $-A$ is the generator of a strongly continuous contraction semigroup on X. Moreover, the semigroup is holomorphic on the sector $\Sigma(\frac{\pi}{2} - \arctan M)$ where M is the constant in the continuity assumption* (C.3).

In particular, the following result deals with generation of holomorphic semigroups for sectorial operators.

Theorem C.9. *Let A be a densely defined operator on a complex Hilbert space X. Assume that A is sectorial, that is,*

$$|\operatorname{Im}(Au, u)| \leq C \operatorname{Re}(Au, u) \text{ for all } u \in D(A),$$

where $C \geq 0$ is a constant. Assume also that there exists $\lambda_0 \in \rho(A)$ with dist$(\lambda_0, \Sigma(\arctan C)) > 0$. Then, $-A$ generates a strongly continuous semigroup which is holomorphic on the sector $\Sigma(\frac{\pi}{2} - \arctan C)$ and such that e^{-zB} is a contraction operator on X for every $z \in \Sigma(\frac{\pi}{2} - \arctan C)$.

List of symbols

$\mathbb{R}^N$	euclidean N-dimensional space;
(x, y)	inner euclidean product between the vectors $x, y \in \mathbb{R}^N$;
$\lvert x\rvert$	euclidean norm of $x \in \mathbb{R}^N$;
$B(r)$	ball centred in 0 of radius r;
$B(x, r)$	ball centred in x of radius r;
χ_E	characteristic function of the set E, i.e.: $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ if $x \notin E$;
$\mathbb{1}$	the function identically equal to 1;
$\Sigma(\delta)$	the sector of $\mathbb{C}$ of angle δ;
$\mathcal{L}(X)$	space of linear and continuous operators of a Banach space X into itself;
$\mathcal{B}(\mathbb{R}^N)$	σ- algebra of borelian sets of $\mathbb{R}^N$;
$C_b(\mathbb{R}^N)$	space of continuous bounded functions of $\mathbb{R}^N$;
$C^\alpha(\mathbb{R}^N)$	space of α-hölderian functions u in $\mathbb{R}^N$, i.e., $u \in C_b(\mathbb{R}^N)$ with $[u]_\alpha := \sup_{x,y\in\mathbb{R}^N; x\neq y} \frac{\lvert u(x)-u(y)\rvert}{\lvert x-y\rvert^\alpha} < +\infty$, with norm $\parallel u \parallel_\alpha := \parallel u \parallel_\infty + [u]_\alpha$;
$C^\alpha_{loc}(\mathbb{R}^N)$	space of functions in $C^\alpha(\Omega)$ for all Ω bounded open subset of $\mathbb{R}^N$;
$C^{1+\frac{\alpha}{2}, 2+\alpha}(\mathbb{R} \times \mathbb{R}^N)$	space of functions ϕ having bounded time derivative and bounded space derivative up to the second order, and $[\phi]_{1+\frac{\alpha}{2},2+\alpha} := [\phi_t]_{\frac{\alpha}{2},\alpha} + \sum_{i,j=1}^N [D_{x_i x_j}\phi]_{\frac{\alpha}{2},\alpha} < +\infty$, with norm $\parallel \Phi \parallel_{1+\frac{\alpha}{2},2+\alpha} = \parallel \Phi \parallel_\infty + \parallel \Phi_t \parallel_\infty + \parallel \nabla_x \Phi \parallel_\infty + \parallel D^2_x \Phi \parallel_\infty + [\phi]_{1+\frac{\alpha}{2},2+\alpha}$;
$C^\infty_c(\mathbb{R}^N)$	space of infinitely many time derivable functions with compact support in $\mathbb{R}^N$;
$L^p(\mathbb{R}^N)$	space of Lebesgue measurable functions u in $\mathbb{R}^N$, with $\lVert u\rVert_p^p := \int_{\mathbb{R}^N} \lvert u(x)\rvert^p \, dx < \infty$;

$\|\cdot\|_{p\to q}$	the norm of operators acting from $L^p(\mathbb{R}^N)$ into $L^q(\mathbb{R}^N)$;
$L^p_{loc}(\mathbb{R}^N)$	space of functions in $L^p(\Omega)$, for all bounded open set $\Omega\subset\mathbb{R}^N$;
$L^p_\mu(\mathbb{R}^N)$	space of measurable functions u in $\mathbb{R}^N$ with respect to the measure μ, with $\|u\|^p_{L^p_\mu} := \int_{\mathbb{R}^N} \|u(x)\|^p\,d\mu < \infty$;
$W^{k,p}(\mathbb{R}^N)$	space of functions $u\in L^p(\mathbb{R}^N)$ with weak derivatives up to order k in $L^p(\mathbb{R}^N)$, with $\|u\|_{W^{k,p}(\mathbb{R}^N)} := \sum_{\|\beta\|\le k}\|D^\beta u\|_p$;
$W^{k,p}_{loc}(\mathbb{R}^N)$	space of functions in $W^{k,p}(\Omega)$, for all bounded open set $\Omega\subset\mathbb{R}^N$;
$H^k(\mathbb{R}^N)$	Sobolev space $W^{k,2}(\mathbb{R}^N)$;
$H^k_{loc}(\mathbb{R}^N)$	space of functions in $H^k(\Omega)$, for all bounded open set $\Omega\subset\mathbb{R}^N$;

where Ω is an open subset of $\mathbb{R}^N$, $1\le p<+\infty$, $k,N\in\mathbb{N}$, $0<\alpha\le 1$, u real valued function.

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