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On fractional probabilistic mean value theorems, fractional counting processes and related results

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Introduction

The present thesis collects the outcomes of the author's research carried out in the research group *Probability Theory and Mathematical Statistics* at the Department of Mathematics, University of Salerno, during the doctoral programme "Mathematics, Physics and Applications". The results are at the interface between Fractional Calculus and Probability Theory. While research in probability and applied fields is now well established and enthusiastically supported, the subject of fractional calculus, i.e. the study of an extension of derivatives and integrals to any arbitrary real or complex order, has achieved widespread popularity only during the past four decades or so, because of its applications in several fields of science, engineering and finance. Indeed, it proves useful in formulating variational problems, mainly due to its non local characteristic, thus providing a very accurate description of reality. The application of the fractional paradigm to probability theory has been carefully but partially explored over the years, especially from the point of view of distribution theory, anomalous diffusion and, more generally, of stochastic processes. In many cases the evolution of the probability distribution of such processes is described by a suitable fractional partial or ordinary differential equation, where the space and/or time derivatives are replaced with their fractional counterparts. The key features that make fractional stochastic processes particularly worthy of attention are, among the others, long-range memory, persistent correlations, path-dependence. For instance, as regards the diffusion equation, fractional derivatives are related to random walks with heavy tails. Fractional derivatives with respect to space describe a super-diffusive behaviour, related to long power-law particle jumps. Time-fractional derivatives, instead, describe a sub-diffusive behaviour, related to long power-law waiting times between particle jumps. However, the investigation of the intersection between fractional calculus and other fields related to probability theory is a relatively young topic and many interesting challenges are often posed. The aim of this dissertation is twofold. On the one hand, we offer a contribution to the consolidated theory of fractional stochastic processes. On the other hand, as a novelty, we extend some classical results of integer-order calculus, both from a probabilistic and a fractional perspective, which, hopefully, will find some new applications in the near future. Moreover, we propose a model for competing risks in survival analysis based on a fractional probability distribution, thus delving into a more statistically oriented framework.

The dissertation is organized as follows.

In Chapter 1 we give an overview about the main ideas that inspire fractional calculus and about the mathematical techniques for dealing with fractional operators and the related special functions and probability distributions.

In order to develop certain fractional probabilistic analogues of Taylor's theorem and mean value theorem, in Chapter 2 we introduce the nth-order fractional equilibrium distribution in terms of the Weyl fractional integral and investigate its main properties. Specifically, we show a characterization result by which the nth-order fractional equilibrium distribution is identical to the starting distribution if and only if it is exponential. The nth-order fractional equilibrium density is then used to prove a fractional probabilistic Taylor's theorem based on derivatives of Riemann-Liouville type. A fractional analogue of the probabilistic mean value theorem is thus developed for pairs of nonnegative random variables ordered according to the survival bounded stochastic order. We also provide some related results, both involving the normalized moments and a fractional extension of the variance, and a formula of interest to actuarial science. In conclusion, we discuss the probabilistic Taylor's theorem based on fractional Caputo derivatives.

In Chapter 3 we consider a fractional counting process with jumps of integer amplitude $1, 2, \dots, k$, whose probabilities satisfy a suitable system of fractional difference-differential equations. We obtain the moment generating function and the probability law of the resulting process in terms of generalized Mittag-Leffler functions. We also discuss two equivalent representations both in terms of a compound fractional Poisson process and of a subordinator governed by a suitable fractional Cauchy problem. The first occurrence time of a jump of fixed amplitude is proved to have the same distribution as the waiting time of the first event of a classical fractional Poisson process, this extending a well-known property of the Poisson process. When $k = 2$ we also express the distribution of the first-passage time of the fractional counting process in an integral form. We then show that the ratios given by the powers of the fractional Poisson process and of the counting process over their means tend to 1 in probability.

In Chapter 4 we propose a generalization of the alternating Poisson process from the point of view of fractional calculus. We consider the system of differential equations governing the state probabilities of the alternating Poisson process and replace the ordinary derivative with a fractional one (in the Caputo sense). This produces a fractional 2-state point process, whose probability mass is expressed in terms of the (two-parameter) Mittag-Leffler function. We then show that it can be recovered also by means of renewal theory arguments. We study the limit state probability, and certain proportions involving the fractional moments of the sub-renewal periods of the process. In order to derive new Mittag-Leffler-like distributions related to the considered process, we then exploit a transformation acting on pairs of stochastically ordered random variables, which is an extension of the equilibrium operator and deserves interest in the analysis of alternating stochastic processes.

In Chapter 5 we analyse a jump-telegraph process by replacing the classical exponential distribution of the interarrival times which separate consecutive velocity changes (and jumps) with a generalized Mittag-Leffler distribution. Such interarrival times constitute the random times of a fractional alternating Poisson process. By means of renewal theory-based arguments we obtain the forward and backward transition densities of the motion in series form, and prove their uniform convergence. Specific attention is then given to the case of jumps with constant size, for which we also obtain the mean of the process. We conclude the chapter by investigating the first-passage time of the process through a constant positive boundary, providing its formal distribution and suitable lower bounds.

Chapter 6 is dedicated to a stochastic model for competing risks involving the Mittag-Leffler distribution, inspired by fractional random growth phenomena. We prove the independence between the time to failure and the cause of failure, and investigate some properties of the related hazard rates and ageing notions. We also face the general problem of identifying the underlying distribution of latent failure times when their joint distribution is expressed in terms of copulas and the time transformed exponential model. The special case concerning the Mittag-Leffler distribution is approached by means of numerical treatment. We finally adapt the proposed model to the case of a random number of independent competing risks. This leads to certain mixtures of Mittag-Leffler distributions, whose parameters are estimated through the method of moments for fractional moments.

Throughout the whole thesis, we refer to the following papers:

Antonio Di Crescenzo and Alessandra Meoli. “On the fractional probabilistic Taylor’s and mean value theorems.” Fractional Calculus and Applied Analysis 19.4 (2016): 921-939. doi: https://doi.org/10.1515/fca-2016-0050

Antonio Di Crescenzo, Barbara Martinucci and Alessandra Meoli. “A fractional counting process and its connection with the Poisson process.” ALEA 13.1 (2016): 291-307.

Antonio Di Crescenzo and Alessandra Meoli. “On a fractional alternating Poisson process.” AIMS Mathematics 1.3 (2016): 212-224. doi: 10.3934/Math.2016.3.212.

Antonio Di Crescenzo and Alessandra Meoli. “On a jump-telegraph process driven by alternating fractional Poisson process.” Submitted.

Antonio Di Crescenzo and Alessandra Meoli. “Competing risks driven by Mittag-Leffler distributions, under copula and time transformed exponential model.” Ricerche di Matematica (2016): 1-21. doi:10.1007/s11587-016-0304-x

Chapter 1

Elements of Fractional Calculus

Fractional calculus is the subfield of mathematical analysis that deals with the generalization of the operators of classical integration and differentiation to any arbitrary real or complex order. The subject is as old as Leibniz–Newton calculus, and dates back to 1695, when a preeminent mathematician of his time and author of the first French treatise on infinitesimal calculus, Guillaume François Antoine, Marquis de l'Hôpital, wrote a letter to Leibniz asking what would happen if the order of differentiation were a real number instead of an integer. Leibniz in a letter dated September 30, 1695 — the exact birthday of fractional calculus — replied: *"It will lead to a paradox, from which one day useful consequences will be drawn."*. However, the effective development of fractional calculus had to wait until 1832, when Liouville defined a fractional derivative by means of the Riemann-Liouville fractional integral. Many great mathematicians contributed to this theory over the years, such as Leibniz, Liouville, Riemann, Abel, Riesz, Weyl. In the present chapter we provide a brief introduction to the main fractional operators and their properties, as well as to some Mittag-Leffler-type functions and related probability distributions that prove to be useful in rest of the dissertation. For a general background in fractional calculus we refer to books [133], [124], [45] and [57]. See also [59] for a brief but comprehensive survey.

1.1 Basic fractional operators

The starting point for the development of the so-called Riemann–Liouville fractional calculus is Cauchy's formula for repeated integration:

$$\begin{aligned} I^n_{a+} f(x) &= \int_a^x \mathrm{d}x_1 \int_a^{x_1} \mathrm{d}x_2 \cdots \int_a^{x_{n-1}} f(x_n)\, \mathrm{d}x_n \\ &= \frac{1}{(n-1)!} \int_a^x (x - x_n)^{n-1} f(x_n)\, \mathrm{d}x_n, \end{aligned} \tag{1.1}$$

where $-\infty < a \leq x < +\infty$ and I^n_{a+}, $n \in \mathbb{N}$, is the multiple integral operator based at a. The function f is assumed to be sufficiently "nice", i.e. locally integrable in the interval considered. Formula (1.1) makes it possible to write any $n-$fold repeated integral by a convolution-type formula and to extend the notion of multiple integral to that of fractional integral by replacing positive integer values of the index n with arbitrary positive values α, using the relation $(n-1)! = \Gamma(n)$. Γ is the Eulerian Gamma function, defined for complex numbers with a positive real part via a convergent improper integral as

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\, \mathrm{d}x.$$

Consequently, the following definition has been proposed.

Definition 1.1.1 (Riemann-Liouville fractional integral)**.** *For any sufficiently well-behaved function f, the fractional integral of order α of f is defined as*

$$I^\alpha_{a+} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y)\, \mathrm{d}y, \qquad a < x < b, \quad \alpha > 0. \tag{1.2}$$

Note that the values of $I^n_{a+} f(x)$ with $n \in \mathbb{N}$ are always finite for $a \leq x < b$, but, while the values $I^\alpha_{a+} f(x)$ for $\alpha > 0$ are finite for $a < x < b$, it may happen that the limit (if it exists) of $I^\alpha_{a+} f(x)$ as $x \to a^+$ is infinite.
For completeness we put $I^0_{a+} := \mathbb{I}$ (Identity operator). The fundamental property of the fractional integrals is the additive index law *(semigroup property)*, according to which

$$I^\alpha_{a+} I^\beta_{a+} = I^{\alpha+\beta}_{a+}, \qquad \alpha, \beta \geq 0. \tag{1.3}$$

Fractional differentiation of any positive real power can be easily defined by combining the standard differential operator with a fractional integral of order between 0 and 1. There are several possibilities and in the following we will describe the two major approaches which provide the basis for two different definitions of the fractional derivative.

Definition 1.1.2 (Riemann-Liouville fractional derivative)**.** *If* $m-1 < \alpha \leq m$, $m \in \mathbb{N}$, *then the Riemann-Liouville derivative of order* α *of* f *is*

$$(D^{\alpha}_{a+}f)(x) := D^m I^{m-\alpha}_{a+} f(x), \qquad a < x < b, \quad \alpha > 0, \tag{1.4}$$

where D^m *is the common derivative of integer order* m*. This is equivalent to*

$$(D^{\alpha}_{a+}f)(x) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \dfrac{\mathrm{d}^m}{\mathrm{d}x^m} \displaystyle\int_a^x (x-t)^{m-\alpha-1} f(t)\mathrm{d}t & \text{if } m-1 < \alpha < m \\ \dfrac{\mathrm{d}^m f}{\mathrm{d}x^m} & \text{if } \alpha = m \end{cases}$$

For completeness, we also define $D^0_{a+} := \mathbb{I}$ *(Identity Operator).*

It can be directly verified that the Riemann-Liouville fractional integration and fractional differentiation operators (1.2) and (1.4) of the power function $f(x) = (x-a)^{\beta-1}$ yield power functions of the same form.

Example 1.1.1. If $\alpha \geq 0$ and $\beta \in \mathbb{C}$, $\Re(\beta) > 0$, then

$$\left(I^{\alpha}_{a+} (t-a)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(x-a)^{\alpha+\beta-1},$$

$$\left(D^{\alpha}_{a+} (t-a)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(x-a)^{\beta-\alpha-1}, \qquad 0 \leq \alpha < 1.$$

In particular, if $\beta = 1$ and $\alpha \geq 0$, $\alpha \notin \mathbb{N}$, then the Riemann-Liouville fractional derivative of a constant is, in general, not equal to zero:

$$\left(D^{\alpha}_{a+} 1\right)(x) = \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)}.$$

As underlined in [57], the fractional derivatives of order α, when α is non-integer, are non-local operators expressed by integer-order derivatives of convolution-type integrals with a weakly singular kernel. Furthermore, they do not necessarily satisfy the analogue of the semigroup property of the fractional integrals, since the base point a plays a "disturbing" role.

By exchanging the order of the derivative and integral operators in (1.4), a second definition of fractional derivative can be proposed.

Definition 1.1.3 (Dzherbashyan–Caputo fractional derivative)**.** *If* $m-1 < \alpha \leq m$, $m \in \mathbb{N}$, *then the Dzherbashyan–Caputo derivative of order* α *of* f *is*

$$({}_{*}D^{\alpha}_{a+}f)(x) = I^{m-\alpha}_{a+} D^m(x), \qquad \alpha > 0, \tag{1.5}$$

that is to say

$$({}_*D^\alpha_{a+}f)(x) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_a^x (x-t)^{m-\alpha-1} \frac{\mathrm{d}^m}{\mathrm{d}t^m} f(t)\mathrm{d}t & \textit{if } m-1<\alpha<m \\ \dfrac{\mathrm{d}^m f}{\mathrm{d}x^m} & \textit{if } \alpha = m \end{cases}$$

For completeness, we also define ${}_*D^0_{a+} := \mathbb{I}$ *(Identity Operator).*

For $m-1 < \alpha < m$ the definition (1.5) is more restrictive than that of Riemann–Liouville, since the absolute integrability of the derivative of order m is needed. Whenever the operator ${}_*D^\alpha_{a+}$ is used, we assume that this condition is met. Another way of defining the Caputo derivative is given by the following theorem.

Theorem 1.1.1. *Let* $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$. *For any sufficiently well-behaved function* f, *the Riemann–Liouville derivative of order* α *of* f *exists almost everywhere and it can be written in terms of the Caputo derivative as*

$$(D^\alpha_{a+}f)(x) = ({}_*D^\alpha_{a+}f)(x) + \sum_{k=0}^{m-1} \frac{(x-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}\left(a^+\right).$$

As a consequence of Theorem 1.1.1, the Caputo derivative can be interpreted as a sort of regularization of the Riemann-Liouville derivative as soon as the values $f^{(k)}\left(a^+\right)$ are finite. Moreover, the Caputo derivative of a constant is always zero. This is one way in which Caputo derivatives are considered to be more well-behaved than Riemann-Liouville ones. See the following example.

Example 1.1.2. *If* $\alpha > 0$, $m-1 < \alpha \le m$ *and* $\Re(\beta) > 0$, *then*

$$\left({}_*D^\alpha_{a+}(t-a)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(x-a)^{\beta-\alpha-1}, \qquad \Re(\beta) > m.$$

In particular,

$$\left({}_*D^\alpha_{a+}1\right)(x) = 0.$$

In Fig. 1.1 the Caputo fractional derivatives of the linear function $f(x) = x$ have been plotted for various choices of α, giving evidence to the fact that fractional derivatives interpolate between successive integer-order derivatives when applied to many kinds of functions.

A key tool for our next investigations is the Laplace transform of the fractional derivatives introduced. Therefore, we highlight the following relations, assuming without loss of generality, that the function f is identically vanishing for $x < 0$. For the Riemann-Liouville fractional derivative the Laplace transform, if it exists,

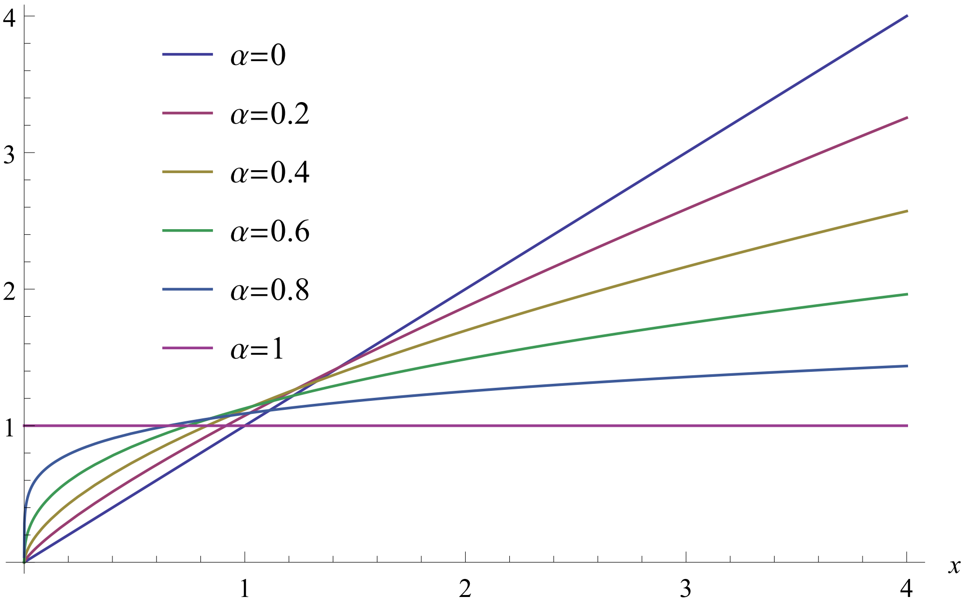

Figure 1.1: Caputo fractional derivatives of the linear function $f(x) = x$ for various choices of α.

requires the knowledge of the (bounded) initial values of the fractional integral $I^{m-\alpha}$ and of its integer derivatives of order $k = 1, 2, \ldots m-1$:

$$\mathcal{L}\left\{(D^{\alpha}f)(x); s\right\} = s^{\alpha}\mathcal{L}\left\{f(x); s\right\} - \sum_{k=0}^{m-1} s^{m-1-k} D^{k} I^{m-\alpha} f\left(0^{+}\right),$$

For the Caputo fractional derivative we need to know the (bounded) initial values of the function and of its integer derivatives of order $k = 1, 2, \ldots m-1$:

$$\mathcal{L}\left\{{}_{*}(D^{\alpha}f)(x); s\right\} = s^{\alpha}\mathcal{L}\left\{f(x); s\right\} - \sum_{k=0}^{m-1} s^{\alpha-1-k} f^{(k)}\left(0^{+}\right). \tag{1.6}$$

Several authors have pointed out the more useful character of the Caputo fractional derivative in the treatment of fractional differential equations in physical applications. In fact, in physical problems, the initial conditions are usually expressed in terms of a given number of bounded values assumed by the field variable and its integer-order derivatives, in spite of the fact that the governing evolution equation may be a generic integro-differential equation.

As underlined in [49], the question of the geometrical or physical interpretation of fractional calculus has been object of investigation for more than 300 years without coming to a conclusion. While classical integral and derivatives can be easily interpreted in terms of areas, tangent lines and planes from a geometric point of view, and in terms of speed and acceleration from a physical point of view, the interpretation of fractional integrals and derivatives was acknowledged as one of the unresolved problems at the First International Conference on Fractional Calculus held in 1974 in New Haven, Connecticut. However, several attempts have been made to provide an interpretation. Among the most interesting ones, we recall Kolokoltsov's analyses of fractional derivatives from a probabilistic point of view [75]. Specifically, the basic Caputo and Riemann-Liouville derivatives of order $\alpha \in (0, 2)$ can be viewed as (regularized) generators of stable Lévy motions interrupted on crossing a boundary. Although the problem is still unanswered, some main features and advantages of fractional operators have been extensively highlighted, such as the properties of non locality and the representation of long memory processes.

A fractional integral over an unbounded interval can also be defined. Specifically, if the function $f(x)$ is locally integrable in $-\infty \leq a < x < +\infty$, and behaves well enough for $x \to +\infty$, the *Weyl fractional integral of order* α *of* f is defined as

$$I_-^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_x^{+\infty} (t-x)^{\alpha-1} f(t)\, \mathrm{d}t, \qquad a < x < +\infty, \quad \alpha > 0. \tag{1.7}$$

Also for the Weyl fractional integral the corresponding *semigroup property* holds:

$$I_-^\alpha I_-^\beta = I_-^{\alpha+\beta}, \quad \alpha, \beta \geq 0, \tag{1.8}$$

where, again for completeness, $I_-^0 := \mathbb{I}$.

1.2 Mittag-Leffler-type functions

The Mittag-Leffler function is so named after the Swedish mathematician who introduced it at the beginning of the last century to answer a classical question of complex analysis, namely to describe the procedure of the analytic continuation of power series outside the radius of their convergence. It was subsequently investigated by Wiman, Pollard, Humbert, Aggarwal and Feller, among the others. The Mittag-Leffler function was re-discovered when its connection to fractional calculus was definitely clear and the community of researchers became aware of its considerable potential in applied sciences and engineering. Indeed, it is possible to naturally express the solution of fractional order differential and integral equations in terms of Mittag-Leffler-type functions. For example, Mittag-Leffler-type functions are related to the solutions of a variety of fractional evolution processes, i.e. phenomena governed by an integro-differential equation containing integrals and/or derivatives of fractional order in time. Moreover, the importance of Mittag-Leffler-type functions in fields of research such as stochastic systems theory, dynamical systems theory, statistical distribution theory, just to name a few, is now widely recognised. In this section we shall consider some Mittag-Leffler-type functions which are relevant for proving the results in the dissertation. For recent advances in Mittag-Leffler-type functions see Lavault [85] and references therein.

The one-parameter Mittag-Leffler function $E_\alpha(z)$ is defined by the following series representation, valid in the whole complex plane

$$E_\alpha(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \qquad \Re(\alpha) > 0,\ z \in \mathbb{C}. \tag{1.9}$$

If $\Re(\alpha) < 0$ the series diverges everywhere on $\mathbb{C} \setminus \{0\}$ and, for $\Re(\alpha) = 0$, its radius of convergence is $R = e^{\pi/2|\Im(z)|}$. The Mittag-Leffler function provides a simple generalization of the exponential function because of the substitution of $n! = \Gamma(n+1)$ with $(\alpha n)! = \Gamma(\alpha n + 1)$. In particular, when $\alpha = 1$ and $\alpha = 2$, we have

$$E_1(z) = e^z, \qquad E_2(z) = \cosh(\sqrt{z}).$$

Many properties of the Mittag-Leffler function can be derived from its integral representation

$$E_\alpha(z) = \frac{1}{2\pi i} \int_C \frac{t^{\alpha-1} e^t}{t^\alpha - z} \mathrm{d}t,$$

where the path of integration C is a loop which starts and ends at $-\infty$ and encircles the circular disc $|t| \leq z^{1/\alpha}$.

A straightforward generalization of $E_\alpha(z)$ is the two-parameter Mittag-Leffler function, obtained by replacing the additive constant 1 in the argument of the Gamma function in (1.9) by an arbitrary complex parameter β:

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \qquad \Re(\alpha) > 0,\ \Re(\beta) > 0,\ z \in \mathbb{C}. \tag{1.10}$$

If both α and β are positive real numbers, the series is convergent for any $z \in \mathbb{C}$; if $\alpha, \beta \in \mathbb{C}$, the conditions of convergence follow the ones for the one-parametric Mittag-Leffler function (1.9). When $\beta = 1$, $E_{\alpha,\beta}(z)$ coincides with the Mittag-Leffler function (1.9):

$$E_{\alpha,1}(z) = E_\alpha(z), \qquad \Re(\alpha) > 0,\ z \in \mathbb{C}.$$

We also recall two other particular cases of (1.10):

$$E_{1,2}(z) = \frac{e^z - 1}{z}, \qquad E_{2,2}(z) = \frac{\sinh(\sqrt{z})}{\sqrt{z}}.$$

$E_\alpha(z)$ and $E_{\alpha,\beta}(z)$ are entire function of $z \in \mathbb{C}$ with order $1/\Re(\alpha)$ and type 1.

Prabhakar [128] introduced the function $E^\gamma_{\alpha,\beta}(z)$ of the form

$$E^\gamma_{\alpha,\beta}(z) := \sum_{r=0}^{\infty} \frac{(\gamma)_r \, z^r}{r! \, \Gamma(\alpha r + \beta)}, \quad z \in \mathbb{C}, \, \gamma \in \mathbb{C}, \, \Re(\alpha) > 0, \, \Re(\beta) > 0, \tag{1.11}$$

where $(\gamma)_k$ is the rising factorial (or Pochhammer symbol), defined as:

$$(\gamma)_k := \gamma(\gamma + 1) \dots (\gamma + k - 1) \quad \text{if} \quad k \in \mathbb{N}, \qquad (\gamma)_0 = 1 \qquad (\gamma \neq 0).$$

It is an entire function of z of order $1/\Re(\alpha)$ and type 1. When $\gamma = 1$ we recover the Mittag-Leffler function (1.10) and for $\gamma = \beta = 1$ we recover the classical Mittag-Leffler function (1.9).
The following formula holds for the Laplace transform of the Mittag-Leffler-type function (1.11)

$$\mathcal{L}\left\{t^{\beta-1} E^\gamma_{\alpha,\beta}(\lambda t^\alpha); s\right\} = \frac{s^{\alpha\gamma - \beta}}{(s^\alpha - \lambda)^\gamma}, \tag{1.12}$$

where $\Re(s) > 0$, $\Re(\beta) > 0$, $\lambda \in \mathbb{C}$, and $|\lambda s^{-\alpha}| < 1$.

Recently Mittag-Leffler functions and distributions have received the attention of mathematicians, statisticians and scientists in physical and chemical sciences. Pillai [122] introduced the Mittag-Leffler distribution in terms of the Mittag-Leffler function (1.9). Indeed, he proved that

$$F_\alpha(x) = 1 - E_\alpha(-x^\alpha), \qquad 0 < \alpha \leq 1,$$

are distribution functions with positive support and with Laplace transform

$$\psi(s) = (1 + s^\alpha)^{-1}, \qquad s \geq 0, \tag{1.13}$$

which is completely monotone for $0 < \alpha \leq 1$. However, in the 60s of the past century, Gnedenko and Kovalenko [77], in their analysis of thinning (or rarefaction) of a renewal process, found, under certain power-law assumptions, the Laplace transform $(1 + s^\alpha)^{-1}$ for the waiting-time density in the infinite thinning limit, but did not identify it as a Mittag-Leffler type function. Moreover, in Balakrishnan [8] the waiting time density with Laplace transform (1.13) plays a distinct role in the context of continuous time random walks, but, again, was not recognized as a Mittag-Leffler-type function. The probability density function corresponding to (1.13) is

$$f_\alpha(x) = x^{\alpha-1} E_{\alpha,\alpha}(-x^\alpha), \qquad x \geq 0.$$

A distribution F different from F_α in scale parameter has Laplace transform $\psi(s) = (1+\lambda s^\alpha)^{-1}$ for some constant $\lambda > 0$, and is also called a Mittag-Leffler distribution. The Mittag-Leffler distribution is a generalization of the exponential distribution, which is recovered for $\alpha = 1$. In the same paper Pillai proved that the Mittag-Leffler distribution is infinitely divisible and geometrically infinitely divisible, and that it is attracted to the stable distribution with exponent α, $0 < \alpha < 1$. Particularly important is the power law asymptotics for $x \to +\infty$:

$$E_\alpha(-x^\alpha) \sim \frac{x^{-\alpha}}{\Gamma(1-\alpha)}, \qquad f_\alpha(x) \sim \frac{\Gamma(\alpha+1)\sin(\alpha\pi)}{\pi} x^{-\alpha-1},$$

in contrast to the exponential decay of $E_1(x) = e^{-x}$. Gorenflo [58] proved the asymptotic long-time equivalence of a generic power law waiting time distribution to the Mittag-Leffler distribution, the waiting time distribution characteristic for a time-fractional continuous time random walk. This asymptotic equivalence is effected by "rescaling" time and "respeeding" the relevant renewal process; then passing to a limit. A suitable relation between the parameters of rescaling and respeeding is needed.
Jose et al. [67] introduced the class of generalized Mittag-Leffler distributions, which involve the generalized Mittag-Leffler function (1.11), denoted by GMLD (α, β). A random variable X with support over $(0, \infty)$ is said to follow the generalized Mittag-Leffler distribution with parameters α and β if its Laplace transform is

$$\psi(s) = (1+s^\alpha)^{-\beta}, \quad 0 < \alpha \leq 1,\, \beta > 0.$$

The corresponding cumulative distribution function is given by

$$F_{\alpha,\beta}(x) = \mathbb{P}(X \leq x) = \sum_{j=0}^{+\infty} \frac{(-1)^j\, \Gamma(\beta+j)\, x^{\alpha(\beta+j)}}{j!\, \Gamma(\beta)\, \Gamma(1+\alpha\beta+\alpha j)} = x^{\alpha\beta} E^\beta_{\alpha,\, \alpha\beta+1}(-x^\alpha).$$

We observe that when $\beta = 1$ we get Pillai's Mittag-Leffler distribution [122], when $\alpha = 1$ we get the gamma distribution, when $\alpha = 1$ and $\beta = 1$ we get the exponential distribution. We now list some properties of generalized Mittag-Leffler distributions:

- If U_α follows the positive stable distribution with Laplace transform $\psi(s) =$

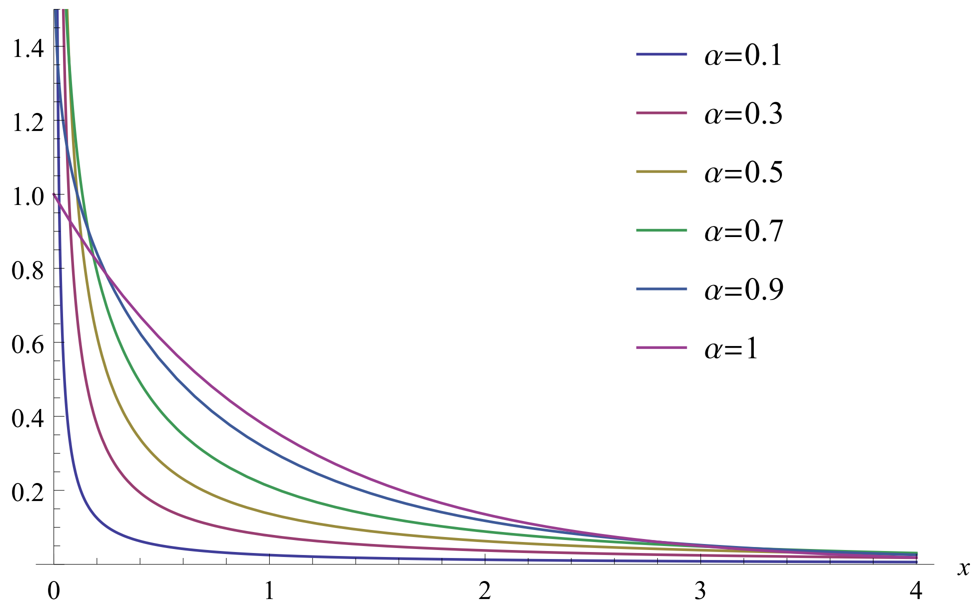

Figure 1.2: Plot of probability density functions of the Mittag–Leffler distribution for various choices of α.

e^{-s^α}, $s > 0$, $0 < \alpha \leq 1$, and if V_β is a random variable, independent of U_α, and following a gamma distribution with Laplace transform $\phi(s) = \left(\frac{1}{1+s}\right)^\beta$, $\beta > 0$, then $X_{\alpha,\beta} = U_\alpha V_\beta^{1/\alpha}$ follows the generalized Mittag-Leffler distribution GMLD (α, β);

- the probability density function of $X_{\alpha,\beta}$ is a mixture of gamma densities;
- Lin [88] has shown that $F_{\alpha,\beta}(x)$ is slowly varying at infinity, for $\alpha \in (0, 1]$ and $\beta > 0$;
- the fractional moments of $X_{\alpha,\beta}$, for $0 < \alpha \leq 1$ and $\beta > 0$, are (cf. [88])

$$E[X^r_{\alpha,\beta}] = \begin{cases} \dfrac{\Gamma(1 - r/\alpha)\Gamma(\beta + r/\alpha)}{\Gamma(1 - r)\Gamma(\beta)} & \text{if } -\alpha\beta < r < \alpha \\ \infty & \text{if } r \leq -\alpha\beta \text{ or } r \geq \alpha. \end{cases}$$

Chapter 2

On the fractional probabilistic Taylor's and mean value theorems

Taylor's theorem is the most important result in differential calculus since it gives a sequence of approximations of a differentiable function in the neighborhood of a given point by polynomials with coefficients depending only on the derivatives of the function at that point. Therefore, given the derivatives of a function at a single point, it is possible to describe the behavior of the function at nearby points. Motivated by the large numbers of its applications, researchers have shown a heightened interest in the extensions of this theorem. For instance, Massey and Whitt [98] derived probabilistic generalizations of the fundamental theorem of calculus and Taylor's theorem by making the argument interval random and expressing the remainder terms by means of iterates of the equilibrium residual-lifetime distribution from the theory of stochastic point processes. Lin [87] modified Massey and Whitt's probabilistic generalization of Taylor's theorem and gave a natural proof by using an explicit form for the density function of the high-order equilibrium distribution. In a similar spirit to these probabilistic extensions of Taylor's theorem, Di Crescenzo [32] gave a probabilistic analogue of the mean value theorem. The previous results have direct applications to queueing and reliability theory. However, probabilistic generalizations are not the only ones. Indeed, fractional Taylor series have been introduced with the idea of approximating non-integer power law functions. Here we recall the most interesting ones. Trujillo et al. [145] established a Riemann-Liouville generalized Taylor's formula, in which the coefficients are expressed in terms of the Riemann-Liouville fractional derivative. On the other hand, Odibat et al. [105] expressed the coefficients of a generalized Taylor's formula in terms of the Caputo fractional derivative. In the aforementioned papers an application of the generalized Taylor's formula to the resolution of fractional differential equations is also shown. Among others, they have been used to study a fractional conservation of mass ([150]

and [106]), a fractional order model for HIV infection [123] and a fractional order model for MINMOD Millennium in order to estimate insraulin sensitivity in glucose–insulin dynamics [25]. Great emphasis has been placed on fractional Lagrange and Cauchy type mean value theorems too (cf. [61], [105], [120], [145], for example). Inspired by such improvements, in the present chapter we propose to unify these two approaches by presenting a fractional probabilistic Taylor's theorem and a fractional probabilistic mean value theorem.

We start by briefly recalling some notions on a generalized Taylor's formula that are pertinent to the next developments. Then, in Section 2.2, after quickly reviewing the notion of equilibrium distribution, we define a fractional extension of the high-order equilibrium distribution. We also give an equivalent version by exploiting the semigroup property of the Weyl fractional integral and derive the explicit expression of the related density function. By means of the Mellin transform we underline the role played by the fractional equilibrium density in characterizing the exponential distribution. In Section 2.3 we prove a fractional probabilistic Taylor's theorem by using the expression of the nth-order fractional equilibrium density. Section 2.4 is devoted to the analysis of a fractional analogue of the probabilistic mean value theorem. We first consider pairs of nonnegative random variables ordered in a suitable way so as to construct a new random variable, say Z_α, which extends the fractional equilibrium operator. The fractional probabilistic mean value theorem indeed is given in terms of Z_α. We also discuss some related results, including a formula of interest to actuarial science. We stress the fact that all the aforementioned results involve derivatives of Riemann-Liouville type. However, in some cases they can be restated also under a different setting. Indeed, in Section 2.5 we conclude the chapter by showing a fractional probabilistic Taylor's theorem in the Caputo sense.

2.1 Background on a generalized Taylor's formula

Let Ω be a real interval and $\alpha \in [0, 1)$. Let $F(\Omega)$ denote the space of Lebesgue measurable functions with domain in Ω and suppose that $x_0 \in \Omega$. Then a function f is called *α-continuous in x_0* if there exists $\lambda \in [0, 1-\alpha)$ for which the function h given by

$$h(x) = |x - x_0|^\lambda f(x)$$

is continuous in x_0. Moreover, f is called *1-continuous in x_0* if it is continuous in x_0, and *α-continuous on Ω* if it is α-continuous in x for every $x \in \Omega$. We denote, for convenience, the class of α-continuous functions on Ω by $C_\alpha(\Omega)$, so that $C_1(\Omega) = C(\Omega)$.

For $a \in \Omega$, a function f is called *a-singular of order* α if

$$\lim_{x \to a} \frac{f(x)}{|x-a|^{\alpha-1}} = k < \infty \quad \text{and} \quad k \neq 0.$$

Let $\alpha \in \mathbb{R}^+$, $a \in \Omega$ and let $F(\Omega)$ denote the space of Lebesgue measurable functions with domain in Ω. In addition, let E be an interval, $E \subset \Omega$, such that $a \leq x$ for every $x \in E$. With regard to the Riemann-Liouville fractional integral (1.2) and to the Riemann-Liouville fractional derivative (1.4) we write

$${}_a\mathbf{I}_\alpha(E) = \left\{ f \in F(\Omega) : I_{a+}^{\alpha} f(x) \text{ exists and it is finite } \forall x \in E \right\}.$$

Furthermore, we denote the *sequential fractional derivative* by

$$D_{a+}^{n\alpha} = \underbrace{D_{a+}^{\alpha} \dots D_{a+}^{\alpha}}_{n \text{ times}}.$$

Recently, Trujillo et al. (cf. Theorem 4.1 of [145]) proved the following result, on which we base our generalization of Taylor's theorem in Section 2.3.

Theorem 2.1.1. *Set* $\alpha \in [0,1]$ *and* $n \in \mathbb{N}$*. Let* g *be a continuous function in* $(a,b]$ *satisfying the following conditions:*

(i) $\forall j = 1, \dots, n$, $D_{a+}^{j\alpha} g \in C((a,b])$ *and* $D_{a+}^{j\alpha} g \in {}_a\mathbf{I}_\alpha([a,b])$*;*

(ii) $D_{a+}^{(n+1)\alpha} g$ *is continuous on* $[a,b]$*;*

(iii) If $\alpha < 1/2$ *then, for each* $j \in \mathbb{N}, 1 \leq j \leq n$*, such that* $(j+1)\alpha < 1$*,* $D_{a+}^{(j+1)\alpha} g(x)$ *is* γ*-continuous in* $x = a$ *for some* γ*,* $1 - (j+1)\alpha \leq \gamma \leq 1$*, or a-singular of order* α*.*

Then, $\forall x \in (a,b]$*,*

$$g(x) = \sum_{j=0}^{n} \frac{c_j (x-a)^{(j+1)\alpha - 1}}{\Gamma((j+1)\alpha)} + R_n(x,a),$$

with

$$R_n(x,a) = \frac{D_{a+}^{(n+1)\alpha} g(\xi)}{\Gamma((n+1)\alpha + 1)} (x-a)^{(n+1)\alpha}, \qquad a \leq \xi \leq x,$$

and

$$c_j = \Gamma(\alpha) \left[(x-a)^{1-\alpha} D_{a+}^{j\alpha} g(x) \right] \left(a^+ \right) = I_{a+}^{1-\alpha} D_{a+}^{j\alpha} g\left(a^+ \right)$$

for each $j \in \mathbb{N}$*,* $0 \leq j \leq n$*.*

2.2 Fractional equilibrium distribution

Let X be a nonnegative random variable with cumulative distribution function $F(x) = \mathbb{P}(X \leq x)$ for $x \geq 0$ and with nonvanishing mean $\mathbb{E}[X] < +\infty$. Define a nonnegative random variable X_e with distribution

$$F_1(x) = \mathbb{P}(X_e \leq x) = \frac{1}{\mathbb{E}[X]} \int_0^x \overline{F}(y)\,\mathrm{d}y, \qquad x \geq 0,$$

or, equivalently, with complementary cumulative distribution function

$$\overline{F}_1(x) = \mathbb{P}(X_e > x) = \frac{1}{\mathbb{E}[X]} \int_x^{+\infty} \overline{F}(y)\,\mathrm{d}y, \qquad x \geq 0,$$

where $\overline{F} = 1 - F$. The distribution F_1 is called *equilibrium distribution with respect to F*. Indeed, if F is the cumulative distribution function of a random interval between renewal epochs of a renewal process, then F_1 is the cumulative distribution function of the random interval to the next renewal epoch from an arbitrary time in equilibrium. Further, suppose $\mathbb{E}[X^2] < +\infty$. Then the equilibrium distribution with respect to F_1 is well-defined and it reads

$$F_2(x) = \mathbb{P}\left(X_e^{(2)} \leq x\right) = \frac{1}{\mathbb{E}[X_e]} \int_0^x \overline{F}_1(y)\,\mathrm{d}y, \qquad x \geq 0.$$

F_2 is known as the second order equilibrium distribution with respect to F. Continuing $n-2$ more iterates of this transformation, it is possible to obtain the nth-order equilibrium distribution with respect to F, denoted by F_n, provided the required moments of X are finite.

Hereafter we introduce a fractional version of the nth-order equilibrium distribution. Let $\alpha \in \mathbb{R}^+$ and let X be a nonnegative random variable with distribution $F(t) = \mathbb{P}(X \leq t)$ for $t \geq 0$ and with moment $\mathbb{E}[X^\alpha] \in (0, +\infty)$. Then we define a random variable $X_\alpha^{(1)}$ whose complementary distribution function is

$$\overline{F}_1^{\alpha}(t) := \mathbb{P}\left(X_\alpha^{(1)} > t\right) = \frac{\Gamma(\alpha+1)}{\mathbb{E}[X^\alpha]} I_-^\alpha \overline{F}(t), \qquad t \geq 0, \tag{2.1}$$

where I_-^α is the Weyl fractional integral (1.7) and $\overline{F} = 1-F$. We call the distribution of $X_\alpha^{(1)}$ *fractional equilibrium distribution with respect to F*. See also Pakes and Navarro [118] and references therein. In Remark 2.2.2 we prove that (2.1) is a legitimate complementary cumulative distribution function.

Remark 2.2.1. Recalling (1.7), from (2.1) we obtain the following suitable probabilistic interpretation of the distribution function of $X_\alpha^{(1)}$ in terms of X. In fact,

$$\mathbb{P}\left(X_\alpha^{(1)} \le t\right) = \frac{\alpha}{\mathbb{E}\left[X^\alpha\right]} \int_0^{+\infty} y^{\alpha-1} \mathbb{P}\left(y < X \le y+t\right) \mathrm{d}y.$$

Further, suppose $\mathbb{E}\left[X^{2\alpha}\right] < +\infty$. Then the *second-order fractional equilibrium distribution with respect to* F is well-defined and its complementary distribution function reads

$$\overline{F}_2^\alpha(t) = \mathbb{P}\left(X_\alpha^{(2)} > t\right) = \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} \frac{\mathbb{E}\left[X^\alpha\right]}{\mathbb{E}\left[X^{2\alpha}\right]} I_-^\alpha \overline{F}_1^\alpha(t), \qquad t \ge 0.$$

Generally, we can recursively define the *nth-order fractional complementary equilibrium distribution with respect to* F by

$$\overline{F}_n^\alpha(t) = \mathbb{P}\left(X_\alpha^{(n)} > t\right) = \frac{\Gamma(n\alpha+1)}{\Gamma((n-1)\alpha+1)} \frac{\mathbb{E}\left[X^{(n-1)\alpha}\right]}{\mathbb{E}\left[X^{n\alpha}\right]} I_-^\alpha \overline{F}_{n-1}^\alpha(t), \qquad t \ge 0,$$

provided that all the moments $\mathbb{E}\left[X^{n\alpha}\right]$, for $n \in \mathbb{N}$, are finite.

Interestingly enough, $\overline{F}_n^\alpha$ can be alternatively expressed in terms of $\overline{F}$. Indeed, the following proposition holds.

Proposition 2.2.1. *Let $\alpha \in \mathbb{R}^+$ and let X be a nonnegative random variable with distribution $F(t)$ for $t \ge 0$. Moreover, suppose that $\mathbb{E}\left[X^{n\alpha}\right] \in (0, +\infty)$, with $n \in \mathbb{N}$. Then the nth-order fractional complementary equilibrium distribution with respect to F reads*

$$\overline{F}_n^\alpha(t) = \frac{\Gamma(n\alpha+1)}{\mathbb{E}\left[X^{n\alpha}\right]} I_-^{n\alpha} \overline{F}(t), \qquad t \ge 0. \tag{2.2}$$

Proof. The proof is by induction on n. In fact, when $n = 1$ formula (2.2) is true due to Definition (2.1). Now let us assume that Eq. (2.2) holds for some n; then, for $t \ge 0$,

$$\begin{aligned}
\overline{F}_{n+1}^\alpha(t) &= \frac{\Gamma((n+1)\alpha+1)}{\Gamma(n\alpha+1)} \frac{\mathbb{E}\left[X^{n\alpha}\right]}{\mathbb{E}\left[X^{(n+1)\alpha}\right]} I_-^\alpha \overline{F}_n^\alpha(t) \\
&= \frac{\Gamma((n+1)\alpha+1)}{\Gamma(n\alpha+1)} \frac{\mathbb{E}\left[X^{n\alpha}\right]}{\mathbb{E}\left[X^{(n+1)\alpha}\right]} I_-^\alpha \frac{\Gamma(n\alpha+1)}{\mathbb{E}\left[X^{n\alpha}\right]} I_-^{n\alpha} \overline{F}(t) \\
&= \frac{\Gamma((n+1)\alpha+1)}{\mathbb{E}\left[X^{(n+1)\alpha}\right]} I_-^{(n+1)\alpha} \overline{F}(t).
\end{aligned}$$

The last equality is valid due to the linearity of the integral and to the semigroup property (1.8). So the validity of Eq. (2.2) for n implies its validity for $n+1$. Therefore it is true for all $n \in \mathbb{N}$. □

In order to obtain the explicit expression of the density function of the nth-order fractional equilibrium distribution, we use the following lemma, which is a generalization of Proposition 4 of [24].

Lemma 2.2.1. *Let X be a nonnegative random variable whose moment of order $n\alpha$ is finite for $\alpha \in \mathbb{R}^+$ and $n \in \mathbb{N}$. Then*

$$\lim_{x \to +\infty} (x-t)^{n\alpha}\, \overline{F}(x) = 0, \quad \forall t \geq 0.$$

Proof. Because the moment of order $n\alpha$ of X is finite, we have

$$\lim_{x \to +\infty} \int_x^{+\infty} y^{n\alpha} \mathrm{d}F(y) = 0. \tag{2.3}$$

Hence, if $0 \leq t \leq x$,

$$\lim_{x \to +\infty} (x-t)^{n\alpha}\, \overline{F}(x) \leq \lim_{x \to +\infty} x^{n\alpha} \overline{F}(x) \leq \lim_{x \to +\infty} \int_x^{+\infty} y^{n\alpha} \mathrm{d}F(y) = 0.$$

The last inequality follows from a generalized Markov inequality:

$$\mathbb{P}(|X| \geq a) \leq \frac{\mathbb{E}(\varphi(|X|))}{\varphi(a)},$$

where φ is a monotonically increasing function for the nonnegative reals, X is a random variable, $a \geq 0$ and $\varphi(a) > 0$. □

Here and throughout the chapter, we denote, for convenience, $(x)_+^{\alpha-1} = (x)^{\alpha-1}\mathbb{1}_{\{x>0\}}$. The following result concerns the probability density function associated with $F_n^{\alpha}(t) := 1 - \overline{F}_n^{\alpha}(t)$.

Proposition 2.2.2. *Let X be a nonnegative random variable with distribution function F and let $\mathbb{E}[X^{n\alpha}] < +\infty$ for some integer $n \geq 1$ and $\alpha \in \mathbb{R}^+$. Then the density function of $X_\alpha^{(n)}$ is*

$$f_n^{\alpha}(t) = \frac{n\alpha \mathbb{E}\left[(X-t)_+^{n\alpha-1}\right]}{\mathbb{E}[X^{n\alpha}]}, \qquad t \geq 0. \tag{2.4}$$

Proof. By virtue of (2.2) and (1.7) we have:

$$\begin{aligned}
\overline{F}_n^{\alpha}(t) &= \frac{\Gamma(n\alpha+1)}{\mathbb{E}[X^{n\alpha}]} I_-^{n\alpha} \overline{F}(t) \\
&= \frac{n\alpha\, \Gamma(n\alpha)}{\mathbb{E}[X^{n\alpha}]} \frac{1}{\Gamma(n\alpha)} \int_t^{+\infty} (x-t)^{n\alpha-1}\, \overline{F}(x)\, \mathrm{d}x.
\end{aligned}$$

Due to integration by parts and making use of Lemma 2.2.1, we have:

$$\begin{aligned}\overline{F}^{\alpha}_{n}(t) &= -\frac{1}{\mathbb{E}\left[X^{n\alpha}\right]}\int_t^{+\infty}(x-t)^{n\alpha}\,\mathrm{d}\overline{F}(x)\\ &= \frac{1}{\mathbb{E}\left[X^{n\alpha}\right]}\int_t^{+\infty}(x-t)^{n\alpha}\,\mathrm{d}F(x)\\ &= \left(\frac{\mathbb{E}\left[(X-t)_+^{n\alpha}\right]}{\mathbb{E}\left[X^{n\alpha}\right]}\right)\\ &= \frac{n\alpha}{\mathbb{E}\left[X^{n\alpha}\right]}\int_t^{+\infty}\mathrm{d}F(x)\int_t^{x}(x-y)^{n\alpha-1}\,\mathrm{d}y\\ &= \frac{n\alpha}{\mathbb{E}\left[X^{n\alpha}\right]}\int_t^{+\infty}\mathrm{d}y\int_y^{+\infty}(x-y)^{n\alpha-1}\,\mathrm{d}F(x)\\ &= \frac{n\alpha}{\mathbb{E}\left[X^{n\alpha}\right]}\int_t^{+\infty}\mathbb{E}\left[(X-y)_+^{n\alpha-1}\right]\mathrm{d}y,\end{aligned} \tag{2.5}$$

this giving the density function (2.4). □

We observe that in Proposition 2.2.2 the random variable X is not necessarily absolutely continuous, unlike $X_\alpha^{(n)}$.

Remark 2.2.2. Formula (2.5) is useful in showing that $\overline{F}^{\alpha}_{n}(t)$ is a proper complementary distribution function for all $n \in \mathbb{N}$. Indeed,

(i) $\overline{F}^{\alpha}_{n}(0) = \left.\frac{\mathbb{E}\left[(X-t)_+^{n\alpha}\right]}{\mathbb{E}[X^{n\alpha}]}\right|_{t=0} = 1$;

(ii) $\overline{F}^{\alpha}_{n}(t)$ is decreasing and continuous in $t \geq 0$;

(iii) $\overline{F}^{\alpha}_{n}(t) \to 0$, when $t \to +\infty$. In fact, due to (2.3), we have

$$\lim_{t\to+\infty}\int_t^{+\infty}(x-t)^{n\alpha}\,\mathrm{d}F(x) \leq \lim_{t\to+\infty}\int_t^{+\infty}x^{n\alpha}\mathrm{d}F(x) = 0.$$

We now prove a characterization result concerning the fractional equilibrium density (2.4). In fact, if X is a nonnegative random variable with probability density function f, the nth-order fractional equilibrium density associated with f coincides with f if and only if X is exponentially distributed. This extends the well-known result concerning case $\alpha = 1$.

Theorem 2.2.1. *Let X be a nonnegative random variable with probability density function f. Then, for every $n \in \mathbb{N}$ and $\alpha \in \mathbb{R}^+$*

$$f^{\alpha}_{n}(t) = f(t),\ t \geq 0, \qquad \Leftrightarrow \qquad X \sim \mathcal{E}(\lambda),$$

where $f_n^{\alpha}(t)$ is the nth-order fractional equilibrium density (2.4) and $\mathcal{E}(\lambda)$ is the exponential distribution with parameter $\lambda \in \mathbb{R}^+$.

Proof. First, let us assume that X is exponentially distributed with parameter λ. Since

$$\mathbb{E}\left[X^{n\alpha}\right] = \frac{\Gamma(n\alpha+1)}{\lambda^{n\alpha}} \quad \text{and} \quad \mathbb{E}\left[(X-t)_+^{n\alpha-1}\right] = \lambda^{1-n\alpha} e^{-t\lambda}\Gamma(n\alpha), \quad t \geq 0,$$

by virtue of (2.4) the assertion "if" is trivially proved. Conversely, suppose that the density and the nth-order fractional equilibrium density of X coincide for every $n \in \mathbb{N}$ and $\alpha \in \mathbb{R}^+$, that is

$$f_n^{\alpha}(t) = f(t), \qquad t \geq 0.$$

Due to (2.4), the last equality can be rewritten as

$$n\alpha \int_t^{+\infty} (x-t)^{n\alpha-1} f(x)\,\mathrm{d}x = f(t) \int_0^{+\infty} x^{n\alpha} f(x)\,\mathrm{d}x,$$

and, on account of (1.7), as

$$\Gamma(n\alpha+1) I_-^{n\alpha} f(t) = f(t) \int_0^{+\infty} x^{n\alpha} f(x)\,\mathrm{d}x.$$

Taking the Mellin transform of both sides of this equation yields the functional equation

$$\frac{f^*(s+n\alpha)}{\Gamma(s+n\alpha)} = \frac{f^*(n\alpha+1)}{\Gamma(n\alpha+1)} \frac{f^*(s)}{\Gamma(s)}, \qquad \Re(s) > 0, \tag{2.6}$$

where

$$f^*(s) = \int_0^{\infty} x^{s-1} f(x)\mathrm{d}x,$$

is the Mellin transform of a function $f(x)$ (cf. (C.3.21) and (C.3.22) of [57]). By reducing Eq. (2.6) to a well-known Cauchy equation, we observe that its nontrivial measurable solution (cf. [63] for instance) is

$$f^*(s) = a^{c(s-1)}\Gamma(s), \qquad a > 0,\ c \in \mathbb{R}.$$

By performing the Mellin inversion, we have

$$f(t) = a^{-c} e^{-a^{-c}t}, \qquad t \geq 0.$$

As a consequence, $X \sim \mathcal{E}(\lambda)$, having set $\lambda = a^{-c}$, and then the "only if" part of the theorem is proved. □

In the next proposition we give the expression of the moments of a random variable following the nth-order fractional equilibrium distribution (2.4).

Proposition 2.2.3. *For $\alpha \in \mathbb{R}^+$ and $n \in \mathbb{N}$, if $\mathbb{E}\left[X^{n\alpha}\right] < \infty$, then*

$$\mathbb{E}\left[\left(X_\alpha^{(n)}\right)^r\right] = \frac{n\alpha\, B(n\alpha, r+1)}{\mathbb{E}\left[X^{n\alpha}\right]}\,\mathbb{E}\left[X^{n\alpha+r}\right], \qquad r \in \mathbb{R}^+, \tag{2.7}$$

where $B(x,y)$ is the Beta function.

Proof. Recalling (2.4), it holds

$$\begin{aligned}
\mathbb{E}\left[\left(X_\alpha^{(n)}\right)^r\right] &= \int_0^{+\infty} t^r f_n^\alpha(t)\,\mathrm{d}t \\
&= \frac{n\alpha}{\mathbb{E}\left[X^{n\alpha}\right]}\int_0^{+\infty} t^r \left(\int_t^{+\infty} (x-t)^{n\alpha-1}\,\mathrm{d}F(x)\right)\mathrm{d}t \\
&= \frac{n\alpha}{\mathbb{E}\left[X^{n\alpha}\right]}\int_0^{+\infty} \mathrm{d}F(x)\left(\int_0^{x} t^r (x-t)^{n\alpha-1}\,\mathrm{d}t\right).
\end{aligned}$$

By applying formula 3.191-4 of [60], i.e.

$$\int_0^u x^{\nu-1}(u-x)^{\mu-1}\,\mathrm{d}x = u^{\mu+\nu-1}B(\mu,\nu), \qquad \Re(\mu) > 0,\ \Re(\nu) > 0,$$

we obtain

$$\begin{aligned}
\mathbb{E}\left[\left(X_\alpha^{(n)}\right)^r\right] &= \frac{n\alpha\, B(n\alpha, r+1)}{\mathbb{E}\left[X^{n\alpha}\right]}\int_0^{+\infty} x^{r+n\alpha}\mathrm{d}F(x) \\
&= \frac{n\alpha\, B(n\alpha, r+1)}{\mathbb{E}\left[X^{n\alpha}\right]}\,\mathbb{E}\left[X^{n\alpha+r}\right].
\end{aligned}$$

□

Clearly, when $\alpha = 1$ the moments (2.7) identify with the expression for the iterated stationary-excess variables given in the Lemma of Massey and Whitt [98] and in Theorem 2.3 of Harkness and Shantaram [62].

2.3 Fractional probabilistic Taylor's theorem

We now derive a probabilistic extension of the Riemann-Liouville generalized Taylor's formula shown in Theorem 2.1.1. For convenience, let us denote

$$I_F = \bigcup_{n=2}^{\infty}\left[0, F^{-1}\left(1-\frac{1}{n}\right)\right], \tag{2.8}$$

the smallest interval containing both 0 and the support of the distribution F. Additionally, without loss of generality, we consider the expansion of a function g about $t = 0$.

Theorem 2.3.1. *Let $0 < \alpha \leq 1$ and let X be a nonnegative random variable with cumulative distribution function F, with moment $\mathbb{E}\left[X^{(n+1)\alpha}\right] < +\infty$ for some integer $n \geq 0$ and moments $\mathbb{E}\left[X^{(j+1)\alpha-1}\right] < +\infty$ for all $j \in \mathbb{N}, 0 \leq j \leq n$. Suppose that g is a function defined on I_F and satisfying the hypoteses (i),(ii) and (iii) of Theorem 2.1.1 in I_F. Assume further $\mathbb{E}\left[\left|D_0^{(n+1)\alpha} g\left(X_\alpha^{(n+1)}\right)\right|\right] < +\infty$. Then $\mathbb{E}\left[g\left(X\right)\right] < +\infty$ and*

$$\mathbb{E}\left[g\left(X\right)\right] = \sum_{j=0}^{n} \frac{c_j}{\Gamma\left(\left(j+1\right)\alpha\right)} \mathbb{E}\left[X^{(j+1)\alpha-1}\right] + \frac{\mathbb{E}\left[X^{(n+1)\alpha}\right]}{\Gamma\left(\left(n+1\right)\alpha+1\right)} \mathbb{E}\left[D_0^{(n+1)\alpha} g\left(X_\alpha^{(n+1)}\right)\right], \quad (2.9)$$

with $c_j = \Gamma(\alpha)[x^{1-\alpha} D_0^{j\alpha} g(x)](0^+)$ for each $j \in \mathbb{N}, 0 \leq j \leq n$, where $X_\alpha^{(n+1)}$ has density (2.4).

Proof. To begin with, we recall a Riemann-Liouville generalized Taylor's formula with integral remainder term (cf. formula (4.1) of [145]), that is, for $x \in I_F$,

$$g(x) = \sum_{j=0}^{n} \frac{c_j x^{(j+1)\alpha-1}}{\Gamma(\left(j+1\right)\alpha)} + R_n(x), \quad (2.10)$$

where

$$R_n(x) = I_0^{(n+1)\alpha} D_0^{(n+1)\alpha} g\left(x\right) = \frac{1}{\Gamma\left(\left(n+1\right)\alpha\right)} \int_0^x \left(x-t\right)^{(n+1)\alpha-1} D_0^{(n+1)\alpha} g\left(t\right) \mathrm{d}t. \quad (2.11)$$

Since $R_n(x)$ is continuous, and hence measurable, on I_F, $R_n(X)$ is a true random variable. Therefore, from (2.10) we have

$$\mathbb{E}\left[g\left(X\right)\right] = \sum_{j=0}^{n} \frac{c_j}{\Gamma\left(\left(j+1\right)\alpha\right)} \mathbb{E}\left[X^{(j+1)\alpha-1}\right] + \mathbb{E}\left[R_n\left(X\right)\right], \quad (2.12)$$

where, from (2.11),

$$\mathbb{E}\left[R_n\left(X\right)\right] = \frac{1}{\Gamma((n+1)\alpha)} \int_0^{+\infty} \mathrm{d}F\left(x\right) \int_0^x D_0^{(n+1)\alpha} g\left(t\right) \left(x-t\right)^{(n+1)\alpha-1} \mathrm{d}t.$$

By making use of Fubini's theorem, the equality above becomes

$$\mathbb{E}\left[R_n\left(X\right)\right]=\frac{1}{\Gamma((n+1)\alpha)}\int_{I_F} D_0^{(n+1)\alpha}g\left(t\right)\mathbb{E}\left[X-t\right]_+^{(n+1)\alpha-1}\mathrm{d}t,$$

and in turn, due to (2.4),

$$\begin{aligned}\mathbb{E}\left[R_n\left(X\right)\right]&=\frac{\mathbb{E}\left[X^{(n+1)\alpha}\right]}{\left(n+1\right)\alpha\,\Gamma\left(\left(n+1\right)\alpha\right)}\int_{I_F} D_0^{(n+1)\alpha}g\left(t\right)f_{n+1}^{\alpha}\left(t\right)\mathrm{d}t\\&=\frac{\mathbb{E}\left[X^{(n+1)\alpha}\right]}{\Gamma\left(\left(n+1\right)\alpha+1\right)}\mathbb{E}\left[D_0^{(n+1)\alpha}g\left(X_\alpha^{(n+1)}\right)\right].\end{aligned} \tag{2.13}$$

Finally, observing that the condition $\mathbb{E}\left[\left|D_0^{(n+1)\alpha}g\left(X_\alpha^{(n+1)}\right)\right|\right]<+\infty$ is equivalent to $\int_{I_F}|D_0^{(n+1)\alpha}g\left(t\right)|\mathbb{E}\left[X-t\right]_+^{(n+1)\alpha-1}\mathrm{d}t<+\infty$, and making use of (2.12) and (2.13), the proof of (2.9) is thus completed. □

Equation (2.9) can be seen as a fractional version of the probabilistic generalization of Taylor's theorem studied in [87] and in [98].

Remark 2.3.1. We observe that for $n=0$ formula (2.9) becomes

$$\mathbb{E}\left[g\left(X\right)\right]=\frac{c_0}{\Gamma\left(\alpha\right)}\mathbb{E}\left[X^{\alpha-1}\right]+\frac{\mathbb{E}\left[X^\alpha\right]}{\Gamma\left(\alpha+1\right)}\mathbb{E}\left[D_0^\alpha g\left(X_\alpha^{(1)}\right)\right], \tag{2.14}$$

with $c_0=\Gamma\left(\alpha\right)\left[x^{1-\alpha}g(x)\right]\left(0^+\right)$, this being useful to prove Theorem 2.4.1 below.

In recent years much attention has been paid to the study of the fractional moments of distributions. See, for instance, [101] and references therein. Motivated by this, in the next corollary we consider the case $g(x)=x^\beta$, $\beta\in\mathbb{R}$. From Theorem 2.3.1 we have the following result.

Corollary 2.3.1. *Let $0<\alpha\leq 1$, $\beta\geq\alpha$ and $n\leq\frac{\beta-\alpha}{\alpha}$, $n\in\mathbb{N}$. Moreover, let X be a nonnegative random variable with cumulative distribution function F, with moment $\mathbb{E}\left[X^{(n+1)\alpha}\right]<+\infty$ and moments $\mathbb{E}\left[X^{(j+1)\alpha-1}\right]<+\infty$ for all $j\in\mathbb{N}, 0\leq j\leq n$. Assume further $\mathbb{E}\left[\left|D_0^{(n+1)\alpha}\left(X_\alpha^{(n+1)}\right)^\beta\right|\right]<+\infty$. Then $\mathbb{E}\left[X^\beta\right]<+\infty$ and*

$$\mathbb{E}\left[X^\beta\right]=\frac{\mathbb{E}\left[X^{(n+1)\alpha}\right]}{\Gamma\left(\left(n+1\right)\alpha+1\right)}\frac{\Gamma(1+\beta)}{\Gamma(1-(n+1)\alpha+\beta)}\mathbb{E}\left[(X_\alpha^{(n+1)})^{\beta-(n+1)\alpha}\right],$$

where $X_\alpha^{(n+1)}$ has density (2.4).

Proof. Since, in general, for $k\in\mathbb{N}$

$$D_0^{k\alpha}x^\beta=\frac{\Gamma(1+\beta)}{\Gamma(1-k\alpha+\beta)}x^{\beta-k\alpha},$$

we have

$$c_j = \Gamma(\alpha)\left[\frac{\Gamma(1+\beta)}{\Gamma(1-j\alpha+\beta)}x^{1-(j+1)\alpha+\beta}\right](0^+) = 0, \qquad 0 \leq j \leq n.$$

Furthermore, assumption $\mathbb{E}\left[\left|D_0^{(n+1)\alpha}\left(X_\alpha^{(n+1)}\right)^\beta\right|\right] < +\infty$ ensures the finiteness of $\mathbb{E}\left[(X_\alpha^{(n+1)})^{\beta-(n+1)\alpha}\right]$. Therefore, formula (2.9) reduces to the sole remainder term, and hence the thesis. □

2.4 Fractional probabilistic mean value theorem

In this section we develop the probabilistic analogue of a fractional mean value theorem. To this purpose we first recall some stochastic orders and introduce a relevant random variable, Z_α.

Let X be a random variable with cumulative distribution function F_X and let $a = \inf\{x|F_X(x) > 0\}$ and $b = \sup\{x|F_X(x) < 1\}$. We set for every real $\alpha > 0$

$$F_X^{(\alpha)}(t) = \begin{cases} \dfrac{\mathbb{E}\left[(t-X)_+^{\alpha-1}\right]}{\Gamma(\alpha)} & \text{if } t > a \\ 0 & \text{if } t \leq a \end{cases}$$

and

$$\overline{F}_X^{(\alpha)}(t) = \begin{cases} \dfrac{\mathbb{E}\left[(X-t)_+^{\alpha-1}\right]}{\Gamma(\alpha)} & \text{if } t < b \\ 0 & \text{if } t \geq b. \end{cases}$$

Ortobelli et al. [115] define a stochastic order as follows:

Definition 2.4.1. *Let X and Y be two random variables. For every $\alpha > 0$, X dominates Y with respect to the α-bounded stochastic dominance order ($X \underset{\alpha}{\overset{b}{\geq}} Y$) if and only if $F_X^{(\alpha)}(t) \leq F_Y^{(\alpha)}(t)$ for every t belonging to supp$\{X,Y\} \equiv [a,b]$, where $a, b \in \overline{\mathbb{R}}$ and $a = \inf\{x|F_X(x)+F_Y(x) > 0\}$, $b = \sup\{x|F_X(x)+F_Y(x) < 2\}$.*

Similarly, Ortobelli et al. [115] define a *survival bounded order* as follows:

Definition 2.4.2. *For every $\alpha > 0$, we write $X \underset{sur\ \alpha}{\overset{a}{\geq}} Y$ if and only if $\overline{F}_X^{(\alpha)}(t) \leq \overline{F}_Y^{(\alpha)}(t)$ for every t belonging to supp$\{X,Y\}$.*

We remark that certain random variables cannot be compared with respect to these orders. For example, Ortobelli et al. [115] proved that for any pair of bounded (from above or/and from below) random variables X and Y that are continuous on the extremes of their support, there is no $\alpha \in (0,1)$ such that $F_X^{(\alpha)}(t) \leq F_Y^{(\alpha)}(t)$

for all $t \in supp\{X, Y\}$. However, although α-bounded orders with $\alpha \in (0,1)$ are not applicable in many cases, they could be useful to rank truncated variables and financial losses, thus resulting of interest from a financial point of view.

We outline that for $\alpha > 1$ the survival bounded order given in Definition 2.4.2 is equivalent to the extension to all real $\alpha > 0$ of the order $\leq_c^\alpha$ defined in 1.7.1 of [141]. Moreover, when $\alpha = 2$, it is equivalent to the increasing convex order $\leq_{icx}$ (cf. Section 4.A of Shaked and Shantikumar [139]).
The following result comes straightforwardly.

Proposition 2.4.1. *Let X and Y be nonnegative random variables such that $\mathbb{E}[X^\alpha] < \mathbb{E}[Y^\alpha] < +\infty$ for some $\alpha > 0$. Then*

$$f_{Z_\alpha}(t) = \alpha \frac{\mathbb{E}\left[(Y-t)_+^{\alpha-1}\right] - \mathbb{E}\left[(X-t)_+^{\alpha-1}\right]}{\mathbb{E}[Y^\alpha] - \mathbb{E}[X^\alpha]}, \qquad t \geq 0, \tag{2.15}$$

is the probability density function of an absolutely continuous nonnegative random variable, say Z_α, if and only if $X \underset{sur\ \alpha}{\overset{0}{\geq}} Y$.

We remark that condition $X \underset{sur\ \alpha}{\overset{0}{\geq}} Y$ ensures that $\mathbb{E}[X^\alpha] \leq \mathbb{E}[Y^\alpha]$ for $\alpha > 0$. Moreover, it is interesting to note that Z_α is necessarily absolutely continuous, in contrast with X and Y.

Example 2.4.1. Let X and Y be exponential random variables having means μ_X and μ_Y, $\mu_Y > \mu_X > 0$, and let $\alpha \geq 1$. From (2.15) we obtain the following expression for the density of Z_α:

$$f_{Z_\alpha}(t) = \frac{\mu_Y^{\alpha-1} e^{-\frac{t}{\mu_Y}} - \mu_X^{\alpha-1} e^{-\frac{t}{\mu_X}}}{\mu_Y^\alpha - \mu_X^\alpha}, \qquad t \geq 0.$$

Example 2.4.2. Let Y be a random variable taking values in $[0, b)$, with $b \in (0, +\infty]$, and let $\mathbb{E}[Y^\alpha]$ and $\mathbb{E}\left[(Y-t)_+^{\alpha-1}\right]$ be finite, $0 \leq t < b$. Furthermore, we define a random variable X with cumulative distribution function

$$F_X(x) := \begin{cases} 0, & x < 0, \\ p + (1-p)F_Y(x), & 0 \leq x \leq b, \\ 1, & x \geq b, \end{cases}$$

where $F_Y(x)$ is the cumulative distribution function of Y and $0 < p \leq 1$. We remark that X can be viewed as a 0-inflated version of Y, i.e. $X = I \cdot Y$, where I

is a Bernoulli r.v. independent of Y. It is easily ascertained that

$$f_{Z_\alpha}(t) = \frac{\alpha \mathbb{E}\left[(Y-t)_+^{\alpha-1}\right]}{\mathbb{E}\left[Y^\alpha\right]} \equiv f_{Y_\alpha^{(1)}}(t) \equiv f_{X_\alpha^{(1)}}(t), \quad t \geq 0.$$

We note that the density of Z_α given in (2.15) is related to the densities of the fractional equilibrium variables $X_\alpha^{(1)}$ and $Y_\alpha^{(1)}$ which, by virtue of (2.4), are respectively given by

$$f_{X_\alpha^{(1)}}(t) = \frac{\alpha \mathbb{E}\left[(X-t)_+^{\alpha-1}\right]}{\mathbb{E}\left[X^\alpha\right]} \quad \text{and} \quad f_{Y_\alpha^{(1)}}(t) = \frac{\alpha \mathbb{E}\left[(Y-t)_+^{\alpha-1}\right]}{\mathbb{E}\left[Y^\alpha\right]}, \quad t \geq 0.$$

Indeed, from (2.15) the following generalized mixture holds:

$$f_{Z_\alpha}(t) = c f_{Y_\alpha^{(1)}}(t) + (1-c) f_{X_\alpha^{(1)}}(t), \tag{2.16}$$

where

$$c = \frac{\mathbb{E}\left[Y^\alpha\right]}{\mathbb{E}\left[Y^\alpha\right] - \mathbb{E}\left[X^\alpha\right]} \geq 1. \tag{2.17}$$

Such representation is useful to find an expression for the moments of Z_α. In fact, from (2.7), (2.16) and (2.17), Proposition 2.4.2 follows immediately.

Proposition 2.4.2. *Let $\alpha \in \mathbb{R}^+$ and suppose that X and Y are two nonnegative random variables such that $\mathbb{E}[X^\alpha] < \mathbb{E}[Y^\alpha] < +\infty$, and $X \underset{sur\ \alpha}{\overset{0}{\geq}} Y$. Then*

$$\mathbb{E}\left[Z_\alpha^r\right] = \frac{\alpha B(\alpha, r+1)}{\mathbb{E}\left[Y^\alpha\right] - \mathbb{E}\left[X^\alpha\right]} \left\{\mathbb{E}\left[Y^{\alpha+r}\right] - \mathbb{E}\left[X^{\alpha+r}\right]\right\}, \qquad r \in \mathbb{R}^+. \tag{2.18}$$

With the notation of 1.C(3) of [96], let $\lambda_\alpha(X)$ denote the normalized moment of a random variable X, i.e.

$$\lambda_\alpha(X) = \frac{\mathbb{E}\left[X^\alpha\right]}{\Gamma(\alpha+1)}, \qquad \alpha > 0. \tag{2.19}$$

We are now ready to prove the main result of this section.

Theorem 2.4.1. *Let $0 < \alpha \leq 1$. Suppose that X and Y are two nonnegative random variables such that $\mathbb{E}[X^\alpha] < \mathbb{E}[Y^\alpha] < +\infty$, and $X \underset{sur\ \alpha}{\overset{0}{\geq}} Y$. Moreover, let Theorem 2.3.1 hold for some function g. Then*

$$\mathbb{E}\left[g(Y)\right] - \mathbb{E}\left[g(X)\right] = \frac{c_0}{\Gamma(\alpha)} \left\{\mathbb{E}\left[Y^{\alpha-1}\right] - \mathbb{E}\left[X^{\alpha-1}\right]\right\} + \left\{\lambda_\alpha(Y) - \lambda_\alpha(X)\right\} \mathbb{E}\left[D_0^\alpha g(Z_\alpha)\right], \tag{2.20}$$

where Z_α is a random variable whose density is defined in (2.15), and $c_0 = \Gamma(\alpha)\left[x^{1-\alpha}g(x)\right](0^+)$.

Proof. By applying Theorem 2.3.1 for $n = 0$, cf. formula (2.14), we have

$$\begin{aligned}\mathbb{E}\left[g\left(Y\right)\right] - \mathbb{E}\left[g\left(X\right)\right] &= \frac{c_0}{\Gamma\left(\alpha\right)}\left\{\mathbb{E}\left[Y^{\alpha-1}\right] - \mathbb{E}\left[X^{\alpha-1}\right]\right\} \\ &+ \frac{1}{\Gamma\left(\alpha+1\right)}\left\{\mathbb{E}\left[Y^\alpha\right]\mathbb{E}\left[D_0^\alpha g\left(Y_\alpha^{(1)}\right)\right] - \mathbb{E}\left[X^\alpha\right]\mathbb{E}\left[D_0^\alpha g\left(X_\alpha^{(1)}\right)\right]\right\}.\end{aligned}$$

From (2.16), (2.17) and (2.19) the theorem is straightforwardly proved. □

Under certain hypotheses, the Lagrange's Theorem guarantees the existence of a mean value belonging to the interval of interest. With regard to Theorem 2.4.1, one might therefore expect that a probabilistic analogue of this relation holds too. However, the relation $X^\alpha \leq_{st} Z_\alpha \leq_{st} Y^\alpha$ does not hold in general. It can be satisfied only when $\mathbb{E}\left[X^\alpha\right] \leq \mathbb{E}\left[Z_\alpha\right] \leq \mathbb{E}\left[Y^\alpha\right]$, which is case (ii) of the next Proposition. For simplicity's sake, if X is a random variable with $\mathbb{E}\left[X^{\alpha+1}\right] < +\infty$, we set

$$V_\alpha\left(X\right) := \mathbb{E}\left[X^{\alpha+1}\right] - \alpha\left(\mathbb{E}\left[X^\alpha\right]\right)^2, \qquad \alpha \in \mathbb{R}^+, \tag{2.21}$$

which turns out to be a fractional extension of the variance of X.

Proposition 2.4.3. *Let $0 < \alpha \leq 1$ and let X and Y satisfy the assumptions of Theorem 2.4.1, with $\mathbb{E}\left[X^{\alpha+1}\right]$ and $\mathbb{E}\left[Y^{\alpha+1}\right]$ finite. Then,*

(i) $\mathbb{E}\left[Z_\alpha\right] \leq \mathbb{E}\left[X^\alpha\right]$
$\Leftrightarrow V_\alpha\left(Y\right) - V_\alpha\left(X\right) \leq -\left\{\mathbb{E}\left[Y^\alpha\right] - \mathbb{E}\left[X^\alpha\right]\right\}\left\{\alpha\mathbb{E}\left[Y^\alpha\right] - \mathbb{E}\left[X^\alpha\right]\right\};$

(ii) $\mathbb{E}\left[X^\alpha\right] \leq \mathbb{E}\left[Z_\alpha\right] \leq \mathbb{E}\left[Y^\alpha\right]$
$$\Leftrightarrow \begin{cases} V_\alpha\left(Y\right) - V_\alpha\left(X\right) \geq -\left\{\mathbb{E}\left[Y^\alpha\right] - \mathbb{E}\left[X^\alpha\right]\right\}\left\{\alpha\mathbb{E}\left[Y^\alpha\right] - \mathbb{E}\left[X^\alpha\right]\right\} \\ V_\alpha\left(Y\right) - V_\alpha\left(X\right) \leq \left\{\mathbb{E}\left[Y^\alpha\right] - \mathbb{E}\left[X^\alpha\right]\right\}\left\{\mathbb{E}\left[Y^\alpha\right] - \alpha\mathbb{E}\left[X^\alpha\right]\right\};\end{cases}$$

(iii) $\mathbb{E}\left[Y^\alpha\right] \leq \mathbb{E}\left[Z_\alpha\right]$
$\Leftrightarrow V_\alpha\left(Y\right) - V_\alpha\left(X\right) \geq \left\{\mathbb{E}\left[Y^\alpha\right] - \mathbb{E}\left[X^\alpha\right]\right\}\left\{\mathbb{E}\left[Y^\alpha\right] - \alpha\mathbb{E}\left[X^\alpha\right]\right\};$

(iv) $\mathbb{E}\left[Z_\alpha\right] = \dfrac{2\alpha}{\alpha+1}\,\dfrac{\mathbb{E}\left[X^\alpha\right] + \mathbb{E}\left[Y^\alpha\right]}{2} \Leftrightarrow V_\alpha(Y) = V_\alpha(X),$

where V_α has been defined in (2.21).

Proof. It follows easily from the identity

$$\frac{\mathbb{E}\left[Z_\alpha\right] - \mathbb{E}\left[X^\alpha\right]}{\mathbb{E}\left[Y^\alpha\right] - \mathbb{E}\left[X^\alpha\right]} = \frac{1}{\alpha+1}\left\{\frac{\alpha\mathbb{E}\left[Y^\alpha\right] - \mathbb{E}\left[X^\alpha\right]}{\mathbb{E}\left[Y^\alpha\right] - \mathbb{E}\left[X^\alpha\right]} + \frac{V_\alpha(Y) - V_\alpha(X)}{\left(\mathbb{E}\left[Y^\alpha\right] - \mathbb{E}\left[X^\alpha\right]\right)^2}\right\},$$

which is a consequence of (2.18) written for $r = 1$. □

Corollary 2.4.1. *Let $0 < \alpha \leq 1$. Suppose that X and Y are two nonnegative random variables such that $\mathbb{E}[X^\alpha] < \mathbb{E}[Y^\alpha] < +\infty$, and $X \underset{sur\ \alpha}{\overset{0}{\geq}} Y$. Moreover, if $\beta > \alpha - 1$, let Theorem 2.3.1 hold for some function $g(x) \sim x^\beta$, $x \to 0^+$. Then*

$$\mathbb{E}[g(Y)] - \mathbb{E}[g(X)] = \{\lambda_\alpha(Y) - \lambda_\alpha(X)\}\,\mathbb{E}[D_0^\alpha g(Z_\alpha)],$$

where Z_α is a random variable whose density is defined in (2.15).

Proof. We observe that the first term in formula (2.20) vanishes, since $c_0 \sim x^{1-\alpha+\beta}\big|_{x=0^+} = 0$, and hence the thesis holds. □

As application, we now show a result of interest to actuarial science. A *deductible* is a treshold amount, denoted d, which must be exceeded by a loss in order for a claim to be paid. If X is the severity random variable representing the size of a single loss event, $X > 0$, and if the deductible is exceeded (that is, if $X > d$), then the amount paid to offset part or all of that loss is $X - d$. Therefore, for $d > 0$, the claim amount random variable X_d is defined to be

$$X_d := (X-d)_+ = \begin{cases} 0, & \text{for } X \leq d, \\ X - d, & \text{for } X > d. \end{cases} \tag{2.22}$$

If a deductible is established, there will be fewer payments than losses, because there are some losses that do not produce payments at all. It is clear from equation (2.22) that X_d has a mixed distribution. In particular, such random variable has an atom at zero representing the absence of payment because the loss did not exceed d. The interested reader is referred to [71] and [73] for further information. Let $b = \sup\{x | F_X(x) < 1\}$. Bearing in mind Definition 2.15, the next Proposition immediately follows from Corollary 2.4.1.

Proposition 2.4.4. *Let $0 < \alpha \leq 1$ and $0 < r < s < b$. With reference to (2.22), suppose that $X_s \underset{sur\ \alpha}{\overset{0}{\geq}} X_r$ and $\lambda_\alpha(X_s) < \lambda_\alpha(X_r) < +\infty$. Moreover, let g satisfy the assumptions of Corollary 2.4.1. Then*

$$\mathbb{E}[g(X_r)] - \mathbb{E}[g(X_s)] = [\lambda_\alpha(X_r) - \lambda_\alpha(X_s)]\mathbb{E}[D_0^\alpha g(Z_\alpha)],$$

where Z_α is a random variable with density

$$f_{Z_\alpha}(z) = \alpha \frac{\mathbb{E}\left[(X_r - z)_+^{\alpha-1}\right] - \mathbb{E}\left[(X_s - z)_+^{\alpha-1}\right]}{\mathbb{E}[X_r^\alpha] - \mathbb{E}[X_s^\alpha]}, \qquad z \geq 0.$$

We conclude this section with the following example.

Example 2.4.3. *Let $0 < \alpha \leq 1$ and $0 < r < s$.*

(i) *Let X be an exponential random variable with parameter λ. Due to (2.19) and Proposition 2.4.4, we have*

$$\mathbb{E}\left[g\left(X_r\right)\right]-\mathbb{E}\left[g\left(X_s\right)\right]=\lambda^{-\alpha}\left(e^{-\lambda r}-e^{-\lambda s}\right)\mathbb{E}\left[D_0^\alpha g\left(Z\right)\right],$$

where Z turns out to be exponentially distributed with parameter λ as well. In this case, it is interesting to note that for $0<r<s$ and $0<u<v$ it results:

$$\frac{\mathbb{E}\left[g\left(X_r\right)\right]-\mathbb{E}\left[g\left(X_s\right)\right]}{\mathbb{E}\left[g\left(X_u\right)\right]-\mathbb{E}\left[g\left(X_v\right)\right]}=\frac{e^{-\lambda r}-e^{-\lambda s}}{e^{-\lambda u}-e^{-\lambda v}},$$

which is independent of g.

(ii) *Now let X be a 2-phase hyperexponential random variable with phase probabilities p and $1-p$, $0<p<1$, and rates λ_1 and λ_2. Similarly, it holds*

$$\begin{aligned}\mathbb{E}\left[g\left(X_r\right)\right]-\mathbb{E}\left[g\left(X_s\right)\right]=&\left\{p\lambda_1^{-\alpha}\left(e^{-\lambda_1 r}-e^{-\lambda_1 s}\right)\right.\\&\left.+\left(1-p\right)\lambda_2^{-\alpha}\left(e^{-\lambda_2 r}-e^{-\lambda_2 s}\right)\right\}\mathbb{E}\left[D_0^\alpha g\left(Z_\alpha\right)\right],\end{aligned}$$

where the density of Z_α is, for $z\geq 0$,

$$f_{Z_\alpha}\left(z\right)=\frac{p\lambda_1^{1-\alpha}e^{-\lambda_1 z}\left(e^{-\lambda_1 r}-e^{-\lambda_1 s}\right)+\left(1-p\right)\lambda_2^{1-\alpha}e^{-\lambda_2 z}\left(e^{-\lambda_2 r}-e^{-\lambda_2 s}\right)}{p\lambda_1^{-\alpha}\left(e^{-\lambda_1 r}-e^{-\lambda_1 s}\right)+\left(1-p\right)\lambda_2^{-\alpha}\left(e^{-\lambda_2 r}-e^{-\lambda_2 s}\right)}.$$

2.5 Concluding remarks

The overall aim of this chapter is to present a novel Taylor's theorem from a probabilistic and a fractional perspective at the same time and to discuss other related findings. It is meaningful to note that, while the coefficients of our formula (2.9) are expressed in terms of the Riemann-Liouville fractional derivative, it is possible to establish a fractional probabilistic Taylor's theorem in the Caputo sense too. We recall that the Caputo derivative, denoted by ${}_*D_{a+}^\alpha$, is defined by exchanging the operators $I_{a+}^{m-\alpha}$ and D^m in the classical definition (1.4). Taking the paper of Odibat et al. [105] as a starting point, the following theorem, which is in some sense the equivalent of Theorem 2.3.1, can be effortlessly proved.

Theorem 2.5.1. *Let $\alpha\in(0,1]$ and let X be a nonnegative random variable with distribution F and moment $\mathbb{E}\left[X^{(n+1)\alpha}\right]<+\infty$ for some integer $n\geq 0$. Assume that g is a function defined on I_F, with I_F defined in (2.8), and suppose that ${}_*D_{0+}^\alpha g(x)\in C\left(I_F\right)$ for $k=0,1,\dots,n+1$ and $\mathbb{E}\left[\left|{}_*D_0^{(n+1)\alpha}g\left(X_\alpha^{(n+1)}\right)\right|\right]<+\infty$. Then $\mathbb{E}\left[g\left(X\right)\right]<$*

$+\infty$ *and*

$$\mathbb{E}\left[g(X)\right] = \sum_{i=0}^{n} \frac{\left({}_*D_{0+}^{i\alpha} f\right)(0)}{\Gamma\left(i\alpha+1\right)} \mathbb{E}\left[X^{i\alpha}\right] + \frac{\mathbb{E}\left[X^{(n+1)\alpha}\right]}{\Gamma\left(\left(n+1\right)\alpha+1\right)} \mathbb{E}\left[{}_*D_0^{(n+1)\alpha} g\left(X_\alpha^{(n+1)}\right)\right],$$

where ${}_*D_{0+}^{n\alpha} = {}_*D_{0+}^{\alpha} \cdot {}_*D_{0+}^{\alpha} \cdots {}_*D_{0+}^{\alpha}$ *(n times) and* $X_\alpha^{(n+1)}$ *has density (2.4).*

Chapter 3

A fractional counting process and its connection with the Poisson process

3.1 Introduction

As recalled by Mainardi et al. [95], a stochastic process $\{N(t), t \geq 0\}$ is a *counting process* if $N(t)$ represents the total number of "events" that have occurred up to time t. The concept of renewal process has been developed to describe the class of counting processes for which the times between successive events (*waiting times*) are independent identically distributed (i.i.d.) nonnegative random variables, obeying a given probability law. It is often assumed that $t = 0$ is a renewal point. Renewal processes have been successfully used to model, *e.g.*, radioactive decay, neural spike trains, failure times in software testing. For more details on renewal theory see the classical books by Cox [27], Feller [50], and Ross [132]. Exponentially distributed waiting times lead to the classical Poisson process, which is Markovian. Indeed, the exponential distribution characterizes processes without memory. However, other waiting time distributions are also relevant in applications, in particular the ones with a fat tail caused by a power law decay of their density. Non-Markovian renewal processes with waiting time distributions described by functions of Mittag-Leffler type, that exhibit a power law decay, have increasingly been attracting attention within the research community. By resorting to different approaches (renewal theory, fractionalization of the governing equation, inverse subordinator), several aspects and definitions on a fractional generalization of the Poisson process have been pointed out by many authors, see for instance Repin and Saichev [131], Laskin [83] and [84], Mainardi et al. [94], Uchaikin et al. [147], Beghin and Orsingher [17] and [18], Meerschaert et al. [100], Politi et al. [125], Leonenko et al. [86].

In addition, counting processes with integer-valued jumps play a major role in many fields of applied probability, since they are useful to describe simultaneous but independent Poisson streams (see Adelson [1] for instance). The case of fractional compound Poisson processes has been investigated by Scalas [137], Beghin and Macci [13], [14] and [15]. Certain fractional growth processes including the possibility of integer-valued jumps have been introduced in Orsingher and Polito [110], Orsingher and Toaldo [113] and Polito and Scalas [126] by suitably time-changing a homogeneous Poisson process. A generalization of the space-time fractional Poisson process involving the Caputo type Saigo differential operator is introduced and its state probabilities are obtained using the Adomian decomposition method in [70]. The relevance of fractional compound Poisson processes in applications in ruin theory and their long-range dependence have been investigated in Biard and Saussereau [20] and [21], and Maheshwari and Vellaisamy [91]. Such processes might also be used in disaster risk management. For instance, Brooks et al. [22] showed that variability of tornado occurrence has increased since the 1970s, due to a decrease in the number of days per year with tornadoes combined with an increase in days with many tornadoes.

In the present chapter we analyse a suitable extension of the fractional Poisson process, say $M^\nu(t)$, which performs k kinds of jumps of amplitude $1, 2, \dots, k$, with rates $\lambda_1, \lambda_2, \dots, \lambda_k$ respectively. Along the same lines as Beghin and Orsingher [18], in Section 3.2 we consider a suitable fractional Cauchy problem whose solution, expressed in terms of a generalized Mittag-Leffler function, represents the probability mass function of $M^\nu(t)$. Besides, we analyse two useful representations for $M^\nu(t)$. We first prove that $M^\nu(t)$ can be expressed as a compound fractional Poisson process, this representation being essential to obtain a waiting time distribution. Then we show that $M^\nu(t)$ can be regarded as a homogeneous Poisson process with k kinds of jumps stopped at a random time. Such random time is the sole component of this subordinating relationship affected by the fractional derivative, since its distribution is obtained from the fundamental solution of a fractional diffusion equation.

In Section 3.4 we face the problem of determining certain waiting time and first-passage-time distributions. Specifically, we evaluate the probability that the first jump of size j, $j = 1, 2, \dots, k$, for the process $M^\nu(t)$ occurs before time $t > 0$. Interestingly, we prove that the first occurrence time of a jump of amplitude j has the same distribution as the waiting time of the first event of the classical fractional Poisson process with parameter λ_j, $j \in \{1, 2, \dots, k\}$. This is an immediate extension of a well-known result. Indeed, for a Poisson process with intensity $\lambda_1 + \lambda_2$ and such that its events are classified as type j via independent Bernoulli trials with probability $\frac{\lambda_j}{\lambda_1+\lambda_2}$, the first occurrence time of an event of type j is distributed as the interarrival time of a Poisson process with intensity λ_j, $j = 1, 2$. In Theorem 3.4.1 we extend this result to the fractional setting. The remarkable difference is that the exponential density characterizing the Poisson process is replaced by a Mittag-Leffler density. In the same section, we also study the distribution of the first passage time of $M^\nu(t)$ to a fixed level when $k = 2$. We express it in an integral form which involves the joint distribution of the fractional Poisson process.

Finally, in Section 3.5 we obtain a formal expression for the moments of $M^\nu(t)$, and show that both the ratios given by the powers of the fractional Poisson process and of the process $M^\nu(t)$ over their means tend to 1 in probability. This result is useful in some applications. In fact, from a physical point of view, it means that the distance between the distributions of such processes at time t and their equilibrium measures is close to 1 until some deterministic 'cutoff time' and is close to 0 shortly after.

In the remaining part of this section we briefly recall some well-known results on the fractional Poisson process which will be used throughout the chapter. The starting point for our investigations is the analysis carried out by Beghin and Orsingher [17] and [18]. They generalise the equation governing the Poisson process by substituting the time-derivative with the fractional derivative in the Caputo sense (1.5) of order $\nu \in (0, 1]$, thus obtaining:

$$\frac{\mathrm{d}p_k^\nu}{\mathrm{d}t^\nu} = -\lambda(p_k - p_{k-1}), \qquad k \geq 0,$$

with initial conditions

$$p_k(0) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \geq 1. \end{cases}$$

and $p_{-1}(t) = 0$. The solution to the Cauchy problem involves the Mittag-Leffler function (1.11). It is expressed as the distribution of a process, denoted by $N_\lambda^\nu(t)$, $t > 0$, and it reads

$$p_k(t) = \mathbb{P}\{N_\lambda^\nu(t) = k\} = (\lambda t^\nu)^k \, E_{\nu,k\nu+1}^{k+1}(-\lambda t^\nu). \tag{3.1}$$

The fractional Poisson process $N_\lambda^\nu(t)$, $t > 0$, represents a renewal process with interarrival times U_j distributed according to the following density, for $j = 1, 2, \ldots$

and $t \in (0, \infty)$

$$f_1^{\nu}(t) = \mathbb{P}\{U_j \in \mathrm{d}\,t\}/\mathrm{d}\,t = \lambda t^{\nu-1} E_{\nu,\nu}(-\lambda t^{\nu}),$$

with Laplace transform

$$\mathcal{L}\{f_1^{\nu}(t); s\} = \frac{\lambda}{s^{\nu} + \lambda}.$$

The density of the waiting time of the kth event, $T_k = \sum_{j=1}^{k} U_j$, possesses the Laplace transform

$$\mathcal{L}\{f_k^{\nu}(t); s\} = \frac{\lambda^k}{(s^{\nu} + \lambda)^k}.$$

Its inverse can be obtained by applying formula (1.12) and can be expressed, as for the probability distribution, in terms of a Mittag-Leffler function as

$$f_k^{\nu}(t) = \mathbb{P}\{T_k \in \mathrm{d}\,t\}/\mathrm{d}\,t = \lambda^k t^{k\nu-1} E_{\nu,k\nu}^{k}(-\lambda t^{\nu}).$$

The corresponding distribution function can be obtained by integration and reads

$$F_k^{\nu}(t) = \mathbb{P}\{T_k < t\} = \lambda^k t^{k\nu} E_{\nu,k\nu+1}^{k}(-\lambda t^{\nu}). \tag{3.2}$$

The moment generating function of the process $N_{\lambda}^{\nu}(t)$, $t > 0$, can be expressed as

$$\mathbb{E}\left[e^{sN_{\lambda}^{\nu}(t)}\right] = E_{\nu,1}\left(\lambda\left(e^{s} - 1\right)t^{\nu}\right), \qquad s \in \mathbb{R}. \tag{3.3}$$

The mean and the variance of $N_{\lambda}^{\nu}(t)$ read

$$\mathbb{E}\left[N_{\lambda}^{\nu}(t)\right] = \frac{\lambda t^{\nu}}{\Gamma(\nu+1)}, \tag{3.4}$$

and

$$\mathrm{Var}\left[N_{\lambda}^{\nu}(t)\right] = \frac{2\left(\lambda t^{\nu}\right)^2}{\Gamma(2\nu+1)} - \frac{\left(\lambda t^{\nu}\right)^2}{\left(\Gamma(\nu+1)\right)^2} + \frac{\lambda t^{\nu}}{\Gamma(\nu+1)}. \tag{3.5}$$

respectively. In general, the analytical expression for the mth order moment of the fractional Poisson process is given by (cf. Laskin [84], Eq. (40))

$$\mathbb{E}\left[\left(N_{\lambda}^{\nu}(t)\right)^m\right] = \sum_{l=0}^{m} S_{\nu}(m,l)\left(\lambda t^{\nu}\right)^l, \tag{3.6}$$

where $S_{\nu}(m,l)$ is the fractional Stirling number, expressed in terms of the standard Stirling number $S(m,l)$ as follows (cf. Laskin [84], Eq. (32)):

$$S_{\nu}(m,l) = \frac{l!}{\Gamma(\nu l+1)} S(m,l) = \frac{1}{\Gamma(\nu l+1)} \sum_{n=0}^{l} (-1)^{l-n} \binom{l}{n} n^m.$$

3.2 A fractional counting process

Let $\{M^1(t); t \geq 0\}$ be a counting process defined by the following rules:

1. $M^1(0) = 0 \quad$ a.s.;
2. $M^1(t)$ has stationary and independent increments;
3. $\mathbb{P}\{M^1(h) = j\} = \lambda_j h + \mathrm{o}(h)$, for $j = 1, 2, \ldots, k$;
4. $\mathbb{P}\{M^1(h) > k\} = \mathrm{o}(h)$,

where $k \in \mathbb{N} \equiv \{1, 2, \ldots\}$ is fixed, and $\lambda_1, \lambda_2, \ldots, \lambda_k > 0$. This is a suitable extension of the Poisson process. We define

$$p_j(t) = \mathbb{P}\left\{M^1(t) = j\right\}, \qquad j \in \mathbb{N}_0 \equiv \{0, 1, 2, \ldots\},$$

and then consider how $p_j(t)$ evolves in some short time period h.

- $p_0(t+h)$ is the probability that no events have taken place at time $t+h$, starting from $t = 0$. This can happen by no events happening until time t and then no events in the interval $[t, t+h]$. This means

$$p_0(t+h) = p_0(t)\left(1 - (\lambda_1 + \ldots + \lambda_k) h\right) + \mathrm{o}(h);$$

- $p_j(t+h)$, $j = 1, 2, \ldots, k-1$, is the probability for j events to happen at time $t+h$. This can happen by $j - r$, $r = 1, 2, \ldots, j$, events happening until time t and r events in the interval $[t, t+h]$, or by j events happening until time t and no events in the interval $[t, t+h]$. This means

$$p_j(t+h) = h \sum_{r=1}^{j} \lambda_r p_{j-r}(t) + p_j(t)\left(1 - (\lambda_1 + \ldots + \lambda_k) h\right) + \mathrm{o}(h);$$

- $p_j(t+h)$, $j = k, k+1, \ldots$, is the probability for j events to happen at time $t+h$. This can happen by $j - r$, $r = 1, 2, \ldots, k$, events happening until time t and r events in the interval $[t, t+h]$, or by j events happening until time t and no events in the interval $[t, t+h]$. This means

$$p_j(t+h) = h \sum_{r=1}^{k} \lambda_r p_{j-r}(t) + p_j(t)\left(1 - (\lambda_1 + \ldots + \lambda_k) h\right) + \mathrm{o}(h).$$

These can be rewritten as

$$\frac{p_0(t+h) - p_0(t)}{h} = -(\lambda_1 + \ldots + \lambda_k) p_0(t) + \frac{\mathrm{o}(h)}{h}$$

$$\frac{p_j\,(t+h)-p_j\,(t)}{h} = \sum_{r=1}^{j} \lambda_r\, p_{j-r}(t) - (\lambda_1 + \ldots + \lambda_k)\, p_j\,(t) + \frac{\mathrm{o}(h)}{h}, \qquad j = 1, \ldots, k-1$$

$$\frac{p_j\,(t+h)-p_j\,(t)}{h} = \sum_{r=1}^{k} \lambda_r\, p_{j-r}(t) - (\lambda_1 + \ldots + \lambda_k)\, p_j\,(t) + \frac{\mathrm{o}(h)}{h}, \qquad j = k, k+1, \ldots$$

Now we take the limit as $h \to 0$, this causing the terms $\mathrm{o}(h)$ to vanish. We obtain the following system of difference-differential equations satisfied by the probability mass function of the process $M^1(t)$:

$$\begin{cases} \dfrac{\mathrm{d}p_0(t)}{\mathrm{d}t} = -\Lambda\, p_0(t) \\ \dfrac{\mathrm{d}p_j(t)}{\mathrm{d}t} = \sum_{r=1}^{j} \lambda_r\, p_{j-r}(t) - \Lambda\, p_j(t), \qquad j = 1, \ldots, k-1 \\ \dfrac{\mathrm{d}p_j(t)}{\mathrm{d}t} = \sum_{r=1}^{k} \lambda_r\, p_{j-r}(t) - \Lambda\, p_j(t), \qquad j = k, k+1, \ldots, \end{cases}$$

for $\Lambda = \lambda_1 + \lambda_2 + \ldots + \lambda_k$, together with the condition

$$p_j(0) = \begin{cases} 1, & j = 0 \\ 0, & j \geq 1. \end{cases}$$

In this chapter we examine a fractional extension of $\{M^1(t); t \geq 0\}$. We obtain a proper probability distribution and explore the main properties of the corresponding fractional process. With reference to the Dzherbashyan-Caputo fractional derivative (1.5) and for all fixed $\nu \in (0, 1]$ and $k \in \mathbb{N}$, let $\{M^\nu(t); t \geq 0\}$ be a counting process, and assume that the probability distribution

$$p_j^\nu(t) = \mathbb{P}\{M^\nu(t) = j\}, \qquad j \in \mathbb{N}_0 \equiv \{0, 1, 2, \ldots\}, \tag{3.7}$$

satisfies the following system of fractional difference-differential equations

$$\begin{cases} \dfrac{\mathrm{d}p_0^\nu(t)}{\mathrm{d}t^\nu} = -\Lambda\, p_0^\nu(t) \\ \dfrac{\mathrm{d}p_j^\nu(t)}{\mathrm{d}t^\nu} = \sum_{r=1}^{j} \lambda_r\, p_{j-r}^\nu(t) - \Lambda\, p_j^\nu(t), \qquad j = 1, 2, \ldots, k-1 \\ \dfrac{\mathrm{d}p_j^\nu(t)}{\mathrm{d}t^\nu} = \sum_{r=1}^{k} \lambda_r\, p_{j-r}^\nu(t) - \Lambda\, p_j^\nu(t), \qquad j = k, k+1, \ldots, \end{cases} \tag{3.8}$$

for $\Lambda = \lambda_1 + \lambda_2 + \ldots + \lambda_k$, together with the condition

$$p_j(0) = \begin{cases} 1, & j = 0 \\ 0, & j \geq 1. \end{cases} \tag{3.9}$$

When $\nu = 1$ system (3.8) identifies with the classical difference-differential equations of the process $M^1(t)$. Furthermore, when $k = 1$ the process $M^\nu(t)$ identifies with the process $N_\lambda^\nu(t)$ considered in Section 3.1.

Hereafter we will obtain the solution to (3.8)-(3.9) in terms of the generalized Mittag-Leffler function (1.11) and show that it represents a true probability distribution. To this purpose, we first obtain the moment generating function of $M^\nu(t)$ in terms of the Mittag-Leffler function.

Proposition 3.2.1. *For all fixed $\nu \in (0,1]$ and $k \in \mathbb{N}$, the moment generating function of $M^\nu(t)$ is given by*

$$\mathbb{E}\left[e^{sM^\nu(t)}\right] = E_{\nu,1}\Big(\sum_{j=1}^{k} \lambda_j \left(e^{js} - 1\right) t^\nu\Big), \qquad t \geq 0, \quad s \in \mathbb{R}. \tag{3.10}$$

Proof. We multiply the jth equation of the system (3.8), $j \in \mathbb{N}$, by z^j. We sum the resulting equations on j and we write the result in the form

$$\frac{\partial}{\partial t^\nu}\left(\sum_{k=0}^{+\infty} z^k p_k^\nu(t)\right) = -\Lambda \sum_{k=0}^{+\infty} z^k p_k^\nu(t) + \sum_{j=1}^{k-1} z^j \sum_{r=1}^{j} \lambda_r p_{j-r}(t) + \sum_{j=k}^{+\infty} z^j \sum_{r=1}^{k} \lambda_r p_{j-r}(t), \tag{3.11}$$

where $\Lambda = \lambda_1 + \cdots + \lambda_k$. We set $G(z,t) := \mathbb{E}\left[z^{M^\nu(t)}\right]$ and then rearrange the summands, so that equation (3.11) can be rewritten as

$$\frac{\partial G(z,t)}{\partial t^\nu} = -\Lambda G(z,t) + \sum_{j=1}^{k} \lambda_j z^j G(z,t).$$

Furthermore, taking into account condition (3.9) too, we have

$$G(z,0) = \sum_{k=0}^{\infty} z^k p_k^\nu(0) = 1,$$

and the probability generating function of the process $M^\nu(t)$ satisfies the Cauchy problem

$$\begin{cases} \dfrac{\partial G(z,t)}{\partial t^\nu} = -\displaystyle\sum_{j=1}^{k} \lambda_j \left(1 - z^j\right) G(z,t) \\ G(z,0) = 1. \end{cases}$$

By adopting a Laplace transform approach we obtain

$$\mathcal{L}\left\{G(z,t); s\right\} = \frac{s^{\nu-1}}{s^\nu + \sum_{j=1}^{k} \lambda_j (1 - z^j)}.$$

Equation (3.10) thus follows recalling formula (1.12) and that $G(e^s, t) = \mathbb{E}\left[e^{sM^\nu(t)}\right]$.

□

We remark that the use of the Caputo fractional derivative allows us to use standard initial conditions in the previous proof (cf. (1.6)).

Let us now show that $M^\nu(t)$ can be expressed as a compound fractional Poisson process.

Proposition 3.2.2. *For all fixed $\nu \in (0, 1]$ we have*

$$M^\nu(t) \stackrel{d}{=} \sum_{i=1}^{N_\Lambda^\nu(t)} X_i, \qquad t \geq 0, \tag{3.12}$$

where $N_\Lambda^\nu(t)$ is a fractional Poisson process, defined as in (3.1), with intensity $\Lambda = \lambda_1 + \lambda_2 + \ldots + \lambda_k$. Moreover, $\{X_n : n \geq 1\}$ is a sequence of i.i.d. random variables, independent of $N_\Lambda^\nu(t)$, such that for any $n \in \mathbb{N}$

$$\mathbb{P}\{X_n = j\} = \frac{\lambda_j}{\Lambda}, \qquad j = 1, 2, \ldots, k, \tag{3.13}$$

and where both $N_\Lambda^\nu(t)$ and X_n depend on the same parameters $\lambda_1, \lambda_2, \ldots, \lambda_k$.

Proof. The moment generating function of $Y(t) := \sum_{i=1}^{N_\Lambda^\nu(t)} X_i$, $t \geq 0$, can be expressed as

$$\begin{aligned}
\mathbb{E}\left[e^{sY(t)}\right] &= \mathbb{E}\left[\mathbb{E}\left[e^{sY(t)} \middle| N_\Lambda^\nu(t)\right]\right] \\
&= \mathbb{E}\left[\left(\mathbb{E}\left[e^{sX_1}\right]\right)^{N_\Lambda^\nu(t)}\right].
\end{aligned}$$

Hence, since

$$\mathbb{E}\left[e^{sX_1}\right] = \frac{1}{\Lambda} \sum_{j=1}^{k} \lambda_j \, e^{js},$$

we have

$$\mathbb{E}\left[e^{sY(t)}\right] = \mathbb{E}\left[e^{N_\Lambda^\nu(t) \ln\left(\frac{1}{\Lambda} \sum_{j=1}^k \lambda_j \, e^{js}\right)}\right].$$

Finally, making use of Equation (3.3) we immediately obtain that the moment generating function of $Y(t)$ identifies with the right-hand side of (3.10). The thesis follows after recalling that a moment generating function uniquely determines a distribution. □

We remark that, due to Proposition 3.2.2, $M^\nu(t)$ can be regarded as a special case of the process defined in Equation (7) of Beghin and Macci [14], under a suitable choice of the probability mass function $(q_k)_{k \geq 1}$ and the parameter λ. Furthermore,

according to Definition 7.1.1 of [19], the process $M^\nu(t)$ is a compound Cox process, since Beghin and Orsingher [18] show that $N_\Lambda^\nu(t)$ is a Cox process with a proper directing measure. Moreover, $M^\nu(t)$ is a compound fractional process, and thus it is neither Markovian nor Lèvy (cf. Scalas [137]).
We are now able to obtain the probability mass function (3.7) of $M^\nu(t)$. Indeed, the following Proposition holds true.

Proposition 3.2.3. *The solution $p_j^\nu(t)$ of the Cauchy problem (3.8)-(3.9), for $j \in \mathbb{N}_0$, $\nu \in (0,1]$ and $t \geq 0$, is given by*

$$p_j^\nu(t) = \sum_{r=0}^{j} \sum_{\substack{\alpha_1+\alpha_2+\ldots+\alpha_k=r \\ \alpha_1+2\alpha_2+\ldots+k\alpha_k=j}} \binom{r}{\alpha_1, \alpha_2, \ldots, \alpha_k} \lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\ldots\lambda_k^{\alpha_k} t^{r\nu} E_{\nu,r\nu+1}^{r+1}(-\Lambda t^\nu). \tag{3.14}$$

Proof. From (3.12) and from a conditioning argument we have

$$p_j^\nu(t) = \mathbb{P}\{M^\nu(t) = j\} = \sum_{r=0}^{j} \mathbb{P}\{X_1 + X_2 + \ldots + X_r = j\} \mathbb{P}\{N_\Lambda^\nu(t) = r\}.$$

Since $X_1, X_2, \ldots, X_r$ are independent and identically distributed (cf. (3.13)), it follows that

$$\mathbb{P}\{X_1 + X_2 + \ldots + X_r = j\} = \sum_{\substack{\alpha_1+\alpha_2+\ldots+\alpha_k=r \\ \alpha_1+2\alpha_2+\ldots+k\alpha_k=j}} \binom{r}{\alpha_1, \alpha_2, \ldots, \alpha_k} \times \left(\frac{\lambda_1}{\Lambda}\right)^{\alpha_1} \left(\frac{\lambda_2}{\Lambda}\right)^{\alpha_2} \ldots \left(\frac{\lambda_k}{\Lambda}\right)^{\alpha_k},$$

where the sum is taken in order to consider all the possible ways of performing r jumps, with α_1 jumps of size 1, …, α_k jumps of size k, and such that the total amplitude, i.e. $\alpha_1 + 2\alpha_2 + \ldots + k\alpha_k$, equals j. Hence, recalling formula (3.1), the proposition follows. □

Proposition 3.2.3 is an extension of Proposition 2 of [39], which studies case $k = 2$. Some plots of probabilities (3.14) are shown in Figure 3.1 and Figure 3.2.
From (3.14) we note that, for $\nu \in (0,1]$,

$$p_0^\nu(t) = E_{\nu,1}(-\Lambda t^\nu), \qquad t \geq 0.$$

Moreover, recalling the definition of the generalized Mittag-Leffler function (1.11) and formula (3.14), we obtain hereafter the distribution of the process $M^\nu(t)$ in the special case $\nu = 1$.

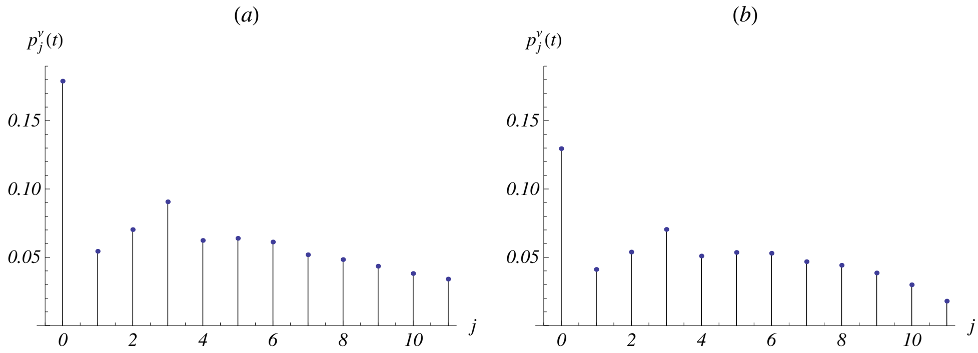

Figure 3.1: Probability distribution of $M^\nu(t)$, given in (3.14), for $j = 0, 1, \dots, 11$, with $k = 3$, $\nu = 0.5$, $\lambda_1 = \lambda_2 = \lambda_3 = 1$, (a) $t = 1$ and (b) $t = 2$. The displayed probability mass is (a) 0.797292 and (b) 0.629278.

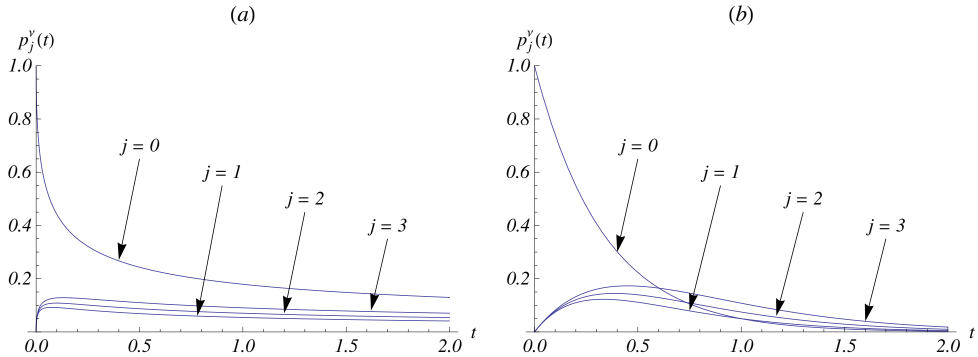

Figure 3.2: Probability distribution of $M^\nu(t)$, given in (3.14), for $0 \le t \le 2$, with $k = 3$, $\lambda_1 = \lambda_2 = \lambda_3 = 1$, (a) $\nu = 0.5$ and (b) $\nu = 1$.

Corollary 3.2.1. *The probability mass function $p_j^1(t)$, for $j \in \mathbb{N}_0$ and $t \ge 0$, is given by*

$$p_j^1(t) = \sum_{r=0}^{j} \sum_{\substack{\alpha_1+\alpha_2+\ldots+\alpha_k=r\\ \alpha_1+2\alpha_2+\ldots+k\alpha_k=j}} \frac{\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\ldots\lambda_k^{\alpha_k}}{\alpha_1!\,\alpha_2!\,\ldots\,\alpha_k!}\, t^r e^{-\Lambda t}. \tag{3.15}$$

3.3 Equivalent representation

We will now examine an interesting relationship between the process $M^\nu(t)$ and the process $M^1(t)$. In fact, we show that the following representation holds:

$$M^\nu(t) \stackrel{d}{=} M^1\left(\mathcal{T}_{2\nu}(t)\right),$$

where $\mathcal{T}_{2\nu}(t)$ is a suitable random process. Thus $M^\nu(t)$ can be considered as a homogeneous Poisson-type counting process with jumps of sizes $1, 2, \dots, k$ stopped at a random time $\mathcal{T}_{2\nu}(t)$.

Let us denote by $g(z,t) = g_{2\nu}(z,t)$ the solution of the Cauchy problem

$$\begin{cases} \frac{\partial^{2\nu} g(z,t)}{\partial t^{2\nu}} = \frac{\partial^2 g(z,t)}{\partial z^2}, & t > 0,\ z \in \mathbb{R} \\ g(z,0) = \delta(z), & 0 < \nu < 1 \\ \left.\frac{\partial g(z,t)}{\partial t}\right|_{t=0} = 0, & \frac{1}{2} < \nu < 1. \end{cases} \tag{3.16}$$

It is well-known that (see [93] and [92])

$$g_{2\nu}(z,t) = \frac{1}{2t^\nu} W_{-\nu,1-\nu}\left(-\frac{|z|}{t^\nu}\right), \quad t > 0,\ z \in \mathbb{R}, \tag{3.17}$$

where

$$W_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!\,\Gamma(\alpha k + \beta)}, \qquad \alpha > -1,\ \beta > 0,\ x \in \mathbb{R}, \tag{3.18}$$

is the Wright function. Let

$$\bar{g}_{2\nu}(z,t) = \begin{cases} 2\,g_{2\nu}(z,t), & z > 0 \\ 0, & z < 0 \end{cases} \tag{3.19}$$

be the folded solution to (3.16), so that negative spatial values are mapped into their positive counterparts. Moreover, let $\mathcal{T}_{2\nu}(t)$ be a random process (independent from the process $M^1(t)$) whose transition density $\mathbb{P}\{\mathcal{T}_{2\nu}(t) \in \mathrm{d}z\}/\mathrm{d}z$ is given in (3.19).

Remark 3.3.1. It has been proved in Orsingher and Beghin [107] that the solution $g_{2\nu}$ to (3.16) can be alternatively expressed as

$$g_{2\nu}(z,t) = \frac{1}{2\Gamma(1-\nu)} \int_0^t (t-w)^{-\nu} f_\nu(w,|z|)\,\mathrm{d}w, \quad z \in \mathbb{R},$$

where $f_\nu(\cdot, y)$ is a stable law $S_\nu(\mu,\beta,\sigma)$ of order ν, with parameters $\mu = 0, \beta = 1$ and $\sigma = \left(z \cos\frac{\pi\nu}{2}\right)^{\frac{1}{\nu}}$.

Proposition 3.3.1. *The process $M^\nu(t)$ and the process $M^1(\mathcal{T}_{2\nu}(t))$ are identically distributed.*

Proof. From (3.7) and (3.19) we have

$$\mathbb{P}\left\{M^1(\mathcal{T}_{2\nu}(t)) = n\right\} = \int_0^\infty p_n^1(z)\,\bar{g}_{2\nu}(z,t)\,\mathrm{d}z.$$

Hence, making use of (3.15) and (3.18) we get

$$\mathbb{P}\left\{M^1\left(\mathcal{T}_{2\nu}\left(t\right)\right)=n\right\}=\sum_{j=0}^{n}\sum_{\substack{\alpha_1+\alpha_2+\ldots+\alpha_k=j\\ \alpha_1+2\alpha_2+\ldots+k\alpha_k=n}}\frac{\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\ldots\lambda_k^{\alpha_k}}{\alpha_1!\,\alpha_2!\,\ldots\,\alpha_k!}$$
$$\times\frac{1}{t^\nu}\int_0^\infty e^{-\Lambda z}\,z^j\,W_{-\nu,1-\nu}\left(-\frac{z}{t^\nu}\right)\mathrm{d}z.$$

For $y=\Lambda z$, the last expression identifies with (3.14) due to the following integral representation of the generalized Mittag-Leffler function in terms of the Wright function, derived by Beghin and Orsingher [18]:

$$E_{\nu,k\nu+1}^{k+1}(-\Lambda t^\nu)=\frac{1}{k!\,\Lambda^{k+1}\,t^{(k+1)\nu}}\int_0^\infty e^{-y}\,y^k\,W_{-\nu,1-\nu}\left(-\frac{y}{\Lambda t^\nu}\right)\mathrm{d}y.$$

This completes the proof. □

Remark 3.3.2. Beghin and Orsingher [18] proved an analogous subordination relationship, i.e.

$$N_\lambda^\nu(t)\overset{d}{=}N_\lambda^1(\mathcal{T}_{2\nu}\left(t\right)),$$

where $N_\lambda^\nu(t)$ is the fractional Poisson process defined in (3.1) and $\mathcal{T}_{2\nu}\left(t\right)$ is the random time defined above.

Remark 3.3.3. By taking $\nu=1/2$, from Proposition 3.3.1 we have that $M^{1/2}\left(t\right)$ and $M^1\left(\mathcal{T}_1\left(t\right)\right)$ are identically distributed. We note that the random time $\mathcal{T}_1\left(t\right)$, $t>0$, is a reflecting Brownian motion. Indeed, in this case equation (3.16) reduces to the heat equation

$$\begin{cases}\frac{\partial g}{\partial t}=\frac{\partial^2 g}{\partial z^2}, & t>0,\ z\in\mathbb{R}\\ g\left(z,0\right)=\delta\left(z\right),\end{cases}$$

and the solution $g_1\left(z,t\right)$ is the density of a Brownian motion $B\left(t\right),t>0$, with infinitesimal variance 2. After folding up the solution, we find the following probability mass

$$\begin{aligned}\mathbb{P}\left\{M^1\left(\mathcal{T}_1\left(t\right)\right)=n\right\}&=\int_0^\infty p_n^1(z)\,\frac{e^{-\frac{z^2}{4t}}}{\sqrt{\pi t}}\mathrm{d}z\\&=\mathbb{P}\left\{M^1\left(\left|B\left(t\right)\right|\right)=n\right\},\end{aligned}$$

so that $M^{1/2}\left(t\right)$ is a jump process at a Brownian time.

Remark 3.3.4. It is worth noticing that both the composition of the fractional Poisson process $N_\lambda^\nu(t)$ defined in (3.1) and of the fractional process $M^\nu(t)$ defined in (3.7) with the random time $\mathcal{T}_{2\nu}(t)$ yield fractional processes of different order, i.e.

$$N_\lambda^\nu(\mathcal{T}_{2\nu}(t)) \stackrel{d}{=} N_\lambda^{\nu^2}(t) \qquad \text{and} \qquad M^\nu(\mathcal{T}_{2\nu}(t)) \stackrel{d}{=} M^{\nu^2}(t).$$

Taking into account the subordinating relations examined in Proposition 3.3.1 and in Remark 3.3.2, this fact follows immediately from Remark 3.1 of [81], since, in general, the composition of two stable subordinators of indices β_1 and β_2 respectively is a stable subordinator of index $\beta_1\beta_2$.

Remark 3.3.5. Bearing in mind Proposition 3.2.2, setting

$$\mathcal{S}_r = \Lambda \cdot \mathbb{E}[X^r] = \sum_{j=1}^{k} j^r \lambda_j, \qquad r = 1, 2,$$

and recalling (3.4) and (3.5), we can more easily compute the mean and the variance of the process. In fact, by Wald's equation we have

$$\begin{aligned}\mathbb{E}[M^\nu(t)] &= \mathbb{E}[X] \cdot \mathbb{E}[N_\Lambda^\nu(t)] \\ &= \frac{\mathcal{S}_1 t^\nu}{\Gamma(\nu+1)}, \qquad t \geq 0.\end{aligned}$$

Moreover, by the law of total variance, we get

$$\begin{aligned}\mathrm{Var}[M^\nu(t)] &= \mathrm{Var}[X] \cdot \mathbb{E}[N_\Lambda^\nu(t)] + (\mathbb{E}[X])^2 \cdot \mathrm{Var}[N_\Lambda^\nu(t)] \\ &= \frac{\mathcal{S}_2 t^\nu}{\Gamma(\nu+1)} + \mathcal{S}_1^2 t^{2\nu} Z(\nu), \qquad t \geq 0,\end{aligned}$$

where

$$Z(\nu) := \frac{1}{\nu}\left(\frac{1}{\Gamma(2\nu)} - \frac{1}{\nu\Gamma^2(\nu)}\right).$$

As a consequence it is not hard to show that $\mathrm{Var}[M^\nu(t)] - \mathbb{E}[M^\nu(t)] > 0$, or, equivalently, that the process $M^\nu(t)$ exhibits overdispersion, since $Z(\nu) > 0$ for all $\nu \in (0, 1)$ and $Z(1) = 0$. Generally speaking, a real-valued random variable Y is said to be overdispersed if $\mathrm{Var}[Y] - \mathbb{E}[Y] > 0$, and a process $Y(\cdot)$ is said to be overdispersed if all the random variables $\{Y(t) : t > 0\}$ are overdispersed. Finally, we point out that a formal expression for the moments of process $M^\nu(t)$ is provided in Lemma 3.5.1.

3.4 Waiting times and first-passage times

We now evaluate the probability distribution function of the waiting time until the first occurrence of a jump of size i, $i = 1, 2, \dots, k$, for the process $M^\nu(t)$. We first observe that the following decomposition holds:

$$M^\nu(t) = \sum_{j=1}^{k} j\, M_j^\nu(t), \qquad t \geq 0,$$

where

$$M_j^\nu(t) := \sum_{i=1}^{N_\Lambda^\nu(t)} \mathbf{1}_{\{X_i=j\}}, \qquad j = 1, 2, \dots, k.$$

In other words, $M_j^\nu(t)$ counts the number of jumps of amplitude j performed by $M^\nu(t)$ in $(0, t]$. We also introduce the random variables

$$H_j := \inf\left\{s > 0 : M_j^\nu(s) = 1\right\} \quad \text{and} \quad G^j \sim \text{Geo}\left(\frac{\lambda_j}{\Lambda}\right), \qquad j = 1, 2, \dots, k.$$

In other words, H_j represents the first occurrence time of a jump of amplitude j for the process $M^\nu(t)$, whereas G^j is a geometric random variable with parameter $\frac{\lambda_j}{\Lambda}$ that describes the order of the first jump of amplitude j in the sequence of jumps of $M^\nu(t)$. We prove that H_j is distributed as the waiting time of the first event of the fractional Poisson process defined in (3.1) with parameter λ_j. Indeed, the following result holds.

Theorem 3.4.1. *Let $j \in \{1, 2, \dots, k\}$. Then*

$$\mathbb{P}\{H_j \leq t\} = \lambda_j t^\nu E_{\nu,\nu+1}\left(-\lambda_j t^\nu\right), \qquad t > 0. \tag{3.20}$$

Proof. By conditioning on G^j, due to Equations (3.12) and (3.2), for $t > 0$,

$$\begin{aligned}
\mathbb{P}\{H_j \leq t\} &= \mathbb{E}_{G^j}\left[\mathbb{P}\left\{H_j \leq t \mid G^j\right\}\right] \\
&= \sum_{n=1}^{+\infty} \mathbb{P}\left\{H_j \leq t \mid G^j = n\right\} \mathbb{P}\left\{G^j = n\right\} \\
&= \sum_{n=1}^{+\infty} F_n^\nu(t) \frac{\lambda_j}{\Lambda}\left(1 - \frac{\lambda_j}{\Lambda}\right)^{n-1} \\
&= \sum_{n=1}^{+\infty} \Lambda^n t^{n\nu} E_{\nu,n\nu+1}^n(-\Lambda t^\nu) \frac{\lambda_j}{\Lambda}\left(1 - \frac{\lambda_j}{\Lambda}\right)^{n-1} \\
&= \lambda_j t^\nu \sum_{n=0}^{+\infty} \Lambda^n t^{n\nu}\left(1 - \frac{\lambda_j}{\Lambda}\right)^n E_{\nu,(n+1)\nu+1}^{n+1}(-\Lambda t^\nu).
\end{aligned}$$

By using formula (2.3.1) of [99], i.e.

$$\frac{1}{\Gamma(\alpha)}\int_0^1 u^{\gamma-1}(1-u)^{\alpha-1}E^{\delta}_{\beta,\gamma}\left(zu^{\beta}\right)\mathrm{d}u = E^{\delta}_{\beta,\gamma+\alpha}(z),$$

(where $Re(\alpha)>0, Re(\beta)>0$ and $Re(\gamma)>0$) for $\alpha=\beta=\nu$, $\gamma=n\nu+1$, $\delta=n+1$ and $z=-\Lambda t^{\nu}$, we get

$$\begin{aligned}\mathbb{P}\{H_j\le t\} &= \frac{\lambda_j t^{\nu}}{\Gamma(\nu)}\sum_{n=0}^{+\infty}\Lambda^n t^{n\nu}\left(1-\frac{\lambda_j}{\Lambda}\right)^n\int_0^1 u^{n\nu}(1-u)^{\nu-1}E^{n+1}_{\nu,n\nu+1}(-\Lambda\, t^{\nu}u^{\nu})\,\mathrm{d}u\\ &= \frac{\lambda_j t^{\nu}}{\Gamma(\nu)}\int_0^1(1-u)^{\nu-1}\sum_{n=0}^{+\infty}\left[\Lambda t^{\nu}\left(1-\frac{\lambda_j}{\Lambda}\right)u^{\nu}\right]^n E^{n+1}_{\nu,n\nu+1}(-\Lambda\, t^{\nu}u^{\nu})\,\mathrm{d}u.\end{aligned}$$

Due to formula (2.30) of [18], i.e.

$$\sum_{n=0}^{+\infty}(\lambda w t^{\nu})^n E^{n+1}_{\nu,\nu n+1}(-\lambda t^{\nu}) = E_{\nu,1}(\lambda(w-1)t^{\nu}), \qquad |w|\le 1,\;\; t>0,$$

we have

$$\mathbb{P}\{H_j\le t\} = \frac{\lambda_j t^{\nu}}{\Gamma(\nu)}\int_0^1(1-u)^{\nu-1}E_{\nu,1}(-\lambda_j\, t^{\nu}u^{\nu})\,\mathrm{d}u.$$

By making use of formula (2.2.14) of [99], i.e.

$$\int_0^1 z^{\beta-1}(1-z)^{\sigma-1}E_{\alpha,\beta}(xz^{\alpha})\,\mathrm{d}z = \Gamma(\sigma)E_{\alpha,\sigma+\beta}(x),$$

(where $\alpha>0$; $\beta,\sigma\in\mathbb{C}$; $Re(\beta)>0$ and $Re(\sigma)>0$), for $\sigma=\alpha=\nu$, $\beta=1$ and $x=-\lambda_j t^{\nu}$, we get

$$\mathbb{P}\{H_j\le t\} = \lambda_j t^{\nu}E_{\nu,\nu+1}(-\lambda_j t^{\nu}), \qquad t\ge 0.$$

Therefore H_j is distributed as the waiting time of the first event of the fractional Poisson process defined in (3.1) (cf. (3.2)). □

The result shown in Theorem 3.4.1 is an immediate extension of the well-known result for the classical Poisson process, i.e. for $\nu=1$, where H_j is exponentially distributed with parameter λ_j.

We will now be concerned with the distribution of the first passage time to a fixed level for the process $M^{\nu}(t)$, denoted as

$$\tau_n = \inf\{s>0 : M^{\nu}(s)=n\}, \qquad n\in\mathbb{N}. \tag{3.21}$$

The following result concerns the case $k = 2$, i.e. when the process $M^\nu(t)$ performs jumps of sizes 1 and 2.

Theorem 3.4.2. *The cumulative distribution function of the first passage time* τ_n *when* $k = 2$ *reads*

$$\mathbb{P}\{\tau_n \le t\} = \sum_{h=n}^{+\infty} \sum_{j=\lceil \frac{h}{2} \rceil}^{h} \sum_{i=1}^{j} \binom{i}{n-i} \binom{j-i}{h-n-j+i} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{2j-h} \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{h-j} \times \int_0^t \mathbb{P}\left\{N^\nu_{\lambda_1+\lambda_2}(t) = j, N^\nu_{\lambda_1+\lambda_2}(s) = i\right\} \mathrm{d}s, \qquad t > 0. \tag{3.22}$$

Proof. Since the process $M^\nu(t)$ performs jumps of size 1 and 2, and has non-independent increments, the computation of the cumulative distribution function of the first passage time (3.21) can be carried out as follows:

$$\begin{aligned}\mathbb{P}\{\tau_n \le t\} &= \sum_{h=n}^{+\infty} \int_0^t \mathbb{P}\{M^\nu(t) = h, M^\nu(s) = n\} \mathrm{d}s \\ &= \sum_{h=n}^{+\infty} \sum_{j=\lceil \frac{h}{2} \rceil}^{h} \sum_{i=1}^{j} \int_0^t \mathbb{P}\left\{M^\nu(t) = h, M^\nu(s) = n \mid N^\nu_{\lambda_1+\lambda_2}(t) = j, N^\nu_{\lambda_1+\lambda_2}(s) = i\right\} \\ &\quad \times \mathbb{P}\left\{N^\nu_{\lambda_1+\lambda_2}(t) = j, N^\nu_{\lambda_1+\lambda_2}(s) = i\right\} \mathrm{d}s.\end{aligned}$$

Making use of Proposition 3.2.2 we have:

$$\begin{aligned}
\mathbb{P}\{\tau_n \le t\} =& \sum_{h=n}^{+\infty} \sum_{j=\lceil \frac{h}{2} \rceil}^{h} \sum_{i=1}^{j} \int_0^t \mathbb{P}\left\{ \sum_{r=1}^{j} X_r = h, \sum_{l=1}^{i} X_l = n \right\} \\
& \times \mathbb{P}\left\{ N_{\lambda_1+\lambda_2}^{\nu}(t) = j, N_{\lambda_1+\lambda_2}^{\nu}(s) = i \right\} \mathrm{d}s \\
=& \sum_{h=n}^{+\infty} \sum_{j=\lceil \frac{h}{2} \rceil}^{h} \sum_{i=1}^{j} \int_0^t \mathbb{P}\left\{ \sum_{l=1}^{i} X_l = n, \sum_{r=i+1}^{j} X_r = h-n \right\} \\
& \times \mathbb{P}\left\{ N_{\lambda_1+\lambda_2}^{\nu}(t) = j, N_{\lambda_1+\lambda_2}^{\nu}(s) = i \right\} \mathrm{d}s \\
=& \sum_{h=n}^{+\infty} \sum_{j=\lceil \frac{h}{2} \rceil}^{h} \sum_{i=1}^{j} \int_0^t \mathbb{P}\left\{ \sum_{l=1}^{i} X_l = n \right\} \mathbb{P}\left\{ \sum_{r=i+1}^{j} X_r = h-n \right\} \\
& \times \mathbb{P}\left\{ N_{\lambda_1+\lambda_2}^{\nu}(t) = j, N_{\lambda_1+\lambda_2}^{\nu}(s) = i \right\} \mathrm{d}s \\
=& \sum_{h=n}^{+\infty} \sum_{j=\lceil \frac{h}{2} \rceil}^{h} \sum_{i=1}^{j} \binom{i}{n-i} \left(\frac{\lambda_1}{\lambda_1+\lambda_2} \right)^{2i-n} \left(\frac{\lambda_2}{\lambda_1+\lambda_2} \right)^{n-i} \\
& \times \binom{j-i}{h-n-j+i} \left(\frac{\lambda_1}{\lambda_1+\lambda_2} \right)^{2j-2i+n-h} \left(\frac{\lambda_2}{\lambda_1+\lambda_2} \right)^{h-n-j+i} \\
& \times \int_0^t \mathbb{P}\left\{ N_{\lambda_1+\lambda_2}^{\nu}(t) = j, N_{\lambda_1+\lambda_2}^{\nu}(s) = i \right\} \mathrm{d}s,
\end{aligned}$$

thus giving Eq. (3.22). □

To the best of our knowledge, the bivariate distribution shown in the right-hand side of (3.22), i.e. $\mathbb{P}\left\{ N_{\lambda_1+\lambda_2}^{\nu}(s) = i, N_{\lambda_1+\lambda_2}^{\nu}(t) = j \right\}$, cannot be expressed in a closed form. Orsingher and Polito [111] derived an expression in terms of Prabhakar integrals, i.e.:

$$\begin{aligned}
& \mathbb{P}\left\{ N_{\lambda_1+\lambda_2}^{\nu}(s) = i, N_{\lambda_1+\lambda_2}^{\nu}(t) = j \right\} \\
&\quad = (\lambda_1+\lambda_2)^j \left(\mathbf{E}^{i}_{\nu,\nu i,-(\lambda_1+\lambda_2);(t-s)+} \left(\mathbf{E}^{j-i}_{\nu,\nu(j-i-1)+1,-(\lambda_1+\lambda_2);(z+s-t)+} \right.\right. \\
&\qquad \left.\left. \times\, y^{\nu-1} E_{\nu,\nu}(-(\lambda_1+\lambda_2) y^{\nu}) \right)(z) \right)(t),
\end{aligned}$$

where

$$\left(\mathbf{E}^{\gamma}_{\rho,\mu,\omega;a+} \phi \right)(x) = \int_a^x (x-t)^{\mu-1} E^{\gamma}_{\rho,\mu}\left(\omega (x-t)^{\rho}\right) \phi(t)\, \mathrm{d}t$$

is the Prabhakar integral (see [128] for details). Politi et al. [125] use the renewal approach as well and evaluate the joint probability given in (3.22) by introducing the random variable Y_i. Such random variable denotes the residual lifetime at s (that is the time to the next epoch) conditional on $N_{\lambda_1+\lambda_2}^{\nu}(s) = i$, i.e. $Y_i \overset{def}{=}$

$\left[\tau_i - s \mid N^{\nu}_{\lambda_1+\lambda_2}(s) = i\right]$ whose cumulative distribution function is denoted by $F_{Y_i}(y)$. Therefore,

$$\begin{aligned}\mathbb{P}\left\{N^{\nu}_{\lambda_1+\lambda_2}(s) = i, N^{\nu}_{\lambda_1+\lambda_2}(t) = j\right\} = \mathbb{P}\left\{N^{\nu}_{\lambda_1+\lambda_2}(t) - N^{\nu}_{\lambda_1+\lambda_2}(s) = j-i \mid N^{\nu}_{\lambda_1+\lambda_2}(s) = i\right\} \\ \times \mathbb{P}\left\{N^{\nu}_{\lambda_1+\lambda_2}(s) = i\right\},\end{aligned}$$

where

$$\begin{aligned}&\mathbb{P}\left\{N^{\nu}_{\lambda_1+\lambda_2}(t) - N^{\nu}_{\lambda_1+\lambda_2}(s) = j-i \mid N^{\nu}_{\lambda_1+\lambda_2}(s) = i\right\} \\ &\qquad = \begin{cases} \int_0^{t-s} \mathbb{P}\left\{N^{\nu}_{\lambda_1+\lambda_2}(t-s-y) = j-i-1\right\} \mathrm{d}F_{Y_i}(y), & \text{if } j-i \geq 1, \\ 1 - F_{Y_i}(t-s), & \text{if } j-i = 0. \end{cases}\end{aligned}$$

As an example, we prove that the two aforementioned expressions coincide in the case $j-i \geq 1$. We recall that the probability density function of Y_i is expressed as (cf. Eq. (21) of [125]):

$$f_{Y_i}(y) = \frac{\int_0^s \mathrm{d}u f_i^{\nu}(u)\, f_1^{\nu}(y+s-u)}{\int_0^s \mathrm{d}u f_i^{\nu}(u)\, [1 - F_1^{\nu}(s-u)]}. \tag{3.23}$$

Density (3.23) can be alternatively expressed as (cf. Section 3.1):

$$f_{Y_i}(y) = \frac{\int_0^s \mathrm{d}u\, \lambda^i u^{i\nu-1} E^i_{\nu,i\nu}(-\lambda u^{\nu})\, \lambda(y+s-u)^{\nu-1} E_{\nu,\nu}(-\lambda(y+s-u)^{\nu})}{\int_0^s \mathrm{d}u\, \lambda^i u^{i\nu-1} E^i_{\nu,i\nu}(-\lambda u^{\nu})\, [1 - \lambda(s-u)^{\nu} E_{\nu,\nu+1}(-\lambda(s-u)^{\nu})]},$$

where we have set $\lambda_1 + \lambda_2 = \lambda$. The denominator in the previous expression can be simplified by recurring to the following relations (cf. (4.2.3) of [57] and cf. Th. 2 of [72]):

$$E_{\alpha,\beta}(z) = \frac{1}{\Gamma(\beta)} + z E_{\alpha,\alpha+\beta}(z),$$

and

$$\int_0^x (x-t)^{\beta-1} E^{\gamma}_{\alpha,\beta}[a(x-t)^{\alpha}] t^{\nu-1} E^{\sigma}_{\alpha,\nu}(at^{\alpha}) \mathrm{d}t = x^{\beta+\nu-1} E^{\gamma+\sigma}_{\alpha,\beta+\nu}(ax^{\alpha}),$$

for $\alpha, \beta, \gamma, a, \nu, \sigma \in \mathbb{C}$ ($\Re(\alpha)$, $\Re(\beta)$, $\Re(\nu) > 0$). Indeed, we get

$$1 - \lambda(s-u)^{\nu} E_{\nu,\nu+1}(-\lambda(s-u)^{\nu}) = E_{\nu,1}(-\lambda(s-u)^{\nu})$$

and

$$\int_0^s \mathrm{d}u\, \lambda^i u^{i\nu-1} E^i_{\nu,i\nu}(-\lambda u^{\nu}) E_{\nu,1}(-\lambda(s-u)^{\nu}) = \lambda^i s^{i\nu} E^{i+1}_{\nu,i\nu+1}(-\lambda s^{\nu}).$$

From (3.1), the bivariate distribution in Politi et al. [125] becomes

$$\int_0^{t-s} \mathrm{d}y \; (\lambda(t-s-y)^\nu)^{j-i-1} \, E^{j-i}_{\nu,(j-i-1)\nu+1} \left(-\lambda(t-s-y)^\nu\right) \\ \times \int_0^s \mathrm{d}u \, \lambda^i u^{i\nu-1} E^i_{\nu,i\nu}(-\lambda u^\nu) \, \lambda(y+s-u)^{\nu-1} E_{\nu,\nu}(-\lambda(y+s-u)^\nu).$$

We change the order of integration; then the substitution $y+s-u=z$ yields

$$\lambda^j \int_0^s u^{i\nu-1} E^i_{\nu,i\nu}(-\lambda u^\nu) \left(\int_{s-u}^{t-u} z^{\nu-1} E_{\nu,\nu}(-\lambda z^\nu) \right. \\ \left. \times \, (t-u-z)^{\nu(j-i-1)} E^{j-i}_{\nu,(j-i-1)\nu+1}(-\lambda(t-u-z)^\nu) \, \mathrm{d}z \right) \mathrm{d}u.$$

The previous formula coincides with the bivariate distribution given in Theorem 2.1 of Orsingher and Polito [111], which, in turn, can be expressed in terms of Prabhakar integrals, as outlined in Remark 2.2 of the same paper.

It is meaningful to stress that when $k=2$ the passage of $M^\nu(t)$ to a level n is not sure. In fact, the process can cross state n without visiting it due to the effect of a jump having size 2.

3.5 Convergence results

For the processes $N^\nu_\lambda(t)$ and $M^\nu(t)$, introduced respectively in (3.1) and in (3.7), we now focus on a property related to their asymptotic behavior as the relevant parameters grow larger.

Proposition 3.5.1. *Let $\nu \in (0,1]$. Then for a fixed $t>0$ we have*

$$\frac{N^\nu_\lambda(t)}{\mathbb{E}\left[N^\nu_\lambda(t)\right]} \xrightarrow[\lambda\to+\infty]{\text{Prob}} 1.$$

Proof. We study the convergence in mean of the random variable $\frac{N^\nu_\lambda(t)}{\mathbb{E}\left[N^\nu_\lambda(t)\right]}$ to 1. Due to the triangle inequality we have

$$\mathbb{E}\left[\left| \frac{N^\nu_\lambda(t)}{\mathbb{E}\left[N^\nu_\lambda(t)\right]} - 1 \right| \right] \leq 2.$$

Therefore, we can apply the dominated convergence theorem and calculate the following limit:

$$\lim_{\lambda\to+\infty}\mathbb{E}\left[\left|\frac{N_\lambda^\nu(t)}{\mathbb{E}\left[N_\lambda^\nu(t)\right]}-1\right|\right]=\lim_{\lambda\to+\infty}\sum_{j=0}^{+\infty}\left|\frac{j}{\frac{\lambda t^\nu}{\Gamma(\nu+1)}}-1\right|(\lambda t^\nu)^j\,E_{\nu,j\nu+1}^{j+1}(-\lambda t^\nu). \tag{3.24}$$

Taking account of the behavior of the generalized Mittag-Leffler function for large z (see [135] for details), i.e.:

$$E_{\alpha,\beta}^{\delta}(z)\sim\mathcal{O}\left(|z|^{-\delta}\right),\qquad |z|>1,$$

we can conclude that limit (3.24) equals 0. This fact proves the proposition since convergence in mean implies convergence in probability. □

The previous result can be extended to a more general setting. Recalling the expression (3.6) for the moments of $N_\lambda^\nu(t)$, the proof of the next proposition is similar to that of Proposition 3.5.1 and thus is omitted.

Proposition 3.5.2. *Let $\nu\in(0,1]$ and $r\in\mathbb{N}$. Then, for a fixed $t>0$,*

$$\frac{\left[N_\lambda^\nu(t)\right]^r}{\mathbb{E}\left\{\left[N_\lambda^\nu(t)\right]^r\right\}}\xrightarrow[\lambda\to+\infty]{\text{Prob}}1.$$

In order to prove an analogous result for $M^\nu(t)$, in the following lemma we give a formal expression for the moments of the process.

Lemma 3.5.1. *The mth order moment of the process $M^\nu(t)$, $t\geq 0$, reads*

$$\begin{aligned}\mathbb{E}\left\{\left[M^\nu(t)\right]^m\right\}=&\sum_{r=0}^{m}\frac{t^{r\nu}}{\Gamma(r\nu+1)}\sum_{i_1+\ldots+i_k=r}\binom{r}{i_1,\ldots,i_k}\lambda_1^{i_1}\ldots\lambda_k^{i_k}\\&\times\sum_{n_1+\ldots+n_k=m}\binom{m}{n_1,\ldots,n_k}\left[\frac{\mathrm{d}^{n_1}}{\mathrm{d}s^{n_1}}\left(e^s-1\right)^{i_1}\ldots\frac{\mathrm{d}^{n_k}}{\mathrm{d}s^{n_k}}\left(e^{ks}-1\right)^{i_k}\right]\Bigg|_{s=0}.\end{aligned} \tag{3.25}$$

Proof. By applying Hoppe's formula in order to evaluate the derivatives of the moment generating function of the process $M^\nu(t)$, cf. (3.10), we have

$$\mathbb{E}\left\{\left[M^\nu(t)\right]^m\right\}=\sum_{r=0}^{m}\frac{\left(E_{\nu,1}(z)\right)^{(r)}\big|_{z=\sum_{j=1}^k\lambda_j(e^{js}-1)t^\nu}}{r!}A_{m,r}\left(\sum_{j=1}^{k}\lambda_j\left(e^{js}-1\right)t^\nu\right)\Bigg|_{s=0}, \tag{3.26}$$

where

$$A_{m,r}\left(\sum_{j=1}^{k}\lambda_j\left(e^{js}-1\right)t^{\nu}\right)=\sum_{h=0}^{r}\binom{r}{h}\left(-\sum_{j=1}^{k}\lambda_j\left(e^{js}-1\right)t^{\nu}\right)^{r-h}\times\frac{\mathrm{d}^m}{\mathrm{d}s^m}\left(\sum_{j=1}^{k}\lambda_j\left(e^{js}-1\right)t^{\nu}\right)^{h}.$$

Since

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^m E^{\gamma}_{\alpha,\beta}(z)=(\gamma)_m\,E^{\gamma+m}_{\alpha,\beta+m\gamma}(z),$$

we get

$$\left(E_{\nu,1}(z)\right)^{(r)}\big|_{z=\sum_{j=1}^{k}\lambda_j(e^{js}-1)t^{\nu}}=r!E^{r+1}_{\nu,r\nu+1}\left(\sum_{j=1}^{k}\lambda_j\left(e^{js}-1\right)t^{\nu}\right).$$

Moreover, recalling definition (1.11),

$$E^{r+1}_{\nu,r\nu+1}\left(\sum_{j=1}^{k}\lambda_j\left(e^{js}-1\right)t^{\nu}\right)\Bigg|_{s=0}=\frac{1}{\Gamma(r\nu+1)}.$$

Therefore, equation (3.26) reduces to

$$\begin{aligned}\mathbb{E}\left\{[M^{\nu}(t)]^m\right\}&=\sum_{r=0}^{m}\frac{1}{\Gamma(r\nu+1)}\,A_{m,r}\left(\sum_{j=1}^{k}\lambda_j\left(e^{js}-1\right)t^{\nu}\right)\Bigg|_{s=0}\\&=\sum_{r=0}^{m}\frac{1}{\Gamma(r\nu+1)}\frac{\mathrm{d}^m}{\mathrm{d}s^m}\left(\sum_{j=1}^{k}\lambda_j\left(e^{js}-1\right)t^{\nu}\right)^{r},\end{aligned}$$

since $\sum_{j=1}^{k}\lambda_j\left(e^{js}-1\right)t^{\nu}|_{s=0}=0$.
The thesis (3.25) then follows observing that

$$\left(\sum_{j=1}^{k}\lambda_j\left(e^{js}-1\right)t^{\nu}\right)^{r}=t^{\nu r}\sum_{i_1+\ldots+i_k=r}\binom{r}{i_1,\ldots,i_k}\lambda_1^{i_1}\left(e^{js}-1\right)^{i_1}\ldots\lambda_k^{i_k}\left(e^{js}-1\right)^{i_k},$$

and then applying the product rule for the mth derivative of an arbitrary number of factors. □

It is now immediate to verify the following result for $M^{\nu}(t)$.

Proposition 3.5.3. *Let $\nu\in(0,1]$ and $m\in\mathbb{N}$. Then, for $i\in\{1,2,\ldots,k\}$ and for a fixed $t>0$, we have*

$$\frac{[M^{\nu}(t)]^m}{\mathbb{E}\left\{[M^{\nu}(t)]^m\right\}}\xrightarrow[\lambda_i\to+\infty]{\text{Prob}}1.$$

Proof. By virtue of (3.25), convergence in probability can be obtained by proving convergence in mean, as in Proposition 3.5.1. □

The results presented in this section are interesting in some physical contexts. We recall that a family of stochastic processes $U^{(\lambda)} = U^{(\lambda)}(t)$ exhibits *cut-off behaviour* at mean times if (see, for instance, Definition 1 of [10])

$$\frac{U^{(\lambda)}}{\mathbb{E}\left[U^{(\lambda)}\right]} \xrightarrow[\lambda\to+\infty]{\text{Prob}} 1.$$

or, equivalently, $\lim_{\lambda\to\infty} \mathbb{P}\left(U^{(\lambda)} > c\mathbb{E}\left[U^{(\lambda)}\right]\right) = 1$ for $c < 1$ and 0 for $c > 0$. As $\lambda \to +\infty$, a suitable distance between the laws $\mathbb{P}\left(U^{(\lambda)}(t) \in \bullet\right)$ and the corresponding invariant measures $\pi^{(\lambda)}(\bullet)$ converges, in macroscopic time units, to a step function centered at *deterministic* times t_λ^{cut}.
Hence, Propositions 3.5.1, 3.5.2 and 3.5.3 show that the processes $[N_\lambda^\nu(t)]^m$ and $[M^\nu(t)]^m$, $m \in \mathbb{N}$, exhibit cut-off behavior at mean times with respect to the relevant parameters or, roughly speaking, that they somehow converge very abruptly to equilibrium.

We finally remark that in this context the sufficient condition given in Proposition 1 of [10] is not useful to prove Proposition 3.5.1, since it holds only when $\nu = 1$.

Chapter 4

On a fractional alternating Poisson process

Alternating renewal processes are special types of renewal processes. Specifically, an alternating renewal process is a stochastic process in which the renewal interval consists of two random subintervals that alternate cyclically. During the first one the process is in mode 1, whilst during the second one the process is in mode 0. For example, consider a repairable system which might periodically be in ON mode (running) or in OFF mode (in repair) for a random time. See Cox [27] for details. An alternating renewal process has been recently used to describe the buildup of perceptual segregation [140]. Other examples can be taken from the fields of inventory control, finance, traffic control, etc.; cf. [152] for more details. If the system starts in state 1 and if a *cycle* consists of a mode-1 and a mode-0 interval, then the process that counts the number of cycles completed up to time t is an alternating renewal process, where returns to state 1 are the arrivals (cycle completions).

Let $\{U_k;\ k = 1, 2, \dots\}$ and $\{D_k;\ k = 1, 2, \dots\}$ be independent sequences of independent copies of two non-negative absolutely continuous random variables U, describing the duration of a mode-1 period, and D, describing the duration of a mode-0 period. Therefore, the k-th cycle is distributed as

$$X_k \stackrel{d}{=} U^{(k)} + D^{(k)}, \tag{4.1}$$

where

$$U^{(k)} = U_1 + U_2 + \cdots + U_k, \quad D^{(k)} = D_1 + D_2 + \cdots + D_k, \qquad k = 1, 2, \dots \tag{4.2}$$

The cumulative distribution functions of U and D are denoted respectively by F_U and F_D, whereas the corresponding complementary cumulative distribution functions are $\overline{F}_U$ and $\overline{F}_D$. If U_k and D_k are exponentially distributed with positive parameters λ and μ, the resulting counting process having interarrival times $U_1, D_1, U_2, D_2, \ldots$ is the *alternating Poisson process* (see, for instance, [82] for details). Equivalently, an alternating Poisson process is a 2-state continuous-time Markov chain, whose state occupancy probabilities satisfy, for $t \geq 0$ and $\lambda, \mu > 0$, the system of equations:

$$\begin{cases} \dfrac{\mathrm{d}\,p_{11}}{\mathrm{d}t} = -\lambda\, p_{11}(t) + \mu\, p_{10}(t) \\ \dfrac{\mathrm{d}\,p_{10}}{\mathrm{d}t} = \lambda\, p_{11}(t) - \mu\, p_{10}(t) \end{cases} \tag{4.3}$$

It is well-known (cf. [82]) that the solutions of system (4.3), subject to the initial conditions $p_{11}(0) = 1$ and $p_{10}(0) = 0$, and normalizing condition $p_{11}(t) + p_{10}(t) = 1$, are

$$p_{11}(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t} \quad \text{and} \quad p_{10}(t) = \frac{\lambda}{\lambda+\mu} - \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}.$$

Specifically, let $Y(t)$, $t \geq 0$, be a stochastic process with state space $\{0, 1\}$. If $Y(t)$ describes the state of the process at time t and $p_{ij}(t) = \mathbb{P}\left(Y(t) = j \,|\, Y(0) = i\right)$, then $p_{11}(t)$ and $p_{10}(t)$ represent respectively the probabilities of being in states 1 and 0 at time t starting from state 1 at $t = 0$. Similarly, the probabilities of being in states 1 and 0 at t starting from state 0 at $t = 0$ are found to be, for $t \geq 0$,

$$p_{01}(t) = \frac{\mu}{\lambda+\mu} - \frac{\mu}{\lambda+\mu} e^{-(\lambda+\mu)t} \quad \text{and} \quad p_{00}(t) = \frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu} e^{-(\lambda+\mu)t}.$$

If we define, for $j \in \{0, 1\}$,

$$\begin{aligned} p_j(t) &= \mathbb{P}\left(Y(t) = j\right) \\ &= \mathbb{P}\left(\text{state } j \text{ occupied at time } t\right) \qquad t \geq 0, \end{aligned}$$

and note that, by a conditioning argument,

$$p_j(t) = p_1(0) p_{1j}(t) + p_0(0) p_{0j}(t),$$

then

$$p_1(t) = \frac{\mu}{\lambda+\mu} + \left[p_1(0) - \frac{\mu}{\lambda+\mu}\right] e^{-(\lambda+\mu)t}$$

and

$$p_0(t) = \frac{\lambda}{\lambda+\mu} + \left[p_0(0) - \frac{\lambda}{\lambda+\mu}\right] e^{-(\lambda+\mu)t}.$$

The Chapter is organized as follows. In Section 4.1, we develop the analysis of the fractional version (in the Caputo sense) of the alternating Poisson process, by determining explicitly the probability law, the renewal function and the renewal density. In Section 4.2, we deal with the asymptotic behaviour of the process, with special attention to the limit probability of being in state 1 as time grows larger, and to similar ratios involving the fractional moments of the renewal variables of the process. Finally, we exploit a suitable transformation of interest in the context of alternating renewal processes aiming to derive new Mittag-Leffler-like distributions.

4.1 Main results

In order to generalize the equations governing the alternating Poisson process, we now replace in (4.3) the time derivative with the fractional derivative in the Caputo sense (1.5) of order $\nu \in (0,1]$, thus obtaining the following system:

$$\begin{cases} \dfrac{\mathrm{d}^\nu p_{11}}{\mathrm{d}t^\nu} = -\lambda\, p_{11}^\nu(t) + \mu\, p_{10}^\nu(t) \\ \dfrac{\mathrm{d}^\nu p_{10}}{\mathrm{d}t^\nu} = \lambda\, p_{11}^\nu(t) - \mu\, p_{10}^\nu(t) \end{cases} \tag{4.4}$$

subject to the initial conditions $p_{11}(0) = 1$ and $p_{10}(0) = 0$, and normalizing condition $p_{11}(t) + p_{10}(t) = 1$. We remark that the use of the Caputo derivative allows us to avoid fractional initial conditions.

Proposition 4.1.1. *The solution of the Cauchy problem (4.4), for $t \geq 0$ and $\nu \in (0,1]$, is given by*

$$p_{11}^\nu(t) = 1 - \lambda t^\nu E_{\nu,\nu+1}(-(\lambda+\mu)t^\nu) \qquad \text{and} \qquad p_{10}^\nu(t) = \lambda t^\nu E_{\nu,\nu+1}(-(\lambda+\mu)t^\nu), \tag{4.5}$$

where $E_{\alpha,\beta}(t)$ is the Mittag-Leffler function (1.11).

Proof. Due to formula (1.6), the Laplace transform of the solution to system (4.4) becomes, for $s > (\lambda+\mu)^{1/\nu}$,

$$\begin{cases} \mathcal{L}\{p_{11}^\nu(t); s\} = \dfrac{s^{\nu-1}}{s^\nu + (\lambda+\mu)} + \mu \dfrac{s^{-1}}{s^\nu + (\lambda+\mu)} \\ \mathcal{L}\{p_{10}^\nu(t); s\} = \lambda \dfrac{s^{-1}}{s^\nu + (\lambda+\mu)} \end{cases} \tag{4.6}$$

System (4.6) can be inverted by using formula (1.12). We obtain

$$\begin{aligned} p_{11}^\nu(t) &= E_{\nu,1}(-(\lambda+\mu)t^\nu) + \mu t^\nu E_{\nu,\nu+1}(-(\lambda+\mu)t^\nu) \\ &= 1 - \lambda t^\nu E_{\nu,\nu+1}(-(\lambda+\mu)t^\nu), \end{aligned}$$

where the last equality follows from formula (4.2.3) of [57], i.e.

$$E_{\alpha,\beta}(z) = \frac{1}{\Gamma(\beta)} + zE_{\alpha,\alpha+\beta}(z). \tag{4.7}$$

Then we invert the second equation in system (4.6) and get

$$p_{10}^{\nu}(t) = \lambda t^{\nu} E_{\nu,\nu+1}(-(\lambda+\mu)t^{\nu}).$$

This completes the proof of (4.5). □

Solutions (4.5) can be interpreted as the probabilities for a *fractional alternating Poisson process* of being in states 1 and 0 at time t starting from state 1 at time $t = 0$. We assume that $t = 0$ is a renewal point. Specifically, if $Y^{\nu}(t)$, $t \geq 0$, is a stochastic process with state space $\{0, 1\}$, then

$$p_{11}^{\nu}(t) = \mathbb{P}\left(Y^{\nu}(t) = 1 \,|\, Y^{\nu}(0) = 1\right)$$

and

$$p_{10}^{\nu}(t) = \mathbb{P}\left(Y^{\nu}(t) = 0 \,|\, Y^{\nu}(0) = 1\right).$$

Similarly to (4.5), we find that the probabilities of being in states 1 and 0 at t starting from state 0 at $t = 0$ are

$$p_{01}^{\nu}(t) = \mu t^{\nu} E_{\nu,\nu+1}(-(\lambda+\mu)t^{\nu})$$

and

$$\begin{aligned} p_{00}^{\nu}(t) &= E_{\nu,1}(-(\lambda+\mu)t^{\nu}) + \lambda t^{\nu} E_{\nu,\nu+1}(-(\lambda+\mu)t^{\nu}) \\ &= 1 - \mu t^{\nu} E_{\nu,\nu+1}(-(\lambda+\mu)t^{\nu}). \end{aligned}$$

By analogy with the non-fractional case, we define

$$\begin{aligned} p_{j}^{\nu}(t) &= \mathbb{P}\left(\text{state } j \text{ occupied at time } t\right) \\ &= p_{1}^{\nu}(0)p_{1j}^{\nu}(t) + p_{0}^{\nu}(0)p_{0j}^{\nu}(t), \end{aligned}$$

so that, if the process starts in state 1 at $t = 0$,

$$p_{1}^{\nu}(t) = p_{11}^{\nu}(t) \quad \text{and} \quad p_{0}^{\nu}(t) = p_{10}^{\nu}(t). \tag{4.8}$$

We point out that whereas the starting alternating Poisson process is Markovian, the new process $Y^{\nu}(t)$ is semi-Markov. Indeed, similarly to other stochastic processes, the “fractionalization” produces persistence or long memory effects.

Such state occupancy probabilities can be recovered also by a different approach. Indeed, we suppose that the random variable U_k (D_k), describing the duration of the kth time interval during which the system is in state 1 (state 0), is equally distributed with a random variable U (D) following a Mittag-Leffler distribution with density

$$f_U(t) = \lambda t^{\nu-1} E_{\nu,\nu}(-\lambda t^\nu), \qquad \left(f_D(t) = \mu t^{\nu-1} E_{\nu,\nu}(-\mu t^\nu)\right), \qquad t > 0,\ 0 < \nu < 1, \tag{4.9}$$

and complementary cumulative distribution function

$$\overline{F}_U(t) = E_{\nu,1}(-\lambda t^\nu), \qquad \left(\overline{F}_D(t) = E_{\nu,1}(-\mu t^\nu)\right), \qquad t > 0,\ 0 < \nu < 1. \tag{4.10}$$

We recall that densities (4.9) are characterized by fat tails, with polynomial decay, and, as a consequence, the mean time spent by the process both in state 1 and in state 0 is infinite.

The probability density function of the first cycle X (cf. Eq. (4.1)), due to the independence of its summands, can be recovered by inverting its Laplace transform:

$$\mathcal{L}_X(s) = \mathcal{L}_U(s)\mathcal{L}_D(s) = \frac{\lambda\mu}{(s^\nu + \lambda)(s^\nu + \mu)}, \tag{4.11}$$

so that, bearing in mind formula (1.12), we recover the following generalized mixture, for $\lambda \neq \mu$:

$$f_X(t) = \frac{\mu}{\mu - \lambda} \lambda t^{\nu-1} E_{\nu,\nu}(-\lambda t^\nu) - \frac{\lambda}{\mu - \lambda} \mu t^{\nu-1} E_{\nu,\nu}(-\mu t^\nu), \qquad t > 0. \tag{4.12}$$

In the next proposition we derive the expression of the renewal function of the considered alternating process.

Proposition 4.1.2. *Let $M(t)$, $t \geq 0$, be the renewal function of an alternating process whose inter-renewal times are distributed as in (4.12). Then*

$$M(t) = \lambda\mu t^{2\nu} E_{\nu,2\nu+1}(-(\lambda+\mu)t^\nu), \qquad t > 0. \tag{4.13}$$

The corresponding renewal density is

$$m(t) = \lambda\mu t^{2\nu-1} E_{\nu,2\nu}(-(\lambda+\mu)t^\nu), \qquad t > 0. \tag{4.14}$$

Proof. With regard to (4.12), the Laplace transform of the renewal function of the considered process, which we call $M(t)$, is (cf. [104])

$$\mathcal{L}\{M(t);s\} = \frac{\mathcal{L}_X(s)}{s(1-\mathcal{L}_X(s))} = \frac{1}{s}\cdot\frac{\lambda\mu}{s^\nu(s^\nu+(\lambda+\mu))}, \tag{4.15}$$

where the last identity follows from (4.11). From Equation (4.15) we infer that the Laplace transform of the corresponding renewal density is

$$\begin{aligned}\mathcal{L}\{m(t)\} = \mathcal{L}\left\{\frac{\mathrm{d}M(t)}{\mathrm{d}t}\right\} &= s\mathcal{L}\{M(t)\} \\ &= \frac{\lambda\mu}{s^\nu(s^\nu+(\lambda+\mu))},\end{aligned}$$

which can be inverted with the help of formula (1.12) in order to obtain

$$m(t) = \frac{\lambda\mu}{\lambda+\mu}\frac{t^{\nu-1}}{\Gamma(\nu)} - \frac{\lambda\mu}{\lambda+\mu}t^{\nu-1}E_{\nu,\nu}(-(\lambda+\mu)t^\nu),$$

this giving (4.14). In addition, the renewal function turns out to be the following:

$$\begin{aligned}M(t) &= \frac{\lambda\mu}{\lambda+\mu}\frac{t^\nu}{\Gamma(\nu+1)} - \frac{\lambda\mu}{\lambda+\mu}t^\nu E_{\nu,\nu+1}(-(\lambda+\mu)t^\nu) \\ &= \lambda\mu t^{2\nu}E_{\nu,2\nu+1}(-(\lambda+\mu)t^\nu),\end{aligned}$$

where the last equality is due to (4.7). The proof of (4.13) is thus complete. □

From the theory of alternating renewal processes (cf. formula (6.66) of [104]), it is known that, for $t \geq 0$,

$$\pi_1(t) = \overline{F}_U(t) + \int_0^t m(t-x)\overline{F}_U(t)\mathrm{d}x,$$

where $\pi_1(t)$ is the probability that at time t the process is in state 1 and $m(t)$ is the renewal density. Such explicit expression for $\pi_1(t)$ is derived by solving a renewal equation. Recalling (4.14) and (4.10), we obtain

$$\begin{aligned}\pi_1(t) &= E_{\nu,1}(-\lambda t^\nu) + \lambda\mu\int_0^t (t-x)^{2\nu-1}E_{\nu,2\nu}(-(\lambda+\mu)(t-x)^\nu)E_{\nu,1}(-\lambda x^\nu)\mathrm{d}x \\ &= E_{\nu,1}(-\lambda t^\nu) - \lambda^2 t^{2\nu}E_{\nu,2\nu+1}(-\lambda t^\nu) + \lambda(\lambda+\mu)t^{2\nu}E_{\nu,2\nu+1}(-(\lambda+\mu)t^\nu) \\ &= 1 - \lambda t^\nu E_{\nu,\nu+1}(-(\lambda+\mu)t^\nu),\end{aligned} \tag{4.16}$$

where the last equality follows from (4.7). Due to (4.5), we observe that probability (4.16) equals the first of (4.8). Therefore, the random times between consecutive events for a fractional alternating Poisson process alternate between two Mittag-

Leffler distributions with parameter λ and μ, respectively. Consequently, the two approaches considered, i.e. the one based on the resolution of the fractional system of equations (4.4), and the one based on renewal theory arguments, lead to two alternating processes with the same one-dimensional distribution.

4.2 Asymptotic behaviour and some transformations

We begin the present section by studying the asymptotic behaviour of the process $Y^\nu(t)$, with reference to $p_1^\nu(t) = \pi_1(t)$.

Proposition 4.2.1. *The limiting probability that the fractional alternating Poisson process is in state 1 is given by*

$$\lim_{t\to+\infty} p_1^\nu(t) = \frac{\mu}{\lambda+\mu}.$$

Proof. From (4.8) we observe that the limiting probability of being in the first phase of the considered process is:

$$\begin{aligned}\lim_{t\to+\infty} p_1^\nu(t) &= \lim_{t\to+\infty} \left(1 - \lambda t^\nu E_{\nu,\nu+1}(-(\lambda+\mu)t^\nu)\right) \\ &= \lim_{t\to+\infty} \frac{\lambda}{\lambda+\mu}\left(\frac{\lambda+\mu}{\lambda} - (\lambda+\mu)\, t^\nu E_{\nu,\nu+1}(-(\lambda+\mu)t^\nu)\right).\end{aligned}$$

It holds that $(\lambda+\mu)\, t^\nu E_{\nu,\nu+1}(-(\lambda+\mu)t^\nu) \overset{t\to+\infty}{\to} 1$, since we are dealing with the probability distribution function of a Mittag-Leffler random variable with parameter $\lambda+\mu$. Hence

$$\begin{aligned}\lim_{t\to+\infty} p_1^\nu(t) &= \frac{\lambda}{\lambda+\mu}\left(\frac{\lambda+\mu}{\lambda} - 1\right) \\ &= \frac{\mu}{\lambda+\mu},\end{aligned}$$

this completing the proof. □

Hereafter we give an alternative proof of Proposition 4.2.1 using the following Tauberian theorem which can be found in Widder [151].

Theorem (Tauberian Theorem)**.** *If $\alpha(t)$ is non-decreasing and such that the integral*

$$f(s) = \int_0^\infty e^{-st} \mathrm{d}\alpha(t)$$

converges for $s > 0$, and if for some non-negative number γ and some constant C

$$f(s) \sim \frac{C}{s^\gamma} \quad \text{as} \quad s \to 0,$$

then

$$\alpha(t) \sim \frac{Ct^\gamma}{\Gamma(\gamma+1)} \quad \text{as} \quad t \to +\infty.$$

Alternative proof of Proposition 4.2.1. Recalling (4.5), the probability of being in state 0 is

$$p_0^\nu(t) = 1 - p_1^\nu(t) = \lambda t^\nu E_{\nu,\nu+1}(-(\lambda+\mu)t^\nu).$$

This is a non-decreasing function on the interval $[0, +\infty)$ such that

$$\int_0^\infty e^{-st}\, \mathrm{d}p_0^\nu(t) = s\, \frac{\lambda s^{-1}}{s^\nu + (\lambda+\mu)} = \frac{\lambda}{s^\nu + (\lambda+\mu)}.$$

Since

$$\frac{\lambda}{s^\nu + (\lambda+\mu)} \to \frac{\lambda}{\lambda+\mu} \quad \text{as} \quad s \to 0,$$

then, due to Tauberian Theorem ($\gamma = 0$),

$$p_0^\nu(t) \to \frac{\lambda}{\lambda+\mu} \quad \text{as} \quad t \to +\infty.$$

Consequently,

$$p_1^\nu(t) \to \frac{\mu}{\lambda+\mu} \quad \text{as} \quad t \to +\infty.$$

□

It is noteworthy to point out that the fractional alternating Poisson process displays the same long-run proportion of time spent in mode 1 as its non fractional counterpart (cf. [82]). Such proportion can be interpreted as the time average of a particle's location for a sufficiently long time. Interestingly, from this fact one gets that weak ergodicity breaking occurs in both cases. This property is usually stated by saying that ensemble average and time average of physical observables, such as the position of the particle, differ, the last one being taken in the long time (infinite) limit. The underlying phase space, however, remains accessible. Moreover, the result presented in Proposition 4.2.3 is in accordance with Theorem 5 of [102], where the limiting distribution of the spent lifetime is presented in the case of infinite mean renewal periods.
We are now concerned with other kinds of proportions involving the fractional moments of the sub-renewal periods of the process $Y^\nu(t)$.

Proposition 4.2.2. *Let U and D be random variables with densities (4.9). Then*

$$\frac{\mathbb{E}[U^q]}{\mathbb{E}[U^q]+\mathbb{E}[D^q]} = \frac{1}{\xi^{q/\nu}+1}, \qquad \xi = \frac{\lambda}{\mu}, \quad 0<q<\nu\le 1.$$

Proof. By [122], the expression for the qth moment, $q<\nu$, of a random variable with density (4.9) is

$$\mathbb{E}[U^q] = \frac{q\pi}{\nu\lambda^{q/\nu}\Gamma(1-q)\sin(q\pi/\nu)}. \tag{4.17}$$

The proof follows by conveniently substituting the expression for the qth moment of D. □

To prove Proposition 4.2.3 below we need the following Lemma (see [88]).

Lemma 4.2.1. *Let X be a positive random variable with Laplace transform ϕ. Then*

$$\mathbb{E}\left[X^r\right] = \frac{r}{\Gamma(1-r)}\int_0^{+\infty} s^{-r-1}\left(1-\phi(s)\right)\mathrm{d}s, \qquad r\in(0,1).$$

With regard to (4.2), we observe that (cf. [18])

$$f_U^k(t) = \mathbb{P}\{U^{(k)}\in dt\}/dt = \lambda^k t^{\nu k-1}E^k_{\nu,\nu k}(-\lambda t^\nu), \quad t>0,\, 0<\nu<1, \tag{4.18}$$

with Laplace transform

$$\mathcal{L}\left\{f_U^k(t);s\right\} = \frac{\lambda^k}{(s^\nu+\lambda)^k}. \tag{4.19}$$

The density and the Laplace transform of $D^{(k)}$ can be obtained from (4.18) and (4.19) respectively, by replacing λ with μ. We are now ready to prove the next proposition, which gives an immediate extension of Proposition 4.2.2.

Proposition 4.2.3. *Let $U^{(k)}$ and $D^{(k)}$ be random variables defined as in (4.2). Then*

$$\frac{\mathbb{E}\left[\left(U^{(k)}\right)^q\right]}{\mathbb{E}\left[\left(U^{(k)}\right)^q\right]+\mathbb{E}\left[\left(D^{(k)}\right)^q\right]} = \frac{1}{\xi^{q/\nu}+1}, \qquad \xi=\frac{\lambda}{\mu}, \quad 0<q<\nu\le 1. \tag{4.20}$$

Proof. From Lemma 4.2.1 and Eq. (4.19), for $q\in(0,1)$,

$$\begin{aligned}\mathbb{E}\left[\left(U^{(k)}\right)^q\right] &= \frac{q}{\Gamma(1-q)}\int_0^{+\infty} s^{-q-1}\left(1-\frac{\lambda^k}{(s^\nu+\lambda)^k}\right)\mathrm{d}s\\ &= \frac{q}{\Gamma(1-q)}\sum_{i=0}^{k-1}\binom{k}{i}\lambda^i\int_0^{+\infty}\frac{s^{\nu(k-i)-q-1}}{(s^\nu+\lambda)^k}\mathrm{d}s,\end{aligned}$$

where the last equality is due to the binomial theorem. By applying formula 3.241-4 of [60], i.e.

$$\int_0^{+\infty} \frac{x^{\mu-1}}{(p+qx^\nu)^{n+1}} \mathrm{d}x = \frac{1}{\nu p^{n+1}} \left(\frac{p}{q}\right)^{\mu/\nu} \frac{\Gamma(\mu/\nu)\Gamma(1+n-\mu/\nu)}{\Gamma(1+n)},$$

where $0 < \frac{\mu}{\nu} < n+1$, $p \neq 0$, $q \neq 0$, we obtain, for $0 < q < \nu \leq 1$,

$$\begin{aligned} \mathbb{E}\left[\left(U^{(k)}\right)^q\right] &= \frac{q}{\Gamma(1-q)} \sum_{i=0}^{k-1} \binom{k}{i} \frac{1}{\nu\lambda^{q/\nu}} \frac{\Gamma(k-i-q/\nu)\Gamma(i+q/\nu)}{\Gamma(k)} \\ &= \frac{q}{\Gamma(1-q)} \frac{1}{\nu\lambda^{q/\nu}} \sum_{i=0}^{k-1} \binom{k}{i} B\left(k-i-q/\nu, i+q/\nu\right), \end{aligned} \tag{4.21}$$

where $B(x,y)$ denotes the Beta function. Observe that, in an analogous way, we can calculate

$$\mathbb{E}\left[\left(D^{(k)}\right)^q\right] = \frac{q}{\Gamma(1-q)} \frac{1}{\nu\mu^{q/\nu}} \sum_{i=0}^{k-1} \binom{k}{i} B\left(k-i-q/\nu, i+q/\nu\right).$$

The thesis thus follows. □

Some plots of the ratio (4.20) are provided in Figure 4.1.

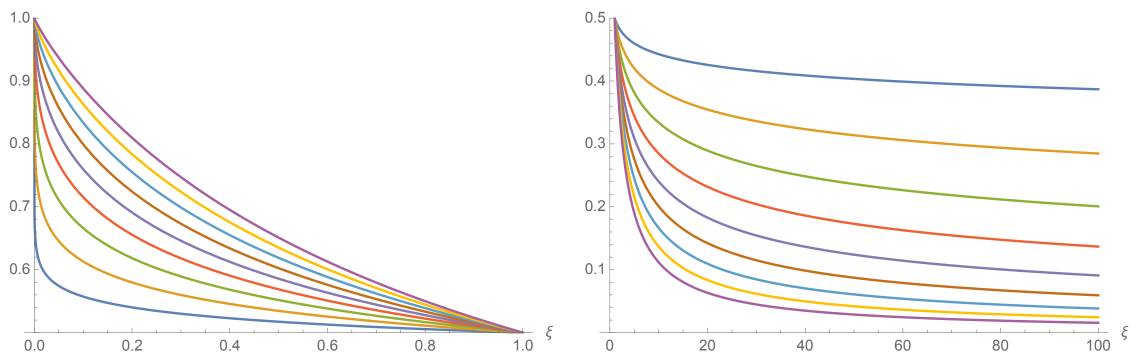

Figure 4.1: The ratio (4.20) is shown on the left for $0 < \xi \leq 1$ and $q/\nu = 0.1, 0.2, \ldots, 0.9$ (from bottom to top), on the right for $1 \leq \xi \leq 100$ and $q/\nu = 0.1, 0.2, \ldots, 0.9$ (from top to bottom).

Hereafter we aim to explore new stochastic models related to the fractional alternating Poisson process. Specifically, with reference to the process $Y^\nu(t)$, we now study a special transformation of the random variables involved. Such transformation, acting on pairs of non-negative random variables having unequal finite means, is an extension of the equilibrium operator. It is of interest since it arises essentially from stochastic processes characterized by two randomly alternating states. In fact, it is suitable to describe the asymptotic behaviour of the corresponding spent lifetime (cf. [32]). In general, if X and Y are non-negative random variables such that $\mathbb{E}[X] < \mathbb{E}[Y] < +\infty$, then

$$f_Z(x) = \frac{\overline{F}_Y(x) - \overline{F}_X(x)}{\mathbb{E}[Y] - \mathbb{E}[X]}, \qquad x \geq 0, \tag{4.22}$$

is the probability density function of an absolutely continuous non-negative random variable Z if and only if $X \leq_{st} Y$, where $\leq_{st}$ is the usual stochastic order (i.e., $X \leq_{st} Y$ if and only if $\overline{F}_X(x) \leq \overline{F}_Y(x)$ for all x). In Eq. (4.22), $\overline{F}_X(x)$ and $\overline{F}_Y(x)$ denote the survival functions of X and Y, respectively. We write $Z \equiv \Psi(X,Y)$ to mean that Z is a random variable with density (4.22).

Example 4.2.1. Let U and D be random variables having Mittag-Leffler densities with parameters λ and μ respectively, expressed by (4.9), and fix a positive real number α, $0 < \alpha < \nu \leq 1$, such that the random variables U^α and D^α have finite means. If $\lambda < \mu$, one has $U \leq_{st} D$ and then $U^\alpha \leq_{st} D^\alpha$. From (4.10), (4.17) and (4.22), the density of $Z \equiv \Psi(U^\alpha, D^\alpha)$ is

$$f_Z(t) = \frac{\nu \lambda^{\alpha/\nu} \mu^{\alpha/\nu} \Gamma(1-\alpha) \sin(\alpha\pi/\nu)}{\alpha\pi \left(\lambda^{\alpha/\nu} - \mu^{\alpha/\nu}\right)} \left(E_{\nu,1}(-\mu t^{\nu/\alpha}) - E_{\nu,1}(-\lambda t^{\nu/\alpha})\right), \qquad t \geq 0. \tag{4.23}$$

Figure 4.2 shows various plots of density (4.23).
Consequently, from the probabilistic mean value theorem given in Theorem 4.1 of [32], if g is a measurable and differentiable function such that $\mathbb{E}\left[g\left(D^\alpha\right)\right]$ and $\mathbb{E}\left[g\left(U^\alpha\right)\right]$ are finite and if its first derivative g' is measurable and Riemann-integrable on the interval $[x,y]$ for all $y \geq x \geq 0$, then $\mathbb{E}\left[g'\left(Z\right)\right]$ is finite and

$$\mathbb{E}\left[g\left(D^\alpha\right)\right] - \mathbb{E}\left[g\left(U^\alpha\right)\right] = \mathbb{E}\left[g'\left(Z\right)\right]\left(\mathbb{E}\left[D^\alpha\right] - \mathbb{E}\left[U^\alpha\right]\right),$$

where Z is a random variable having density (4.23).

Example 4.2.2. Let us consider the random variables $U^{(1)}$ and $U^{(2)}$ (cf. (4.2)), with densities (4.9) and (4.18) respectively. Again, we fix a positive real number α, with $0 < \alpha < \nu \leq 1$, such that both random variables involved, i.e. $\left(U^{(1)}\right)^\alpha$ and $\left(U^{(2)}\right)^\alpha$ have finite first order moments. Since $U^{(1)} \leq_{st} U^{(2)}$, and then $\left(U^{(1)}\right)^\alpha \leq_{st} \left(U^{(2)}\right)^\alpha$, we can study the transformation Ψ acting on $\left(U^{(1)}\right)^\alpha$ and $\left(U^{(2)}\right)^\alpha$. The complementary cumulative distribution functions of $U^{(1)}$ and $U^{(2)}$ are expressed in terms of the generalized Mittag-Leffler function (1.11), since (cf. (4.10))

$$\mathbb{P}\left(U^{(1)} > t\right) = 1 - \lambda t^\nu E_{\nu,\nu+1}(-\lambda t^\nu), \qquad t \geq 0 \tag{4.24}$$

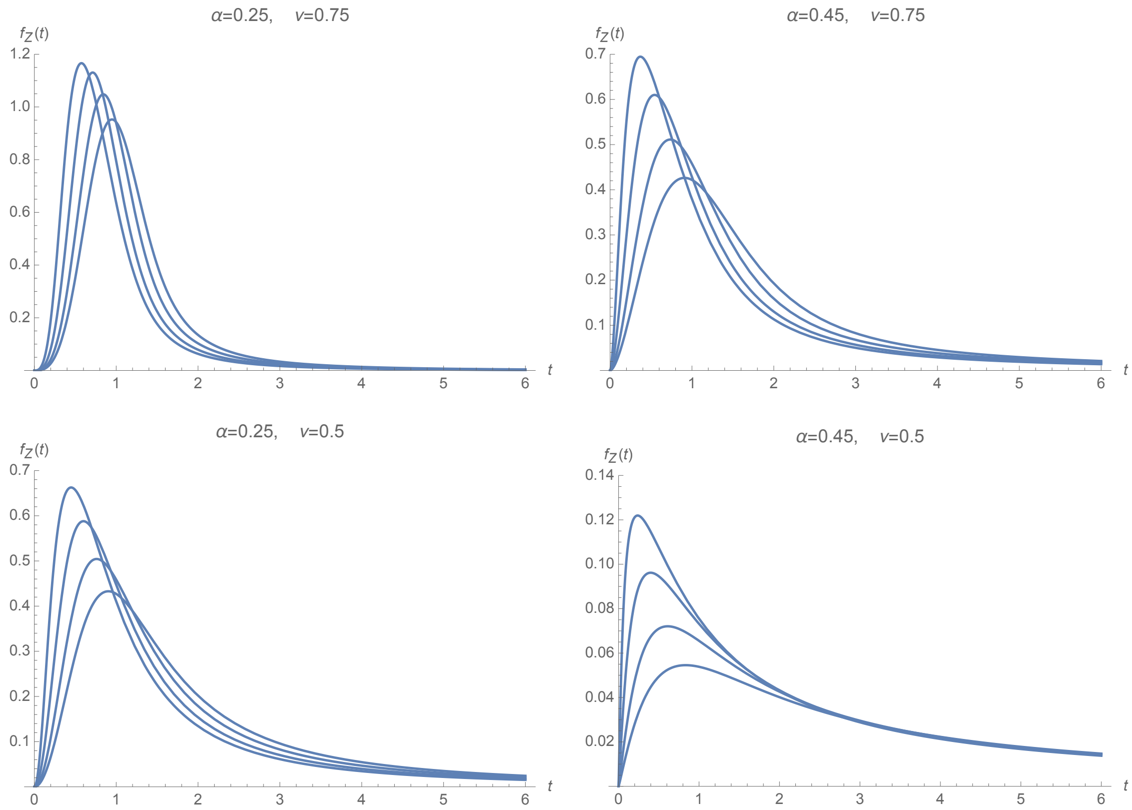

Figure 4.2: Density (4.23) for various choices of α and ν, with $\lambda = 1$ and $\mu = 1.01, 2, 5, 15$ (from bottom to top near the origin).

and (cf. [18])

$$\mathbb{P}\left(U^{(2)} > t\right) = 1 - \lambda^2 t^{2\nu} E^2_{\nu,2\nu+1}(-\lambda t^\nu), \qquad t \geq 0. \tag{4.25}$$

Recalling that (cf. formula (5.1.12) of [57]) if α, β, $\gamma \in \mathbb{C}$ and $Re\ \alpha > 0$, $Re\ \beta > 0$, $Re\ \beta - \alpha > 0$

$$zE^\gamma_{\alpha,\beta} = E^\gamma_{\alpha,\beta-\alpha} - E^{\gamma-1}_{\alpha,\beta-\alpha}, \tag{4.26}$$

the following equality holds:

$$E_{\nu,\nu+1}(-\lambda t^\nu) = \lambda t^\nu E^2_{\nu,2\nu+1}(-\lambda t^\nu) + E^2_{\nu,\nu+1}(-\lambda t^\nu), \qquad t \geq 0,$$

and then

$$\lambda t^\nu E_{\nu,\nu+1}(-\lambda t^\nu) = \lambda^2 t^{2\nu} E^2_{\nu,2\nu+1}(-\lambda t^\nu) + \lambda t^\nu E^2_{\nu,\nu+1}(-\lambda t^\nu), \qquad t \geq 0. \tag{4.27}$$

Due to formula (5.1.14) of [57], i.e.

$$\alpha E^2_{\alpha,\beta} = E_{\alpha,\beta-1} - (1 + \alpha - \beta) E_{\alpha,\beta}$$

if α, $\beta \in \mathbb{C}$ and $Re\ \alpha > 0$, $Re\ \beta > 1$, then

$$E^2_{\nu,\nu+1}(-\lambda t^\nu) = \frac{1}{\nu} E_{\nu,\nu}(-\lambda t^\nu). \tag{4.28}$$

By using (4.28) into (4.27), we get

$$\lambda t^\nu E_{\nu,\nu+1}(-\lambda t^\nu) = \lambda^2 t^{2\nu} E^2_{\nu,2\nu+1}(-\lambda t^\nu) + \frac{\lambda t^\nu}{\nu} E_{\nu,\nu}(-\lambda t^\nu), \qquad t \geq 0, \tag{4.29}$$

and the function $t^\nu E_{\nu,\nu}(-\lambda t^\nu)$ is positive due to the complete monotonicity of $t^{\nu-1} E_{\nu,\nu}(-\lambda t^\nu)$ (cf. (5.1.10) of [57]). Consequently, recalling (4.24) and (4.25), from (4.29) we obtain

$$\begin{aligned}
& 1 - \lambda^2 t^{2\nu} E^2_{\nu,2\nu+1}(-\lambda t^\nu) \geq 1 - \lambda t^\nu E_{\nu,\nu+1}(-\lambda t^\nu) \\
\Longleftrightarrow \quad & \mathbb{P}\left(U^{(2)} > t\right) \geq \mathbb{P}\left(U^{(1)} > t\right) \\
\Longleftrightarrow \quad & U^{(1)} \leq_{st} U^{(2)} \\
\Longleftrightarrow \quad & \left(U^{(1)}\right)^\alpha \leq_{st} \left(U^{(2)}\right)^\alpha .
\end{aligned}$$

Hence, if $Z \equiv \Psi\left(\left(U^{(1)}\right)^\alpha, \left(U^{(2)}\right)^\alpha\right)$, from (4.17) and (4.21) we have, for $t \geq 0$,

$$f_Z(t) = \frac{\Gamma(1-\alpha)\nu^2 \lambda^{\alpha/\nu} \sin(\alpha\pi/\nu)}{\alpha^2 \pi} \left(\lambda t^{\nu/\alpha} E_{\nu,\nu+1}(-\lambda t^{\nu/\alpha}) - \lambda^2 t^{2\nu/\alpha} E^2_{\nu,2\nu+1}(-\lambda t^{\nu/\alpha})\right).$$

It follows that, making use of (4.26), for $0 < \alpha < \nu \leq 1$ and $\lambda > 0$ we obtain

$$f_Z(t) = \frac{\Gamma(1-\alpha)\nu^2 \lambda^{\alpha/\nu} \sin(\alpha\pi/\nu)}{\alpha^2 \pi} \lambda t^{\nu/\alpha} E^2_{\nu,\nu+1}(-\lambda t^{\nu/\alpha}), \qquad t \geq 0. \tag{4.30}$$

Again, from Theorem 4.1 of [32], if g is a suitable function and Z is a random variable with density (4.30), then

$$\mathbb{E}\left[g\left(\left(U^{(2)}\right)^\alpha\right)\right] - \mathbb{E}\left[g\left(\left(U^{(1)}\right)^\alpha\right)\right] = \mathbb{E}\left[g'(Z)\right]\left(\mathbb{E}\left[\left(U^{(2)}\right)^\alpha\right] - \mathbb{E}\left[\left(U^{(1)}\right)^\alpha\right]\right).$$

4.3 Concluding remarks

In this chapter we have studied a generalization of the alternating Poisson process from the point of view of fractional calculus. In the system of differential equations governing the state occupancy probabilities for the alternating Poisson process we replace the ordinary derivative with the Caputo one, thus endowing the process with persistent memory. We obtain the probability mass function of a fractional alternating Poisson process and then show that it can be recovered also by means of renewal theory arguments. Furthermore, we provide results for the behaviour of some quantities characterizing the process under examination and derive new Mittag-Leffler-like distributions of interest in the context of alternating renewal processes.

Chapter 5

On a jump-telegraph process driven by alternating fractional Poisson process

5.1 Introduction

The (integrated) telegraph process is a continuous-time stochastic process that describes a random motion on the real line. The motion has finite (constant) velocity $c > 0$, and its direction is reversed at every event of a homogeneous Poisson process with intensity λ. The transition density $p(x, t)$ of the telegraph process satisfies a second-order (hyperbolic) telegraph equation (see the seminal articles by Goldstein [56] and Kac [68]), namely

$$c^2 \frac{\partial^2 p}{\partial x^2} = \frac{\partial^2 p}{\partial t^2} + 2\lambda \frac{\partial p}{\partial t}.$$

Under suitable conditions, the aforementioned equation tends asymptotically to the heat diffusion equation. In other words, the transition density of the telegraph process tends to the transition density of the one-dimensional Brownian motion, the former being more general but more difficult to deal with than the latter. The telegraph process has been introduced to overcome the serious limitations of the Brownian motion in the realistic representation of real random motions, that is to say the infinite speed at which a particle travels and the non-differentiability of the trajectory (which implies total absence of inertia). Various extensions of the telegraph process have been proposed in the literature towards motions characterized by two or more than two velocities, or by random velocities, or with velocity changes governed by an alternating renewal process. The telegraph process and its generalizations have been widely applied in biomathematics and in queueing theory (see

[55], [116] and Section 1 of [33]).

Motions with deterministic or random jumps along the alternating direction at each velocity reversal have been studied in detail (cf. [34] and [38]), also with a special focus on some general rescaling properties [89]. Damped versions of the telegraph process have been considered in [129] and [37] in the presence and in absence of jumps, respectively. A jump-telegraph process is interesting for the purposes of financial modelling. For the sake of brevity, we only mention two works: the paper by Ratanov [130], in which the author proposes a new generalisation of the jump-telegraph process with variable velocities and jumps, and then applies this construction to markets modelling; and the recent book by Kolesnik and Ratanov (see [74] and references therein), which gives a thorough investigation on the telegraph process and its applications to option pricing. Estimation procedures for the standard and geometric telegraph process ([29], [30], [66]), and for a Brownian motion governed by a telegraph process [127], have been recently provided under the hypothesis of discrete-time sampling.

In the last decades a number of works have appeared analysing processes governed by (space)-time fractional telegraph equations, obtained by replacing the ordinary derivatives in the telegraph equation by suitable fractional derivatives (cf. [107] and [114]). The key features of the resulting processes include long-range memory, path-dependence, non Markovian properties, anomalous diffusion behaviour. Masoliver [97] justified on physical grounds the fractional telegraph equation. Special forms of fractional telegraph equations with rational order are studied in [16]. Another approach is adopted in [17], in which the authors propose a finite-velocity planar random motion whose changes of direction occur at times spaced by a fractional Poisson process. In general, fractional calculus is useful in computing probability distribution functions with fat tails. In the recent past pure jump fractional processes have attracted great attention. Just to mention a few examples, [18] illustrates various results on the fractional Poisson process and also focuses on certain higher-order extensions, whilst a fractional counting process with multiple jumps has been studied in [40] (Chapter 3 of the dissertation). See also [126] for a generalization of the space-fractional Poisson process. Birth, birth-death and death processes have been investigated in [108], [109] and [112] respectively.

In the light of the previous investigations, and aiming to construct a more general model that takes into account both the occurrence of jumps and the fractional nature, in this chapter we propose and study a one-dimensional jump-telegraph process with deterministic jumps occurring at velocity changes, and with intertimes governed by a fractional alternating counting process that has been studied in detail in [42] and in Chapter 4. We obtain the probability law of the new process, which is given in a series form involving the generalized Mittag-Leffler function (1.11). We also discuss the uniform convergence of the distribution.

We devote special attention to the case of jumps having constant size. It turns out that the structure of the solution (see Proposition 5.3.1) is quite similar to that obtained in [38], even though the shape of the relevant density is qualitatively different, due to the special form of the underlying generalized Mittag-Leffler distributions. We also obtain the mean of the process in the special case of identically distributed upward and downward intertimes, and compare it to the means of other fractional processes. We stress that the mean results in the sum of two terms. The first term is linear in time and refers to the alternating component of the motion. The second term is a power of time, where the exponent ν is the "tail" index of the underlying generalized Mittag-Leffler distribution, and is related to the jump component of the motion.

An interesting but difficult problem is the determination of the first-passage-time distribution through a constant boundary. This problem for an asymmetric telegraph process has been treated in [90], where the distributions of the first-passage times are described by using the Laplace transforms and their inversions. We provide a formal expression of the first-passage-time distribution (in series form) by conditioning on the number of jumps. The formal expression is finally used to provide suitable lower bounds.

5.2 Probability law of the jump-telegraph process

Let $\{(X_t, V_t), t \geq 0\}$ denote a jump-telegraph process, where X_t and V_t represent respectively the position and the velocity of a particle running on the real line. The motion is performed starting at the origin at time 0, with two alternating constant velocities, say $-v, c$. The initial velocity can be either $-v$ or c. The velocities change at random times, which are the epochs of an alternating counting process $\{N_t, t \geq 0\}$. A jump occurs at each velocity change, the jump's displacement being $\alpha_k > 0$ (upward jump) or $-\beta_k < 0$ (downward jump) if it follows the kth period of forward or backward motion, respectively.

Formally, the process is described by the following stochastic equations, for $t > 0$ and $V_0 \in \{-v, c\}$:

$$X_t = \int_0^t V_s \, \mathrm{d}s + \sum_{k=1}^{N_t} \left[\frac{\alpha_k - \beta_k}{2} - \operatorname{sgn}(V_0) \, \frac{\alpha_k + \beta_k}{2} \, (-1)^k \right], \tag{5.1}$$

$$V_t = \frac{c-v}{2} + \left[V_0 - \left(\frac{c-v}{2} \right) \right] (-1)^{N_t}. \tag{5.2}$$

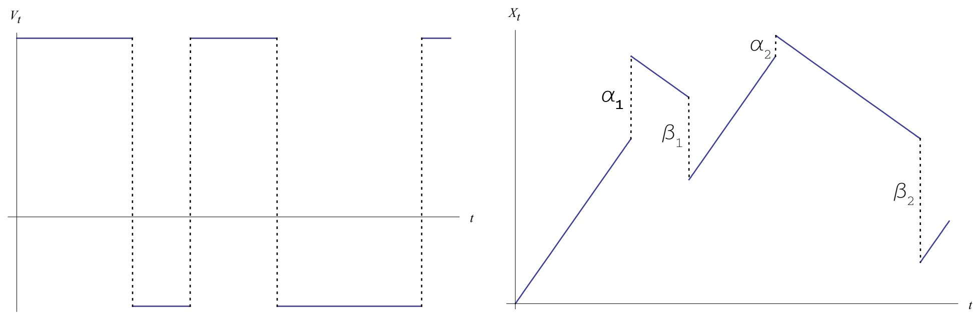

Figure 5.1: Left panel: a sample path of V_t. Right panel: the corresponding sample path of X_t; the displacement of each jump is also indicated.

We assume that $\{U_k\}_{k\in\mathbb{N}}$ and $\{D_k\}_{k\in\mathbb{N}}$ are independent sequences of independent copies of the nonnegative random variables U and D, which describe the duration of the kth random period in which the motion proceeds forward or backward, respectively. Note that the interarrival random times of the alternating counting process $\{N_t, t \geq 0\}$ are $U_1, D_1, U_2, D_2, \ldots$ (resp. $D_1, U_1, D_2, U_2, \ldots$) when the initial velocity is positive (resp. negative).

In a previous paper [38] the case of Erlang-distributed random periods U and D and deterministic jumps has been studied in detail. As a novelty, in the present chapter we investigate the case when the random times U and D separating consecutive velocity changes (and jumps) follow a Mittag-Leffler distribution with parameters (λ, ν) and (μ, ν), respectively. With reference to the function (1.10) introduced in Section 1.2, for parameters λ, $\mu > 0$ and $0 < \nu < 1$, we consider the probability density functions (PDF's)

$$f_U(t) = \lambda t^{\nu-1} E_{\nu,\nu}(-\lambda t^\nu), \qquad f_D(t) = \mu t^{\nu-1} E_{\nu,\nu}(-\mu t^\nu), \qquad t > 0, \tag{5.3}$$

and the complementary cumulative distribution functions

$$\overline{F}_U(t) = \mathbb{P}(U > t) = E_{\nu,1}(-\lambda t^\nu), \qquad \overline{F}_D(t) = \mathbb{P}(D > t) = E_{\nu,1}(-\mu t^\nu), \qquad t > 0. \tag{5.4}$$

We recall that for $\nu = 1$ formulas (5.3) and (5.4) lead to exponential distributions. Furthermore, for $k \in \mathbb{N}$ the probability density functions of

$$U^{(k)} = U_1 + U_2 + \cdots + U_k, \qquad D^{(k)} = D_1 + D_2 + \cdots + D_k, \tag{5.5}$$

are respectively (cf. Eq. (2.19) of [18]),

$$f_U^{(k)}(t) = \lambda^k t^{\nu k-1} E^k_{\nu,\nu k}(-\lambda t^\nu), \qquad f_D^{(k)}(t) = \mu^k t^{\nu k-1} E^k_{\nu,\nu k}(-\mu t^\nu), \qquad t > 0, \tag{5.6}$$

where $E^k_{\nu,\nu k}(\cdot)$ has been defined in (1.11). Note that the distributions given in (5.6) can be viewed both as generalized Erlang distributions and as generalized Mittag-Leffler distributions (see Jose *et al.* [67]). They are also named Positive Linnik distributions (cf. [26] and [117]). Moreover, such distributions are involved in the analysis of the fractional Poisson process and its extensions (cf. [94], [18] and [40]).

It is worth pointing out that, under the assumptions (5.3) and (5.4), the process $\{N_t, t \geq 0\}$ constitutes the fractional alternating Poisson process investigated in [42] (Chapter 4 of the thesis). Let us now introduce the following forward and backward transition PDF's, for $x \in \mathbb{R}$, $t > 0$ and $y \in \{-v, c\}$:

$$\begin{aligned} f(x,t\,|\,y)\mathrm{d}x &= \mathbb{P}[X_t \in \mathrm{d}x, V_t = c\,|\,X_0 = 0, V_0 = y], \\ b(x,t\,|\,y)\mathrm{d}x &= \mathbb{P}[X_t \in \mathrm{d}x, V_t = -v\,|\,X_0 = 0, V_0 = y]. \end{aligned}$$

The probability law of (X_t, V_t) has an absolutely continuous component

$$p(x,t\,|\,y) = f(x,t\,|\,y) + b(x,t\,|\,y), \tag{5.7}$$

and a discrete component

$$\mathbb{P}[X_t = yt, V_t = y\,|\,X_0 = 0, V_0 = y].$$

In order to provide the formal expression of the above functions, we denote by

$$\alpha^{(k)} = \alpha_1 + \alpha_2 + \cdots + \alpha_k \qquad \big(\beta^{(k)} = \beta_1 + \beta_2 + \cdots + \beta_k\big)$$

the total amplitude of the first k upward (downward) jumps. Moreover, we set

$$I_{j,k}(x,t) := \begin{cases} 1 & \text{if } -vt + \alpha^{(j)} - \beta^{(k)} < x < ct + \alpha^{(j)} - \beta^{(k)} \\ 0 & \text{otherwise} \end{cases} \qquad x \in \mathbb{R},\ t > 0,$$

and

$$\tau_* = \frac{vt + x}{c + v}, \qquad \theta_k = \frac{\alpha^{(k)} - \beta^{(k)}}{c + v}, \qquad \eta_k = \frac{\alpha^{(k+1)} - \beta^{(k)}}{c + v}. \tag{5.8}$$

Theorem 5.2.1. *The probability law of (X_t, V_t), $t > 0$, conditional on positive initial velocity c is given by the discrete component*

$$\mathbb{P}[X_t = ct, V_t = c\,|\,X_0 = 0, V_0 = c] = E_{\nu,1}(-\lambda t^\nu), \tag{5.9}$$

and by the absolutely continuous component for the forward and backward transition

PDF's, for $x \in \mathbb{R}$:

$$f(x,t\,|\,c) = \sum_{k=1}^{+\infty} \left\{ \frac{I_{k,k}(x,t)}{c+v} \mu^k \left(\frac{ct-x}{c+v} + \theta_k \right)^{\nu k - 1} E^k_{\nu,\nu k} \left(-\mu \left(\frac{ct-x}{c+v} + \theta_k \right)^{\nu} \right) \right.$$
$$\left. \times \lambda^k \left(\frac{vt+x}{c+v} - \theta_k \right)^{\nu k} E^{k+1}_{\nu,\nu k+1} \left(-\lambda \left(\frac{vt+x}{c+v} - \theta_k \right)^{\nu} \right) \right\}, \qquad (5.10)$$

$$b(x,t\,|\,c) = \sum_{k=0}^{+\infty} \left\{ \frac{I_{k+1,k}(x,t)}{c+v} \lambda^{k+1} \left(\frac{vt+x}{c+v} - \eta_k \right)^{\nu(k+1)-1} \right.$$
$$\times E^{k+1}_{\nu,\nu(k+1)} \left(-\lambda \left(\frac{vt+x}{c+v} - \eta_k \right)^{\nu} \right)$$
$$\left. \times \mu^k \left(\frac{ct-x}{c+v} + \eta_k \right)^{\nu k} E^{k+1}_{\nu,\nu k+1} \left(-\mu \left(\frac{ct-x}{c+v} + \eta_k \right)^{\nu} \right) \right\}, \qquad (5.11)$$

where the function $E^{k+1}_{\nu,\nu k+1}(\cdot)$ has been defined in (1.11).

Proof. Since $\mathbb{P}[X_t = ct, V_t = c\,|\,X_0 = 0, V_0 = c] = \overline{F}_U(t)$, Eq. (5.9) follows immediately from Eq. (5.4). In Theorem 2.1 of [38] the following general expressions have been proved for $t > 0$ and $x \in \mathbb{R}$:

$$f(x,t\,|\,c) = \sum_{k=1}^{+\infty} \frac{I_{k,k}(x,t)}{c+v} f_D^{(k)}(t - \tau_* + \theta_k) \int_{t-\tau_*+\theta_k}^{t} f_U^{(k)}(s - t + \tau_* - \theta_k) \overline{F}_U(t-s)\,\mathrm{d}s, \qquad (5.12)$$

$$b(x,t\,|\,c) = \frac{I_{1,0}(x,t)}{c+v} f_U(\tau_* - \eta_0) \overline{F}_D(t - \tau_* + \eta_0)$$
$$+ \sum_{k=1}^{+\infty} \frac{I_{k+1,k}(x,t)}{c+v} f_U^{(k+1)}(\tau_* - \eta_k) \int_{\tau_*-\eta_k}^{t} f_D^{(k)}(s - \tau_* + \eta_k) \overline{F}_D(t-s)\,\mathrm{d}s, \qquad (5.13)$$

where τ_*, θ_k and η_k have been defined in (5.8). Therefore, densities (5.10) and (5.11) can be obtained from Eqs. (5.3), (5.4) and (5.6) and noting that the integrals in the right-hand side of (5.12) and (5.13) can be computed by means of (cf. Th. 2 of [72])

$$\int_0^x (x-t)^{\beta-1} E^{\gamma}_{\alpha,\beta}[a(x-t)^{\alpha}] t^{\nu-1} E^{\sigma}_{\alpha,\nu}(at^{\alpha}) \mathrm{d}t = x^{\beta+\nu-1} E^{\gamma+\sigma}_{\alpha,\beta+\nu}(ax^{\alpha}),$$

for $\alpha, \beta, \gamma, a, \nu, \sigma \in \mathbb{C}$ ($\Re(\alpha)$, $\Re(\beta)$, $\Re(\nu) > 0$). □

Theorem 5.2.2. *The series on the right-hand side of Eqs. (5.10) and (5.11) are uniformly convergent for $x \in \mathbb{R}$ and for fixed $t > 0$.*

Proof. Set, for $x \in \mathbb{R}$, $k \in \mathbb{N}$ and $t > 0$,

$$\begin{aligned} f_k(x,t\,|\,c) &:= \mathbb{P}[X_t \in \mathrm{d}x,\, V_t = c,\, N_t = k \,|\, X_0 = 0,\, V_0 = c] \\ &= \frac{I_{k,k}(x,t)}{c+v}\mu^k \left(\frac{ct-x}{c+v}+\theta_k\right)^{\nu k-1} E^k_{\nu,\nu k}\left(-\mu\left(\frac{ct-x}{c+v}+\theta_k\right)^{\nu}\right) \\ &\qquad \times \lambda^k \left(\frac{vt+x}{c+v}-\theta_k\right)^{\nu k} E^{k+1}_{\nu,\nu k+1}\left(-\lambda\left(\frac{vt+x}{c+v}-\theta_k\right)^{\nu}\right). \end{aligned}$$

The forward transition PDF (5.10) can thus be split as

$$f(x,t\,|\,c) = \sum_{k=1}^{k^*-1} f_k(x,t\,|\,c) + \sum_{k=k^*}^{+\infty} f_k(x,t\,|\,c),$$

where $k^* \in \mathbb{N}$ is determined by the Archimedean property of the real numbers so that $\nu k^* > 1$. In general, for all $k \in \mathbb{N}$,

$$f_k(x,t\,|\,c) \le \frac{I_{k,k}(x,t)}{c+v}\mu^k \left(\frac{ct-x}{c+v}+\theta_k\right)^{\nu k-1} E^k_{\nu,\nu k}\left(-\mu\left(\frac{ct-x}{c+v}+\theta_k\right)^{\nu}\right).$$

In fact, the generalized Mittag-Leffler function $E^{k+1}_{\nu,\nu k+1}(-\lambda t^\nu)$, $k \ge 0$, suitably normalized by the factor $(\lambda t^\nu)^k$, represents a proper probability distribution (see [18]). If $k > k^*$, for fixed $t > 0$ the function $\left(\frac{ct-x}{c+v}+\theta_k\right)^{\nu k-1}$ is monotonically decreasing in $x \in \left(-vt+\alpha^{(k)}-\beta^{(k)}, ct+\alpha^{(k)}-\beta^{(k)}\right)$. Consequently, we have:

$$\left(\frac{ct-x}{c+v}+\theta_k\right)^{\nu k-1} \le \left(\frac{ct-x}{c+v}+\theta_k\right)^{\nu k-1}\Bigg|_{x=-vt+\alpha^{(k)}-\beta^{(k)}} = t^{\nu k-1}.$$

Moreover, we note that the function $E^k_{\nu,\nu k}\left(-\mu\left(\frac{ct-x}{c+v}+\theta_k\right)^{\nu}\right)$ is monotonically increasing in $x \in \left(-vt+\alpha^{(k)}-\beta^{(k)}, ct+\alpha^{(k)}-\beta^{(k)}\right)$ (cf. Section 2.3 of [99]), so that

$$E^k_{\nu,\nu k}\left(-\mu\left(\frac{ct-x}{c+v}+\theta_k\right)^{\nu}\right) \le E^k_{\nu,\nu k}\left(-\mu\left(\frac{ct-x}{c+v}+\theta_k\right)^{\nu}\right)\Bigg|_{x=ct+\alpha^{(k)}-\beta^{(k)}} = \frac{1}{\Gamma(\nu k)}.$$

The forward PDF thus satisfies the following relation:

$$\begin{aligned}
f(x,t\,|\,c) &\leq \sum_{k=1}^{k^*-1} f_k(x,t\,|\,c) + \sum_{k=k^*}^{+\infty} \frac{I_{k,k}(x,t)}{c+v}\mu^k \frac{t^{\nu k-1}}{\Gamma(\nu k)} \\
&\leq \frac{t^{-1}}{c+v}\sum_{k=k^*}^{+\infty} \frac{(\mu t^\nu)^k}{\Gamma(\nu k)} \\
&= \frac{t^{-1}}{c+v}\sum_{r=0}^{+\infty} \frac{(\mu t^\nu)^{r+k^*}}{\Gamma(\nu(r+k^*))} \\
&= \frac{t^{-1}}{c+v} E_{\nu,\nu k^*}(\mu t^\nu).
\end{aligned}$$

Uniform convergence then is due to Weierstrass M-test. □

Remark 5.2.1. *Due to symmetry, if $V_0 = -v$ the probability law of (X_t, V_t) can be obtained from Theorem 5.2.1 by interchanging f with b, U with D, c with v, x with $-x$ and α_i with β_i for all $i \in \mathbb{N}$, thus having*

$$\begin{aligned}
f(x,t\,|-v) = \sum_{k=0}^{+\infty} \Bigg\{ & \frac{I_{k,k+1}(x,t)}{c+v}\mu^{k+1}\left(\frac{ct-x}{c+v} - \tilde{\eta}_k\right)^{\nu(k+1)-1} \\
& \times E^{k+1}_{\nu,\nu(k+1)}\left(-\mu\left(\frac{ct-x}{c+v} - \tilde{\eta}_k\right)^\nu\right) \\
& \times \lambda^k \left(\frac{vt+x}{c+v} + \tilde{\eta}_k\right)^{\nu k} E^{k+1}_{\nu,\nu k+1}\left(-\lambda\left(\frac{vt+x}{c+v} + \tilde{\eta}_k\right)^\nu\right)\Bigg\}
\end{aligned}$$

and

$$\begin{aligned}
b(x,t\,|-v) = \sum_{k=1}^{+\infty} \Bigg\{ & \frac{I_{k,k}(x,t)}{c+v}\lambda^k\left(\frac{vt+x}{c+v} - \theta_k\right)^{\nu k-1} E^{k}_{\nu,\nu k}\left(-\lambda\left(\frac{vt+x}{c+v} - \theta_k\right)^\nu\right) \\
& \times \mu^k \left(\frac{ct-x}{c+v} + \theta_k\right)^{\nu k} E^{k+1}_{\nu,\nu k+1}\left(-\mu\left(\frac{ct-x}{c+v} + \theta_k\right)^\nu\right)\Bigg\},
\end{aligned}$$

where

$$\tilde{\eta}_k = \frac{\beta^{(k+1)} - \alpha^{(k)}}{c+v} = \frac{\beta_{k+1}}{c+v} - \theta_k.$$

Corollary 5.2.1. *If the initial velocity is random, i.e. V_0 is either c or $-v$ with equal probability, we obtain*

$$\begin{aligned}
\mathbb{P}[X_t = ct\,|\,X_0 = 0] &= \tfrac{1}{2}\,E_{\nu,1}(-\lambda t^\nu), \\
\mathbb{P}[X_t = -vt\,|\,X_0 = 0] &= \tfrac{1}{2}\,E_{\nu,1}(-\mu t^\nu),
\end{aligned}
\qquad V_0 = \begin{cases} c & \textit{w.p. 1/2}, \\ -v & \textit{w.p. 1/2}. \end{cases} \tag{5.14}$$

Furthermore, from (5.7) the transition PDF of X_t is

$$\begin{aligned} p(x,t) := \mathbb{P}[X_t \in \mathrm{d}x \,|\, X_0 = 0] &= \frac{1}{2}\left[p(x,t\,|\,c) + p(x,t\,|\,-v)\right] \\ &= \frac{1}{2}\left[f(x,t\,|\,c) + b(x,t\,|\,c) + f(x,t\,|\,-v) + b(x,t\,|\,-v)\right], \end{aligned} \tag{5.15}$$

where the forward and backward transition PDF's conditional on initial velocity are given in Theorem 5.2.1 and Remark 5.2.1.

5.3 Constant jump sizes

We now focus on the special case when all the jumps have equal constant amplitude, say α. Hereafter we obtain the explicit expression of the density $p(x,t)$ defined in (5.15).

Proposition 5.3.1. *Let $\alpha_k = \beta_k = \alpha > 0$ for all $k \in \mathbb{N}$, and let U and D be Mittag-Leffler distributed with parameters (λ, ν) and (μ, ν), respectively. Let $\mathbb{P}(V_0 = c) = \mathbb{P}(V_0 = -v) = 1/2$. The probability law of X_t is characterized by the discrete component indicated in (5.14), and by the absolutely continuous component $p(x,t)$ specified hereafter.*
(i) If $0 < t < \alpha/(c+v)$ then

$$p(x,t) = \begin{cases} \varphi_{-1}(x,t), & -vt-\alpha < x < ct-\alpha \\ \varphi_0(x,t), & -vt < x < ct \\ \varphi_1(x,t), & -vt+\alpha < x < ct+\alpha \\ 0 & \textit{otherwise}, \end{cases}$$

(ii) if $\alpha/(c+v) \le t < 2\alpha/(c+v)$ then

$$p(x,t) = \begin{cases} \varphi_{-1}(x,t), & -vt-\alpha < x < -vt \\ \varphi_{-1}(x,t) + \varphi_0(x,t), & -vt < x < ct-\alpha \\ \varphi_0(x,t), & ct-\alpha < x < -vt+\alpha \\ \varphi_0(x,t) + \varphi_1(x,t), & -vt+\alpha < x < ct \\ \varphi_1(x,t), & ct < x < ct+\alpha \\ 0 & \textit{otherwise}, \end{cases}$$

(iii) if $t \geq 2\alpha/(c+v)$ then

$$p(x,t) = \begin{cases} \varphi_{-1}(x,t), & -vt-\alpha < x < -vt \\ \varphi_{-1}(x,t)+\varphi_0(x,t), & -vt < x < -vt+\alpha \\ \varphi_{-1}(x,t)+\varphi_0(x,t)+\varphi_1(x,t), & -vt+\alpha < x < ct-\alpha \\ \varphi_0(x,t)+\varphi_1(x,t), & ct-\alpha < x < ct \\ \varphi_1(x,t), & ct < x < ct+\alpha \\ 0 & \textit{otherwise}, \end{cases}$$

with

$$\varphi_{-1}(x,t) = \frac{1}{2\,(c+v)} \sum_{k=0}^{+\infty} \Bigg\{ \mu^{k+1} \left(\frac{ct-x-\alpha}{c+v}\right)^{\nu(k+1)-1} E^{k+1}_{\nu,\nu(k+1)} \left(-\mu \left(\frac{ct-x-\alpha}{c+v}\right)^{\nu}\right) \\ \times \lambda^{k} \left(\frac{vt+x+\alpha}{c+v}\right)^{\nu k} E^{k+1}_{\nu,\nu k+1} \left(-\lambda \left(\frac{vt+x+\alpha}{c+v}\right)^{\nu}\right)\Bigg\},$$

$$\varphi_{0}(x,t) = \frac{1}{2\,(c+v)} \sum_{k=0}^{+\infty} \Bigg\{ \mu^{k+1} \left(\frac{ct-x}{c+v}\right)^{\nu(k+1)-1} E^{k+1}_{\nu,\nu(k+1)} \left(-\mu \left(\frac{ct-x}{c+v}\right)^{\nu}\right) \\ \times \lambda^{k+1} \left(\frac{vt+x}{c+v}\right)^{\nu(k+1)} E^{k+2}_{\nu,\nu(k+1)+1} \left(-\lambda \left(\frac{vt+x}{c+v}\right)^{\nu}\right) \\ + \lambda^{k+1} \left(\frac{vt+x}{c+v}\right)^{\nu(k+1)-1} E^{k+1}_{\nu,\nu(k+1)} \left(-\lambda \left(\frac{vt+x}{c+v}\right)^{\nu}\right) \\ \times \mu^{k+1} \left(\frac{ct-x}{c+v}\right)^{\nu(k+1)} E^{k+2}_{\nu,\nu(k+1)+1} \left(-\mu \left(\frac{ct-x}{c+v}\right)^{\nu}\right)\Bigg\}$$

and

$$\varphi_{1}(x,t) = \frac{1}{2\,(c+v)} \sum_{k=0}^{+\infty} \Bigg\{ \lambda^{k+1} \left(\frac{vt+x-\alpha}{c+v}\right)^{\nu(k+1)-1} E^{k+1}_{\nu,\nu(k+1)} \left(-\lambda \left(\frac{vt+x-\alpha}{c+v}\right)^{\nu}\right) \\ \times \mu^{k} \left(\frac{ct-x+\alpha}{c+v}\right)^{\nu k} E^{k+1}_{\nu,\nu k+1} \left(-\mu \left(\frac{ct-x+\alpha}{c+v}\right)^{\nu}\right)\Bigg\}.$$

Proof. It is a straightforward consequence of Eq. (5.15), by recalling that assumption $\alpha_k = \beta_k = \alpha > 0$, $k \in \mathbb{N}$, yields $\theta_k = 0$ and $\eta_k = \tilde{\eta}_k = \alpha/(c+v)$. □

It is interesting to note that the functions φ_i, $i = -1, 0, 1$, have a specific probabilistic meaning. Indeed, φ_{-1} (φ_1) represents a measure of the sample-paths of the process that perform a number of downward (upward) jumps that is one more the upward (downward) jumps in interval $(0, t]$, whereas φ_0 refers to the case when the number of upward and downward jumps concide.

We also remark that Proposition 5.3.1 is an immediate extension of Proposition 4.1 of [38], which concerns the case of exponentially distributed interarrival times. In Figures 5.2÷5.7 we show some plots of density $p(x, t)$ obtained in Proposition 5.3.1 for three choices of ν and two choices of the switching intensities. The Mittag-Leffler function with three parameters has been evaluated by means of the MATLAB® routine:

Garrappa, Roberto. The Mittag-Leffler function.
https://it.mathworks.com/matlabcentral/fileexchange/48154-the-mittag-leffler-function?requestedDomain=www.mathworks.com
MATLAB Central File Exchange. Updated December 07, 2015.

We first observe that the vertical asymptotes of density $p(x, t)$ are due to the singular behaviour at 0^+ of the Mittag-Leffler distribution. Similarly to the case of exponentially distributed interarrival times (cf. Figures 2 and 3 of [38]), at the beginning of the motion the probability mass is concentrated in a neighbourhood of the origin and of $\pm\alpha$ (due to the occurrence of a small number of jumps). As time grows larger, the singularities are shifted towards the endpoints of the spatial interval and the effect of further velocity changes and jumps makes the density smoother and smoother, so that the probability mass is spread over the whole diffusion domain.

Let us now analyse the mean displacement of the particle, described by X_t, when the upward and downward jumps are constant, and possibly different.

Proposition 5.3.2. *Let $\alpha_j = \alpha$ and $\beta_j = \beta$ for all $j \in \mathbb{N}$, and let both U and D be Mittag-Leffler distributed with parameters (λ, ν). Let $\mathbb{P}(V_0 = c) = \mathbb{P}(V_0 = -v) = 1/2$. Then, for $t > 0$ we have*

$$\mathbb{E}(X_t | X_0 = 0) = \frac{1}{2}\left[(c - v)\, t + (\alpha - \beta)\, \frac{\lambda t^{\nu}}{\Gamma(\nu + 1)}\right]. \tag{5.16}$$

Proof. The proof is similar to that of Proposition 4.5 of [38]. Indeed, denoting by $\mathbb{E}_y(\cdot)$ the mean conditional on $V_0 = y \in \{-v, c\}$, from (5.1) we need to compute

$$\begin{aligned}\mathbb{E}(X_t | X_0 = 0) = \frac{1}{2}\Bigg[&\int_0^t \mathbb{E}_c(V_s)\,\mathrm{d}s + \int_0^t \mathbb{E}_{-v}(V_s)\,\mathrm{d}s \\ &+ \mathbb{E}_c\left(\sum_{k=1}^{N_t} w_k\right) + \mathbb{E}_{-v}\left(\sum_{k=1}^{N_t} w_k\right)\Bigg].\end{aligned}$$

By setting

$$p_r(t) = \mathbb{P}(N_t = r),$$

we have (cf. (2.33) and Remark 3.4 of [18])

$$\begin{aligned}\mathbb{E}_c\left(\sum_{k=1}^{N_t} w_k\right) &= \frac{\alpha-\beta}{2}\sum_{n=0}^{+\infty} n\, p_n(t) - \frac{\alpha+\beta}{2}\sum_{n=0}^{+\infty} p_{2n+1}(t) \\ &= \frac{\alpha-\beta}{2}\frac{\lambda t^\nu}{\Gamma(\nu+1)} + \frac{\alpha+\beta}{2}\lambda t^\nu E_{\nu,\nu+1}\left(-2\lambda t^\nu\right). \end{aligned} \tag{5.17}$$

Moreover, due to formula (2.30) of [18], we have

$$\mathbb{E}\left[(-1)^{N_\lambda(s)}\right] = E_{\nu,1}\left(-2\lambda s^\nu\right). \tag{5.18}$$

(See also Ferraro et al. [51], where the autocovariance function for a random telegraph signal of Mittag-Leffler type is studied). Then, from (5.2),

$$\int_0^t \mathbb{E}_c(V_s)\,\mathrm{d}s = \frac{c-v}{2}t + \frac{c+v}{2}tE_{\nu,2}(-2\lambda t^\nu). \tag{5.19}$$

Similarly to Eqs. (5.17) and (5.19), we have:

$$\mathbb{E}_{-v}\left(\sum_{k=1}^{N_t} w_k\right) = \frac{\alpha-\beta}{2}\frac{\lambda t^\nu}{\Gamma(\nu+1)} - \frac{\alpha+\beta}{2}\lambda t^\nu E_{\nu,\nu+1}\left(-2\lambda t^\nu\right)$$

and

$$\int_0^t \mathbb{E}_{-v}(V_s)\,\mathrm{d}s = \frac{c-v}{2}t - \frac{c+v}{2}tE_{\nu,2}(-2\lambda t^\nu). \tag{5.20}$$

Therefore, the thesis follows from Eqs. (5.17)-(5.20). □

Let us now discuss some features of the mean given in (5.16). First of all, we note that the linear term $(c-v)t$ has to be attributed to the alternating motion, the term $(\alpha-\beta)\lambda t^\nu/\Gamma(\nu+1)$ is related to the jump component of the motion, characterized by Mittag-Leffler distributed intertimes, and the factor 1/2 is due to the random initial velocity V_0. Eq. (5.16) is in agreement with Eq. (39) of [38] for the jump-telegraph process with exponentially distributed intertimes, that we recover for $\nu = 1$.

In Table 5.1 we provide the mean of some fractional stochastic processes that can be compared with the conditional mean of X_t determined in (5.16). We note that cases (vi), (vii) and (viii) refer to birth-death type processes with n_0 progenitors. Moreover, the fractional Poisson process (i) becomes identical to X_t for $c = v = 0$ (no telegraph component), $\alpha = 1$ and $\beta = -1$ (upward jumps of size 1 occur at every event of the underlying fractional Poisson process). A similar remark holds for the process with multiple jumps (iv).The mean of the jump-telegraph process X_t, under the assumptions of Proposition 5.3.2, vanishes in the symmetric case when $c = v$ and $\alpha = \beta$, as well as for the symmetric fractional (telegraph) process studied in [107] (case (ix) of Table 5.1). Figure 5.8 shows the means considered in cases (i)÷(viii) of Table 5.1, for two choices of ν, with $\lambda = 1$, $\mu = 0.5$ and $n_0 = 10$, and for $k = 3$ and $\lambda_1 = \lambda_2 = \lambda_3 = 1$ in case (iv). The asymptotic behaviour of the means given in Table 5.1 is finally provided in Table 5.2, obtained thanks to formulas (4.4.16) of Gorenflo *et al.* [57] and (A.3) of [134]. In case (viii) of Table 5.2, $\gamma \simeq 0.577216$ is the Euler's constant and $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function, i.e. the logarithmic derivative of the gamma function.

Table 5.1: Expected values of some processes of interest

	fractional process	mean value	ref.
(i)	Poisson process	$\frac{\lambda t^\nu}{\Gamma(\nu+1)}$	[18]
(ii)	alternative Poisson process	$\frac{\lambda t}{\nu} \frac{E_{\nu,\nu}(\lambda t)}{E_{\nu,1}(\lambda t)}$	[17]
(iii)	2nd-order Poisson process	$\frac{\lambda t^\nu}{2\Gamma(\nu+1)} - \frac{\lambda t^\nu}{2} E_{\nu,\nu+1}(-2\lambda t^\nu)$	[18]
(iv)	Poisson process with jumps $1, \dots, k$	$\frac{\sum_{j=1}^{k} j\lambda_j t^\nu}{\Gamma(\nu+1)}$	[40]
(v)	linear birth process	$E_{\nu,1}(\lambda t^\nu)$	[108]
(vi)	linear birth-death process	$n_0 E_{\nu,1}((\lambda-\mu)t^\nu)$	[109]
(vii)	linear death process	$n_0 E_{\nu,1}(-\mu t^\nu)$	[112]
(viii)	sublinear death process	$\sum_{j=1}^{n_0} \binom{n_0+1}{j+1} (-1)^{j+1} E_{\nu,1}(-\mu j t^\nu)$	[112]
(ix)	telegraph process	0	[107]

Table 5.2: Asymptotic behaviour of the means of some processes of interest

(i)	$\frac{\lambda t^\nu}{\Gamma(\nu+1)}$	(v)	$\frac{\exp\{\lambda^{1/\nu} t\}}{\nu}$
(ii)	$\frac{(\lambda t)^{1/\nu}}{\nu}$	(vi)	$n_0 \frac{\exp\{(\lambda-\mu)^{1/\nu} t\}}{\nu}$, for $\lambda > \mu$
(iii)	$\frac{\lambda t^\nu}{2\Gamma(\nu+1)} - \frac{1}{4}$	(vii)	$n_0 \frac{1}{\Gamma(1-\nu)\mu t^\nu}$
(iv)	$\frac{\sum_{j=1}^{k} j\lambda_j t^\nu}{\Gamma(\nu+1)}$	(viii)	$\frac{1}{\Gamma(1-\nu)\mu t^\nu}(n_0+1)[\gamma - 1 + \psi(n+2)]$

We conclude this section with the following theorem, which gives the conditional distribution of X_t in absence of jumps, recovered by applying Theorem 2.1 of [33]. We omit the proof since it proceeds similarly to that of Theorem 5.2.1.

Theorem 5.3.1. *Let U and D have Mittag-Leffler distribution with parameters (λ, ν) and (μ, ν) respectively, and let $\alpha = \beta = 0$. For all $t > 0$ we have*

$$\mathbb{P}[X_t = ct, V_t = c \,|\, X_0 = 0, V_0 = c] = E_{\nu,1}(-\lambda t^\nu);$$

moreover, for $-vt < x < ct$ it holds

$$f(x,t\,|\,c) = \frac{1}{c+v}\sum_{k=1}^{+\infty}\left\{\mu^k\left(\frac{ct-x}{c+v}\right)^{\nu k-1} E^k_{\nu,\nu k}\left(-\mu\left(\frac{ct-x}{c+v}\right)^{\nu}\right)\times\right.$$
$$\left.\times\lambda^k\left(\frac{vt+x}{c+v}\right)^{\nu k} E^{k+1}_{\nu,\nu k+1}\left(-\lambda\left(\frac{vt+x}{c+v}\right)^{\nu}\right)\right\},$$

$$b(x,t\,|\,c) = \frac{1}{c+v}\sum_{k=0}^{+\infty}\left\{\lambda^{k+1}\left(\frac{vt+x}{c+v}\right)^{\nu(k+1)-1} E^{k+1}_{\nu,\nu(k+1)}\left(-\lambda\left(\frac{vt+x}{c+v}\right)^{\nu}\right)\times\right.$$
$$\left.\times\mu^k\left(\frac{ct-x}{c+v}\right)^{\nu k} E^{k+1}_{\nu,\nu k+1}\left(-\mu\left(\frac{ct-x}{c+v}\right)^{\nu}\right)\right\},$$

and

$$b(x,t\,|\,-v) = \frac{1}{c+v}\sum_{k=1}^{+\infty}\left\{\lambda^k\left(\frac{vt+x}{c+v}\right)^{\nu k-1} E^k_{\nu,\nu k}\left(-\lambda\left(\frac{vt+x}{c+v}\right)^{\nu}\right)\times\right.$$
$$\left.\times\mu^k\left(\frac{ct-x}{c+v}\right)^{\nu k} E^{k+1}_{\nu,\nu k+1}\left(-\mu\left(\frac{ct-x}{c+v}\right)^{\nu}\right)\right\},$$

$$f(x,t\,|\,-v) = \frac{1}{c+v}\sum_{k=0}^{+\infty}\left\{\mu^{k+1}\left(\frac{ct-x}{c+v}\right)^{\nu(k+1)-1} E^{k+1}_{\nu,\nu(k+1)}\left(-\mu\left(\frac{ct-x}{c+v}\right)^{\nu}\right)\times\right.$$
$$\left.\times\lambda^k\left(\frac{vt+x}{c+v}\right)^{\nu k} E^{k+1}_{\nu,\nu k+1}\left(-\lambda\left(\frac{vt+x}{c+v}\right)^{\nu}\right)\right\}.$$

5.4 First-passage-time problem

In this section we study the distribution of the (upward) first-passage time of X_t through a constant barrier, say $\gamma > 0$, conditional on the initial state

$$\tau_\gamma = \inf\{t \geq 0 : X_t \geq \gamma\}, \qquad X_0 = 0,\, V_0 = c. \tag{5.21}$$

The downward case can be treated similarly. Hereafter we express the probability distribution of (5.21) in terms of the following subdensity functions:

$$g_\gamma(t;n)\mathrm{d}t := \mathbb{P}(\tau_\gamma \in \mathrm{d}t, N_t = n) = \frac{\mathrm{d}}{\mathrm{d}s}\mathbb{P}(\tau_\gamma \leq s, N_t = n)\Big|_{s=t}, \qquad n \in \mathbb{N}.$$

Proposition 5.4.1. *For $t > 0$ it holds:*

$$\mathbb{P}\left(\tau_\gamma \in \mathrm{d}t\right) = E_{\nu,1}(-\lambda t^\nu)\delta_{\frac{\gamma}{c}}(\mathrm{d}t) + \sum_{k=0}^{+\infty} g_\gamma(t; 2k+1)\mathrm{d}t + \sum_{k=1}^{+\infty} g_\gamma(t; 2k)\mathrm{d}t, \tag{5.22}$$

where

$$\begin{aligned} g_\gamma(t; 2k+1)\mathrm{d}t = \mathbb{P}\Big(&cU^{(i)} - vD^{(i-1)} + \alpha^{(i)} - \beta^{(i-1)} < \gamma,\, i = 1, \dots, k, \\ &U^{(k+1)} + D^k \in \mathrm{d}t,\, cU^{(k+1)} - vD^{(k)} + \alpha^{(k+1)} - \beta^{(k)} \geq \gamma\Big), \end{aligned} \tag{5.23}$$

and

$$\begin{aligned} g_\gamma(t; 2k)\mathrm{d}t = \mathbb{P}\Big(&cU^{(i)} - vD^{(i-1)} + \alpha^{(i)} - \beta^{(i-1)} < \gamma,\, i = 1, \dots, k, \\ &\frac{\gamma - \alpha^{(k)} + \beta^{(k)} + (c+v)D^{(k)}}{c} \in \mathrm{d}t\Big). \end{aligned} \tag{5.24}$$

Proof. By the law of total probability, we can express the conditional distribution of τ_γ in the form

$$\mathbb{P}\left(\tau_\gamma \in \mathrm{d}t\right) = \mathbb{P}\left(U_1 > t\right)\delta_{\frac{\gamma}{c}}(\mathrm{d}t) + \sum_{j=1}^{+\infty} \mathbb{P}\left(\tau_\gamma \in \mathrm{d}t, N_t = j\right), \tag{5.25}$$

where $\delta_{\gamma/c}$ is the Dirac's delta measure at γ/c corresponding to the motion without any direction switchings. Moreover, in Eq. (5.25), the series in the right-hand side represents the absolutely continuous part of the distribution with the condition of at least one direction reversal, and N_t is the fractional alternating Poisson process introduced in Section 5.2. We also recall that U_1 has a Mittag-Leffler distribution with parameters (λ, ν). We consider two cases, namely when N_t is odd, and when N_t is even. If by time t there have been $2k+1$, $k \geq 0$, changes of direction ($k+1$ upward and k backward), then the particle crosses level γ for the first time owing to the effect of the $(k+1)$th upward jump. If by time t there have been $2k, k \geq 1$, changes of direction (k upward and k backward), then the first passage of the particle through level γ is due to the effect of the upward motion after the last renewal event. Recalling (5.5), this implies that density (5.25) becomes

$$\mathbb{P}\left(\tau_\gamma \in \mathrm{d}t\right) = \mathbb{P}\left(U_1 > t\right)\delta_{\frac{\gamma}{c}}(\mathrm{d}t) + \sum_{k=0}^{+\infty} g_\gamma(t; 2k+1)\mathrm{d}t + \sum_{k=1}^{+\infty} g_\gamma(t; 2k)\mathrm{d}t,$$

where $g_\gamma(t; 2k+1)\mathrm{d}t$ and $g_\gamma(t; 2k)\mathrm{d}t$ have been expressed in (5.23) and (5.24). The final expression (5.22) thus follows. □

We remark that formula (5.22) is formally effective, but the determination of an explicit form of $g_\gamma(t;k)$ is hard to be obtained when k is large. Nevertheless, the above result is useful since

$$\hat{g}_\gamma(t;k) := \sum_{i=1}^{k} g_\gamma(t;i), \qquad k \in \mathbb{N}, \tag{5.26}$$

constitutes a sequence of increasing lower bounds for $\mathbb{P}\left(\tau_\gamma \in \mathrm{d}t\right)$ when k grows. In conclusion, Fig. 5.9 shows some plots of $\hat{g}_\gamma(t;k)$, for $k = 1, 2, 3$ and for various choices of the parameters involved, these giving lower bounds for the distribution (5.25).

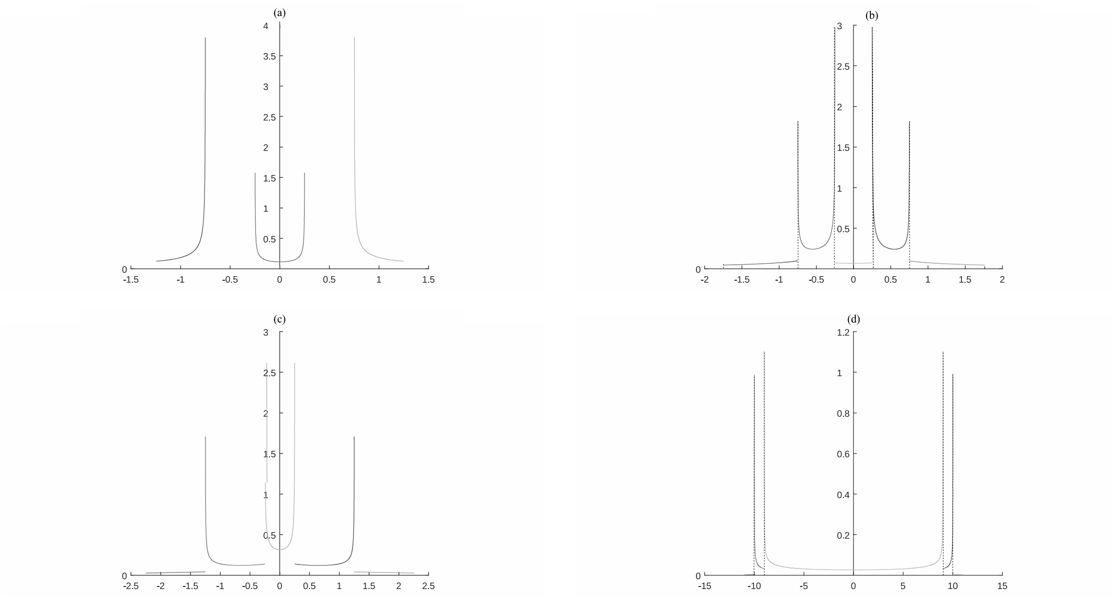

Figure 5.2: Plots of density $p(x,t)$ obtained in Proposition 5.3.1, for $c = v = 1$, $\lambda = \mu = 1$, $\alpha = 1$ and $\nu = 0.5$, when (a) $t = 0.25$, (b) $t = 0.75$, (c) $t = 1.25$, (d) $t = 10$.

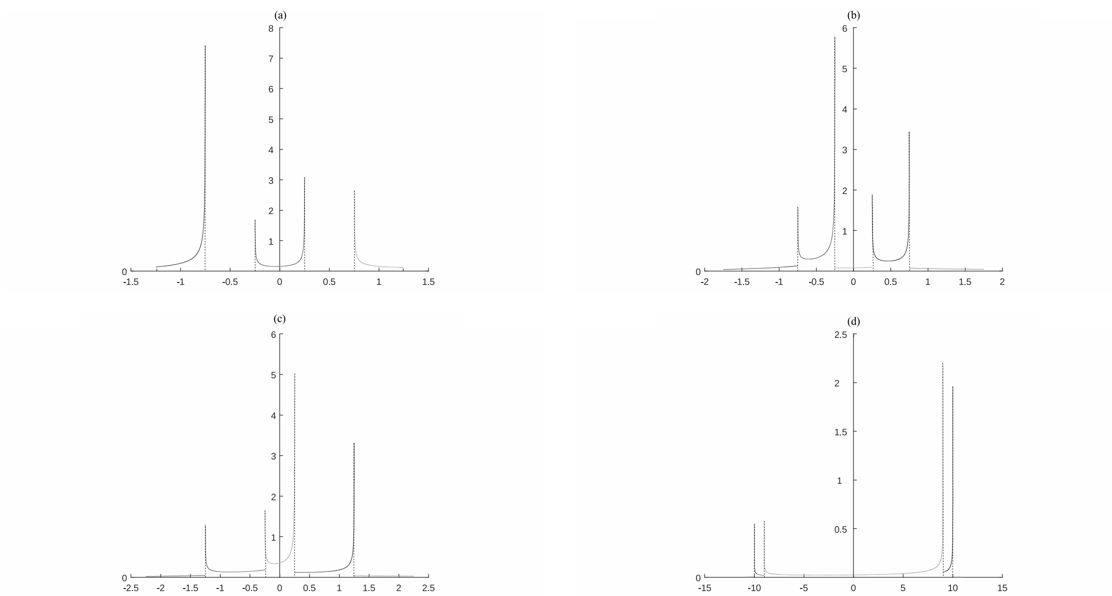

Figure 5.3: As in Fig. 5.2, with $\mu = 2$.

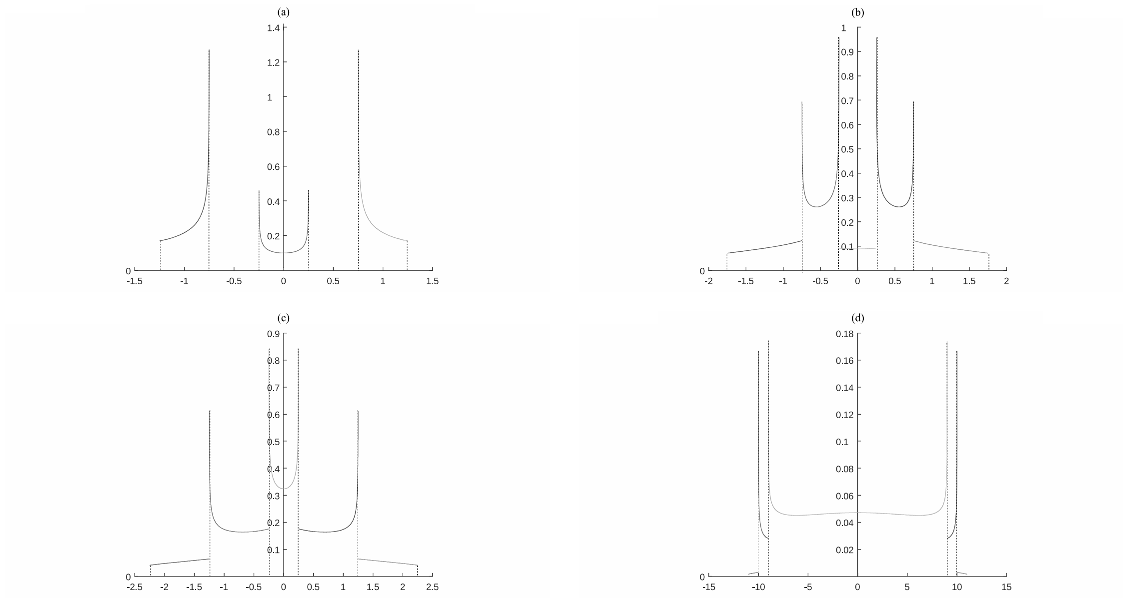

Figure 5.4: Plots of density $p(x,t)$ obtained in Proposition 5.3.1, for $c = v = 1$, $\lambda = \mu = 1$, $\alpha = 1$ and $\nu = 0.7$, when (a) $t = 0.25$, (b) $t = 0.75$, (c) $t = 1.25$, (d) $t = 10$.

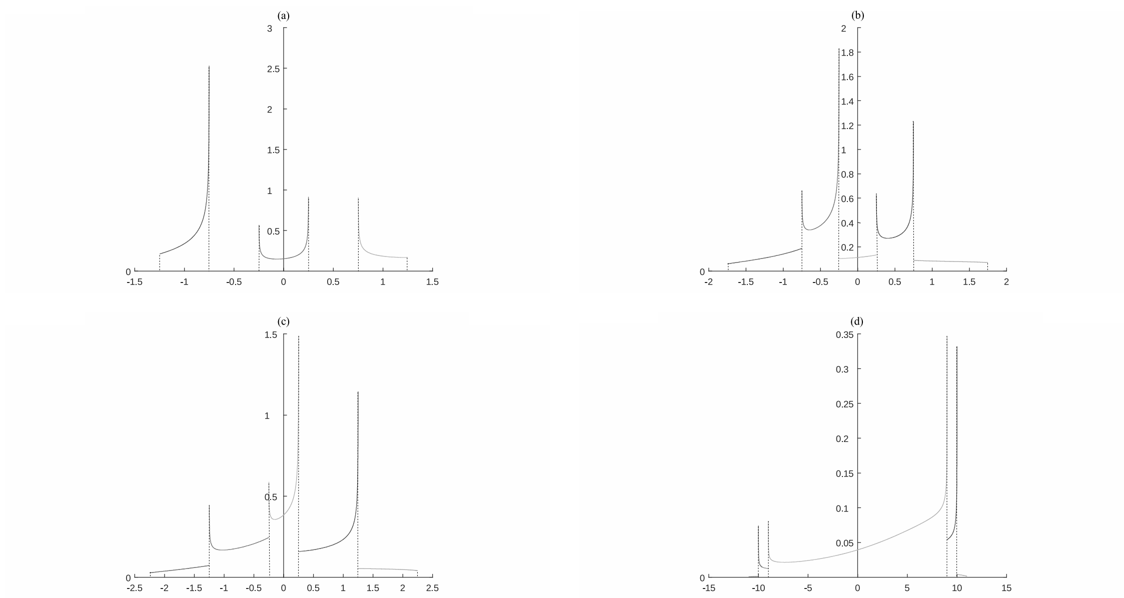

Figure 5.5: As in Fig. 5.4, with $\mu = 2$.

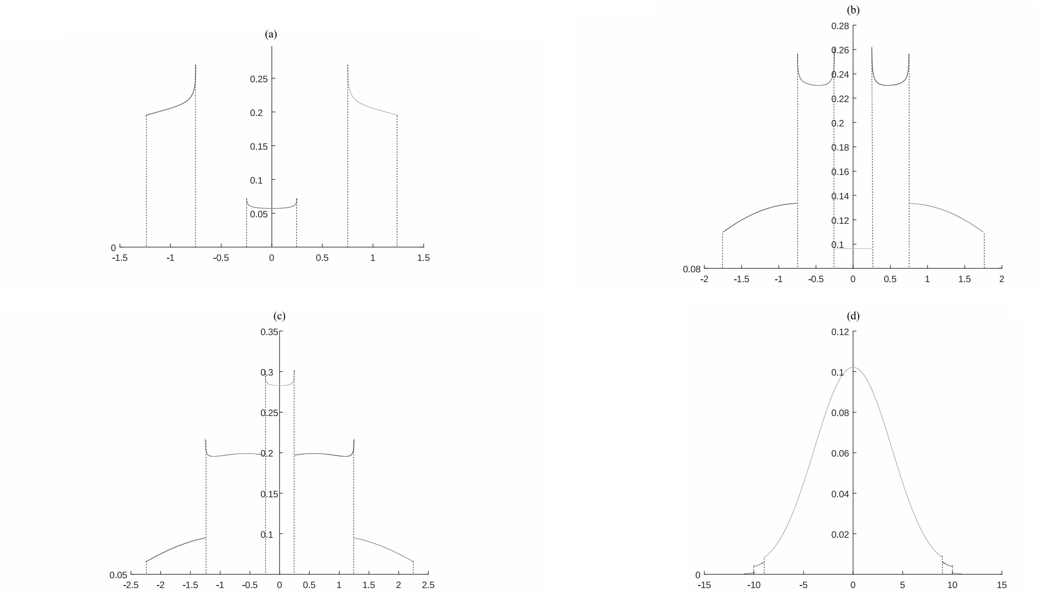

Figure 5.6: Plots of density $p(x,t)$ obtained in Proposition 5.3.1, for $c = v = 1$, $\lambda = \mu = 1$, $\alpha = 1$ and $\nu = 0.95$, when (a) $t = 0.25$, (b) $t = 0.75$, (c) $t = 1.25$, (d) $t = 10$.

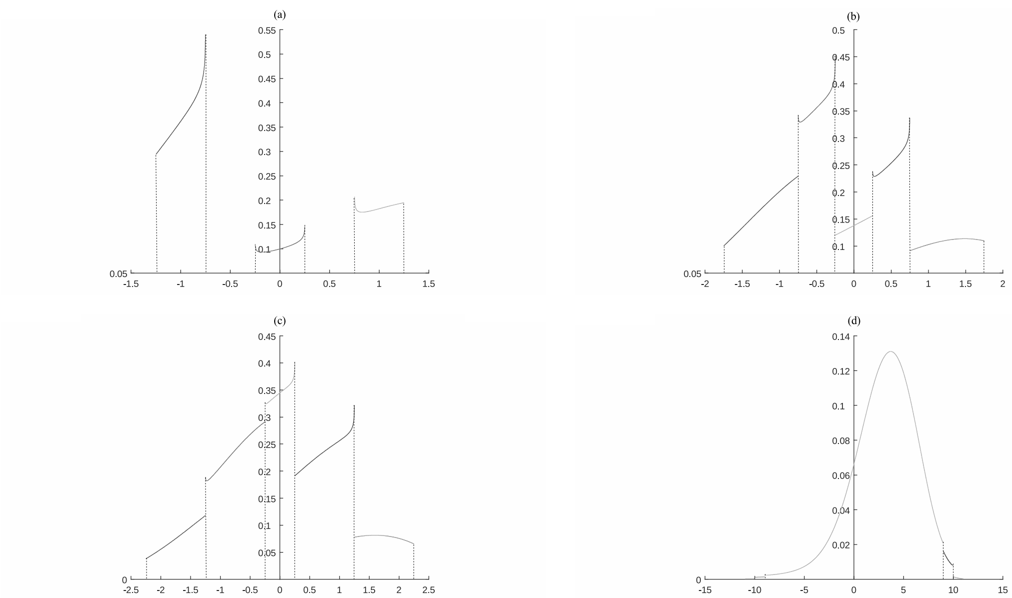

Figure 5.7: As in Fig. 5.6, with $\mu = 2$.

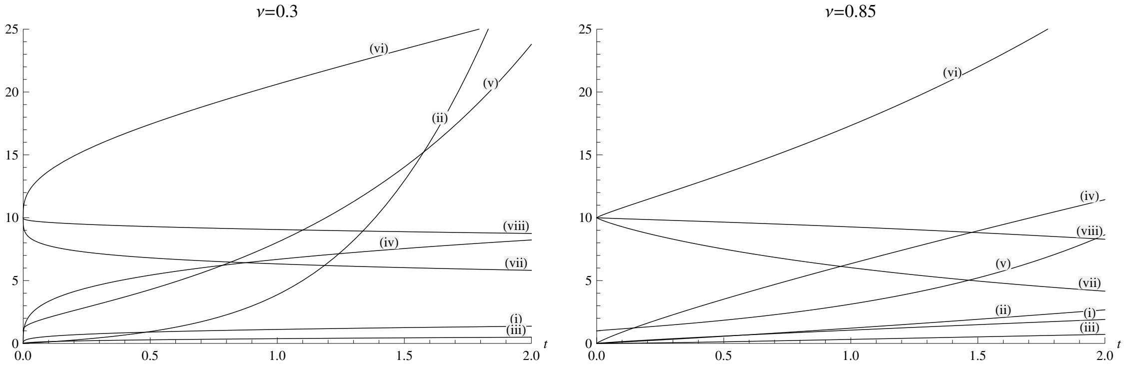

Figure 5.8: Expected values of the processes considered in Table 5.1.

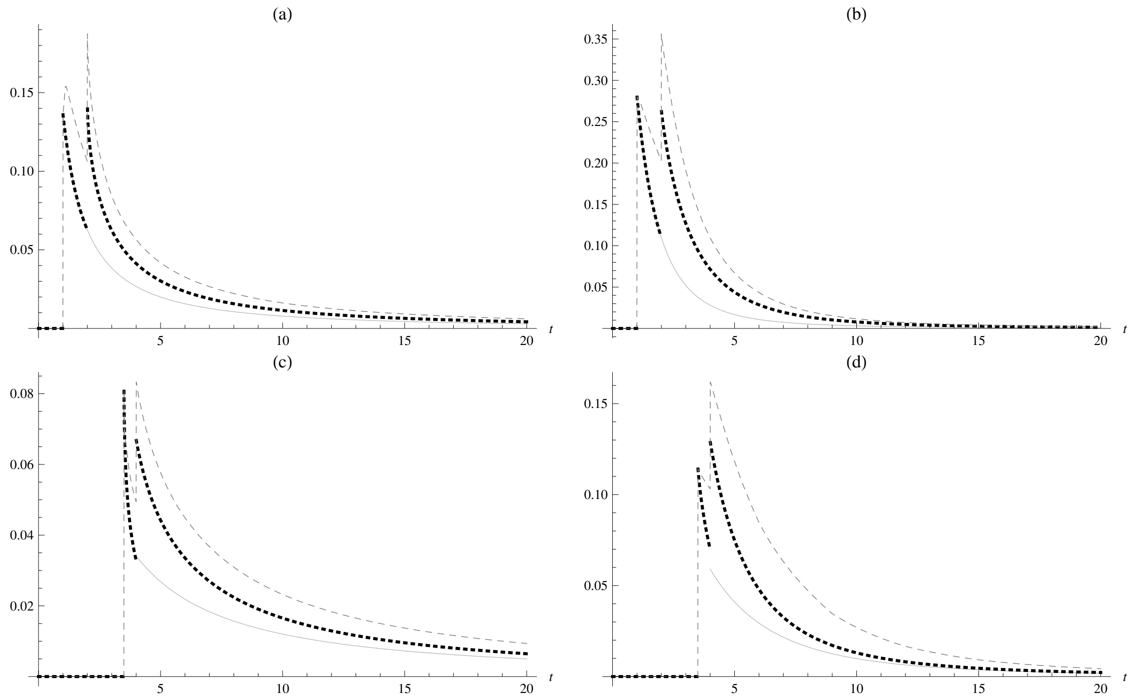

Figure 5.9: Plots of bounds (5.26) for $\gamma = 2$, $\alpha = \beta = 1$, $c = v = 1$, $\lambda = \mu = 1$ and (a) $\nu = 0.5$, (b) $\nu = 0.85$, and for $\gamma = 10$, $\alpha = 2$, $\beta = 1$, $c = 2$, $v = 1$, $\lambda = 2$, $\mu = 1$ and (c) $\nu = 0.5$, (d) $\nu = 0.85$. The cases $k = 1$ (continuous line), $k = 2$ (dotted line), and $k = 3$ (dashed line) are considered.

Chapter 6

Competing risks driven by Mittag-Leffler distributions, under copula and time transformed exponential model

6.1 Introduction

As recalled in Section 1.2, the *Mittag-Leffler distribution* was introduced by Pillai in [122] in terms of the Mittag-Leffler function as a fractional generalization of the exponential distribution. Its main features are: being singular at zero, completely monotonic, long-tailed and geometrically infinitely divisible. Disparate phenomena follow power law distributions in their tails. For example, there is increasing evidence that the timing of many human activities, ranging from financial market transactions and communication to entertainment and work patterns, follow non-Poisson statistics, characterized by bursts of rapidly occurring events separated by long periods of inactivity. See Engle and Russel [48] and Barabasi [9] and references therein, for instance. A recent contribution to the issue of the stationarity of the inter-event power-law distributions has been given by Gandica et al. [54] Over the years, many researchers have shown interest in deepening and generalizing Pillai's results (cf. for instance [52], [88], [2], [78] and [76]). The Mittag-Leffler distribution has been found to be useful in a variety of applications. For example, in the fundamental paper of Hilfer and Anton [64] the correspondence between continuous time random walks (CTRWs) with Mittag-Leffler distributed waiting times and the time-fractional diffusion equation is highlighted. CTRWs, under suitable hypothesis, are successfully used to model normal and anomalous diffusion phenomena in physics (e.g. cf. [149]

for an insight on the role of the Mittag-Leffler distribution in the Cole-Cole relaxation phenomena), in finance and economics (cf. [136] and references therein for a minireview on the topic), in queueing theory (cf. [65] for a work on sales forecast and planning). At the start of the 21st century many papers began to appear about the fractional generalization of the pure and compound Poisson processes, replacing the exponential waiting time distribution by a distribution given via a Mittag-Leffer function with modified argument (see, for instance, [94], [17] and [100]). Therefore, the direct involvement of such probability distribution in different areas of modern science stimulates us to propose a competing risks model governed by Mittag-Leffler distribution, even showing a connection between the quantities of interest and certain fractional stochastic processes, so to be able to deal with heavy-tailed data. In the past decades some papers have appeared in this direction (e.g. [119] and [28]). We mention that the competing risks approach is also adopted in some contexts of Economics and Finance dealing with data observations containing severe outliers (see, for instance, [3]). The setting of competing risks models based on heavy-tailed distributions, such as the Mittag-Leffler one, is thus welcome.

Several authors considered the problem of establishing or testing the independence between the competing cause and the failure time, this being a relevant issue in this field (cf. for instance [47] and [31]). To this extent, it is interesting to note that the proposed model exhibits the above-mentioned independence property. We also remark that, even though in some applications such independence is unusual, certain stochastic models properly include this property. See, for instance, the competing risks model considered in [36], in which system failures are due to shock models governed by a bivariate Poisson process.

We also investigate a competing cause setting in which the actual number of competing causes is a latent discrete random variable. Many researchers have shown interest in this scenario, since it turns out to be useful, among the other things, to describe cure rate models (cf. [6] and [7]) and from a finantial point of view (see [4] and [5]). The problem of identifying the distribution of the risks under copula functions and the time transformed exponential model is also considered in the more general case of arbitrary underlying distributions.

The chapter is organized as follows. In Section 6.2 we recall some essential aspects of the competing risks model. In Section 6.3 we describe a Mittag-Leffler distribution-based model, focusing on the quantities of interest and showing a relation with fractional random growth phenomena. We also prove the independence between the time to failure and the cause of failure, and show that hazard rates cross when one of the parameters varies. Then we restrict our attention to some ageing properties of the lifetimes involved in the competing risks model. In Section 6.4 we face the problem of identifying the distribution of failure times when their joint distribution is expressed by means of copulas and the time transformed exponential model. Some special cases regarding the Mittag-Leffler distribution-based model are treated numerically. In Section 6.5 we adapt the model studied in Section 6.3 to the case of a random number of independent competing risks. Even in this setting we are able to show an interesting relationship with a fractional stochastic process. As a case study, we consider a certain mixture of Mittag-Leffler distributions. An estimation method for the parameters of such distribution, based on fractional moments, is implemented.

6.2 Background on competing risks

Consider a competing risks problem in which a subject is exposed to n causes of failure, with $n \in \mathbb{N}$, and the occurrence of one of these will prevent any other competing event from ever happening. Let $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ be a vector of non-negative and not necessarily independent random variables, where X_i describes the lifetime of the subject when its failure is due to the i-th risk, $i \in \{1, 2, \ldots, n\}$. Setting $\mathbf{x} := (x_1, x_2, \ldots, x_n)$, we denote by

$$F(\mathbf{x}) = \mathbb{P}(X_1 \le x_1, X_2 \le x_2, \ldots, X_n \le x_n)$$

the joint cumulative distribution function of $\mathbf{X}$ and assume that it is absolutely continuous, so that $\mathbb{P}(X_i = X_j) = 0$ for all $i \neq j$. Moreover, let

$$\overline{F}(\mathbf{x}) = \mathbb{P}(X_1 > x_1, X_2 > x_2, \ldots, X_n > x_n) \tag{6.1}$$

be the survival function of $\mathbf{X}$. Let $T = \min(X_1, X_2, \ldots, X_n)$ be the observable lifetime and δ the competing event, i.e. $\delta = i$ if and only if $T = X_i$, $(i = 1, 2, \ldots, n)$. Their distributions can be expressed in terms of the so-called *sub-distribution* and *sub-survival* functions:

$$F_i^*(x) = \mathbb{P}(T \le x, \delta = i), \qquad \overline{F}_i^*(x) = \mathbb{P}(T > x, \delta = i), \tag{6.2}$$

with $x \ge 0$ and $i = 1, 2, \ldots, n$. Indeed, from (6.2) and from the law of total probability we have, for $x \ge 0$,

$$F_T(x) = \mathbb{P}(T \le x) = \sum_{i=1}^{n} F_i^*(x), \qquad \overline{F}_T(x) = \mathbb{P}(T > x) = \sum_{i=1}^{n} \overline{F}_i^*(x), \tag{6.3}$$

and

$$\mathbb{P}(\delta = i) = \overline{F}_i^*(0). \tag{6.4}$$

We remark that from (6.1) and the second of (6.2) one has the following property of the *sub-survival* functions $\overline{F}_i^*(x)$, $i = 1, 2, \dots, n$ that will be used later:

$$\frac{\mathrm{d}}{\mathrm{d}x}\overline{F}_i^*(x) = \frac{\partial}{\partial x_i}\overline{F}(\mathbf{x})\Big|_{x_1=\dots=x_n=x}. \tag{6.5}$$

Such property can be proved as follows (cf. [23], and [146] for an alternative proof). Since $F(\mathbf{x})$ is absolutely continuous, there exists a function $f(x_1, x_2, \dots, x_n)$ such that

$$\overline{F}(\mathbf{x}) = \mathbb{P}(X_1 > x_1, X_2 > x_2, \dots, X_n > x_n) = \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} f(s_1, s_2, \dots, s_n)\mathrm{d}s_1 \dots \mathrm{d}s_n.$$

Therefore,

$$\begin{aligned}
\overline{F}_i^*(x) &= \mathbb{P}(T > x, \delta = i) \\
&= \mathbb{P}(\min(X_1, X_2, \dots, X_n) > x, \delta = i) \\
&= \mathbb{P}(X_k > x, X_i \le X_k \,\forall k) \\
&= \mathbb{P}(X_i > x, X_k \ge X_i \,\forall k) \\
&= \int_x^{+\infty} \left\{ \int_{x_i}^{+\infty} \cdots \int_{x_i}^{+\infty} f(s_1, s_2, \dots, s_n) \prod_{k \neq i} \mathrm{d}s_k \right\} \mathrm{d}s_i \\
&= \int_x^{+\infty} \left\{ -\frac{\partial}{\partial x_i}\overline{F}(\mathbf{x})\Big|_{x_k = x_i \,\forall k} \right\} \mathrm{d}s_i.
\end{aligned}$$

Consequently,

$$-\frac{\mathrm{d}}{\mathrm{d}x}\overline{F}_i^*(x) = -\frac{\partial}{\partial x_i}\overline{F}(\mathbf{x})\Big|_{x_1=\dots=x_n=x}.$$

For $i = 1, 2, \dots, n$ and $x \ge 0$, let us introduce the cause specific hazard rate (CSHR) corresponding to the i-th cause of failure:

$$h_i(x) = \lim_{\tau \to 0^+} \frac{1}{\tau}\,\mathbb{P}(x < T \le x + \tau, \delta = i \,|\, T > x).$$

We observe that $\overline{F}_T(x) = \overline{F}(x, \dots, x)$, $x \ge 0$. In fact, to be alive at time x, all of the potential failure times have to exceed x. From (6.5), the CSHR can be expressed as

$$h_i(x) = -\frac{1}{\overline{F}(x, \dots, x)}\,\frac{\partial}{\partial x_i}\overline{F}(\mathbf{x})\Big|_{x_1=\dots=x_n=x} = -\frac{1}{\overline{F}_T(x)}\,\frac{\partial}{\partial x}\,\overline{F}_i^*(x).$$

Therefore, the following identities hold:

$$F_i^*(x) = \int_0^x h_i(s)\,\overline{F}_T(s)\,\mathrm{d}s, \qquad \overline{F}_i^*(x) = \int_x^{+\infty} h_i(s)\,\overline{F}_T(s)\,\mathrm{d}s, \tag{6.6}$$

for $x \geq 0$ and $i = 1, 2, \ldots, n$. Moreover, the hazard rate of T is given by

$$h_T(x) = -\frac{\mathrm{d}}{\mathrm{d}x} \ln \overline{F}_T(x) = \sum_{i=1}^n h_i(x), \qquad x \geq 0. \tag{6.7}$$

We remark that the sub-survival functions can be expressed as

$$\overline{F}_i^*(x) = \overline{F}_i^*(0)\,\exp\left\{-\int_0^x r_i(s)\,\mathrm{d}s\right\}, \qquad x \geq 0,$$

where, for $x \geq 0$ and $i = 1, 2, \ldots, n$,

$$r_i(x) = -\frac{\mathrm{d}}{\mathrm{d}x}\ln \overline{F}_i^*(x) = \lim_{\tau \to 0^+} \frac{1}{\tau}\,\mathbb{P}\left(x < T \leq x + \tau \,|\, T > x, \delta = i\right). \tag{6.8}$$

Indeed,

$$\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}x}\ln \overline{F}_i^*(x) &= -\frac{1}{\overline{F}_i^*(x)}\frac{\mathrm{d}}{\mathrm{d}x}\overline{F}_i^*(x) \\
&= -\frac{1}{\overline{F}_i^*(x)} \lim_{\tau \to 0^+} \frac{\overline{F}_i^*(x+\tau) - \overline{F}_i^*(x)}{\tau} \\
&= -\frac{1}{\overline{F}_i^*(x)} \lim_{\tau \to 0^+} \frac{\mathbb{P}(T > x+\tau, \delta = i) - \mathbb{P}(T > x, \delta = i)}{\tau} \\
&= \frac{1}{\overline{F}_i^*(x)} \lim_{\tau \to 0^+} \frac{\mathbb{P}(T > x, \delta = i) - \mathbb{P}(T > x+\tau, \delta = i)}{\tau} \\
&= \lim_{\tau \to 0^+} \frac{1}{\tau} \frac{\mathbb{P}(x < T \leq x+\tau,\, \delta = i)}{\mathbb{P}(T > x, \delta = i)} \\
&= \lim_{\tau \to 0^+} \frac{1}{\tau} \frac{\mathbb{P}(x < T \leq x+\tau,\, T > x,\, \delta = i)}{\mathbb{P}(T > x, \delta = i)} \\
&= \lim_{\tau \to 0^+} \frac{1}{\tau}\,\mathbb{P}\left(x < T \leq x+\tau \,|\, T > x, \delta = i\right).
\end{aligned}$$

Moreover, recalling (6.8), if $x \geq 0$,

$$\begin{aligned}
\overline{F}_i^*(0)\, \exp\left\{-\int_0^x r_i(s)\,\mathrm{d}s\right\} &= \overline{F}_i^*(0)\, \exp\left\{\int_0^x \frac{\mathrm{d}}{\mathrm{d}s}\ln \overline{F}_i^*(s)\,\mathrm{d}s\right\} \\
&= \overline{F}_i^*(0)\, \exp\left\{\ln \overline{F}_i^*(s)\Big|_0^x\right\} \\
&= \overline{F}_i^*(0) \exp\left\{\ln \frac{\overline{F}_i^*(x)}{\overline{F}_i^*(0)}\right\} \\
&= \overline{F}_i^*(x).
\end{aligned}$$

We note that, in general, $h_T(x) \not\equiv \sum_{i=1}^n r_i(x)$.
As is well known, the random lifetime T of an item is said to be NBU [NWU] (new better [worse] than used) if $\overline{F}_T(t+x) \leq [\geq] \overline{F}_T(t)\,\overline{F}_T(x)$ for all $x, t \geq 0$. This means that the probability that an item of age t survives for an additional duration x is less [greater] than, or equal to, the probability that a brand new item survives for a duration x, whatever x and t. This ageing notion was extended in [35] to the framework of the competing risks model. Indeed, the random lifetime X_i, $i \in \{1, 2, \dots, n\}$, is NBU* [NWU*] if and only if

$$\overline{F}_i^*(x \,|\, t) \equiv \frac{\overline{F}_i^*(t+x)}{\overline{F}_T(t)} \leq [\geq] \overline{F}_i^*(x) \qquad \forall\, x \geq 0, \;\; t \in \mathcal{T},$$

where $\mathcal{T} = \{s \geq 0 \,:\, \overline{F}_T(s) > 0\}$ and

$$\overline{F}_i^*(x \,|\, t) = \mathbb{P}(T > t + x\,,\, \delta = i \,|\, T > t), \qquad x \geq 0,\, t \in \mathcal{T}.$$

In other words, the probability that an item of age t survives for an additional duration x and that the cause of failure will be the i-th, is less [greater] than or equal to the probability that a brand new item survives for a duration x and that the cause of failure will be the i-th, for all x and t.

6.3 A Mittag-Leffler distribution-based model

Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be positive parameters and let $\Lambda_n := \lambda_1 + \lambda_2 + \cdots + \lambda_n$. With reference to (6.2), we suppose that the i-th sub-distribution function, for $x \geq 0$ and $i \in \{1, 2, \dots, n\}$, reads

$$F_i^*(x) = \mathbb{P}(T \leq x, \delta = i) = \lambda_i x^\nu E_{\nu,\nu+1}(-\Lambda_n x^\nu), \tag{6.9}$$

where $E_{\alpha,\beta}(x)$ is the (two-parameter) Mittag-Leffler function (1.10).

In Fig. 6.1 we show some plots of the sub-distribution functions given in (6.9) in the presence of two competing risks. We generated such plots by making use of the *Mathematica*® built-in function `Plot`. The Mittag-Leffler function with two parameters is implemented in the Wolfram Language as `MittagLefflerE[a,b,z]`. We see that $F_1^*(t)$ $[F_2^*(t)]$ is increasing [decreasing] when λ_1 increases, for fixed t and λ_2. In the following proposition we see that T follows the Mittag-Leffler distribution.

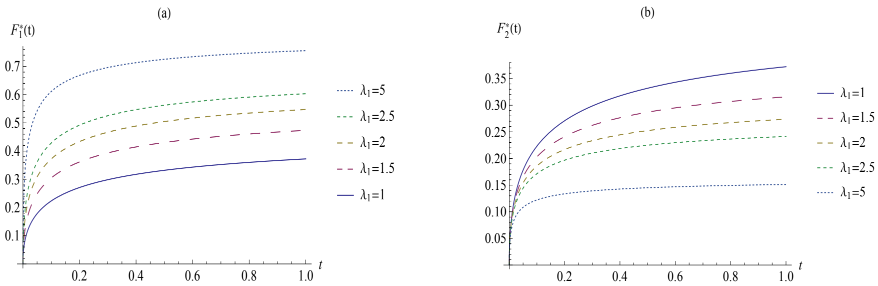

Figure 6.1: Sub-distribution functions given in (6.9), for $n = 2$, $\nu = 0.5$ and $\lambda_2 = 1$: (a) $F_1^*(t)$ and (b) $F_2^*(t)$.

Proposition 6.3.1. *For the model specified in (6.9), the probability of failure due to the i-th risk, and the probability of failure before time x read respectively*

$$\mathbb{P}(\delta = i) = \frac{\lambda_i}{\Lambda_n}, \qquad i \in \{1, 2, \dots, n\}$$

and

$$F_T(x) = \mathbb{P}(T \le x) = \Lambda_n x^\nu E_{\nu,\nu+1}(-\Lambda_n x^\nu), \qquad x \ge 0. \tag{6.10}$$

Proof. Due to the asymptotic behaviour of the Mittag-Leffler function (cf. [135]), i.e.

$$E_{\alpha,\beta}(z) \sim \mathcal{O}\left(|z|^{-1}\right), \qquad |z| > 1,$$

we obtain from (6.9), for $i \in \{1, 2, \dots, n\}$,

$$\mathbb{P}(\delta = i) = \lim_{x\to+\infty} F_i^*(x) = \frac{\lambda_i}{\Lambda_n}. \tag{6.11}$$

In other words, the competing risks are classified as type i via independent Bernoulli trials with probability $\frac{\lambda_i}{\Lambda_n}$, $i \in \{1, 2, \dots, n\}$. Furthermore, from the first of (6.3) it is immediate to obtain the distribution of the observable lifetime (6.10). □

With regard to Proposition 2.2 of [42], the time to failure T has the same distribution as the interarrival time of the fractional growth process with n kinds of jumps

$$M^\nu(t) \overset{d}{=} \sum_{k=1}^{N^\nu_{\Lambda_n}(t)} X_k, \qquad t \geq 0,$$

where $N^\nu_{\Lambda_n}(t)$ is a fractional Poisson process with intensity Λ_n (see [18] for details), and $\{X_k : k \geq 1\}$ is a sequence of independent random variables, independent of $N^\nu_{\Lambda_n}(t)$ given the parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$, and such that, for any positive integer k,

$$X_k \overset{d}{=} X = i, \quad \text{w.p. } \frac{\lambda_i}{\Lambda_n}, \quad i \in \{1, \ldots, n\}. \tag{6.12}$$

As it turns out, the probability of failure due to the i-th risk (6.11) is the same as the probability (6.12) of occurrence of a jump of size i.
We recall here the expression for the q-th moment, $q < \nu$, of the random lifetime T having distribution (6.10). Such formula was derived in [122]:

$$\mathbb{E}[T^q] = \frac{q\pi}{\nu \Lambda_n^{q/\nu} \Gamma(1-q) \sin(q\pi/\nu)}, \qquad q < \nu.$$

We remark that the moments of T of order greater than ν are infinite.
In Fig. 6.2 we show some plots of the distribution function $F_T(x)$ given in (6.10) and of the corresponding survival function,

$$\overline{F}_T(x) = \mathbb{P}(T > x) = 1 - \Lambda_n x^\nu E_{\nu,\nu+1}(-\Lambda_n x^\nu), \qquad x \geq 0, \tag{6.13}$$

when $n = 2$. The expression for the sub-survival functions given in the second

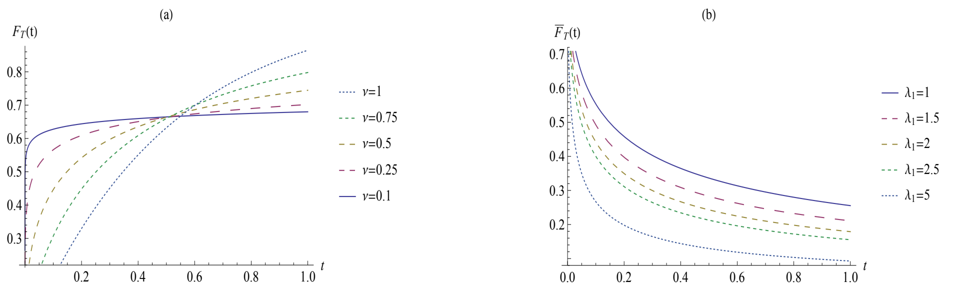

Figure 6.2: Distribution functions (a) given in (6.10) for $\lambda_1 = \lambda_2 = 1$ and survival functions (b) given in (6.13), for $\nu = 0.5$ and $\lambda_2 = 1$, $n = 2$.

of (6.2) can be easily derived. In fact, from the law of total probability we can express the probability of failure due to the i-th risk as the sum of the i-th sub-distribution function and the i-th sub-survival function, i.e.

$$F_i^*(x) + \overline{F}_i^*(x) = \mathbb{P}(\delta = i), \qquad i = 1, 2, \ldots, n,$$

so that, from (6.11) and (6.9),

$$\overline{F}_i^*(x) = \frac{\lambda_i}{\Lambda_n} - \lambda_i x^\nu E_{\nu,\nu+1}(-\Lambda_n x^\nu), \qquad i = 1, 2, \dots, n. \tag{6.14}$$

In Fig. 6.3 some plots of the sub-distribution and sub-survival functions given in (6.9) and (6.14) are shown.

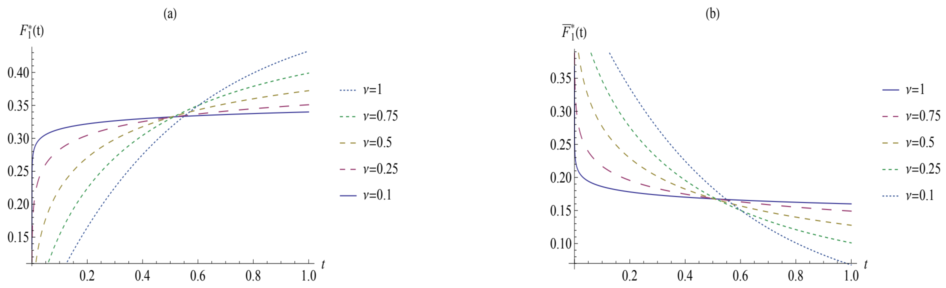

Figure 6.3: Sub-distribution and sub-survival functions given in (6.9) and (6.14) respectively, when $n = 2$, for $\lambda_1 = \lambda_2 = 1$, (a) $F_1^*(t) = F_2^*(t)$ and (b) $\overline{F}_1^*(t) = \overline{F}_2^*(t)$.

Proposition 6.3.2. *For the model (6.9), the cause specific hazard rate corresponding to the i-th cause of failure, $i \in \{1, 2, \dots, n\}$, and the overall hazard rate from all causes read, for $x \geq 0$, respectively*

$$h_i(x) = \frac{\lambda_i x^{\nu-1} E_{\nu,\nu}(-\Lambda_n x^\nu)}{1 - \Lambda_n x^\nu E_{\nu,\nu+1}(-\Lambda_n x^\nu)} \tag{6.15}$$

and

$$h_T(x) = \frac{\Lambda_n x^{\nu-1} E_{\nu,\nu}(-\Lambda_n x^\nu)}{1 - \Lambda_n x^\nu E_{\nu,\nu+1}(-\Lambda_n x^\nu)}. \tag{6.16}$$

Proof. The expression for the cause specific hazard rate corresponding to the i-th cause of failure is derived owing to the first of (6.6), to (6.9) and (6.13) and applying the fundamental theorem of calculus. From (6.7), expression (6.16) easily follows. □

In Fig. 6.4 and in Fig. 6.5 some plots of the hazard rates (6.16) and of the cause specific hazard rates (6.15) are displayed, limited to the case of two competing risks. We now focus on the independence between the failure time and the cause of failure. Indeed, the following proposition holds.

Proposition 6.3.3. *With respect to the model (6.9), the observable lifetime T and the cause of failure δ prove to be independent.*

Proof. Owing to (6.10) and (6.11), the i-th sub-distribution function (6.9) $F_i^*(x)$ factorizes as $F_T(x)\mathbb{P}(\delta = i)$, $i \in \{1, 2, \dots, n\}$, $x \geq 0$.

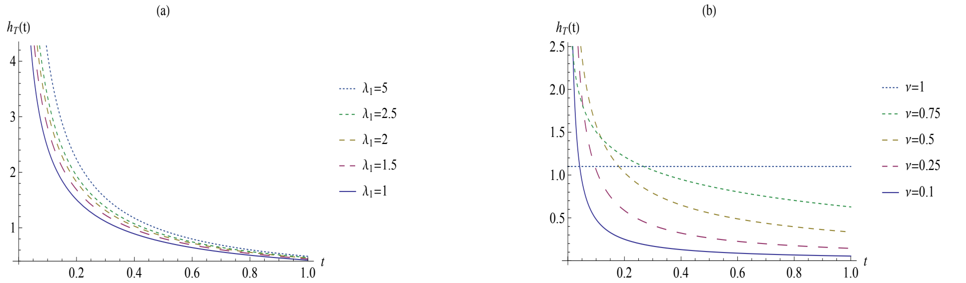

Figure 6.4: Hazard rates given in (6.16), when $n = 2$, for (a) $\nu = 0.5$ and $\lambda_2 = 1$, (b) $\lambda_1 = \lambda_2 = 1$.

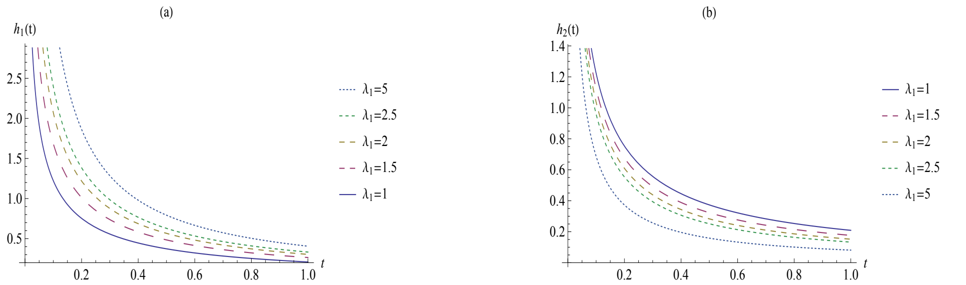

Figure 6.5: Cause specific hazard rates given in (6.15), for $n = 2$, $\nu = 0.5$ and $\lambda_2 = 1$: (a) $h_1(t)$ and (b) $h_2(t)$.

Equivalently, we can observe that T and δ are independent since the cause specific hazard rates $h_1(x), \ldots, h_n(x)$ of $X_1, \ldots, X_n$, whose expression is provided by (6.15), are proportional to each other. □

Remark 6.3.1. With reference to Remark 2.1 of [35], the independence of T and δ ensures that $r_1(t) = r_2(t) = \cdots = r_n(t) = h_T(t)$ for all $t \geq 0$, where $r_i(t)$, $i = 1, 2, \ldots, n$, and $h_T(t)$ have been expressed in (6.8) and (6.16), respectively.

Remark 6.3.2. Crossing hazard rate functions are interesting for survival analysis. Indeeed, one is interested in comparing two hazard rate functions for evaluating a treatment effect over a period of time. In relation to the model (6.9), from (6.16) we have that the $h_T(x)$'s intersect when the parameter ν varies (cf. Fig. 6.4(b)). To fix ideas, let $0 < \nu_1 < \nu_2 \leq 1$ and let $h_T^1(x)$ and $h_T^2(x)$ be the two corresponding hazard rates, that is to say:

$$h_T^1(x) = \frac{\Lambda_n x^{\nu_1 - 1} E_{\nu_1, \nu_1}(-\Lambda_n x^{\nu_1})}{1 - \Lambda_n x^{\nu_1} E_{\nu_1, \nu_1 + 1}(-\Lambda_n x^{\nu_1})}, \qquad x \geq 0,$$

and

$$h_T^2(x) = \frac{\Lambda_n x^{\nu_2 - 1} E_{\nu_2, \nu_2}(-\Lambda_n x^{\nu_2})}{1 - \Lambda_n x^{\nu_2} E_{\nu_2, \nu_2 + 1}(-\Lambda_n x^{\nu_2})}, \qquad x \geq 0.$$

By resorting to the asymptotic representation of the Mittag-Leffler function as $t \to 0$

and $t \to +\infty$ (cf. [144] and Remark 2.3 of [18]), we have the following:

$$\psi(x) := h_T^2(x) - h_T^1(x) \sim \begin{cases} \dfrac{\Lambda_n}{x^{1-\nu_2}}\left(\dfrac{1}{\Gamma(\nu_2)} - \dfrac{1}{\Gamma(\nu_1)\, x^{\nu_2-\nu_1}}\right), & x \to 0^+, \\ \dfrac{\nu_2 - \nu_1}{x}, & x \to +\infty. \end{cases}$$

Due to continuity and Bolzano's Theorem, since $\psi(0^+) = -\infty$ and $\psi(x) \to 0^+$ as $x \to +\infty$, it follows that $\psi(x)$ must be 0 at some point, this showing that $h_T^1(x)$ and $h_T^2(x)$ intersect at least once.

We conclude this section with an interesting result concerning an ageing notion of the random lifetimes $X_1, X_2, \ldots, X_n$.

Proposition 6.3.4. *As for the model (6.9), the random lifetimes* $X_1, X_2, \ldots, X_n$ *are NWU*.*

Proof. As pointed out in [39], the time to failure T has the decreasing likelihood ratio (DLR) property. Therefore, the random lifetime T belongs to the NWU class too. Hence, since T and δ are independent, the result straightforwardly holds due to Theorem 3.2 of [35]. □

6.4 Identifiability problem under copula and TTE models

This section is devoted to the identification of the underlying distribution of latent failure times and is inspired by the work of [69]. Identifiability, however, is not a recent topic (see [138] and [148] and references therein to get an insight into the problem). We highlight that the results presented hereafter have general validity and that in Subsection 6.4.1 we will show an application to the model (6.9). Specifically, we now tackle the problem of evaluating the so-called *net* survival functions $\overline{F}_i(x) := \mathbb{P}(X_i > x)$, for $x \geq 0$ and $i = 1, \ldots, n$. We note that $\overline{F}_i(x)$ is the marginal survival function, due to i-th cause alone, associated with the joint multivariate survival function (6.1). We will obtain estimates of the net survival functions $\overline{F}_i(x)$ on the basis of the sub-survival functions $\overline{F}_i^*(x)$ defined in (6.2) by solving a system of non-linear differential equations, which connects the two sets of functions. The downside is that we need to impose a certain dependence structure to characterize the joint distribution of the random vector $\mathbf{X} = (X_1, \ldots, X_n)$. One way to do so is, for instance, to use copulas, as outlined in Theorem 6 of [23]. In probabilistic terms, a function $C : [0,1]^n \to [0,1]$ is a n-dimensional copula if C is a joint cumulative distribution function of a n-dimensional random vector on the unit cube $[0,1]^n$ with uniform marginals. The copula C contains all information on the dependence structure between the components of the random vector, whereas the marginal cumulative distribution functions contain all information on the marginal distributions. Let us fix a copula function $C(u_1,\ldots,u_n)$ for the joint distribution of $\mathbf{X}$.

Theorem. *If $C(u_1,\ldots,u_n)$ is differentiable with respect to $u_i \in (0,1)$ and $\overline{F}_i(x_i)$ is differentiable with respect to $x_i > 0$ for all $i = 1,\ldots,n$, then*

$$\frac{\mathrm{d}}{\mathrm{d}x}\overline{F}_i^*(x) = C_i[\overline{F}_1(x),\ldots,\overline{F}_n(x)]\,\frac{\mathrm{d}}{\mathrm{d}x}\overline{F}_i(x), \tag{6.17}$$

where $C_i(u_1,\ldots,u_n) = \frac{\partial}{\partial u_i}C(u_1,\ldots,u_n)$.

In order to evaluate the functions of interest, we only need to specify a suitable copula, assign a value to its parameters, give estimates of the sub-survival functions and then solve the system numerically.
Since we focus on the bivariate case in the sequel, we first define in analytic terms a two-dimensional copula.

Definition. A *copula* is a function $C : [0,1]^2 \to [0,1]$ with the following properties:

1. For every $x, y \in [0,1]$,
$$C(x,0) = 0 = C(0,y)$$
and
$$C(x,1) = u \ \text{ and } \ C(1,y) = y;$$
2. For every $x_1, x_2, y_1, y_2 \in [0,1]$ such that $x_1 \le x_2$ and $y_1 \le y_2$,
$$C(x_2,y_2) - C(x_2,y_1) - C(x_1,y_2) + C(x_1,y_1) \ge 0.$$

We now aim to show a case in which X_1 and X_2 turn out to be identically distributed. Let $C = C(x,y)$ be a bivariate copula. The following result ensures the existence of the partial derivatives $\partial C(x,y)/\partial x$ and $\partial C(x,y)/\partial y$ for almost all x and y, respectively.

Theorem (Theorem 2.2.7 of [103])**.** *Let C be a copula. For any $y \in [0,1]$, the partial derivative $\partial C(x,y)/\partial x$ exists for almost all x, and for such y and x,*

$$0 \le \frac{\partial}{\partial x}C(x,y) \le 1.$$

Similarly, for any $x \in [0,1]$, the partial derivative $\partial C(x,y)/\partial y$ exists for almost all y, and for such x and y,

$$0 \leq \frac{\partial}{\partial y} C(x,y) \leq 1.$$

Furthermore, the functions $x \mapsto \partial C(x,y)/\partial y$ and $y \mapsto \partial C(x,y)/\partial x$ are defined and nondecreasing almost everywhere on $[0,1]$.

We are interested, as we shall see later, in the equality of such derivatives, which is attained if and only if $C(x,y) = C(x+y)$. However, the class of copulas satisfying this property is quite restricted, as the following lemma shows.

Lemma 6.4.1. *The Fréchet-Hoeffding lower bound*

$$C(x,y) = \max\{0, x+y-1\}, \qquad 0 \leq x, y \leq 1, \tag{6.18}$$

is the only copula which depends exclusively on the sum of its arguments.

Proof. It follows directly from the properties which define a copula. □

If we choose the copula function (6.18) in system (6.17), under the hypothesis of independence of T and δ, the distributions of the risks X_i, $i = 1, 2$, prove to be identical, as is shown in the following Proposition.

Proposition 6.4.1. *Let the joint distribution of $\mathbf{X} = (X_1, X_2)$ be governed by the Fréchet-Hoeffding copula (6.18). If T, the observable lifetime, and δ, the cause of failure, are independent, then X_1 and X_2 are identically distributed.*

Proof. Due to the hypotesis of independence, we have $\overline{F}_i^*(x) = \overline{F}_i^*(0)\overline{F}_T(x)$, $i = 1, 2$. From (6.17), by observing that, due to Lemma 6.4.1, the partial derivatives of (6.18) coincide, we have

$$\frac{\overline{F}_1(x)}{\overline{F}_2(x)} = \frac{\overline{F}_1^*(0)}{\overline{F}_2^*(0)}, \qquad x \geq 0.$$

By taking $x = 0$ in the latter identity, the thesis immediately follows. □

Another way to impose a dependence structure is to assume that the random lifetimes X_1 and X_2 follow the time transformed exponential (TTE) model. Specifically, we assume that the joint survival function of X_1 and X_2 may be expressed in the following way:

$$\overline{F}(\mathbf{x}) = \mathbb{P}(X_1 > x_1, X_2 > x_2) = \overline{W}[R_1(x_1) + R_2(x_2)], \qquad x_1, x_2 \geq 0, \tag{6.19}$$

where $\overline{W} : [0, +\infty) \to [0,1]$ is a continuous, convex, and strictly decreasing survival function, such that $\overline{W}(0) = 1$ and $\lim_{x \to +\infty} \overline{W}(x) = 0$, and where $R_i : [0, +\infty) \to$

$[0, +\infty)$ is a continuous and strictly increasing function, such that $R_i(0) = 0$ and $\lim_{x \to +\infty} R_i(x) = +\infty$ for $i = 1, 2$. Functions $\overline{W}$ and R_i, $i = 1, 2$, provide the time transform and the accumulated hazards, respectively. The marginal survival functions are given by

$$\overline{F}_1(x_1) = \overline{W}[R_1(x_1)], \ x_1 \geq 0, \qquad \overline{F}_2(x_2) = \overline{W}[R_2(x_2)], \ x_2 \geq 0. \tag{6.20}$$

The classical TTE model refers to the case in which the accumulated hazards R_1 and R_2 in Eq. (6.19) are identical (cf. [11] and [121] for further details on this model and its applications). Nevertheless, similarly as in [44], here we consider the more general case of unequal accumulated hazards, so that X_1 and X_2 have different marginal survival functions due to (6.20). As in [44], the following notation will be used for the model (6.19): $\mathbf{X} \sim \mathrm{TTE}(\overline{W}, R_1, R_2)$. The TTE model is important in survival analysis, in that it allows us to separate dependence from ageing properties. For the TTE model we can prove a result analogous to (6.17). Indeed, the following theorem holds.

Theorem 6.4.1. *For the TTE model (6.19), if $\overline{W}(x)$ is differentiable with respect to x, $x \in [0, +\infty)$, and $R_i(x)$ is differentiable with respect to $x > 0$ for $i = 1, 2$, then*

$$\frac{\mathrm{d}}{\mathrm{d}x} \overline{F}_i^*(x) = \overline{W}'[R_1(x) + R_2(x)] \frac{\mathrm{d}}{\mathrm{d}x} R_i(x), \tag{6.21}$$

where $\overline{W}'(x) = \frac{\mathrm{d}}{\mathrm{d}x} \overline{W}(x)$.

Proof. From (6.5) and (6.19), along with the chain rule, we straightforwardly obtain the result after having set $x_1 = x_2 = x$. □

Again, (6.21) gives a nonlinear system of two differential equations where the functions $R_i(x)$ can be solved if the time transform $\overline{W}(x)$ and the sub-survival functions are specified. In Subsection 6.4.1 we consider the case when the sub-survival functions are provided by (6.14).

With reference to Theorem 6.4.1, arguments similar to those of Proposition 6.4.1 can be used to prove the following Corollary.

Corollary 6.4.1. *For the TTE model (6.19), if T and δ are independent, then $R_1(x)$ and $R_2(x)$ are proportional, i.e.*

$$R_1(x) = \frac{\overline{F}_1^*(0)}{\overline{F}_2^*(0)} R_2(x), \qquad x \geq 0.$$

We now prove an interesting result concerning families of survival functions depending on a rate parameter $c > 0$.

Theorem 6.4.2. *Let $\overline{W}_c$, $c > 0$, be a parametric family of continuous, convex and strictly decreasing survival functions such that $\overline{W}_c(x) = S(cx)$, $x \geq 0$, for a proper survival function S. If $\mathbf{X}$ satisfies two versions of the TTE model (6.19), namely $\mathbf{X} \sim \mathrm{TTE}(\overline{W}_c, R_1, R_2)$ and $\mathbf{X} \sim \mathrm{TTE}(\overline{W}_{\tilde{c}}, \tilde{R}_1, \tilde{R}_2)$, where $c \neq \tilde{c}$ and the pairs of accumulated hazards (R_1, R_2) and $(\tilde{R}_1, \tilde{R}_2)$ are possibly different, then the marginal survival functions of $\mathbf{X}$ are independent of the rate parameter c.*

Proof. For two positive real numbers c and $\tilde{c}$, such that $c \neq \tilde{c}$, from Theorem 6.4.1 we have for $i = 1, 2$ and $x \geq 0$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\overline{F}_i^*(x) &= S'(c[R_1(x) + R_2(x)])\, c\, \frac{\mathrm{d}}{\mathrm{d}x}R_i(x), \\ \frac{\mathrm{d}}{\mathrm{d}x}\overline{F}_i^*(x) &= S'(\tilde{c}[\tilde{R}_1(x) + \tilde{R}_2(x)])\, \tilde{c}\, \frac{\mathrm{d}}{\mathrm{d}x}\tilde{R}_i(x). \end{aligned} \tag{6.22}$$

By combining these two expressions, we get

$$\frac{\mathrm{d}}{\mathrm{d}x}\, S(c[R_1(x) + R_2(x)]) = \frac{\mathrm{d}}{\mathrm{d}x}\, S(\tilde{c}[\tilde{R}_1(x) + \tilde{R}_2(x)]),$$

and then, recalling that S is a proper survival function, one gets

$$c[R_1(x) + R_2(x)] = \tilde{c}[\tilde{R}_1(x) + \tilde{R}_2(x)].$$

From these facts, and making use of system (6.22), we can infer that

$$cR_i(x) = \tilde{c}\tilde{R}_i(x), \qquad i = 1, 2. \tag{6.23}$$

Under the given assumption, the marginal survival functions associated with $\overline{W}_c(x)$ and $\overline{W}_{\tilde{c}}(x)$ are given, for $i = 1, 2$, respectively by

$$\overline{F}_{i,c}(x) = \overline{W}_c(R_i(x)) = S(cR_i(x))$$

and

$$\overline{F}_{i,\tilde{c}}(x) = \overline{W}_{\tilde{c}}(\tilde{R}_i(x)) = S(\tilde{c}\tilde{R}_i(x)).$$

From (6.23) we thus have $\overline{F}_{i,c}(x) = \overline{F}_{i,\tilde{c}}(x)$, and the proof follows. □

6.4.1 Special cases

In this subsection we solve the systems (6.17) and (6.21) by adopting a numerical approach, having specified the copula function C and the time transform $\overline{W}$, respectively. With regard to model (6.9), the sub-survival functions are provided by (6.14), where the value of the parameters are set as follows: $\lambda_1 = 1$, $\lambda_2 = 3$, $\nu = 0.7$. We make use of the *Mathematica*® built-in function `NDSolve`, which solves numerically systems of differential equations and produces solutions $\overline{F}_1(x)$ and $\overline{F}_2(x)$, and $R_1(x)$ and $R_2(x)$.

As far as the system (6.17) is concerned, the choice of the right copula is a delicate task. Generally, one takes a parametric family of copulas among many existing others and fit it to the data by estimating the parameters of the family. Usually, such parameters control the strength of dependence between the variables of interest. In a competing risks setting, however, the estimation of the copula parameter(s) is not possible, since we do not have a set of pairwise observations of the failure times X_1 and X_2, but only one observable failure time, i.e. $\min(X_1, X_2)$. In our study, such parameters will be considered free, but in general they could be deduced from available knowledge about the degree of pairwise association between the two competing risks, expressed, for example, in terms of Kendall's τ. We will make use of *comprehensive* copulas, that is to say copulas that can capture the various degrees of association between the failure times X_1 and X_2, from extreme positive to extreme negative dependence. In particular, for the model (6.9), with reference to the representation (6.17), we explore the Gaussian copula, the Clayton copula and the Plackett copula as alternatives, since they belong to different families with different properties (cf., for instance, [103]). The Gaussian copula allows us to create a family of bivariate normal distributions with a specified correlation coefficient. It belongs to the class of Elliptical copulas, which are the copulas of elliptical distributions. The class of elliptical distributions provides a source of multivariate distributions which share many of the tractable properties of the multivariate normal distribution and enables modelling of multivariate extremes and other forms of nonnormal dependences. Gaussian copulas do not have neither upper nor lower tail dependence. However, elliptical copulas do not have closed form expressions and are restricted to have radial symmetry. The Clayton copula belongs to the class of Archimedean copulas, which are characterized by a suitable generator. We chose this copula in the set of the multivariate Archimedean copulas because it is easy to compute. Moreover, the Gaussian copula tends to form elliptic groups, whereas the copula of Clayton will tend to form groups "with pear shape", this being due to the property of lower tail dependence. The Plackett copula is constructed from the Plackett family of distributions. It is neither Archimedean nor Elliptical, and it has no tail dependence. In Fig. 6.6 we show some plots of the net survival functions $\overline{F}_1(x)$ and $\overline{F}_2(x)$, $x \geq 0$, corresponding to the following values of Kendall's τ: 0.85, 0.35, -0.35 and -0.85. As for the TTE model (6.19), in Figs 6.7, 6.8 and 6.9 we show some plots of the net survival functions $\overline{F}_1(x)$ and $\overline{F}_2(x)$, $x \geq 0$, obtained via numerical treatment of system (6.21), corresponding to three different choices of the time transform $\overline{W}(x)$: a power law, the Gompertz law and the exponential law respectively, in order to modulate different dependence properties. The power law leads to a proportional hazard model with a Gamma distribution as mixing distribution; the Gompertz law may be used to model negative dependence and the exponential law leads to independent laws (cf. [46]). The functions R_1 and R_2 have been determined numerically, and then the corresponding analytical expressions are not available.

6.5 Random number of competing risks

In the present section we consider a more general setting within which the failure of the subject is due to a random number of independent competing risks. This situation is of interest, among other things, in finance and biomedical studies. Indeed, Artikis and Artikis and Artikis et al. proposed in [4] and in [5] respectively, stochastic discounting models providing risk managers and analysts with valuable information for making optimal decisions in the environment of a random number of independent, competing and catastrophic risks. In the same environment, Balakrishnan et al., [6] and [7], considered a cure rate model and analyzed a real data set on cutaneous melanoma.

Let us now turn to the mathematical structure of the model. We suppose that the failure of an item is subject to a random number N of independent competing risks, with N taking values in $\mathcal{S}$, where $\mathcal{S} \subseteq \mathbb{N}$. In this case, we shall refer to the observable pairs (T_N, δ_N), where T_N is the time of failure of the item and δ_N describes the cause or type of failure, in the presence of a random number N of causes. We again assume that failure may be due to a single cause. The distributions of T_N and δ_N conditional on $N = n$ are identical to those of the first one of (6.3) and (6.4), respectively. Hence, the distribution function of T_N and the probability mass function of δ_N can be expressed respectively as follows:

$$\mathbb{P}(T_N \leq x) = \sum_{n \in \mathcal{S}} \mathbb{P}(T_n \leq x)\mathbb{P}(N = n), \qquad x \geq 0,$$
$$\mathbb{P}(\delta_N = i) = \sum_{n \geq i;\, n \in \mathcal{S}} \mathbb{P}(\delta_n = i)\mathbb{P}(N = n), \qquad i \in \mathcal{S},$$

where T_n and δ_n refer to the case of n fixed causes. As for the fractional model presented in Section 6.3, recalling Proposition 6.3.1 we thus have

$$\mathbb{P}(T_N \leq x) = \sum_{n \in \mathcal{S}} \Lambda_n x^{\nu} E_{\nu,\nu+1}(-\Lambda_n x^{\nu})\mathbb{P}(N = n), \qquad x \geq 0, \tag{6.24}$$

and

$$\mathbb{P}(\delta_N = i) = \sum_{n \geq i;\, n \in \mathcal{S}} \frac{\lambda_i}{\Lambda_n} \mathbb{P}(N = n), \qquad i \in \mathcal{S}. \tag{6.25}$$

Specifically, we point out that the distribution function (6.24) turns out to be a mixture distribution. Recalling that the distribution of T_n is DLR, and that the DLR property is closed under mixtures (cf. [12]), from (6.24) we immediately have that T_N is DLR, and thus NWU, too.
We now present some examples by specializing the probability mass function of N and with a suitable choice of the parameters λ_i. In fact, we set $\lambda_i := \lambda i$, for $i \in \mathcal{S}$, this being of interest since, in general, λi represents the hazard rate of a series system with i independent and exponentially distributed components, each with parameter λ. One gets $\Lambda_n = \lambda_1 + \cdots + \lambda_n = \lambda\,(1 + \cdots + n) = \lambda\,\dfrac{n(n+1)}{2}$.

Example 6.5.1. *(Discrete uniform distribution.)* We have, for $\mathcal{S} = \{1, \dots, n\}$,

$$\mathbb{P}(N = h) = \frac{1}{n}, \qquad h \in \{1, \dots, n\},$$

so that from (6.24) and (6.25)

$$\mathbb{P}(T_N \leq x) = \frac{\lambda}{2n} \sum_{h=1}^{n} h(h+1) x^{\nu} E_{\nu,\nu+1}\left(-\lambda \frac{h(h+1)}{2} x^{\nu}\right), \qquad x \geq 0,$$

and

$$\begin{aligned}\mathbb{P}(\delta_N = i) &= \frac{2i}{n} \sum_{h=i}^{n} \frac{1}{h(h+1)} \\ &= 2\,\frac{n+1-i}{n(n+1)}, \qquad i \in \{1, \dots, n\}.\end{aligned}$$

Example 6.5.2. *(Truncated geometric distribution.)* We have, for $\mathcal{S} = \{1, \dots, n\}$,

$$\mathbb{P}(N = h) = \frac{p(1-p)^{h-1}}{1-(1-p)^n}, \qquad h \in \{1, \dots, n\},\ p \in (0,1).$$

Expressions (6.24) and (6.25) become respectively, for $x \geq 0$,

$$\mathbb{P}(T_N \leq x) = \frac{\lambda p}{2\,[1-(1-p)^n]} \sum_{h=1}^{n} h(h+1)(1-p)^{h-1} x^{\nu} E_{\nu,\nu+1}\left(-\lambda \frac{h(h+1)}{2} x^{\nu}\right)$$

and

$$\mathbb{P}(\delta_N = i) = \frac{2ip}{1-(1-p)^n} \sum_{h=i}^{n} \frac{(1-p)^{h-1}}{h(h+1)}, \qquad i \in \{1, \dots, n\}.$$

Example 6.5.3. *(Fractional Poisson distribution.)* With reference to Remark 2.5

of [17], we assume that the random number of competing risks depends on time and is represented by a fractional Poisson process $N_\nu^\lambda(t)$ with intensity λ, so that

$$\mathbb{P}(N_\nu^\lambda(t) = k) = \frac{1}{k!}\frac{\mathrm{d}^k}{\mathrm{d}s^k}\left[s^{k-1}E_{\nu,1}\left(-\frac{\lambda t^\nu}{s}\right)\right]_{s=1}, \qquad k \geq 1,$$

where $E_{\alpha,\beta}(x)$ is the Mittag-Leffler function (1.10). Therefore, the competing causes happen to be countably infinite. If $X_1, \ldots, X_{N_\nu^\lambda(t)}$ are i.i.d. random variables with probability distribution function $F(z)$ describing the lifetime of the subject when its failure is due to the i-th risk, then, for $z \geq 0$,

$$\mathbb{P}\left(T_{N_\nu^\lambda(t)} < z\right) = \mathbb{P}\left(\min_{1\leq j\leq N_\nu^\lambda(t)} X_j < z \,\middle|\, N_\nu^\lambda(t) \geq 1\right) = \frac{1 - E_{\nu,1}(-\lambda t^\nu F(z))}{1 - E_{\nu,1}(-\lambda t^\nu)},$$

and

$$\mathbb{P}\left(\delta_{N_\nu^\lambda(t)} = i\right) = \sum_{n\geq i}\frac{2i}{n(n+1)}\mathbb{P}(N_\nu^\lambda(t) = n), \qquad i \geq 1.$$

6.5.1 Estimates and simulation results

In conclusion, in this section we develop a procedure for estimating the parameters of the mixture distribution (6.24) of the random lifetime T. We adapt the approach based on fractional moments proposed in [79]. Specifically, it is meaningful to give an accurate estimate of the probabilities (6.25) since $\mathbb{P}(\delta_N = i)$ is essential to assess the model with a random number of causes.
For simplicity's sake, we consider a situation where a unit can fail due to up to three competing causes, i.e. $\mathcal{S} = \{1, 2, 3\}$. The probability density function and the fractional moments of T_N read respectively

$$f_{T_N}(t) = \sum_{n=1}^{3} p_n \Lambda_n x^{\nu-1} E_{\nu,\nu}(-\Lambda_n x^\nu) \tag{6.26}$$

and

$$E[T_N^q] = \frac{q\pi}{\nu\Gamma(1-q)\sin(q\pi/\nu)}\sum_{n=1}^{3}\frac{p_n}{\Lambda_n^{q/\nu}}, \qquad q < \nu, \tag{6.27}$$

where $p_n = \mathbb{P}(N = n)$, $n = 1, 2, 3$.

Example 6.5.4. In order to perform a statistical analysis, we simulate a random sample of size 10^4 from distribution (6.26). Along the lines of [79], this is done by simulating each of the 3 components of the mixture by taking into account that the Mittag-Leffler distribution can be equivalently represented as a scale mixture of exponential distributions. To this aim we set $\lambda_1 = 1$, $\lambda_2 = 5$, $\lambda_3 = 10$, $\nu = 0.75$,

$p_1 = 0.6$ and $p_2 = 0.3$.
A chi-square goodness of fit test at the 0.05 significance level, considering 10 classes, has been also conducted in order to compare the observed sample distribution with the theoretical density (6.26) having the parameter values assigned as before. The value of the test statistic turns out to be 3.702, which is less than the critical value $\chi^2_{0.05;3} = 7.815$, so that the data are consistent with the theoretical density (6.26). The results of the simulation are presented in Fig. 6.10, where the histogram provided by the simulated data is compared with the theoretical density (6.26).
Moreover, in order to perform the analysis in the presence of an unknown source of randomness, we assume that for each observation the parameter ν is perturbed from uniform noise, so that it is sampled independently, uniformly in the interval $[0.55, 0.95]$. Formula (6.27) allows us to exploit the special version of the method of moments estimators, involving the fractional moments, proposed in [79], for the unknown parameters, i.e. λ_1, λ_2, λ_3, ν, and the probabilities p_1 and p_2. In order to apply such method, we choose six values $q_i = \left(\frac{1}{2}\right)^{2i-1}$, $i = 1, \ldots, 6$, representing the order of the moments. Furthermore, the estimates of the parameters have been obtained by replacing (6.27) with its sample counterpart and solving the resulting equations with the MATLAB® function `lsqnonlin`, which is suitable for nonlinear least-squares problems. The estimates of the parameters are shown in the second row of Table 6.1.

Table 6.1: Parameter values

	λ_1	λ_2	λ_3	ν	p_1	p_2
Assigned parameters	1	5	10	0.75	0.6	0.3
Estimated values	0.9856	4.9990	9.9998	0.7580	0.6051	0.2668

For completeness, we remark that a Mittag-Leffler random number can be expressed through a suitable inversion formula as follows (see Kozubowski and Rachev [80]):

$$\tau_\nu = -\gamma_t \log u \left(\frac{\sin(\nu\pi)}{\tan(\nu\pi z)} - \cos(\nu\pi) \right)^{\frac{1}{\nu}},$$

where $u, z \in (0, 1)$ are independent uniform random numbers, γ_t is the scale parameter, and τ_ν is a Mittag-Leffler random number. Fulger et al. [53] found it numerically convenient to use Mittag-Leffler random numbers generated according to the previous equation in the Monte Carlo simulation of uncoupled continuous-time random walks. Moreover, Mittag-Leffler random numbers can be generated by means of the MATLAB® routine:

Germano, Guido, et al. Mittag-Leffler random number generator.
https://it.mathworks.com/matlabcentral/fileexchange/19392-mittag-leffler-random-number-generator
MATLAB Central File Exchange. Updated April 04, 2016.

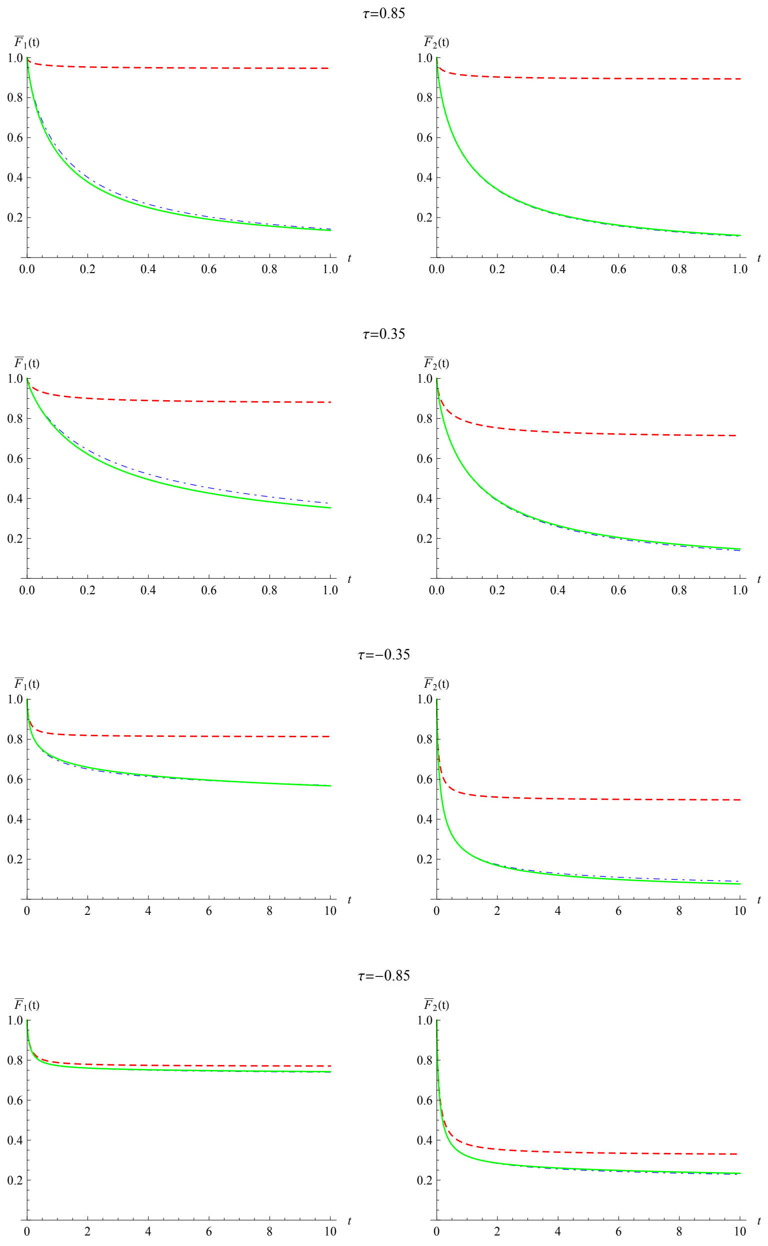

Figure 6.6: Survival functions $\overline{F}_1(x)$ (left panel) and $\overline{F}_2(x)$ (right panel), $x \geq 0$, for the sub-survival functions (6.14), with $\lambda_1 = 1$, $\lambda_2 = 3$, $\nu = 0.7$, and corresponding to the Plackett copula (dashed line), to the Gaussian copula (dot-dashed line) and to the Clayton copula (continuous line) within model (6.17).

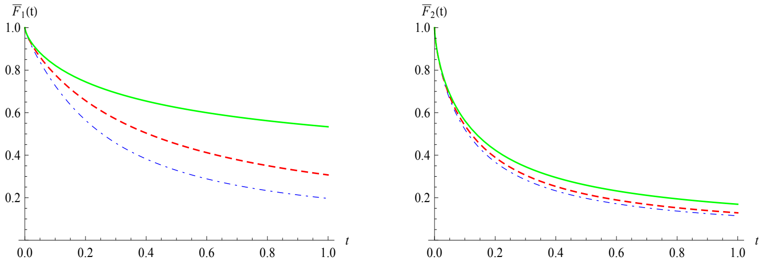

Figure 6.7: Survival functions $\overline{F}_1(x)$ (left panel) and $\overline{F}_2(x)$ (right panel), $x \geq 0$, for the sub-survival functions (6.14), with $\lambda_1 = 1$, $\lambda_2 = 3$, $\nu = 0.7$, and corresponding to the time transform $\overline{W}(x) = \frac{1}{(1+x)^c}$, with $c = 0.5$ (dot-dashed line), $c = 1$ (dashed line) and $c = 10$ (continuous line) within model (6.19).

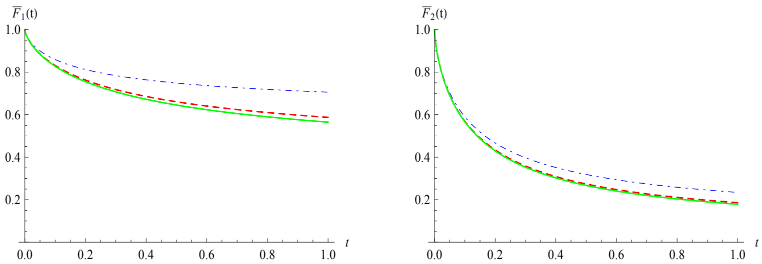

Figure 6.8: Survival functions $\overline{F}_1(x)$ (left panel) and $\overline{F}_2(x)$ (right panel), $x \geq 0$, for the sub-survival functions (6.14), with $\lambda_1 = 1$, $\lambda_2 = 3$, $\nu = 0.7$, and corresponding to the time transform $\overline{W}(x) = e^{-\eta(e^x - 1)}$, with $\eta = 1$ (dot-dashed line), $\eta = 10$ (dashed line) and $\eta = 100$ (continuous line) within model (6.19).

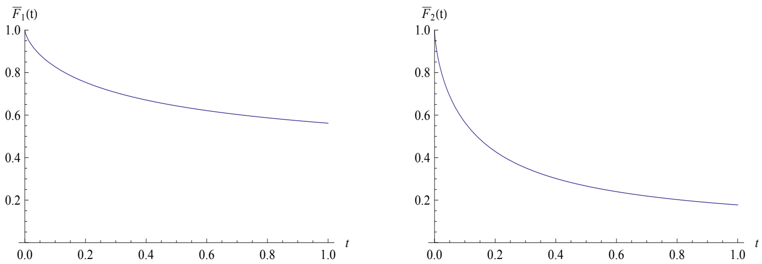

Figure 6.9: Survival functions $\overline{F}_1(x)$ (left panel) and $\overline{F}_2(x)$ (right panel), $x \geq 0$, for the sub-survival functions (6.14), with $\lambda_1 = 1$, $\lambda_2 = 3$, $\nu = 0.7$, and corresponding to the time transform $\overline{W}(x) = e^{-x}$ within model (6.19).

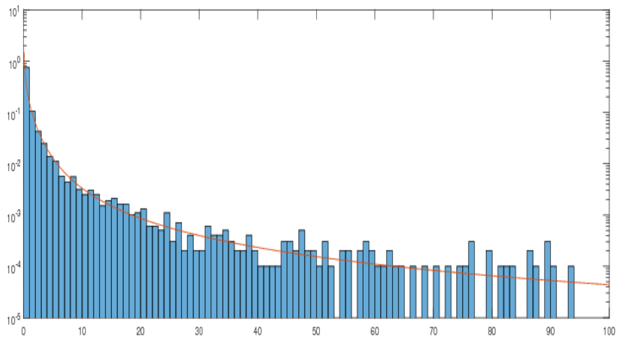

Figure 6.10: Theoretical density (6.26) for $\lambda_1 = 1$, $\lambda_2 = 5$, $\lambda_3 = 10$, $\nu = 0.75$ and $(p_1, p_2, p_3) = (0.6, 0.3, 0.1)$, and histogram of the simulated sample of size 10000.

Conclusions and future developments

In the present thesis we explored some connections between Probability Theory and Fractional Calculus. While the former is a relatively old subject, the latter is a branch of Mathematical Analysis that has been receiving some attention among the community of researchers only recently. Despite its novelty, it has been successfully applied to study phenomena in physics, chemistry, robotics, finance, engineering, just to name a few, because of its ability to take into account the history and non-local distributed effects. This allows scientists to describe the complexity of nature better than integer-order calculus. Encouraged by the growing interest in this discipline, and driven by natural curiosity, we faced some interesting mathematical challenges in the following direction.

First, we introduced the nth-order fractional equilibrium distribution in order to develop certain fractional probabilistic analogues of Taylor's theorem and mean value theorem; then, we discussed other related findings. Afterwards, we investigated Poisson-type and fractional Poisson-type processes subject to multiple jumps. In particular, we obtained and analyzed the probability distribution function, discussed some equivalent representations, studied the behaviour of waiting times and first-passage times and proved some convergence results. We then studied a generalization of the alternating Poisson process from the point of view of fractional calculus, providing results for the behaviour of some quantities which characterize the process under examination and deriving new Mittag-Leffler-like distributions of interest in the context of alternating renewal processes. The random times of a fractional alternating Poisson process have been used to describe the interarrival times separating consecutive velocity changes of a generalized jump-telegraph process. Among others, we obtained the probability law of the new process, devoted special attention to the case of jumps having constant size and provided a formal expression of the first-passage-time distribution through a constant boundary. The last chapter deals with the specification and the analysis of a stochastic model for competing risks involving the Mittag-Leffler distribution, both from a theoretical and from a numerical point of view.

Future research work could deal with:

- the "fractionalization" of some topics and models in reliability theory and survival analysis, including ageing notions of random lifetimes, comparisons based on stochastic orders, and relative ageing of distributions, following the lines of Tapiero and Vallois [143] and [142], and continuing to pursue a path adopted in Di Crescenzo and Meoli [43] and [41];
- the integration of such theoretical design with the peculiarities of the datasets effectively available (from biology and from engineering), fitting the model equations to the data, validating or detecting deficiencies in the models, conducting statistical analyses;
- the definition of a fractional model for the somatic evolution of cancer which generalizes the Luria-Delbrück model. Microbiologist Salvador Luria and theoretical physicist Max Delbrück in 1943 investigated mutations dynamics in exponentially growing microbial populations and observed that virus-resistant mutants emerge randomly, and not in response to selection, during the birth events. Since then, many mathematical models inspired by the Luria–Delbrück fluctuation test were developed to understand the emergence of drug resistance in bacterial colonies and in malignant tumors. The proposed project is currently being devoloped in collaboration with the Computational Biology Group at the Department of Biosystems Science and Engineering, ETH Zürich, directed by Prof. Dr. Niko Beerenwinkel.

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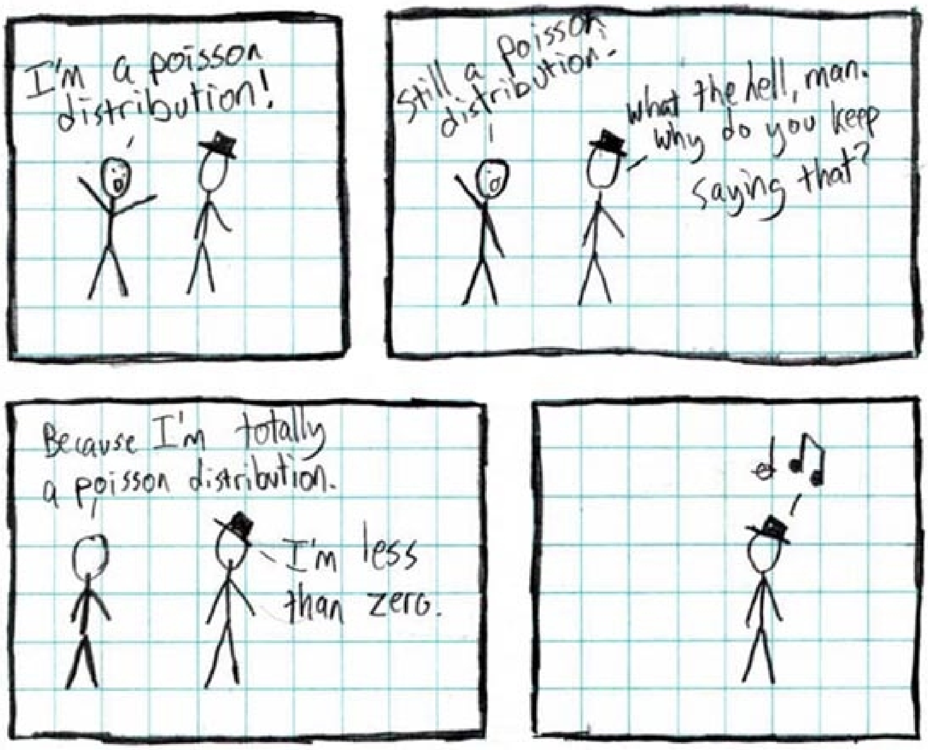

Title text: Poisson distributions have no value over negative numbers

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