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"Dynamics, Identification and Control of Multibody Systems"

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CONTENTS

1. INTRODUCTION

The central goal of this work is to put in an unified framework Dynamics, Identification and Control of multibody systems. A multibody system is a mechanical system constituted of interconnected rigid and deformable components which can undergo large translational and rotational displacements. The description of the motion of multibody systems is the leitmotif of Multibody Dynamics [1]. On the other hand, System Identification is the art of determining a mathematical model of a physical system by combining information obtained from experimental data with that derived from an a priori knowledge [2]. In addition, the System Identification methods can be successfully employed to refine a multibody model obtained from fundamental principles of Dynamics by using experimental data. In particular, applied System Identification methods allows to get modal parameters of a dynamical system using force and vibration measurements. On the other hand, the raison d'etre of Control Theory is to study how to design a control system which can influence the dynamic of a mechanical system in order to make it behave in a desirable manner [3], [4]. Consequently, it is intuitive to understand that these three seemingly unconnected subjects (Multibody Dynamics, System Identification, Control Theory) are actually strongly linked together. Therefore, the study of one of these subjects cannot be separated from the study of the other two. The structure of this works represents an attempt to encompass the essence of Multibody Dynamics, System Identification and Control Theory. In the first chapter (Multibody Dynamics) a synthesis of the most important principles and techniques to derive the equations of motion of multibody systems is presented. In this chapter a particular attention is devoted to the fundamental problem of constrained Dynamics [5],

[6] and to the finite element formulation of flexible multibody Dynamics [7]. In the second chapter (System Identification) a synthesis of the most important methodologies to obtain modal parameters of a dynamical system using force and vibration measurements is presented. In this chapter a particular interest is addressed to the Egensystem Realization Algorithm with Data Correlation (ERA/DC) using Observer/Kalman Filter Identification Method (OKID) [8] and to the method to construct physical models from identified state-space representations (MKR) [9], [10], [11]. In the third chapter (Control Theory) a synthesis of the most important algorithm to design a feedback control system based on a state observer is presented. In this chapter a particular attention is devoted to the Linear Quadratic Gaussian control method (LQG) [12]. Finally, in the last chapter (Case Study: Active Control of a Three-story Building Model) a case-study is analysed. The case study examined is a three-story building model with a pendulum hinged on the third floor [13], [14]. The motivations of this choice can be summarized in two points. First, the three-story frame, in spite of its simplicity, is a mechanical system whose dynamical behaviour is qualitatively similar to complex flexible structures. Therefore, all methods able to derive the equations of motion of multibody systems, all algorithms capable to identify the modal parameters of structural systems, and all strategies adequate to perform active vibration control of mechanical systems can be identically used in order to obtain qualitatively similar results. Second, the three-story building model, by virtue of its simplicity, is a mechanical system which can be quite simply assembled in laboratory making relatively little effort in order to perform a quick and easy-to-test experimental analysis [15]. In particular, a lumped parameter model and a finite element model of the three-story frame have been developed. Subsequently, a data-driven model relative to the system under test has been developed exploiting System Identification techniques. In particular, the Eigensystem Realization Algorithm with Data Correlation using Observer/Kalman Filter Identification method (ERA/DC OKID) [8] and the Numerical Algorithm for Subspace Identification (N4SID) [16] have been used to determine two different state-space models of the structural system using

experimental input and output measurements. In addition, the algorithm to determine a physical model from the identified sate-space representation (MKR) [9], [10], [11] has been used to obtain two different second-order mechanical models of the three-story frame. Subsequently, the design of a Linear Quadratic Gaussian regulator (LQG) [12] has been performed using the previously identified physical model of the system under test. The effectiveness of this controller has been tested in the worst-case scenario in which the system is excited by an external force whose harmonic content is close to the first three system natural frequencies. Finally, a new control algorithm for nonlinear underactuated mechanical systems affected by uncertainties (EUK-EKF) is proposed. The control problem of nonlinear underactuated mechanical system forced with nonholonomic constraints is the main object of many recent researches [17], [18], [19]. In analogy with the Linear Quadratic Gaussian regulation method (LQG) [12], the proposed algorithm represents the extension of the Udwadia-Kalaba control method (UK) [5], [6], [20], [21] to underactuated mechanical systems disturbed by noise. This extension is performed combining the extended Udwadia-Kalaba control method (EUK), which is the extension of the Udwadia-Kalaba control method (UK) [5], [6], [20], [21] to underactuated mechanical systems, with the well-known extended Kalman filter estimation method (EKF) [12]. Even in this case, the effectiveness of the combined algorithms (EUK-EKF) has been tested in the worst-case scenario in which the system is excited by an external force whose harmonic content is close to the first three system natural frequencies.

2. MULTIBODY DYNAMICS

2.1. INTRODUCTION

The motion of mechanical systems has been the central subject of some of the oldest research performed by the pioneers of Physics. From their work has developed over the centuries the vast field of knowledge commonly known as Mechanics, which is mainly composed of two parts: Kinematics and Dynamics. The word “kinematic” originates form the Greek word “κίνημα” which literally means “movement” whereas the word “dynamic” originates from the Greek word “δυναμις” which literally means “force” [1]. Indeed, Kinematics is the branch of Mechanics which studies the geometric description of motion without considering the causes that generate it. On the other hand, Dynamics is the branch of Mechanics which studies the causes of motion and how it takes place. In Dynamics the concept of force is introduced as the cause of the motion of bodies and the principal purpose of Dynamics is to formulate a mathematical model of a mechanical system starting from the basic principles of Physics in order to quantitatively describe the relationship between causes and effects, namely between forces and motion. The mathematical model of motion consists in appropriate differential equations whose solution resolves the central problem of Dynamics: predict the movement of a general mechanical system knowing its initial conditions and the forces acting on it. In general, note that the knowledge of only the position coordinates at a given instant of time is not sufficient to determine the mechanical state of a material system, which makes it impossible to predict the configuration of the system in the immediate future. On the contrary, if both the position and velocity coordinates are known in a given instant of time, then the mechanical configuration of the system is entirely determined and, in principle, it is possible to predict its future motion. Physically this means that the knowledge of the state variables in a fixed moment of time uniquely defines the value of the acceleration in same instant of time. The equations that mathematically link the position, velocity and acceleration coordinates to the forces which physically produce motion are called equations of motion. From a mathematical viewpoint, these equations are typically second order ordinary differential equations not necessarily linear. The integration of the equations of motion allows to theoretically determine the behavior of a mechanical system in terms of its motion as a function of time. In this chapter a synthesis of the most important principles and techniques to derive the equations of motion of multibody systems is presented.

2.2. ELEMENTS OF ANALYTICAL DYNAMICS

2.2.1. INTRODUCTION

The central problem of Dynamics consists in determining the motion of a mechanical system knowing the initial conditions and the forces acting on the system itself. To solve the central problem of Dynamics it is first necessary to derive the equations of motion of the system under examination. In the following sections some basic elements of analytical Dynamics are introduced. The starting point is Newton's second law of Dynamics, which represents the most fundamental law of Mechanics $\left[2\right]$. Then from Newton's second law D'alembert principle is derived, which paves the way to lagrangian Dynamics, and next the Lagrange equations are deduced from D'Alembert principle $\left[3\right]$, $\left[4\right]$. Afterwards, another fundamental principle of Mechanics is introduced, namely Hamilton principle of least action, and some basic elements of Calculus of Variation are briefly mentioned $\left[5\right]$. Subsequently, some modern techniques to derive the equations of motion of mechanical systems are concisely explained, such as Gibbs-Appel equations and Kane equations $\left[6\right]$. Finally, a fundamental principle of Mechanics perfectly equivalent to D'Alembert principle is introduced, namely Gausss principle of least constraint $\left[7\right]$.

2.2.2. NEWTON SECOND LAW

Consider a particle of mass m whose position is represented by the $\mathbb{R}^3$ vector $\mathbf{r}(t)$ function of time t. From simple geometrical considerations, it is straightforward to deduce that the velocity vector $\mathbf{v}(t)$ of the particle is equal to the first time derivative of the position vector $\mathbf{r}(t)$ whereas the acceleration vector $\mathbf{a}(t)$ is equal to the second time derivative of the same vector $\mathbf{r}(t)$. According to Newton second law of Mechanics, the resultant force $\mathbf{F}(t)$ acting on the particle is equal to the time rate of change of the linear momentum vector $\mathbf{p}(t)$ of the particle:

$$\mathbf{F}(t) = \dot{\mathbf{p}}(t) \tag{2.1}$$

Where the linear momentum vector $\mathbf{p}(t)$ of the particle is defined as:

$$\begin{aligned}\mathbf{p}(t) &= m\mathbf{v}(t) = \\ &= m\dot{\mathbf{r}}(t)\end{aligned} \tag{2.2}$$

This equation can also be restated in the form of dynamic equilibrium:

$$\mathbf{F}(t) - \dot{\mathbf{p}}(t) = \mathbf{0} \tag{2.3}$$

Where the second term on the left hand side can be interpreted as the resultant of the inertia forces acting on the particle. If the mass m of the particle is constant, then the second law of Dynamics can be rewritten as follows:

$$\begin{aligned}\mathbf{F}(t) &= \dot{\mathbf{p}}(t) = \\ &= m\dot{\mathbf{v}}(t) = \\ &= m\mathbf{a}(t) = \\ &= m\ddot{\mathbf{r}}(t)\end{aligned} \tag{2.4}$$

Now consider a set S of n_p particles of constant mass m^i for $i = 1, 2, \ldots, n_p$. For each particle of the set Newton second law holds:

$$\mathbf{F}^i(t) = \dot{\mathbf{p}}^i(t) \tag{2.5}$$

If mutual distance between the particles of the set is forced to remain constant, then the set is named rigid system. However, the particles of the set can also be linked together or to the ground in a different way. In any case, the effect of the constraints on the particles rebounds on Newton law creating some constraints forces whose resultant is $\mathbf{F}_c^i(t)$. Hence, the resultant force $\mathbf{F}^i(t)$ acting on the particles of the set can be decomposed into the sum of the resultant of the external active forces $\mathbf{F}_e^i(t)$ and the resultant of the constraint forces $\mathbf{F}_c^i(t)$. Therefore, Newton second law become:

$$\mathbf{F}_e^i(t) + \mathbf{F}_c^i(t) = m^i \ddot{\mathbf{r}}^i(t) \tag{2.6}$$

Even in this case the second law of Dynamics can be seen as a dynamic equilibrium:

$$\mathbf{F}_e^i(t) + \mathbf{F}_c^i(t) - m^i \ddot{\mathbf{r}}^i(t) = \mathbf{0} \tag{2.7}$$

Where the last term on the left hand side is equal to the resultant of the inertia forces acting on the particle.

2.2.3. D'ALEMBERT PRINCIPLE

Consider the dynamic equilibrium equations of a set of particles. In order to formulate D'Alembert principle, the virtual operator δ must be introduced first. At this stage, the virtual operator δ can be treated identically to the differential operator d except that the former does not operate on the time variable t, that is considered fixed. According to this definition, an arbitrary virtual displacement $\delta \mathbf{r}^i(t)$ of the particle i can be introduced and multiplied for the dynamic equilibrium equations:

$$\left(\mathbf{F}_e^i(t) + \mathbf{F}_c^i(t) - m^i \ddot{\mathbf{r}}^i(t)\right)^T \delta \mathbf{r}^i(t) = 0 \quad , \quad \forall \delta \mathbf{r}^i(t) \tag{2.8}$$

This is a scalar equation written in terms of virtual work of the resultant forces acting on a generic particle of the system. If this equation holds for every arbitrary virtual displacements $\delta \mathbf{r}^i(t)$, then it is perfectly equivalent to the second law of Dynamics, that is a vector equation. Now a summation on every particle of the system can be performed to get:

$$\begin{aligned}
&\sum_{i=1}^{n_p}\left(\left(\mathbf{F}_e^i(t)+\mathbf{F}_c^i(t)-m^i\ddot{\mathbf{r}}^i(t)\right)^T \delta\mathbf{r}^i(t)\right)= \\
&=\sum_{i=1}^{n_p}\left(\mathbf{F}_e^{iT}(t)\delta\mathbf{r}^i(t)\right)+\sum_{i=1}^{n_p}\left(\mathbf{F}_c^{iT}(t)\delta\mathbf{r}^i(t)\right)+\sum_{i=1}^{n_p}\left(-m^i\ddot{\mathbf{r}}^{iT}(t)\delta\mathbf{r}^i(t)\right)= \\
&=\delta W_e(t)+\delta W_c(t)+\delta W_i(t)= \\
&=0 \quad , \quad \forall\delta\mathbf{r}^i(t)
\end{aligned} \tag{2.9}$$

In the last equality the total virtual work of the external forces, constraint forces and inertia forces can be identified:

$$\delta W_e(t)=\sum_{i=1}^{n_p}\left(\mathbf{F}_e^{iT}(t)\delta\mathbf{r}^i(t)\right) \tag{2.10}$$

$$\delta W_c(t)=\sum_{i=1}^{n_p}\left(\mathbf{F}_c^{iT}(t)\delta\mathbf{r}^i(t)\right) \tag{2.11}$$

$$\delta W_i(t)=\sum_{i=1}^{n_p}\left(-m^i\ddot{\mathbf{r}}^{iT}(t)\delta\mathbf{r}^i(t)\right) \tag{2.12}$$

These virtual works are incremental expressions rather than differential of functions. The typical assumption of Classical Mechanics is that the constraints do no work and therefore they are called workless constraints:

$$\delta W_c(t)=0 \tag{2.13}$$

Using this assumption the D'Alembert principle can be obtained:

$$\delta W_e(t)+\delta W_i(t)=0 \quad , \quad \forall\delta\mathbf{r}^i(t) \tag{2.14}$$

This equation claims that for any virtual displacements of the system particles compatible with the constraints, the sum of the total virtual works performed by external forces and inertia forces is equal to zero.

2.2.4. LAGRANGIAN DYNAMICS

Consider a system of n_p particles subjected to a set of n_c workless constraints. It is clear that not all the particle coordinates are independent because of the presence of the constraints. Indeed, the actual number of independent coordinates is:

$$n = 3n_p - n_c \tag{2.15}$$

The independent coordinates are customary called degrees of freedom of the system. From a geometrical viewpoint, a coordinate transformation can be introduced in order to express the position vectors $\mathbf{r}^i(t)$ of the system particles in terms of a set of n generalized independent coordinates $\mathbf{q}(t)$ that can also lack of an obvious physical meaning. These coordinates are named lagrangian coordinates whereas the position coordinates are sometimes called physical coordinates to distinguish them from generalized coordinates. Hence, there is a mathematical vector function which represents the relation between system physical coordinates and lagrangian coordinates:

$$\mathbf{r}^i(t) = \mathbf{r}^i(\mathbf{q}(t)) \tag{2.16}$$

Using this relation the virtual displacement of the generic physical coordinate vector can be expressed in terms of the virtual change of lagrangian coordinates as follows:

$$\begin{aligned}\delta\mathbf{r}^i(t) = \delta\mathbf{r}^i(\mathbf{q}(t)) = \\ = \frac{\partial\mathbf{r}^i(t)}{\partial\mathbf{q}(t)}\delta\mathbf{q}(t) = \\ = \mathbf{J}^i(t)\delta\mathbf{q}(t)\end{aligned} \tag{2.17}$$

Where $\mathbf{J}^i(t)$ is a $\mathbb{R}^{3\times n}$ jacobian transformation matrix. Thanks to this relation it is possible to express the virtual works of both external and inertial forces in term of the generalized force vectors called lagrangian components:

$$\begin{aligned}\delta W_e(t) &= \sum_{i=1}^{n_p}\left(\mathbf{F}_e^{iT}(t)\delta\mathbf{r}^i(t)\right) = \\ &= \sum_{i=1}^{n_p}\left(\mathbf{F}_e^{iT}(t)\mathbf{J}^i(t)\delta\mathbf{q}(t)\right) = \\ &= \left(\sum_{i=1}^{n_p}\left(\mathbf{F}_e^{iT}(t)\mathbf{J}^i(t)\right)\right)\delta\mathbf{q}(t) = \\ &= \mathbf{Q}_e^T(t)\delta\mathbf{q}(t)\end{aligned} \tag{2.18}$$

$$\begin{aligned}\delta W_i(t) &= \sum_{i=1}^{n_p}\left(-m^i\ddot{\mathbf{r}}^{iT}(t)\delta\mathbf{r}^i(t)\right) = \\ &= \sum_{i=1}^{n_p}\left(-m^i\ddot{\mathbf{r}}^{iT}(t)\mathbf{J}^i(t)\delta\mathbf{q}(t)\right) = \\ &= \left(\sum_{i=1}^{n_p}\left(-m^i\ddot{\mathbf{r}}^{iT}(t)\mathbf{J}^i(t)\right)\right)\delta\mathbf{q}(t) = \\ &= \mathbf{Q}_i^T(t)\delta\mathbf{q}(t)\end{aligned} \tag{2.19}$$

Where the $\mathbb{R}^n$ lagrangian components vectors of external and inertial forces are respectively defined as:

$$\mathbf{Q}_e(t) = \sum_{i=1}^{n_p} \left(\mathbf{J}^{iT}(t) \mathbf{F}_e^i(t) \right) \tag{2.20}$$

$$\mathbf{Q}_i(t) = \sum_{i=1}^{n_p} \left(\mathbf{J}^{iT}(t) \left(-m^i \ddot{\mathbf{r}}^i(t) \right) \right) \tag{2.21}$$

Finally, the D'Alembert principle can be restated by using lagrangian coordinates in the following way:

$$\left(\mathbf{Q}_e(t) + \mathbf{Q}_i(t) \right)^T \delta\mathbf{q}(t) = 0 \quad , \quad \forall \delta\mathbf{q}(t) \tag{2.22}$$

Observing that the generalized coordinates are assumed to be independent coordinates, the D'Alembert principle in lagrangian coordinates is the following:

$$\mathbf{Q}_e(t) + \mathbf{Q}_i(t) = \mathbf{0} \tag{2.23}$$

This set of equations represents the system equations of motion expressed in terms of lagrangian coordinates.

2.2.5. LAGRANGE EQUATIONS

Lagrange equations are a mathematical device able to derive system equations of motion. One method to get Lagrange equations is to start from D'Alembert principle in lagrangian coordinates: this method consist in expressing the lagrangian component of inertia forces by using a physical quantity called kinetic energy $T^i(t)$. Kinetic energy is a form of mechanical energy possessed by a body only because of its motion and, in the case of a particle, it is defined as:

$$T^i(t) = \frac{1}{2} m^i \dot{\mathbf{r}}^{iT}(t) \dot{\mathbf{r}}^i(t) \tag{2.24}$$

Thereby, the kinetic energy of a material system $T(t)$ is simply the sum of the single kinetic energy of each particle:

$$
\begin{aligned}
T(t) &= \sum_{i=1}^{n_p} T^i(t) \\
&= \sum_{i=1}^{n_p} \left(\frac{1}{2} m^i \dot{\mathbf{r}}^{iT}(t) \dot{\mathbf{r}}^i(t) \right)
\end{aligned}
\tag{2.25}
$$

On the other hand, an useful observation is to note that jacobian matrix can also be computed from the time derivative of physical coordinates and lagrangian coordinates:

$$
\begin{aligned}
\mathbf{J}^i(t) &= \frac{\partial \mathbf{r}^i(t)}{\partial \mathbf{q}(t)} = \\
&= \frac{\partial \dot{\mathbf{r}}^i(t)}{\partial \dot{\mathbf{q}}(t)}
\end{aligned}
\tag{2.26}
$$

According to these observations, the lagrangian component of inertia forces can be computed in the following manner:

$$
\begin{aligned}
\mathbf{Q}_i(t) &= \sum_{i=1}^{n_p} \left(\mathbf{J}^{iT}(t) \left(-m^i \ddot{\mathbf{r}}^i(t) \right) \right) = \\
&= \sum_{i=1}^{n_p} \left(\left(\frac{\partial \mathbf{r}^i(t)}{\partial \mathbf{q}(t)} \right)^T \left(-m^i \ddot{\mathbf{r}}^i(t) \right) \right) = \\
&= \sum_{i=1}^{n_p} \left(\left(\frac{\partial \dot{\mathbf{r}}^i(t)}{\partial \dot{\mathbf{q}}(t)} \right)^T \left(-m^i \ddot{\mathbf{r}}^i(t) \right) \right) = \\
&= -\frac{d}{dt} \left(\frac{\partial T(t)}{\partial \dot{\mathbf{q}}(t)} \right)^T + \left(\frac{\partial T(t)}{\partial \mathbf{q}(t)} \right)^T
\end{aligned}
\tag{2.27}
$$

Consequently, Lagrange equations are:

$$\frac{d}{dt}\left(\frac{\partial T(t)}{\partial \dot{\mathbf{q}}(t)}\right)^T - \left(\frac{\partial T(t)}{\partial \mathbf{q}(t)}\right)^T = \mathbf{Q}_e(t) \tag{2.28}$$

Lagrange equations can also be written in slightly different forms. In fact, first note that external forces acting on the system particles can be separated in external conservative forces and external non-conservative forces:

$$\mathbf{F}_e^i(t) = \mathbf{F}_{e,c}^i(t) + \mathbf{F}_{e,nc}^i(t) \tag{2.29}$$

As a result, the total virtual work of external forces can be divided in two parts:

$$\delta W_e(t) = \delta W_{e,c}(t) + \delta W_{e,nc}(t) \tag{2.30}$$

Similarly to the previous case, the virtual work of external non-conservative forces can be calculated in terms of the lagrangian component:

$$\begin{aligned} \delta W_{e,nc}(t) &= \sum_{i=1}^{n_p}\left(\mathbf{F}_{e,nc}^{i\,T}(t)\delta\mathbf{r}^i(t)\right) = \\ &= \sum_{i=1}^{n_p}\left(\mathbf{F}_{e,nc}^{i\,T}(t)\mathbf{J}^i(t)\delta\mathbf{q}(t)\right) = \\ &= \left(\sum_{i=1}^{n_p}\left(\mathbf{F}_e^{i\,T}(t)\mathbf{J}^i(t)\right)\right)\delta\mathbf{q}(t) = \\ &= \mathbf{Q}_{e,nc}^T(t)\delta\mathbf{q}(t) \end{aligned} \tag{2.31}$$

Where:

$$\mathbf{Q}_{e,nc}(t) = \sum_{i=1}^{n_p}\left(\mathbf{J}^{i\,T}(t)\mathbf{F}_{e,nc}^i(t)\right) \tag{2.32}$$

On the other hand, the total virtual works of conservative forces can be expressed in terms of system potential energy $U(t)$. Potential energy is a form of mechanical energy possessed by a body only because of its position in a conservative force field. In the case of a set of particles, the virtual work of conservative forces can be rewritten as:

$$\begin{aligned}\delta W_{e,c}(t) &= \sum_{i=1}^{n_p}\left(\mathbf{F}_{e,c}^{i\,T}(t)\delta\mathbf{r}^i(t)\right) = \\ &= \sum_{i=1}^{n_p}\left(-\delta U^i(t)\right) = \\ &= -\delta U(t)\end{aligned} \tag{2.33}$$

Where:

$$U(t) = \sum_{i=1}^{n_p} U^i(t) \tag{2.34}$$

In addition, the virtual change in potential energy can be restated using the lagrangian component of conservative forces in the following way:

$$\begin{aligned}\delta U(t) &= \frac{\partial U(t)}{\partial \mathbf{q}(t)}\delta\mathbf{q}(t) = \\ &= \mathbf{Q}_{e,c}^T(t)\delta\mathbf{q}(t)\end{aligned} \tag{2.35}$$

Where:

$$\mathbf{Q}_{e,c}(t) = -\left(\frac{\partial U(t)}{\partial \mathbf{q}(t)}\right)^T \tag{2.36}$$

Finally, Lagrange equations can be rewritten as:

$$\frac{d}{dt}\left(\frac{\partial T(t)}{\partial \dot{\mathbf{q}}(t)}\right)^T - \left(\frac{\partial T(t)}{\partial \mathbf{q}(t)}\right)^T + \left(\frac{\partial U(t)}{\partial \mathbf{q}(t)}\right)^T = \mathbf{Q}_{e,nc}(t) \tag{2.37}$$

Now a new physical quantity can be introduced, namely the system lagrangian $L(t)$. The lagrangian is defined as the difference between the kinetic energy and the potential energy of the system:

$$L(t) = T(t) - U(t) \tag{2.38}$$

Using this definition, and noting that the potential energy is not a function of the derivative of generalized coordinates, it is easy to prove that Lagrange equations can be expressed as:

$$\frac{d}{dt}\left(\frac{\partial L(t)}{\partial \dot{\mathbf{q}}(t)}\right)^T - \left(\frac{\partial L(t)}{\partial \mathbf{q}(t)}\right)^T = \mathbf{Q}_{e,nc}(t) \tag{2.39}$$

A great advantage in using Lagrange equations is working with scalar physical quantities, such as kinetic energy $T(t)$ and potential energy $U(t)$, to develop the equations of motion instead of working with vector quantities, like forces and accelerations, that are necessary to apply Newton second law.

2.2.6. HAMILTON PRINCIPLE

One of the most basic principle of Classical Mechanics is Hamilton principle. This principle, also known as Hamilton principle of least action, is based on the techniques of the Calculus of variations and can be used as a valid mathematical tool to derive the equations of motion of mechanical systems. Consider a system of particles S whose configuration at time t is univocally identified by the generalized independent coordinate vector $\mathbf{q}(t)$. Assume that the system is evolving during a time span included between two specific instants t_0 and t_f from the configuration state vector $\mathbf{q}(t_0)$ to the configuration state vector $\mathbf{q}(t_f)$. Hamilton principle asserts that between all the possible paths, compatible with constraints, that the configuration vector $\mathbf{q}(t)$ can follow in its evolution during the time span between the instants t_0 and t_f, the one which actually materializes is that that minimize the time definite integral of the system lagrangian $L(t)$, also named the action, that is:

$$\int_{t_0}^{t_f} L(t)dt \tag{2.40}$$

The solution of this minimization problem can be found considering a perturbation, namely a virtual change, $\delta\mathbf{q}(t)$ of the true path followed by the configuration vector $\mathbf{q}(t)$ in its time evolution and assuming that the true path and the perturbed path always coincide at the time instants t_0 and t_f. That is to say:

$$\delta\mathbf{q}(t_0) = \delta\mathbf{q}(t_f) = \mathbf{0} \tag{2.41}$$

According to this method it is possible to find a stationary value of the so called action functional:

$$\delta\int_{t_0}^{t_f} L(t)dt = 0 \tag{2.42}$$

At this stage, by using the definition of the lagrangian function and the formula of integration by parts, the perturbation of the action functional become:

$$
\begin{aligned}
&\delta\int_{t_0}^{t_f} L(t)dt = \int_{t_0}^{t_f} \delta L(t)dt = \\
&= \int_{t_0}^{t_f} \delta\left(T(t) - U(t)\right)dt = \\
&= \int_{t_0}^{t_f} \delta T(t) - \delta U(t)dt = \\
&= \int_{t_0}^{t_f} \delta T(\mathbf{q}(t), \dot{\mathbf{q}}(t), t) - \delta U(\mathbf{q}(t), t)dt = \\
&= \int_{t_0}^{t_f} \frac{\partial T(t)}{\partial \mathbf{q}(t)} \delta\mathbf{q}(t) + \frac{\partial T(t)}{\partial \dot{\mathbf{q}}(t)} \delta\dot{\mathbf{q}}(t) - \frac{\partial U(t)}{\partial \mathbf{q}(t)} \delta\mathbf{q}(t)dt = \\
&= \left[\frac{\partial T(t)}{\partial \dot{\mathbf{q}}(t)} \delta\mathbf{q}(t)\right]_{t_0}^{t_f} + \int_{t_0}^{t_f} \left(-\frac{d}{dt}\frac{\partial T(t)}{\partial \dot{\mathbf{q}}(t)} + \frac{\partial T(t)}{\partial \mathbf{q}(t)} - \frac{\partial U(t)}{\partial \mathbf{q}(t)}\right)\delta\mathbf{q}(t)dt = \\
&= \left[\frac{\partial T(t)}{\partial \dot{\mathbf{q}}(t)} \delta\mathbf{q}(t)\right]_{t_0}^{t_f} + \int_{t_0}^{t_f} \left(-\frac{d}{dt}\left(\frac{\partial T(t)}{\partial \dot{\mathbf{q}}(t)}\right)^T + \left(\frac{\partial T(t)}{\partial \mathbf{q}(t)}\right)^T - \left(\frac{\partial U(t)}{\partial \mathbf{q}(t)}\right)^T\right)^T \delta\mathbf{q}(t)dt = \\
&= 0 \quad , \quad \forall \delta\mathbf{q}(t)
\end{aligned}
\tag{2.43}
$$

Observing that all the lagrangian coordinates are independent, each quantity in the time integral can be independently taken equal to zero:

$$
\begin{cases}
\delta\mathbf{q}(t_0) = \delta\mathbf{q}(t_f) = \mathbf{0} \\
\dfrac{d}{dt}\left(\dfrac{\partial T(t)}{\partial \dot{\mathbf{q}}(t)}\right)^T - \left(\dfrac{\partial T(t)}{\partial \mathbf{q}(t)}\right)^T + \left(\dfrac{\partial U(t)}{\partial \mathbf{q}(t)}\right)^T = \mathbf{0}
\end{cases}
\tag{2.44}
$$

These differential equations are the well-known Euler-Lagrange equations. The solution of this set of differential equations corresponds to the minimum of the action integral. Hamilton principle can be modified in order to include the effect of non-conservative external forces on motion. This modified principle is named extended Hamilton principle and can be mathematically stated through the following stationary problem:

$$\delta\int_{t_0}^{t_f} L(t)dt + \int_{t_0}^{t_f} \delta W_{e,nc}(t)dt = 0 \tag{2.45}$$

Where $\delta W_{e,nc}(t)$ is the virtual work of the external non-conservative forces. As expected, it is straightforward to prove that the final result are Lagrange equations of motion:

$$\frac{d}{dt}\left(\frac{\partial T(t)}{\partial \dot{\mathbf{q}}(t)}\right)^T - \left(\frac{\partial T(t)}{\partial \mathbf{q}(t)}\right)^T + \left(\frac{\partial U(t)}{\partial \mathbf{q}(t)}\right)^T = \mathbf{Q}_{e,nc}(t) \tag{2.46}$$

It is worth saying that both D'Alembert principle and Hamilton principle are powerful physical-mathematical tool to derive the equations of motion of all kind of mechanical systems but conceptually the latter can be logically deduced from the former.

2.2.7. GIBBS-APPELL EQUATIONS

Gibbs-Appell equations represent another useful and effective mathematical technique to obtain the equations of motion of mechanical systems. These equations are based on the so-called Gibbs-Appell function $G^i(t)$ that, for a single particle, can be defined as follows:

$$G^i(t) = \frac{1}{2} m^i \ddot{\mathbf{r}}^{iT}(t)\ddot{\mathbf{r}}^i(t) \tag{2.47}$$

Consequently, the Gibbs-Appell function of the whole system $G(t)$ can be easily computed as the sum of the single particle Gibbs-Appell function. Indeed:

$$\begin{aligned} G(t) &= \sum_{i=1}^{n_p} G^i(t) = \\ &= \sum_{i=1}^{n_p}\left(\frac{1}{2} m^i \ddot{\mathbf{r}}^{iT}(t)\ddot{\mathbf{r}}^i(t)\right) \end{aligned} \tag{2.48}$$

A possible strategy to obtain Gibbs-Appell equations is to leverage D'Alembert principle written in lagrangian coordinates. To do that, a useful observation is to note that jacobian transformation matrix can also be computed from the second time derivative of physical coordinates and lagrangian coordinates:

$$\begin{aligned}\mathbf{J}^i(t) &= \frac{\partial \mathbf{r}^i(t)}{\partial \mathbf{q}(t)} = \\ &= \frac{\partial \ddot{\mathbf{r}}^i(t)}{\partial \ddot{\mathbf{q}}(t)}\end{aligned} \tag{2.49}$$

According to these observations, the lagrangian component of inertia forces can be computed in the following manner:

$$\begin{aligned}\mathbf{Q}_i(t) &= \sum_{i=1}^{n_p}\left(\mathbf{J}^{iT}(t)\left(-m^i\ddot{\mathbf{r}}^i(t)\right)\right) = \\ &= \sum_{i=1}^{n_p}\left(\left(\frac{\partial \mathbf{r}^i(t)}{\partial \mathbf{q}(t)}\right)^T\left(-m^i\ddot{\mathbf{r}}^i(t)\right)\right) = \\ &= \sum_{i=1}^{n_p}\left(\left(\frac{\partial \ddot{\mathbf{r}}^i(t)}{\partial \ddot{\mathbf{q}}(t)}\right)^T\left(-m^i\ddot{\mathbf{r}}^{iT}(t)\right)\right) = \\ &= -\left(\frac{\partial G(t)}{\partial \ddot{\mathbf{q}}(t)}\right)^T\end{aligned} \tag{2.50}$$

Consequently, Gibbs-Appell equations can be expressed as:

$$\left(\frac{\partial G(t)}{\partial \ddot{\mathbf{q}}(t)}\right)^T = \mathbf{Q}_e(t) \tag{2.51}$$

Similar to Lagrange equations, Gibbs-Appell equations allow one to get the equations of motion of a mechanical system by using a scalar function, namely the system Gibbs-Appell function $G(t)$, instead of working with vector quantities.

2.2.8. KANE EQUATIONS

Kane equations are sophisticated mathematical tool which permits to obtain the equations of motion of mechanical systems. The simplest way to derive Kane equations is to deduce them from D'Alembert principle expressed in lagrangian coordinates. First, for a single particle, Kane function can be defined as:

$$\mathbf{K}^i(t) = \mathbf{J}^{iT}(t) m^i \ddot{\mathbf{r}}^i(t) \tag{2.52}$$

As a consequence, Kane function of the whole system $\mathbf{K}(t)$ is simply the sum of every single particle function. Indeed:

$$\begin{aligned} \mathbf{K}(t) &= \sum_{i=1}^{n_p} \mathbf{K}^i(t) = \\ &= \sum_{i=1}^{n_p} \left(\mathbf{J}^{iT}(t) m^i \ddot{\mathbf{r}}^i(t) \right) \end{aligned} \tag{2.53}$$

By using this definition, the lagrangian component of inertia forces can be rewritten as follows:

$$\begin{aligned} \mathbf{Q}_i(t) &= \sum_{i=1}^{n_p} \left(\mathbf{J}^{iT}(t) \left(-m^i \ddot{\mathbf{r}}^i(t) \right) \right) = \\ &= -\mathbf{K}(t) \end{aligned} \tag{2.54}$$

Therefore, Kane equations can be written as:

$$\mathbf{K}(t) = \mathbf{Q}_e(t) \tag{2.55}$$

It is straightforward to understand that, according to the preceding definitions, to write down Kane equations it is necessary to compute explicitly the jacobian transformation matrix.

2.2.9. GAUSS PRINCIPLE

Gauss principle, also known as the principle of least constraint, is a fundamental principle of Classical Mechanics perfectly equivalent to D'Alembert principle. This principle states that among all the accelerations that a mechanical system can have which are compatible with constraints, the ones that actually materialize are those that present the minimum deviation from the free accelerations in a least-square sense. Consider a material system S . If the particles of the system have no constraint in their evolution in time, the free acceleration of a generic particle can be computed as:

$$\ddot{\mathbf{r}}_e^i(t) = \frac{\mathbf{F}_e^i(t)}{m^i} \tag{2.56}$$

Let $\ddot{\mathbf{r}}^i(t)$ be the actual acceleration of a system particle due to the presence of some constraint forces. The Gauss function $Z^i(t)$ of the material point is define as:

$$Z^i(t) = \frac{1}{2} m^i \left(\ddot{\mathbf{r}}^i(t) - \ddot{\mathbf{r}}_e^i(t)\right)^T \left(\ddot{\mathbf{r}}^i(t) - \ddot{\mathbf{r}}_e^i(t)\right) \tag{2.57}$$

Obviously, the Gauss function of the whole system $Z(t)$ is merely the sum of the single Gausss function of each particle of the set. Indeed:

$$\begin{aligned} Z(t) &= \sum_{i=1}^{n_p} Z^i(t) = \\ &= \sum_{i=1}^{n_p} \left(\frac{1}{2} m^i \left(\ddot{\mathbf{r}}^i(t) - \ddot{\mathbf{r}}_e^i(t)\right)^T \left(\ddot{\mathbf{r}}^i(t) - \ddot{\mathbf{r}}_e^i(t)\right) \right) \end{aligned} \tag{2.58}$$

One way to obtain Gauss principle is to leverage on D'Alembert principle in lagrangian coordinates:

$$
\begin{aligned}
\mathbf{Q}_e(t)+\mathbf{Q}_i(t) &= \sum_{i=1}^{n_p}\left(\mathbf{J}^{iT}(t)\left(\mathbf{F}_e^i(t)-m^i\ddot{\mathbf{r}}^i(t)\right)\right)= \\
&= \sum_{i=1}^{n_p}\left(\left(\frac{\partial \mathbf{r}^i(t)}{\partial \mathbf{q}(t)}\right)^T m^i\left(\frac{\mathbf{F}_e^i(t)}{m^i}-\ddot{\mathbf{r}}^i(t)\right)\right)= \\
&= \sum_{i=1}^{n_p}\left(\left(\frac{\partial \ddot{\mathbf{r}}^i(t)}{\partial \ddot{\mathbf{q}}(t)}\right)^T m^i\left(\ddot{\mathbf{r}}_e^i(t)-\ddot{\mathbf{r}}^i(t)\right)\right)= \\
&= -\sum_{i=1}^{n_p}\left(\left(\frac{\partial \ddot{\mathbf{r}}^i(t)}{\partial \ddot{\mathbf{q}}(t)}\right)^T m^i\left(\ddot{\mathbf{r}}^i(t)-\ddot{\mathbf{r}}_e^i(t)\right)\right)= \\
&= -\left(\frac{\partial Z(t)}{\partial \ddot{\mathbf{q}}(t)}\right)^T = \\
&= \mathbf{0}
\end{aligned}
\tag{2.59}
$$

Setting the last expression equal to zero, Gauss principle is obtained:

$$
\left(\frac{\partial Z(t)}{\partial \ddot{\mathbf{q}}(t)}\right)^T = \mathbf{0} \tag{2.60}
$$

In analogy with Lagrange equations and Gibbs-Appell equations, Gauss principle is a mighty mathematical methods that allows one to get the equations of motion of a mechanical system by using the scalar physical quantity $Z(t)$ called Gauss function. As can be intuitively understood, there is a strong physical-mathematical link between Gauss principle, Gibbs-Appell equations and Lagrange equations.

2.3. THE FUNDAMENTAL PROBLEM OF CONSTRAINED DYNAMICS

2.3.1. INTRODUCTION

The central problem of constrained Dynamics consists in determining the motion of a constraint mechanical system knowing the initial conditions and the forces acting on the system itself. Unlike the case of unconstrained Dynamics, in this case the constraint forces are further unknowns. In the following sections the central problem of constrained motion is addressed and solved according to the formulation proposed by Udwadia and Kalaba $\left[8\right]$.

2.3.2. HOLONOMIC AND NONHOLOMIC CONSTRAINTS

The equations of motion of mechanical system can be analytically deduced from the basic principles of Dynamics. Indeed, Lagrange equations of motion are:

$$\frac{d}{dt}\left(\frac{\partial L(t)}{\partial \dot{\mathbf{q}}(t)}\right)^T - \left(\frac{\partial L(t)}{\partial \mathbf{q}(t)}\right)^T = \mathbf{Q}_{e,nc}(t) \tag{2.61}$$

These equations have been developed assuming that all the n generalized coordinates $\mathbf{q}(t)$ are independent from each other, namely by using an embedding technique. This method identifies the configuration of the system through a minimal set of coordinates and, using the hypothesis of workless constrains, produces a set of differential equations which does not exhibit the generalized constraint forces. If an augmented formulation is used instead, then a larger configuration $\mathbf{q}(t)$ vector is used which, for instance in the case of a material system, can be made of the n_p physical coordinates of the particles. It is intuitive to understand that the generalized constraint forces will influence the equations of motion:

$$\frac{d}{dt}\left(\frac{\partial L(t)}{\partial \dot{\mathbf{q}}(t)}\right)^T - \left(\frac{\partial L(t)}{\partial \mathbf{q}(t)}\right)^T = \mathbf{Q}_{e,nc}(t) + \mathbf{Q}^c(t) \tag{2.62}$$

Where $\mathbf{Q}^c(t)$ is a $\mathbb{R}^n$ vector of the generalized constraint forces (hereafter, for simplicity, the dimension of the set of redundant coordinates will be indicated as n). This vector is a ulterior unknown of the central problem of constraint Dynamics. Indeed, it is possible to express the generalized constraint forces by using the constraint equations and leveraging on Lagrange multipliers technique $[5]$. Constraint equations are a set of algebraic equations that links together the generalized coordinates vector. Basically, constraints can be classified in two type: holonomic constraints and nonholonomic constraints. According to the traditional acceptation, holonomic constraints are characterized by a set of algebraic equations which can be integrated and reduced to the following form:

$$\mathbf{f}(\mathbf{q}(t), t) = \mathbf{0} \tag{2.63}$$

Where $\mathbf{f}(t)$ is a $\mathbb{R}^{m_f}$ vector function of only the generalized coordinates vector $\mathbf{q}(t)$. Holonomic constraints are also referred to as kinematic constraints. On the other hand, nonholonomic constraints are characterized by a set of nonintegrable algebraic equations which involve also the time derivatives of lagrangian coordinates. For the sake of simplicity, nonholonomic constrains can be distinguished in velocity nonholonomic constraints and acceleration nonholonomic constraints. The equations of velocity nonholonomic constraints involves the first time derivative of the configuration vector and can be defined as:

$$\mathbf{g}(\mathbf{q}(t), \dot{\mathbf{q}}(t), t) = \mathbf{0} \tag{2.64}$$

Where $\mathbf{g}(t)$ is a $\mathbb{R}^{m_g}$ nonintegrable vector function of generalized position vector $\mathbf{q}(t)$ and generalized velocity vector $\dot{\mathbf{q}}(t)$. Moreover, the equations of acceleration nonholonomic constraints involves the first and the second time derivative of the configuration vector and can be defined as:

$$\mathbf{h}(\mathbf{q}(t), \dot{\mathbf{q}}(t), \ddot{\mathbf{q}}(t), t) = \mathbf{0} \tag{2.65}$$

Where $\mathbf{h}(t)$ is a $\mathbb{R}^{m_h}$ nonintegrable vector function of generalized position vector $\mathbf{q}(t)$, generalized velocity vector $\dot{\mathbf{q}}(t)$ and generalized acceleration vector $\ddot{\mathbf{q}}(t)$. On the whole, the total number of constraint equations n_c is:

$$n_c = m_f + m_g + m_h \tag{2.66}$$

Where m_f, m_g and m_h are respectively the number of holonomic constraint equations, velocity nonholonomic constraint equations and acceleration nonholonomic constraint equations. It can be proved [9], [10] that the generalized constraints forces can be expressed in terms of the constraint equations by the Lagrange multiplies method as follows:

$$\mathbf{Q}^c(t) = \left(\frac{\partial \mathbf{f}(t)}{\partial \mathbf{q}(t)}\right)^T \boldsymbol{\lambda}_f(t) + \left(\frac{\partial \mathbf{g}(t)}{\partial \dot{\mathbf{q}}(t)}\right)^T \boldsymbol{\lambda}_g(t) + \left(\frac{\partial \mathbf{h}(t)}{\partial \ddot{\mathbf{q}}(t)}\right)^T \boldsymbol{\lambda}_h(t) \tag{2.67}$$

Where $\boldsymbol{\lambda}_f(t)$, $\boldsymbol{\lambda}_g(t)$ and $\boldsymbol{\lambda}_h(t)$ are $\mathbb{R}^{m_f}$, $\mathbb{R}^{m_g}$ and $\mathbb{R}^{m_h}$ vectors, respectively, named Lagrange multipliers and correspond to the holonomic and nonholonomic constraint equations. Using this result, generalized constraints forces can be adjoined to Lagrange equations:

$$\begin{aligned} &\frac{d}{dt}\left(\frac{\partial L(t)}{\partial \dot{\mathbf{q}}(t)}\right)^T - \left(\frac{\partial L(t)}{\partial \mathbf{q}(t)}\right)^T = \mathbf{Q}_{e,nc}(t) + \\ &+ \left(\frac{\partial \mathbf{f}(t)}{\partial \mathbf{q}(t)}\right)^T \boldsymbol{\lambda}_f(t) + \left(\frac{\partial \mathbf{g}(t)}{\partial \dot{\mathbf{q}}(t)}\right)^T \boldsymbol{\lambda}_g(t) + \left(\frac{\partial \mathbf{h}(t)}{\partial \ddot{\mathbf{q}}(t)}\right)^T \boldsymbol{\lambda}_h(t) \end{aligned} \tag{2.68}$$

This is a set of n differential equations but the unknowns are the n generalized coordinates $\mathbf{q}(t)$ plus the n_c Lagrange multipliers $\boldsymbol{\lambda}(t)$. The problem can be mathematically closed only including the n_c algebraic constraint equations and solving the whole resulting system:

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial L(t)}{\partial \dot{\mathbf{q}}(t)}\right)^T - \left(\dfrac{\partial L(t)}{\partial \mathbf{q}(t)}\right)^T = \mathbf{Q}_{e,nc}(t) + \\ + \left(\dfrac{\partial \mathbf{f}(t)}{\partial \mathbf{q}(t)}\right)^T \boldsymbol{\lambda}_f(t) + \left(\dfrac{\partial \mathbf{g}(t)}{\partial \dot{\mathbf{q}}(t)}\right)^T \boldsymbol{\lambda}_g(t) + \left(\dfrac{\partial \mathbf{h}(t)}{\partial \ddot{\mathbf{q}}(t)}\right)^T \boldsymbol{\lambda}_h(t) \\ \mathbf{f}(\mathbf{q}(t),t) = \mathbf{0} \\ \mathbf{g}(\mathbf{q}(t),\dot{\mathbf{q}}(t),t) = \mathbf{0} \\ \mathbf{h}(\mathbf{q}(t),\dot{\mathbf{q}}(t),\ddot{\mathbf{q}}(t),t) = \mathbf{0} \end{cases} \tag{2.69}$$

These equations represent the general equations of motion of a discrete constrained mechanical system.

2.3.3. EQUATIONS OF MOTION OF CONSTRAINED MECHANICAL SYSTEMS

Assuming that the acceleration constraint vector $\mathbf{h}(t)$ is a linear function of generalized accelerations $\ddot{\mathbf{q}}(t)$, it is possible to obtain an explicit solution of the problem of constrained Dynamics in the sense that the generalized constrained acceleration $\ddot{\mathbf{q}}(t)$ and the Lagrange multipliers $\boldsymbol{\lambda}(t)$ can be computed explicitly. Before doing that, it is necessary to express the equations of motion and the constraint equations in a different form. It can be simply proved $[11]$ that the equations of motion of a discrete constrained mechanical systems can always be rewritten in this form:

$$\mathbf{M}(\mathbf{q}(t))\ddot{\mathbf{q}}(t) = \mathbf{Q}(\mathbf{q}(t),\dot{\mathbf{q}}(t),t) + \mathbf{Q}^c(\mathbf{q}(t),\dot{\mathbf{q}}(t),t) \tag{2.70}$$

Where $\mathbf{M}(t)$ is a $\mathbb{R}^{n\times n}$ generalized mass matrix and $\mathbf{Q}(t)$ is a $\mathbb{R}^{n}$ generalized force vector. On the other hand, it is necessary to take the second time derivative of the holonomic constraint equations. The time derivative of the kinematic constraint equations yields:

$$\begin{aligned}\dot{\mathbf{f}}(t) &= \frac{d}{dt}\mathbf{f}(\mathbf{q}(t),t) = \\ &= \frac{\partial \mathbf{f}(\mathbf{q}(t),t)}{\partial \mathbf{q}(t)}\dot{\mathbf{q}}(t) + \frac{\partial \mathbf{f}(\mathbf{q}(t),t)}{\partial t} = \\ &= \mathbf{f}_{\mathbf{q}}(t)\dot{\mathbf{q}}(t) + \mathbf{f}_{t}(t) = \\ &= \mathbf{0}\end{aligned} \tag{2.71}$$

Taking the second time derivative of the holonomic constraint equations yields:

$$\begin{aligned}\ddot{\mathbf{f}}(t) &= \frac{d}{dt}\left(\mathbf{f}_{\mathbf{q}}(t)\dot{\mathbf{q}}(t) + \mathbf{f}_{t}(t)\right) = \\ &= \frac{d}{dt}\left(\frac{\partial \mathbf{f}(\mathbf{q}(t),t)}{\partial \mathbf{q}(t)}\dot{\mathbf{q}}(t) + \frac{\partial \mathbf{f}(\mathbf{q}(t),t)}{\partial t}\right) = \\ &= \frac{d}{dt}\left(\frac{\partial \mathbf{f}(\mathbf{q}(t),t)}{\partial \mathbf{q}(t)}\right)\dot{\mathbf{q}}(t) + \frac{\partial \mathbf{f}(\mathbf{q}(t),t)}{\partial \mathbf{q}(t)}\ddot{\mathbf{q}}(t) + \frac{d}{dt}\left(\frac{\partial \mathbf{f}(\mathbf{q}(t),t)}{\partial t}\right) = \\ &= \left(\frac{\partial}{\partial \mathbf{q}(t)}\left(\frac{\partial \mathbf{f}(\mathbf{q}(t),t)}{\partial \mathbf{q}(t)}\right)\dot{\mathbf{q}}(t)\right)\dot{\mathbf{q}}(t) + \frac{\partial^2 \mathbf{f}(\mathbf{q}(t),t)}{\partial t\partial \mathbf{q}(t)}\dot{\mathbf{q}}(t) + \frac{\partial \mathbf{f}(\mathbf{q}(t),t)}{\partial \mathbf{q}(t)}\ddot{\mathbf{q}}(t) \\ &\quad + \frac{\partial^2 \mathbf{f}(\mathbf{q}(t),t)}{\partial \mathbf{q}(t)\partial t}\dot{\mathbf{q}}(t) + \frac{\partial^2 \mathbf{f}(\mathbf{q}(t),t)}{\partial t^2} = \\ &= \frac{\partial}{\partial \mathbf{q}(t)}\left(\frac{\partial \mathbf{f}(t)}{\partial \mathbf{q}(t)}\dot{\mathbf{q}}(t)\right)\dot{\mathbf{q}}(t) + \frac{\partial \mathbf{f}(t)}{\partial \mathbf{q}(t)}\ddot{\mathbf{q}}(t) + 2\frac{\partial^2 \mathbf{f}(t)}{\partial \mathbf{q}(t)\partial t}\dot{\mathbf{q}}(t) + \frac{\partial^2 \mathbf{f}(t)}{\partial t^2} = \\ &= \left(\mathbf{f}_{\mathbf{q}}(t)\dot{\mathbf{q}}(t)\right)_{\mathbf{q}}\dot{\mathbf{q}}(t) + \mathbf{f}_{\mathbf{q}}(t)\ddot{\mathbf{q}}(t) + 2\mathbf{f}_{t,\mathbf{q}}(t)\dot{\mathbf{q}}(t) + \mathbf{f}_{t,t}(t) = \\ &= \mathbf{0}\end{aligned}$$

(2.72)

This vector equation can be rearranged as:

$$\mathbf{f}_{\mathbf{q}}(t)\ddot{\mathbf{q}}(t) = \mathbf{Q}_f(t) \tag{2.73}$$

Where $\mathbf{Q}_f(t)$ is a $\mathbb{R}^n$ vector function defined as follows:

$$\begin{aligned}
\mathbf{Q}_f(t) &= \mathbf{Q}_f(\mathbf{q}(t), \dot{\mathbf{q}}(t), t) = \\
&= -\frac{\partial}{\partial \mathbf{q}(t)}\left(\frac{\partial \mathbf{f}(t)}{\partial \mathbf{q}(t)}\dot{\mathbf{q}}(t)\right)\dot{\mathbf{q}}(t) - 2\frac{\partial^2 \mathbf{f}(t)}{\partial \mathbf{q}(t)\partial t}\dot{\mathbf{q}}(t) - \frac{\partial^2 \mathbf{f}(t)}{\partial t^2} = \\
&= -\left(\mathbf{f}_{\mathbf{q}}(t)\dot{\mathbf{q}}(t)\right)_{\mathbf{q}}\dot{\mathbf{q}}(t) - 2\mathbf{f}_{t,\mathbf{q}}(t)\dot{\mathbf{q}}(t) - \mathbf{f}_{t,t}(t)
\end{aligned} \tag{2.74}$$

This vector equation represents the second time derivative of the holonomic constraint equations acting on the system $[12]$. The time derivative of nonholonomic constraint equations is:

$$\begin{aligned}
\dot{\mathbf{g}}(t) &= \frac{d}{dt}\mathbf{g}(\mathbf{q}(t), \dot{\mathbf{q}}(t), t) = \\
&= \frac{\partial \mathbf{g}(\mathbf{q}(t), \dot{\mathbf{q}}(t), t)}{\partial \mathbf{q}(t)}\dot{\mathbf{q}}(t) + \frac{\partial \mathbf{g}(\mathbf{q}(t), \dot{\mathbf{q}}(t), t)}{\partial \dot{\mathbf{q}}(t)}\ddot{\mathbf{q}}(t) + \frac{\partial \mathbf{g}(\mathbf{q}(t), \dot{\mathbf{q}}(t), t)}{\partial t} = \\
&= \mathbf{g}_{\mathbf{q}}(t)\dot{\mathbf{q}}(t) + \mathbf{g}_{\dot{\mathbf{q}}}(t)\ddot{\mathbf{q}}(t) + \mathbf{g}_t(t) = \\
&= \mathbf{0}
\end{aligned} \tag{2.75}$$

This vector equations can be rearranged as follows:

$$\mathbf{g}_{\dot{\mathbf{q}}}(t)\ddot{\mathbf{q}}(t) = \mathbf{Q}_g(t) \tag{2.76}$$

Where $\mathbf{Q}_g(t)$ is a $\mathbb{R}^n$ vector function defined as:

$$\begin{aligned}\mathbf{Q}_g(t) = \mathbf{Q}_g(\mathbf{q}(t),\dot{\mathbf{q}}(t),t) = \\ = -\frac{\partial \mathbf{g}(\mathbf{q}(t),\dot{\mathbf{q}}(t),t)}{\partial \mathbf{q}(t)}\dot{\mathbf{q}}(t) - \frac{\partial \mathbf{g}(\mathbf{q}(t),\dot{\mathbf{q}}(t),t)}{\partial t} = \\ = -\mathbf{g}_{\mathbf{q}}(t)\dot{\mathbf{q}}(t) - \mathbf{g}_t(t)\end{aligned} \tag{2.77}$$

Assume that the generalized acceleration involved in the nonholonomic constraint equations is a linear function:

$$\mathbf{D}(\mathbf{q}(t),\dot{\mathbf{q}}(t),t)\ddot{\mathbf{q}}(t) = \mathbf{Q}_h(\mathbf{q}(t),\dot{\mathbf{q}}(t),t) \tag{2.78}$$

Where $\mathbf{D}(t)$ is a $\mathbb{R}^{m_h \times n}$ matrix function and $\mathbf{Q}_h(t)$ is a $\mathbb{R}^{m_h}$ vector function. Since all the constraint equations are now linear in the generalized coordinates vector $\ddot{\mathbf{q}}(t)$, it can be simply proved $[8]$ that they can be all rearranged in an unique compact form as follows:

$$\mathbf{A}(\mathbf{q}(t),\dot{\mathbf{q}}(t),t)\ddot{\mathbf{q}}(t) = \mathbf{b}(\mathbf{q}(t),\dot{\mathbf{q}}(t),t) \tag{2.79}$$

Where $\mathbf{A}(t)$ is a $\mathbb{R}^{n_c \times n}$ matrix function defined as:

$$\mathbf{A}(t) = \begin{bmatrix} \dfrac{\partial \mathbf{f}(t)}{\partial \mathbf{q}(t)} \\ \dfrac{\partial \mathbf{g}(t)}{\partial \dot{\mathbf{q}}(t)} \\ \mathbf{D}(t) \\ \end{bmatrix} \tag{2.80}$$

And $\mathbf{b}(t)$ is a $\mathbb{R}^{n_c}$ vector function defined as:

$$\mathbf{b}(t) = \begin{bmatrix} \mathbf{Q}_f(t) \\ \mathbf{Q}_g(t) \\ \mathbf{Q}_h(t) \end{bmatrix} \tag{2.81}$$

According to this mathematical reformulation, the fundamental problem of constrained Dynamics can be restated as follows:

$$\begin{cases} \mathbf{M}(t)\ddot{\mathbf{q}}(t) = \mathbf{Q}(t) + \mathbf{A}^T(t)\boldsymbol{\lambda}(t) \\ \mathbf{A}(t)\ddot{\mathbf{q}}(t) = \mathbf{b}(t) \end{cases} \tag{2.82}$$

Where the generalized constrained vector $\mathbf{Q}^c(t)$ has been expressed as a function of the constraint equations through Lagrange multipliers rule.

2.3.4. FUNDAMENTAL EQUATIONS OF CONSTRAINED DYNAMICS

The fundamental equations of constrained Dynamics were originally developed in the field of analytical Dynamics by Udwadia and Kalaba $[8]$. Indeed, the constrained acceleration vector and the Lagrange multipliers vector can be obtained explicitly solving the fundamental problem of constrained Dynamics. To do that, some algebraic manipulation of the equations of motion of constrained mechanical systems must be performed. The basic observation is that these equations are linear in the generalized acceleration vector $[12]$ and therefore a matrix notation can be used to get:

$$\begin{bmatrix} \mathbf{M}(t) & -\mathbf{A}^T(t) \\ \mathbf{A}(t) & \mathbf{O} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}(t) \\ \boldsymbol{\lambda}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{Q}(t) \\ \mathbf{b}(t) \end{bmatrix} \tag{2.83}$$

Now one method to solve this matrix equation for $\ddot{\mathbf{q}}(t)$ and $\boldsymbol{\lambda}(t)$ is using the matrix inversion lemma exploiting the block structure of the generalized mass and constraints matrix $[13]$ to get:

$$\begin{bmatrix} \mathbf{M}(t) & -\mathbf{A}^T(t) \\ \mathbf{A}(t) & \mathbf{O} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{M}^{-1}(t) + \mathbf{B}_1(t)\mathbf{H}(t)\mathbf{B}_2(t) & -\mathbf{B}_1(t)\mathbf{H}(t) \\ -\mathbf{H}(t)\mathbf{B}_2(t) & \mathbf{H}(t) \end{bmatrix} \quad (2.84)$$

Where $\mathbf{B}_1(t)$, $\mathbf{B}_2(t)$ and $\mathbf{H}(t)$ are, respectively, $\mathbb{R}^{n\times n_c}$, $\mathbb{R}^{n_c\times n}$ and $\mathbb{R}^{n_c\times n_c}$ matrices define as:

$$\mathbf{B}_1(t) = -\mathbf{M}^{-1}(t)\mathbf{A}^T(t) \quad (2.85)$$

$$\mathbf{B}_2(t) = \mathbf{A}(t)\mathbf{M}^{-1}(t) \quad (2.86)$$

$$\begin{aligned} \mathbf{H}(t) &= \left(\mathbf{B}_2(t)\mathbf{A}^T(t)\right)^{-1} = \\ &= \left(-\mathbf{A}(t)\mathbf{B}_1(t)\right)^{-1} = \\ &= \left(\mathbf{A}(t)\mathbf{M}^{-1}(t)\mathbf{A}^T(t)\right)^{-1} \end{aligned} \quad (2.87)$$

By using this block matrix inversion lemma an explicit solution for the generalized acceleration vector and Lagrange multipliers vector can be found:

$$\begin{cases} \ddot{\mathbf{q}}(t) = \left(\mathbf{M}^{-1}(t) + \mathbf{B}_1(t)\mathbf{H}(t)\mathbf{B}_2(t)\right)\mathbf{Q}(t) - \mathbf{B}_1(t)\mathbf{H}(t)\mathbf{b}(t) \\ \boldsymbol{\lambda}(t) = -\mathbf{H}(t)\mathbf{B}_2(t)\mathbf{Q}(t) + \mathbf{H}(t)\mathbf{b}(t) \end{cases} \quad (2.88)$$

Manipulating mathematically this solution a deep physical insights can be found $[8]$. As a results, the fundamental equations of constrained Dynamics are deduced:

$$\begin{cases} \ddot{\mathbf{q}}(t) = \mathbf{a}(t) + \mathbf{B}(t)\boldsymbol{\lambda}(t) = \\ \qquad = \mathbf{a}(t) + \mathbf{F}(t)\mathbf{e}(t) \\ \boldsymbol{\lambda}(t) = \mathbf{H}(t)\mathbf{e}(t) \end{cases} \quad (2.89)$$

Where $\mathbf{B}(t)$ is a $\mathbb{R}^{n\times n_c}$ matrix defined as:

$$\mathbf{B}(t) = \mathbf{M}^{-1}(t)\mathbf{A}^T(t) \tag{2.90}$$

Here $\mathbf{a}(t)$ is a $\mathbb{R}^n$ vector defined as:

$$\mathbf{a}(t) = \mathbf{M}^{-1}(t)\mathbf{Q}(t) \tag{2.91}$$

And $\mathbf{a}_c(t)$ is a $\mathbb{R}^n$ vector defined as:

$$\mathbf{a}_c(t) = \mathbf{F}(t)\mathbf{e}(t) \tag{2.92}$$

Where $\mathbf{F}(t)$ is a $\mathbb{R}^{n \times n_c}$ matrix defined as:

$$\begin{aligned} \mathbf{F}(t) &= \mathbf{B}(t)\mathbf{H}(t) = \\ &= \mathbf{M}^{-1}(t)\mathbf{A}^T(t)\left(\mathbf{A}(t)\mathbf{M}^{-1}(t)\mathbf{A}^T(t)\right)^{-1} \end{aligned} \tag{2.93}$$

And $\mathbf{e}(t)$ is a $\mathbb{R}^{n_c}$ vector defined as:

$$\mathbf{e}(t) = \mathbf{b}(t) - \mathbf{A}(t)\mathbf{a}(t) \tag{2.94}$$

These vectors and matrices have a profound physical interpretation [8], [14]: the vector $\mathbf{a}(t)$ is the free system acceleration vector, that is to say the generalized acceleration the system would have if there were no constraints. The vector $\mathbf{e}(t)$ is the vector error that measures how much the free accelerations vectors $\mathbf{a}(t)$ violates the actual constraints acting on the system. Moreover, the matrix $\mathbf{F}(t)$ is a feedback matrix which, once multiplied by the acceleration error $\mathbf{e}(t)$, allows to express the system actual constrained acceleration $\ddot{\mathbf{q}}(t)$ as the sum of the free acceleration $\mathbf{a}(t)$ and a feedback term $\mathbf{F}(t)\mathbf{e}(t)$ which represents the acceleration $\mathbf{a}_c(t)$ induced to the system by the action of the constraints. Finally, the matrix $\mathbf{H}(t)$ is a proportional matrix that, once multiplied for the acceleration error $\mathbf{e}(t)$, allows to calculate explicitly the Lagrange multipliers $\boldsymbol{\lambda}(t)$. It is noteworthy to point out that the fact that the Lagrange multipliers $\boldsymbol{\lambda}(t)$ have been computed explicitly allows to compute directly the generalized constrained vector $\mathbf{Q}^c(t)$ too $[8]$, $[14]$. Indeed:

$$\begin{aligned} \mathbf{Q}^c(t) &= \mathbf{A}^T(t)\boldsymbol{\lambda}(t) = \\ &= \mathbf{A}^T(t)\mathbf{H}(t)\mathbf{e}(t) \end{aligned} \tag{2.95}$$

The consequences of this results are twofold. The first one is the logic fact that through this formula it is actually possible to predict the generalized constraint forces acting on a mechanical system as a function of the system state $[8]$. The second consequences is not so obvious: by using this formula the inverse Dynamics problem can be solved in a simple and elegant way $[14]$. Indeed, if the constrained equations do not correspond to actual physical constrains acting on the system, they can be assumed to be virtual constraints which must be satisfied by the system. In this way the vector of generalized constrained forces become a vector of generalized control actions that are necessary to force the system state to follow a specified path. In addition, it is intuitive to understand that the constraint equations cannot always be satisfied in the sense that not any kind of constraint equations can be effectively followed by the system state. Indeed, it can be proved that $[15]$ only the constraint equations which make the following matrix of full rank can be actually implemented:

$$\mathbf{M}_c(t) = \begin{bmatrix} \mathbf{M}(t) \\ \mathbf{A}(t) \end{bmatrix} \tag{2.96}$$

This matrix can be interpreted as a generalized controllabity matrix relative to nonlinear mechanical system. It noteworthy to point out that when the constraint matrix $\mathbf{A}(t)$ has not full rank, but at the same time the generalized controllability matrix $\mathbf{M}_c(t)$ has full rank, it can be proved $[8]$, $[14]$ that the solution of the fundamental problem of constrained Dynamics can be found simply replacing the matrix inverse operation with the Moore-Penrose pseudoinverse in the computation of the matrix $\mathbf{H}(t)$. Indeed:

$$\mathbf{H}(t) = \left(\mathbf{A}(t)\mathbf{M}^{-1}(t)\mathbf{A}^{T}(t)\right)^{+} \tag{2.97}$$

This means that even in presence of contradictory constraint equations, that is when the constraint matrix $\mathbf{A}(t)$ has not full rank, the constrained acceleration of the mechanical system exists and it is unique. Clearly, in this case the evolution of the constrained system satisfies the constraint equations only in a least-squares sense. In addition, it can be proved $[8]$ that in this case the Lagrange multipliers are not unique but are defined ut to an arbitrary $\mathbb{R}^m$ vector function:

$$\boldsymbol{\lambda}(t) = \mathbf{A}(t)\mathbf{A}^{+}(t)\mathbf{H}(t)\mathbf{e}(t) + \left(\mathbf{I} - \mathbf{A}(t)\mathbf{A}^{+}(t)\right)\mathbf{h}(t) \quad , \quad \forall \mathbf{h}(t) \in \mathbb{R}^m \tag{2.98}$$

Where $\mathbf{h}(t)$ is an arbitrary vector function. On the other hand, it noteworthy to point out that when the mass matrix $\mathbf{M}(t)$ has not full rank, but at the same time the generalized controllability matrix $\mathbf{M}_c(t)$ has full rank, it can be proved $[15]$, $[16]$ that the solution of the fundamental problem of constrained Dynamics can be found simply replacing in every computation respectively the mass matrix $\mathbf{M}(t)$ and the lagrangian component of generalized forces $\mathbf{Q}(t)$ with the following quantities:

$$\mathbf{M}_A(t) = \mathbf{M}(t) + \mathbf{A}^{+}(t)\mathbf{A}(t) \tag{2.99}$$

$$\mathbf{Q}_b(t) = \mathbf{Q}(t) + \mathbf{A}^{+}(t)\mathbf{b}(t) \tag{2.100}$$

This means that even in presence of a singular mass matrix the constrained acceleration of the mechanical system exists, it is unique and it can be computed explicitly.

2.4. VIBRATION OF DISCRETE AND CONTINUOUS SYSTEMS

2.4.1. INTRODUCTION

Mechanical system can be modelled in different forms. Basically, two fundamental category can be distinguished: discrete mechanical systems and continuous mechanical systems. From a mathematical viewpoint, the former are systems whose equations of motion can be represented by ordinary differential equations (ODE) whereas the latter are systems whose equations of motion can be modelled as partial differential equations (PDE). From a physical point of view, discrete systems are mechanical systems which can be modelled as lumped mass systems, that is to say this type of systems can be represented by an equivalent system which consists of some bulky elements that can be considered rigid with specified inertia properties whereas the other elements can be assumed elastic elements with negligible inertia effects. Therefore, the motion of discrete system can be described by a set of n coupled ordinary differential equations, one for each degree of freedom [13], [17], [18], [19], [20]. On the other hand, continuous systems are systems that consist of structural components which have distributed mass and elasticity and therefore their motion can be adequately represented only by partial differential equations which involve variables that depend on time as well as spatial coordinates [13], [17], [18], [19], [20]. Indeed, continuous systems have an infinite number of degrees of freedom. The following sections concern the vibration of discrete and continuous mechanical systems, namely systems whose equations of motion are linear. In particular, by using the Euler-Lagrange equations the general equations of motion relative to discrete multiple degrees of freedom system and relative to monodimensional continuous systems are both derived.

2.4.2. EQUATIONS OF MOTION OF MULTIPLE DEGREES OF FREEDOM SYSTEMS

Consider a discrete linear mechanical system composed of n particles connected by a set of linear elastic elements such as springs. Let $\mathbf{x}(t)$ be a $\mathbb{R}^n$ vector representing the displacements of the system particles. The kinetic energy $T(t)$ and the potential energy $U(t)$ of the system can be written in matrix notation as:

$$T(t) = \frac{1}{2}\dot{\mathbf{x}}^T(t)\mathbf{M}\dot{\mathbf{x}}(t) \tag{2.101}$$

$$U(t) = \frac{1}{2}\mathbf{x}^T(t)\mathbf{K}\mathbf{x}(t) \tag{2.102}$$

Where $\mathbf{M}$ and $\mathbf{K}$ are $\mathbb{R}^{n\times n}$ matrices representing respectively the system mass and stiffness matrices. The equations of motion of this multiple degrees of freedom mechanical system can be found by Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial T(t)}{\partial \dot{\mathbf{q}}(t)}\right)^T - \left(\frac{\partial T(t)}{\partial \mathbf{q}(t)}\right)^T + \left(\frac{\partial U(t)}{\partial \mathbf{q}(t)}\right)^T = \mathbf{Q}_{e,nc}(t) \tag{2.103}$$

Now assume that part of the virtual work done by nonconservative forces can be derived from the so-called Rayleigh's dissipation function $V(t)$:

$$V(t) = \frac{1}{2}\dot{\mathbf{x}}^T(t)\mathbf{R}\dot{\mathbf{x}}(t) \tag{2.104}$$

Where $\mathbf{R}$ is a $\mathbb{R}^{n\times n}$ damping matrix. Systems whose damping can be modelled through a quadratic Rayleigh's dissipation function are often referred to as linear viscously damped systems. Assume that the remaining part of nonconservative virtual work is performed by an generalized external force vector denoted with $\mathbf{F}(t)$. Lagrange equations yield:

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{R}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{F}(t) \tag{2.105}$$

These are the general equations of motion of a linear multiple degrees of freedom mechanical system $[21]$.

2.4.3. FREE VIBRATION OF MULTIPLE DEGREES OF FREEDOM SYSTEMS

In the case of free vibration of undamped linear discrete systems, the equations of motion reduce to:

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{0} \tag{2.106}$$

The solution of these differential equations can be found supposing that the displacement vector assumes the following form $[13]$, $[17]$, $[18]$, $[19]$, $[20]$:

$$\mathbf{x}(t) = \boldsymbol{\varphi} e^{\lambda_c t} \tag{2.107}$$

Where $\boldsymbol{\varphi}$ is an $\mathbb{R}^n$ unknown vector and λ_c is an unknown scalar. The assumed solution must satisfy the equations of motion and therefore, to impose it, the supposed solution can be put into the equation of motion in order to get:

$$\left(\lambda_c^2 \mathbf{M} + \mathbf{K}\right)\boldsymbol{\varphi} = \mathbf{0} \tag{2.108}$$

This is an eigenvalue problem that can be restated in the standard form as:

$$\mathbf{M}^{-1}\mathbf{K}\boldsymbol{\varphi} = -\lambda_c^2 \boldsymbol{\varphi} \tag{2.109}$$

The results of this problem is a set of $2n$ complex conjugate eigenvalues:

$$\begin{cases} \lambda_{c,2j-1} = -\mathbf{i}\omega_{n,j} \\ \lambda^*_{c,2j-1} = \mathbf{i}\omega_{n,j} \end{cases} \quad , \quad j = 1, 2, \dots, n \tag{2.110}$$

These eigenvalues $\lambda_{c,j}$ correspond to a set of n natural frequencies $\omega_{n,j}$ and to a set of n eigenvectors $\boldsymbol{\varphi}_j$ which represent the system mode shapes, namely the system principal modes of vibration. Indeed, the general solution of the undamped free vibration of a multiple degrees of freedom system can be written as a linear combination of the normal modes as follows:

$$\begin{aligned} \mathbf{x}(t) &= \sum_{j=1}^{n} \left(C_{2j-1}\boldsymbol{\varphi}_{2j-1} e^{\lambda_{c,2j-1}t} + C^*_{2j-1}\boldsymbol{\varphi}^*_{2j-1} e^{\lambda^*_{c,2j-1}t} \right) = \\ &= \sum_{j=1}^{n} \left(C_{2j-1}\boldsymbol{\varphi}_{2j-1} e^{-\mathbf{i}\omega_{n,2j-1}t} + C^*_{2j-1}\boldsymbol{\varphi}_{2j-1} e^{\mathbf{i}\omega_{n,2j-1}t} \right) \end{aligned} \tag{2.111}$$

Where the constants C_j can be determined by using the initial conditions.

2.4.4. FORCED VIBRATION OF MULTIPLE DEGREES OF FREEDOM SYSTEMS

In the case of forced vibration of undamped linear discrete systems, the equations of motion reduce to:

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{F}(t) \tag{2.112}$$

Define the $\mathbb{R}^{n\times n}$ modal matrix $\boldsymbol{\Phi}$ as a matrix whose columns are the system eigenvectors:

$$\boldsymbol{\Phi} = \begin{bmatrix} \boldsymbol{\varphi}_1 & \boldsymbol{\varphi}_2 & \dots & \boldsymbol{\varphi}_{n-1} & \boldsymbol{\varphi}_n \end{bmatrix} \tag{2.113}$$

Consider the following coordinate transformation [13], [17], [18], [19], [20]:

$$\mathbf{x}(t) = \mathbf{\Phi}\mathbf{q}(t) \tag{2.114}$$

Where the transformation matrix is precisely the system modal matrix $\mathbf{\Phi}$. This transformation is referred as modal transformation and the coordinate vector $\mathbf{q}(t)$ is a $\mathbb{R}^n$ vector of modal coordinates. The equations of motion of an undamped forced system can be transformed in modal coordinates to get:

$$\mathbf{M\Phi}\ddot{\mathbf{q}}(t) + \mathbf{K\Phi q}(t) = \mathbf{F}(t) \tag{2.115}$$

Premultiplying this equation by $\mathbf{\Phi}^T$ yields:

$$\mathbf{\Phi}^T\mathbf{M\Phi}\ddot{\mathbf{q}}(t) + \mathbf{\Phi}^T\mathbf{K\Phi q}(t) = \mathbf{\Phi}^T\mathbf{F}(t) \tag{2.116}$$

At this stage, an important property of normal modes can be used: the orthogonality of mode shapes $[13]$, $[17]$, $[18]$, $[19]$, $[20]$. This mathematical property can be stated as:

$$\begin{aligned}\boldsymbol{\varphi}_j^T\mathbf{M}\boldsymbol{\varphi}_h &= \delta_{j,h} m_{m,h} = \\ &= \begin{cases} 0 \quad , \quad j \neq h \\ m_{m,j} \quad , \quad j = h \end{cases}\end{aligned} \tag{2.117}$$

$$\begin{aligned}\boldsymbol{\varphi}_j^T\mathbf{K}\boldsymbol{\varphi}_h &= \delta_{j,h} k_{m,h} = \\ &= \begin{cases} 0 \quad , \quad j \neq h \\ k_{m,j} \quad , \quad j = h \end{cases}\end{aligned} \tag{2.118}$$

According to this property, the products of mass and stiffness matrices with modal matrix result to be $\mathbb{R}^{n\times n}$ diagonal matrices:

$$\begin{aligned}\mathbf{M}_m &= \mathbf{\Phi}^T\mathbf{M\Phi} = \\ &= diag(m_{m,1}, m_{m,2}, \ldots, m_{m,n-1}, m_{m,n})\end{aligned} \tag{2.119}$$

$$\mathbf{K}_m = \mathbf{\Phi}^T \mathbf{K} \mathbf{\Phi} = \\ = diag(k_{m,1}, k_{m,2}, \ldots, k_{m,n-1}, k_{m,n}) \tag{2.120}$$

Where $m_{m,j}$ and $k_{m,j}$ are respectively the modal mass and the modal stiffness of the system. Since the modal matrices $\mathbf{M}_m$ and $\mathbf{K}_m$ are diagonal matrices, the equations of motion expressed in modal coordinates are decoupled. Indeed:

$$\mathbf{M}_m \ddot{\mathbf{q}}(t) + \mathbf{K}_m \mathbf{q}(t) = \mathbf{Q}(t) \tag{2.121}$$

Where the modal force vector $\mathbf{Q}(t)$ is a $\mathbb{R}^n$ vector defined as:

$$\mathbf{Q}(t) = \mathbf{\Phi}^T \mathbf{F}(t) \tag{2.122}$$

It is easy to prove [13], [17], [18], [19], [20] that if the system is proportionally damped, that is to say if the damping matrix $\mathbf{R}$ can be expressed as a linear combination of the mass and stiffness matrices:

$$\mathbf{R} = \alpha \mathbf{M} + \beta \mathbf{K} \tag{2.123}$$

Then the modal decoupling of the equations of motion can still be performed to get:

$$\mathbf{M}_m \ddot{\mathbf{q}}(t) + \mathbf{R}_m \dot{\mathbf{q}}(t) + \mathbf{K}_m \mathbf{q}(t) = \mathbf{Q}(t) \tag{2.124}$$

Where $\mathbf{R}_m$ is a $\mathbb{R}^{n \times n}$ modal damping matrix and it is computed as a linear combination of the modal mass and stiffness matrices:

$$\mathbf{R}_m = \alpha \mathbf{M}_m + \beta \mathbf{K}_m \tag{2.125}$$

The equations of motion expressed in modal coordinates can be written in scalar form as:

$$m_{m,j}\ddot{q}_j(t)+r_{m,j}\dot{q}_j(t)+k_{m,j}q_j(t)=Q_{m,j}(t) \quad , \quad j=1,2,\ldots,n \quad (2.126)$$

Where $r_{m,j}$ is the system modal damping. These ordinary differential equations are decoupled: each equation behaves like a damped harmonic oscillator which vibrates according to one of each system natural frequency. Therefore, each equation can be independently solved by using Duhamel principle $[13]$, $[17]$, $[18]$, $[19]$, $[20]$:

$$\begin{aligned} q_j(t) &= k_{m,j}q_j(0)g_{m,j}(t)+m_{m,j}\dot{q}_j(0)h_{m,j}(t) \\ &\quad + \int_0^t Q_{m,j}(\tau)h_{m,j}(t-\tau)d\tau \quad , \quad j=1,2,\ldots,n \end{aligned} \quad (2.127)$$

Where the functions $g_{m,j}(t)$ and $h_{m,j}(t)$ are defined as:

$$g_{m,j}(t)=\frac{1}{k_{m,j}}e^{-\xi_j\omega_{n,j}t}\left(\cos(\omega_{d,j}t)+\frac{\xi_j\omega_{n,j}}{\omega_{d,j}}\sin(\omega_{d,j}t)\right) \quad , \quad j=1,2,\ldots,n$$
(2.128)

$$h_{m,j}(t)=\frac{1}{m_{m,j}}e^{-\xi_j\omega_{n,j}t}\,\frac{\sin(\omega_{d,j}t)}{\omega_{d,j}} \quad , \quad j=1,2,\ldots,n \quad (2.129)$$

Where $\omega_{n,j}$ and $\omega_{d,j}$ are respectively the damped and undamped natural frequencies of the system whereas ξ_j are the system damping ratios. The functions $g_{m,j}(t)$ and $h_{m,j}(t)$ have a remarkable physical interpretation: they are respectively the system response to a step down function and to an impulse function.

2.4.5. MODAL TRUNCATION METHOD

The modal truncation method simplify the equations of motion of a multiple degrees of freedom system considering only the significant mode shapes. Consider a multiple degrees of freedom system:

$$\mathbf{M}\ddot{\mathbf{x}}(t)+\mathbf{R}\dot{\mathbf{x}}(t)+\mathbf{K}\mathbf{x}(t)=\mathbf{F}(t) \tag{2.130}$$

Assume that the system is proportionally damped:

$$\mathbf{R}=\alpha\mathbf{M}+\beta\mathbf{K} \tag{2.131}$$

If the generalized eternal forces $\mathbf{F}(t)$ are periodic functions, they can be expressed with a set of oscillating functions by using a Fourier series $[13]$, $[17]$, $[18]$, $[19]$, $[20]$:

$$\mathbf{F}(t)=\frac{1}{2}\mathbf{a}_0+\sum_{k=1}^{\infty}\left(\mathbf{a}_k\cos(kt)+\mathbf{b}_k\sin(kt)\right) \tag{2.132}$$

Where $\mathbf{a}_0$, $\mathbf{a}_k$ and $\mathbf{b}_k$ are $\mathbb{R}^n$ constant vectors which can be computed as follows:

$$\mathbf{a}_0=\frac{1}{\pi}\int_{-\pi}^{\pi}\mathbf{F}(t)dt \tag{2.133}$$

$$\mathbf{a}_k=\frac{1}{\pi}\int_{-\pi}^{\pi}\mathbf{F}(t)\cos(kt)dt \quad , \quad k=1,2,\dots \tag{2.134}$$

$$\mathbf{b}_k=\frac{1}{\pi}\int_{-\pi}^{\pi}\mathbf{F}(t)\sin(kt)dt \quad , \quad k=1,2,\dots \tag{2.135}$$

This Fourier series can be approximated with a finite summation in order to retain only the terms which are close to the system natural frequencies. In this way, the external force vector $\mathbf{F}(t)$ can be written as a sum of few sinusoidal functions. As a consequence, only n_t modes of vibration of the system appear to be significant instead of the whole set. Therefore, an elimination of insignificant mode shapes can be performed by using a coordinate reduction technique [13], [17], [18], [19], [20]:

$$\mathbf{x}(t) = \mathbf{\Phi}_t \mathbf{q}_t(t) \tag{2.136}$$

Where $\mathbf{\Phi}_t$ is a $\mathbb{R}^{n \times n_t}$ truncated eigenvector matrix and $\mathbf{q}_t(t)$ is a $\mathbb{R}^{n_t}$ truncated modal vector made of only the relevant mode shapes and modal coordinates. Consequently, the system equations of motion can be approximated with truncated modal coordinates to yield:

$$\mathbf{M}_t \ddot{\mathbf{q}}_t(t) + \mathbf{R}_t \dot{\mathbf{q}}_t(t) + \mathbf{K}_t \mathbf{q}_t(t) = \mathbf{Q}_t(t) \tag{2.137}$$

Where $\mathbf{M}_t$, $\mathbf{R}_t$ and $\mathbf{K}_t$ are $\mathbb{R}^{n_t \times n_t}$ diagonal matrices and $\mathbf{Q}_t(t)$ is a $\mathbb{R}^{n_t}$ vector defined as follows:

$$\begin{aligned} \mathbf{M}_t &= \mathbf{\Phi}_t^T \mathbf{M} \mathbf{\Phi}_t = \\ &= diag(m_{m,1}, m_{m,2}, \ldots, m_{m,n_t-1}, m_{m,n_t}) \end{aligned} \tag{2.138}$$

$$\begin{aligned} \mathbf{K}_t &= \mathbf{\Phi}_t^T \mathbf{K} \mathbf{\Phi}_t = \\ &= diag(k_{m,1}, k_{m,2}, \ldots, k_{m,n_t-1}, k_{m,n_t}) \end{aligned} \tag{2.139}$$

$$\mathbf{Q}_t(t) = \mathbf{\Phi}_t^T \mathbf{F}(t) \tag{2.140}$$

These equations represent a set of n_t decoupled ordinary differential equations.

2.4.6. EQUATIONS OF MOTION OF MONODIMENSIONAL DISTRIBUTED PARAMETER SYSTEMS

Hamilton principle can be used to derive Lagrange equations of continuous systems with monodimensional distributed parameters $\left[19\right]$. The general form of Hamilton principle is:

$$\delta\int_{t_0}^{t_f} L(t)dt + \int_{t_0}^{t_f} \delta W_{e,nc}(t)dt = 0 \tag{2.141}$$

Where $L(t)$ is the system lagrangian and $\delta W_{e,nc}(t)$ is the virtual work of external nonconservative forces. In the case of a continuous system with monodimensional distributed parameters, the kinetic energy and potential energy can be expressed as:

$$T(t) = \int_0^l \bar{T}(x,t)dx \tag{2.142}$$

$$U(t) = \int_0^l \bar{U}(x,t)dx \tag{2.143}$$

Where $\bar{T}(x,t)$ and $\bar{U}(x,t)$ are the kinetic energy and potential energy density functions. These functions assume a different form according to the type of monodimensional continuous system in analysis, such as rods or beams for instance. In any case, the extended Hamilton principle yields:

$$
\begin{aligned}
&\delta\int_{t_0}^{t_f} L(t)dt + \int_{t_0}^{t_f} \delta W_{e,nc}(t)dt = \int_{t_0}^{t_f} \delta L(t)dt + \int_{t_0}^{t_f} \delta W_{e,nc}(t)dt = \\
&= \int_{t_0}^{t_f} \delta\left(T(t) - U(t)\right)dt + \int_{t_0}^{t_f} \delta W_{e,nc}(t)dt = \\
&= \int_{t_0}^{t_f} \left(\delta T(t) - \delta U(t)\right)dt + \int_{t_0}^{t_f} \delta W_{e,nc}(t)dt = \\
&= \int_{t_0}^{t_f} \left(\delta\int_0^l \bar{T}(x,t)dx - \delta\int_0^l \bar{U}(x,t)dx\right)dt + \int_{t_0}^{t_f} \delta W_{e,nc}(t)dt = \\
&= \int_{t_0}^{t_f}\int_0^l \delta\bar{T}(q(x,t), q_t(x,t), x, t)dxdt + \\
&\quad - \int_{t_0}^{t_f}\int_0^l \delta\bar{U}(q(x,t), q_x(x,t), q_{xx}(x,t), x, t)dxdt + \\
&\quad + \int_{t_0}^{t_f}\int_0^l \bar{Q}_{e,nc}(x,t)\delta q(x,t)dxdt = \\
&= \int_{t_0}^{t_f}\int_0^l \left(\frac{\partial\bar{T}(x,t)}{\partial q(x,t)}\delta q(x,t) + \frac{\partial\bar{T}(x,t)}{\partial q_t(x,t)}\delta q_t(x,t)\right)dxdt + \\
&\quad - \int_{t_0}^{t_f}\int_0^l \left(\frac{\partial\bar{U}(x,t)}{\partial q(x,t)}\delta q(x,t) + \frac{\partial\bar{U}(x,t)}{\partial q_x(x,t)}\delta q_x(x,t)\right)dxdt + \\
&\quad - \int_{t_0}^{t_f}\int_0^l \frac{\partial\bar{U}(x,t)}{\partial q_{xx}(x,t)}\delta q_{xx}(x,t)dxdt + \\
&\quad + \int_{t_0}^{t_f}\int_0^l \bar{Q}_{e,nc}(x,t)\delta q(x,t)dxdt = \\
&= \int_{t_0}^{t_f}\int_0^l \left(\frac{\partial\bar{T}(x,t)}{\partial q(x,t)}\delta q(x,t) + \frac{\partial\bar{T}(x,t)}{\partial q_t(x,t)}\frac{\partial}{\partial t}\left(\delta q(x,t)\right)\right)dxdt + \\
&\quad + \int_{t_0}^{t_f}\int_0^l \left(-\frac{\partial\bar{U}(x,t)}{\partial q(x,t)}\delta q(x,t) - \frac{\partial\bar{U}(x,t)}{\partial q_x(x,t)}\delta\left(\frac{\partial q(x,t)}{\partial x}\right)\right)dxdt + \\
&\quad - \int_{t_0}^{t_f}\int_0^l \frac{\partial\bar{U}(x,t)}{\partial q_{xx}(x,t)}\delta\left(\frac{\partial^2 q(x,t)}{\partial x^2}\right)dxdt + \\
&\quad + \int_{t_0}^{t_f}\int_0^l \bar{Q}_{e,nc}(x,t)\delta q(x,t)dxdt = 0
\end{aligned}
\tag{2.144}
$$

This formula can be further transformed integrating by parts:

$$
\begin{aligned}
&\int_0^l \left[\frac{\partial \bar{T}(x,t)}{\partial q_t(x,t)} \delta q(x,t) \right]_{t_0}^{t_f} dx + \\
&+\int_{t_0}^{t_f} \int_0^l \left(\frac{\partial \bar{T}(x,t)}{\partial q(x,t)} \delta q(x,t) - \frac{\partial^2 \bar{T}(x,t)}{\partial t \partial q_t(x,t)} \delta q(x,t) \right) dxdt + \\
&-\int_{t_0}^{t_f} \left[\frac{\partial \bar{U}(x,t)}{\partial q_x(x,t)} \delta q(x,t) \right]_0^l dt - \int_{t_0}^{t_f} \left[\frac{\partial \bar{U}(x,t)}{\partial q_{xx}(x,t)} \delta \left(\frac{\partial q(x,t)}{\partial x} \right) \right]_0^l dt + \\
&+\int_{t_0}^{t_f} \int_0^l \left(-\frac{\partial \bar{U}(x,t)}{\partial q(x,t)} \delta q(x,t) + \frac{\partial^2 \bar{U}(x,t)}{\partial x \partial q_x(x,t)} \delta q(x,t) \right) dxdt + \\
&+\int_{t_0}^{t_f} \int_0^l \left(\frac{\partial^2 \bar{U}(x,t)}{\partial x \partial q_{xx}(x,t)} \delta \left(\frac{\partial q(x,t)}{\partial x} \right) \right) dxdt + \\
&+\int_{t_0}^{t_f} \int_0^l \bar{Q}_{e,nc}(x,t) \delta q(x,t) dxdt = \\
&= \int_0^l \left[\frac{\partial \bar{T}(x,t)}{\partial q_t(x,t)} \delta q(x,t) \right]_{t_0}^{t_f} dx + \\
&+\int_{t_0}^{t_f} \int_0^l \left(\frac{\partial \bar{T}(x,t)}{\partial q(x,t)} - \frac{\partial^2 \bar{T}(x,t)}{\partial t \partial q_t(x,t)} \right) \delta q(x,t) dxdt + \\
&-\int_{t_0}^{t_f} \left[\frac{\partial \bar{U}(x,t)}{\partial q_x(x,t)} \delta q(x,t) \right]_0^l dt - \int_{t_0}^{t_f} \left[\frac{\partial \bar{U}(x,t)}{\partial q_{xx}(x,t)} \delta \left(\frac{\partial q(x,t)}{\partial x} \right) \right]_0^l dt + \\
&+\int_{t_0}^{t_f} \left[\frac{\partial^2 \bar{U}(x,t)}{\partial x \partial q_{xx}(x,t)} \delta q(x,t) \right]_0^l dt + \\
&+\int_{t_0}^{t_f} \int_0^l \left(-\frac{\partial \bar{U}(x,t)}{\partial q(x,t)} \delta q(x,t) + \frac{\partial^2 \bar{U}(x,t)}{\partial x \partial q_x(x,t)} \delta q(x,t) \right) dxdt + \\
&-\int_{t_0}^{t_f} \int_0^l \frac{\partial^3 \bar{U}(x,t)}{\partial x^2 \partial q_{xx}(x,t)} \delta q(x,t) dxdt + \int_{t_0}^{t_f} \int_0^l \bar{Q}_{e,nc}(x,t) \delta q(x,t) dxdt = 0
\end{aligned}
\tag{2.145}
$$

Finally, this formula can be rewritten as follows:

$$
\begin{aligned}
&\int_0^l \left[\frac{\partial \bar{T}(x,t)}{\partial q_t(x,t)} \delta q(x,t) \right]_{t_0}^{t_f} dx + \\
&+\int_{t_0}^{t_f} \int_0^l \left(\frac{\partial \bar{T}(x,t)}{\partial q(x,t)} - \frac{\partial^2 \bar{T}(x,t)}{\partial t \partial q_t(x,t)} \right) \delta q(x,t) dx dt + \\
&-\int_{t_0}^{t_f} \left[\frac{\partial \bar{U}(x,t)}{\partial q_x(x,t)} \delta q(x,t) \right]_0^l dt - \int_{t_0}^{t_f} \left[\frac{\partial \bar{U}(x,t)}{\partial q_{xx}(x,t)} \delta \left(\frac{\partial q(x,t)}{\partial x} \right) \right]_0^l dt + \\
&+\int_{t_0}^{t_f} \left[\frac{\partial^2 \bar{U}(x,t)}{\partial x \partial q_{xx}(x,t)} \delta q(x,t) \right]_0^l dt + \\
&+\int_{t_0}^{t_f} \int_0^l \left(-\frac{\partial \bar{U}(x,t)}{\partial q(x,t)} \delta q(x,t) + \frac{\partial^2 \bar{U}(x,t)}{\partial x \partial q_x(x,t)} \delta q(x,t) \right) dx dt + \\
&-\int_{t_0}^{t_f} \int_0^l \frac{\partial^3 \bar{U}(x,t)}{\partial x^2 \partial q_{xx}(x,t)} \delta q(x,t) dx dt + \int_{t_0}^{t_f} \int_0^l \bar{Q}_{e,nc}(x,t) \delta q(x,t) dx dt = \\
&= \int_0^l \left[\frac{\partial \bar{T}(x,t)}{\partial q_t(x,t)} \delta q(x,t) \right]_{t_0}^{t_f} dx + \\
&+\int_{t_0}^{t_f} \int_0^l \left(\frac{\partial \bar{T}(x,t)}{\partial q(x,t)} - \frac{\partial^2 \bar{T}(x,t)}{\partial t \partial q_t(x,t)} \right) \delta q(x,t) dx dt + \\
&+\int_{t_0}^{t_f} \left[\left(\frac{\partial^2 \bar{U}(x,t)}{\partial x \partial q_{xx}(x,t)} - \frac{\partial \bar{U}(x,t)}{\partial q_x(x,t)} \right) \delta q(x,t) \right]_0^l dt + \\
&-\int_{t_0}^{t_f} \left[\frac{\partial \bar{U}(x,t)}{\partial q_{xx}(x,t)} \delta \left(\frac{\partial q(x,t)}{\partial x} \right) \right]_0^l dt + \\
&+\int_{t_0}^{t_f} \int_0^l \left(-\frac{\partial \bar{U}(x,t)}{\partial q(x,t)} + \frac{\partial^2 \bar{U}(x,t)}{\partial x \partial q_x(x,t)} - \frac{\partial^3 \bar{U}(x,t)}{\partial x^2 \partial q_{xx}(x,t)} \right) \delta q(x,t) dx dt + \\
&+\int_{t_0}^{t_f} \int_0^l \bar{Q}_{e,nc}(x,t) \delta q(x,t) dx dt = 0 \quad , \quad \forall \delta q(x,t)
\end{aligned} \tag{2.146}
$$

The integrating term in the last equality can be set equal to zero because the virtual change of the configuration variable $\delta q(x,t)$ is arbitrary. Indeed:

$$\begin{cases} \delta q(x,t_0) = \delta q(x,t_f) = 0 \\ \left. \left(\left(\dfrac{\partial^2 \bar{U}(x,t)}{\partial x \partial q_{xx}(x,t)} - \dfrac{\partial \bar{U}(x,t)}{\partial q_x(x,t)} \right) \delta q(x,t) \right) \right|_0 = \\ = \left. \left(\left(\dfrac{\partial^2 \bar{U}(x,t)}{\partial x \partial q_{xx}(x,t)} - \dfrac{\partial \bar{U}(x,t)}{\partial q_x(x,t)} \right) \delta q(x,t) \right) \right|_l = 0 \\ \left. \left(\dfrac{\partial \bar{U}(x,t)}{\partial q_{xx}(x,t)} \delta \left(\dfrac{\partial q(x,t)}{\partial x} \right) \right) \right|_0 = \left. \left(\dfrac{\partial \bar{U}(x,t)}{\partial q_{xx}(x,t)} \delta \left(\dfrac{\partial q(x,t)}{\partial x} \right) \right) \right|_l = 0 \\ \dfrac{\partial^2 \bar{T}(x,t)}{\partial t \partial q_t(x,t)} - \dfrac{\partial \bar{T}(x,t)}{\partial q(x,t)} + \dfrac{\partial^3 \bar{U}(x,t)}{\partial x^2 \partial q_{xx}(x,t)} - \dfrac{\partial^2 \bar{U}(x,t)}{\partial x \partial q_x(x,t)} + \dfrac{\partial \bar{U}(x,t)}{\partial q(x,t)} = \bar{Q}_{e,nc}(x,t) \end{cases}$$

(2.147)

These equations are Lagrange equations for one-dimensional continuous systems $[22]$. It is meaningful to point out that these equations represent the complete set of systems differential equations of motion with all admissible boundary conditions. Consider now the transversal vibrations of beams $[22]$. In this case, the kinetic and potential energy density functions are:

$$\bar{T}(x,t) = \frac{1}{2} \rho A \left(\frac{\partial v(x,t)}{\partial t} \right)^2 \tag{2.148}$$

$$\bar{U}(x,t) = \frac{1}{2} E I_z \left(\frac{\partial^2 v(x,t)}{\partial x^2} \right)^2 \tag{2.149}$$

Where $v(x,t)$ is the transversal displacement function of the beam, ρ is the mass density, A is the cross-sectional area, E is Young elasticity modulus and I_z is the area moment of inertia of the beam. As a consequence, in the case of transversal vibrations of beams, Lagrange equations for one-dimensional continuous systems yields:

$$\begin{cases} \left.\left(\left(EI_z \dfrac{\partial^3 v(x,t)}{\partial x^3}\right)\delta v(x,t)\right)\right|_0 = \left.\left(\left(EI_z \dfrac{\partial^3 v(x,t)}{\partial x^3}\right)\delta v(x,t)\right)\right|_l = 0 \\ \left.\left(\left(EI_z \dfrac{\partial^2 v(x,t)}{\partial x^2}\right)\delta\left(\dfrac{\partial v(x,t)}{\partial x}\right)\right)\right|_0 = \left.\left(\left(EI_z \dfrac{\partial^2 v(x,t)}{\partial x^2}\right)\delta\left(\dfrac{\partial v(x,t)}{\partial x}\right)\right)\right|_l = 0 \\ \rho A \dfrac{\partial^2 v(x,t)}{\partial t^2} + EI_z \dfrac{\partial^4 v(x,t)}{\partial x^4} = f(x,t) \end{cases}$$

(2.150)

Where $f(x,t)$ is a distributed external force per unit length. This is the partial differential equation for the transversal vibration of beams assuming simple ends boundary conditions, such as free ends, fixed ends or simply supported ends.

2.4.7. FREE VIBRATION OF BEAMS

Consider now the free vibration of beams. In this case, the equation of reduces to:

$$\frac{\partial^2 v}{\partial t^2}(x,t) + c^2 \frac{\partial^4 v}{\partial x^4}(x,t) = 0 \tag{2.151}$$

Where the constant c is defined as:

$$c = \sqrt{\frac{EI_z}{\rho A}} \tag{2.152}$$

This is the Euler-Lagrange dynamic equation for Euler-Bernoulli beams $[23]$. This equation can be easily solved through the method of separation of variable $[13]$, $[17]$, $[18]$, $[19]$, $[20]$. According to this method, the solution $v(x,t)$ is assumed to be the product of two different functions, one $\phi(x)$ which depends only on the space independent variable x and the other one $q(t)$ which depends only on time variable t. Indeed:

$$v(x,t) = \phi(x)q(t) \tag{2.153}$$

Substituting this equation in the equation of motion leads to:

$$\phi(x)\ddot{q}(t) + c^2\phi^{IV}(x)q(t) = 0 \tag{2.154}$$

This equation can be rearranged in order to separate it in two parts: the former is a function only of time, the latter is a function only of space. This implies that:

$$\frac{\ddot{q}(t)}{q(t)} = -c^2\frac{\phi^{IV}(x)}{\phi(x)} = -\omega_n^2 \tag{2.155}$$

Where ω_n is a nonnegative constant to be determined. The last equation leads to two distinct ordinary differential equations in time and space:

$$\ddot{q}(t) + \omega_n^2 q(t) = 0 \tag{2.156}$$

$$\phi^{IV}(x) - \eta^4\phi(x) = 0 \tag{2.157}$$

Where η is a constant defined as:

$$\eta = \sqrt{\frac{\omega_n}{c}} \tag{2.158}$$

The first equation is the classic differential equation of the harmonic oscillator whereas the second equation is an ordinary homogeneous linear differential equations with constant coefficients. These equations can be easily solved to yield:

$$q(t) = A\sin(\omega_n t) + B\cos(\omega_n t) \tag{2.159}$$

$$\phi(x) = C\sin(\eta x) + D\cos(\eta x) + \bar{C}\sinh(\eta x) + \bar{D}\cosh(\eta x) \tag{2.160}$$

The constants A and B are indeterminate constants which can be found through initial conditions whereas the indeterminate constants C, D, $\bar{C}$ and $\bar{D}$ must be found by using boundary conditions. In any case, boundary conditions yields to an algebraic homogeneous systems of this type:

$$\mathbf{A}(\omega_n)\mathbf{b} = \mathbf{0} \tag{2.161}$$

Where the matrix $\mathbf{A}(\omega_n)$ is a function of the unknown constant ω_n and the vector $\mathbf{b}$ is made of the unknown constants C, D, $\bar{C}$ and $\bar{D}$. To avoid trivial solutions, the determinant of the matrix $\mathbf{A}(\omega_n)$ must be set equal to zero:

$$\det(\mathbf{A}(\omega_n)) = 0 \tag{2.162}$$

This equation is called frequency equation because its roots are the system eigenvalues or natural frequencies $\omega_{n,j}$, which are infinite:

$$\omega_{n,j} \quad , \quad j = 1,2,3,... \tag{2.163}$$

Substituting each natural frequencies in the algebraic equations for the unknown constants C, D, $\bar{C}$ and $\bar{D}$ leads to a corresponding set of mode shapes:

$$\phi_j(x) \quad , \quad j = 1,2,3,... \tag{2.164}$$

As a consequence, the solution of the free vibration of beams is an infinite linear combination space-dependent eigenfunction $\phi_j(x)$ and time-dependent modal coordinates $q_j(t)$ which can be expressed as follows:

$$
\begin{aligned}
v(x,t) &= \sum_{j=1}^{\infty}\left(\phi_j(x)q_j(t)\right) = \\
&= \sum_{j=1}^{\infty}\left(\left(C_j \sin(\eta_j x) + D_j \cos(\eta_j x) + \bar{C}_j \sinh(\eta_j x) + \bar{D}_j \cosh(\eta_j x)\right) A_j \sin(\omega_{n,j} t)\right) + \\
&+ \sum_{j=1}^{\infty}\left(\left(C_j \sin(\eta_j x) + D_j \cos(\eta_j x) + \bar{C}_j \sinh(\eta_j x) + \bar{D}_j \cosh(\eta_j x)\right) B_j \cos(\omega_{n,j} t)\right)
\end{aligned}
\tag{2.165}
$$

Where the unknown constants A_j and B_j can be determined using the set of initial conditions.

2.4.8. FORCED VIBRATIONS OF BEAMS

Consider the case of forced bending response of beams to an applied distributed force $f(x,t)$. The equation of motion is:

$$\rho A \frac{\partial^2 v}{\partial t^2}(x,t) + EI_z \frac{\partial^4 v}{\partial x^4}(x,t) = f(x,t) \tag{2.166}$$

This equation can be solved leveraging on the property of orthogonality of the eigenfunctions [13], [17], [18], [19], [20]. This mathematical property can be stated as:

$$
\begin{aligned}
\int_0^l \rho A \phi_j(x)\phi_h(x)dx &= \delta_{j,h} m_{m,h} = \\
&= \begin{cases} 0 & , \quad j \neq h \\ m_{m,j} & , \quad j = h \end{cases}
\end{aligned}
\tag{2.167}
$$

$$\int_0^l EI_z \phi_j^{II}(x)\phi_h^{II}(x)dx = \delta_{j,h} k_{m,h} = \\ = \begin{cases} 0 & , \quad j \neq h \\ k_{m,j} & , \quad j = h \end{cases} \tag{2.168}$$

Where the hypothesis of simple end boundary conditions is assumed. Indeed, consider the space integral of the equation of motion multiplied for a virtual change of displacement function $\delta v(x,t)$:

$$\int_0^l \rho A \frac{\partial^2 v}{\partial t^2}(x,t)\delta v(x,t)dx + \int_0^l EI_z \frac{\partial^4 v}{\partial x^4}(x,t)\delta v(x,t)dx = \\ = \int_0^l f(x,t)\delta v(x,t)dx \tag{2.169}$$

According to the orthogonality eigenfunctions property, this equation leads to:

$$m_{m,j}\ddot{q}_j(t) + k_{m,j}q_j(t) = Q_{m,j}(t) \quad , \quad j = 1,2,3,\ldots \tag{2.170}$$

Where $m_{m,j}$ and $k_{m,j}$ are respectively the modal mass and the modal stiffness of the system and the set of generalized lagrangian components $Q_{m,j}(t)$ relative to the external force function $f(x,t)$ are defined as:

$$Q_{m,j}(t) = \int_0^l f(x,t)\phi_j(x)dx \quad , \quad j = 1,2,3,\ldots \tag{2.171}$$

Consequently, a set of infinite decoupled equations each of which behaves like an harmonic oscillator has been obtained. Similarly to the case of discrete systems, the solution of this set of differential equations can be easily found by Duhamel integral:

$$q_j(t) = k_{m,j} q_j(0) g_{m,j}(t) + m_{m,j} \dot{q}_j(0) h_{m,j}(t) + \int_0^t Q_{m,j}(\tau) h_{m,j}(t-\tau) d\tau \quad , \quad j = 1, 2, 3, \dots \tag{2.172}$$

Where the functions $g_{m,j}(t)$ and $h_{m,j}(t)$ are defined as:

$$g_{m,j}(t) = \frac{1}{k_{m,j}} \cos(\omega_{n,j} t) \quad , \quad j = 1, 2, 3, \dots \tag{2.173}$$

$$h_{m,j}(t) = \frac{1}{m_{m,j}} \frac{\sin(\omega_{n,j} t)}{\omega_{n,j}} \quad , \quad j = 1, 2, 3, \dots \tag{2.174}$$

These functions represent the modal responses of undamped beams respectively to a step down function and to an impulse function. It is worth to note that the equations of motion relative to viscously damped beams can be easily obtained from these equations by induction $[24]$.

2.4.9. ASSUMED MODES METHOD

The assumed modes method can be seen as the continuous counterpart of modal truncation method. According to this method, the shape of deformation of the continuous systems is approximated using a set of assumed shape functions $[13]$, $[17]$, $[18]$, $[19]$, $[20]$. In analogy to the case of multiple degrees of freedom systems, consider a continuous beam whose only the first n_t mode shapes are significant. The displacement function $v(x,t)$ can be approximated according to this assumption:

$$v(x,t) = \boldsymbol{\varphi}_t^T(x) \mathbf{q}_t(t) \tag{2.175}$$

Where $\boldsymbol{\varphi}_t(x)$ is a vector containing the first n_t mode shapes and $\mathbf{q}_t(t)$ is a vector containing the first n_t modal coordinates. Indeed:

$$\boldsymbol{\varphi}_t(x) = \begin{bmatrix} \phi_1(x) \\ \phi_2(x) \\ \vdots \\ \phi_{n_t}(x) \end{bmatrix} \tag{2.176}$$

$$\mathbf{q}_t(x) = \begin{bmatrix} q_1(x) \\ q_2(x) \\ \vdots \\ q_{n_t}(x) \end{bmatrix} \tag{2.177}$$

By using this assumption, the kinetic energy and the potential energy of beams can be written as:

$$\begin{aligned} T(t) &= \frac{1}{2}\int_0^l \rho A\left(\frac{\partial v}{\partial t}(x,t)\right)^2 dx = \\ &= \frac{1}{2}\dot{\mathbf{q}}_t^T(t)\left(\int_0^l \rho A \boldsymbol{\varphi}_t(x)\boldsymbol{\varphi}_t^T(x)dx\right)\dot{\mathbf{q}}_t(t) = \\ &= \frac{1}{2}\dot{\mathbf{q}}_t^T(t)\mathbf{M}_t\dot{\mathbf{q}}_t(t) \end{aligned} \tag{2.178}$$

$$\begin{aligned} U(t) &= \frac{1}{2}\int_0^l EI_z\left(\frac{\partial^2 v}{\partial x^2}(x,t)\right)^2 dx = \\ &= \frac{1}{2}\mathbf{q}_t^T(t)\left(\int_0^l EI_z \boldsymbol{\varphi}_t^{II}(x)\boldsymbol{\varphi}_t^{II\,T}(x)dx\right)\mathbf{q}_t(t) = \\ &= \frac{1}{2}\mathbf{q}_t^T(t)\mathbf{K}_t\mathbf{q}_t(t) \end{aligned} \tag{2.179}$$

Where $\mathbf{M}_t$ and $\mathbf{K}_t$ are $\mathbb{R}^{n_t \times n_t}$ diagonal matrices corresponding to system modal mass and modal stiffness. These matrices are defined as follows:

$$\mathbf{M}_t = \int_0^l \rho A \boldsymbol{\varphi}_t(x)\boldsymbol{\varphi}_t^T(x)dx = \\ = diag(m_{m,1}, m_{m,2}, \ldots, m_{m,n_t-1}, m_{m,n_t}) \tag{2.180}$$

$$\mathbf{K}_t = \int_0^l EI_z \boldsymbol{\varphi}_t^{II}(x)\boldsymbol{\varphi}_t^{II\,T}(x)dx = \\ = diag(k_{m,1}, k_{m,2}, \ldots, k_{m,n_t-1}, k_{m,n_t}) \tag{2.181}$$

The effect of eternal force function $f(x,t)$ can be accounted for by using the virtual work to yield:

$$\mathbf{Q}_t(t) = \int_0^l f(x,t)\boldsymbol{\varphi}_t(x)dx \tag{2.182}$$

Where $\mathbf{Q}_t(t)$ is a $\mathbb{R}^{n_t}$ vector of the lagrangian component of eternal force function. Consequently, the equations of motion can be approximated through assumed modes method in order to yield:

$$\mathbf{M}_t\ddot{\mathbf{q}}_t(t) + \mathbf{K}_t\mathbf{q}_t(t) = \mathbf{Q}_t(t) \tag{2.183}$$

These equations are a set of decoupled ordinary differential equations and they are very similar to those relative to discrete systems obtained using the modal truncation method. Indeed, these equations represent an equivalent finite-dimensional model for the infinite degrees of freedom continuous systems.

2.5. KINEMATICS AND DYNAMICS OF RIGID MULTIBODY SYSTEMS

2.5.1. INTRODUCTION

The purpose of the following sections is to develop methods for the kinematic and dynamic analysis of multibody systems which consist of interconnected rigid components [25], [26]. On the other hand, the analysis of flexible multibody systems has been postponed to the subsequent chapters. The approach followed here was originally developed by Shabana [11], [12], [13]. Basic to any study of multibody systems is the understanding of the motion of the different bodies and components that form the system, namely the subsystem kinematics. When dealing with rigid body system, the kinematics of the body is completely described by the kinematics of a frame coordinate system which is rigidly connected to a point of the body. This frame of reference is formed of three orthogonal axes and it is referred to as floating reference. Therefore, the local position of a particle on the body can be described in terms of fixed components along the axes of this moving coordinates system. Besides, Chasles theorems states that the displacement of a rigid frame can be described by a translation and a rotation about an instantaneous axes of rotation. Hence, it is fundamental to understand the mathematical description of rotation in space. Once that an adequate kinematic description of the system configuration has been obtained, the equations of motion that model the dynamics of rigid body systems can be derived by using Lagrange equations. Indeed, an effective systematic technique can be developed to derive the mass matrix and the quadratic velocity vector of multibody systems. To do that, it necessary to compute a set of inertia shape integrals which represent the total mass, the moment of mass and the inertia matrix of the rigid bodies [11], [12], [13].

2.5.2. REFERENCE FRAMES KINEMATICS

Consider a set of n_b rigid bodies connected with different type of mechanical joints. The spatial configuration of this multibody system can be described setting one inertial frame of reference and a floating frame of reference for each rigid body. Let $\mathbf{r}^i(P^i,t)$ be the position vector of a generic particle P^i on the body i. This vector can be expressed as the sum of global position vector $\mathbf{R}^i(t)$ of the origin O^i of the body reference and the position vector $\mathbf{u}^i(P^i)$ of point P^i with respect to O^i. Indeed:

$$\mathbf{r}^i(P^i,t)=\mathbf{R}^i(t)+\mathbf{u}^i(P^i) \tag{2.184}$$

Where $\mathbf{r}^i(P^i,t)$, $\mathbf{R}^i(t)$ and $\mathbf{u}^i(P^i)$ are $\mathbb{R}^3$ vectors whose components are referred to the global frame of reference. The components of the position vector $\mathbf{u}^i(P^i)$ can be referred to the body frame of reference using the rotation matrix $\mathbf{A}^i(t)$:

$$\mathbf{u}^i(P^i)=\mathbf{A}^i(t)\overline{\mathbf{u}}^i(P^i) \tag{2.185}$$

Where $\overline{\mathbf{u}}^i(P^i)$ is a $\mathbb{R}^3$ vector whose components represents the position of point P^i referred to the body floating frame of reference and $\mathbf{A}^i(t)$ is a $\mathbb{R}^{3\times 3}$ rotation matrix. Hence, the position of a particle P^i on the body i can be expressed as:

$$\mathbf{r}^i(P^i,t)=\mathbf{R}^i(t)+\mathbf{A}^i(t)\overline{\mathbf{u}}^i(P^i) \tag{2.186}$$

This vector equation represents a fundamental formula for multibody system analysis. The rotation matrix $\mathbf{A}^i(t)$ can be computed by Rodriguez formula:

$$\begin{aligned}\mathbf{A}^i(t)&=e^{\tilde{\mathbf{v}}^i(t)\vartheta^i(t)}=\\&=\mathbf{I}+\sin(\vartheta^i(t))\tilde{\mathbf{v}}^i(t)+2\sin^2(\frac{\vartheta^i(t)}{2})\tilde{\mathbf{v}}^{i\,2}(t)\end{aligned} \tag{2.187}$$

In this formula $\vartheta^i(t)$ is the instantaneous rotation angle and $\tilde{\mathbf{v}}^i(t)$ is a $\mathbb{R}^{3\times3}$ skew symmetric matrix obtained from the $\mathbb{R}^3$ unit vector $\mathbf{v}^i(t)$ corresponding to the direction of the instantaneous rotation axis. Indeed:

$$\tilde{\mathbf{v}}^i(t) = \begin{bmatrix} 0 & -v_3^i(t) & v_2^i(t) \\ v_3^i(t) & 0 & -v_1^i(t) \\ -v_2^i(t) & v_1^i(t) & 0 \end{bmatrix} \tag{2.188}$$

It is straightforward to note that the rotation matrix $\mathbf{A}^i(t)$ is an orthogonal matrix, that is the inverse of rotation matrix is equal to its transposed. Indeed:

$$\begin{aligned} \mathbf{A}^{i\,-1}(t) &= e^{-\tilde{\mathbf{v}}^i(t)\vartheta^i(t)} = \\ &= \mathbf{I} - \sin(\vartheta^i(t))\tilde{\mathbf{v}}^i(t) + 2\sin^2(\frac{\vartheta^i(t)}{2})\tilde{\mathbf{v}}^{i\,2}(t) = \\ &= e^{\tilde{\mathbf{v}}^{i\,T}(t)\vartheta^i(t)} = \\ &= \mathbf{A}^{i\,T}(t) \end{aligned} \tag{2.189}$$

Instead of using the rotation angle $\vartheta^i(t)$ and the direction vector $\mathbf{v}^i(t)$, the rotation matrix $\mathbf{A}^i(t)$ can be also expressed in other forms according to the set of rotation parameters used. Consider the set of Euler's parameters which are defined as:

$$\begin{cases} \theta_0^i(t) = \cos(\frac{\vartheta^i(t)}{2}) \\ \theta_1^i(t) = v_1^i(t)\sin(\frac{\vartheta^i(t)}{2}) \\ \theta_2^i(t) = v_2^i(t)\sin(\frac{\vartheta^i(t)}{2}) \\ \theta_3^i(t) = v_3^i(t)\sin(\frac{\vartheta^i(t)}{2}) \end{cases} \tag{2.190}$$

These parameters can be grouped in a $\mathbb{R}^4$ vector $\boldsymbol{\theta}^i(t)$. Indeed:

$$\boldsymbol{\theta}^i(t) = \begin{bmatrix} \theta_0^i(t) \\ \theta_1^i(t) \\ \theta_2^i(t) \\ \theta_3^i(t) \end{bmatrix} \tag{2.191}$$

Euler's parameters are not all independent coordinates, as can be noted from their definition. Indeed, Euler's parameters satisfy the following equation:

$$\boldsymbol{\theta}^{iT}(t)\boldsymbol{\theta}^i(t) - 1 = 0 \tag{2.192}$$

By using Euler's parameters the rotation matrix $\mathbf{A}^i(t)$ can be rewritten as:

$$\mathbf{A}^i(t) = \mathbf{E}^i(t)\bar{\mathbf{E}}^{iT}(t) \tag{2.193}$$

Where the matrices $\mathbf{E}^i(t)$ and $\bar{\mathbf{E}}^i(t)$ are $\mathbb{R}^{3\times4}$ matrices which can be computed through Euler's parameters vector $\boldsymbol{\theta}^i(t)$:

$$\mathbf{E}^i(t) = \begin{bmatrix} -\theta_1^i(t) & \theta_0^i(t) & -\theta_3^i(t) & \theta_2^i(t) \\ -\theta_2^i(t) & \theta_3^i(t) & \theta_0^i(t) & -\theta_1^i(t) \\ -\theta_3^i(t) & -\theta_2^i(t) & \theta_1^i(t) & \theta_0^i(t) \end{bmatrix} \tag{2.194}$$

$$\bar{\mathbf{E}}^i(t) = \begin{bmatrix} -\theta_1^i(t) & \theta_0^i(t) & \theta_3^i(t) & -\theta_2^i(t) \\ -\theta_2^i(t) & -\theta_3^i(t) & \theta_0^i(t) & \theta_1^i(t) \\ -\theta_3^i(t) & \theta_2^i(t) & -\theta_1^i(t) & \theta_0^i(t) \end{bmatrix} \tag{2.195}$$

These matrices are also useful to compute the angular velocity vector referred to the global coordinate system $\boldsymbol{\omega}^i(t)$ and the angular velocity vector referred to the local coordinate system $\bar{\boldsymbol{\omega}}^i(t)$ by using the time derivative of the Euler's parameters vector $\boldsymbol{\theta}^i(t)$. Indeed:

$$\boldsymbol{\omega}^i(t) = \mathbf{G}^i(t)\dot{\boldsymbol{\theta}}^i(t) \tag{2.196}$$

$$\bar{\boldsymbol{\omega}}^i(t) = \bar{\mathbf{G}}^i(t)\dot{\boldsymbol{\theta}}^i(t) \tag{2.197}$$

Where the matrices $\mathbf{G}^i(t)$ and $\bar{\mathbf{G}}^i(t)$ are $\mathbb{R}^{3\times 4}$ matrices which can be easily computed through the matrices $\mathbf{E}^i(t)$ and $\bar{\mathbf{E}}^i(t)$:

$$\mathbf{G}^i(t) = 2\mathbf{E}^i(t) \tag{2.198}$$

$$\bar{\mathbf{G}}^i(t) = 2\bar{\mathbf{E}}^i(t) \tag{2.199}$$

On the other hand, it can be proved that the time derivative of rotation matrix $\mathbf{A}^i(t)$ can be computed by using the rotation matrix itself and a skew matrix corresponding to the angular velocity vector referred to global or local coordinate systems:

$$\begin{aligned}\dot{\mathbf{A}}^i(t) &= \tilde{\boldsymbol{\omega}}^i(t)\mathbf{A}^i(t) = \\ &= \mathbf{A}^i(t)\tilde{\bar{\boldsymbol{\omega}}}^i(t)\end{aligned} \tag{2.200}$$

At this stage, consider as generalized coordinate vector a $\mathbb{R}^7$ vector $\mathbf{q}^i(t)$ for each body which is formed by the position of the origin of the body $\mathbf{R}^i(t)$ and the Euler's parameters $\boldsymbol{\theta}^i(t)$ representing the rotation of the body:

$$\mathbf{q}^i(t) = \begin{bmatrix}\mathbf{R}^i(t)\\ \boldsymbol{\theta}^i(t)\end{bmatrix} \tag{2.201}$$

By using all previous definition the time derivative of the position of a generic particle on the body i can be computed as:

$$\begin{aligned}\dot{\mathbf{r}}^i(P^i,t) &= \dot{\mathbf{R}}^i(t) + \dot{\mathbf{A}}^i(t)\bar{\mathbf{u}}^i(P^i) = \\ &= \dot{\mathbf{R}}^i(t) + \mathbf{A}^i(t)\tilde{\bar{\boldsymbol{\omega}}}^i(t)\bar{\mathbf{u}}^i(P^i) = \\ &= \dot{\mathbf{R}}^i(t) - \mathbf{A}^i(t)\tilde{\bar{\mathbf{u}}}^i(P^i)\bar{\boldsymbol{\omega}}^i(t) = \\ &= \dot{\mathbf{R}}^i(t) - \mathbf{A}^i(t)\tilde{\bar{\mathbf{u}}}^i(P^i)\bar{\mathbf{G}}^i(t)\dot{\boldsymbol{\theta}}^i(t) = \\ &= \begin{bmatrix}\mathbf{I} & -\mathbf{A}^i(t)\tilde{\bar{\mathbf{u}}}^i(P^i)\bar{\mathbf{G}}^i(t)\end{bmatrix}\begin{bmatrix}\dot{\mathbf{R}}^i(t) \\ \dot{\boldsymbol{\theta}}^i(t)\end{bmatrix} = \\ &= \mathbf{L}^i(P^i,t)\dot{\mathbf{q}}^i(t)\end{aligned} \tag{2.202}$$

Where $\mathbf{L}^i(P^i,t)$ is a $\mathbb{R}^{3\times 7}$ matrix defined as:

$$\begin{aligned}\mathbf{L}^i(P^i,t) &= \begin{bmatrix}\mathbf{L}^i_R(P^i,t) & \mathbf{L}^i_\theta(P^i,t)\end{bmatrix} = \\ &= \begin{bmatrix}\mathbf{I} & -\mathbf{A}^i(t)\tilde{\bar{\mathbf{u}}}^i(P^i)\bar{\mathbf{G}}^i(t)\end{bmatrix}\end{aligned} \tag{2.203}$$

This matrix is a function of the reference coordinate vector $\mathbf{q}^i(t)$ and depends on the particle P^i under consideration. It is remarkable to note that the virtual change of the position vector $\mathbf{r}^i(P^i,t)$ can be computed in the same way by using the matrix $\mathbf{L}^i(P^i,t)$:

$$\delta\mathbf{r}^i(P^i,t) = \mathbf{L}^i(P^i,t)\delta\mathbf{q}^i(t) \tag{2.204}$$

Indeed, this matrix is a jacobian transformation matrix which mathematically describe the relation between the physical coordinates vector $\mathbf{r}^i(P^i,t)$ and the lagrangian coordinates vector $\mathbf{q}^i(t)$.

2.5.3. MASS MATRIX OF RIGID BODIES

Once that the kinematic description of motion has been obtained, the mass matrix of the generic rigid body i can be computed using the definition of kinetic energy $T^i(t)$. Indeed:

$$
\begin{aligned}
T^i(t) &= \frac{1}{2}\int_{\Omega^i} \rho^i \dot{\mathbf{r}}^{iT}(P^i,t)\dot{\mathbf{r}}^i(P^i,t)dV^i = \\
&= \frac{1}{2}\dot{\mathbf{q}}^{iT}(t)\int_{\Omega^i} \rho^i \mathbf{L}^{iT}(P^i,t)\mathbf{L}^i(P^i,t)dV^i\dot{\mathbf{q}}^i(t) = \\
&= \frac{1}{2}\dot{\mathbf{q}}^{iT}(t)\mathbf{M}^i(t)\dot{\mathbf{q}}^i(t)
\end{aligned}
\tag{2.205}
$$

Where ρ^i and Ω^i are respectively the mass density and the volume of body i and $\mathbf{M}^i(t)$ is a $\mathbb{R}^{7\times 7}$ matrix representing the body i mass matrix. The body mass matrix can be computed as:

$$
\begin{aligned}
\mathbf{M}^i(t) &= \int_{\Omega^i} \rho^i \mathbf{L}^{iT}(P^i,t)\mathbf{L}^i(P^i,t)dV^i = \\
&= \int_{\Omega^i} \rho^i \begin{bmatrix} \mathbf{L}_R^{iT}(P^i,t) \\ \mathbf{L}_\theta^{iT}(P^i,t) \end{bmatrix} \begin{bmatrix} \mathbf{L}_R^i(P^i,t) & \mathbf{L}_\theta^i(P^i,t) \end{bmatrix} dV^i = \\
&= \begin{bmatrix} \mathbf{m}_{R,R}^i(t) & \mathbf{m}_{R,\theta}^i(t) \\ \mathbf{m}_{\theta,R}^i(t) & \mathbf{m}_{\theta,\theta}^i(t) \end{bmatrix}
\end{aligned}
\tag{2.206}
$$

Where $\mathbf{m}_{R,R}^i(t)$, $\mathbf{m}_{R,\theta}^i(t)$ and $\mathbf{m}_{\theta,\theta}^i(t)$ are respectively $\mathbb{R}^{3\times 3}$, $\mathbb{R}^{3\times 4}$ and $\mathbb{R}^{4\times 4}$ symmetric matrices which can be computed explicitly. Indeed, the mass submatrix $\mathbf{m}_{R,R}^i(t)$ can be computed as:

$$\begin{aligned}\mathbf{m}^i_{R,R}(t) &= \int_{\Omega^i} \rho^i \mathbf{L}^{iT}_R(P^i,t)\mathbf{L}^i_R(P^i,t)dV^i = \\ &= \int_{\Omega^i} \rho^i \mathbf{I} dV^i = \\ &= m^i \mathbf{I}\end{aligned} \tag{2.207}$$

Where m^i is the total mass of body i. The mass submatrix $\mathbf{m}^i_{R,\theta}(t)$ can be computed in this way:

$$\begin{aligned}\mathbf{m}^i_{R,\theta}(t) &= \int_{\Omega^i} \rho^i \mathbf{L}^{iT}_R(P^i,t)\mathbf{L}^i_\theta(P^i,t)dV^i = \\ &= -\int_{\Omega^i} \rho^i \mathbf{A}^i(t)\tilde{\bar{\mathbf{u}}}^i(P^i)\bar{\mathbf{G}}^i(t)dV^i = \\ &= -\mathbf{A}^i(t)\int_{\Omega^i} \rho^i \tilde{\bar{\mathbf{u}}}^i(P^i)dV^i\bar{\mathbf{G}}^i(t) = \\ &= -\mathbf{A}^i(t)\tilde{\bar{\mathbf{U}}}^i\bar{\mathbf{G}}^i(t) = \\ &= \mathbf{m}^{iT}_{\theta,R}(t)\end{aligned} \tag{2.208}$$

Where the $\mathbb{R}^{3\times 3}$ matrix $\tilde{\bar{\mathbf{U}}}^i$ is a skew symmetric matrix defined as:

$$\begin{aligned}\tilde{\bar{\mathbf{U}}}^i &= \int_{\Omega^i} \rho^i \tilde{\bar{\mathbf{u}}}^i(P^i)dV^i = \\ &= Skew\left(\int_{\Omega^i} \rho^i \bar{\mathbf{u}}^i(P^i)dV^i\right) = \\ &= Skew\left(\bar{\mathbf{U}}^i\right)\end{aligned} \tag{2.209}$$

Where the $\mathbb{R}^3$ vector $\bar{\mathbf{U}}^i$ can be computed by the following integral:

$$\bar{\mathbf{U}}^i = \int_{\Omega^i} \rho^i \bar{\mathbf{u}}^i(P)dV^i \tag{2.210}$$

From this definition is straightforward to note that if the origin of the local frame of reference coincides with the centre of mass of body i, then the mass submatrix $\mathbf{m}^i_{R,\theta}(t)$ is equal to zero. Finally, the mass submatrix $\mathbf{m}^i_{\theta,\theta}(t)$ can be computed as:

$$\begin{aligned}
\mathbf{m}^i_{\theta,\theta}(t) &= \int_{\Omega^{i,j}} \rho^i \mathbf{L}^{iT}_{\theta}(P^i,t)\mathbf{L}^i_{\theta}(P^i,t)dV^i = \\
&= \int_{\Omega^i} \rho^i \overline{\mathbf{G}}^{iT}(t)\tilde{\overline{\mathbf{u}}}^{iT}(P^i)\mathbf{A}^{iT}(t)\mathbf{A}^i(t)\tilde{\overline{\mathbf{u}}}^i(P^i)\overline{\mathbf{G}}^i(t)dV^i = \\
&= \overline{\mathbf{G}}^{iT}(t)\int_{\Omega^i} \rho^{i,j}\tilde{\overline{\mathbf{u}}}^{iT}(P^i)\tilde{\overline{\mathbf{u}}}^i(P^i)dV^i\overline{\mathbf{G}}^i(t) = \\
&= \overline{\mathbf{G}}^{iT}(t)\overline{\mathbf{I}}^i_{\theta,\theta}\overline{\mathbf{G}}^i(t)
\end{aligned} \tag{2.211}$$

Where the orthogonality property of rotation matrix $\mathbf{A}^i(t)$ has been used. The $\mathbb{R}^{3\times 3}$ matrix $\overline{\mathbf{I}}^i_{\theta,\theta}$ is the inertia matrix of body i which is defined as:

$$\begin{aligned}
\overline{\mathbf{I}}^i_{\theta,\theta} &= \int_{\Omega^i} \rho^i \tilde{\overline{\mathbf{u}}}^{iT}(P^i)\tilde{\overline{\mathbf{u}}}^i(P^i)dV^i = \\
&= \int_{\Omega^i} \rho^i \left(\overline{\mathbf{u}}^{iT}(P^i)\overline{\mathbf{u}}^i(P^i)\mathbf{I} - \overline{\mathbf{u}}^i(P^i)\overline{\mathbf{u}}^{iT}(P^i)\right)dV^i = \\
&= \begin{bmatrix}
\left(\overline{\mathbf{I}}^i_{\theta,\theta}\right)_{1,1} & \left(\overline{\mathbf{I}}^i_{\theta,\theta}\right)_{1,2} & \left(\overline{\mathbf{I}}^i_{\theta,\theta}\right)_{1,3} \\
\left(\overline{\mathbf{I}}^i_{\theta,\theta}\right)_{2,1} & \left(\overline{\mathbf{I}}^i_{\theta,\theta}\right)_{2,2} & \left(\overline{\mathbf{I}}^i_{\theta,\theta}\right)_{2,3} \\
\left(\overline{\mathbf{I}}^i_{\theta,\theta}\right)_{3,1} & \left(\overline{\mathbf{I}}^i_{\theta,\theta}\right)_{3,2} & \left(\overline{\mathbf{I}}^i_{\theta,\theta}\right)_{3,3}
\end{bmatrix}
\end{aligned} \tag{2.212}$$

The components of inertia matrix $\overline{\mathbf{I}}^i_{\theta,\theta}$ can be computed as:

$$\begin{aligned}
\left(\overline{\mathbf{I}}^i_{\theta,\theta}\right)_{1,1} &= \int_{\Omega^i} \rho^i \left(\left(\overline{u}^i_2(P^i)\right)^2 + \left(\overline{u}^i_3(P^i)\right)^2\right)dV^i = \\
&= \int_{\Omega^i} \rho^i \left(\overline{y}^{i\,2}(P^i) + \overline{z}^{i\,2}(P^i)\right)dV^i
\end{aligned} \tag{2.213}$$

$$\left(\overline{\mathbf{I}}_{\theta,\theta}^{i}\right)_{1,2} = -\int_{\Omega^i} \rho^i \overline{u}_1^i(P^i)\overline{u}_2^i(P^i)dV^i = \\ = -\int_{\Omega^i} \rho^i \overline{x}^i(P^i)\overline{y}^i(P^i)dV^i = \\ = \left(\overline{\mathbf{I}}_{\theta,\theta}^{i}\right)_{2,1} \tag{2.214}$$

$$\left(\overline{\mathbf{I}}_{\theta,\theta}^{i}\right)_{1,3} = -\int_{\Omega^i} \rho^i \overline{u}_1^i(P^i)\overline{u}_3^i(P^i)dV^i = \\ = -\int_{\Omega^i} \rho^i \overline{x}^i(P^i)\overline{z}^i(P^i)dV^i = \\ = \left(\overline{\mathbf{I}}_{\theta,\theta}^{i}\right)_{3,1} \tag{2.215}$$

$$\left(\overline{\mathbf{I}}_{\theta,\theta}^{i}\right)_{2,2} = \int_{\Omega^i} \rho^i \left(\left(\overline{u}_3^i(P^i)\right)^2 + \left(\overline{u}_1^i(P^i)\right)^2\right)dV^i = \\ = \int_{\Omega^i} \rho^i \left(\overline{z}^{i\,2}(P^i) + \overline{x}^{i\,2}(P^i)\right)dV^i \tag{2.216}$$

$$\left(\overline{\mathbf{I}}_{\theta,\theta}^{i}\right)_{2,3} = -\int_{\Omega^i} \rho^i \overline{u}_3^i(P^i)\overline{u}_3^i(P^i)dV^i = \\ = -\int_{\Omega^i} \rho^i \overline{y}^i(P^i)\overline{z}^i(P^i)dV^i = \\ = \left(\overline{\mathbf{I}}_{\theta,\theta}^{i}\right)_{3,2} \tag{2.217}$$

$$\left(\overline{\mathbf{I}}_{\theta,\theta}^{i}\right)_{3,3} = \int_{\Omega^i} \rho^i \left(\left(\overline{u}_1^i(P^i)\right)^2 + \left(\overline{u}_2^i(P^i)\right)^2\right)dV^i = \\ = \int_{\Omega^i} \rho^i \left(\overline{x}^{i\,2}(P^i) + \overline{y}^{i\,2}(P^i)\right)dV^i \tag{2.218}$$

Where the $\left(\overline{\mathbf{I}}_{\theta,\theta}^{i}\right)_{j,k}$ elements with $j = k$ are called mass moment of inertia and the $\left(\overline{\mathbf{I}}_{\theta,\theta}^{i}\right)_{j,k}$ elements with $j \neq k$ are called mass products of inertia. Once that all mass submatrix $\mathbf{m}_{R,R}^{i}(t)$, $\mathbf{m}_{R,\theta}^{i}(t)$ and $\mathbf{m}_{\theta,\theta}^{i}(t)$ of the mass matrix $\mathbf{M}^i(t)$ of the rigid body i has been computed, the kinetic energy $T^i(t)$ of the same body can be written as follows:

$$
\begin{aligned}
T^i(t) &= \frac{1}{2}\dot{\mathbf{q}}^{iT}(t)\mathbf{M}^i(t)\dot{\mathbf{q}}^i(t) = \\
&= \frac{1}{2}\begin{bmatrix}\dot{\mathbf{R}}^{iT}(t) & \dot{\boldsymbol{\theta}}^{iT}(t)\end{bmatrix}\begin{bmatrix}\mathbf{m}^i_{R,R}(t) & \mathbf{m}^i_{R,\theta}(t)\\ \mathbf{m}^i_{\theta,R}(t) & \mathbf{m}^i_{\theta,\theta}(t)\end{bmatrix}\begin{bmatrix}\dot{\mathbf{R}}^i(t)\\ \dot{\boldsymbol{\theta}}^i(t)\end{bmatrix} = \\
&= \frac{1}{2}\dot{\mathbf{R}}^{iT}(t)\mathbf{m}^i_{R,R}(t)\dot{\mathbf{R}}^i(t) + \dot{\mathbf{R}}^{iT}(t)\mathbf{m}^i_{R,\theta}(t)\dot{\boldsymbol{\theta}}^i(t) + \frac{1}{2}\dot{\boldsymbol{\theta}}^{iT}(t)\mathbf{m}^i_{\theta,\theta}(t)\dot{\boldsymbol{\theta}}^i(t)
\end{aligned}
\tag{2.219}
$$

The kinetic energy $T^i(t)$ of rigid body i results to be a function of the system configuration $\mathbf{q}^i(t)$ and of its time derivative $\dot{\mathbf{q}}^i(t)$. It is worth noting that to derive the expression of the kinetic energy of the rigid body i a set of shape integrals corresponding to the total mass m^i, the moment of mass and the inertia matrix $\left(\bar{\mathbf{I}}^i_{\theta,\theta}\right)_{j,k}$ of the same body must be previously computed.

2.5.4. DYNAMIC EQUATIONS OF RIGID MULTIBODY SYSTEMS

Up to this point, all configuration coordinates $\mathbf{q}^i(t)$ of rigid body i has been considered as independent coordinates. Obviously, this is not the general case of a rigid multibody system which is typically formed of a set of rigid bodies mutually interconnected. Therefore, if it is required to use this nonminimal set of configuration coordinate $\mathbf{q}^i(t)$ to derive system equations of motion, then the actions of the constraints must be considered in the dynamic equations. Indeed, consider that the generic body i of the set is forced to satisfy the following constraint equations written in the standard form:

$$
\mathbf{A}^i(t)\ddot{\mathbf{q}}^i(t) = \mathbf{b}^i(t) \tag{2.220}
$$

Where $\mathbf{A}^i(t)$ is a $\mathbb{R}^{n_c^i \times 7}$ constraint matrix and $\mathbf{b}^i(t)$ is a $\mathbb{R}^{n_c^i}$ constraint vector. (Note that the constraint matrix $\mathbf{A}^i(t)$ relative to body i has been denoted with the same symbol of the rotation matrix $\mathbf{A}^i(t)$ of body i). These equations are a set of algebraic constraint equations written in the standard form and encompass all kind of constraints acting on the system, such as mechanical joints as well as specific constraints which derive from the definition of Euler's parameters. Consequently, Lagrange equations assume the following form:

$$\frac{d}{dt}\left(\frac{\partial T^i(t)}{\partial \dot{\mathbf{q}}^i(t)}\right)^T - \left(\frac{\partial T^i(t)}{\partial \mathbf{q}^i(t)}\right)^T = \mathbf{Q}_e^i(t) + \mathbf{Q}_c^i(t) \tag{2.221}$$

Where the $\mathbb{R}^7$ vector $\mathbf{Q}_e^i(t)$ is the vector of generalized external forces and $\mathbf{Q}_c^i(t)$ is a $\mathbb{R}^7$ vector representing the generalized constraint forces. The term on the left hand side of Lagrange equations is equal to the negative of lagrangian components of inertia forces $\mathbf{Q}_i^i(t)$ of body i and it can be explicitly computed by using the previous expression of kinetic energy $T^i(t)$ based on the expression of mass matrix $\mathbf{M}^i(t)$. Indeed:

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{\partial T^i(t)}{\partial \dot{\mathbf{q}}^i(t)}\right)^T - \left(\frac{\partial T^i(t)}{\partial \mathbf{q}^i(t)}\right)^T = \\
&= \frac{d}{dt}\left(\frac{\partial}{\partial \dot{\mathbf{q}}^i(t)}\left(\frac{1}{2}\dot{\mathbf{q}}^{iT}(t)\mathbf{M}^i(t)\dot{\mathbf{q}}^i(t)\right)\right)^T - \left(\frac{\partial T^i(t)}{\partial \mathbf{q}^i(t)}\right)^T = \\
&= \frac{d}{dt}\left(\dot{\mathbf{q}}^{iT}(t)\mathbf{M}^i(t)\right)^T - \left(\frac{\partial T^i(t)}{\partial \mathbf{q}^i(t)}\right)^T = \\
&= \mathbf{M}^i(t)\ddot{\mathbf{q}}^i(t) + \dot{\mathbf{M}}^i(t)\dot{\mathbf{q}}^i(t) - \left(\frac{\partial T^i(t)}{\partial \mathbf{q}^i(t)}\right)^T = \\
&= \mathbf{M}^i(t)\ddot{\mathbf{q}}^i(t) - \mathbf{Q}_v^i(t)
\end{aligned} \tag{2.222}$$

Where the vector $\mathbf{Q}_v^i(t)$ is a $\mathbb{R}^7$ vector defined as:

$$\mathbf{Q}_v^i(t) = \left(\frac{\partial T^i(t)}{\partial \mathbf{q}^i(t)}\right)^T - \dot{\mathbf{M}}^i(t)\dot{\mathbf{q}}^i(t) \tag{2.223}$$

The vector $\mathbf{Q}_v^i(t)$ is called quadratic velocity vector and it contains the gyroscopic and Coriolis force components. Since the kinetic energy $T^i(t)$ and the mass matrix $\mathbf{M}^i(t)$ of the generic rigid body i has been computed explicitly, the quadratic velocity vector $\mathbf{Q}_v^i(t)$ can be computed in a direct way. Indeed, the time derivative of mass matrix can be rewritten as:

$$\dot{\mathbf{M}}^i(t) = \begin{bmatrix} \dot{\mathbf{m}}_{R,R}^i(t) & \dot{\mathbf{m}}_{R,\theta}^i(t) \\ \dot{\mathbf{m}}_{\theta,R}^i(t) & \dot{\mathbf{m}}_{\theta,\theta}^i(t) \end{bmatrix} \tag{2.224}$$

Where the time derivative of the mass submatrices $\mathbf{m}_{R,R}^i(t)$, $\mathbf{m}_{R,\theta}^i(t)$ and $\mathbf{m}_{\theta,\theta}^i(t)$ can be computed as follows:

$$\begin{aligned} \dot{\mathbf{m}}_{R,R}^i(t) &= \frac{d}{dt}\left(m^i\mathbf{I}\right) = \\ &= \mathbf{O} \end{aligned} \tag{2.225}$$

$$\dot{\mathbf{m}}_{R,\theta}^i(t) = \mathbf{O} \tag{2.226}$$

$$\begin{aligned} \dot{\mathbf{m}}_{\theta,\theta}^i(t) &= \frac{d}{dt}\left(\bar{\mathbf{G}}^{iT}(t)\bar{\mathbf{I}}_{\theta,\theta}^i\bar{\mathbf{G}}^i(t)\right) = \\ &= \dot{\bar{\mathbf{G}}}^{iT}(t)\bar{\mathbf{I}}_{\theta,\theta}^i\bar{\mathbf{G}}^i(t) + \bar{\mathbf{G}}^{iT}(t)\bar{\mathbf{I}}_{\theta,\theta}^i\dot{\bar{\mathbf{G}}}^i(t) \end{aligned} \tag{2.227}$$

Where the origin of the local frame of reference is assumed to coincide with the body centre of mass. Furthermore, the first term that form the quadratic velocity vector $\mathbf{Q}_v^i(t)$ can be expressed as:

$$-\dot{\mathbf{M}}^i(t)\dot{\mathbf{q}}^i(t) = -\begin{bmatrix} \dot{\mathbf{m}}^i_{R,R}(t) & \dot{\mathbf{m}}^i_{R,\theta}(t) \\ \dot{\mathbf{m}}^i_{\theta,R}(t) & \dot{\mathbf{m}}^i_{\theta,\theta}(t) \end{bmatrix} \begin{bmatrix} \dot{\mathbf{R}}^i(t) \\ \dot{\boldsymbol{\theta}}^i(t) \end{bmatrix} =$$
$$= -\begin{bmatrix} \dot{\mathbf{m}}^i_{R,R}(t)\dot{\mathbf{R}}^i(t) + \dot{\mathbf{m}}^i_{R,\theta}(t)\dot{\boldsymbol{\theta}}^i(t) \\ \dot{\mathbf{m}}^i_{\theta,R}(t)\dot{\mathbf{R}}^i(t) + \dot{\mathbf{m}}^i_{\theta,\theta}(t)\dot{\boldsymbol{\theta}}^i(t) \end{bmatrix} = \quad (2.228)$$
$$= -\begin{bmatrix} \left(\dot{\mathbf{M}}^i(t)\dot{\mathbf{q}}^i(t)\right)_R \\ \left(\dot{\mathbf{M}}^i(t)\dot{\mathbf{q}}^i(t)\right)_\theta \end{bmatrix}$$

Where from the previous expression it can be deduced that:

$$\left(\dot{\mathbf{M}}^i(t)\dot{\mathbf{q}}^i(t)\right)_R = \dot{\mathbf{m}}^i_{R,R}(t)\dot{\mathbf{R}}^i(t) + \dot{\mathbf{m}}^i_{R,\theta}(t)\dot{\boldsymbol{\theta}}^i(t) = \\ = \mathbf{0} \quad (2.229)$$

$$\left(\dot{\mathbf{M}}^i(t)\dot{\mathbf{q}}^i(t)\right)_\theta = \dot{\mathbf{m}}^i_{\theta,R}(t)\dot{\mathbf{R}}^i(t) + \dot{\mathbf{m}}^i_{\theta,\theta}(t)\dot{\boldsymbol{\theta}}^i(t) =$$
$$= \left(\dot{\bar{\mathbf{G}}}^{iT}(t)\bar{\mathbf{I}}^i_{\theta,\theta}\bar{\mathbf{G}}^i(t) + \bar{\mathbf{G}}^{iT}(t)\bar{\mathbf{I}}^i_{\theta,\theta}\dot{\bar{\mathbf{G}}}^i(t)\right)\dot{\boldsymbol{\theta}}^i(t) =$$
$$= \dot{\bar{\mathbf{G}}}^{iT}(t)\bar{\mathbf{I}}^i_{\theta,\theta}\bar{\mathbf{G}}^i(t)\dot{\boldsymbol{\theta}}^i(t) + \bar{\mathbf{G}}^{iT}(t)\bar{\mathbf{I}}^i_{\theta,\theta}\dot{\bar{\mathbf{G}}}^i(t)\dot{\boldsymbol{\theta}}^i(t) = \quad (2.230)$$
$$= \dot{\bar{\mathbf{G}}}^{iT}(t)\bar{\mathbf{I}}^i_{\theta,\theta}\bar{\mathbf{G}}^i(t)\dot{\boldsymbol{\theta}}^i(t) =$$
$$= -\dot{\bar{\mathbf{G}}}^{iT}(t)\bar{\mathbf{I}}^i_{\theta,\theta}\dot{\bar{\mathbf{G}}}^i(t)\boldsymbol{\theta}^i(t)$$

Where the following two identities has been used:

$$\dot{\bar{\mathbf{G}}}^i(t)\dot{\boldsymbol{\theta}}^i(t) = \mathbf{0} \quad (2.231)$$

$$\bar{\mathbf{G}}^i(t)\dot{\boldsymbol{\theta}}^i(t) = -\dot{\bar{\mathbf{G}}}^i(t)\boldsymbol{\theta}^i(t) \quad (2.232)$$

On the other hand, the second term which form the quadratic velocity vector $\mathbf{Q}^i_v(t)$ can be computed as:

$$\left(\frac{\partial T^i(t)}{\partial \mathbf{q}^i(t)}\right)^T = \begin{bmatrix} \left(\dfrac{\partial T^i(t)}{\partial \mathbf{R}^i(t)}\right)^T \\ \left(\dfrac{\partial T^i(t)}{\partial \boldsymbol{\theta}^i(t)}\right)^T \end{bmatrix} \tag{2.233}$$

Where from the previous expression it can be deduced that:

$$\begin{aligned}
\frac{\partial T^i(t)}{\partial \mathbf{R}^i(t)} &= \frac{\partial}{\partial \mathbf{R}^i(t)}\left(\frac{1}{2}\dot{\mathbf{R}}^{iT}(t)\mathbf{m}^i_{R,R}(t)\dot{\mathbf{R}}^i(t)\right) + \\
&\quad + \frac{\partial}{\partial \mathbf{R}^i(t)}\left(\dot{\mathbf{R}}^{iT}(t)\mathbf{m}^i_{R,\theta}(t)\dot{\boldsymbol{\theta}}^i(t)\right) + \\
&\quad + \frac{\partial}{\partial \mathbf{R}^i(t)}\left(\frac{1}{2}\dot{\boldsymbol{\theta}}^{iT}(t)\mathbf{m}^i_{\theta,\theta}(t)\dot{\boldsymbol{\theta}}^i(t)\right) = \\
&= \frac{\partial}{\partial \mathbf{R}^i(t)}\left(\frac{1}{2}\dot{\mathbf{R}}^{iT}(t)\left(m^i\mathbf{I}\right)\dot{\mathbf{R}}^i(t)\right) + \\
&\quad + \frac{\partial}{\partial \mathbf{R}^i(t)}\left(\frac{1}{2}\dot{\boldsymbol{\theta}}^{iT}(t)\left(\bar{\mathbf{G}}^{iT}(t)\bar{\mathbf{I}}^i_{\theta,\theta}(t)\bar{\mathbf{G}}^i(t)\right)\dot{\boldsymbol{\theta}}^i(t)\right) = \\
&= \mathbf{0}
\end{aligned} \tag{2.234}$$

$$\begin{aligned}
\frac{\partial T^i(t)}{\partial \boldsymbol{\theta}^i(t)} &= \frac{\partial}{\partial \boldsymbol{\theta}^i(t)}\left(\frac{1}{2}\dot{\mathbf{R}}^{iT}(t)\mathbf{m}^i_{R,R}(t)\dot{\mathbf{R}}^i(t)\right)+ \\
&+\frac{\partial}{\partial \boldsymbol{\theta}^i(t)}\left(\frac{1}{2}\dot{\boldsymbol{\theta}}^{iT}(t)\mathbf{m}^i_{\theta,\theta}(t)\dot{\boldsymbol{\theta}}^i(t)\right)+ \\
&+\frac{\partial}{\partial \boldsymbol{\theta}^i(t)}\left(\dot{\mathbf{R}}^{iT}(t)\mathbf{m}^i_{R,\theta}(t)\dot{\boldsymbol{\theta}}^i(t)\right)= \\
&=\frac{\partial}{\partial \boldsymbol{\theta}^i(t)}\left(\frac{1}{2}\dot{\mathbf{R}}^{iT}(t)\left(m^i\mathbf{I}\right)\dot{\mathbf{R}}^i(t)\right)+ \\
&+\frac{\partial}{\partial \boldsymbol{\theta}^i(t)}\left(\frac{1}{2}\dot{\boldsymbol{\theta}}^{iT}(t)\left(\bar{\mathbf{G}}^{iT}(t)\bar{\mathbf{I}}^i_{\theta,\theta}(t)\bar{\mathbf{G}}^i(t)\right)\dot{\boldsymbol{\theta}}^i(t)\right)= \\
&=\frac{\partial}{\partial \boldsymbol{\theta}^i(t)}\left(\frac{1}{2}\boldsymbol{\theta}^{iT}(t)\dot{\bar{\mathbf{G}}}^{iT}(t)\bar{\mathbf{I}}^i_{\theta,\theta}(t)\dot{\bar{\mathbf{G}}}^i(t)\boldsymbol{\theta}^i(t)\right)= \\
&=\boldsymbol{\theta}^{iT}(t)\dot{\bar{\mathbf{G}}}^{iT}(t)\bar{\mathbf{I}}^i_{\theta,\theta}(t)\dot{\bar{\mathbf{G}}}^i(t)
\end{aligned} \tag{2.235}$$

Consequently, the quadratic velocity vector $\mathbf{Q}^i_v(t)$ for rigid body i can be explicitly computed as:

$$\begin{aligned}
\mathbf{Q}^i_v(t) &= -\begin{bmatrix}\left(\dot{\mathbf{M}}^i(t)\dot{\mathbf{q}}^i(t)\right)_R \\ \left(\dot{\mathbf{M}}^i(t)\dot{\mathbf{q}}^i(t)\right)_\theta\end{bmatrix}+\begin{bmatrix}\left(\dfrac{\partial T^i(t)}{\partial \mathbf{R}^i(t)}\right)^T \\ \left(\dfrac{\partial T^i(t)}{\partial \boldsymbol{\theta}^i(t)}\right)^T\end{bmatrix}= \\
&=\begin{bmatrix}\mathbf{0} \\ 2\dot{\bar{\mathbf{G}}}^{iT}(t)\bar{\mathbf{I}}^i_{\theta,\theta}(t)\dot{\bar{\mathbf{G}}}^i(t)\boldsymbol{\theta}^i(t)\end{bmatrix}= \\
&=\begin{bmatrix}\mathbf{0} \\ -2\dot{\bar{\mathbf{G}}}^{iT}(t)\bar{\mathbf{I}}^i_{\theta,\theta}(t)\bar{\mathbf{G}}^i(t)\dot{\boldsymbol{\theta}}^i(t)\end{bmatrix}
\end{aligned} \tag{2.236}$$

Finally, the equations of motion of rigid body i can be expressed as:

$$\mathbf{M}^i(t)\ddot{\mathbf{q}}^i(t) = \mathbf{Q}_v^i(t) + \mathbf{Q}_e^i(t) + \mathbf{Q}_c^i(t) \tag{2.237}$$

These dynamic equations can be easily assembled to derive the equations of motion of the whole rigid multibody system to yield:

$$\mathbf{M}(t)\ddot{\mathbf{q}}(t) = \mathbf{Q}_v(t) + \mathbf{Q}_e(t) + \mathbf{Q}_c(t) \tag{2.238}$$

Where the configuration vector $\mathbf{q}(t)$ represents the total $\mathbb{R}^{7n_b}$ vector of the rigid system generalized coordinates and is defined as:

$$\mathbf{q}(t) = \begin{bmatrix} \mathbf{q}^1(t) \\ \mathbf{q}^2(t) \\ \vdots \\ \mathbf{q}^{n_b}(t) \end{bmatrix} \tag{2.239}$$

The matrix $\mathbf{M}(t)$ is the global $\mathbb{R}^{7n_b \times 7n_b}$ mass matrix of the multibody system and it can be easily assembled as:

$$\mathbf{M}(t) = \begin{bmatrix} \mathbf{M}^1(t) & \mathbf{O} & \dots & \mathbf{O} \\ \mathbf{O} & \mathbf{M}^2(t) & \dots & \mathbf{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{O} & \mathbf{O} & \dots & \mathbf{M}^{n_b}(t) \end{bmatrix} \tag{2.240}$$

The $\mathbb{R}^{7n_b}$ vectors $\mathbf{Q}_v(t)$, $\mathbf{Q}_e(t)$ and $\mathbf{Q}_c(t)$ are lagrangian component vectors which represent respectively the generalized gyroscopic and Coriolis forces, the generalized external forces and the generalized constraint forces. These vectors can be simply assembled as:

$$\mathbf{Q}_v(t) = \begin{bmatrix} \mathbf{Q}_v^1(t) \\ \mathbf{Q}_v^2(t) \\ \vdots \\ \mathbf{Q}_v^{n_b}(t) \end{bmatrix} \tag{2.241}$$

$$\mathbf{Q}_e(t) = \begin{bmatrix} \mathbf{Q}_e^1(t) \\ \mathbf{Q}_e^2(t) \\ \vdots \\ \mathbf{Q}_e^{n_b}(t) \end{bmatrix} \tag{2.242}$$

$$\mathbf{Q}_c(t) = \begin{bmatrix} \mathbf{Q}_c^1(t) \\ \mathbf{Q}_c^2(t) \\ \vdots \\ \mathbf{Q}_c^{n_b}(t) \end{bmatrix} \tag{2.243}$$

On the other hand, the algebraic constraint equations can be assembled in a similar manner to yield:

$$\mathbf{A}(t)\ddot{\mathbf{q}}(t) = \mathbf{b}(t) \tag{2.244}$$

Where $\mathbf{A}(t)$ is a $\mathbb{R}^{n_c \times 7n_b}$ matrix representing the total constrain matrix and it can be directly computed as:

$$\mathbf{A}(t) = \begin{bmatrix} \mathbf{A}^1(t) & \mathbf{O} & \dots & \mathbf{O} \\ \mathbf{O} & \mathbf{A}^2(t) & \dots & \mathbf{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{O} & \mathbf{O} & \dots & \mathbf{A}^{n_b}(t) \end{bmatrix} \tag{2.245}$$

The vector $\mathbf{b}(t)$ is a $\mathbb{R}^{n_c}$ vector corresponding to the global constraint vector and it can be assembled as follows:

$$\mathbf{b}(t) = \begin{bmatrix} \mathbf{b}^1(t) \\ \mathbf{b}^2(t) \\ \vdots \\ \mathbf{b}^{n_b}(t) \end{bmatrix} \tag{2.246}$$

Finally, the set of equation of motion and constraint equations which describe the dynamic of a general rigid multibody system is:

$$\begin{cases} \mathbf{M}(t)\ddot{\mathbf{q}}(t) = \mathbf{Q}_v(t) + \mathbf{Q}_e(t) + \mathbf{Q}_c(t) \\ \mathbf{A}(t)\ddot{\mathbf{q}}(t) = \mathbf{b}(t) \end{cases} \tag{2.247}$$

These equation can be explicitly solved to get the generalized acceleration vector $\ddot{\mathbf{q}}(t)$ and the generalized constraint vector $\mathbf{Q}_c(t)$ in order to obtain the fundamental equations of constrained Dynamics.

2.6. THE FINITE ELEMENT METHOD

2.6.1. INTRODUCTION

The finite element method is a powerful numerical method that is commonly used to describe the motion of complex structural systems $[27]$, $[28]$. Indeed, when a mechanical system is composed of continuous components with complex geometry, it is difficult, or even impossible, to correctly model them by using the analytical techniques which leads to partial differential equations (PDE). Therefore, in this case the finite element method can be used to transform the structural system equations of motion from partial differential equations (PDE) to ordinary differential equations (ODE). This can be done discretizing the structure into relatively small regions called elements which are rigidly interconnected at selected nodal points. The deformation within each element can then be described by approximating functions, such as polynomials. The coefficients of these polynomials are defined in terms of physical coordinates called nodal coordinates which describe the displacements and slopes of selected nodal points on element. Afterwards, the displacement of the element can be expressed using the separation of variables as the product of space-dependent shape functions and time-dependent nodal coordinates. Using the assumed displacement field, the kinetic and strain energy of each element can be developed and the finite-element mass and stiffness matrices of the whole structure can be computed to yield the system equations of motion. The following sections concern the formulation of mass and stiffness matrices of continuous systems using the finite element method and assuming that the system does not undergo large deformation or rigid body motion, namely when the reference motion is not allowed. The approach followed here was originally developed by Shabana $[11]$, $[12]$, $[13]$. Similarly to the preceding case of rigid body systems, it can be showed that the mass and stiffness matrices of structural systems can be derived by finite element method once that a set of shape integrals has been computed. The general case of multibody systems which contain rigid and structural components that exhibit large reference motion will be described in the next chapter $[11]$, $[12]$, $[13]$.

2.6.2. ASSUMED DISPLACEMET FIELD

Consider a structural system composed of n_b flexible bodies mutually interconnected and linked to the ground. Assume that each body i of the system is discretized in n_e^i elements. For the generic element j of body i, the element displacement field can be decomposed by the product of a space-dependent global shape function and a time-dependent nodal vector $\mathbf{q}_f^{i,j}(t)$ in oerder to yield:

$$\bar{\mathbf{u}}_f^{i,j}(P^{i,j},t) = \mathbf{S}_g^{i,j}(P^{i,j})\mathbf{q}_f^{i,j}(t) \tag{2.248}$$

Where $\bar{\mathbf{u}}_f^{i,j}(P^{i,j},t)$ is a $\mathbb{R}^3$ vector function representing the element displacement field, $\mathbf{S}_g^{i,j}(P^{i,j})$ is a $\mathbb{R}^{3\times n_f^{i,j}}$ matrix function representing the global shape function and $\mathbf{q}_f^{i,j}(t)$ is a $\mathbb{R}^{n_f^{i,j}}$ vector function corresponding to the vector of nodal coordinates. The element displacement vector, the global shape function and the element vector of nodal coordinates are all defined in the global reference system. The global shape function $\mathbf{S}_g^{i,j}(P^{i,j})$ can be computed by using the local shape function $\mathbf{S}^{i,j}(P^{i,j})$ as:

$$\mathbf{S}_g^{i,j}(P^{i,j}) = \mathbf{C}^{i,j}\mathbf{S}^{i,j}(P^{i,j})\bar{\mathbf{C}}^{i,j} \tag{2.249}$$

Where $\mathbf{C}^{i,j}$ and $\bar{\mathbf{C}}^{i,j}$ are respectively $\mathbb{R}^{3\times n_f^{i,j}}$ and $\mathbb{R}^{n_f^{i,j}\times n_f^{i,j}}$ rotation matrices. For instance, the local shape function of a bidimensional beam element subjected to longitudinal and transversal vibration is:

$$\mathbf{S}^{i,j}(P^{i,j}) = \begin{bmatrix} \mathbf{S}_1^{i,j}(P^{i,j}) \\ \mathbf{S}_2^{i,j}(P^{i,j}) \end{bmatrix} \tag{2.250}$$

Where $\mathbf{S}_1^{i,j}(P^{i,j})$ and $\mathbf{S}_2^{i,j}(P^{i,j})$ are the following vector functions:

$$\mathbf{S}_1^{i,j}(P^{i,j}) = \begin{bmatrix} 1-\xi^{i,j} & 0 & 0 & \xi^{i,j} & 0 & 0 \end{bmatrix} \tag{2.251}$$

$$\mathbf{S}_2^{i,j\,T}(P^{i,j}) = \\ = \begin{bmatrix} 0 \\ 1-3\xi^{i,j\,2}+2\xi^{i,j\,3} \\ L^{i,j}\left(\xi^{i,j}-2\xi^{i,j\,2}+\xi^{i,j\,3}\right) \\ 0 \\ 3\xi^{i,j\,2}-2\xi^{i,j\,3} \\ L^{i,j}\left(-\xi^{i,j\,2}+\xi^{i,j\,3}\right) \end{bmatrix} \tag{2.252}$$

Where $\xi^{i,j}$ is a dimensionless spatial coordinate defined as:

$$\xi^{i,j} = \frac{x^{i,j}}{L^{i,j}} \tag{2.253}$$

In the case of the two-dimensional beam element, the rotation matrices $\mathbf{C}^{i,j}$ and $\bar{\mathbf{C}}^{i,j}$ are defined as:

$$\mathbf{C}^{i,j} = \begin{bmatrix} \cos(\alpha^{i,j}) & -\sin(\alpha^{i,j}) & 0 \\ \sin(\alpha^{i,j}) & \cos(\alpha^{i,j}) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.254}$$

$$\bar{\mathbf{C}}^{i,j} = \begin{bmatrix} \mathbf{C}^{i,j\,T} & \mathbf{O} \\ \mathbf{O} & \mathbf{C}^{i,j\,T} \end{bmatrix} \tag{2.255}$$

Where $\alpha^{i,j}$ is the rotation angle between the local reference frame of element j of body i and the inertial reference system. Once that the representation of the displacement field has been obtained, the connectivity conditions between the elements of each body must be applied. The connectivity conditions require that when two elements are rigidly connected at a nodal point, the coordinate of this point must be the same for the two elements. Therefore, a new nodal coordinate vector $\mathbf{q}_{f,v}^{i}(t)$ can be defined for each body to represent the total body nodal coordinates. Hence, the coordinate vector of each element $\mathbf{q}_{f}^{i,j}(t)$ can be expressed using the body coordinate vector $\mathbf{q}_{f,v}^{i}(t)$ by a Boolean matrix to yield:

$$\mathbf{q}_{f}^{i,j}(t) = \mathbf{B}_{c}^{i,j}\mathbf{q}_{f,v}^{i}(t) \tag{2.256}$$

Where $\mathbf{q}_{f,v}^{i}(t)$ is a $\mathbb{R}^{n_{f,v}^{i}}$ vector and $\mathbf{B}_{c}^{i,j}$ is a $\mathbb{R}^{n_{f}^{i,j}\times n_{f,v}^{i}}$ Boolean matrix. The Boolean matrix $\mathbf{B}_{c}^{i,j}$ is used to represent the internal kinematic constraints of each element j of body i. On the other hand, another Boolean matrix $\mathbf{B}_{e}^{i}$ can be used for each body i of the system to represent the external kinematic constraint acting on the system. Indeed, the vector of total nodal coordinate $\mathbf{q}_{f,v}^{i}(t)$ can be expressed as a function of the vector of free coordinate $\mathbf{q}_{f}^{i}(t)$ of body i by using the Boolean matrix of external constraints $\mathbf{B}_{e}^{i}$ to yield:

$$\mathbf{q}_{f,v}^{i}(t) = \mathbf{B}_{e}^{i}\mathbf{q}_{f}^{i}(t) \tag{2.257}$$

Where $\mathbf{q}_{f}^{i}(t)$ is a $\mathbb{R}^{n_{f}^{i}}$ vector and $\mathbf{B}_{e}^{i}$ is a $\mathbb{R}^{n_{f,v}^{i}\times n_{f}^{i}}$ Boolean matrix. According to the preceding definitions, the displacement vector of element j of body i can be rewritten as follows:

$$\begin{aligned}\bar{\mathbf{u}}_{f}^{i,j}(P^{i,j},t) &= \mathbf{S}_{g}^{i,j}(P^{i,j})\mathbf{q}_{f}^{i,j}(t) = \\ &= \mathbf{S}_{g}^{i,j}(P^{i,j})\mathbf{B}_{c}^{i,j}\mathbf{q}_{f,v}^{i}(t) = \\ &= \mathbf{S}_{g}^{i,j}(P^{i,j})\mathbf{B}_{c}^{i,j}\mathbf{B}_{e}^{i}\mathbf{q}_{f}^{i}(t) = \\ &= \mathbf{N}^{i,j}(P^{i,j})\mathbf{B}_{e}^{i}\mathbf{q}_{f}^{i}(t)\end{aligned} \tag{2.258}$$

Where $\mathbf{N}^{i,j}(P^{i,j})$ is a $\mathbb{R}^{3\times n^i_{f,v}}$ matrix sometimes referred to as compact shape function which is defined as:

$$\mathbf{N}^{i,j}(P^{i,j})=\mathbf{S}^{i,j}_g(P^{i,j})\mathbf{B}^{i,j}_c \tag{2.259}$$

Using this expression it is possible to compute the displacement vector of the element j of body i by the vector of the free nodal coordinate of the whole body. Thus, the time derivative of the displacement field can be computed by the time derivative of free nodal coordinates:

$$\dot{\bar{\mathbf{u}}}^{i,j}_f(P^{i,j},t)=\mathbf{N}^{i,j}(P^{i,j})\mathbf{B}^i_e\dot{\mathbf{q}}^i_f(t) \tag{2.260}$$

From this formula it is straightforward to deduce the relation between the virtual change of the displacement field and the virtual change of the free nodal coordinate vector to yield:

$$\delta\bar{\mathbf{u}}^{i,j}_f(P^{i,j},t)=\mathbf{N}^{i,j}(P^{i,j})\mathbf{B}^i_e\delta\mathbf{q}^i_f(t) \tag{2.261}$$

Indeed, the matrix $\mathbf{N}^{i,j}(P^{i,j})\mathbf{B}^i_e$ corresponds to the jacobian transformation matrix which mathematically describe the relation between the physical position coordinates and the lagrangian configuration coordinates.

2.6.3. MASS MATRIX OF STRUCTURAL ELEMENTS

The formulation of the mass matrix corresponding to the element j of body i can be performed by using the definition of kinetic energy $T^{i,j}(t)$ of the same element. Indeed:

$$
\begin{aligned}
&T^{i,j}(t) = \frac{1}{2}\int_{\Omega^{i,j}} \rho^{i,j}\dot{\bar{\mathbf{u}}}_f^{i,jT}(P^{i,j},t)\dot{\bar{\mathbf{u}}}_f^{i,j}(P^{i,j},t)dV^{i,j} = \\
&= \frac{1}{2}\int_{\Omega^{i,j}} \rho^{i,j}\dot{\mathbf{q}}_f^{iT}(t)\mathbf{B}_e^{iT}\mathbf{N}^{i,jT}(P^{i,j})\mathbf{N}^{i,j}(P^{i,j})\mathbf{B}_e^{i}\dot{\mathbf{q}}_f^{i}(t)dV^{i,j} = \\
&= \frac{1}{2}\dot{\mathbf{q}}_f^{iT}(t)\mathbf{B}_e^{iT}\int_{\Omega^{i,j}} \rho^{i,j}\mathbf{N}^{i,jT}(P^{i,j})\mathbf{N}^{i,j}(P^{i,j})dV^{i,j}\mathbf{B}_e^{i}\dot{\mathbf{q}}_f^{i}(t) = \\
&= \frac{1}{2}\dot{\mathbf{q}}_f^{iT}(t)\mathbf{B}_e^{iT}\int_{\Omega^{i,j}} \rho^{i,j}\mathbf{B}_c^{i,jT}\bar{\mathbf{C}}^{i,jT}\mathbf{S}^{i,jT}(P^{i,j})\mathbf{C}^{i,jT}\mathbf{C}^{i,j}\mathbf{S}^{i,j}(P^{i,j})\bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j}dV^{i,j}\mathbf{B}_e^{i}\dot{\mathbf{q}}_f^{i}(t) = \\
&= \frac{1}{2}\dot{\mathbf{q}}_f^{iT}(t)\mathbf{B}_e^{iT}\mathbf{B}_c^{i,jT}\bar{\mathbf{C}}^{i,jT}\int_{\Omega^{i,j}} \rho^{i,j}\mathbf{S}^{i,jT}(P^{i,j})\mathbf{S}^{i,j}(P^{i,j})dV^{i,j}\bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j}\mathbf{B}_e^{i}\dot{\mathbf{q}}_f^{i}(t) = \\
&= \frac{1}{2}\dot{\mathbf{q}}_f^{iT}(t)\mathbf{B}_e^{iT}\mathbf{B}_c^{i,jT}\bar{\mathbf{C}}^{i,jT}\int_{\Omega^{i,j}} \rho^{i,j}\mathbf{S}_1^{i,jT}(P^{i,j})\mathbf{S}_1^{i,j}(P^{i,j})dV^{i,j}\bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j}\mathbf{B}_e^{i}\dot{\mathbf{q}}_f^{i}(t) + \\
&\quad + \frac{1}{2}\dot{\mathbf{q}}_f^{iT}(t)\mathbf{B}_e^{iT}\mathbf{B}_c^{i,jT}\bar{\mathbf{C}}^{i,jT}\int_{\Omega^{i,j}} \rho^{i,j}\mathbf{S}_2^{i,jT}(P^{i,j})\mathbf{S}_2^{i,j}(P^{i,j})dV^{i,j}\bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j}\mathbf{B}_e^{i}\dot{\mathbf{q}}_f^{i}(t) + \\
&\quad + \frac{1}{2}\dot{\mathbf{q}}_f^{iT}(t)\mathbf{B}_e^{iT}\mathbf{B}_c^{i,jT}\bar{\mathbf{C}}^{i,jT}\int_{\Omega^{i,j}} \rho^{i,j}\mathbf{S}_3^{i,jT}(P^{i,j})\mathbf{S}_3^{i,j}(P^{i,j})dV^{i,j}\bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j}\mathbf{B}_e^{i}\dot{\mathbf{q}}_f^{i}(t) = \\
&= \frac{1}{2}\dot{\mathbf{q}}_f^{iT}(t)\mathbf{B}_e^{iT}\mathbf{B}_c^{i,jT}\bar{\mathbf{C}}^{i,jT}\left(\bar{\mathbf{S}}_{1,1}^{i,j} + \bar{\mathbf{S}}_{2,2}^{i,j} + \bar{\mathbf{S}}_{3,3}^{i,j}\right)\bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j}\mathbf{B}_e^{i}\dot{\mathbf{q}}_f^{i}(t) = \\
&= \frac{1}{2}\dot{\mathbf{q}}_f^{iT}(t)\mathbf{B}_e^{iT}\mathbf{B}_c^{i,jT}\bar{\mathbf{C}}^{i,jT}\mathbf{S}_{f,f}^{i,j}\bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j}\mathbf{B}_e^{i}\dot{\mathbf{q}}_f^{i}(t) = \\
&= \frac{1}{2}\dot{\mathbf{q}}_f^{iT}(t)\mathbf{B}_e^{iT}\bar{\mathbf{J}}_{f,f}^{i,j}\mathbf{B}_e^{i}\dot{\mathbf{q}}_f^{i}(t) = \\
&= \frac{1}{2}\dot{\mathbf{q}}_f^{iT}(t)\mathbf{M}_{f,f}^{i,j}\dot{\mathbf{q}}_f^{i}(t)
\end{aligned}
\tag{2.262}
$$

Where $\rho^{i,j}$ and $\Omega^{i,j}$ are respectively the mass density and the volume of element j of the body i and the local shape function has been expressed as:

$$\mathbf{S}^{i,j}(P^{i,j}) = \begin{bmatrix} \mathbf{S}_1^{i,j}(P^{i,j}) \\ \mathbf{S}_2^{i,j}(P^{i,j}) \\ \mathbf{S}_3^{i,j}(P^{i,j}) \end{bmatrix} \tag{2.263}$$

Consequently, the final expression of the mass matrix $\mathbf{M}_{f,f}^{i,j}$ of the element j of body i is:

$$\mathbf{M}_{f,f}^{i,j} = \mathbf{B}_e^{iT} \overline{\mathbf{J}}_{f,f}^{i,j} \mathbf{B}_e^{i} \tag{2.264}$$

This matrix is a $\mathbb{R}^{n_f^i \times n_f^i}$ symmetric matrix whereas $\overline{\mathbf{J}}_{f,f}^{i,j}$ is a $\mathbb{R}^{n_{f,v}^i \times n_{f,v}^i}$ symmetric matrix defined as:

$$\overline{\mathbf{J}}_{f,f}^{i,j} = \mathbf{B}_c^{i,j\,T} \overline{\mathbf{C}}^{i,j\,T} \mathbf{S}_{f,f}^{i,j} \overline{\mathbf{C}}^{i,j} \mathbf{B}_c^{i,j} \tag{2.265}$$

Where $\mathbf{S}_{f,f}^{i,j}$ is a $\mathbb{R}^{n_f^{i,j} \times n_f^{i,j}}$ symmetric matrix defined as follows:

$$\mathbf{S}_{f,f}^{i,j} = \overline{\mathbf{S}}_{1,1}^{i,j} + \overline{\mathbf{S}}_{2,2}^{i,j} + \overline{\mathbf{S}}_{3,3}^{i,j} \tag{2.266}$$

Where the $\mathbb{R}^{n_f^{i,j} \times n_f^{i,j}}$ symmetric matrices $\overline{\mathbf{S}}_{1,1}^{i,j}$, $\overline{\mathbf{S}}_{2,2}^{i,j}$ and $\overline{\mathbf{S}}_{3,3}^{i,j}$ come from the integration of the local shape function and are defined as:

$$\overline{\mathbf{S}}_{1,1}^{i,j} = \int_{\Omega^{i,j}} \rho^{i,j} \mathbf{S}_1^{i,j\,T}(P^{i,j}) \mathbf{S}_1^{i,j}(P^{i,j}) dV^{i,j} \tag{2.267}$$

$$\overline{\mathbf{S}}_{2,2}^{i,j} = \int_{\Omega^{i,j}} \rho^{i,j} \mathbf{S}_2^{i,j\,T}(P^{i,j}) \mathbf{S}_2^{i,j}(P^{i,j}) dV^{i,j} \tag{2.268}$$

$$\overline{\mathbf{S}}_{3,3}^{i,j} = \int_{\Omega^{i,j}} \rho^{i,j} \mathbf{S}_3^{i,j\,T}(P^{i,j}) \mathbf{S}_3^{i,j}(P^{i,j}) dV^{i,j} \tag{2.269}$$

These matrices represent the inertia shape integrals required to compute explicitly the mass matrix $\mathbf{M}_{f,f}^{i,j}$ of the flexible element j of the body i of the system.

2.6.4. STIFFNESS MATRIX OF STRUCTURAL ELEMENTS

The formulation of the stiffness matrix of the element j of the body i of the system can be achieved by the definition of the elastic strain energy $U^{i,j}(t)$ of the same element. To do that, it is preliminary required to obtain an expression of element stress field and deformation field based on configuration coordinates. Assuming Voigt notation, the deformation field can be computed in a matrix form by using the linear strain-displacement equations to yield:

$$\begin{aligned}\boldsymbol{\varepsilon}^{i,j}(P^{i,j},t) &= \mathbf{D}^{i,j}\overline{\mathbf{u}}_f^{i,j}(P^{i,j},t) = \\ &= \mathbf{D}^{i,j}\mathbf{N}^{i,j}(P^{i,j})\mathbf{B}_e^i\mathbf{q}_f^i(t)\end{aligned} \tag{2.270}$$

Where $\boldsymbol{\varepsilon}^{i,j}(P^{i,j},t)$ is a $\mathbb{R}^6$ vector representing the deformation field of element j of body i and $\mathbf{D}^{i,j}$ is a differential matrix operator which are defined as:

$$\boldsymbol{\varepsilon}^{i,j}(P^{i,j},t) = \begin{bmatrix} \varepsilon_{x^{i,j}x^{i,j}}^{i,j}(P^{i,j},t) \\ \varepsilon_{y^{i,j}y^{i,j}}^{i,j}(P^{i,j},t) \\ \varepsilon_{z^{i,j}z^{i,j}}^{i,j}(P^{i,j},t) \\ \varepsilon_{x^{i,j}y^{i,j}}^{i,j}(P^{i,j},t) \\ \varepsilon_{y^{i,j}z^{i,j}}^{i,j}(P^{i,j},t) \\ \varepsilon_{z^{i,j}x^{i,j}}^{i,j}(P^{i,j},t) \end{bmatrix} \tag{2.271}$$

$$\mathbf{D}^{i,j} = \begin{bmatrix} \dfrac{\partial}{\partial x^{i,j}} & 0 & 0 \\ 0 & \dfrac{\partial}{\partial y^{i,j}} & 0 \\ 0 & 0 & \dfrac{\partial}{\partial z^{i,j}} \\ \dfrac{\partial}{\partial y^{i,j}} & \dfrac{\partial}{\partial x^{i,j}} & 0 \\ 0 & \dfrac{\partial}{\partial z^{i,j}} & \dfrac{\partial}{\partial y^{i,j}} \\ \dfrac{\partial}{\partial z^{i,j}} & 0 & \dfrac{\partial}{\partial x^{i,j}} \end{bmatrix} \tag{2.272}$$

The action of the differential matrix operator $\mathbf{D}^{i,j}$ on the element compact shape function $\mathbf{N}^{i,j}(P^{i,j})$ can be developed to yield:

$$\mathbf{D}^{i,j}\mathbf{N}^{i,j}(P^{i,j}) = \mathbf{D}^{i,j}\mathbf{C}^{i,j}\mathbf{S}^{i,j}(P^{i,j})\overline{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j} =$$

$$= \mathbf{D}^{i,j}\begin{bmatrix} \mathbf{C}_{*1}^{i,j} & \mathbf{C}_{*2}^{i,j} & \mathbf{C}_{*3}^{i,j} \end{bmatrix}\begin{bmatrix} \mathbf{S}_1^{i,j}(P^{i,j}) \\ \mathbf{S}_2^{i,j}(P^{i,j}) \\ \mathbf{S}_3^{i,j}(P^{i,j}) \end{bmatrix}\overline{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j} =$$

$$= \mathbf{D}^{i,j}\left(\mathbf{C}_{*1}^{i,j}\mathbf{S}_1^{i,j}(P^{i,j}) + \mathbf{C}_{*2}^{i,j}\mathbf{S}_2^{i,j}(P^{i,j}) + \mathbf{C}_{*3}^{i,j}\mathbf{S}_3^{i,j}(P^{i,j})\right)\overline{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j} =$$

$$= \begin{bmatrix} \frac{\partial}{\partial x^{i,j}} & 0 & 0 \\ 0 & \frac{\partial}{\partial y^{i,j}} & 0 \\ 0 & 0 & \frac{\partial}{\partial z^{i,j}} \\ \frac{\partial}{\partial y^{i,j}} & \frac{\partial}{\partial x^{i,j}} & 0 \\ 0 & \frac{\partial}{\partial z^{i,j}} & \frac{\partial}{\partial y^{i,j}} \\ \frac{\partial}{\partial z^{i,j}} & 0 & \frac{\partial}{\partial x^{i,j}} \end{bmatrix}\begin{bmatrix} C_{1,1}^{i,j}\mathbf{S}_1^{i,j}(P^{i,j}) + C_{1,2}^{i,j}\mathbf{S}_2^{i,j}(P^{i,j}) + C_{1,3}^{i,j}\mathbf{S}_3^{i,j}(P^{i,j}) \\ C_{2,1}^{i,j}\mathbf{S}_1^{i,j}(P^{i,j}) + C_{2,2}^{i,j}\mathbf{S}_2^{i,j}(P^{i,j}) + C_{2,3}^{i,j}\mathbf{S}_3^{i,j}(P^{i,j}) \\ C_{3,1}^{i,j}\mathbf{S}_1^{i,j}(P^{i,j}) + C_{3,2}^{i,j}\mathbf{S}_2^{i,j}(P^{i,j}) + C_{3,3}^{i,j}\mathbf{S}_3^{i,j}(P^{i,j}) \end{bmatrix}\overline{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j} =$$

$$= \begin{bmatrix} C_{1,1}^{i,j}\mathbf{S}_{1,x}^{i,j}(P^{i,j}) + C_{1,2}^{i,j}\mathbf{S}_{2,x}^{i,j}(P^{i,j}) + C_{1,3}^{i,j}\mathbf{S}_{3,x}^{i,j}(P^{i,j}) \\ C_{2,1}^{i,j}\mathbf{S}_{1,y}^{i,j}(P^{i,j}) + C_{2,2}^{i,j}\mathbf{S}_{2,y}^{i,j}(P^{i,j}) + C_{2,3}^{i,j}\mathbf{S}_{3,y}^{i,j}(P^{i,j}) \\ C_{3,1}^{i,j}\mathbf{S}_{1,z}^{i,j}(P^{i,j}) + C_{3,2}^{i,j}\mathbf{S}_{2,z}^{i,j}(P^{i,j}) + C_{3,3}^{i,j}\mathbf{S}_{3,z}^{i,j}(P^{i,j}) \\ C_{1,1}^{i,j}\mathbf{S}_{1,y}^{i,j}(P^{i,j}) + C_{1,2}^{i,j}\mathbf{S}_{2,y}^{i,j}(P^{i,j}) + C_{1,3}^{i,j}\mathbf{S}_{3,y}^{i,j}(P^{i,j}) + \\ + C_{2,1}^{i,j}\mathbf{S}_{1,x}^{i,j}(P^{i,j}) + C_{2,2}^{i,j}\mathbf{S}_{2,x}^{i,j}(P^{i,j}) + C_{2,3}^{i,j}\mathbf{S}_{3,x}^{i,j}(P^{i,j}) \\ C_{2,1}^{i,j}\mathbf{S}_{1,z}^{i,j}(P^{i,j}) + C_{2,2}^{i,j}\mathbf{S}_{2,z}^{i,j}(P^{i,j}) + C_{2,3}^{i,j}\mathbf{S}_{3,z}^{i,j}(P^{i,j}) + \\ + C_{3,1}^{i,j}\mathbf{S}_{1,y}^{i,j}(P^{i,j}) + C_{3,2}^{i,j}\mathbf{S}_{2,y}^{i,j}(P^{i,j}) + C_{3,3}^{i,j}\mathbf{S}_{3,y}^{i,j}(P^{i,j}) \\ C_{1,1}^{i,j}\mathbf{S}_{1,z}^{i,j}(P^{i,j}) + C_{1,2}^{i,j}\mathbf{S}_{2,z}^{i,j}(P^{i,j}) + C_{1,3}^{i,j}\mathbf{S}_{3,z}^{i,j}(P^{i,j}) + \\ + C_{3,1}^{i,j}\mathbf{S}_{1,x}^{i,j}(P^{i,j}) + C_{3,2}^{i,j}\mathbf{S}_{2,x}^{i,j}(P^{i,j}) + C_{3,3}^{i,j}\mathbf{S}_{3,x}^{i,j}(P^{i,j}) \end{bmatrix}\overline{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j}$$

(2.273)

This expression can be rewritten in a more compact form to yield:

$$\begin{bmatrix} C_{1,1}^{i,j}\mathbf{S}_{1,x}^{i,j}(P^{i,j})+C_{1,2}^{i,j}\mathbf{S}_{2,x}^{i,j}(P^{i,j})+C_{1,3}^{i,j}\mathbf{S}_{3,x}^{i,j}(P^{i,j}) \\ C_{2,1}^{i,j}\mathbf{S}_{1,y}^{i,j}(P^{i,j})+C_{2,2}^{i,j}\mathbf{S}_{2,y}^{i,j}(P^{i,j})+C_{2,3}^{i,j}\mathbf{S}_{3,y}^{i,j}(P^{i,j}) \\ C_{3,1}^{i,j}\mathbf{S}_{1,z}^{i,j}(P^{i,j})+C_{3,2}^{i,j}\mathbf{S}_{2,z}^{i,j}(P^{i,j})+C_{3,3}^{i,j}\mathbf{S}_{3,z}^{i,j}(P^{i,j}) \\ C_{1,1}^{i,j}\mathbf{S}_{1,y}^{i,j}(P^{i,j})+C_{1,2}^{i,j}\mathbf{S}_{2,y}^{i,j}(P^{i,j})+C_{1,3}^{i,j}\mathbf{S}_{3,y}^{i,j}(P^{i,j})+ \\ +C_{2,1}^{i,j}\mathbf{S}_{1,x}^{i,j}(P^{i,j})+C_{2,2}^{i,j}\mathbf{S}_{2,x}^{i,j}(P^{i,j})+C_{2,3}^{i,j}\mathbf{S}_{3,x}^{i,j}(P^{i,j}) \\ C_{2,1}^{i,j}\mathbf{S}_{1,z}^{i,j}(P^{i,j})+C_{2,2}^{i,j}\mathbf{S}_{2,z}^{i,j}(P^{i,j})+C_{2,3}^{i,j}\mathbf{S}_{3,z}^{i,j}(P^{i,j})+ \\ +C_{3,1}^{i,j}\mathbf{S}_{1,y}^{i,j}(P^{i,j})+C_{3,2}^{i,j}\mathbf{S}_{2,y}^{i,j}(P^{i,j})+C_{3,3}^{i,j}\mathbf{S}_{3,y}^{i,j}(P^{i,j}) \\ C_{1,1}^{i,j}\mathbf{S}_{1,z}^{i,j}(P^{i,j})+C_{1,2}^{i,j}\mathbf{S}_{2,z}^{i,j}(P^{i,j})+C_{1,3}^{i,j}\mathbf{S}_{3,z}^{i,j}(P^{i,j})+ \\ +C_{3,1}^{i,j}\mathbf{S}_{1,x}^{i,j}(P^{i,j})+C_{3,2}^{i,j}\mathbf{S}_{2,x}^{i,j}(P^{i,j})+C_{3,3}^{i,j}\mathbf{S}_{3,x}^{i,j}(P^{i,j}) \end{bmatrix} \bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j} =$$

$$= \begin{bmatrix} C_{1,1}^{i,j} & C_{1,2}^{i,j} & C_{1,3}^{i,j} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ C_{2,1}^{i,j} & C_{2,2}^{i,j} & C_{2,3}^{i,j} \\ 0 & 0 & 0 \\ C_{3,1}^{i,j} & C_{3,2}^{i,j} & C_{3,3}^{i,j} \end{bmatrix} \begin{bmatrix} \mathbf{S}_{1,x}^{i,j}(P^{i,j}) \\ \mathbf{S}_{2,x}^{i,j}(P^{i,j}) \\ \mathbf{S}_{3,x}^{i,j}(P^{i,j}) \end{bmatrix} \bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j} + \begin{bmatrix} 0 & 0 & 0 \\ C_{2,1}^{i,j} & C_{2,2}^{i,j} & C_{2,3}^{i,j} \\ 0 & 0 & 0 \\ C_{1,1}^{i,j} & C_{1,2}^{i,j} & C_{1,3}^{i,j} \\ C_{3,1}^{i,j} & C_{3,2}^{i,j} & C_{3,3}^{i,j} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{S}_{1,y}^{i,j}(P^{i,j}) \\ \mathbf{S}_{2,y}^{i,j}(P^{i,j}) \\ \mathbf{S}_{3,y}^{i,j}(P^{i,j}) \end{bmatrix} \bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j} +$$

$$+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ C_{3,1}^{i,j} & C_{3,2}^{i,j} & C_{3,3}^{i,j} \\ 0 & 0 & 0 \\ C_{2,1}^{i,j} & C_{2,2}^{i,j} & C_{2,3}^{i,j} \\ C_{1,1}^{i,j} & C_{1,2}^{i,j} & C_{1,3}^{i,j} \end{bmatrix} \begin{bmatrix} \mathbf{S}_{1,z}^{i,j}(P^{i,j}) \\ \mathbf{S}_{2,z}^{i,j}(P^{i,j}) \\ \mathbf{S}_{3,z}^{i,j}(P^{i,j}) \end{bmatrix} \bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j} =$$

$$= \left(\bar{\bar{\mathbf{C}}}_x^{i,j}\mathbf{S}_x^{i,j}(P^{i,j}) + \bar{\bar{\mathbf{C}}}_y^{i,j}\mathbf{S}_y^{i,j}(P^{i,j}) + \bar{\bar{\mathbf{C}}}_z^{i,j}\mathbf{S}_z^{i,j}(P^{i,j}) \right) \bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j} \tag{2.274}$$

Where the rotation matrix $\mathbf{C}^{i,j}$ has been expressed as follows:

$$\mathbf{C}^{i,j} = \begin{bmatrix} \mathbf{C}^{i,j}_{*1} & \mathbf{C}^{i,j}_{*2} & \mathbf{C}^{i,j}_{*3} \end{bmatrix} \tag{2.275}$$

Therefore, the final expression of the action of the matrix differential operator on the compact shape function is:

$$\mathbf{D}^{i,j}\mathbf{N}^{i,j}(P^{i,j}) = \left(\bar{\bar{\mathbf{C}}}^{i,j}_x \mathbf{S}^{i,j}_x(P^{i,j}) + \bar{\bar{\mathbf{C}}}^{i,j}_y \mathbf{S}^{i,j}_y(P^{i,j}) + \bar{\bar{\mathbf{C}}}^{i,j}_z \mathbf{S}^{i,j}_z(P^{i,j})\right)\bar{\mathbf{C}}^{i,j}\mathbf{B}^{i,j}_c \tag{2.276}$$

Where $\bar{\bar{\mathbf{C}}}^{i,j}_x$, $\bar{\bar{\mathbf{C}}}^{i,j}_y$ and $\bar{\bar{\mathbf{C}}}^{i,j}_z$ are $\mathbb{R}^{6\times 3}$ matrices whose components are the components of the rotation matrix $\mathbf{C}^{i,j}$ which are defined as:

$$\bar{\bar{\mathbf{C}}}^{i,j}_x = \begin{bmatrix} C^{i,j}_{1,1} & C^{i,j}_{1,2} & C^{i,j}_{1,3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ C^{i,j}_{2,1} & C^{i,j}_{2,2} & C^{i,j}_{2,3} \\ 0 & 0 & 0 \\ C^{i,j}_{3,1} & C^{i,j}_{3,2} & C^{i,j}_{3,3} \end{bmatrix} \tag{2.277}$$

$$\bar{\bar{\mathbf{C}}}^{i,j}_y = \begin{bmatrix} 0 & 0 & 0 \\ C^{i,j}_{2,1} & C^{i,j}_{2,2} & C^{i,j}_{2,3} \\ 0 & 0 & 0 \\ C^{i,j}_{1,1} & C^{i,j}_{1,2} & C^{i,j}_{1,3} \\ C^{i,j}_{3,1} & C^{i,j}_{3,2} & C^{i,j}_{3,3} \\ 0 & 0 & 0 \end{bmatrix} \tag{2.278}$$

$$\bar{\bar{\mathbf{C}}}_z^{i,j} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ C_{3,1}^{i,j} & C_{3,2}^{i,j} & C_{3,3}^{i,j} \\ 0 & 0 & 0 \\ C_{2,1}^{i,j} & C_{2,2}^{i,j} & C_{2,3}^{i,j} \\ C_{1,1}^{i,j} & C_{1,2}^{i,j} & C_{1,3}^{i,j} \end{bmatrix} \tag{2.279}$$

For instance, in the case of a two-dimensional system, these matrices assumes the following form:

$$\bar{\bar{\mathbf{C}}}_x^{i,j} = \begin{bmatrix} \cos(\alpha^{i,j}) & -\sin(\alpha^{i,j}) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sin(\alpha^{i,j}) & \cos(\alpha^{i,j}) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.280}$$

$$\bar{\bar{\mathbf{C}}}_y^{i,j} = \begin{bmatrix} 0 & 0 & 0 \\ \sin(\alpha^{i,j}) & \cos(\alpha^{i,j}) & 0 \\ 0 & 0 & 0 \\ \cos(\alpha^{i,j}) & -\sin(\alpha^{i,j}) & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.281}$$

$$\bar{\bar{\mathbf{C}}}_z^{i,j} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ \sin(\alpha^{i,j}) & \cos(\alpha^{i,j}) & 0 \\ \cos(\alpha^{i,j}) & -\sin(\alpha^{i,j}) & 0 \end{bmatrix} \tag{2.282}$$

Besides, the matrices $\mathbf{S}_x^{i,j}(P^{i,j})$, $\mathbf{S}_y^{i,j}(P^{i,j})$ and $\mathbf{S}_z^{i,j}(P^{i,j})$ are simply the space derivative of the shape function $\mathbf{S}^{i,j}(P^{i,j})$. Indeed:

$$\begin{aligned} \mathbf{S}_x^{i,j}(P^{i,j}) &= \frac{\partial \mathbf{S}^{i,j}(P^{i,j})}{\partial x^{i,j}} = \\ &= \begin{bmatrix} \mathbf{S}_{1,x}^{i,j}(P^{i,j}) \\ \mathbf{S}_{2,x}^{i,j}(P^{i,j}) \\ \mathbf{S}_{3,x}^{i,j}(P^{i,j}) \end{bmatrix} \end{aligned} \tag{2.283}$$

$$\begin{aligned} \mathbf{S}_y^{i,j}(P^{i,j}) &= \frac{\partial \mathbf{S}^{i,j}(P^{i,j})}{\partial y^{i,j}} = \\ &= \begin{bmatrix} \mathbf{S}_{1,y}^{i,j}(P^{i,j}) \\ \mathbf{S}_{2,y}^{i,j}(P^{i,j}) \\ \mathbf{S}_{3,y}^{i,j}(P^{i,j}) \end{bmatrix} \end{aligned} \tag{2.284}$$

$$\begin{aligned} \mathbf{S}_z^{i,j}(P^{i,j}) &= \frac{\partial \mathbf{S}^{i,j}(P^{i,j})}{\partial z^{i,j}} = \\ &= \begin{bmatrix} \mathbf{S}_{1,z}^{i,j}(P^{i,j}) \\ \mathbf{S}_{2,z}^{i,j}(P^{i,j}) \\ \mathbf{S}_{3,z}^{i,j}(P^{i,j}) \end{bmatrix} \end{aligned} \tag{2.285}$$

On the other hand, the stress field can be computed by the constitutive equations of the linear elastic material according to Voigt notation in order to yield:

$$\begin{aligned}\boldsymbol{\sigma}^{i,j}(P^{i,j},t) &= \mathbf{E}^{i,j}\boldsymbol{\varepsilon}^{i,j}(P^{i,j},t) = \\ &= \mathbf{E}^{i,j}\mathbf{D}^{i,j}\mathbf{N}^{i,j}(P^{i,j})\mathbf{B}_e^i\mathbf{q}_f^i(t)\end{aligned} \tag{2.286}$$

Where $\boldsymbol{\sigma}^{i,j}(P^{i,j},t)$ is a $\mathbb{R}^6$ vector representing the stress field of element j of body i and $\mathbf{E}^{i,j}$ is the $\mathbb{R}^{6\times 6}$ matrix of elastic coefficients. According to Voigt notation, the stress field can be written as follows:

$$\boldsymbol{\sigma}^{i,j}(P^{i,j},t) = \begin{bmatrix} \sigma^{i,j}_{x^{i,j}x^{i,j}}(P^{i,j},t) \\ \sigma^{i,j}_{y^{i,j}y^{i,j}}(P^{i,j},t) \\ \sigma^{i,j}_{z^{i,j}z^{i,j}}(P^{i,j},t) \\ \sigma^{i,j}_{x^{i,j}y^{i,j}}(P^{i,j},t) \\ \sigma^{i,j}_{y^{i,j}z^{i,j}}(P^{i,j},t) \\ \sigma^{i,j}_{z^{i,j}x^{i,j}}(P^{i,j},t) \end{bmatrix} \tag{2.287}$$

For instance, in the case of homogeneous isotropic linear elastic material the matrix of elastic coefficients $\mathbf{E}^{i,j}$ become:

$$\mathbf{E}^{i,j} = \frac{E^{i,j}}{\left(1+\nu^{i,j}\right)\left(1-2\nu^{i,j}\right)} \begin{bmatrix} 1-\nu^{i,j} & \nu^{i,j} & \nu^{i,j} & 0 & 0 & 0 \\ \nu^{i,j} & 1-\nu^{i,j} & \nu^{i,j} & 0 & 0 & 0 \\ \nu^{i,j} & \nu^{i,j} & 1-\nu^{i,j} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu^{i,j}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu^{i,j}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu^{i,j}}{2} \end{bmatrix} \tag{2.288}$$

Where $E^{i,j}$ and $\nu^{i,j}$ are respectively the Young elasticity modulus and Poisson ratio of elastic element j of body i. At this stage, the stiffness matrix of element j of body i can be computed by using the definition of the strain energy $U^{i,j}(t)$:

$$\begin{aligned} U^{i,j}(t) &= \frac{1}{2}\int_{\Omega^{i,j}} \boldsymbol{\sigma}^{i,jT}(P^{i,j},t)\boldsymbol{\varepsilon}^{i,j}(P^{i,j},t)dV^{i,j} = \\ &= \frac{1}{2}\int_{\Omega^{i,j}} \mathbf{q}_f^{iT}(t)\mathbf{B}_e^{iT}\left(\mathbf{D}^{i,j}\mathbf{N}^{i,j}(P^{i,j})\right)^T \mathbf{E}^{i,j}\mathbf{D}^{i,j}\mathbf{N}^{i,j}(P^{i,j})\mathbf{B}_e^i\mathbf{q}_f^i(t)dV^{i,j} = \\ &= \frac{1}{2}\mathbf{q}_f^{iT}(t)\mathbf{B}_e^{iT}\int_{\Omega^{i,j}}\left(\mathbf{D}^{i,j}\mathbf{N}^{i,j}(P^{i,j})\right)^T \mathbf{E}^{i,j}\mathbf{D}^{i,j}\mathbf{N}^{i,j}(P^{i,j})dV^{i,j}\mathbf{B}_e^i\mathbf{q}_f^i(t) = \\ &= \frac{1}{2}\mathbf{q}_f^{iT}(t)\mathbf{B}_e^{iT}\bar{\mathbf{V}}_{f,f}^{i,j}\mathbf{B}_e^i\mathbf{q}_f^i(t) = \\ &= \frac{1}{2}\mathbf{q}_f^{iT}(t)\mathbf{K}_{f,f}^{i,j}\mathbf{q}_f^i(t) \end{aligned} \tag{2.289}$$

Therefore, the final expression of the stiffness matrix $\mathbf{K}_{f,f}^{i,j}$ of the element j of body i is:

$$\mathbf{K}_{f,f}^{i,j} = \mathbf{B}_e^{iT} \bar{\mathbf{V}}_{f,f}^{i,j} \mathbf{B}_e^i \tag{2.290}$$

This matrix is a $\mathbb{R}^{n_f^i \times n_f^i}$ symmetric matrix whereas $\bar{\mathbf{V}}_{f,f}^{i,j}$ is a $\mathbb{R}^{n_{f,v}^i \times n_{f,v}^i}$ symmetric matrix defined as:

$$
\begin{aligned}
\bar{\mathbf{V}}^{i,j}_{f,f} &= \int_{\Omega^{i,j}} \left(\mathbf{D}^{i,j}\mathbf{N}^{i,j}(P^{i,j})\right)^T \mathbf{E}^{i,j}\mathbf{D}^{i,j}\mathbf{N}^{i,j}(P^{i,j})dV^{i,j} = \\
&= \mathbf{B}^{i,j\,T}_c \bar{\mathbf{C}}^{i,j\,T} \int_{\Omega^{i,j}} \mathbf{S}^{i,j\,T}_x(P^{i,j})\bar{\bar{\mathbf{C}}}^{i,j\,T}_x \mathbf{E}^{i,j}\bar{\bar{\mathbf{C}}}^{i,j}_x \mathbf{S}^{i,j}_x(P^{i,j})dV^{i,j}\bar{\mathbf{C}}^{i,j}\mathbf{B}^{i,j}_c + \\
&\quad + \mathbf{B}^{i,j\,T}_c \bar{\mathbf{C}}^{i,j\,T} \int_{\Omega^{i,j}} \mathbf{S}^{i,j\,T}_x(P^{i,j})\bar{\bar{\mathbf{C}}}^{i,j\,T}_x \mathbf{E}^{i,j}\bar{\bar{\mathbf{C}}}^{i,j}_y \mathbf{S}^{i,j}_y(P^{i,j})dV^{i,j}\bar{\mathbf{C}}^{i,j}\mathbf{B}^{i,j}_c + \\
&\quad + \mathbf{B}^{i,j\,T}_c \bar{\mathbf{C}}^{i,j\,T} \int_{\Omega^{i,j}} \mathbf{S}^{i,j\,T}_x(P^{i,j})\bar{\bar{\mathbf{C}}}^{i,j\,T}_x \mathbf{E}^{i,j}\bar{\bar{\mathbf{C}}}^{i,j}_z \mathbf{S}^{i,j}_z(P^{i,j})dV^{i,j}\bar{\mathbf{C}}^{i,j}\mathbf{B}^{i,j}_c + \\
&\quad + \mathbf{B}^{i,j\,T}_c \bar{\mathbf{C}}^{i,j\,T} \int_{\Omega^{i,j}} \mathbf{S}^{i,j\,T}_y(P^{i,j})\bar{\bar{\mathbf{C}}}^{i,j\,T}_y \mathbf{E}^{i,j}\bar{\bar{\mathbf{C}}}^{i,j}_x \mathbf{S}^{i,j}_x(P^{i,j})dV^{i,j}\bar{\mathbf{C}}^{i,j}\mathbf{B}^{i,j}_c + \\
&\quad + \mathbf{B}^{i,j\,T}_c \bar{\mathbf{C}}^{i,j\,T} \int_{\Omega^{i,j}} \mathbf{S}^{i,j\,T}_y(P^{i,j})\bar{\bar{\mathbf{C}}}^{i,j\,T}_y \mathbf{E}^{i,j}\bar{\bar{\mathbf{C}}}^{i,j}_y \mathbf{S}^{i,j}_y(P^{i,j})dV^{i,j}\bar{\mathbf{C}}^{i,j}\mathbf{B}^{i,j}_c + \\
&\quad + \mathbf{B}^{i,j\,T}_c \bar{\mathbf{C}}^{i,j\,T} \int_{\Omega^{i,j}} \mathbf{S}^{i,j\,T}_y(P^{i,j})\bar{\bar{\mathbf{C}}}^{i,j\,T}_y \mathbf{E}^{i,j}\bar{\bar{\mathbf{C}}}^{i,j}_z \mathbf{S}^{i,j}_z(P^{i,j})dV^{i,j}\bar{\mathbf{C}}^{i,j}\mathbf{B}^{i,j}_c + \\
&\quad + \mathbf{B}^{i,j\,T}_c \bar{\mathbf{C}}^{i,j\,T} \int_{\Omega^{i,j}} \mathbf{S}^{i,j\,T}_z(P^{i,j})\bar{\bar{\mathbf{C}}}^{i,j\,T}_z \mathbf{E}^{i,j}\bar{\bar{\mathbf{C}}}^{i,j}_x \mathbf{S}^{i,j}_x(P^{i,j})dV^{i,j}\bar{\mathbf{C}}^{i,j}\mathbf{B}^{i,j}_c + \\
&\quad + \mathbf{B}^{i,j\,T}_c \bar{\mathbf{C}}^{i,j\,T} \int_{\Omega^{i,j}} \mathbf{S}^{i,j\,T}_z(P^{i,j})\bar{\bar{\mathbf{C}}}^{i,j\,T}_z \mathbf{E}^{i,j}\bar{\bar{\mathbf{C}}}^{i,j}_y \mathbf{S}^{i,j}_y(P^{i,j})dV^{i,j}\bar{\mathbf{C}}^{i,j}\mathbf{B}^{i,j}_c + \\
&\quad + \mathbf{B}^{i,j\,T}_c \bar{\mathbf{C}}^{i,j\,T} \int_{\Omega^{i,j}} \mathbf{S}^{i,j\,T}_z(P^{i,j})\bar{\bar{\mathbf{C}}}^{i,j\,T}_z \mathbf{E}^{i,j}\bar{\bar{\mathbf{C}}}^{i,j}_z \mathbf{S}^{i,j}_z(P^{i,j})dV^{i,j}\bar{\mathbf{C}}^{i,j}\mathbf{B}^{i,j}_c = \\
&= \mathbf{B}^{i,j\,T}_c \bar{\mathbf{C}}^{i,j\,T}\bar{\bar{\mathbf{S}}}^{i,j}_{1,1}\bar{\mathbf{C}}^{i,j}\mathbf{B}^{i,j}_c + \mathbf{B}^{i,j\,T}_c \bar{\mathbf{C}}^{i,j\,T}\bar{\bar{\mathbf{S}}}^{i,j}_{1,2}\bar{\mathbf{C}}^{i,j}\mathbf{B}^{i,j}_c + \mathbf{B}^{i,j\,T}_c \bar{\mathbf{C}}^{i,j\,T}\bar{\bar{\mathbf{S}}}^{i,j}_{1,3}\bar{\mathbf{C}}^{i,j}\mathbf{B}^{i,j}_c + \\
&\quad + \mathbf{B}^{i,j\,T}_c \bar{\mathbf{C}}^{i,j\,T}\bar{\bar{\mathbf{S}}}^{i,j}_{2,1}\bar{\mathbf{C}}^{i,j}\mathbf{B}^{i,j}_c + \mathbf{B}^{i,j\,T}_c \bar{\mathbf{C}}^{i,j\,T}\bar{\bar{\mathbf{S}}}^{i,j}_{2,2}\bar{\mathbf{C}}^{i,j}\mathbf{B}^{i,j}_c + \mathbf{B}^{i,j\,T}_c \bar{\mathbf{C}}^{i,j\,T}\bar{\bar{\mathbf{S}}}^{i,j}_{2,3}\bar{\mathbf{C}}^{i,j}\mathbf{B}^{i,j}_c + \\
&\quad + \mathbf{B}^{i,j\,T}_c \bar{\mathbf{C}}^{i,j\,T}\bar{\bar{\mathbf{S}}}^{i,j}_{3,1}\bar{\mathbf{C}}^{i,j}\mathbf{B}^{i,j}_c + \mathbf{B}^{i,j\,T}_c \bar{\mathbf{C}}^{i,j\,T}\bar{\bar{\mathbf{S}}}^{i,j}_{3,2}\bar{\mathbf{C}}^{i,j}\mathbf{B}^{i,j}_c + \mathbf{B}^{i,j\,T}_c \bar{\mathbf{C}}^{i,j\,T}\bar{\bar{\mathbf{S}}}^{i,j}_{3,3}\bar{\mathbf{C}}^{i,j}\mathbf{B}^{i,j}_c = \\
&= \mathbf{B}^{i,j\,T}_c \bar{\mathbf{C}}^{i,j\,T}\left(\bar{\bar{\mathbf{S}}}^{i,j}_{1,1} + \bar{\bar{\mathbf{S}}}^{i,j}_{2,2} + \bar{\bar{\mathbf{S}}}^{i,j}_{3,3}\right)\bar{\mathbf{C}}^{i,j}\mathbf{B}^{i,j}_c + \mathbf{B}^{i,j\,T}_c \bar{\mathbf{C}}^{i,j\,T}\left(\bar{\bar{\mathbf{S}}}^{i,j}_{1,2} + \bar{\bar{\mathbf{S}}}^{i,j}_{2,1}\right)\bar{\mathbf{C}}^{i,j}\mathbf{B}^{i,j}_c + \\
&\quad + \mathbf{B}^{i,j\,T}_c \bar{\mathbf{C}}^{i,j\,T}\left(\bar{\bar{\mathbf{S}}}^{i,j}_{1,3} + \bar{\bar{\mathbf{S}}}^{i,j}_{3,1}\right)\bar{\mathbf{C}}^{i,j}\mathbf{B}^{i,j}_c + \mathbf{B}^{i,j\,T}_c \bar{\mathbf{C}}^{i,j\,T}\left(\bar{\bar{\mathbf{S}}}^{i,j}_{2,3} + \bar{\bar{\mathbf{S}}}^{i,j}_{3,2}\right)\bar{\mathbf{C}}^{i,j}\mathbf{B}^{i,j}_c = \\
&= \mathbf{B}^{i,j\,T}_c \bar{\mathbf{C}}^{i,j\,T}\bar{\bar{\mathbf{S}}}^{i,j}_{f,f}\bar{\mathbf{C}}^{i,j}\mathbf{B}^{i,j}_c + \mathbf{B}^{i,j\,T}_c \bar{\mathbf{C}}^{i,j\,T}\bar{\bar{\mathbf{S}}}^{i,j}_{f,1}\bar{\mathbf{C}}^{i,j}\mathbf{B}^{i,j}_c + \\
&\quad + \mathbf{B}^{i,j\,T}_c \bar{\mathbf{C}}^{i,j\,T}\bar{\bar{\mathbf{S}}}^{i,j}_{f,2}\bar{\mathbf{C}}^{i,j}\mathbf{B}^{i,j}_c + \mathbf{B}^{i,j\,T}_c \bar{\mathbf{C}}^{i,j\,T}\bar{\bar{\mathbf{S}}}^{i,j}_{f,3}\bar{\mathbf{C}}^{i,j}\mathbf{B}^{i,j}_c = \\
&= \bar{\bar{\mathbf{J}}}^{i,j}_{f,f} + \bar{\bar{\mathbf{J}}}^{i,j}_{f,1} + \bar{\bar{\mathbf{J}}}^{i,j}_{f,2} + \bar{\bar{\mathbf{J}}}^{i,j}_{f,3}
\end{aligned} \tag{2.291}
$$

Where $\bar{\bar{\mathbf{J}}}^{i,j}_{f,f}$, $\bar{\bar{\mathbf{J}}}^{i,j}_{f,1}$, $\bar{\bar{\mathbf{J}}}^{i,j}_{f,2}$ and $\bar{\bar{\mathbf{J}}}^{i,j}_{f,3}$ are $\mathbb{R}^{n^i_{f,v} \times n^i_{f,v}}$ symmetric matrices defined as follows:

$$\bar{\bar{\mathbf{J}}}^{i,j}_{f,f} = \mathbf{B}^{i,j\,T}_c \bar{\mathbf{C}}^{i,j\,T} \bar{\bar{\mathbf{S}}}^{i,j}_{f,f} \bar{\mathbf{C}}^{i,j} \mathbf{B}^{i,j}_c \tag{2.292}$$

$$\bar{\bar{\mathbf{J}}}^{i,j}_{f,1} = \mathbf{B}^{i,j\,T}_c \bar{\mathbf{C}}^{i,j\,T} \bar{\bar{\mathbf{S}}}^{i,j}_{f,1} \bar{\mathbf{C}}^{i,j} \mathbf{B}^{i,j}_c \tag{2.293}$$

$$\bar{\bar{\mathbf{J}}}^{i,j}_{f,2} = \mathbf{B}^{i,j\,T}_c \bar{\mathbf{C}}^{i,j\,T} \bar{\bar{\mathbf{S}}}^{i,j}_{f,2} \bar{\mathbf{C}}^{i,j} \mathbf{B}^{i,j}_c \tag{2.294}$$

$$\bar{\bar{\mathbf{J}}}^{i,j}_{f,3} = \mathbf{B}^{i,j\,T}_c \bar{\mathbf{C}}^{i,j\,T} \bar{\bar{\mathbf{S}}}^{i,j}_{f,3} \bar{\mathbf{C}}^{i,j} \mathbf{B}^{i,j}_c \tag{2.295}$$

Where $\bar{\bar{\mathbf{S}}}^{i,j}_{f,f}$, $\bar{\bar{\mathbf{S}}}^{i,j}_{f,1}$, $\bar{\bar{\mathbf{S}}}^{i,j}_{f,2}$ and $\bar{\bar{\mathbf{S}}}^{i,j}_{f,3}$ are $\mathbb{R}^{n^{i,j}_f \times n^{i,j}_f}$ symmetric matrices defined as:

$$\bar{\bar{\mathbf{S}}}^{i,j}_{f,f} = \bar{\bar{\mathbf{S}}}^{i,j}_{1,1} + \bar{\bar{\mathbf{S}}}^{i,j}_{2,2} + \bar{\bar{\mathbf{S}}}^{i,j}_{3,3} \tag{2.296}$$

$$\bar{\bar{\mathbf{S}}}^{i,j}_{f,1} = \bar{\bar{\mathbf{S}}}^{i,j}_{2,3} + \bar{\bar{\mathbf{S}}}^{i,j}_{3,2} \tag{2.297}$$

$$\bar{\bar{\mathbf{S}}}^{i,j}_{f,2} = \bar{\bar{\mathbf{S}}}^{i,j}_{3,1} + \bar{\bar{\mathbf{S}}}^{i,j}_{1,3} \tag{2.298}$$

$$\bar{\bar{\mathbf{S}}}^{i,j}_{f,3} = \bar{\bar{\mathbf{S}}}^{i,j}_{1,2} + \bar{\bar{\mathbf{S}}}^{i,j}_{2,1} \tag{2.299}$$

Where $\bar{\bar{\mathbf{S}}}^{i,j}_{1,1}$, $\bar{\bar{\mathbf{S}}}^{i,j}_{1,2}$, $\bar{\bar{\mathbf{S}}}^{i,j}_{1,3}$, $\bar{\bar{\mathbf{S}}}^{i,j}_{2,1}$, $\bar{\bar{\mathbf{S}}}^{i,j}_{2,2}$, $\bar{\bar{\mathbf{S}}}^{i,j}_{2,3}$, $\bar{\bar{\mathbf{S}}}^{i,j}_{3,1}$, $\bar{\bar{\mathbf{S}}}^{i,j}_{3,2}$ and $\bar{\bar{\mathbf{S}}}^{i,j}_{3,3}$ are $\mathbb{R}^{n^{i,j}_f \times n^{i,j}_f}$ matrices defined as:

$$\bar{\bar{\mathbf{S}}}^{i,j}_{1,1} = \int_{\Omega^{i,j}} \mathbf{S}^{i,j\,T}_x (P^{i,j}) \bar{\bar{\mathbf{C}}}^{i,j\,T}_x \mathbf{E}^{i,j} \bar{\bar{\mathbf{C}}}^{i,j}_x \mathbf{S}^{i,j}_x (P^{i,j}) dV^{i,j} \tag{2.300}$$

$$\bar{\bar{\mathbf{S}}}^{i,j}_{1,2} = \int_{\Omega^{i,j}} \mathbf{S}^{i,j\,T}_x (P^{i,j}) \bar{\bar{\mathbf{C}}}^{i,j\,T}_x \mathbf{E}^{i,j} \bar{\bar{\mathbf{C}}}^{i,j}_y \mathbf{S}^{i,j}_y (P^{i,j}) dV^{i,j} \tag{2.301}$$

$$\bar{\bar{\mathbf{S}}}^{i,j}_{1,3} = \int_{\Omega^{i,j}} \mathbf{S}^{i,j\,T}_x (P^{i,j}) \bar{\bar{\mathbf{C}}}^{i,j\,T}_x \mathbf{E}^{i,j} \bar{\bar{\mathbf{C}}}^{i,j}_z \mathbf{S}^{i,j}_z (P^{i,j}) dV^{i,j} \tag{2.302}$$

$$\bar{\bar{\mathbf{S}}}_{2,1}^{i,j} = \int_{\Omega^{i,j}} \mathbf{S}_y^{i,j\,T}(P^{i,j})\bar{\bar{\mathbf{C}}}_y^{i,j\,T}\mathbf{E}^{i,j}\bar{\bar{\mathbf{C}}}_x^{i,j}\mathbf{S}_x^{i,j}(P^{i,j})dV^{i,j} \tag{2.303}$$

$$\bar{\bar{\mathbf{S}}}_{2,2}^{i,j} = \int_{\Omega^{i,j}} \mathbf{S}_y^{i,j\,T}(P^{i,j})\bar{\bar{\mathbf{C}}}_y^{i,j\,T}\mathbf{E}^{i,j}\bar{\bar{\mathbf{C}}}_y^{i,j}\mathbf{S}_y^{i,j}(P^{i,j})dV^{i,j} \tag{2.304}$$

$$\bar{\bar{\mathbf{S}}}_{2,3}^{i,j} = \int_{\Omega^{i,j}} \mathbf{S}_y^{i,j\,T}(P^{i,j})\bar{\bar{\mathbf{C}}}_y^{i,j\,T}\mathbf{E}^{i,j}\bar{\bar{\mathbf{C}}}_z^{i,j}\mathbf{S}_z^{i,j}(P^{i,j})dV^{i,j} \tag{2.305}$$

$$\bar{\bar{\mathbf{S}}}_{3,1}^{i,j} = \int_{\Omega^{i,j}} \mathbf{S}_z^{i,j\,T}(P^{i,j})\bar{\bar{\mathbf{C}}}_z^{i,j\,T}\mathbf{E}^{i,j}\bar{\bar{\mathbf{C}}}_x^{i,j}\mathbf{S}_x^{i,j}(P^{i,j})dV^{i,j} \tag{2.306}$$

$$\bar{\bar{\mathbf{S}}}_{3,2}^{i,j} = \int_{\Omega^{i,j}} \mathbf{S}_z^{i,j\,T}(P^{i,j})\bar{\bar{\mathbf{C}}}_z^{i,j\,T}\mathbf{E}^{i,j}\bar{\bar{\mathbf{C}}}_y^{i,j}\mathbf{S}_y^{i,j}(P^{i,j})dV^{i,j} \tag{2.307}$$

$$\bar{\bar{\mathbf{S}}}_{3,3}^{i,j} = \int_{\Omega^{i,j}} \mathbf{S}_z^{i,j\,T}(P^{i,j})\bar{\bar{\mathbf{C}}}_z^{i,j\,T}\mathbf{E}^{i,j}\bar{\bar{\mathbf{C}}}_z^{i,j}\mathbf{S}_z^{i,j}(P^{i,j})dV^{i,j} \tag{2.308}$$

These matrices are the elastic shape integrals required to compute explicitly the mass matrix $\mathbf{K}_{f,f}^{i,j}$ of the flexible element j of the body i of the system.

2.6.5. DYNAMIC EQUATIONS OF STRUCTURAL SYSTEMS

To derive the equations of motion of the structural system, it is necessary first to compute the mass and stiffness matrices of the whole bodies from the same matrices corresponding to the structural elements. This can be easily done by summing the kinetic energy $T^{i,j}(t)$ and the strain energy $U^{i,j}(t)$ of each element. Indeed, from the definition of the kinetic energy $T^{i}(t)$ of the body i the mass matrix can be obtained:

$$\begin{aligned} T^{i}(t) &= \sum_{j=1}^{n_e^i} T^{i,j}(t) = \\ &= \sum_{j=1}^{n_e^i} \frac{1}{2} \dot{\mathbf{q}}_f^{iT}(t) \mathbf{M}_{f,f}^{i,j} \dot{\mathbf{q}}_f^{i}(t) = \\ &= \frac{1}{2} \dot{\mathbf{q}}_f^{iT}(t) \left(\sum_{j=1}^{n_e^i} \mathbf{M}_{f,f}^{i,j} \right) \dot{\mathbf{q}}_f^{i}(t) = \\ &= \frac{1}{2} \dot{\mathbf{q}}_f^{iT}(t) \mathbf{M}_{f,f}^{i} \dot{\mathbf{q}}_f^{i}(t) \end{aligned} \tag{2.309}$$

Where the $\mathbb{R}^{n_f^i \times n_f^i}$ matrix $\mathbf{M}_{f,f}^{i}$ represents the mass matrix of the body i and it is defined as:

$$\begin{aligned} \mathbf{M}_{f,f}^{i} &= \sum_{j=1}^{n_e^i} \mathbf{M}_{f,f}^{i,j} = \\ &= \sum_{j=1}^{n_e^i} \left(\mathbf{B}_e^{iT} \overline{\mathbf{J}}_{f,f}^{i,j} \mathbf{B}_e^{i} \right) = \\ &= \mathbf{B}_e^{iT} \left(\sum_{j=1}^{n_e^i} \overline{\mathbf{J}}_{f,f}^{i,j} \right) \mathbf{B}_e^{i} = \\ &= \mathbf{B}_e^{iT} \overline{\mathbf{J}}_{f,f}^{i} \mathbf{B}_e^{i} \end{aligned} \tag{2.310}$$

Where $\overline{\mathbf{J}}_{f,f}^{i}$ is a $\mathbb{R}^{n_f^i \times n_f^i}$ symmetric matrix and it can be computed as follows:

$$\begin{aligned} \overline{\mathbf{J}}_{f,f}^{i} &= \sum_{j=1}^{n_e^i} \overline{\mathbf{J}}_{f,f}^{i,j} = \\ &= \sum_{j=1}^{n_e^i} \left(\mathbf{B}_c^{i,jT} \overline{\mathbf{C}}^{i,jT} \overline{\mathbf{S}}_{f,f}^{i,j} \overline{\mathbf{C}}^{i,j} \mathbf{B}_c^{i,j} \right) \end{aligned} \tag{2.311}$$

On the other hand, from the definition of the strain energy $U^i(t)$ of the body i the stiffness matrix can be obtained:

$$\begin{aligned} U^i(t) &= \sum_{j=1}^{n_e^i} U^{i,j}(t) = \\ &= \sum_{j=1}^{n_e^i}\left(\frac{1}{2}\mathbf{q}_f^{iT}(t)\mathbf{K}_{f,f}^{i,j}\mathbf{q}_f^i(t)\right) = \\ &= \frac{1}{2}\mathbf{q}_f^{iT}(t)\left(\sum_{j=1}^{n_e^i}\mathbf{K}_{f,f}^{i,j}\right)\mathbf{q}_f^i(t) = \\ &= \frac{1}{2}\mathbf{q}_f^{iT}(t)\mathbf{K}_{f,f}^{i}\mathbf{q}_f^i(t) \end{aligned} \tag{2.312}$$

Where the $\mathbb{R}^{n_f^i \times n_f^i}$ matrix $\mathbf{K}_{f,f}^i$ represents the stiffness matrix of the body i and it is defined as:

$$\begin{aligned} \mathbf{K}_{f,f}^i &= \sum_{j=1}^{n_e^i}\mathbf{K}_{f,f}^{i,j} = \\ &= \sum_{j=1}^{n_e^i}\left(\mathbf{B}_e^{iT}\bar{\mathbf{V}}_{f,f}^{i,j}\mathbf{B}_e^i\right) = \\ &= \mathbf{B}_e^{iT}\left(\sum_{j=1}^{n_e^i}\bar{\mathbf{V}}_{f,f}^{i,j}\right)\mathbf{B}_e^i = \\ &= \mathbf{B}_e^{iT}\bar{\mathbf{V}}_{f,f}^{i}\mathbf{B}_e^i \end{aligned} \tag{2.313}$$

Where $\bar{\mathbf{V}}_{f,f}^i$ is a $\mathbb{R}^{n_f^i \times n_f^i}$ symmetric matrix and it can be computed as follows:

$$\begin{aligned}\bar{\mathbf{V}}^{i}_{f,f} &= \sum_{j=1}^{n_e^i} \bar{\mathbf{V}}^{i,j}_{f,f} = \\ &= \sum_{j=1}^{n_e^i} \left(\bar{\bar{\mathbf{J}}}^{i,j}_{f,f} + \bar{\bar{\mathbf{J}}}^{i,j}_{f,1} + \bar{\bar{\mathbf{J}}}^{i,j}_{f,2} + \bar{\bar{\mathbf{J}}}^{i,j}_{f,3} \right) = \\ &= \bar{\bar{\mathbf{J}}}^{i}_{f,f} + \bar{\bar{\mathbf{J}}}^{i}_{f,1} + \bar{\bar{\mathbf{J}}}^{i}_{f,2} + \bar{\bar{\mathbf{J}}}^{i}_{f,3}\end{aligned} \tag{2.314}$$

Where $\bar{\bar{\mathbf{J}}}^{i}_{f,f}$, $\bar{\bar{\mathbf{J}}}^{i}_{f,1}$, $\bar{\bar{\mathbf{J}}}^{i}_{f,2}$ and $\bar{\bar{\mathbf{J}}}^{i}_{f,3}$ are $\mathbb{R}^{n^i_{f,v} \times n^i_{f,v}}$ symmetric matrices defined as:

$$\begin{aligned}\bar{\bar{\mathbf{J}}}^{i}_{f,f} &= \sum_{j=1}^{n_e^i} \bar{\bar{\mathbf{J}}}^{i,j}_{f,f} = \\ &= \sum_{j=1}^{n_e^i} \left(\mathbf{B}_c^{i,j\,T} \bar{\mathbf{C}}^{i,j\,T} \bar{\bar{\mathbf{S}}}^{i,j}_{f,f} \bar{\mathbf{C}}^{i,j} \mathbf{B}_c^{i,j} \right)\end{aligned} \tag{2.315}$$

$$\begin{aligned}\bar{\bar{\mathbf{J}}}^{i}_{f,1} &= \sum_{j=1}^{n_e^i} \bar{\bar{\mathbf{J}}}^{i,j}_{f,1} = \\ &= \sum_{j=1}^{n_e^i} \left(\mathbf{B}_c^{i,j\,T} \bar{\mathbf{C}}^{i,j\,T} \bar{\bar{\mathbf{S}}}^{i,j}_{f,1} \bar{\mathbf{C}}^{i,j} \mathbf{B}_c^{i,j} \right)\end{aligned} \tag{2.316}$$

$$\begin{aligned}\bar{\bar{\mathbf{J}}}^{i}_{f,2} &= \sum_{j=1}^{n_e^i} \bar{\bar{\mathbf{J}}}^{i,j}_{f,2} = \\ &= \sum_{j=1}^{n_e^i} \left(\mathbf{B}_c^{i,j\,T} \bar{\mathbf{C}}^{i,j\,T} \bar{\bar{\mathbf{S}}}^{i,j}_{f,2} \bar{\mathbf{C}}^{i,j} \mathbf{B}_c^{i,j} \right)\end{aligned} \tag{2.317}$$

$$\begin{aligned}\bar{\bar{\mathbf{J}}}^{i}_{f,3} &= \sum_{j=1}^{n_e^i} \bar{\bar{\mathbf{J}}}^{i,j}_{f,3} = \\ &= \sum_{j=1}^{n_e^i} \left(\mathbf{B}_c^{i,j\,T} \bar{\mathbf{C}}^{i,j\,T} \bar{\bar{\mathbf{S}}}^{i,j}_{f,3} \bar{\mathbf{C}}^{i,j} \mathbf{B}_c^{i,j} \right)\end{aligned} \tag{2.318}$$

Once that the mass matrix $\mathbf{M}^i_{f,f}$ and the stiffness matrix $\mathbf{K}^i_{f,f}$ of body i has been computed, the system equations of motion can be derived by using Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial T^i(t)}{\partial \dot{\mathbf{q}}^i_f(t)}\right)^T - \left(\frac{\partial T^i(t)}{\partial \mathbf{q}^i_f(t)}\right)^T + \left(\frac{\partial U^i(t)}{\partial \mathbf{q}^i_f(t)}\right)^T = \mathbf{Q}^i_{e,nc}(t) \tag{2.319}$$

Note that it is assumed that the bodies of the set in analysis does not exhibit large deformation and rigid body motion. Consequently, it is also supposed that every body of the system is not linked or constrained in some way to each other. Therefore, the Lagrange equations can be applied without considering the effect of some generalized constraint action to yield:

$$\mathbf{M}^i_{f,f}\ddot{\mathbf{q}}^i_f(t) + \mathbf{K}^i_{f,f}\mathbf{q}^i_f(t) = \mathbf{Q}^i_{e,nc}(t) \tag{2.320}$$

Where $\mathbf{Q}^i_{e,nc}(t)$ is a $\mathbb{R}^{n^i_f}$ vector representing the generalized external nonconservative forces applied on the system which dynamically couple the bodies of the set. Finally, these dynamic equations can be easily assembled to derive the equations of motion of the whole structural system to yield:

$$\mathbf{M}_{f,f}\ddot{\mathbf{q}}_f(t) + \mathbf{K}_{f,f}\mathbf{q}_f(t) = \mathbf{Q}_{e,nc}(t) \tag{2.321}$$

Where the configuration vector $\mathbf{q}_f(t)$ represents the total $\mathbb{R}^{n_f}$ vector of the structural system generalized coordinates and is defined as:

$$\mathbf{q}_f(t) = \begin{bmatrix} \mathbf{q}^1_f(t) \\ \mathbf{q}^2_f(t) \\ \vdots \\ \mathbf{q}^{n_f}_f(t) \end{bmatrix} \tag{2.322}$$

The matrices $\mathbf{M}_{f,f}$ and $\mathbf{K}_{f,f}$ are respectively the global $\mathbb{R}^{n_f \times n_f}$ mass and stiffness matrices of the structural system which can be easily assembled as:

$$\mathbf{M}_{f,f} = \begin{bmatrix} \mathbf{M}_{f,f}^1 & \mathbf{O} & \dots & \mathbf{O} \\ \mathbf{O} & \mathbf{M}_{f,f}^2 & \dots & \mathbf{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{O} & \mathbf{O} & \dots & \mathbf{M}_{f,f}^{n_f} \end{bmatrix} \tag{2.323}$$

$$\mathbf{K}_{f,f} = \begin{bmatrix} \mathbf{K}_{f,f}^1 & \mathbf{O} & \dots & \mathbf{O} \\ \mathbf{O} & \mathbf{K}_{f,f}^2 & \dots & \mathbf{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{O} & \mathbf{O} & \dots & \mathbf{K}_{f,f}^{n_f} \end{bmatrix} \tag{2.324}$$

The $\mathbb{R}^{n_f}$ vector $\mathbf{Q}_{e,nc}(t)$ is the lagrangian component vectors which represent the generalized external nonconservative forces acting on the whole structural system and it can be simply assembled as:

$$\mathbf{Q}_{e,nc}(t) = \begin{bmatrix} \mathbf{Q}_{e,nc}^1(t) \\ \mathbf{Q}_{e,nc}^2(t) \\ \vdots \\ \mathbf{Q}_{e,nc}^{n_f}(t) \end{bmatrix} \tag{2.325}$$

Finally, consider the proportional damping hypothesis:

$$\mathbf{R}_{f,f} = \alpha \mathbf{M}_{f,f} + \beta \mathbf{K}_{f,f} \tag{2.326}$$

Where $\mathbf{R}_{f,f}$ is a $\mathbb{R}^{n_f \times n_f}$ matrix representing the system damping matrix. The equations of motion of the structural system slightly modifies to yield:

$$\mathbf{M}_{f,f}\ddot{\mathbf{q}}_f(t)+\mathbf{R}_{f,f}\dot{\mathbf{q}}_f(t)+\mathbf{K}_{f,f}\mathbf{q}_f(t)=\mathbf{Q}_{e,nc}(t) \tag{2.327}$$

These equations can be easily solved to numerically find the damped vibration of a structural system with complex geometry.

2.7. FINITE ELEMENT FORMULATION OF FLEXIBLE MULTIBODY DYNAMICS

2.7.1. INTRODUCTION

In the following sections the general case of the motion of a multibody system composed of rigid and deformable elements is analysed [29]. The approach followed here was originally developed by Shabana [11], [12], [13]. To describe the system kinematics the floating frame of reference formulation is used. This formulation allows to combine the systematic method for the derivation of the equations of motion of rigid multibody systems with the classic finite element methods to deduce the equations of motion of flexible multibody systems. In the floating frame of reference formulation the configuration of each body of the system is described by using two sets of coordinates: references coordinates and elastic coordinates. The former define the location and orientation of a given body reference in respect to a fixed inertial frame whereas the latter describe the elastic deformation of system elements in respect to the corresponding body reference. Therefore, the position of an arbitrary point on the deformable body is represented by using a couple set of reference and elastic coordinates. As a legacy of classic finite element methods, the floating frame of reference formulation considers as nodal coordinates the infinitesimal rotations of nodes and consequently it can be used in the assumption of large reference frame and small elastic deformation with respect to the flexible body reference. Using the concept of the intermediate element coordinate system, a nonlinear formulation that leads to exact modelling of the rigid body motion for elements whose coordinates are defined in terms of infinitesimal rotations can be developed. The intermediate element coordinate system is defined as a reference system whose origin is rigidly attached to the origin of the body coordinate system, its orientation is fixed with respect to the body coordinate system and it is initially oriented with its axes parallel to the axes of the element coordinate system. The floating frame of reference formulation yields to an inertial coupling between the reference motion and the elastic deformation which reverberates in a coupled nonlinear formulation of the mass matrix. Similarly to the case of rigid multibody systems, to derive the system mass matrix it is necessary to compute preliminary a set of inertia shape integrals $[11]$, $[12]$, $[13]$. In addition, the quadratic velocity vector which contains the gyroscopic and Coriolis force components appears in the equations of motion of flexible multibody systems because the mass matrix and the kinetic energy of the bodies result to be functions of the configuration vector. On the other hand, the stiffness matrix results to be a constant matrix whose derivation is identical to that of classic finite element methods. Therefore, a set of elastic shape integrals is necessary to compute explicitly the stiffness matrix $[11]$, $[12]$, $[13]$. Finally, mechanical joints in the flexible multibody system are formulated by using a set of nonlinear algebraic constraint equations which can be adjoined to the system differential equations of motion by using Lagrange multiplayers rule.

2.7.2. FLOATING FRAME OF REFERENCE FORMULATION

Consider a flexible multibody system composed of n_b bodies each one discretized in n_e^i elements. According to the floating frame of reference formulation, the configuration of the generic body i of the system is represented by a set of reference coordinates and a set of elastic coordinates. Indeed:

$$\mathbf{q}^i(t) = \begin{bmatrix} \mathbf{q}_r^i(t) \\ \mathbf{q}_f^i(t) \end{bmatrix} \tag{2.328}$$

Where $\mathbf{q}^i(t)$ is a $\mathbb{R}^{n^i}$ vector which describes the configuration of body i, $\mathbf{q}_r^i(t)$ is a $\mathbb{R}^7$ vector representing the reference coordinates corresponding to the body i and $\mathbf{q}_f^i(t)$ is a $\mathbb{R}^{n_f^i}$ vector representing the elastic coordinates corresponding to the body i. Indeed, by using Euler's parameters to represent the orientation of the body frame of reference i, the reference coordinates vector $\mathbf{q}_r^i(t)$ is defined as:

$$\mathbf{q}_r^i(t) = \begin{bmatrix} \mathbf{R}^i(t) \\ \boldsymbol{\theta}^i(t) \end{bmatrix} \tag{2.329}$$

Where $\mathbf{R}^i(t)$ is a $\mathbb{R}^3$ vector corresponding to the origin position of the body reference system i in respect to the inertial frame of reference and $\boldsymbol{\theta}^i(t)$ is a $\mathbb{R}^4$ vector which contains the Euler's parameters that describe the orientation of the body reference i in respect to the inertial frame of reference. The first step to describe the system kinematics is to express the position of an arbitrary material point of the system as a function of the generalized configuration vector $\mathbf{q}^i(t)$. Thus, consider for the generic element j of flexible body i the position vector of the material point $P^{i,j}$ in respect to the body frame of reference. The position vector of this point can be vectorially decomposed by the sum of the position vector of the point $P^{i,j}$ referred to the undeformed state of the body and the elastic displacement vector of point $P^{i,j}$ measured from the undeformed configuration. This yields to:

$$\bar{\mathbf{u}}^{i,j}(P^{i,j},t) = \bar{\mathbf{u}}_o^{i,j}(P^{i,j}) + \bar{\mathbf{u}}_f^{i,j}(P^{i,j},t) \tag{2.330}$$

Where $\bar{\mathbf{u}}^{i,j}(P^{i,j},t)$ is a $\mathbb{R}^3$ vector representing the position of point $P^{i,j}$ in respect to the body frame of reference, $\bar{\mathbf{u}}_o^{i,j}(P^{i,j})$ is a $\mathbb{R}^3$ vector corresponding to the position of point $P^{i,j}$ in the undeformed configuration and $\bar{\mathbf{u}}_f^{i,j}(P^{i,j},t)$ is a $\mathbb{R}^3$ vector representing the elastic displacement of point $P^{i,j}$. According the classic finite elements methods, the elastic displacement field $\bar{\mathbf{u}}_f^{i,j}(P^{i,j},t)$ can be decomposed by the product of a space-dependent global shape function and a time-dependent nodal vector to yield:

$$\begin{aligned}\bar{\mathbf{u}}_f^{i,j}(P^{i,j},t) &= \mathbf{S}_g^{i,j}(P^{i,j})\mathbf{q}_f^{i,j}(t) = \\ &= \mathbf{S}_g^{i,j}(P^{i,j})\mathbf{B}_c^{i,j}\mathbf{q}_{f,v}^{i}(t) = \\ &= \mathbf{S}_g^{i,j}(P^{i,j})\mathbf{B}_c^{i,j}\mathbf{B}_e^{i}\mathbf{q}_f^{i}(t) = \\ &= \mathbf{N}^{i,j}(P^{i,j})\mathbf{B}_e^{i}\mathbf{q}_f^{i}(t)\end{aligned} \tag{2.331}$$

Where $\mathbf{S}_g^{i,j}(P^{i,j})$ is a $\mathbb{R}^{3\times n_f^{i,j}}$ matrix representing the global shape function, $\mathbf{q}_f^{i,j}(t)$ is a $\mathbb{R}^{n_f^{i,j}}$ vector function corresponding to the vector of nodal coordinates, $\mathbf{B}_c^{i,j}$ is a $\mathbb{R}^{n_f^{i,j}\times n_{f,v}^{i}}$ Boolean matrix representing the internal kinematic constraints of each element j of body i, $\mathbf{q}_{f,v}^{i}(t)$ is a $\mathbb{R}^{n_{f,v}^{i}}$ nodal coordinate vector, $\mathbf{B}_e^{i}$ is a $\mathbb{R}^{n_{f,v}^{i}\times n_f^{i}}$ Boolean matrix corresponding to the external kinematic constraints of each body i, and $\mathbf{N}^{i,j}(P^{i,j})$ is a $\mathbb{R}^{3\times n_{f,v}^{i}}$ matrix representing the compact shape function. On the other hand, the position vector of point $P^{i,j}$ respect to the undeformed configuration can also be expressed by using the compact shape function as follows:

$$\begin{aligned}\bar{\mathbf{u}}_o^{i,j}(P^{i,j}) &= \mathbf{S}_g^{i,j}(P^{i,j})\mathbf{B}_c^{i,j}\mathbf{q}_o^{i} = \\ &= \mathbf{N}^{i,j}(P^{i,j})\mathbf{q}_o^{i}\end{aligned} \tag{2.332}$$

Where $\mathbf{q}_o^{i}$ is a $\mathbb{R}^{n_{f,v}^{i}}$ vector containing the value of the nodal coordinates of body i in the undeformed state. Therefore the position of point $P^{i,j}$ referred to the body reference can be expressed as follows:

$$\begin{aligned}\bar{\mathbf{u}}^{i,j}(P^{i,j},t) &= \bar{\mathbf{u}}_o^{i,j}(P^{i,j}) + \bar{\mathbf{u}}_f^{i,j}(P^{i,j},t) = \\ &= \mathbf{N}^{i,j}(P^{i,j})\mathbf{q}_o^{i} + \mathbf{N}^{i,j}(P^{i,j})\mathbf{B}_e^{i}\mathbf{q}_f^{i}(t) = \\ &= \mathbf{N}^{i,j}(P^{i,j})\left(\mathbf{q}_o^{i} + \mathbf{B}_e^{i}\mathbf{q}_f^{i}(t)\right) = \\ &= \mathbf{N}^{i,j}(P^{i,j})\mathbf{q}_n^{i}(t)\end{aligned} \tag{2.333}$$

Where $\mathbf{q}_n^i(t)$ is a $\mathbb{R}^{n_{f,v}^i}$ vector of nodal coordinates defined as:

$$\mathbf{q}_n^i(t) = \mathbf{q}_o^i + \mathbf{B}_e^i \mathbf{q}_f^i(t) \tag{2.334}$$

Afterwards, the position of the material point $P^{i,j}$ can be referred to the inertial frame of reference to yield:

$$\begin{aligned}\mathbf{r}^{i,j}(P^{i,j},t) &= \mathbf{R}^i(t) + \mathbf{A}^i(t)\bar{\mathbf{u}}^{i,j}(P^{i,j},t) = \\ &= \mathbf{R}^i(t) + \mathbf{A}^i(t)\mathbf{N}^{i,j}(P^{i,j})\mathbf{q}_n^i(t) = \\ &= \mathbf{R}^i(t) + \mathbf{A}^i(t)\mathbf{N}^{i,j}(P^{i,j})\left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right)\end{aligned} \tag{2.335}$$

Where $\mathbf{r}^{i,j}(P^{i,j},t)$ is a $\mathbb{R}^3$ vector representing the position of point $P^{i,j}$ referred to the inertial frame of reference and $\mathbf{A}^i(t)$ is a $\mathbb{R}^{3\times 3}$ rotation matrix which defines the orientation of the body reference i respect to the inertial frame of reference. The time derivative of position vector $\mathbf{r}^{i,j}(P^{i,j},t)$ can be computed by using the time derivative of the configuration vector $\mathbf{q}^i(t)$ as:

$$\begin{aligned}\dot{\mathbf{r}}^{i,j}(P^{i,j},t) &= \dot{\mathbf{R}}^i(t) + \dot{\mathbf{A}}^i(t)\bar{\mathbf{u}}^{i,j}(P^{i,j},t) + \mathbf{A}^i(t)\dot{\bar{\mathbf{u}}}^{i,j}(P^{i,j},t) = \\ &= \dot{\mathbf{R}}^i(t) + \mathbf{A}^i(t)\tilde{\bar{\boldsymbol{\omega}}}^i(t)\bar{\mathbf{u}}^{i,j}(P^{i,j},t) + \mathbf{A}^i(t)\mathbf{N}^{i,j}(P^{i,j})\dot{\mathbf{q}}_n^i(t) = \\ &= \dot{\mathbf{R}}^i(t) - \mathbf{A}^i(t)\tilde{\bar{\mathbf{u}}}^{i,j}(P^{i,j},t)\bar{\boldsymbol{\omega}}^i(t) + \\ &\quad + \mathbf{A}^i(t)\left(\mathbf{N}^{i,j}(P^{i,j})\frac{d}{dt}\left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right)\right) = \\ &= \dot{\mathbf{R}}^i(t) - \mathbf{A}^i(t)\tilde{\bar{\mathbf{u}}}^{i,j}(P^{i,j},t)\bar{\mathbf{G}}^i(t)\dot{\boldsymbol{\theta}}^i(t) + \mathbf{A}^i(t)\mathbf{N}^{i,j}(P^{i,j})\mathbf{B}_e^i\dot{\mathbf{q}}_f^i(t) = \\ &= \begin{bmatrix}\mathbf{I} & -\mathbf{A}^i(t)\tilde{\bar{\mathbf{u}}}^{i,j}(P^{i,j},t)\bar{\mathbf{G}}^i(t) & \mathbf{A}^i(t)\mathbf{N}^{i,j}(P^{i,j})\mathbf{B}_e^i\end{bmatrix}\begin{bmatrix}\dot{\mathbf{R}}^i(t) \\ \dot{\boldsymbol{\theta}}^i(t) \\ \dot{\mathbf{q}}_f^i(t)\end{bmatrix} = \\ &= \mathbf{L}^{i,j}(P^{i,j},t)\dot{\mathbf{q}}^i(t)\end{aligned} \tag{2.336}$$

Where $\mathbf{L}^{i,j}(P^{i,j},t)$ is a $\mathbb{R}^{3\times n^i}$ matrix defined as follows:

$$\begin{aligned}\mathbf{L}^{i,j}(P^{i,j},t) &= \begin{bmatrix}\mathbf{L}_R^{i,j}(P^{i,j},t) & \mathbf{L}_\theta^{i,j}(P^{i,j},t) & \mathbf{L}_{q_f}^{i,j}(P^{i,j},t)\end{bmatrix} = \\ &= \begin{bmatrix}\mathbf{I} & -\mathbf{A}^i(t)\tilde{\bar{\mathbf{u}}}^{i,j}(P^{i,j},t)\bar{\mathbf{G}}^i(t) & \mathbf{A}^i(t)\mathbf{N}^{i,j}(P^{i,j})\mathbf{B}_e^i\end{bmatrix}\end{aligned} \tag{2.337}$$

This matrix is a function of the reference coordinate vector $\mathbf{q}^i(t)$ and depends on the particle $P^{i,j}$ under consideration. It is remarkable to note that the virtual change of the position vector $\mathbf{r}^{i,j}(P^{i,j},t)$ can be computed in the same way by using the matrix $\mathbf{L}^{i,j}(P^{i,j},t)$:

$$\delta\mathbf{r}^{i,j}(P^{i,j},t) = \mathbf{L}^{i,j}(P^{i,j},t)\delta\mathbf{q}^i(t) \tag{2.338}$$

Indeed, this matrix is a jacobian transformation matrix which mathematically describe the relation between the physical coordinates vector $\mathbf{r}^{i,j}(P^{i,j},t)$ and the lagrangian coordinates vector $\mathbf{q}^i(t)$.

2.7.3. MASS MATRIX OF SYSTEM ELEMENTS

Once that the kinematic description of motion has been obtained, the mass matrix of the generic element j of rigid body i can be computed using the definition of kinetic energy $T^{i,j}(t)$. Indeed:

$$\begin{aligned}T^{i,j}(t) &= \frac{1}{2}\int_{\Omega^{i,j}}\rho^{i,j}\dot{\mathbf{r}}^{i,jT}(P^{i,j},t)\dot{\mathbf{r}}^{i,j}(P^{i,j},t)dV^{i,j} = \\ &= \frac{1}{2}\dot{\mathbf{q}}^{iT}(t)\int_{\Omega^{i,j}}\rho^{i,j}\mathbf{L}^{i,jT}(P^{i,j},t)\mathbf{L}^{i,j}(P^{i,j},t)dV^{i,j}\dot{\mathbf{q}}^i(t) = \\ &= \frac{1}{2}\dot{\mathbf{q}}^{iT}(t)\mathbf{M}^{i,j}(t)\dot{\mathbf{q}}^i(t)\end{aligned} \tag{2.339}$$

Where $\rho^{i,j}$ and $\Omega^{i,j}$ are respectively the mass density and the volume of element j body i and $\mathbf{M}^{i,j}(t)$ is a $\mathbb{R}^{n^i \times n^i}$ matrix representing the mass matrix of the same element. The mass matrix of system elements can be computed as:

$$
\begin{aligned}
&\mathbf{M}^{i,j}(t) = \int_{\Omega^{i,j}} \rho^{i,j} \mathbf{L}^{i,j\,T}(P^{i,j},t)\mathbf{L}^{i,j}(P^{i,j},t)dV^{i,j} = \\
&= \int_{\Omega^{i,j}} \rho^{i,j} \begin{bmatrix} \mathbf{L}_R^{i,j\,T}(P^{i,j},t) \\ \mathbf{L}_\theta^{i,j\,T}(P^{i,j},t) \\ \mathbf{L}_{q_f}^{i,j\,T}(P^{i,j},t) \end{bmatrix} \begin{bmatrix} \mathbf{L}_R^{i,j}(P^{i,j},t) & \mathbf{L}_\theta^{i,j}(P^{i,j},t) & \mathbf{L}_{q_f}^{i,j}(P^{i,j},t) \end{bmatrix} dV^{i,j} = \\
&= \begin{bmatrix} \mathbf{m}_{R,R}^{i,j}(t) & \mathbf{m}_{R,\theta}^{i,j}(t) & \mathbf{m}_{R,q_f}^{i,j}(t) \\ \mathbf{m}_{\theta,R}^{i,j}(t) & \mathbf{m}_{\theta,\theta}^{i,j}(t) & \mathbf{m}_{\theta,q_f}^{i,j}(t) \\ \mathbf{m}_{q_f,R}^{i,j}(t) & \mathbf{m}_{q_f,\theta}^{i,j}(t) & \mathbf{m}_{q_f,q_f}^{i,j}(t) \end{bmatrix}
\end{aligned}
\tag{2.340}
$$

Where $\mathbf{m}_{R,R}^{i,j}(t)$, $\mathbf{m}_{R,\theta}^{i,j}(t)$, $\mathbf{m}_{R,q_f}^{i,j}(t)$, $\mathbf{m}_{\theta,\theta}^{i,j}(t)$, $\mathbf{m}_{\theta,q_f}^{i,j}(t)$ and $\mathbf{m}_{q_f,q_f}^{i,j}(t)$ are respectively $\mathbb{R}^{3\times3}$, $\mathbb{R}^{3\times4}$, $\mathbb{R}^{3\times n_f^i}$, $\mathbb{R}^{4\times4}$, $\mathbb{R}^{4\times n_f^i}$ and $\mathbb{R}^{n_f^i \times n_f^i}$ symmetric matrices which can be computed explicitly. Indeed, the mass submatrix $\mathbf{m}_{R,R}^{i,j}(t)$ can be computed as:

$$
\begin{aligned}
\mathbf{m}_{R,R}^{i,j}(t) &= \int_{\Omega^{i,j}} \rho^{i,j} \mathbf{L}_R^{i,j\,T}(P^{i,j},t)\mathbf{L}_R^{i,j}(P^{i,j},t)dV^{i,j} = \\
&= \int_{\Omega^{i,j}} \rho^{i,j} \mathbf{I} dV^{i,j} = \\
&= m^{i,j}\mathbf{I}
\end{aligned}
\tag{2.341}
$$

Where $m^{i,j}$ is the total mass of element j of body i. The mass submatrix $\mathbf{m}_{R,\theta}^{i,j}(t)$ can be computed in the following way:

$$\begin{aligned}\mathbf{m}_{R,\theta}^{i,j}(t) &= \int_{\Omega^{i,j}} \rho^{i,j} \mathbf{L}_R^{i,jT}(P^{i,j},t)\mathbf{L}_\theta^{i,j}(P^{i,j},t)dV^{i,j} = \\ &= -\int_{\Omega^{i,j}} \rho^{i,j} \mathbf{A}^i(t)\tilde{\bar{\mathbf{u}}}^{i,j}(P^{i,j},t)\bar{\mathbf{G}}^i(t)dV^{i,j} = \\ &= -\mathbf{A}^i(t)\int_{\Omega^{i,j}} \rho^{i,j}\tilde{\bar{\mathbf{u}}}^{i,j}(P^{i,j},t)dV^{i,j}\bar{\mathbf{G}}^i(t) = \\ &= -\mathbf{A}^i(t)\tilde{\bar{\mathbf{U}}}^{i,j}(t)\bar{\mathbf{G}}^i(t) = \\ &= \mathbf{m}_{\theta,R}^{i,jT}(t)\end{aligned} \tag{2.342}$$

Where $\tilde{\bar{\mathbf{U}}}^{i,j}(t)$ is a $\mathbb{R}^{3\times3}$ skew symmetric matrix defined as:

$$\begin{aligned}\tilde{\bar{\mathbf{U}}}^{i,j}(t) &= \int_{\Omega^{i,j}} \rho^{i,j}\tilde{\bar{\mathbf{u}}}^{i,j}(P^{i,j},t)dV^{i,j} = \\ &= Skew\left(\int_{\Omega^{i,j}} \rho^{i,j}\bar{\mathbf{u}}^{i,j}(P^{i,j},t)dV^{i,j}\right) = \\ &= Skew\left(\int_{\Omega^{i,j}} \rho^{i,j}\mathbf{N}^{i,j}(P^{i,j})\left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right)dV^{i,j}\right) = \\ &= Skew\left(\int_{\Omega^{i,j}} \rho^{i,j}\mathbf{S}_g^{i,j}(P^{i,j})\mathbf{B}_c^{i,j}\left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right)dV^{i,j}\right) = \\ &= Skew\left(\int_{\Omega^{i,j}} \rho^{i,j}\mathbf{C}^{i,j}\mathbf{S}^{i,j}(P^{i,j})\bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j}dV^{i,j}\left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right)\right) = \\ &= Skew\left(\mathbf{C}^{i,j}\int_{\Omega^{i,j}} \rho^{i,j}\mathbf{S}^{i,j}(P^{i,j})dV^{i,j}\bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j}\left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right)\right) = \\ &= Skew\left(\mathbf{C}^{i,j}\bar{\mathbf{S}}^{i,j}\bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j}\left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right)\right) = \\ &= Skew\left(\bar{\mathbf{N}}^{i,j}\left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right)\right) = \\ &= Skew\left(\bar{\mathbf{N}}^{i,j}\mathbf{q}_n^i(t)\right)\end{aligned} \tag{2.343}$$

Where $\bar{\mathbf{N}}^{i,j}$ is a $\mathbb{R}^{3\times n_{f,v}^{i,j}}$ matrix defined as:

$$\bar{\mathbf{N}}^{i,j} = \mathbf{C}^{i,j}\bar{\mathbf{S}}^{i,j}\bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j} \tag{2.344}$$

Where $\bar{\mathbf{S}}^{i,j}$ is a $\mathbb{R}^{3\times n_f^{i,j}}$ matrix that can be computed as follows:

$$\bar{\mathbf{S}}^{i,j}=\int_{\Omega^{i,j}}\rho^{i,j}\mathbf{S}^{i,j}(P^{i,j})dV^{i,j} \tag{2.345}$$

The mass submatrix $\mathbf{m}^{i,j}_{R,q_f}(t)$ is defined as:

$$\begin{aligned}
\mathbf{m}^{i,j}_{R,q_f}(t)&=\int_{\Omega^{i,j}}\rho^{i,j}\mathbf{L}^{i,jT}_{R}(P^{i,j},t)\mathbf{L}^{i,j}_{q_f}(P^{i,j},t)dV^{i,j}=\\
&=\int_{\Omega^{i,j}}\rho^{i,j}\mathbf{A}^{i}(t)\mathbf{N}^{i,j}(P^{i,j})\mathbf{B}^{i}_{e}dV^{i,j}=\\
&=\mathbf{A}^{i}(t)\int_{\Omega^{i,j}}\rho^{i,j}\mathbf{S}^{i,j}_{g}(P^{i,j})\mathbf{B}^{i,j}_{c}dV^{i,j}\mathbf{B}^{i}_{e}=\\
&=\mathbf{A}^{i}(t)\int_{\Omega^{i,j}}\rho^{i,j}\mathbf{C}^{i,j}\mathbf{S}^{i,j}(P^{i,j})\bar{\mathbf{C}}^{i,j}\mathbf{B}^{i,j}_{c}dV^{i,j}\mathbf{B}^{i}_{e}=\\
&=\mathbf{A}^{i}(t)\mathbf{C}^{i,j}\int_{\Omega^{i,j}}\rho^{i,j}\mathbf{S}^{i,j}(P^{i,j})dV^{i,j}\bar{\mathbf{C}}^{i,j}\mathbf{B}^{i,j}_{c}\mathbf{B}^{i}_{e}=\\
&=\mathbf{A}^{i}(t)\mathbf{C}^{i,j}\bar{\mathbf{S}}^{i,j}\bar{\mathbf{C}}^{i,j}\mathbf{B}^{i,j}_{c}\mathbf{B}^{i}_{e}=\\
&=\mathbf{A}^{i}(t)\bar{\mathbf{N}}^{i,j}\mathbf{B}^{i}_{e}=\\
&=\mathbf{m}^{i,jT}_{q_f,R}(t)
\end{aligned} \tag{2.346}$$

The mass submatrix $\mathbf{m}^{i,j}_{\theta,\theta}(t)$ can be computed in the following way:

$$\begin{aligned}
\mathbf{m}^{i,j}_{\theta,\theta}(t)&=\int_{\Omega^{i,j}}\rho^{i,j}\mathbf{L}^{i,jT}_{\theta}(P^{i,j},t)\mathbf{L}^{i,j}_{\theta}(P^{i,j},t)dV^{i,j}=\\
&=\int_{\Omega^{i,j}}\rho^{i,j}\bar{\mathbf{G}}^{iT}(t)\tilde{\bar{\mathbf{u}}}^{i,jT}(P^{i,j},t)\mathbf{A}^{iT}(t)\mathbf{A}^{i}(t)\tilde{\bar{\mathbf{u}}}^{i,j}(P^{i,j},t)\bar{\mathbf{G}}^{i}(t)dV^{i,j}=\\
&=\bar{\mathbf{G}}^{iT}(t)\int_{\Omega^{i,j}}\rho^{i,j}\tilde{\bar{\mathbf{u}}}^{i,jT}(P^{i,j},t)\tilde{\bar{\mathbf{u}}}^{i,j}(P^{i,j},t)dV^{i,j}\bar{\mathbf{G}}^{i}(t)=\\
&=\bar{\mathbf{G}}^{iT}(t)\bar{\mathbf{I}}^{i,j}_{\theta,\theta}(t)\bar{\mathbf{G}}^{i}(t)
\end{aligned} \tag{2.347}$$

Where the orthogonality property of rotation matrix $\mathbf{A}^{i}(t)$ has been used. The $\mathbb{R}^{3\times3}$ matrix $\bar{\mathbf{I}}^{i,j}_{\theta,\theta}(t)$ is the inertia matrix of element j of body i which is defined as:

$$\bar{\mathbf{I}}_{\theta,\theta}^{i,j}(t) = \int_{\Omega^{i,j}} \rho^{i,j} \tilde{\bar{\mathbf{u}}}^{i,jT}(P^{i,j},t)\tilde{\bar{\mathbf{u}}}^{i,j}(P^{i,j},t)dV^{i,j} =$$

$$= \int_{\Omega^{i,j}} \rho^{i,j} \left(\bar{\mathbf{u}}^{i,jT}(P^{i,j},t)\bar{\mathbf{u}}^{i,j}(P^{i,j},t)\mathbf{I} - \bar{\mathbf{u}}^{i,j}(P^{i,j},t)\bar{\mathbf{u}}^{i,jT}(P^{i,j},t) \right) dV^{i,j} =$$

$$= \begin{bmatrix} \left(\bar{\mathbf{I}}_{\theta,\theta}^{i,j}(t)\right)_{1,1} & \left(\bar{\mathbf{I}}_{\theta,\theta}^{i,j}(t)\right)_{1,2} & \left(\bar{\mathbf{I}}_{\theta,\theta}^{i,j}(t)\right)_{1,3} \\ \left(\bar{\mathbf{I}}_{\theta,\theta}^{i,j}(t)\right)_{2,1} & \left(\bar{\mathbf{I}}_{\theta,\theta}^{i,j}(t)\right)_{2,2} & \left(\bar{\mathbf{I}}_{\theta,\theta}^{i,j}(t)\right)_{2,3} \\ \left(\bar{\mathbf{I}}_{\theta,\theta}^{i,j}(t)\right)_{3,1} & \left(\bar{\mathbf{I}}_{\theta,\theta}^{i,j}(t)\right)_{3,2} & \left(\bar{\mathbf{I}}_{\theta,\theta}^{i,j}(t)\right)_{3,3} \end{bmatrix} \tag{2.348}$$

It is worth to point out that in this case of flexible multibody systems the inertia matrix $\bar{\mathbf{I}}_{\theta,\theta}^{i,j}(t)$ results to be a function of system configuration vector and consequently it changes in time. Indeed, the components of this matrix can be computed as:

$$
\begin{aligned}
&\left(\bar{\mathbf{I}}_{\theta,\theta}^{i,j}(t)\right)_{1,1} = \int_{\Omega^{i,j}} \rho^{i,j}\left(\left(\bar{u}_2^{i,j}(P^{i,j},t)\right)^2 + \left(\bar{u}_3^{i,j}(P^{i,j},t)\right)^2\right) dV^{i,j} = \\
&= \int_{\Omega^{i,j}} \rho^{i,j}\left(\bar{u}_2^{i,j}(P^{i,j},t)\right)^2 dV^{i,j} + \int_{\Omega^{i,j}} \rho^{i,j}\left(\bar{u}_3^{i,j}(P^{i,j},t)\right)^2 dV^{i,j} = \\
&= \int_{\Omega^{i,j}} \rho^{i,j}\mathbf{q}_n^{iT}(t)\mathbf{N}_2^{i,jT}(P^{i,j})\mathbf{N}_2^{i,j}(P^{i,j})\mathbf{q}_n^{i}(t)dV^{i,j} + \\
&+ \int_{\Omega^{i,j}} \rho^{i,j}\mathbf{q}_n^{iT}(t)\mathbf{N}_3^{i,jT}(P^{i,j})\mathbf{N}_3^{i,j}(P^{i,j})\mathbf{q}_n^{i}(t)dV^{i,j} = \\
&= \mathbf{q}_n^{iT}(t)\int_{\Omega^{i,j}} \rho^{i,j}\mathbf{B}_c^{i,jT}\bar{\mathbf{C}}^{i,jT}\mathbf{S}_2^{i,jT}(P^{i,j})\mathbf{C}^{i,jT}\mathbf{C}^{i,j}\mathbf{S}_2^{i,j}(P^{i,j})\bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j}dV^{i,j}\mathbf{q}_n^{i}(t) + \\
&+ \mathbf{q}_n^{iT}(t)\int_{\Omega^{i,j}} \rho^{i,j}\mathbf{B}_c^{i,jT}\bar{\mathbf{C}}^{i,jT}\mathbf{S}_3^{i,jT}(P^{i,j})\mathbf{C}^{i,jT}\mathbf{C}^{i,j}\mathbf{S}_3^{i,j}(P^{i,j})\bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j}dV^{i,j}\mathbf{q}_n^{i}(t) = \\
&= \mathbf{q}_n^{iT}(t)\mathbf{B}_c^{i,jT}\bar{\mathbf{C}}^{i,jT}\int_{\Omega^{i,j}} \rho^{i,j}\mathbf{S}_2^{i,jT}(P^{i,j})\mathbf{S}_2^{i,j}(P^{i,j})dV^{i,j}\bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j}\mathbf{q}_n^{i}(t) + \\
&+ \mathbf{q}_n^{iT}(t)\mathbf{B}_c^{i,jT}\bar{\mathbf{C}}^{i,jT}\int_{\Omega^{i,j}} \rho^{i,j}\mathbf{S}_3^{i,jT}(P^{i,j})\mathbf{S}_3^{i,j}(P^{i,j})dV^{i,j}\bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j}\mathbf{q}_n^{i}(t) = \\
&= \mathbf{q}_n^{iT}(t)\mathbf{B}_c^{i,jT}\bar{\mathbf{C}}^{i,jT}\bar{\mathbf{S}}_{2,2}^{i,j}\bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j}\mathbf{q}_n^{i}(t) + \\
&+ \mathbf{q}_n^{iT}(t)\mathbf{B}_c^{i,jT}\bar{\mathbf{C}}^{i,jT}\bar{\mathbf{S}}_{3,3}^{i,j}\bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j}\mathbf{q}_n^{i}(t) = \\
&= \mathbf{q}_n^{iT}(t)\mathbf{B}_c^{i,jT}\bar{\mathbf{C}}^{i,jT}\left(\bar{\mathbf{S}}_{2,2}^{i,j} + \bar{\mathbf{S}}_{3,3}^{i,j}\right)\bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j}\mathbf{q}_n^{i}(t) = \\
&= \left(\mathbf{q}_o^{i} + \mathbf{B}_e^{i}\mathbf{q}_f^{i}(t)\right)^T \bar{\mathbf{J}}_{1,1}^{i,j}\left(\mathbf{q}_o^{i} + \mathbf{B}_e^{i}\mathbf{q}_f^{i}(t)\right)
\end{aligned}
\tag{2.349}
$$

$$
\begin{aligned}
&\left(\bar{\mathbf{I}}_{\theta,\theta}^{i,j}(t)\right)_{1,2} = -\int_{\Omega^{i,j}} \rho^{i,j}\bar{u}_1^{i,j}(P^{i,j},t)\bar{u}_2^{i,j}(P^{i,j},t)dV^{i,j} = \\
&= -\int_{\Omega^{i,j}} \rho^{i,j}\mathbf{q}_n^{iT}(t)\mathbf{N}_1^{i,jT}(P^{i,j})\mathbf{N}_2^{i,j}(P^{i,j})\mathbf{q}_n^i(t)dV^{i,j} = \\
&= -\mathbf{q}_n^{iT}(t)\int_{\Omega^{i,j}} \rho^{i,j}\mathbf{B}_c^{i,jT}\bar{\mathbf{C}}^{i,jT}\mathbf{S}_1^{i,jT}(P^{i,j})\mathbf{C}^{i,jT}\mathbf{C}^{i,j}\mathbf{S}_2^{i,j}(P^{i,j})\bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j}dV^{i,j}\mathbf{q}_n^i(t) = \\
&= -\mathbf{q}_n^{iT}(t)\mathbf{B}_c^{i,jT}\bar{\mathbf{C}}^{i,jT}\int_{\Omega^{i,j}} \rho^{i,j}\mathbf{S}_1^{i,jT}(P^{i,j})\mathbf{S}_2^{i,j}(P^{i,j})dV^{i,j}\bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j}\mathbf{q}_n^i(t) = \\
&= -\mathbf{q}_n^{iT}(t)\mathbf{B}_c^{i,jT}\bar{\mathbf{C}}^{i,jT}\bar{\mathbf{S}}_{1,2}^{i,j}\bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j}\mathbf{q}_n^i(t) = \\
&= \left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right)^T \bar{\mathbf{J}}_{1,2}^{i,j}\left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right) = \\
&= \left(\bar{\mathbf{I}}_{\theta,\theta}^{i,j}(t)\right)_{2,1}
\end{aligned}
\tag{2.350}
$$

$$
\begin{aligned}
&\left(\bar{\mathbf{I}}_{\theta,\theta}^{i,j}(t)\right)_{1,3} = -\int_{\Omega^{i,j}} \rho^{i,j}\bar{u}_1^{i,j}(P^{i,j},t)\bar{u}_3^{i,j}(P^{i,j},t)dV^{i,j} = \\
&= -\int_{\Omega^{i,j}} \rho^{i,j}\mathbf{q}_n^{iT}(t)\mathbf{N}_1^{i,jT}(P^{i,j})\mathbf{N}_3^{i,j}(P^{i,j})\mathbf{q}_n^i(t)dV^{i,j} = \\
&= -\mathbf{q}_n^{iT}(t)\int_{\Omega^{i,j}} \rho^{i,j}\mathbf{B}_c^{i,jT}\bar{\mathbf{C}}^{i,jT}\mathbf{S}_1^{i,jT}(P^{i,j})\mathbf{C}^{i,jT}\mathbf{C}^{i,j}\mathbf{S}_3^{i,j}(P^{i,j})\bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j}dV^{i,j}\mathbf{q}_n^i(t) = \\
&= -\mathbf{q}_n^{iT}(t)\mathbf{B}_c^{i,jT}\bar{\mathbf{C}}^{i,jT}\int_{\Omega^{i,j}} \rho^{i,j}\mathbf{S}_1^{i,jT}(P^{i,j})\mathbf{S}_3^{i,j}(P^{i,j})dV^{i,j}\bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j}\mathbf{q}_n^i(t) = \\
&= -\mathbf{q}_n^{iT}(t)\mathbf{B}_c^{i,jT}\bar{\mathbf{C}}^{i,jT}\bar{\mathbf{S}}_{1,3}^{i,j}\bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j}\mathbf{q}_n^i(t) = \\
&= \left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right)^T \bar{\mathbf{J}}_{1,3}^{i,j}\left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right) = \\
&= \left(\bar{\mathbf{I}}_{\theta,\theta}^{i,j}(t)\right)_{3,1}
\end{aligned}
\tag{2.351}
$$

$$
\begin{aligned}
&\left(\bar{\mathbf{I}}_{\theta,\theta}^{i,j}(t)\right)_{2,2} = \int_{\Omega^{i,j}} \rho^{i,j}\left(\left(\bar{u}_3^{i,j}(P^{i,j},t)\right)^2 + \left(\bar{u}_1^{i,j}(P^{i,j},t)\right)^2\right) dV^{i,j} = \\
&= \int_{\Omega^{i,j}} \rho^{i,j}\left(\bar{u}_3^{i,j}(P^{i,j},t)\right)^2 dV^{i,j} + \int_{\Omega^{i,j}} \rho^{i,j}\left(\bar{u}_1^{i,j}(P^{i,j},t)\right)^2 dV^{i,j} = \\
&= \int_{\Omega^{i,j}} \rho^{i,j} \mathbf{q}_n^{iT}(t)\mathbf{N}_3^{i,jT}(P^{i,j})\mathbf{N}_3^{i,j}(P^{i,j})\mathbf{q}_n^i(t)dV^{i,j} + \\
&+ \int_{\Omega^{i,j}} \rho^{i,j} \mathbf{q}_n^{iT}(t)\mathbf{N}_1^{i,jT}(P^{i,j})\mathbf{N}_1^{i,j}(P^{i,j})\mathbf{q}_n^i(t)dV^{i,j} = \\
&= \mathbf{q}_n^{iT}(t)\int_{\Omega^{i,j}} \rho^{i,j}\mathbf{B}_c^{i,jT}\bar{\mathbf{C}}^{i,jT}\mathbf{S}_3^{i,jT}(P^{i,j})\mathbf{C}^{i,jT}\mathbf{C}^{i,j}\mathbf{S}_3^{i,j}(P^{i,j})\bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j}dV^{i,j}\mathbf{q}_n^i(t) + \\
&+ \mathbf{q}_n^{iT}(t)\int_{\Omega^{i,j}} \rho^{i,j}\mathbf{B}_c^{i,jT}\bar{\mathbf{C}}^{i,jT}\mathbf{S}_1^{i,jT}(P^{i,j})\mathbf{C}^{i,jT}\mathbf{C}^{i,j}\mathbf{S}_1^{i,j}(P^{i,j})\bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j}dV^{i,j}\mathbf{q}_n^i(t) = \\
&= \mathbf{q}_n^{iT}(t)\mathbf{B}_c^{i,jT}\bar{\mathbf{C}}^{i,jT}\int_{\Omega^{i,j}} \rho^{i,j}\mathbf{S}_3^{i,jT}(P^{i,j})\mathbf{S}_3^{i,j}(P^{i,j})dV^{i,j}\bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j}\mathbf{q}_n^i(t) + \\
&+ \mathbf{q}_n^{iT}(t)\mathbf{B}_c^{i,jT}\bar{\mathbf{C}}^{i,jT}\int_{\Omega^{i,j}} \rho^{i,j}\mathbf{S}_1^{i,jT}(P^{i,j})\mathbf{S}_1^{i,j}(P^{i,j})dV^{i,j}\bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j}\mathbf{q}_n^i(t) = \\
&= \mathbf{q}_n^{iT}(t)\mathbf{B}_c^{i,jT}\bar{\mathbf{C}}^{i,jT}\bar{\mathbf{S}}_{3,3}^{i,j}\bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j}\mathbf{q}_n^i(t) + \\
&+ \mathbf{q}_n^{iT}(t)\mathbf{B}_c^{i,jT}\bar{\mathbf{C}}^{i,jT}\bar{\mathbf{S}}_{1,1}^{i,j}\bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j}\mathbf{q}_n^i(t) = \\
&= \mathbf{q}_n^{iT}(t)\mathbf{B}_c^{i,jT}\bar{\mathbf{C}}^{i,jT}\left(\bar{\mathbf{S}}_{3,3}^{i,j} + \bar{\mathbf{S}}_{1,1}^{i,j}\right)\bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j}\mathbf{q}_n^i(t) = \\
&= \left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right)^T \bar{\mathbf{J}}_{2,2}^{i,j}\left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right)
\end{aligned}
\tag{2.352}
$$

$$\begin{aligned}
&\left(\bar{\mathbf{I}}_{\theta,\theta}^{i,j}(t)\right)_{2,3}=-\int_{\Omega^{i,j}}\rho^{i,j}\bar{u}_2^{i,j}(P^{i,j},t)\bar{u}_3^{i,j}(P^{i,j},t)dV^{i,j}=\\
&=-\int_{\Omega^{i,j}}\rho^{i,j}\mathbf{q}_n^{iT}(t)\mathbf{N}_2^{i,jT}(P^{i,j})\mathbf{N}_3^{i,j}(P^{i,j})\mathbf{q}_n^i(t)dV^{i,j}=\\
&=-\mathbf{q}_n^{iT}(t)\int_{\Omega^{i,j}}\rho^{i,j}\mathbf{B}_c^{i,jT}\bar{\mathbf{C}}^{i,jT}\mathbf{S}_2^{i,jT}(P^{i,j})\mathbf{C}^{i,jT}\mathbf{C}^{i,j}\mathbf{S}_3^{i,j}(P^{i,j})\bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j}dV^{i,j}\mathbf{q}_n^i(t)=\\
&=-\mathbf{q}_n^{iT}(t)\mathbf{B}_c^{i,jT}\bar{\mathbf{C}}^{i,jT}\int_{\Omega^{i,j}}\rho^{i,j}\mathbf{S}_2^{i,jT}(P^{i,j})\mathbf{S}_3^{i,j}(P^{i,j})dV^{i,j}\bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j}\mathbf{q}_n^i(t)=\\
&=-\mathbf{q}_n^{iT}(t)\mathbf{B}_c^{i,jT}\bar{\mathbf{C}}^{i,jT}\bar{\mathbf{S}}_{2,3}^{i,j}\bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j}\mathbf{q}_n^i(t)=\\
&=\left(\mathbf{q}_o^i+\mathbf{B}_e^i\mathbf{q}_f^i(t)\right)^T\bar{\mathbf{J}}_{2,3}^{i,j}\left(\mathbf{q}_o^i+\mathbf{B}_e^i\mathbf{q}_f^i(t)\right)=\\
&=\left(\bar{\mathbf{I}}_{\theta,\theta}^{i,j}(t)\right)_{3,2}
\end{aligned}\tag{2.353}$$

$$
\begin{aligned}
&\left(\overline{\mathbf{I}}^{i,j}_{\theta,\theta}(t)\right)_{3,3} = \int_{\Omega^{i,j}} \rho^{i,j}\left(\left(\overline{u}^{i,j}_1(P^{i,j},t)\right)^2 + \left(\overline{u}^{i,j}_2(P^{i,j},t)\right)^2\right) dV^{i,j} = \\
&= \int_{\Omega^{i,j}} \rho^{i,j}\left(\overline{u}^{i,j}_1(P^{i,j},t)\right)^2 dV^{i,j} + \int_{\Omega^{i,j}} \rho^{i,j}\left(\overline{u}^{i,j}_2(P^{i,j},t)\right)^2 dV^{i,j} = \\
&= \int_{\Omega^{i,j}} \rho^{i,j}\mathbf{q}^{iT}_n(t)\mathbf{N}^{i,jT}_1(P^{i,j})\mathbf{N}^{i,j}_1(P^{i,j})\mathbf{q}^i_n(t)dV^{i,j} + \\
&\quad + \int_{\Omega^{i,j}} \rho^{i,j}\mathbf{q}^{iT}_n(t)\mathbf{N}^{i,jT}_2(P^{i,j})\mathbf{N}^{i,j}_2(P^{i,j})\mathbf{q}^i_n(t)dV^{i,j} = \\
&= \mathbf{q}^{iT}_n(t)\int_{\Omega^{i,j}} \rho^{i,j}\mathbf{B}^{i,jT}_c\overline{\mathbf{C}}^{i,jT}\mathbf{S}^{i,jT}_1(P^{i,j})\mathbf{C}^{i,jT}\mathbf{C}^{i,j}\mathbf{S}^{i,j}_1(P^{i,j})\overline{\mathbf{C}}^{i,j}\mathbf{B}^{i,j}_c dV^{i,j}\mathbf{q}^i_n(t) + \\
&+ \mathbf{q}^{iT}_n(t)\int_{\Omega^{i,j}} \rho^{i,j}\mathbf{B}^{i,jT}_c\overline{\mathbf{C}}^{i,jT}\mathbf{S}^{i,jT}_2(P^{i,j})\mathbf{C}^{i,jT}\mathbf{C}^{i,j}\mathbf{S}^{i,j}_2(P^{i,j})\overline{\mathbf{C}}^{i,j}\mathbf{B}^{i,j}_c dV^{i,j}\mathbf{q}^i_n(t) = \\
&= \mathbf{q}^{iT}_n(t)\mathbf{B}^{i,jT}_c\overline{\mathbf{C}}^{i,jT}\int_{\Omega^{i,j}} \rho^{i,j}\mathbf{S}^{i,jT}_1(P^{i,j})\mathbf{S}^{i,j}_1(P^{i,j})dV^{i,j}\overline{\mathbf{C}}^{i,j}\mathbf{B}^{i,j}_c\mathbf{q}^i_n(t) + \\
&+ \mathbf{q}^{iT}_n(t)\mathbf{B}^{i,jT}_c\overline{\mathbf{C}}^{i,jT}\int_{\Omega^{i,j}} \rho^{i,j}\mathbf{S}^{i,jT}_2(P^{i,j})\mathbf{S}^{i,j}_2(P^{i,j})dV^{i,j}\overline{\mathbf{C}}^{i,j}\mathbf{B}^{i,j}_c\mathbf{q}^i_n(t) = \\
&= \mathbf{q}^{iT}_n(t)\mathbf{B}^{i,jT}_c\overline{\mathbf{C}}^{i,jT}\overline{\mathbf{S}}^{i,j}_{1,1}\overline{\mathbf{C}}^{i,j}\mathbf{B}^{i,j}_c\mathbf{q}^i_n(t) + \\
&+ \mathbf{q}^{iT}_n(t)\mathbf{B}^{i,jT}_c\overline{\mathbf{C}}^{i,jT}\overline{\mathbf{S}}^{i,j}_{2,2}\overline{\mathbf{C}}^{i,j}\mathbf{B}^{i,j}_c\mathbf{q}^i_n(t) = \\
&= \mathbf{q}^{iT}_n(t)\mathbf{B}^{i,jT}_c\overline{\mathbf{C}}^{i,jT}\left(\overline{\mathbf{S}}^{i,j}_{1,1} + \overline{\mathbf{S}}^{i,j}_{2,2}\right)\overline{\mathbf{C}}^{i,j}\mathbf{B}^{i,j}_c\mathbf{q}^i_n(t) = \\
&= \left(\mathbf{q}^i_o + \mathbf{B}^i_e\mathbf{q}^i_f(t)\right)^T\overline{\mathbf{J}}^{i,j}_{3,3}\left(\mathbf{q}^i_o + \mathbf{B}^i_e\mathbf{q}^i_f(t)\right)
\end{aligned}
\tag{2.354}
$$

Where $\overline{\mathbf{J}}^{i,j}_{1,1}$, $\overline{\mathbf{J}}^{i,j}_{1,2}$, $\overline{\mathbf{J}}^{i,j}_{1,3}$, $\overline{\mathbf{J}}^{i,j}_{2,1}$, $\overline{\mathbf{J}}^{i,j}_{2,2}$, $\overline{\mathbf{J}}^{i,j}_{2,3}$, $\overline{\mathbf{J}}^{i,j}_{3,1}$, $\overline{\mathbf{J}}^{i,j}_{3,2}$ and $\overline{\mathbf{J}}^{i,j}_{3,3}$ are $\mathbb{R}^{n^i_{f,v}\times n^i_{f,v}}$ symmetric matrices defined as:

$$
\overline{\mathbf{J}}^{i,j}_{1,1} = \mathbf{B}^{i,jT}_c\overline{\mathbf{C}}^{i,jT}\left(\overline{\mathbf{S}}^{i,j}_{2,2} + \overline{\mathbf{S}}^{i,j}_{3,3}\right)\overline{\mathbf{C}}^{i,j}\mathbf{B}^{i,j}_c \tag{2.355}
$$

$$
\begin{aligned}
\overline{\mathbf{J}}^{i,j}_{1,2} &= -\mathbf{B}^{i,jT}_c\overline{\mathbf{C}}^{i,jT}\overline{\mathbf{S}}^{i,j}_{1,2}\overline{\mathbf{C}}^{i,j}\mathbf{B}^{i,j}_c = \\
&= \overline{\mathbf{J}}^{i,jT}_{2,1}
\end{aligned}
\tag{2.356}
$$

$$\begin{aligned}\bar{\mathbf{J}}_{1,3}^{i,j} &= -\mathbf{B}_c^{i,jT}\bar{\mathbf{C}}^{i,jT}\bar{\mathbf{S}}_{1,3}^{i,j}\bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j} = \\ &= \bar{\mathbf{J}}_{3,1}^{i,jT}\end{aligned} \tag{2.357}$$

$$\bar{\mathbf{J}}_{2,2}^{i,j} = \mathbf{B}_c^{i,jT}\bar{\mathbf{C}}^{i,jT}\left(\bar{\mathbf{S}}_{3,3}^{i,j} + \bar{\mathbf{S}}_{1,1}^{i,j}\right)\bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j} \tag{2.358}$$

$$\begin{aligned}\bar{\mathbf{J}}_{2,3}^{i,j} &= -\mathbf{B}_c^{i,jT}\bar{\mathbf{C}}^{i,jT}\bar{\mathbf{S}}_{2,3}^{i,j}\bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j} = \\ &= \bar{\mathbf{J}}_{3,2}^{i,jT}\end{aligned} \tag{2.359}$$

$$\bar{\mathbf{J}}_{3,3}^{i,j} = \mathbf{B}_c^{i,jT}\bar{\mathbf{C}}^{i,jT}\left(\bar{\mathbf{S}}_{1,1}^{i,j} + \bar{\mathbf{S}}_{2,2}^{i,j}\right)\bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j} \tag{2.360}$$

Where $\bar{\mathbf{S}}_{1,1}^{i,j}$, $\bar{\mathbf{S}}_{1,2}^{i,j}$, $\bar{\mathbf{S}}_{1,3}^{i,j}$, $\bar{\mathbf{S}}_{2,1}^{i,j}$, $\bar{\mathbf{S}}_{2,2}^{i,j}$, $\bar{\mathbf{S}}_{2,3}^{i,j}$, $\bar{\mathbf{S}}_{3,1}^{i,j}$, $\bar{\mathbf{S}}_{3,2}^{i,j}$ and $\bar{\mathbf{S}}_{3,3}^{i,j}$ are $\mathbb{R}^{n_f^{i,j}\times n_f^{i,j}}$ symmetric matrices defined as:

$$\bar{\mathbf{S}}_{1,1}^{i,j} = \int_{\Omega^{i,j}} \rho^{i,j}\mathbf{S}_1^{i,jT}(P^{i,j})\mathbf{S}_1^{i,j}(P^{i,j})dV^{i,j} \tag{2.361}$$

$$\begin{aligned}\bar{\mathbf{S}}_{1,2}^{i,j} &= \int_{\Omega^{i,j}} \rho^{i,j}\mathbf{S}_1^{i,jT}(P^{i,j})\mathbf{S}_2^{i,j}(P^{i,j})dV^{i,j} = \\ &= \bar{\mathbf{S}}_{2,1}^{i,jT}\end{aligned} \tag{2.362}$$

$$\begin{aligned}\bar{\mathbf{S}}_{1,3}^{i,j} &= \int_{\Omega^{i,j}} \rho^{i,j}\mathbf{S}_1^{i,jT}(P^{i,j})\mathbf{S}_3^{i,j}(P^{i,j})dV^{i,j} = \\ &= \bar{\mathbf{S}}_{3,1}^{i,jT}\end{aligned} \tag{2.363}$$

$$\bar{\mathbf{S}}_{2,2}^{i,j} = \int_{\Omega^{i,j}} \rho^{i,j}\mathbf{S}_2^{i,jT}(P^{i,j})\mathbf{S}_2^{i,j}(P^{i,j})dV^{i,j} \tag{2.364}$$

$$\begin{aligned}\bar{\mathbf{S}}_{2,3}^{i,j} &= \int_{\Omega^{i,j}} \rho^{i,j}\mathbf{S}_2^{i,jT}(P^{i,j})\mathbf{S}_3^{i,j}(P^{i,j})dV^{i,j} = \\ &= \bar{\mathbf{S}}_{3,2}^{i,jT}\end{aligned} \tag{2.365}$$

$$\bar{\mathbf{S}}_{3,3}^{i,j} = \int_{\Omega^{i,j}} \rho^{i,j} \mathbf{S}_3^{i,jT}(P^{i,j}) \mathbf{S}_3^{i,j}(P^{i,j}) dV^{i,j} \tag{2.366}$$

The mass submatrix $\mathbf{m}_{\theta,q_f}^{i,j}(t)$ can be computed as:

$$\begin{aligned}
\mathbf{m}_{\theta,q_f}^{i,j}(t) &= \int_{\Omega^{i,j}} \rho^{i,j} \mathbf{L}_{\theta}^{i,jT}(P^{i,j},t) \mathbf{L}_{q_f}^{i,j}(P^{i,j},t) dV^{i,j} = \\
&= -\int_{\Omega^{i,j}} \rho^{i,j} \bar{\mathbf{G}}^{iT}(t) \tilde{\bar{\mathbf{u}}}^{i,jT}(P^{i,j},t) \mathbf{A}^{iT}(t) \mathbf{A}^{i}(t) \mathbf{N}^{i,j}(P^{i,j}) \mathbf{B}_e^i dV^{i,j} = \\
&= -\bar{\mathbf{G}}^{iT}(t) \int_{\Omega^{i,j}} \rho^{i,j} \tilde{\bar{\mathbf{u}}}^{i,jT}(P^{i,j},t) \mathbf{N}^{i,j}(P^{i,j}) dV^{i,j} \mathbf{B}_e^i = \\
&= \bar{\mathbf{G}}^{iT}(t) \int_{\Omega^{i,j}} \rho^{i,j} \tilde{\bar{\mathbf{u}}}^{i,j}(P^{i,j},t) \mathbf{N}^{i,j}(P^{i,j}) dV^{i,j} \mathbf{B}_e^i = \\
&= \bar{\mathbf{G}}^{iT}(t) \mathbf{H}^{i,j}(t) \mathbf{B}_e^i = \\
&= \mathbf{m}_{\theta,q_f}^{i,jT}(t)
\end{aligned} \tag{2.367}$$

Where $\mathbf{H}^{i,j}(t)$ is a $\mathbb{R}^{3\times n_{f,v}^i}$ matrix defined as:

$$
\begin{aligned}
&\mathbf{H}^{i,j}(t) = \int_{\Omega^{i,j}} \rho^{i,j}\, \tilde{\bar{\mathbf{u}}}^{i,j}(P^{i,j},t)\mathbf{N}^{i,j}(P^{i,j})dV^{i,j} = \\
&= \int_{\Omega^{i,j}} \rho^{i,j} \begin{bmatrix} 0 & -\bar{u}_3^{i,j}(P^{i,j},t) & \bar{u}_2^{i,j}(P^{i,j},t) \\ \bar{u}_3^{i,j}(P^{i,j},t) & 0 & -\bar{u}_1^{i,j}(P^{i,j},t) \\ -\bar{u}_2^{i,j}(P^{i,j},t) & \bar{u}_1^{i,j}(P^{i,j},t) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{N}_1^{i,j}(P^{i,j}) \\ \mathbf{N}_2^{i,j}(P^{i,j}) \\ \mathbf{N}_3^{i,j}(P^{i,j}) \end{bmatrix} dV^{i,j} = \\
&= \begin{bmatrix} -\int_{\Omega^{i,j}} \rho^{i,j}\bar{u}_3^{i,j}(P^{i,j},t)\mathbf{N}_2^{i,j}(P^{i,j})dV^{i,j} + \int_{\Omega^{i,j}} \rho^{i,j}\bar{u}_2^{i,j}(P^{i,j},t)\mathbf{N}_3^{i,j}(P^{i,j})dV^{i,j} \\ \int_{\Omega^{i,j}} \rho^{i,j}\bar{u}_3^{i,j}(P^{i,j},t)\mathbf{N}_1^{i,j}(P^{i,j})dV^{i,j} - \int_{\Omega^{i,j}} \rho^{i,j}\bar{u}_1^{i,j}(P^{i,j},t)\mathbf{N}_3^{i,j}(P^{i,j})dV^{i,j} \\ -\int_{\Omega^{i,j}} \rho^{i,j}\bar{u}_2^{i,j}(P^{i,j},t)\mathbf{N}_1^{i,j}(P^{i,j})dV^{i,j} + \int_{\Omega^{i,j}} \rho^{i,j}\bar{u}_1^{i,j}(P^{i,j},t)\mathbf{N}_2^{i,j}(P^{i,j})dV^{i,j} \end{bmatrix} = \\
&= \begin{bmatrix} -\mathbf{q}_n^{iT}(t)\int_{\Omega^{i,j}} \rho^{i,j}\mathbf{N}_3^{i,jT}(P^{i,j})\mathbf{N}_2^{i,j}(P^{i,j})dV^{i,j} + \\ +\mathbf{q}_n^{iT}(t)\int_{\Omega^{i,j}} \rho^{i,j}\mathbf{N}_2^{i,jT}(P^{i,j})\mathbf{N}_3^{i,j}(P^{i,j})dV^{i,j} \\ \mathbf{q}_n^{iT}(t)\int_{\Omega^{i,j}} \rho^{i,j}\mathbf{N}_3^{i,jT}(P^{i,j})\mathbf{N}_1^{i,j}(P^{i,j})dV^{i,j} + \\ -\mathbf{q}_n^{iT}(t)\int_{\Omega^{i,j}} \rho^{i,j}\mathbf{N}_1^{i,jT}(P^{i,j})\mathbf{N}_3^{i,j}(P^{i,j})dV^{i,j} \\ -\mathbf{q}_n^{iT}(t)\int_{\Omega^{i,j}} \rho^{i,j}\mathbf{N}_2^{i,jT}(P^{i,j})\mathbf{N}_1^{i,j}(P^{i,j})dV^{i,j} + \\ +\mathbf{q}_n^{iT}(t)\int_{\Omega^{i,j}} \rho^{i,j}\mathbf{N}_1^{i,jT}(P^{i,j})\mathbf{N}_2^{i,j}(P^{i,j})dV^{i,j} \end{bmatrix} \\
&= \begin{bmatrix} -\mathbf{q}_n^{iT}(t)\int_{\Omega^{i,j}} \rho^{i,j}\mathbf{B}_c^{i,jT}\bar{\mathbf{C}}^{i,jT}\mathbf{S}_3^{i,jT}(P^{i,j})\mathbf{C}^{i,jT}\mathbf{C}^{i,j}\mathbf{S}_2^{i,j}(P^{i,j})\bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j}dV^{i,j} \\ \mathbf{q}_n^{iT}(t)\int_{\Omega^{i,j}} \rho^{i,j}\mathbf{B}_c^{i,jT}\bar{\mathbf{C}}^{i,jT}\mathbf{S}_3^{i,jT}(P^{i,j})\mathbf{C}^{i,jT}\mathbf{C}^{i,j}\mathbf{S}_1^{i,j}(P^{i,j})\bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j}dV^{i,j} \\ -\mathbf{q}_n^{iT}(t)\int_{\Omega^{i,j}} \rho^{i,j}\mathbf{B}_c^{i,jT}\bar{\mathbf{C}}^{i,jT}\mathbf{S}_2^{i,jT}(P^{i,j})\mathbf{C}^{i,jT}\mathbf{C}^{i,j}\mathbf{S}_1^{i,j}(P^{i,j})\bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j}dV^{i,j} \end{bmatrix} + \\
&+ \begin{bmatrix} \mathbf{q}_n^{iT}(t)\int_{\Omega^{i,j}} \rho^{i,j}\mathbf{B}_c^{i,jT}\bar{\mathbf{C}}^{i,jT}\mathbf{S}_2^{i,jT}(P^{i,j})\mathbf{C}^{i,jT}\mathbf{C}^{i,j}\mathbf{S}_3^{i,j}(P^{i,j})\bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j}dV^{i,j} \\ -\mathbf{q}_n^{iT}(t)\int_{\Omega^{i,j}} \rho^{i,j}\mathbf{B}_c^{i,jT}\bar{\mathbf{C}}^{i,jT}\mathbf{S}_1^{i,jT}(P^{i,j})\mathbf{C}^{i,jT}\mathbf{C}^{i,j}\mathbf{S}_3^{i,j}(P^{i,j})\bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j}dV^{i,j} \\ \mathbf{q}_n^{iT}(t)\int_{\Omega^{i,j}} \rho^{i,j}\mathbf{B}_c^{i,jT}\bar{\mathbf{C}}^{i,jT}\mathbf{S}_1^{i,jT}(P^{i,j})\mathbf{C}^{i,jT}\mathbf{C}^{i,j}\mathbf{S}_2^{i,j}(P^{i,j})\bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j}dV^{i,j} \end{bmatrix}
\end{aligned}
\tag{2.368}
$$

This expression can be further simplified to yield:

$$\begin{bmatrix} -\mathbf{q}_n^{iT}(t)\mathbf{B}_c^{i,jT}\overline{\mathbf{C}}^{i,jT}\int_{\Omega^{i,j}}\rho^{i,j}\mathbf{S}_3^{i,jT}(P^{i,j})\mathbf{S}_2^{i,j}(P^{i,j})dV^{i,j}\overline{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j} \\ \mathbf{q}_n^{iT}(t)\mathbf{B}_c^{i,jT}\overline{\mathbf{C}}^{i,jT}\int_{\Omega^{i,j}}\rho^{i,j}\mathbf{S}_3^{i,jT}(P^{i,j})\mathbf{S}_1^{i,j}(P^{i,j})dV^{i,j}\overline{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j} \\ -\mathbf{q}_n^{iT}(t)\mathbf{B}_c^{i,jT}\overline{\mathbf{C}}^{i,jT}\int_{\Omega^{i,j}}\rho^{i,j}\mathbf{S}_2^{i,jT}(P^{i,j})\mathbf{S}_1^{i,j}(P^{i,j})dV^{i,j}\overline{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j} \end{bmatrix} +$$

$$+\begin{bmatrix} \mathbf{q}_n^{iT}(t)\mathbf{B}_c^{i,jT}\overline{\mathbf{C}}^{i,jT}\int_{\Omega^{i,j}}\rho^{i,j}\mathbf{S}_2^{i,jT}(P^{i,j})\mathbf{S}_3^{i,j}(P^{i,j})dV^{i,j}\overline{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j} \\ -\mathbf{q}_n^{iT}(t)\mathbf{B}_c^{i,jT}\overline{\mathbf{C}}^{i,jT}\int_{\Omega^{i,j}}\rho^{i,j}\mathbf{S}_1^{i,jT}(P^{i,j})\mathbf{S}_3^{i,j}(P^{i,j})dV^{i,j}\overline{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j} \\ \mathbf{q}_n^{iT}(t)\mathbf{B}_c^{i,jT}\overline{\mathbf{C}}^{i,jT}\int_{\Omega^{i,j}}\rho^{i,j}\mathbf{S}_1^{i,jT}(P^{i,j})\mathbf{S}_2^{i,j}(P^{i,j})dV^{i,j}\overline{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j} \end{bmatrix} =$$

$$= \begin{bmatrix} \mathbf{q}_n^{iT}(t)\left(-\mathbf{B}_c^{i,jT}\overline{\mathbf{C}}^{i,jT}\overline{\mathbf{S}}_{3,2}^{i,j}\overline{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j} + \mathbf{B}_c^{i,jT}\overline{\mathbf{C}}^{i,jT}\overline{\mathbf{S}}_{2,3}^{i,j}\overline{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j}\right) \\ \mathbf{q}_n^{iT}(t)\left(\mathbf{B}_c^{i,jT}\overline{\mathbf{C}}^{i,jT}\overline{\mathbf{S}}_{3,1}^{i,j}\overline{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j} - \mathbf{B}_c^{i,jT}\overline{\mathbf{C}}^{i,jT}\overline{\mathbf{S}}_{1,3}^{i,j}\overline{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j}\right) \\ \mathbf{q}_n^{iT}(t)\left(-\mathbf{B}_c^{i,jT}\overline{\mathbf{C}}^{i,jT}\overline{\mathbf{S}}_{2,1}^{i,j}\overline{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j} + \mathbf{B}_c^{i,jT}\overline{\mathbf{C}}^{i,jT}\overline{\mathbf{S}}_{1,2}^{i,j}\overline{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j}\right) \end{bmatrix} =$$

$$= \begin{bmatrix} \mathbf{q}_n^{iT}(t)\mathbf{B}_c^{i,jT}\overline{\mathbf{C}}^{i,jT}\left(\overline{\mathbf{S}}_{2,3}^{i,j} - \overline{\mathbf{S}}_{3,2}^{i,j}\right)\overline{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j} \\ \mathbf{q}_n^{iT}(t)\mathbf{B}_c^{i,jT}\overline{\mathbf{C}}^{i,jT}\left(\overline{\mathbf{S}}_{3,1}^{i,j} - \overline{\mathbf{S}}_{1,3}^{i,j}\right)\overline{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j} \\ \mathbf{q}_n^{iT}(t)\mathbf{B}_c^{i,jT}\overline{\mathbf{C}}^{i,jT}\left(\overline{\mathbf{S}}_{1,2}^{i,j} - \overline{\mathbf{S}}_{2,1}^{i,j}\right)\overline{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j} \end{bmatrix} =$$

$$= \begin{bmatrix} \left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right)^T\left(\overline{\mathbf{J}}_{3,2}^{i,j} - \overline{\mathbf{J}}_{2,3}^{i,j}\right) \\ \left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right)^T\left(\overline{\mathbf{J}}_{1,3}^{i,j} - \overline{\mathbf{J}}_{3,1}^{i,j}\right) \\ \left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right)^T\left(\overline{\mathbf{J}}_{2,1}^{i,j} - \overline{\mathbf{J}}_{1,2}^{i,j}\right) \end{bmatrix}$$

(2.369)

Where the compact shape function has been written as follows:

$$\mathbf{N}^{i,j}(P^{i,j}) = \begin{bmatrix} \mathbf{N}_1^{i,j}(P^{i,j}) \\ \mathbf{N}_2^{i,j}(P^{i,j}) \\ \mathbf{N}_3^{i,j}(P^{i,j}) \end{bmatrix} \tag{2.370}$$

The mass submatrix $\mathbf{m}_{q_f,q_f}^{i,j}(t)$ can be computed as:

$$
\begin{aligned}
&\mathbf{m}_{q_f,q_f}^{i,j}(t)=\int_{\Omega^{i,j}}\rho^{i,j}\mathbf{L}_{q_f}^{i,jT}(P^{i,j},t)\mathbf{L}_{q_f}^{i,j}(P^{i,j},t)dV^{i,j}=\\
&=\int_{\Omega^{i,j}}\rho^{i,j}\mathbf{B}_e^{iT}\mathbf{N}^{i,jT}(P^{i,j})\mathbf{A}^{iT}(t)\mathbf{A}^{i}(t)\mathbf{N}^{i,j}(P^{i,j})\mathbf{B}_e^{i}dV^{i,j}=\\
&=\mathbf{B}_e^{iT}\int_{\Omega^{i,j}}\rho^{i,j}\mathbf{N}^{i,jT}(P^{i,j})\mathbf{N}^{i,j}(P^{i,j})dV^{i,j}\mathbf{B}_e^{i}=\\
&=\mathbf{B}_e^{iT}\int_{\Omega^{i,j}}\rho^{i,j}\mathbf{B}_c^{i,jT}\overline{\mathbf{C}}^{i,jT}\mathbf{S}^{i,jT}(P^{i,j})\mathbf{C}^{i,jT}\mathbf{C}^{i,j}\mathbf{S}^{i,j}(P^{i,j})\overline{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j}dV^{i,j}\mathbf{B}_e^{i}=\\
&=\mathbf{B}_e^{iT}\mathbf{B}_c^{i,jT}\overline{\mathbf{C}}^{i,jT}\int_{\Omega^{i,j}}\rho^{i,j}\mathbf{S}^{i,jT}(P^{i,j})\mathbf{S}^{i,j}(P^{i,j})dV^{i,j}\overline{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j}\mathbf{B}_e^{i}=\\
&=\mathbf{B}_e^{iT}\mathbf{B}_c^{i,jT}\overline{\mathbf{C}}^{i,jT}\int_{\Omega^{i,j}}\rho^{i,j}\mathbf{S}_1^{i,jT}(P^{i,j})\mathbf{S}_1^{i,j}(P^{i,j})dV^{i,j}\overline{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j}\mathbf{B}_e^{i}+\\
&\quad+\mathbf{B}_e^{iT}\mathbf{B}_c^{i,jT}\overline{\mathbf{C}}^{i,jT}\int_{\Omega^{i,j}}\rho^{i,j}\mathbf{S}_2^{i,jT}(P^{i,j})\mathbf{S}_2^{i,j}(P^{i,j})dV^{i,j}\overline{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j}\mathbf{B}_e^{i}+\\
&\quad+\mathbf{B}_e^{iT}\mathbf{B}_c^{i,jT}\overline{\mathbf{C}}^{i,jT}\int_{\Omega^{i,j}}\rho^{i,j}\mathbf{S}_3^{i,jT}(P^{i,j})\mathbf{S}_3^{i,j}(P^{i,j})dV^{i,j}\overline{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j}\mathbf{B}_e^{i}=\\
&=\mathbf{B}_e^{iT}\mathbf{B}_c^{i,jT}\overline{\mathbf{C}}^{i,jT}\left(\overline{\mathbf{S}}_{1,1}^{i,j}+\overline{\mathbf{S}}_{2,2}^{i,j}+\overline{\mathbf{S}}_{3,3}^{i,j}\right)\overline{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j}\mathbf{B}_e^{i}=\\
&=\mathbf{B}_e^{iT}\mathbf{B}_c^{i,jT}\overline{\mathbf{C}}^{i,jT}\overline{\mathbf{S}}_{f,f}^{i,j}\overline{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j}\mathbf{B}_e^{i}=\\
&=\frac{1}{2}\mathbf{B}_e^{iT}\left(\overline{\mathbf{J}}_{1,1}^{i,j}+\overline{\mathbf{J}}_{2,2}^{i,j}+\overline{\mathbf{J}}_{3,3}^{i,j}\right)\mathbf{B}_e^{i}=\\
&=\mathbf{B}_e^{iT}\overline{\mathbf{J}}_{f,f}^{i,j}\mathbf{B}_e^{i}
\end{aligned}
\tag{2.371}
$$

Where $\overline{\mathbf{J}}_{f,f}^{i,j}$ is a $\mathbb{R}^{n_{f,v}^{i}\times n_{f,v}^{i}}$ symmetric matrix defined as:

$$
\begin{aligned}
\overline{\mathbf{J}}_{f,f}^{i,j}&=\frac{1}{2}\left(\overline{\mathbf{J}}_{1,1}^{i,j}+\overline{\mathbf{J}}_{2,2}^{i,j}+\overline{\mathbf{J}}_{3,3}^{i,j}\right)=\\
&=\mathbf{B}_c^{i,jT}\overline{\mathbf{C}}^{i,jT}\overline{\mathbf{S}}_{f,f}^{i,j}\overline{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j}
\end{aligned}
\tag{2.372}
$$

Where $\overline{\mathbf{S}}_{f,f}^{i,j}$ is a $\mathbb{R}^{n_f^{i,j}\times n_f^{i,j}}$ symmetric matrix defined as follows:

$$\bar{\mathbf{S}}^{i,j}_{f,f} = \bar{\mathbf{S}}^{i,j}_{1,1} + \bar{\mathbf{S}}^{i,j}_{2,2} + \bar{\mathbf{S}}^{i,j}_{3,3} \tag{2.373}$$

These are the mathematical expressions that allow to compute explicitly all mass submatrix $\mathbf{m}^{i,j}_{R,R}(t)$, $\mathbf{m}^{i,j}_{R,\theta}(t)$, $\mathbf{m}^{i,j}_{R,q_f}(t)$, $\mathbf{m}^{i,j}_{\theta,\theta}(t)$, $\mathbf{m}^{i,j}_{\theta,q_f}(t)$ and $\mathbf{m}^{i,j}_{q_f,q_f}(t)$ relative to the mass matrix $\mathbf{M}^{i,j}(t)$ of element j of the flexible body i. It is worth noting that to compute all the mass submatrix it is necessary to evaluate previously the following sets of inertia shape integrals:

$$\begin{aligned} \mathbf{I}^{(1)\,i,j} &= m^{i,j}\mathbf{I} = \\ &= \int_{\Omega^{i,j}} \rho^{i,j}\, dV^{i,j}\mathbf{I} \end{aligned} \tag{2.374}$$

$$\begin{aligned} \mathbf{I}^{(2)\,i,j} &= \bar{\mathbf{S}}^{i,j} = \\ &= \int_{\Omega^{i,j}} \rho^{i,j}\mathbf{S}^{i,j}(P^{i,j})dV^{i,j} \end{aligned} \tag{2.375}$$

$$\begin{aligned} \mathbf{I}^{(3)\,i,j}_{h,k} &= \bar{\mathbf{S}}^{i,j}_{h,k} = \\ &= \int_{\Omega^{i,j}} \rho^{i,j}\mathbf{S}^{i,jT}_{h}(P^{i,j})\mathbf{S}^{i,j}_{k}(P^{i,j})dV^{i,j} \end{aligned} \tag{2.376}$$

These sets of inertia shape integrals can be computed in advance and then they can be used to write the equations of motion of flexible multibody systems.

2.7.4. MASS MATRIX OF SYSTEM BODIES

Once that all mass submatrices has been obtained, the mass matrix of the generic flexible body i can be computed using the additive property of kinetic energy $T^i(t)$. Indeed:

$$\begin{aligned} T^i(t) &= \sum_{j=1}^{n_e^i} T^{i,j}(t) = \\ &= \sum_{j=1}^{n_e^i} \left(\frac{1}{2} \dot{\mathbf{q}}^{iT}(t) \mathbf{M}^{i,j}(t) \dot{\mathbf{q}}^i(t) \right) = \\ &= \frac{1}{2} \dot{\mathbf{q}}^{iT}(t) \left(\sum_{j=1}^{n_e^i} \mathbf{M}^{i,j}(t) \right) \dot{\mathbf{q}}^i(t) = \\ &= \frac{1}{2} \dot{\mathbf{q}}^{iT}(t) \mathbf{M}^i(t) \dot{\mathbf{q}}^i(t) \end{aligned} \tag{2.377}$$

Where $\mathbf{M}^i(t)$ is a $\mathbb{R}^{n^i \times n^i}$ matrix representing the mass matrix of flexible body i defined as:

$$\begin{aligned} \mathbf{M}^i(t) &= \sum_{j=1}^{n_e^i} \mathbf{M}^{i,j}(t) = \\ &= \begin{bmatrix} \mathbf{m}^i_{R,R}(t) & \mathbf{m}^i_{R,\theta}(t) & \mathbf{m}^i_{R,q_f}(t) \\ \mathbf{m}^i_{\theta,R}(t) & \mathbf{m}^i_{\theta,\theta}(t) & \mathbf{m}^i_{\theta,q_f}(t) \\ \mathbf{m}^i_{q_f,R}(t) & \mathbf{m}^i_{q_f,\theta}(t) & \mathbf{m}^i_{q_f,q_f}(t) \end{bmatrix} \end{aligned} \tag{2.378}$$

Where $\mathbf{m}^i_{R,R}(t)$, $\mathbf{m}^i_{R,\theta}(t)$, $\mathbf{m}^i_{R,q_f}(t)$, $\mathbf{m}^i_{\theta,\theta}(t)$, $\mathbf{m}^i_{\theta,q_f}(t)$ and $\mathbf{m}^i_{q_f,q_f}(t)$ are respectively $\mathbb{R}^{3\times3}$, $\mathbb{R}^{3\times4}$, $\mathbb{R}^{3\times n_f^i}$, $\mathbb{R}^{4\times4}$, $\mathbb{R}^{4\times n_f^i}$ and $\mathbb{R}^{n_f^i\times n_f^i}$ symmetric matrices which can be computed explicitly. Indeed, the mass submatrix $\mathbf{m}^i_{R,R}(t)$ can be computed as:

$$\begin{aligned}\mathbf{m}^i_{R,R}(t) &= \sum_{j=1}^{n_e^i} \mathbf{m}^{i,j}_{R,R}(t) = \\ &= \sum_{j=1}^{n_e^i} m^{i,j}\mathbf{I} = \\ &= m^i\mathbf{I}\end{aligned} \tag{2.379}$$

Where m^i is the total mass of flexible body i. The mass submatrix $\mathbf{m}^i_{R,\theta}(t)$ can be computed in the following way:

$$\begin{aligned}\mathbf{m}^i_{R,\theta}(t) &= \sum_{j=1}^{n_e^i} \mathbf{m}^{i,j}_{R,\theta}(t) = \\ &= -\sum_{j=1}^{n_e^i} \left(\mathbf{A}^i(t)\tilde{\bar{\mathbf{U}}}^{i,j}(t)\bar{\mathbf{G}}^i(t)\right) = \\ &= -\mathbf{A}^i(t)\left(\sum_{j=1}^{n_e^i} \tilde{\bar{\mathbf{U}}}^{i,j}(t)\right)\bar{\mathbf{G}}^i(t) = \\ &= -\mathbf{A}^i(t)\tilde{\bar{\mathbf{U}}}^i(t)\bar{\mathbf{G}}^i(t) = \\ &= \mathbf{m}^{i\,T}_{\theta,R}(t)\end{aligned} \tag{2.380}$$

Where $\tilde{\bar{\mathbf{U}}}^i(t)$ is a $\mathbb{R}^{3\times 3}$ skew symmetric matrix defined as:

$$\begin{aligned}
\tilde{\tilde{\mathbf{U}}}^i(t) &= \sum_{j=1}^{n_e^i} \tilde{\tilde{\mathbf{U}}}^{i,j}(t) = \\
&= \sum_{j=1}^{n_e^i} \left(Skew\left(\bar{\mathbf{N}}^{i,j} \left(\mathbf{q}_o^i + \mathbf{B}_e^i \mathbf{q}_f^i(t) \right) \right) \right) = \\
&= Skew\left(\left(\sum_{j=1}^{n_e^i} \bar{\mathbf{N}}^{i,j} \right) \left(\mathbf{q}_o^i + \mathbf{B}_e^i \mathbf{q}_f^i(t) \right) \right) = \\
&= Skew\left(\bar{\mathbf{N}}^i \left(\mathbf{q}_o^i + \mathbf{B}_e^i \mathbf{q}_f^i(t) \right) \right) = \\
&= Skew\left(\bar{\mathbf{N}}^i \mathbf{q}_n^i(t) \right)
\end{aligned} \tag{2.381}$$

Where $\bar{\mathbf{N}}^i$ is a $\mathbb{R}^{3\times n_{f,v}^i}$ matrix defined as:

$$\bar{\mathbf{N}}^i = \sum_{j=1}^{n_e^i} \bar{\mathbf{N}}^{i,j} \tag{2.382}$$

The mass submatrix $\mathbf{m}_{R,q_f}^i(t)$ is defined as:

$$\begin{aligned}
\mathbf{m}_{R,q_f}^i(t) &= \sum_{j=1}^{n_e^i} \mathbf{m}_{R,q_f}^{i,j}(t) = \\
&= \sum_{j=1}^{n_e^i} \left(\mathbf{A}^i(t) \bar{\mathbf{N}}^{i,j} \mathbf{B}_e^i \right) = \\
&= \mathbf{A}^i(t) \left(\sum_{j=1}^{n_e^i} \bar{\mathbf{N}}^{i,j} \right) \mathbf{B}_e^i = \\
&= \mathbf{A}^i(t) \bar{\mathbf{N}}^i \mathbf{B}_e^i = \\
&= \mathbf{m}_{q_f,R}^{i\,T}(t)
\end{aligned} \tag{2.383}$$

The mass submatrix $\mathbf{m}_{\theta,\theta}^i(t)$ is defined as:

$$\begin{aligned}\mathbf{m}^{i}_{\theta,\theta}(t) &= \sum_{j=1}^{n_e^i} \mathbf{m}^{i,j}_{\theta,\theta}(t) = \\ &= \sum_{j=1}^{n_e^i}\left(\bar{\mathbf{G}}^{i\,T}(t)\bar{\mathbf{I}}^{i,j}_{\theta,\theta}(t)\bar{\mathbf{G}}^{i}(t)\right) = \\ &= \bar{\mathbf{G}}^{i\,T}(t)\left(\sum_{j=1}^{n_e^i} \bar{\mathbf{I}}^{i,j}_{\theta,\theta}(t)\right)\bar{\mathbf{G}}^{i}(t) = \\ &= \bar{\mathbf{G}}^{i\,T}(t)\bar{\mathbf{I}}^{i}_{\theta,\theta}(t)\bar{\mathbf{G}}^{i}(t)\end{aligned} \tag{2.384}$$

Where $\bar{\mathbf{I}}^{i}_{\theta,\theta}(t)$ is a $\mathbb{R}^{3\times 3}$ representing the inertia matrix of flexible body i and it is defined as:

$$\begin{aligned}\bar{\mathbf{I}}^{i}_{\theta,\theta}(t) &= \sum_{j=1}^{n_e^i} \bar{\mathbf{I}}^{i,j}_{\theta,\theta}(t) = \\ &= \begin{bmatrix} \left(\bar{\mathbf{I}}^{i}_{\theta,\theta}(t)\right)_{1,1} & \left(\bar{\mathbf{I}}^{i}_{\theta,\theta}(t)\right)_{1,2} & \left(\bar{\mathbf{I}}^{i}_{\theta,\theta}(t)\right)_{1,3} \\ \left(\bar{\mathbf{I}}^{i}_{\theta,\theta}(t)\right)_{2,1} & \left(\bar{\mathbf{I}}^{i}_{\theta,\theta}(t)\right)_{2,2} & \left(\bar{\mathbf{I}}^{i}_{\theta,\theta}(t)\right)_{2,3} \\ \left(\bar{\mathbf{I}}^{i}_{\theta,\theta}(t)\right)_{3,1} & \left(\bar{\mathbf{I}}^{i}_{\theta,\theta}(t)\right)_{3,2} & \left(\bar{\mathbf{I}}^{i}_{\theta,\theta}(t)\right)_{3,3} \end{bmatrix}\end{aligned} \tag{2.385}$$

It is worth to point out that in this case of flexible multibody systems the inertia matrix $\bar{\mathbf{I}}^{i}_{\theta,\theta}(t)$ results to be a function of system configuration vector and consequently it changes in time. Indeed, the components of this matrix can be computed as:

$$
\begin{aligned}
\left(\bar{\mathbf{I}}^{i}_{\theta,\theta}(t)\right)_{1,1} &= \sum_{j=1}^{n_e^i}\left(\bar{\mathbf{I}}^{i,j}_{\theta,\theta}(t)\right)_{1,1} = \\
&= \sum_{j=1}^{n_e^i}\left(\left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right)^T \bar{\mathbf{J}}^{i,j}_{1,1}\left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right)\right) = \\
&= \left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right)^T \left(\sum_{j=1}^{n_e^i}\bar{\mathbf{J}}^{i,j}_{1,1}\right)\left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right) = \\
&= \left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right)^T \bar{\mathbf{J}}^{i}_{1,1}\left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right) = \\
&= \mathbf{q}_n^{i\,T}(t)\bar{\mathbf{J}}^{i}_{1,1}\mathbf{q}_n^i(t)
\end{aligned}
\tag{2.386}
$$

$$
\begin{aligned}
\left(\bar{\mathbf{I}}^{i}_{\theta,\theta}(t)\right)_{1,2} &= \sum_{j=1}^{n_e^i}\left(\bar{\mathbf{I}}^{i,j}_{\theta,\theta}(t)\right)_{1,2} = \\
&= \sum_{j=1}^{n_e^i}\left(\left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right)^T \bar{\mathbf{J}}^{i,j}_{1,2}\left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right)\right) = \\
&= \left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right)^T \left(\sum_{j=1}^{n_e^i}\bar{\mathbf{J}}^{i,j}_{1,2}\right)\left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right) = \\
&= \left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right)^T \bar{\mathbf{J}}^{i}_{1,2}\left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right) = \\
&= \mathbf{q}_n^{i\,T}(t)\bar{\mathbf{J}}^{i}_{1,2}\mathbf{q}_n^i(t) = \\
&= \left(\bar{\mathbf{I}}^{i}_{\theta,\theta}(t)\right)_{2,1}
\end{aligned}
\tag{2.387}
$$

$$\begin{aligned}
\left(\bar{\mathbf{I}}^{i}_{\theta,\theta}(t)\right)_{1,3} &= \sum_{j=1}^{n_e^i}\left(\bar{\mathbf{I}}^{i,j}_{\theta,\theta}(t)\right)_{1,3} = \\
&= \sum_{j=1}^{n_e^i}\left(\left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right)^T \bar{\mathbf{J}}^{i,j}_{1,3}\left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right)\right) = \\
&= \left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right)^T\left(\sum_{j=1}^{n_e^i}\bar{\mathbf{J}}^{i,j}_{1,3}\right)\left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right) = \\
&= \left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right)^T\bar{\mathbf{J}}^{i}_{1,3}\left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right) = \\
&= \mathbf{q}_n^{i\,T}(t)\bar{\mathbf{J}}^{i}_{1,3}\mathbf{q}_n^i(t) = \\
&= \left(\bar{\mathbf{I}}^{i}_{\theta,\theta}(t)\right)_{3,1}
\end{aligned} \tag{2.388}$$

$$\begin{aligned}
\left(\bar{\mathbf{I}}^{i}_{\theta,\theta}(t)\right)_{2,2} &= \sum_{j=1}^{n_e^i}\left(\bar{\mathbf{I}}^{i,j}_{\theta,\theta}(t)\right)_{2,2} = \\
&= \sum_{j=1}^{n_e^i}\left(\left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right)^T \bar{\mathbf{J}}^{i,j}_{2,2}\left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right)\right) = \\
&= \left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right)^T\left(\sum_{j=1}^{n_e^i}\bar{\mathbf{J}}^{i,j}_{2,2}\right)\left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right) = \\
&= \left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right)^T\bar{\mathbf{J}}^{i}_{2,2}\left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right) = \\
&= \mathbf{q}_n^{i\,T}(t)\bar{\mathbf{J}}^{i}_{2,2}\mathbf{q}_n^i(t)
\end{aligned} \tag{2.389}$$

$$\begin{aligned}
\left(\overline{\mathbf{I}}^{i}_{\theta,\theta}(t)\right)_{2,3} &= \sum_{j=1}^{n_e^i}\left(\overline{\mathbf{I}}^{i,j}_{\theta,\theta}(t)\right)_{2,3} = \\
&= \sum_{j=1}^{n_e^i}\left(\left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right)^T \overline{\mathbf{J}}^{i,j}_{2,3}\left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right)\right) = \\
&= \left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right)^T \left(\sum_{j=1}^{n_e^i}\overline{\mathbf{J}}^{i,j}_{2,3}\right)\left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right) = \\
&= \left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right)^T \overline{\mathbf{J}}^{i}_{2,3}\left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right) = \\
&= \mathbf{q}_n^{i\,T}(t)\overline{\mathbf{J}}^{i}_{2,3}\mathbf{q}_n^i(t) = \\
&= \left(\overline{\mathbf{I}}^{i}_{\theta,\theta}(t)\right)_{3,2}
\end{aligned} \tag{2.390}$$

$$\begin{aligned}
\left(\overline{\mathbf{I}}^{i}_{\theta,\theta}(t)\right)_{3,3} &= \sum_{j=1}^{n_e^i}\left(\overline{\mathbf{I}}^{i,j}_{\theta,\theta}(t)\right)_{3,3} = \\
&= \sum_{j=1}^{n_e^i}\left(\left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right)^T \overline{\mathbf{J}}^{i,j}_{3,3}\left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right)\right) = \\
&= \left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right)^T \left(\sum_{j=1}^{n_e^i}\overline{\mathbf{J}}^{i,j}_{3,3}\right)\left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right) = \\
&= \left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right)^T \overline{\mathbf{J}}^{i}_{3,3}\left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right) = \\
&= \mathbf{q}_n^{i\,T}(t)\overline{\mathbf{J}}^{i}_{3,3}\mathbf{q}_n^i(t)
\end{aligned} \tag{2.391}$$

Where $\overline{\mathbf{J}}^i_{1,1}$, $\overline{\mathbf{J}}^i_{1,2}$, $\overline{\mathbf{J}}^i_{1,3}$, $\overline{\mathbf{J}}^i_{2,1}$, $\overline{\mathbf{J}}^i_{2,2}$, $\overline{\mathbf{J}}^i_{2,3}$, $\overline{\mathbf{J}}^i_{3,1}$, $\overline{\mathbf{J}}^i_{3,2}$ and $\overline{\mathbf{J}}^i_{3,3}$ are $\mathbb{R}^{n^i_{f,v}\times n^i_{f,v}}$ symmetric matrices defined as:

$$\begin{aligned}
\overline{\mathbf{J}}^i_{1,1} &= \sum_{j=1}^{n_e^i}\overline{\mathbf{J}}^{i,j}_{1,1} = \\
&= \sum_{j=1}^{n_e^i}\left(\mathbf{B}_c^{i,j\,T}\overline{\mathbf{C}}^{i,j\,T}\left(\overline{\mathbf{S}}^{i,j}_{2,2} + \overline{\mathbf{S}}^{i,j}_{3,3}\right)\overline{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j}\right)
\end{aligned} \tag{2.392}$$

$$
\begin{aligned}
\bar{\mathbf{J}}^{i}_{1,2} &= \sum_{j=1}^{n_e^i} \bar{\mathbf{J}}^{i,j}_{1,2} = \\
&= -\sum_{j=1}^{n_e^i} \left(\mathbf{B}_c^{i,j\,T} \bar{\mathbf{C}}^{i,j\,T} \bar{\mathbf{S}}^{i,j}_{1,2} \bar{\mathbf{C}}^{i,j} \mathbf{B}_c^{i,j} \right) = \\
&= \bar{\mathbf{J}}^{i\,T}_{2,1}
\end{aligned}
\tag{2.393}
$$

$$
\begin{aligned}
\bar{\mathbf{J}}^{i}_{1,3} &= \sum_{j=1}^{n_e^i} \bar{\mathbf{J}}^{i,j}_{1,3} = \\
&= -\sum_{j=1}^{n_e^i} \left(\mathbf{B}_c^{i,j\,T} \bar{\mathbf{C}}^{i,j\,T} \bar{\mathbf{S}}^{i,j}_{1,3} \bar{\mathbf{C}}^{i,j} \mathbf{B}_c^{i,j} \right) = \\
&= \bar{\mathbf{J}}^{i\,T}_{3,1}
\end{aligned}
\tag{2.394}
$$

$$
\begin{aligned}
\bar{\mathbf{J}}^{i}_{2,2} &= \sum_{j=1}^{n_e^i} \bar{\mathbf{J}}^{i,j}_{2,2} = \\
&= \sum_{j=1}^{n_e^i} \left(\mathbf{B}_c^{i,j\,T} \bar{\mathbf{C}}^{i,j\,T} \left(\bar{\mathbf{S}}^{i,j}_{3,3} + \bar{\mathbf{S}}^{i,j}_{1,1} \right) \bar{\mathbf{C}}^{i,j} \mathbf{B}_c^{i,j} \right)
\end{aligned}
\tag{2.395}
$$

$$
\begin{aligned}
\bar{\mathbf{J}}^{i}_{2,3} &= \sum_{j=1}^{n_e^i} \bar{\mathbf{J}}^{i,j}_{2,3} = \\
&= -\sum_{j=1}^{n_e^i} \left(\mathbf{B}_c^{i,j\,T} \bar{\mathbf{C}}^{i,j\,T} \bar{\mathbf{S}}^{i,j}_{2,3} \bar{\mathbf{C}}^{i,j} \mathbf{B}_c^{i,j} \right) = \\
&= \bar{\mathbf{J}}^{i\,T}_{3,2}
\end{aligned}
\tag{2.396}
$$

$$
\begin{aligned}
\bar{\mathbf{J}}^{i}_{3,3} &= \sum_{j=1}^{n_e^i} \bar{\mathbf{J}}^{i,j}_{3,3} = \\
&= \sum_{j=1}^{n_e^i} \left(\mathbf{B}_c^{i,j\,T} \bar{\mathbf{C}}^{i,j\,T} \left(\bar{\mathbf{S}}^{i,j}_{1,1} + \bar{\mathbf{S}}^{i,j}_{2,2} \right) \bar{\mathbf{C}}^{i,j} \mathbf{B}_c^{i,j} \right)
\end{aligned}
\tag{2.397}
$$

The mass submatrix $\mathbf{m}^i_{\theta,q_f}(t)$ can be computed as:

$$\begin{aligned}\mathbf{m}^i_{\theta,q_f}(t) &= \sum_{j=1}^{n_e^i} \mathbf{m}^{i,j}_{\theta,q_f}(t) = \\ &= \sum_{j=1}^{n_e^i} \left(\bar{\mathbf{G}}^{iT}(t)\mathbf{H}^{i,j}(t)\mathbf{B}^i_e \right) = \\ &= \bar{\mathbf{G}}^{iT}(t) \left(\sum_{j=1}^{n_e} \mathbf{H}^{i,j}(t) \right) \mathbf{B}^i_e = \\ &= \bar{\mathbf{G}}^{iT}(t)\mathbf{H}^i(t)\mathbf{B}^i_e\end{aligned} \tag{2.398}$$

Where $\mathbf{H}^i(t)$ is a $\mathbb{R}^{3\times n^i_{f,v}}$ matrix defined as:

$$\mathbf{H}^i(t)=\sum_{j=1}^{n_e^i}\mathbf{H}^{i,j}(t)=$$
$$=\begin{bmatrix}\sum_{j=1}^{n_e^i}\left(\left(\mathbf{q}_o^i+\mathbf{B}_e^i\mathbf{q}_f^i(t)\right)^T\left(\overline{\mathbf{J}}_{3,2}^{i,j}-\overline{\mathbf{J}}_{2,3}^{i,j}\right)\right)\\ \sum_{j=1}^{n_e^i}\left(\left(\mathbf{q}_o^i+\mathbf{B}_e^i\mathbf{q}_f^i(t)\right)^T\left(\overline{\mathbf{J}}_{1,3}^{i,j}-\overline{\mathbf{J}}_{3,1}^{i,j}\right)\right)\\ \sum_{j=1}^{n_e^i}\left(\left(\mathbf{q}_o^i+\mathbf{B}_e^i\mathbf{q}_f^i(t)\right)^T\left(\overline{\mathbf{J}}_{2,1}^{i,j}-\overline{\mathbf{J}}_{1,2}^{i,j}\right)\right)\end{bmatrix}=$$
$$=\begin{bmatrix}\left(\mathbf{q}_o^i+\mathbf{B}_e^i\mathbf{q}_f^i(t)\right)^T\sum_{j=1}^{n_e^i}\left(\overline{\mathbf{J}}_{3,2}^{i,j}-\overline{\mathbf{J}}_{2,3}^{i,j}\right)\\ \left(\mathbf{q}_o^i+\mathbf{B}_e^i\mathbf{q}_f^i(t)\right)^T\sum_{j=1}^{n_e^i}\left(\overline{\mathbf{J}}_{1,3}^{i,j}-\overline{\mathbf{J}}_{3,1}^{i,j}\right)\\ \left(\mathbf{q}_o^i+\mathbf{B}_e^i\mathbf{q}_f^i(t)\right)^T\sum_{j=1}^{n_e^i}\left(\overline{\mathbf{J}}_{2,1}^{i,j}-\overline{\mathbf{J}}_{1,2}^{i,j}\right)\end{bmatrix}=$$
$$=\begin{bmatrix}\mathbf{q}_n^{iT}(t)\left(\overline{\mathbf{J}}_{3,2}^{i}-\overline{\mathbf{J}}_{2,3}^{i}\right)\\ \mathbf{q}_n^{iT}(t)\left(\overline{\mathbf{J}}_{1,3}^{i}-\overline{\mathbf{J}}_{3,1}^{i}\right)\\ \mathbf{q}_n^{iT}(t)\left(\overline{\mathbf{J}}_{2,1}^{i}-\overline{\mathbf{J}}_{1,2}^{i}\right)\end{bmatrix} \tag{2.399}$$

The mass submatrix $\mathbf{m}_{q_f,q_f}^i(t)$ can be computed as:

$$\begin{aligned}
\mathbf{m}^i_{q_f,q_f}(t) = \sum_{j=1}^{n_e^i} \mathbf{m}^{i,j}_{q_f,q_f}(t) = \\
= \sum_{j=1}^{n_e^i} \left(\mathbf{B}_e^{iT} \overline{\mathbf{J}}^{i,j}_{f,f} \mathbf{B}_e^i \right) = \\
= \mathbf{B}_e^{iT} \left(\sum_{j=1}^{n_e^i} \overline{\mathbf{J}}^{i,j}_{f,f} \right) \mathbf{B}_e^i = \\
= \mathbf{B}_e^{iT} \overline{\mathbf{J}}^i_{f,f} \mathbf{B}_e^i
\end{aligned} \tag{2.400}$$

Where $\overline{\mathbf{J}}^i_{f,f}$ is a $\mathbb{R}^{n^i_{f,v} \times n^i_{f,v}}$ symmetric matrix defined as:

$$\begin{aligned}
\overline{\mathbf{J}}^i_{f,f} = \sum_{j=1}^{n_e^i} \overline{\mathbf{J}}^{i,j}_{f,f} = \\
= \frac{1}{2} \sum_{j=1}^{n_e^i} \left(\overline{\mathbf{J}}^{i,j}_{1,1} + \overline{\mathbf{J}}^{i,j}_{2,2} + \overline{\mathbf{J}}^{i,j}_{3,3} \right) = \\
= \sum_{j=1}^{n_e^i} \left(\mathbf{B}_c^{i,j\,T} \overline{\mathbf{C}}^{i,j\,T} \overline{\mathbf{S}}^{i,j}_{f,f} \overline{\mathbf{C}}^{i,j} \mathbf{B}_c^{i,j} \right) = \\
= \sum_{j=1}^{n_e^i} \left(\mathbf{B}_c^{i,j\,T} \overline{\mathbf{C}}^{i,j\,T} \left(\overline{\mathbf{S}}^{i,j}_{1,1} + \overline{\mathbf{S}}^{i,j}_{2,2} + \overline{\mathbf{S}}^{i,j}_{3,3} \right) \overline{\mathbf{C}}^{i,j} \mathbf{B}_c^{i,j} \right)
\end{aligned} \tag{2.401}$$

These are the mathematical expressions that allow to compute explicitly all mass submatrix $\mathbf{m}^i_{R,R}(t)$, $\mathbf{m}^i_{R,\theta}(t)$, $\mathbf{m}^i_{R,q_f}(t)$, $\mathbf{m}^i_{\theta,\theta}(t)$, $\mathbf{m}^i_{\theta,q_f}(t)$ and $\mathbf{m}^i_{q_f,q_f}(t)$ relative to the mass matrix $\mathbf{M}^i(t)$ of the flexible body i. Indeed, the kinetic energy $T^i(t)$ corresponding to the flexible body i can be expressed by using the mass submatrices to yield:

$$T^i(t) = \frac{1}{2}\dot{\mathbf{q}}^{iT}(t)\mathbf{M}^i(t)\dot{\mathbf{q}}^i(t) =$$

$$= \frac{1}{2}\begin{bmatrix}\dot{\mathbf{R}}^{iT}(t) & \dot{\boldsymbol{\theta}}^{iT}(t) & \dot{\mathbf{q}}_f^{iT}(t)\end{bmatrix}\begin{bmatrix}\mathbf{m}^i_{R,R}(t) & \mathbf{m}^i_{R,\theta}(t) & \mathbf{m}^i_{R,q_f}(t) \\ \mathbf{m}^i_{\theta,R}(t) & \mathbf{m}^i_{\theta,\theta}(t) & \mathbf{m}^i_{\theta,q_f}(t) \\ \mathbf{m}^i_{q_f,R}(t) & \mathbf{m}^i_{q_f,\theta}(t) & \mathbf{m}^i_{q_f,q_f}(t)\end{bmatrix}\begin{bmatrix}\dot{\mathbf{R}}^i(t) \\ \dot{\boldsymbol{\theta}}^i(t) \\ \dot{\mathbf{q}}_f^i(t)\end{bmatrix} =$$

$$= \frac{1}{2}\dot{\mathbf{R}}^{iT}(t)\mathbf{m}^i_{R,R}(t)\dot{\mathbf{R}}^i(t) + \dot{\mathbf{R}}^{iT}(t)\mathbf{m}^i_{R,\theta}(t)\dot{\boldsymbol{\theta}}^i(t) + \dot{\mathbf{R}}^{iT}(t)\mathbf{m}^i_{R,q_f}(t)\dot{\mathbf{q}}_f^i(t) +$$
$$+ \frac{1}{2}\dot{\boldsymbol{\theta}}^{iT}(t)\mathbf{m}^i_{\theta,\theta}(t)\dot{\boldsymbol{\theta}}^i(t) + \dot{\boldsymbol{\theta}}^{iT}(t)\mathbf{m}^i_{\theta,q_f}(t)\dot{\mathbf{q}}_f^i(t) + \frac{1}{2}\dot{\mathbf{q}}_f^{iT}(t)\mathbf{m}^i_{q_f,q_f}(t)\dot{\mathbf{q}}_f^i(t)$$
(2.402)

This expression can be used to compute the quadratic velocity vector.

2.7.5. STIFFNESS MATRIX OF SYSTEM ELEMENTS

The formulation of the stiffness matrix of the element j of the flexible body i can be achieved by the definition of the elastic strain energy $U^{i,j}(t)$ of the same element. First, note that only the elastic coordinates $\mathbf{q}_f^i(t)$ are involved in the computation of the strain energy $U^{i,j}(t)$. Indeed, the elastic coordinate vector $\mathbf{q}_f^i(t)$ can be simply recovered from the configuration coordinate vector $\mathbf{q}^i(t)$ by using a Boolean matrix as follows:

$$\mathbf{q}_f^i(t) = \begin{bmatrix}\mathbf{O}_{n_f^i,3} & \mathbf{O}_{n_f^i,4} & \mathbf{I}_{n_f^i,n_f^i}\end{bmatrix}\begin{bmatrix}\mathbf{R}^i(t) \\ \boldsymbol{\theta}^i(t) \\ \mathbf{q}_f^i(t)\end{bmatrix} = \qquad (2.403)$$
$$= \mathbf{B}_f^i\mathbf{q}^i(t)$$

Where $\mathbf{B}_f^i$ is a $\mathbb{R}^{n_f^i \times n^i}$ Boolean matrix defined as:

$$\mathbf{B}_f^i = \begin{bmatrix} \mathbf{O}_{n_f^i,3} & \mathbf{O}_{n_f^i,4} & \mathbf{I}_{n_f^i,n_f^i} \end{bmatrix} \tag{2.404}$$

Therefore, the elastic displacement field can be written as:

$$\begin{aligned} \bar{\mathbf{u}}_f^{i,j}(P^{i,j},t) &= \mathbf{N}^{i,j}(P^{i,j})\mathbf{B}_e^i\mathbf{q}_f^i(t) = \\ &= \mathbf{N}^{i,j}(P^{i,j})\mathbf{B}_e^i\mathbf{B}_f^i\mathbf{q}^i(t) \end{aligned} \tag{2.405}$$

To compute the elastic strain energy $U^{i,j}(t)$ it is preliminary required to obtain an expression of element stress field and deformation field based on configuration coordinates. Assuming Voigt notation, the deformation field can be computed in a matrix form by using the linear strain-displacement equations to yield:

$$\begin{aligned} \boldsymbol{\varepsilon}^{i,j}(P^{i,j},t) &= \mathbf{D}^{i,j}\bar{\mathbf{u}}_f^{i,j}(P^{i,j},t) = \\ &= \mathbf{D}^{i,j}\mathbf{N}^{i,j}(P^{i,j})\mathbf{B}_e^i\mathbf{B}_f^i\mathbf{q}^i(t) \end{aligned} \tag{2.406}$$

Where $\boldsymbol{\varepsilon}^{i,j}(P^{i,j},t)$ is a $\mathbb{R}^6$ vector representing the deformation field of element j of flexible body i and $\mathbf{D}^{i,j}$ is a $\mathbb{R}^{6\times3}$ differential matrix operator. Similarly to the case of classic finite element formulation, the result of the differential matrix operator $\mathbf{D}^{i,j}$ on the element compact shape function $\mathbf{N}^{i,j}(P^{i,j})$ can be explicitly developed to yield:

$$\mathbf{D}^{i,j}\mathbf{N}^{i,j}(P^{i,j}) = \left(\bar{\bar{\mathbf{C}}}_x^{i,j}\mathbf{S}_x^{i,j}(P^{i,j}) + \bar{\bar{\mathbf{C}}}_y^{i,j}\mathbf{S}_y^{i,j}(P^{i,j}) + \bar{\bar{\mathbf{C}}}_z^{i,j}\mathbf{S}_z^{i,j}(P^{i,j})\right)\bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j} \tag{2.407}$$

Where $\bar{\bar{\mathbf{C}}}_x^{i,j}$, $\bar{\bar{\mathbf{C}}}_y^{i,j}$ and $\bar{\bar{\mathbf{C}}}_z^{i,j}$ are $\mathbb{R}^{6\times3}$ matrices whose components are the components of the rotation matrix $\mathbf{C}^{i,j}$ and $\mathbf{S}_x^{i,j}(P^{i,j})$, $\mathbf{S}_y^{i,j}(P^{i,j})$ and $\mathbf{S}_z^{i,j}(P^{i,j})$ are simply the space derivative of the shape function $\mathbf{S}^{i,j}(P^{i,j})$. On the other hand, the stress field can be computed through the constitutive equations of the homogeneous isotropic linear elastic material according to Voigt notation to yield:

$$\begin{aligned}\boldsymbol{\sigma}^{i,j}(P^{i,j},t) &= \mathbf{E}^{i,j}\boldsymbol{\varepsilon}^{i,j}(P^{i,j},t) = \\ &= \mathbf{E}^{i,j}\mathbf{N}^{i,j}(P^{i,j})\mathbf{B}_e^i\mathbf{B}_f^i\mathbf{q}^i(t)\end{aligned} \tag{2.408}$$

Where $\boldsymbol{\sigma}^{i,j}(P^{i,j},t)$ is a $\mathbb{R}^6$ vector representing the stress field of element j of body i and $\mathbf{E}^{i,j}$ is the $\mathbb{R}^{6\times6}$ matrix of elastic coefficients of the same element. At this stage, the stiffness matrix of element j of body i can be computed by using the definition of the strain energy $U^{i,j}(t)$. Indeed:

$$\begin{aligned}&U^{i,j}(t) = \frac{1}{2}\int_{\Omega^{i,j}} \boldsymbol{\sigma}^{i,jT}(P^{i,j},t)\boldsymbol{\varepsilon}^{i,j}(P^{i,j},t)dV^{i,j} = \\ &= \frac{1}{2}\int_{\Omega^{i,j}} \mathbf{q}^{iT}(t)\mathbf{B}_f^{iT}\mathbf{B}_e^{iT}\left(\mathbf{D}^{i,j}\mathbf{N}^{i,j}(P^{i,j})\right)^T \mathbf{E}^{i,j}\mathbf{D}^{i,j}\mathbf{N}^{i,j}(P^{i,j})\mathbf{B}_e^i\mathbf{B}_f^i\mathbf{q}^i(t)dV^{i,j} = \\ &= \frac{1}{2}\mathbf{q}^{iT}(t)\mathbf{B}_f^{iT}\mathbf{B}_e^{iT}\int_{\Omega^{i,j}}\left(\mathbf{D}^{i,j}\mathbf{N}^{i,j}(P^{i,j})\right)^T \mathbf{E}^{i,j}\mathbf{D}^{i,j}\mathbf{N}^{i,j}(P^{i,j})dV^{i,j}\mathbf{B}_e^i\mathbf{B}_f^i\mathbf{q}^i(t) = \\ &= \frac{1}{2}\mathbf{q}^{iT}(t)\mathbf{B}_f^{iT}\mathbf{B}_e^{iT}\bar{\mathbf{V}}_{f,f}^{i,j}\mathbf{B}_e^i\mathbf{B}_f^i\mathbf{q}^i(t) = \\ &= \frac{1}{2}\mathbf{q}^{iT}(t)\mathbf{K}^{i,j}\mathbf{q}^i(t)\end{aligned} \tag{2.409}$$

Therefore, the final expression of the stiffness matrix $\mathbf{K}^{i,j}$ of the element j of body i is:

$$\mathbf{K}^{i,j} = \mathbf{B}_f^{iT}\mathbf{B}_e^{iT}\bar{\mathbf{V}}_{f,f}^{i,j}\mathbf{B}_e^i\mathbf{B}_f^i \tag{2.410}$$

This matrix is a $\mathbb{R}^{n_f^i\times n_f^i}$ matrix whereas $\bar{\mathbf{V}}_{f,f}^{i,j}$ is a $\mathbb{R}^{n_{f,v}^i\times n_{f,v}^i}$ matrix can be computed as:

$$\begin{aligned}\bar{\mathbf{V}}_{f,f}^{i,j} &= \int_{\Omega^{i,j}} \left(\mathbf{D}^{i,j}\mathbf{N}^{i,j}(P^{i,j})\right)^T \mathbf{E}^{i,j}\mathbf{D}^{i,j}\mathbf{N}^{i,j}(P^{i,j})dV^{i,j} = \\ &= \bar{\bar{\mathbf{J}}}_{f,f}^{i,j} + \bar{\bar{\mathbf{J}}}_{f,1}^{i,j} + \bar{\bar{\mathbf{J}}}_{f,2}^{i,j} + \bar{\bar{\mathbf{J}}}_{f,3}^{i,j}\end{aligned} \tag{2.411}$$

Where $\bar{\bar{\mathbf{J}}}_{f,f}^{i,j}$, $\bar{\bar{\mathbf{J}}}_{f,1}^{i,j}$, $\bar{\bar{\mathbf{J}}}_{f,2}^{i,j}$ and $\bar{\bar{\mathbf{J}}}_{f,3}^{i,j}$ are $\mathbb{R}^{n_{f,v}^i \times n_{f,v}^i}$ symmetric matrices defined as:

$$\bar{\bar{\mathbf{J}}}_{f,f}^{i,j} = \mathbf{B}_c^{i,j\,T}\bar{\mathbf{C}}^{i,j\,T}\bar{\bar{\mathbf{S}}}_{f,f}^{i,j}\bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j} \tag{2.412}$$

$$\bar{\bar{\mathbf{J}}}_{f,1}^{i,j} = \mathbf{B}_c^{i,j\,T}\bar{\mathbf{C}}^{i,j\,T}\bar{\bar{\mathbf{S}}}_{f,1}^{i,j}\bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j} \tag{2.413}$$

$$\bar{\bar{\mathbf{J}}}_{f,2}^{i,j} = \mathbf{B}_c^{i,j\,T}\bar{\mathbf{C}}^{i,j\,T}\bar{\bar{\mathbf{S}}}_{f,2}^{i,j}\bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j} \tag{2.414}$$

$$\bar{\bar{\mathbf{J}}}_{f,3}^{i,j} = \mathbf{B}_c^{i,j\,T}\bar{\mathbf{C}}^{i,j\,T}\bar{\bar{\mathbf{S}}}_{f,3}^{i,j}\bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j} \tag{2.415}$$

Where $\bar{\bar{\mathbf{S}}}_{f,f}^{i,j}$, $\bar{\bar{\mathbf{S}}}_{f,1}^{i,j}$, $\bar{\bar{\mathbf{S}}}_{f,2}^{i,j}$ and $\bar{\bar{\mathbf{S}}}_{f,3}^{i,j}$ are $\mathbb{R}^{n_f^{i,j} \times n_f^{i,j}}$ symmetric matrices defined as follows:

$$\bar{\bar{\mathbf{S}}}_{f,f}^{i,j} = \bar{\bar{\mathbf{S}}}_{1,1}^{i,j} + \bar{\bar{\mathbf{S}}}_{2,2}^{i,j} + \bar{\bar{\mathbf{S}}}_{3,3}^{i,j} \tag{2.416}$$

$$\bar{\bar{\mathbf{S}}}_{f,1}^{i,j} = \bar{\bar{\mathbf{S}}}_{2,3}^{i,j} + \bar{\bar{\mathbf{S}}}_{3,2}^{i,j} \tag{2.417}$$

$$\bar{\bar{\mathbf{S}}}_{f,2}^{i,j} = \bar{\bar{\mathbf{S}}}_{3,1}^{i,j} + \bar{\bar{\mathbf{S}}}_{1,3}^{i,j} \tag{2.418}$$

$$\bar{\bar{\mathbf{S}}}_{f,3}^{i,j} = \bar{\bar{\mathbf{S}}}_{1,2}^{i,j} + \bar{\bar{\mathbf{S}}}_{2,1}^{i,j} \tag{2.419}$$

Where $\bar{\bar{\mathbf{S}}}_{1,1}^{i,j}$, $\bar{\bar{\mathbf{S}}}_{1,2}^{i,j}$, $\bar{\bar{\mathbf{S}}}_{1,3}^{i,j}$, $\bar{\bar{\mathbf{S}}}_{2,1}^{i,j}$, $\bar{\bar{\mathbf{S}}}_{2,2}^{i,j}$, $\bar{\bar{\mathbf{S}}}_{2,3}^{i,j}$, $\bar{\bar{\mathbf{S}}}_{3,1}^{i,j}$, $\bar{\bar{\mathbf{S}}}_{3,2}^{i,j}$ and $\bar{\bar{\mathbf{S}}}_{3,3}^{i,j}$ are $\mathbb{R}^{n_f^{i,j} \times n_f^{i,j}}$ matrices defined as:

$$\overline{\overline{\mathbf{S}}}_{1,1}^{i,j} = \int_{\Omega^{i,j}} \mathbf{S}_x^{i,j\,T}(P^{i,j}) \overline{\overline{\mathbf{C}}}_x^{i,j\,T} \mathbf{E}^{i,j} \overline{\overline{\mathbf{C}}}_x^{i,j} \mathbf{S}_x^{i,j}(P^{i,j}) dV^{i,j} \tag{2.420}$$

$$\overline{\overline{\mathbf{S}}}_{1,2}^{i,j} = \int_{\Omega^{i,j}} \mathbf{S}_x^{i,j\,T}(P^{i,j}) \overline{\overline{\mathbf{C}}}_x^{i,j\,T} \mathbf{E}^{i,j} \overline{\overline{\mathbf{C}}}_y^{i,j} \mathbf{S}_y^{i,j}(P^{i,j}) dV^{i,j} \tag{2.421}$$

$$\overline{\overline{\mathbf{S}}}_{1,3}^{i,j} = \int_{\Omega^{i,j}} \mathbf{S}_x^{i,j\,T}(P^{i,j}) \overline{\overline{\mathbf{C}}}_x^{i,j\,T} \mathbf{E}^{i,j} \overline{\overline{\mathbf{C}}}_z^{i,j} \mathbf{S}_z^{i,j}(P^{i,j}) dV^{i,j} \tag{2.422}$$

$$\overline{\overline{\mathbf{S}}}_{2,1}^{i,j} = \int_{\Omega^{i,j}} \mathbf{S}_y^{i,j\,T}(P^{i,j}) \overline{\overline{\mathbf{C}}}_y^{i,j\,T} \mathbf{E}^{i,j} \overline{\overline{\mathbf{C}}}_x^{i,j} \mathbf{S}_x^{i,j}(P^{i,j}) dV^{i,j} \tag{2.423}$$

$$\overline{\overline{\mathbf{S}}}_{2,2}^{i,j} = \int_{\Omega^{i,j}} \mathbf{S}_y^{i,j\,T}(P^{i,j}) \overline{\overline{\mathbf{C}}}_y^{i,j\,T} \mathbf{E}^{i,j} \overline{\overline{\mathbf{C}}}_y^{i,j} \mathbf{S}_y^{i,j}(P^{i,j}) dV^{i,j} \tag{2.424}$$

$$\overline{\overline{\mathbf{S}}}_{2,3}^{i,j} = \int_{\Omega^{i,j}} \mathbf{S}_y^{i,j\,T}(P^{i,j}) \overline{\overline{\mathbf{C}}}_y^{i,j\,T} \mathbf{E}^{i,j} \overline{\overline{\mathbf{C}}}_z^{i,j} \mathbf{S}_z^{i,j}(P^{i,j}) dV^{i,j} \tag{2.425}$$

$$\overline{\overline{\mathbf{S}}}_{3,1}^{i,j} = \int_{\Omega^{i,j}} \mathbf{S}_z^{i,j\,T}(P^{i,j}) \overline{\overline{\mathbf{C}}}_z^{i,j\,T} \mathbf{E}^{i,j} \overline{\overline{\mathbf{C}}}_x^{i,j} \mathbf{S}_x^{i,j}(P^{i,j}) dV^{i,j} \tag{2.426}$$

$$\overline{\overline{\mathbf{S}}}_{3,2}^{i,j} = \int_{\Omega^{i,j}} \mathbf{S}_z^{i,j\,T}(P^{i,j}) \overline{\overline{\mathbf{C}}}_z^{i,j\,T} \mathbf{E}^{i,j} \overline{\overline{\mathbf{C}}}_y^{i,j} \mathbf{S}_y^{i,j}(P^{i,j}) dV^{i,j} \tag{2.427}$$

$$\overline{\overline{\mathbf{S}}}_{3,3}^{i,j} = \int_{\Omega^{i,j}} \mathbf{S}_z^{i,j\,T}(P^{i,j}) \overline{\overline{\mathbf{C}}}_z^{i,j\,T} \mathbf{E}^{i,j} \overline{\overline{\mathbf{C}}}_z^{i,j} \mathbf{S}_z^{i,j}(P^{i,j}) dV^{i,j} \tag{2.428}$$

Where $\overline{\overline{\mathbf{C}}}_x^{i,j}$, $\overline{\overline{\mathbf{C}}}_y^{i,j}$ and $\overline{\overline{\mathbf{C}}}_z^{i,j}$ are $\mathbb{R}^{6\times 3}$ matrices whose components are the components of the rotation matrix $\mathbf{C}^{i,j}$ and $\mathbf{S}_x^{i,j}(P^{i,j})$, $\mathbf{S}_y^{i,j}(P^{i,j})$ and $\mathbf{S}_z^{i,j}(P^{i,j})$ are simply the space derivative of the shape function $\mathbf{S}^{i,j}(P^{i,j})$. These matrices are the elastic shape integrals required to compute explicitly the mass matrix $\mathbf{K}^{i,j}$ of the flexible element j of the body i of the system. Indeed, the fourth set of integrals required to write the equations of motion of flexible multibody systems is the following set of elastic shape integrals:

$$\begin{aligned}
\mathbf{I}^{(4)\,i,j}_{h,k} &= \bar{\bar{\mathbf{S}}}^{i,j}_{h,k} = \\
&= \int_{\Omega^{i,j}} \mathbf{S}^{i,j\,T}_{x^h}(P^{i,j}) \bar{\bar{\mathbf{C}}}^{i,j\,T}_h \mathbf{E}^{i,j} \bar{\bar{\mathbf{C}}}^{i,j}_k \mathbf{S}^{i,j}_{x^k}(P^{i,j}) dV^{i,j} = \\
&= \int_{\Omega^{i,j}} \left(\frac{\partial \mathbf{S}^{i,j\,T}(P^{i,j})}{\partial x^h} \right) \bar{\bar{\mathbf{C}}}^{i,j\,T}_h \mathbf{E}^{i,j} \bar{\bar{\mathbf{C}}}^{i,j}_k \left(\frac{\partial \mathbf{S}^{i,j}(P^{i,j})}{\partial x^k} \right) dV^{i,j}
\end{aligned} \tag{2.429}$$

These sets of elastic shape integrals can be computed in advance and then they can be used to write the equations of motion of flexible multibody systems.

2.7.6. STIFFNESS MATRIX OF SYSTEM BODIES

The stiffness matrix of the generic flexible body i can be computed using the additive property of stain energy $U^i(t)$. Indeed:

$$\begin{aligned}
U^i(t) &= \sum_{j=1}^{n_e^i} U^{i,j}(t) = \\
&= \sum_{j=1}^{n_e^i} \left(\frac{1}{2} \mathbf{q}^{i\,T}(t) \mathbf{K}^{i,j} \mathbf{q}^i(t) \right) = \\
&= \frac{1}{2} \mathbf{q}^{i\,T}(t) \left(\sum_{j=1}^{n_e^i} \mathbf{K}^{i,j} \right) \mathbf{q}^i(t) = \\
&= \frac{1}{2} \mathbf{q}^{i\,T}(t) \mathbf{K}^i \mathbf{q}^i(t)
\end{aligned} \tag{2.430}$$

Where $\mathbf{K}^i$ is a $\mathbb{R}^{n^i \times n^i}$ matrix representing the stiffness matrix of flexible body i defined as:

$$\begin{aligned}
\mathbf{K}^{i} &= \sum_{j=1}^{n_e^i} \mathbf{K}^{i,j} = \\
&= \sum_{j=1}^{n_e^i} \left(\mathbf{B}_f^{i\,T} \mathbf{B}_e^{i\,T} \bar{\mathbf{V}}_{f,f}^{i,j} \mathbf{B}_e^{i} \mathbf{B}_f^{i} \right) = \\
&= \mathbf{B}_f^{i\,T} \mathbf{B}_e^{i\,T} \left(\sum_{j=1}^{n_e^i} \bar{\mathbf{V}}_{f,f}^{i,j} \right) \mathbf{B}_e^{i} \mathbf{B}_f^{i} = \\
&= \mathbf{B}_f^{i\,T} \mathbf{B}_e^{i\,T} \bar{\mathbf{V}}_{f,f}^{i} \mathbf{B}_e^{i} \mathbf{B}_f^{i}
\end{aligned} \tag{2.431}$$

Where $\bar{\mathbf{V}}_{f,f}^{i}$ is a $\mathbb{R}^{n_f^i \times n_f^i}$ symmetric matrix and it can be computed as follows:

$$\begin{aligned}
\bar{\mathbf{V}}_{f,f}^{i} &= \sum_{j=1}^{n_e^i} \bar{\mathbf{V}}_{f,f}^{i,j} = \\
&= \sum_{j=1}^{n_e^i} \left(\bar{\bar{\mathbf{J}}}_{f,f}^{i,j} + \bar{\bar{\mathbf{J}}}_{f,1}^{i,j} + \bar{\bar{\mathbf{J}}}_{f,2}^{i,j} + \bar{\bar{\mathbf{J}}}_{f,3}^{i,j} \right) = \\
&= \bar{\bar{\mathbf{J}}}_{f,f}^{i} + \bar{\bar{\mathbf{J}}}_{f,1}^{i} + \bar{\bar{\mathbf{J}}}_{f,2}^{i} + \bar{\bar{\mathbf{J}}}_{f,3}^{i}
\end{aligned} \tag{2.432}$$

Where $\bar{\bar{\mathbf{J}}}_{f,f}^{i}$, $\bar{\bar{\mathbf{J}}}_{f,1}^{i}$, $\bar{\bar{\mathbf{J}}}_{f,2}^{i}$ and $\bar{\bar{\mathbf{J}}}_{f,3}^{i}$ are $\mathbb{R}^{n_{f,v}^i \times n_{f,v}^i}$ symmetric matrices defined as:

$$\begin{aligned}
\bar{\bar{\mathbf{J}}}_{f,f}^{i} &= \sum_{j=1}^{n_e^i} \bar{\bar{\mathbf{J}}}_{f,f}^{i,j} = \\
&= \sum_{j=1}^{n_e^i} \left(\mathbf{B}_c^{i,j\,T} \bar{\mathbf{C}}^{i,j\,T} \bar{\bar{\mathbf{S}}}_{f,f}^{i,j} \bar{\mathbf{C}}^{i,j} \mathbf{B}_c^{i,j} \right)
\end{aligned} \tag{2.433}$$

$$\bar{\bar{\mathbf{J}}}^{i}_{f,1} = \sum_{j=1}^{n_e^i} \bar{\bar{\mathbf{J}}}^{i,j}_{f,1} = = \sum_{j=1}^{n_e^i} \left(\mathbf{B}_c^{i,j\,T} \bar{\mathbf{C}}^{i,j\,T} \bar{\bar{\mathbf{S}}}^{i,j}_{f,1} \bar{\mathbf{C}}^{i,j} \mathbf{B}_c^{i,j} \right) \tag{2.434}$$

$$\bar{\bar{\mathbf{J}}}^{i}_{f,2} = \sum_{j=1}^{n_e^i} \bar{\bar{\mathbf{J}}}^{i,j}_{f,2} = = \sum_{j=1}^{n_e^i} \left(\mathbf{B}_c^{i,j\,T} \bar{\mathbf{C}}^{i,j\,T} \bar{\bar{\mathbf{S}}}^{i,j}_{f,2} \bar{\mathbf{C}}^{i,j} \mathbf{B}_c^{i,j} \right) \tag{2.435}$$

$$\bar{\bar{\mathbf{J}}}^{i}_{f,3} = \sum_{j=1}^{n_e^i} \bar{\bar{\mathbf{J}}}^{i,j}_{f,3} = = \sum_{j=1}^{n_e^i} \left(\mathbf{B}_c^{i,j\,T} \bar{\mathbf{C}}^{i,j\,T} \bar{\bar{\mathbf{S}}}^{i,j}_{f,3} \bar{\mathbf{C}}^{i,j} \mathbf{B}_c^{i,j} \right) \tag{2.436}$$

These are the mathematical expressions that allow to compute explicitly the stiffness matrix $\mathbf{K}^i$ of the flexible body i. Note that according the floating frame of reference formulation the stiffness matrix $\mathbf{K}^i$ of a generic flexible body i is not a function of time.

2.7.7. QUADRATIC VELOCITY VECTOR

The next step to derive the equations of motion of flexible multibody systems is to compute the quadratic velocity vector. This vector is defined as:

$$\mathbf{Q}_v^i(t) = \left(\frac{\partial T^i(t)}{\partial \mathbf{q}^i(t)} \right)^T - \dot{\mathbf{M}}^i(t)\dot{\mathbf{q}}^i(t) \tag{2.437}$$

To compute the first term on the right hand side is necessary to evaluate the time derivative of mass matrix $\mathbf{M}^i(t)$. This yields to:

$$\dot{\mathbf{M}}^i(t) = \begin{bmatrix} \dot{\mathbf{m}}^i_{R,R}(t) & \dot{\mathbf{m}}^i_{R,\theta}(t) & \dot{\mathbf{m}}^i_{R,q_f}(t) \\ \dot{\mathbf{m}}^i_{\theta,R}(t) & \dot{\mathbf{m}}^i_{\theta,\theta}(t) & \dot{\mathbf{m}}^i_{\theta,q_f}(t) \\ \dot{\mathbf{m}}^i_{q_f,R}(t) & \dot{\mathbf{m}}^i_{q_f,\theta}(t) & \dot{\mathbf{m}}^i_{q_f,q_f}(t) \end{bmatrix} \tag{2.438}$$

The time derivative of mass submatrix $\mathbf{m}^i_{R,R}(t)$ yields to:

$$\begin{aligned} \dot{\mathbf{m}}^i_{R,R}(t) &= \frac{d}{dt}\left(m^i\mathbf{I}\right) = \\ &= \mathbf{O} \end{aligned} \tag{2.439}$$

The time derivative of mass submatrix $\mathbf{m}^i_{R,\theta}(t)$ can be computed as:

$$\begin{aligned} &\dot{\mathbf{m}}^i_{R,\theta}(t) = \frac{d}{dt}\left(-\mathbf{A}^i(t)\tilde{\bar{\mathbf{U}}}^i(t)\bar{\mathbf{G}}^i(t)\right) = \\ &= -\dot{\mathbf{A}}^i(t)\tilde{\bar{\mathbf{U}}}^i(t)\bar{\mathbf{G}}^i(t) - \mathbf{A}^i(t)\dot{\tilde{\bar{\mathbf{U}}}}^i(t)\bar{\mathbf{G}}^i(t) - \mathbf{A}^i(t)\tilde{\bar{\mathbf{U}}}^i(t)\dot{\bar{\mathbf{G}}}^i(t) = \\ &= -\mathbf{A}^i(t)\tilde{\bar{\boldsymbol{\omega}}}^i(t)\tilde{\bar{\mathbf{U}}}^i(t)\bar{\mathbf{G}}^i(t) - \mathbf{A}^i(t)\dot{\tilde{\bar{\mathbf{U}}}}^i(t)\bar{\mathbf{G}}^i(t) - \mathbf{A}^i(t)\tilde{\bar{\mathbf{U}}}^i(t)\dot{\bar{\mathbf{G}}}^i(t) = \\ &= -\frac{1}{2}\mathbf{A}^i(t)\bar{\mathbf{G}}^i(t)\dot{\bar{\mathbf{G}}}^{iT}(t)\tilde{\bar{\mathbf{U}}}^i(t)\bar{\mathbf{G}}^i(t) - \mathbf{A}^i(t)\dot{\tilde{\bar{\mathbf{U}}}}^i(t)\bar{\mathbf{G}}^i(t) - \mathbf{A}^i(t)\tilde{\bar{\mathbf{U}}}^i(t)\dot{\bar{\mathbf{G}}}^i(t) = \\ &= \dot{\mathbf{m}}^{iT}_{\theta,R}(t) \end{aligned} \tag{2.440}$$

Where the following matrix identity has been used:

$$\begin{aligned} \tilde{\bar{\boldsymbol{\omega}}}^i(t) &= \mathbf{A}^{iT}(t)\dot{\mathbf{A}}^i(t) = \\ &= 2\bar{\mathbf{E}}^i(t)\dot{\bar{\mathbf{E}}}^{iT}(t) = \\ &= -\frac{1}{2}\dot{\bar{\mathbf{G}}}^i(t)\bar{\mathbf{G}}^{iT}(t) = \\ &= \frac{1}{2}\bar{\mathbf{G}}^i(t)\dot{\bar{\mathbf{G}}}^{iT}(t) \end{aligned} \tag{2.441}$$

The time derivative of the matrix $\tilde{\bar{\mathbf{U}}}^i(t)$ can be computed as:

$$\begin{aligned}\dot{\tilde{\bar{\mathbf{U}}}}^i(t) &= \frac{d}{dt}\left(Skew\left(\bar{\mathbf{N}}^i\left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right)\right)\right) = \\ &= Skew\left(\bar{\mathbf{N}}^i\mathbf{B}_e^i\dot{\mathbf{q}}_f^i(t)\right)\end{aligned} \tag{2.442}$$

The time derivative of mass submatrix $\mathbf{m}_{R,q_f}^i(t)$ can be computed as:

$$\begin{aligned}\dot{\mathbf{m}}_{R,q_f}^i(t) &= \frac{d}{dt}\left(\mathbf{A}^i(t)\bar{\mathbf{N}}^i\mathbf{B}_e^i\right) = \\ &= \dot{\mathbf{A}}^i(t)\bar{\mathbf{N}}^i\mathbf{B}_e^i = \\ &= \mathbf{A}^i(t)\tilde{\bar{\boldsymbol{\omega}}}^i(t)\bar{\mathbf{N}}^i\mathbf{B}_e^i = \\ &= \frac{1}{2}\mathbf{A}^i(t)\bar{\mathbf{G}}^i(t)\dot{\bar{\mathbf{G}}}^{iT}(t)\bar{\mathbf{N}}^i\mathbf{B}_e^i = \\ &= \dot{\mathbf{m}}_{q_f,R}^{iT}(t)\end{aligned} \tag{2.443}$$

The time derivative of mass submatrix $\mathbf{m}_{\theta,\theta}^i(t)$ can be computed as:

$$\begin{aligned}\dot{\mathbf{m}}_{\theta,\theta}^i(t) &= \frac{d}{dt}\left(\bar{\mathbf{G}}^{iT}(t)\bar{\mathbf{I}}_{\theta,\theta}^i(t)\bar{\mathbf{G}}^i(t)\right) = \\ &= \dot{\bar{\mathbf{G}}}^{iT}(t)\bar{\mathbf{I}}_{\theta,\theta}^i\bar{\mathbf{G}}^i(t) + \bar{\mathbf{G}}^{iT}(t)\dot{\bar{\mathbf{I}}}_{\theta,\theta}^i(t)\bar{\mathbf{G}}^i(t) + \bar{\mathbf{G}}^{iT}(t)\bar{\mathbf{I}}_{\theta,\theta}^i\dot{\bar{\mathbf{G}}}^i(t)\end{aligned} \tag{2.444}$$

Where the time derivative of the inertia matrix $\bar{\mathbf{I}}_{\theta,\theta}^i(t)$ can be computed as:

$$\dot{\bar{\mathbf{I}}}^{i}_{\theta,\theta}(t)=\begin{bmatrix}\left(\dot{\bar{\mathbf{I}}}^{i}_{\theta,\theta}(t)\right)_{1,1} & \left(\dot{\bar{\mathbf{I}}}^{i}_{\theta,\theta}(t)\right)_{1,2} & \left(\dot{\bar{\mathbf{I}}}^{i}_{\theta,\theta}(t)\right)_{1,3}\\ \left(\dot{\bar{\mathbf{I}}}^{i}_{\theta,\theta}(t)\right)_{2,1} & \left(\dot{\bar{\mathbf{I}}}^{i}_{\theta,\theta}(t)\right)_{2,2} & \left(\dot{\bar{\mathbf{I}}}^{i}_{\theta,\theta}(t)\right)_{2,3}\\ \left(\dot{\bar{\mathbf{I}}}^{i}_{\theta,\theta}(t)\right)_{3,1} & \left(\dot{\bar{\mathbf{I}}}^{i}_{\theta,\theta}(t)\right)_{3,2} & \left(\dot{\bar{\mathbf{I}}}^{i}_{\theta,\theta}(t)\right)_{3,3}\end{bmatrix} \tag{2.445}$$

Indeed, the components of this matrix can be computed as:

$$\begin{aligned}\left(\dot{\bar{\mathbf{I}}}^{i}_{\theta,\theta}(t)\right)_{1,1} &= \frac{d}{dt}\left(\left(\mathbf{q}^{i}_{o}+\mathbf{B}^{i}_{e}\mathbf{q}^{i}_{f}(t)\right)^{T}\bar{\mathbf{J}}^{i}_{1,1}\left(\mathbf{q}^{i}_{o}+\mathbf{B}^{i}_{e}\mathbf{q}^{i}_{f}(t)\right)\right)=\\ &=\dot{\mathbf{q}}^{iT}_{f}(t)\mathbf{B}^{iT}_{e}\bar{\mathbf{J}}^{i}_{1,1}\left(\mathbf{q}^{i}_{o}+\mathbf{B}^{i}_{e}\mathbf{q}^{i}_{f}(t)\right)+\left(\mathbf{q}^{i}_{o}+\mathbf{B}^{i}_{e}\mathbf{q}^{i}_{f}(t)\right)^{T}\bar{\mathbf{J}}^{i}_{1,1}\mathbf{B}^{i}_{e}\dot{\mathbf{q}}^{i}_{f}(t)=\\ &=\dot{\mathbf{q}}^{iT}_{f}(t)\mathbf{B}^{iT}_{e}\bar{\mathbf{J}}^{iT}_{1,1}\left(\mathbf{q}^{i}_{o}+\mathbf{B}^{i}_{e}\mathbf{q}^{i}_{f}(t)\right)+\left(\mathbf{q}^{i}_{o}+\mathbf{B}^{i}_{e}\mathbf{q}^{i}_{f}(t)\right)^{T}\bar{\mathbf{J}}^{i}_{1,1}\mathbf{B}^{i}_{e}\dot{\mathbf{q}}^{i}_{f}(t)=\\ &=2\left(\mathbf{q}^{i}_{o}+\mathbf{B}^{i}_{e}\mathbf{q}^{i}_{f}(t)\right)^{T}\bar{\mathbf{J}}^{i}_{1,1}\mathbf{B}^{i}_{e}\dot{\mathbf{q}}^{i}_{f}(t)=\\ &=2\mathbf{q}^{iT}_{n}(t)\bar{\mathbf{J}}^{i}_{1,1}\mathbf{B}^{i}_{e}\dot{\mathbf{q}}^{i}_{f}(t)\end{aligned} \tag{2.446}$$

$$\begin{aligned}\left(\dot{\bar{\mathbf{I}}}^{i}_{\theta,\theta}(t)\right)_{1,2} &= \frac{d}{dt}\left(\left(\mathbf{q}^{i}_{o}+\mathbf{B}^{i}_{e}\mathbf{q}^{i}_{f}(t)\right)^{T}\bar{\mathbf{J}}^{i}_{1,2}\left(\mathbf{q}^{i}_{o}+\mathbf{B}^{i}_{e}\mathbf{q}^{i}_{f}(t)\right)\right)=\\ &=\dot{\mathbf{q}}^{iT}_{f}(t)\mathbf{B}^{iT}_{e}\bar{\mathbf{J}}^{i}_{1,2}\left(\mathbf{q}^{i}_{o}+\mathbf{B}^{i}_{e}\mathbf{q}^{i}_{f}(t)\right)+\left(\mathbf{q}^{i}_{o}+\mathbf{B}^{i}_{e}\mathbf{q}^{i}_{f}(t)\right)^{T}\bar{\mathbf{J}}^{i}_{1,2}\mathbf{B}^{i}_{e}\dot{\mathbf{q}}^{i}_{f}(t)=\\ &=2\left(\mathbf{q}^{i}_{o}+\mathbf{B}^{i}_{e}\mathbf{q}^{i}_{f}(t)\right)^{T}\bar{\mathbf{J}}^{i}_{1,2}\mathbf{B}^{i}_{e}\dot{\mathbf{q}}^{i}_{f}(t)=\\ &=2\mathbf{q}^{iT}_{n}(t)\bar{\mathbf{J}}^{i}_{1,2}\mathbf{B}^{i}_{e}\dot{\mathbf{q}}^{i}_{f}(t)=\\ &=\left(\dot{\bar{\mathbf{I}}}^{i}_{\theta,\theta}(t)\right)_{2,1}\end{aligned} \tag{2.447}$$

$$
\begin{aligned}
\left(\dot{\bar{\mathbf{I}}}^i_{\theta,\theta}(t)\right)_{1,3} &= \left(\mathbf{q}^i_o + \mathbf{B}^i_e \mathbf{q}^i_f(t)\right)^T \bar{\mathbf{J}}^i_{1,3} \left(\mathbf{q}^i_o + \mathbf{B}^i_e \mathbf{q}^i_f(t)\right) = \\
&= \dot{\mathbf{q}}^{iT}_f(t) \mathbf{B}^{iT}_e \bar{\mathbf{J}}^i_{1,3} \left(\mathbf{q}^i_o + \mathbf{B}^i_e \mathbf{q}^i_f(t)\right) + \left(\mathbf{q}^i_o + \mathbf{B}^i_e \mathbf{q}^i_f(t)\right)^T \bar{\mathbf{J}}^i_{1,3} \mathbf{B}^i_e \dot{\mathbf{q}}^i_f(t) = \\
&= 2\left(\mathbf{q}^i_o + \mathbf{B}^i_e \mathbf{q}^i_f(t)\right)^T \bar{\mathbf{J}}^i_{1,3} \mathbf{B}^i_e \dot{\mathbf{q}}^i_f(t) = \\
&= 2\mathbf{q}^{iT}_n(t) \bar{\mathbf{J}}^i_{1,3} \mathbf{B}^i_e \dot{\mathbf{q}}^i_f(t) = \\
&= \left(\dot{\bar{\mathbf{I}}}^i_{\theta,\theta}(t)\right)_{3,1}
\end{aligned}
\tag{2.448}
$$

$$
\begin{aligned}
\left(\dot{\bar{\mathbf{I}}}^i_{\theta,\theta}(t)\right)_{2,2} &= \left(\mathbf{q}^i_o + \mathbf{B}^i_e \mathbf{q}^i_f(t)\right)^T \bar{\mathbf{J}}^i_{2,2} \left(\mathbf{q}^i_o + \mathbf{B}^i_e \mathbf{q}^i_f(t)\right) = \\
&= \dot{\mathbf{q}}^{iT}_f(t) \mathbf{B}^{iT}_e \bar{\mathbf{J}}^i_{2,2} \left(\mathbf{q}^i_o + \mathbf{B}^i_e \mathbf{q}^i_f(t)\right) + \left(\mathbf{q}^i_o + \mathbf{B}^i_e \mathbf{q}^i_f(t)\right)^T \bar{\mathbf{J}}^i_{2,2} \mathbf{B}^i_e \dot{\mathbf{q}}^i_f(t) = \\
&= \dot{\mathbf{q}}^{iT}_f(t) \mathbf{B}^{iT}_e \bar{\mathbf{J}}^{iT}_{2,2} \left(\mathbf{q}^i_o + \mathbf{B}^i_e \mathbf{q}^i_f(t)\right) + \left(\mathbf{q}^i_o + \mathbf{B}^i_e \mathbf{q}^i_f(t)\right)^T \bar{\mathbf{J}}^i_{2,2} \mathbf{B}^i_e \dot{\mathbf{q}}^i_f(t) = \\
&= 2\left(\mathbf{q}^i_o + \mathbf{B}^i_e \mathbf{q}^i_f(t)\right)^T \bar{\mathbf{J}}^i_{2,2} \mathbf{B}^i_e \dot{\mathbf{q}}^i_f(t) = \\
&= 2\mathbf{q}^{iT}_n(t) \bar{\mathbf{J}}^i_{2,2} \mathbf{B}^i_e \dot{\mathbf{q}}^i_f(t)
\end{aligned}
\tag{2.449}
$$

$$
\begin{aligned}
\left(\dot{\bar{\mathbf{I}}}^i_{\theta,\theta}(t)\right)_{2,3} &= \left(\mathbf{q}^i_o + \mathbf{B}^i_e \mathbf{q}^i_f(t)\right)^T \bar{\mathbf{J}}^i_{2,3} \left(\mathbf{q}^i_o + \mathbf{B}^i_e \mathbf{q}^i_f(t)\right) = \\
&= \dot{\mathbf{q}}^{iT}_f(t) \mathbf{B}^{iT}_e \bar{\mathbf{J}}^i_{2,3} \left(\mathbf{q}^i_o + \mathbf{B}^i_e \mathbf{q}^i_f(t)\right) + \left(\mathbf{q}^i_o + \mathbf{B}^i_e \mathbf{q}^i_f(t)\right)^T \bar{\mathbf{J}}^i_{2,3} \mathbf{B}^i_e \dot{\mathbf{q}}^i_f(t) = \\
&= 2\left(\mathbf{q}^i_o + \mathbf{B}^i_e \mathbf{q}^i_f(t)\right)^T \bar{\mathbf{J}}^i_{2,3} \mathbf{B}^i_e \dot{\mathbf{q}}^i_f(t) = \\
&= 2\mathbf{q}^{iT}_n(t) \bar{\mathbf{J}}^i_{2,3} \mathbf{B}^i_e \dot{\mathbf{q}}^i_f(t) = \\
&= \left(\dot{\bar{\mathbf{I}}}^i_{\theta,\theta}(t)\right)_{3,2}
\end{aligned}
\tag{2.450}
$$

$$\begin{aligned}
\left(\dot{\bar{\mathbf{I}}}^i_{\theta,\theta}(t)\right)_{3,3} &= \frac{d}{dt}\left(\left(\mathbf{q}^i_o + \mathbf{B}^i_e\mathbf{q}^i_f(t)\right)^T \bar{\mathbf{J}}^i_{3,3}\left(\mathbf{q}^i_o + \mathbf{B}^i_e\mathbf{q}^i_f(t)\right)\right) = \\
&= \dot{\mathbf{q}}^{iT}_f(t)\mathbf{B}^{iT}_e\bar{\mathbf{J}}^i_{3,3}\left(\mathbf{q}^i_o + \mathbf{B}^i_e\mathbf{q}^i_f(t)\right) + \left(\mathbf{q}^i_o + \mathbf{B}^i_e\mathbf{q}^i_f(t)\right)^T \bar{\mathbf{J}}^i_{3,3}\mathbf{B}^i_e\dot{\mathbf{q}}^i_f(t) = \\
&= \dot{\mathbf{q}}^{iT}_f(t)\mathbf{B}^{iT}_e\bar{\mathbf{J}}^{iT}_{3,3}\left(\mathbf{q}^i_o + \mathbf{B}^i_e\mathbf{q}^i_f(t)\right) + \left(\mathbf{q}^i_o + \mathbf{B}^i_e\mathbf{q}^i_f(t)\right)^T \bar{\mathbf{J}}^i_{3,3}\mathbf{B}^i_e\dot{\mathbf{q}}^i_f(t) = \\
&= 2\left(\mathbf{q}^i_o + \mathbf{B}^i_e\mathbf{q}^i_f(t)\right)^T \bar{\mathbf{J}}^i_{3,3}\mathbf{B}^i_e\dot{\mathbf{q}}^i_f(t) = \\
&= 2\mathbf{q}^{iT}_n(t)\bar{\mathbf{J}}^i_{3,3}\mathbf{B}^i_e\dot{\mathbf{q}}^i_f(t)
\end{aligned} \tag{2.451}$$

The time derivative of the mass submatrix $\mathbf{m}^i_{\theta,q_f}(t)$ can be computed as follows:

$$\begin{aligned}
\dot{\mathbf{m}}^i_{\theta,q_f}(t) &= \frac{d}{dt}\left(\bar{\mathbf{G}}^{iT}(t)\mathbf{H}^i(t)\mathbf{B}^i_e\right) = \\
&= \dot{\bar{\mathbf{G}}}^{iT}(t)\mathbf{H}^i(t)\mathbf{B}^i_e + \bar{\mathbf{G}}^{iT}(t)\dot{\mathbf{H}}^i(t)\mathbf{B}^i_e = \\
&= \dot{\mathbf{m}}^{iT}_{q_f,\theta}(t)
\end{aligned} \tag{2.452}$$

Where the time derivative of the matrix $\mathbf{H}^i(t)$ can be evaluated as:

$$\dot{\mathbf{H}}^i(t) = \frac{d}{dt}\begin{bmatrix} \left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right)^T\left(\overline{\mathbf{J}}_{3,2}^i - \overline{\mathbf{J}}_{2,3}^i\right) \\ \left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right)^T\left(\overline{\mathbf{J}}_{1,3}^i - \overline{\mathbf{J}}_{3,1}^i\right) \\ \left(\mathbf{q}_o^i + \mathbf{B}_e^i\mathbf{q}_f^i(t)\right)^T\left(\overline{\mathbf{J}}_{2,1}^i - \overline{\mathbf{J}}_{1,2}^i\right) \end{bmatrix} = \\ = \begin{bmatrix} \left(\mathbf{B}_e^i\dot{\mathbf{q}}_f^i(t)\right)^T\left(\overline{\mathbf{J}}_{3,2}^i - \overline{\mathbf{J}}_{2,3}^i\right) \\ \left(\mathbf{B}_e^i\dot{\mathbf{q}}_f^i(t)\right)^T\left(\overline{\mathbf{J}}_{1,3}^i - \overline{\mathbf{J}}_{3,1}^i\right) \\ \left(\mathbf{B}_e^i\dot{\mathbf{q}}_f^i(t)\right)^T\left(\overline{\mathbf{J}}_{2,1}^i - \overline{\mathbf{J}}_{1,2}^i\right) \end{bmatrix} = \\ = \begin{bmatrix} \dot{\mathbf{q}}_f^{iT}(t)\mathbf{B}_e^{iT}\left(\overline{\mathbf{J}}_{3,2}^i - \overline{\mathbf{J}}_{2,3}^i\right) \\ \dot{\mathbf{q}}_f^{iT}(t)\mathbf{B}_e^{iT}\left(\overline{\mathbf{J}}_{1,3}^i - \overline{\mathbf{J}}_{3,1}^i\right) \\ \dot{\mathbf{q}}_f^{iT}(t)\mathbf{B}_e^{iT}\left(\overline{\mathbf{J}}_{2,1}^i - \overline{\mathbf{J}}_{1,2}^i\right) \end{bmatrix} \tag{2.453}$$

The time derivative of mass submatrix $\mathbf{m}_{q_f,q_f}^i(t)$ yields to:

$$\dot{\mathbf{m}}_{q_f,q_f}^i(t) = \frac{d}{dt}\left(\mathbf{B}_e^{iT}\overline{\mathbf{J}}_{f,f}^i\mathbf{B}_e^i\right) = \\ = \mathbf{O} \tag{2.454}$$

By using the time derivative of mass matrix $\mathbf{M}^i(t)$ the first term of quadratic velocity vector $\mathbf{Q}_v^i(t)$ can be evaluated as:

$$-\dot{\mathbf{M}}^i(t)\dot{\mathbf{q}}^i(t) = -\begin{bmatrix} \dot{\mathbf{m}}^i_{R,R}(t) & \dot{\mathbf{m}}^i_{R,\theta}(t) & \dot{\mathbf{m}}^i_{R,q_f}(t) \\ \dot{\mathbf{m}}^i_{\theta,R}(t) & \dot{\mathbf{m}}^i_{\theta,\theta}(t) & \dot{\mathbf{m}}^i_{\theta,q_f}(t) \\ \dot{\mathbf{m}}^i_{q_f,R}(t) & \dot{\mathbf{m}}^i_{q_f,\theta}(t) & \dot{\mathbf{m}}^i_{q_f,q_f}(t) \end{bmatrix} \begin{bmatrix} \dot{\mathbf{R}}^i(t) \\ \dot{\boldsymbol{\theta}}^i(t) \\ \dot{\mathbf{q}}^i_f(t) \end{bmatrix} =$$

$$= -\begin{bmatrix} \dot{\mathbf{m}}^i_{R,R}(t)\dot{\mathbf{R}}^i(t) + \dot{\mathbf{m}}^i_{R,\theta}(t)\dot{\boldsymbol{\theta}}^i(t) + \dot{\mathbf{m}}^i_{R,q_f}(t)\dot{\mathbf{q}}^i_f(t) \\ \dot{\mathbf{m}}^i_{\theta,R}(t)\dot{\mathbf{R}}^i(t) + \dot{\mathbf{m}}^i_{\theta,\theta}(t)\dot{\boldsymbol{\theta}}^i(t) + \dot{\mathbf{m}}^i_{\theta,q_f}(t)\dot{\mathbf{q}}^i_f(t) \\ \dot{\mathbf{m}}^i_{q_f,R}(t)\dot{\mathbf{R}}^i(t) + \dot{\mathbf{m}}^i_{q_f,\theta}(t)\dot{\boldsymbol{\theta}}^i(t) + \dot{\mathbf{m}}^i_{q_f,q_f}(t)\dot{\mathbf{q}}^i_f(t) \end{bmatrix} =$$

$$= -\begin{bmatrix} \left(\dot{\mathbf{M}}^i(t)\dot{\mathbf{q}}^i(t)\right)_R \\ \left(\dot{\mathbf{M}}^i(t)\dot{\mathbf{q}}^i(t)\right)_\theta \\ \left(\dot{\mathbf{M}}^i(t)\dot{\mathbf{q}}^i(t)\right)_{q_f} \end{bmatrix}$$

(2.455)

Where the matrix components $\left(\dot{\mathbf{M}}^i(t)\dot{\mathbf{q}}^i(t)\right)_R$, $\left(\dot{\mathbf{M}}^i(t)\dot{\mathbf{q}}^i(t)\right)_\theta$ and $\left(\dot{\mathbf{M}}^i(t)\dot{\mathbf{q}}^i(t)\right)_{q_f}$ are respectively $\mathbb{R}^3$, $\mathbb{R}^4$ and $\mathbb{R}^{n^i_f}$ vectors defined as:

$$
\begin{aligned}
&\left(\dot{\mathbf{M}}^i(t)\dot{\mathbf{q}}^i(t)\right)_R = \dot{\mathbf{m}}^i_{R,R}(t)\dot{\mathbf{R}}^i(t) + \dot{\mathbf{m}}^i_{R,\theta}(t)\dot{\boldsymbol{\theta}}^i(t) + \dot{\mathbf{m}}^i_{R,q_f}(t)\dot{\mathbf{q}}^i_f(t) = \\
&= \left(-\frac{1}{2}\mathbf{A}^i(t)\bar{\mathbf{G}}^i(t)\dot{\bar{\mathbf{G}}}^{iT}(t)\tilde{\bar{\mathbf{U}}}^i(t)\bar{\mathbf{G}}^i(t)\right)\dot{\boldsymbol{\theta}}^i(t) + \\
&\quad + \left(-\mathbf{A}^i(t)\dot{\tilde{\bar{\mathbf{U}}}}^i(t)\bar{\mathbf{G}}^i(t) - \mathbf{A}^i(t)\tilde{\bar{\mathbf{U}}}^i(t)\dot{\bar{\mathbf{G}}}^i(t)\right)\dot{\boldsymbol{\theta}}^i(t) + \\
&\quad + \frac{1}{2}\mathbf{A}^i(t)\bar{\mathbf{G}}^i(t)\dot{\bar{\mathbf{G}}}^{iT}(t)\bar{\mathbf{N}}^i\mathbf{B}^i_e\dot{\mathbf{q}}^i_f(t) = \\
&= -\frac{1}{2}\mathbf{A}^i(t)\bar{\mathbf{G}}^i(t)\dot{\bar{\mathbf{G}}}^{iT}(t)\tilde{\bar{\mathbf{U}}}^i(t)\bar{\mathbf{G}}^i(t)\dot{\boldsymbol{\theta}}^i(t) + \\
&\quad - \mathbf{A}^i(t)\dot{\tilde{\bar{\mathbf{U}}}}^i(t)\bar{\mathbf{G}}^i(t)\dot{\boldsymbol{\theta}}^i(t) - \mathbf{A}^i(t)\tilde{\bar{\mathbf{U}}}^i(t)\dot{\bar{\mathbf{G}}}^i(t)\dot{\boldsymbol{\theta}}^i(t) + \\
&\quad + \frac{1}{2}\mathbf{A}^i(t)\bar{\mathbf{G}}^i(t)\dot{\bar{\mathbf{G}}}^{iT}(t)\bar{\mathbf{N}}^i\mathbf{B}^i_e\dot{\mathbf{q}}^i_f(t) = \\
&= -\frac{1}{2}\mathbf{A}^i(t)\bar{\mathbf{G}}^i(t)\dot{\bar{\mathbf{G}}}^{iT}(t)\tilde{\bar{\mathbf{U}}}^i(t)\bar{\mathbf{G}}^i(t)\dot{\boldsymbol{\theta}}^i(t) - \mathbf{A}^i(t)\dot{\tilde{\bar{\mathbf{U}}}}^i(t)\bar{\mathbf{G}}^i(t)\dot{\boldsymbol{\theta}}^i(t) + \\
&\quad + \frac{1}{2}\mathbf{A}^i(t)\bar{\mathbf{G}}^i(t)\dot{\bar{\mathbf{G}}}^{iT}(t)\bar{\mathbf{N}}^i\mathbf{B}^i_e\dot{\mathbf{q}}^i_f(t)
\end{aligned}
\tag{2.456}
$$

$$
\begin{aligned}
&\left(\dot{\mathbf{M}}^{i}(t)\dot{\mathbf{q}}^{i}(t)\right)_{\theta} = \dot{\mathbf{m}}^{i}_{\theta,R}(t)\dot{\mathbf{R}}^{i}(t) + \dot{\mathbf{m}}^{i}_{\theta,\theta}(t)\dot{\boldsymbol{\theta}}^{i}(t) + \dot{\mathbf{m}}^{i}_{\theta,q_f}(t)\dot{\mathbf{q}}^{i}_{f}(t) = \\
&= \left(-\frac{1}{2}\bar{\mathbf{G}}^{iT}(t)\tilde{\bar{\mathbf{U}}}^{iT}(t)\dot{\bar{\mathbf{G}}}^{i}(t)\bar{\mathbf{G}}^{iT}(t)\mathbf{A}^{iT}(t)\right)\dot{\mathbf{R}}^{i}(t) + \\
&\quad + \left(-\bar{\mathbf{G}}^{iT}(t)\dot{\tilde{\bar{\mathbf{U}}}}^{iT}(t)\mathbf{A}^{iT}(t) - \dot{\bar{\mathbf{G}}}^{iT}(t)\tilde{\bar{\mathbf{U}}}^{iT}(t)\mathbf{A}^{iT}(t)\right)\dot{\mathbf{R}}^{i}(t) + \\
&\quad + \left(\dot{\bar{\mathbf{G}}}^{iT}(t)\bar{\mathbf{I}}^{i}_{\theta,\theta}\bar{\mathbf{G}}^{i}(t) + \bar{\mathbf{G}}^{iT}(t)\dot{\bar{\mathbf{I}}}^{i}_{\theta,\theta}(t)\bar{\mathbf{G}}^{i}(t) + \bar{\mathbf{G}}^{iT}(t)\bar{\mathbf{I}}^{i}_{\theta,\theta}\dot{\bar{\mathbf{G}}}^{i}(t)\right)\dot{\boldsymbol{\theta}}^{i}(t) + \\
&\quad + \left(\dot{\bar{\mathbf{G}}}^{iT}(t)\mathbf{H}^{i}(t)\mathbf{B}^{i}_{e} + \bar{\mathbf{G}}^{iT}(t)\dot{\mathbf{H}}^{i}(t)\mathbf{B}^{i}_{e}\right)\dot{\mathbf{q}}^{i}_{f}(t) = \\
&= -\frac{1}{2}\bar{\mathbf{G}}^{iT}(t)\tilde{\bar{\mathbf{U}}}^{iT}(t)\dot{\bar{\mathbf{G}}}^{i}(t)\bar{\mathbf{G}}^{iT}(t)\mathbf{A}^{iT}(t)\dot{\mathbf{R}}^{i}(t) - \bar{\mathbf{G}}^{iT}(t)\dot{\tilde{\bar{\mathbf{U}}}}^{iT}(t)\mathbf{A}^{iT}(t)\dot{\mathbf{R}}^{i}(t) + \\
&\quad - \dot{\bar{\mathbf{G}}}^{iT}(t)\tilde{\bar{\mathbf{U}}}^{iT}(t)\mathbf{A}^{iT}(t)\dot{\mathbf{R}}^{i}(t) + \dot{\bar{\mathbf{G}}}^{iT}(t)\bar{\mathbf{I}}^{i}_{\theta,\theta}\bar{\mathbf{G}}^{i}(t)\dot{\boldsymbol{\theta}}^{i}(t) + \bar{\mathbf{G}}^{iT}(t)\dot{\bar{\mathbf{I}}}^{i}_{\theta,\theta}\bar{\mathbf{G}}^{i}(t)\dot{\boldsymbol{\theta}}^{i}(t) + \\
&\quad + \bar{\mathbf{G}}^{iT}(t)\bar{\mathbf{I}}^{i}_{\theta,\theta}\dot{\bar{\mathbf{G}}}^{i}(t)\dot{\boldsymbol{\theta}}^{i}(t) + \dot{\bar{\mathbf{G}}}^{iT}(t)\mathbf{H}^{i}(t)\mathbf{B}^{i}_{e}\dot{\mathbf{q}}^{i}_{f}(t) + \bar{\mathbf{G}}^{iT}(t)\dot{\mathbf{H}}^{i}(t)\mathbf{B}^{i}_{e}\dot{\mathbf{q}}^{i}_{f}(t) = \\
&= -\frac{1}{2}\bar{\mathbf{G}}^{iT}(t)\tilde{\bar{\mathbf{U}}}^{iT}(t)\dot{\bar{\mathbf{G}}}^{i}(t)\bar{\mathbf{G}}^{iT}(t)\mathbf{A}^{iT}(t)\dot{\mathbf{R}}^{i}(t) - \bar{\mathbf{G}}^{iT}(t)\dot{\tilde{\bar{\mathbf{U}}}}^{iT}(t)\mathbf{A}^{iT}(t)\dot{\mathbf{R}}^{i}(t) + \\
&\quad - \dot{\bar{\mathbf{G}}}^{iT}(t)\tilde{\bar{\mathbf{U}}}^{iT}(t)\mathbf{A}^{iT}(t)\dot{\mathbf{R}}^{i}(t) + \dot{\bar{\mathbf{G}}}^{iT}(t)\bar{\mathbf{I}}^{i}_{\theta,\theta}\bar{\mathbf{G}}^{i}(t)\dot{\boldsymbol{\theta}}^{i}(t) + \bar{\mathbf{G}}^{iT}(t)\dot{\bar{\mathbf{I}}}^{i}_{\theta,\theta}\bar{\mathbf{G}}^{i}(t)\dot{\boldsymbol{\theta}}^{i}(t) + \\
&\quad + \dot{\bar{\mathbf{G}}}^{iT}(t)\mathbf{H}^{i}(t)\mathbf{B}^{i}_{e}\dot{\mathbf{q}}^{i}_{f}(t) + \bar{\mathbf{G}}^{iT}(t)\dot{\mathbf{H}}^{i}(t)\mathbf{B}^{i}_{e}\dot{\mathbf{q}}^{i}_{f}(t)
\end{aligned}
\tag{2.457}
$$

$$
\begin{aligned}
&\left(\dot{\mathbf{M}}^i(t)\dot{\mathbf{q}}^i(t)\right)_{q_f} = \dot{\mathbf{m}}^i_{q_f,R}(t)\dot{\mathbf{R}}^i(t) + \dot{\mathbf{m}}^i_{q_f,\theta}(t)\dot{\boldsymbol{\theta}}^i(t) + \dot{\mathbf{m}}^i_{q_f,q_f}(t)\dot{\mathbf{q}}^i_f(t) = \\
&= \frac{1}{2}\mathbf{B}_e^{iT}\overline{\mathbf{N}}^{iT}\dot{\overline{\mathbf{G}}}^i(t)\overline{\mathbf{G}}^{iT}(t)\mathbf{A}^{iT}(t)\dot{\mathbf{R}}^i(t) + \\
&\quad + \left(\mathbf{B}_e^{iT}\mathbf{H}^{iT}(t)\dot{\overline{\mathbf{G}}}^i(t) + \mathbf{B}_e^{iT}\dot{\mathbf{H}}^{iT}(t)\overline{\mathbf{G}}^i(t)\right)\dot{\boldsymbol{\theta}}^i(t) = \\
&= \frac{1}{2}\mathbf{B}_e^{iT}\overline{\mathbf{N}}^{iT}\dot{\overline{\mathbf{G}}}^i(t)\overline{\mathbf{G}}^{iT}(t)\mathbf{A}^{iT}(t)\dot{\mathbf{R}}^i(t) + \mathbf{B}_e^{iT}\mathbf{H}^{iT}(t)\dot{\overline{\mathbf{G}}}^i(t)\dot{\boldsymbol{\theta}}^i(t) + \\
&\quad + \mathbf{B}_e^{iT}\dot{\mathbf{H}}^{iT}(t)\overline{\mathbf{G}}^i(t)\dot{\boldsymbol{\theta}}^i(t) = \\
&= \frac{1}{2}\mathbf{B}_e^{iT}\overline{\mathbf{N}}^{iT}\dot{\overline{\mathbf{G}}}^i(t)\overline{\mathbf{G}}^{iT}(t)\mathbf{A}^{iT}(t)\dot{\mathbf{R}}^i(t) + \mathbf{B}_e^{iT}\dot{\mathbf{H}}^{iT}(t)\overline{\mathbf{G}}^i(t)\dot{\boldsymbol{\theta}}^i(t)
\end{aligned}
\tag{2.458}
$$

Where the following matrix identity has been used:

$$\dot{\overline{\mathbf{G}}}^i(t)\dot{\boldsymbol{\theta}}^i(t) = \mathbf{0} \tag{2.459}$$

On the other hand, the second term of quadratic velocity vector $\mathbf{Q}_v^i(t)$ can be written as follows:

$$
\left(\frac{\partial T^i(t)}{\partial \mathbf{q}^i(t)}\right)^T = \begin{bmatrix} \left(\dfrac{\partial T^i(t)}{\partial \mathbf{R}^i(t)}\right)^T \\ \left(\dfrac{\partial T^i(t)}{\partial \boldsymbol{\theta}^i(t)}\right)^T \\ \left(\dfrac{\partial T^i(t)}{\partial \mathbf{q}^i_f(t)}\right)^T \end{bmatrix} \tag{2.460}
$$

Where the matrix components $\left(\dfrac{\partial T^i(t)}{\partial \mathbf{R}^i(t)}\right)^T$, $\left(\dfrac{\partial T^i(t)}{\partial \boldsymbol{\theta}^i(t)}\right)^T$ and $\left(\dfrac{\partial T^i(t)}{\partial \mathbf{q}_f^i(t)}\right)^T$ are respectively $\mathbb{R}^3$, $\mathbb{R}^4$ and $\mathbb{R}^{n_f^i}$ vectors defined as:

$$
\begin{aligned}
\frac{\partial T^i(t)}{\partial \mathbf{R}^i(t)} &= \frac{\partial}{\partial \mathbf{R}^i(t)}\left(\frac{1}{2}\dot{\mathbf{R}}^{iT}(t)\mathbf{m}^i_{R,R}(t)\dot{\mathbf{R}}^i(t)\right)+\frac{\partial}{\partial \mathbf{R}^i(t)}\left(\frac{1}{2}\dot{\boldsymbol{\theta}}^{iT}(t)\mathbf{m}^i_{\theta,\theta}(t)\dot{\boldsymbol{\theta}}^i(t)\right)+\\
&+\frac{\partial}{\partial \mathbf{R}^i(t)}\left(\frac{1}{2}\dot{\mathbf{q}}_f^{iT}(t)\mathbf{m}^i_{q_f,q_f}(t)\dot{\mathbf{q}}_f^i(t)\right)+\frac{\partial}{\partial \mathbf{R}^i(t)}\left(\dot{\mathbf{R}}^{iT}(t)\mathbf{m}^i_{R,\theta}(t)\dot{\boldsymbol{\theta}}^i(t)\right)+\\
&+\frac{\partial}{\partial \mathbf{R}^i(t)}\left(\dot{\mathbf{R}}^{iT}(t)\mathbf{m}^i_{R,q_f}(t)\dot{\mathbf{q}}_f^i(t)\right)+\frac{\partial}{\partial \mathbf{R}^i(t)}\left(\dot{\boldsymbol{\theta}}^{iT}(t)\mathbf{m}^i_{\theta,q_f}(t)\dot{\mathbf{q}}_f^i(t)\right)=\\
&=\frac{\partial}{\partial \mathbf{R}^i(t)}\left(\frac{1}{2}\dot{\mathbf{R}}^{iT}(t)\left(m^i\mathbf{I}\right)\dot{\mathbf{R}}^i(t)\right)+\\
&+\frac{\partial}{\partial \mathbf{R}^i(t)}\left(\frac{1}{2}\dot{\boldsymbol{\theta}}^{iT}(t)\left(\bar{\mathbf{G}}^{iT}(t)\bar{\mathbf{I}}^i_{\theta,\theta}(t)\bar{\mathbf{G}}^i(t)\right)\dot{\boldsymbol{\theta}}^i(t)\right)+\\
&+\frac{\partial}{\partial \mathbf{R}^i(t)}\left(\frac{1}{2}\dot{\mathbf{q}}_f^{iT}(t)\left(\mathbf{B}_e^{iT}\bar{\mathbf{J}}^i_{f,f}\mathbf{B}_e^i\right)\dot{\mathbf{q}}_f^i(t)\right)+\\
&+\frac{\partial}{\partial \mathbf{R}^i(t)}\left(\dot{\mathbf{R}}^{iT}(t)\left(-\mathbf{A}^i(t)\bar{\tilde{\mathbf{U}}}^i(t)\bar{\mathbf{G}}^i(t)\right)\dot{\boldsymbol{\theta}}^i(t)\right)+\\
&+\frac{\partial}{\partial \mathbf{R}^i(t)}\left(\dot{\mathbf{R}}^{iT}(t)\left(\mathbf{A}^i(t)\bar{\mathbf{N}}^i\mathbf{B}_e^i\right)\dot{\mathbf{q}}_f^i(t)\right)+\\
&+\frac{\partial}{\partial \mathbf{R}^i(t)}\left(\dot{\boldsymbol{\theta}}^{iT}(t)\left(\bar{\mathbf{G}}^{iT}(t)\mathbf{H}^i(t)\mathbf{B}_e^i\right)\dot{\mathbf{q}}_f^i(t)\right)=\\
&=\mathbf{0}
\end{aligned}
\tag{2.461}
$$

$$
\begin{aligned}
\frac{\partial T^i(t)}{\partial \boldsymbol{\theta}^i(t)} &= \frac{\partial}{\partial \boldsymbol{\theta}^i(t)}\left(\frac{1}{2}\dot{\mathbf{R}}^{iT}(t)\mathbf{m}^i_{R,R}(t)\dot{\mathbf{R}}^i(t)\right)+\frac{\partial}{\partial \boldsymbol{\theta}^i(t)}\left(\frac{1}{2}\dot{\boldsymbol{\theta}}^{iT}(t)\mathbf{m}^i_{\theta,\theta}(t)\dot{\boldsymbol{\theta}}^i(t)\right)+\\
&\quad+\frac{\partial}{\partial \boldsymbol{\theta}^i(t)}\left(\frac{1}{2}\dot{\mathbf{q}}^{iT}_f(t)\mathbf{m}^i_{q_f,q_f}(t)\dot{\mathbf{q}}^i_f(t)\right)+\frac{\partial}{\partial \boldsymbol{\theta}^i(t)}\left(\dot{\mathbf{R}}^{iT}(t)\mathbf{m}^i_{R,\theta}(t)\dot{\boldsymbol{\theta}}^i(t)\right)+\\
&\quad+\frac{\partial}{\partial \boldsymbol{\theta}^i(t)}\left(\dot{\mathbf{R}}^{iT}(t)\mathbf{m}^i_{R,q_f}(t)\dot{\mathbf{q}}^i_f(t)\right)+\frac{\partial}{\partial \boldsymbol{\theta}^i(t)}\left(\dot{\boldsymbol{\theta}}^{iT}(t)\mathbf{m}^i_{\theta,q_f}(t)\dot{\mathbf{q}}^i_f(t)\right)=\\
&=\frac{\partial}{\partial \boldsymbol{\theta}^i(t)}\left(\frac{1}{2}\dot{\mathbf{R}}^{iT}(t)\left(m^i\mathbf{I}\right)\dot{\mathbf{R}}^i(t)\right)+\frac{\partial}{\partial \boldsymbol{\theta}^i(t)}\left(\frac{1}{2}\dot{\boldsymbol{\theta}}^{iT}(t)\left(\bar{\mathbf{G}}^{iT}(t)\bar{\mathbf{I}}^i_{\theta,\theta}(t)\bar{\mathbf{G}}^i(t)\right)\dot{\boldsymbol{\theta}}^i(t)\right)+\\
&\quad+\frac{\partial}{\partial \boldsymbol{\theta}^i(t)}\left(\frac{1}{2}\dot{\mathbf{q}}^{iT}_f(t)\left(\mathbf{B}^{iT}_e\bar{\mathbf{J}}^i_{f,f}\mathbf{B}^i_e\right)\dot{\mathbf{q}}^i_f(t)\right)+\\
&\quad+\frac{\partial}{\partial \boldsymbol{\theta}^i(t)}\left(\dot{\mathbf{R}}^{iT}(t)\left(-\mathbf{A}^i(t)\tilde{\bar{\mathbf{U}}}^i(t)\bar{\mathbf{G}}^i(t)\right)\dot{\boldsymbol{\theta}}^i(t)\right)+\\
&\quad+\frac{\partial}{\partial \boldsymbol{\theta}^i(t)}\left(\dot{\mathbf{R}}^{iT}(t)\left(\mathbf{A}^i(t)\bar{\mathbf{N}}^i\mathbf{B}^i_e\right)\dot{\mathbf{q}}^i_f(t)\right)+\frac{\partial}{\partial \boldsymbol{\theta}^i(t)}\left(\dot{\boldsymbol{\theta}}^{iT}(t)\left(\bar{\mathbf{G}}^{iT}(t)\mathbf{H}^i(t)\mathbf{B}^i_e\right)\dot{\mathbf{q}}^i_f(t)\right)=\\
&=\frac{\partial}{\partial \boldsymbol{\theta}^i(t)}\left(\frac{1}{2}\boldsymbol{\theta}^{iT}(t)\dot{\bar{\mathbf{G}}}^{iT}(t)\bar{\mathbf{I}}^i_{\theta,\theta}(t)\dot{\bar{\mathbf{G}}}^i(t)\boldsymbol{\theta}^i(t)\right)+\\
&\quad+\frac{\partial}{\partial \boldsymbol{\theta}^i(t)}\left(\dot{\mathbf{R}}^{iT}(t)\mathbf{A}^i(t)\tilde{\bar{\mathbf{U}}}^i(t)\dot{\bar{\mathbf{G}}}^i(t)\boldsymbol{\theta}^i(t)\right)+\\
&\quad+\frac{\partial}{\partial \boldsymbol{\theta}^i(t)}\left(\dot{\mathbf{R}}^{iT}(t)\mathbf{A}^i(t)\bar{\mathbf{N}}^i\mathbf{B}^i_e\dot{\mathbf{q}}^i_f(t)\right)-\frac{\partial}{\partial \boldsymbol{\theta}^i(t)}\left(\boldsymbol{\theta}^{iT}(t)\dot{\bar{\mathbf{G}}}^{iT}(t)\mathbf{H}^i(t)\mathbf{B}^i_e\dot{\mathbf{q}}^i_f(t)\right)=\\
&=\boldsymbol{\theta}^{iT}(t)\dot{\bar{\mathbf{G}}}^{iT}(t)\bar{\mathbf{I}}^i_{\theta,\theta}(t)\dot{\bar{\mathbf{G}}}^i(t)+\dot{\mathbf{R}}^{iT}(t)\frac{\partial \mathbf{A}^i(t)}{\partial \boldsymbol{\theta}^i(t)}\tilde{\bar{\mathbf{U}}}^i(t)\dot{\bar{\mathbf{G}}}^i(t)\boldsymbol{\theta}^i(t)+\\
&\quad+\dot{\mathbf{R}}^{iT}(t)\mathbf{A}^i(t)\tilde{\bar{\mathbf{U}}}^i(t)\dot{\bar{\mathbf{G}}}^i(t)+\dot{\mathbf{R}}^{iT}(t)\frac{\partial \mathbf{A}^i(t)}{\partial \boldsymbol{\theta}^i(t)}\bar{\mathbf{N}}^i\mathbf{B}^i_e\dot{\mathbf{q}}^i_f(t)-\dot{\bar{\mathbf{G}}}^{iT}(t)\mathbf{H}^i(t)\mathbf{B}^i_e\dot{\mathbf{q}}^i_f(t)=\\
&=\boldsymbol{\theta}^{iT}(t)\dot{\bar{\mathbf{G}}}^{iT}(t)\bar{\mathbf{I}}^i_{\theta,\theta}(t)\dot{\bar{\mathbf{G}}}^i(t)+\dot{\mathbf{R}}^{iT}(t)\mathbf{A}^i_{\boldsymbol{\theta}^i}(t)\tilde{\bar{\mathbf{U}}}^i(t)\dot{\bar{\mathbf{G}}}^i(t)\boldsymbol{\theta}^i(t)+\\
&\quad+\dot{\mathbf{R}}^{iT}(t)\mathbf{A}^i(t)\tilde{\bar{\mathbf{U}}}^i(t)\dot{\bar{\mathbf{G}}}^i(t)+\dot{\mathbf{R}}^{iT}(t)\mathbf{A}^i_{\boldsymbol{\theta}^i}(t)\bar{\mathbf{N}}^i\mathbf{B}^i_e\dot{\mathbf{q}}^i_f(t)-\dot{\bar{\mathbf{G}}}^{iT}(t)\mathbf{H}^i(t)\mathbf{B}^i_e\dot{\mathbf{q}}^i_f(t)
\end{aligned}
\tag{2.462}
$$

$$\begin{aligned}
\frac{\partial T^i(t)}{\partial \mathbf{q}_f^i(t)} &= \frac{\partial}{\partial \mathbf{q}_f^i(t)}\left(\frac{1}{2}\dot{\mathbf{R}}^{iT}(t)\mathbf{m}_{R,R}^i(t)\dot{\mathbf{R}}^i(t)\right)+\frac{\partial}{\partial \mathbf{q}_f^i(t)}\left(\frac{1}{2}\dot{\boldsymbol{\theta}}^{iT}(t)\mathbf{m}_{\theta,\theta}^i(t)\dot{\boldsymbol{\theta}}^i(t)\right)+\\
&+\frac{\partial}{\partial \mathbf{q}_f^i(t)}\left(\frac{1}{2}\dot{\mathbf{q}}_f^{iT}(t)\mathbf{m}_{q_f,q_f}^i(t)\dot{\mathbf{q}}_f^i(t)\right)+\frac{\partial}{\partial \mathbf{q}_f^i(t)}\left(\dot{\mathbf{R}}^{iT}(t)\mathbf{m}_{R,\theta}^i(t)\dot{\boldsymbol{\theta}}^i(t)\right)+\\
&+\frac{\partial}{\partial \mathbf{q}_f^i(t)}\left(\dot{\mathbf{R}}^{iT}(t)\mathbf{m}_{R,q_f}^i(t)\dot{\mathbf{q}}_f^i(t)\right)+\frac{\partial}{\partial \mathbf{q}_f^i(t)}\left(\dot{\boldsymbol{\theta}}^{iT}(t)\mathbf{m}_{\theta,q_f}^i(t)\dot{\mathbf{q}}_f^i(t)\right)=\\
&=\frac{\partial}{\partial \mathbf{q}_f^i(t)}\left(\frac{1}{2}\dot{\mathbf{R}}^{iT}(t)\left(m^i\mathbf{I}\right)\dot{\mathbf{R}}^i(t)\right)+\frac{\partial}{\partial \mathbf{q}_f^i(t)}\left(\frac{1}{2}\dot{\boldsymbol{\theta}}^{iT}(t)\left(\bar{\mathbf{G}}^{iT}(t)\bar{\mathbf{I}}_{\theta,\theta}^i(t)\bar{\mathbf{G}}^i(t)\right)\dot{\boldsymbol{\theta}}^i(t)\right)+\\
&+\frac{\partial}{\partial \mathbf{q}_f^i(t)}\left(\frac{1}{2}\dot{\mathbf{q}}_f^{iT}(t)\left(\mathbf{B}_e^{iT}\bar{\mathbf{J}}_{f,f}^i\mathbf{B}_e^i\right)\dot{\mathbf{q}}_f^i(t)\right)+\\
&+\frac{\partial}{\partial \mathbf{q}_f^i(t)}\left(\dot{\mathbf{R}}^{iT}(t)\left(-\mathbf{A}^i(t)\tilde{\bar{\mathbf{U}}}^i(t)\bar{\mathbf{G}}^i(t)\right)\dot{\boldsymbol{\theta}}^i(t)\right)+\\
&+\frac{\partial}{\partial \mathbf{q}_f^i(t)}\left(\dot{\mathbf{R}}^{iT}(t)\left(\mathbf{A}^i(t)\bar{\mathbf{N}}^i\mathbf{B}_e^i\right)\dot{\mathbf{q}}_f^i(t)\right)+\frac{\partial}{\partial \mathbf{q}_f^i(t)}\left(\dot{\boldsymbol{\theta}}^{iT}(t)\left(\bar{\mathbf{G}}^{iT}(t)\mathbf{H}^i(t)\mathbf{B}_e^i\right)\dot{\mathbf{q}}_f^i(t)\right)=\\
&=\frac{1}{2}\dot{\boldsymbol{\theta}}^{iT}(t)\bar{\mathbf{G}}^{iT}(t)\frac{\partial\bar{\mathbf{I}}_{\theta,\theta}^i(t)}{\partial \mathbf{q}_f^i(t)}\bar{\mathbf{G}}^i(t)\dot{\boldsymbol{\theta}}^i(t)-\dot{\mathbf{R}}^{iT}(t)\mathbf{A}^i(t)\frac{\partial\tilde{\bar{\mathbf{U}}}^i(t)}{\partial \mathbf{q}_f^i(t)}\bar{\mathbf{G}}^i(t)\dot{\boldsymbol{\theta}}^i(t)+\\
&+\dot{\boldsymbol{\theta}}^{iT}(t)\bar{\mathbf{G}}^{iT}(t)\frac{\partial\mathbf{H}^i(t)}{\partial \mathbf{q}_f^i(t)}\mathbf{B}_e^i\dot{\mathbf{q}}_f^i(t)=\\
&=\frac{1}{2}\dot{\boldsymbol{\theta}}^{iT}(t)\bar{\mathbf{G}}^{iT}(t)\left(\bar{\mathbf{I}}_{\theta,\theta}^i(t)\right)_{\mathbf{q}_f^i}\bar{\mathbf{G}}^i(t)\dot{\boldsymbol{\theta}}^i(t)-\dot{\mathbf{R}}^{iT}(t)\mathbf{A}^i(t)\tilde{\bar{\mathbf{U}}}_{\mathbf{q}_f^i}^i(t)\bar{\mathbf{G}}^i(t)\dot{\boldsymbol{\theta}}^i(t)+\\
&+\dot{\boldsymbol{\theta}}^{iT}(t)\bar{\mathbf{G}}^{iT}(t)\mathbf{H}_{\mathbf{q}_f^i}^i(t)\mathbf{B}_e^i\dot{\mathbf{q}}_f^i(t)
\end{aligned} \tag{2.463}$$

Where the components of the matrix derivative $\left(\bar{\mathbf{I}}_{\theta,\theta}^i(t)\right)_{\mathbf{q}_f^i}$, $\tilde{\bar{\mathbf{U}}}_{\mathbf{q}_f^i}^i(t)$ and $\mathbf{H}_{\mathbf{q}_f^i}^i(t)$ can be respectively evaluated as:

$$
\begin{aligned}
&\frac{\partial\left(\bar{\mathbf{I}}^{i}_{\theta,\theta}(t)\right)_{h,k}}{\partial q^{i}_{f,l}} = \frac{\partial}{\partial q^{i}_{f,l}}\left(\left(\mathbf{q}^{i}_{o}+\mathbf{B}^{i}_{e}\mathbf{q}^{i}_{f}(t)\right)^{T}\bar{\mathbf{J}}^{i}_{h,k}\left(\mathbf{q}^{i}_{o}+\mathbf{B}^{i}_{e}\mathbf{q}^{i}_{f}(t)\right)\right)=\\
&=\frac{\partial}{\partial q^{i}_{f,l}}\left(\mathbf{q}^{iT}_{o}\bar{\mathbf{J}}^{i}_{h,k}\mathbf{q}^{i}_{o}+\mathbf{q}^{iT}_{o}\bar{\mathbf{J}}^{i}_{h,k}\mathbf{B}^{i}_{e}\mathbf{q}^{i}_{f}(t)+\mathbf{q}^{iT}_{f}(t)\mathbf{B}^{iT}_{e}\bar{\mathbf{J}}^{i}_{h,k}\mathbf{q}^{i}_{o}+\mathbf{q}^{iT}_{f}(t)\mathbf{B}^{iT}_{e}\bar{\mathbf{J}}^{i}_{h,k}\mathbf{B}^{i}_{e}\mathbf{q}^{i}_{f}(t)\right)=\\
&=\mathbf{q}^{iT}_{o}\bar{\mathbf{J}}^{i}_{h,k}\mathbf{B}^{i}_{e}\frac{\partial\mathbf{q}^{i}_{f}(t)}{\partial q^{i}_{f,l}}+\frac{\partial\mathbf{q}^{iT}_{f}(t)}{\partial q^{i}_{f,l}}\mathbf{B}^{iT}_{e}\bar{\mathbf{J}}^{i}_{h,k}\mathbf{q}^{i}_{o}+2\mathbf{q}^{iT}_{f}(t)\mathbf{B}^{iT}_{e}\bar{\mathbf{J}}^{i}_{h,k}\mathbf{B}^{i}_{e}\frac{\partial\mathbf{q}^{i}_{f}(t)}{\partial q^{i}_{f,l}}=\\
&=\mathbf{q}^{iT}_{o}\bar{\mathbf{J}}^{i}_{h,k}\mathbf{B}^{i}_{e}\mathbf{c}^{i}_{l}+\mathbf{c}^{iT}_{l}\mathbf{B}^{iT}_{e}\bar{\mathbf{J}}^{i}_{h,k}\mathbf{q}^{i}_{o}+2\mathbf{q}^{iT}_{f}(t)\mathbf{B}^{iT}_{e}\bar{\mathbf{J}}^{i}_{h,k}\mathbf{B}^{i}_{e}\mathbf{c}^{i}_{l}
\end{aligned}
\tag{2.464}
$$

$$
\begin{aligned}
\frac{\partial\tilde{\bar{\mathbf{U}}}^{i}(t)}{\partial q^{i}_{f,l}} &= \frac{\partial}{\partial q^{i}_{f,l}}\left(Skew\left(\bar{\mathbf{N}}^{i}\left(\mathbf{q}^{i}_{o}+\mathbf{B}^{i}_{e}\mathbf{q}^{i}_{f}(t)\right)\right)\right)=\\
&=Skew\left(\frac{\partial}{\partial q^{i}_{f,l}}\left(\bar{\mathbf{N}}^{i}\mathbf{q}^{i}_{o}+\bar{\mathbf{N}}^{i}\mathbf{B}^{i}_{e}\mathbf{q}^{i}_{f}(t)\right)\right)=\\
&=Skew\left(\bar{\mathbf{N}}^{i}\mathbf{B}^{i}_{e}\frac{\partial\mathbf{q}^{i}_{f}(t)}{\partial q^{i}_{f,l}}\right)=\\
&=Skew\left(\bar{\mathbf{N}}^{i}\mathbf{B}^{i}_{e}\mathbf{c}^{i}_{l}\right)
\end{aligned}
\tag{2.465}
$$

$$\frac{\partial \mathbf{H}^i(t)}{\partial q^i_{f,l}} = \begin{bmatrix} \frac{\partial}{\partial q^i_{f,l}}\left(\left(\mathbf{q}^i_o + \mathbf{B}^i_e \mathbf{q}^i_f(t)\right)^T \left(\overline{\mathbf{J}}^i_{3,2} - \overline{\mathbf{J}}^i_{2,3}\right)\right) \\ \frac{\partial}{\partial q^i_{f,l}}\left(\left(\mathbf{q}^i_o + \mathbf{B}^i_e \mathbf{q}^i_f(t)\right)^T \left(\overline{\mathbf{J}}^i_{1,3} - \overline{\mathbf{J}}^i_{3,1}\right)\right) \\ \frac{\partial}{\partial q^i_{f,l}}\left(\left(\mathbf{q}^i_o + \mathbf{B}^i_e \mathbf{q}^i_f(t)\right)^T \left(\overline{\mathbf{J}}^i_{2,1} - \overline{\mathbf{J}}^i_{1,2}\right)\right) \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{\partial \mathbf{q}^{iT}_f(t)}{\partial q^i_{f,l}} \mathbf{B}^{iT}_e \left(\overline{\mathbf{J}}^i_{3,2} - \overline{\mathbf{J}}^i_{2,3}\right) \\ \frac{\partial \mathbf{q}^{iT}_f(t)}{\partial q^i_{f,l}} \mathbf{B}^{iT}_e \left(\overline{\mathbf{J}}^i_{1,3} - \overline{\mathbf{J}}^i_{3,1}\right) \\ \frac{\partial \mathbf{q}^{iT}_f(t)}{\partial q^i_{f,l}} \mathbf{B}^{iT}_e \left(\overline{\mathbf{J}}^i_{2,1} - \overline{\mathbf{J}}^i_{1,2}\right) \end{bmatrix} =$$

$$= \begin{bmatrix} \mathbf{c}^{iT}_l \mathbf{B}^{iT}_e \left(\overline{\mathbf{J}}^i_{3,2} - \overline{\mathbf{J}}^i_{2,3}\right) \\ \mathbf{c}^{iT}_l \mathbf{B}^{iT}_e \left(\overline{\mathbf{J}}^i_{1,3} - \overline{\mathbf{J}}^i_{3,1}\right) \\ \mathbf{c}^{iT}_l \mathbf{B}^{iT}_e \left(\overline{\mathbf{J}}^i_{2,1} - \overline{\mathbf{J}}^i_{1,2}\right) \end{bmatrix} \tag{2.466}$$

Where $\mathbf{c}^i_l$ is a $\mathbb{R}^{n^i_f}$ constant vector defined as:

$$\mathbf{c}^i_l = \frac{\partial \mathbf{q}^i_f(t)}{\partial q^i_{f,l}} \tag{2.467}$$

A this stage, all the terms required to evaluate the quadratic velocity vector $\mathbf{Q}^i_v(t)$ has been explicitly computed.

2.7.8. DYNAMIC EQUATIONS OF FLEXIBLE MULTIBODY SYSTEMS

To derive the mass matrix $\mathbf{M}^i(t)$, the stiffness matrix $\mathbf{K}^i$ and the quadratic velocity vector $\mathbf{Q}_v^i(t)$ of flexible body i all configuration coordinates $\mathbf{q}^i(t)$ has been considered as independent coordinates. Obviously, this is not the general case of a flexible multibody system which is typically formed of a set of flexible bodies mutually interconnected. Therefore the actions of the constraints must be considered in the dynamic equations as generalized constraint forces. Indeed, consider that the generic body i of the system is forced to satisfy the following constraint equations written in the standard form:

$$\mathbf{A}^i(t)\ddot{\mathbf{q}}^i(t) = \mathbf{b}^i(t) \tag{2.468}$$

Where $\mathbf{A}^i(t)$ is a $\mathbb{R}^{n_c^i \times n^i}$ constraint matrix and $\mathbf{b}^i(t)$ is a $\mathbb{R}^{n_c^i}$ constraint vector. (Note that the constraint matrix $\mathbf{A}^i(t)$ relative to body i has been denoted with the same symbol of the rotation matrix $\mathbf{A}^i(t)$ of body i). These equations are a set of algebraic constraint equations written in the standard form and encompass all kind of constraints acting on the system, such as mechanical joints as well as specific constraints which derive from the definition of Euler parameters. Consequently, Lagrange equations take the following form:

$$\frac{d}{dt}\left(\frac{\partial T^i(t)}{\partial \dot{\mathbf{q}}^i(t)}\right)^T - \left(\frac{\partial T^i(t)}{\partial \mathbf{q}^i(t)}\right)^T + \left(\frac{\partial U^i(t)}{\partial \mathbf{q}^i(t)}\right)^T = \mathbf{Q}_{e,nc}^i(t) + \mathbf{Q}_c^i(t) \tag{2.469}$$

Where $\mathbf{Q}_{e,nc}^i(t)$ is a $\mathbb{R}^{n^i}$ vector representing the vector of generalized external nonconservative forces and $\mathbf{Q}_c^i(t)$ is a $\mathbb{R}^{n^i}$ vector representing the generalized constraint forces. The first two terms on the left hand side of Lagrange equations is equal to the negative of lagrangian components of inertia forces $\mathbf{Q}_i^i(t)$ of body i and it can be explicitly computed by using the expression of kinetic energy $T^i(t)$ based on the expression of mass matrix $\mathbf{M}^i(t)$. Indeed:

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{\partial T^i(t)}{\partial \dot{\mathbf{q}}^i(t)}\right)^T - \left(\frac{\partial T^i(t)}{\partial \mathbf{q}^i(t)}\right)^T = -\mathbf{Q}_i^i(t) = \\
&= \frac{d}{dt}\left(\frac{\partial}{\partial \dot{\mathbf{q}}^i(t)}\left(\frac{1}{2}\dot{\mathbf{q}}^{iT}(t)\mathbf{M}^i(t)\dot{\mathbf{q}}^i(t)\right)\right)^T - \left(\frac{\partial T^i(t)}{\partial \mathbf{q}^i(t)}\right)^T = \\
&= \frac{d}{dt}\left(\dot{\mathbf{q}}^{iT}(t)\mathbf{M}^i(t)\right)^T - \left(\frac{\partial T^i(t)}{\partial \mathbf{q}^i(t)}\right)^T = \\
&= \mathbf{M}^i(t)\ddot{\mathbf{q}}^i(t) + \dot{\mathbf{M}}^i(t)\dot{\mathbf{q}}^i(t) - \left(\frac{\partial T^i(t)}{\partial \mathbf{q}^i(t)}\right)^T = \\
&= \mathbf{M}^i(t)\ddot{\mathbf{q}}^i(t) - \mathbf{Q}_v^i(t)
\end{aligned}
\tag{2.470}
$$

Where $\mathbf{Q}_v^i(t)$ is the $\mathbb{R}^{n^i}$ quadratic velocity vector defined as:

$$
\mathbf{Q}_v^i(t) = \left(\frac{\partial T^i(t)}{\partial \mathbf{q}^i(t)}\right)^T - \dot{\mathbf{M}}^i(t)\dot{\mathbf{q}}^i(t) \tag{2.471}
$$

On the other hand, the last term on the left hand side of Lagrange equations is equal to the opposite of the lagrangian components of conservative elastic forces $\mathbf{Q}_k^i(t)$ of body i and therefore it can be explicitly computed by using the expression of potential energy $U^i(t)$ based on the expression of stiffness matrix $\mathbf{K}^i$. Indeed:

$$\begin{aligned}&\left(\frac{\partial U^i(t)}{\partial \mathbf{q}^i(t)}\right)^T = -\mathbf{Q}_k^i(t) = \\ &= \left(\frac{\partial}{\partial \mathbf{q}^i(t)}\left(\frac{1}{2}\mathbf{q}^{iT}(t)\mathbf{K}^i\mathbf{q}^i(t)\right)\right)^T = \\ &= \left(\mathbf{q}^{iT}(t)\mathbf{K}^i\right)^T = \\ &= \mathbf{K}^i\mathbf{q}^i(t)\end{aligned} \tag{2.472}$$

Consequently, the equations of motion of flexible body i can be expressed as:

$$\mathbf{M}^i(t)\ddot{\mathbf{q}}^i(t) + \mathbf{K}^i\mathbf{q}^i(t) = \mathbf{Q}_v^i(t) + \mathbf{Q}_{e,nc}^i(t) + \mathbf{Q}_c^i(t) \tag{2.473}$$

These dynamic equations can be easily assembled to derive the equations of motion of the whole flexible multibody system formed of n_b bodies to yield:

$$\mathbf{M}(t)\ddot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{Q}_v(t) + \mathbf{Q}_{e,nc}(t) + \mathbf{Q}_c(t) \tag{2.474}$$

Where the configuration vector $\mathbf{q}(t)$ represents the total $\mathbb{R}^n$ vector of the system lagrangian coordinates and is defined as follows:

$$\mathbf{q}(t) = \begin{bmatrix} \mathbf{q}^1(t) \\ \mathbf{q}^2(t) \\ \vdots \\ \mathbf{q}^{n_b}(t) \end{bmatrix} \tag{2.475}$$

The matrix $\mathbf{M}(t)$ is the global $\mathbb{R}^{n\times n}$ mass matrix of the flexible multibody system and it can be easily assembled as:

$$\mathbf{M}(t) = \begin{bmatrix} \mathbf{M}^1(t) & \mathbf{O} & \dots & \mathbf{O} \\ \mathbf{O} & \mathbf{M}^2(t) & \dots & \mathbf{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{O} & \mathbf{O} & \dots & \mathbf{M}^{n_b}(t) \end{bmatrix} \tag{2.476}$$

The matrix $\mathbf{K}$ is the global $\mathbb{R}^{n \times n}$ stiffness matrix of the flexible multibody system and it can be easily assembled as:

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}^1 & \mathbf{O} & \dots & \mathbf{O} \\ \mathbf{O} & \mathbf{K}^1 & \dots & \mathbf{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{O} & \mathbf{O} & \dots & \mathbf{K}^{n_b} \end{bmatrix} \tag{2.477}$$

The $\mathbb{R}^n$ vectors $\mathbf{Q}_v(t)$, $\mathbf{Q}_{e,nc}(t)$ and $\mathbf{Q}_c(t)$ are lagrangian component vectors which represent respectively the generalized gyroscopic and Coriolis forces, the generalized external nonconservative forces and the generalized constraint forces. These vectors can be simply assembled as:

$$\mathbf{Q}_v(t) = \begin{bmatrix} \mathbf{Q}_v^1(t) \\ \mathbf{Q}_v^2(t) \\ \vdots \\ \mathbf{Q}_v^{n_b}(t) \end{bmatrix} \tag{2.478}$$

$$\mathbf{Q}_{e,nc}(t) = \begin{bmatrix} \mathbf{Q}_{e,nc}^1(t) \\ \mathbf{Q}_{e,nc}^2(t) \\ \vdots \\ \mathbf{Q}_{e,nc}^{n_b}(t) \end{bmatrix} \tag{2.479}$$

$$\mathbf{Q}_c(t) = \begin{bmatrix} \mathbf{Q}_c^1(t) \\ \mathbf{Q}_c^2(t) \\ \vdots \\ \mathbf{Q}_c^{n_b}(t) \end{bmatrix} \tag{2.480}$$

On the other hand, the algebraic constraint equations can be assembled in a similar manner to yield:

$$\mathbf{A}(t)\ddot{\mathbf{q}}(t) = \mathbf{b}(t) \tag{2.481}$$

Where $\mathbf{A}(t)$ is a $\mathbb{R}^{n_c \times n}$ matrix representing the total constrain matrix and it can be directly computed as:

$$\mathbf{A}(t) = \begin{bmatrix} \mathbf{A}^1(t) & \mathbf{O} & \dots & \mathbf{O} \\ \mathbf{O} & \mathbf{A}^2(t) & \dots & \mathbf{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{O} & \mathbf{O} & \dots & \mathbf{A}^{n_b}(t) \end{bmatrix} \tag{2.482}$$

The vector $\mathbf{b}(t)$ is a $\mathbb{R}^{n_c}$ vector representing to the global constraint vector and it can be directly assembled as:

$$\mathbf{b}(t) = \begin{bmatrix} \mathbf{b}^1(t) \\ \mathbf{b}^2(t) \\ \vdots \\ \mathbf{b}^{n_b}(t) \end{bmatrix} \tag{2.483}$$

Finally, the set of equation of motion and constraint equations which describe the dynamic of a general flexible multibody system is:

$$\begin{cases} \mathbf{M}(t)\ddot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{Q}_v(t) + \mathbf{Q}_{e,nc}(t) + \mathbf{Q}_c(t) \\ \mathbf{A}(t)\ddot{\mathbf{q}}(t) = \mathbf{b}(t) \end{cases} \tag{2.484}$$

It is worth noting that, unlike the global mass matrix $\mathbf{M}(t)$, the global stiffness matrix $\mathbf{K}$ results to be a constant matrix. These equation can be explicitly solved to get the generalized acceleration vector $\ddot{\mathbf{q}}(t)$ and the generalized constraint vector $\mathbf{Q}_c(t)$ in order to obtain the fundamental equations of constrained Dynamics.

3. SYSTEM IDENTIFICATION

3.1. INTRODUCTION

System identification is the art of determining a mathematical model of a physical system by combining information obtained from experimental data with that derived from an a priori knowledge $[1]$. There are several types of system identification algorithms in relation to different goals one wants to pursue $[2]$. In mechanical engineering, applied system identification allows to get modal parameters of a dynamical system using force and vibration measurements $[3]$. These parameters are typically used to design optimal control laws whereas in the field of structural health monitoring they are used to detect and evaluate system damage $[4]$. A very powerful algorithm to perform system identification is Eigensystem Realization Algorithm with Data Correlation using Observer/Kalman Filter Identification Method (ERA/DC OKID). This method was originally developed by Juang $[5]$, $[6]$. This numerical procedure is able to construct a state-space representation of a mechanical system starting from input and output measurements even in presence of process and measurement noise. Another important algorithm is the Numerical Algorithm for State Space Subspace System Identification (N4SID). This method was originally developed by Van Overschee and De Moor $[7]$. On the other hand, when all degrees of freedom are instrumented with a force and/or an acceleration transducer, an efficient numerical procedure can be implemented to construct a second-order models of the mechanical system starting from state-space representations

(MKR) [8], [9], [10]. Experimental investigations show that the Eigensystem Realization Algorithm with Data Correlation using Observer/Kalman Filter Identification Method (ERA/DC OKID), as well as Numerical Algorithm for State Space Subspace System Identification (N4SID), correctly determines system natural frequencies and damping ratios [11]. On the other hand, the method to construct second-order models from state-space representations (MKR) properly identifies mass and stiffness matrices but it fails in esteeming damping matrix because actual measurements are never noise-free [12], [13]. Nevertheless, if the real system is lightly damped, an efficient procedure can be developed to identifying in a direct way system damping matrix from state-space realization by assuming proportional damping hypothesis [12], [13].

3.2. STATE SPACE REPRESENTATION

Consider a linear mechanical system with multiple degrees of freedom. The system equations of motion are a set of n_2 coupled second-order differential equations, where n_2 is the number of system independent coordinates. These equations can be expressed in matrix notation as:

$$\mathbf{M}\ddot{\mathbf{x}}(t)+\mathbf{R}\dot{\mathbf{x}}(t)+\mathbf{K}\mathbf{x}(t)=\mathbf{F}(t) \tag{3.1}$$

Where $\mathbf{x}(t)$ is a $\mathbb{R}^{n_2}$ vector representing the system generalized displacement vector which describe the system dynamic, $\mathbf{M}$, $\mathbf{K}$ and $\mathbf{R}$ are $\mathbb{R}^{n_2\times n_2}$ matrices representing respectively the system mass, stiffness and damping matrices and $\mathbf{F}(t)$ is a $\mathbb{R}^{n_2}$ vector of external applied forces. In practical application not all the degrees of freedom are excited by an external force and therefore the vector of forcing functions $\mathbf{F}(t)$ is typically expressed as a linear combination of an input vector. Indeed:

$$\mathbf{F}(t)=\mathbf{B}_2\mathbf{u}(t) \tag{3.2}$$

Where $\mathbf{u}(t)$ is a $\mathbb{R}^r$ input vector and $\mathbf{B}_2$ is a $\mathbb{R}^{n_2\times r}$ matrix characterizing the location and the type of inputs. On the other hand, in control problem there is another set of equations describing the output quantities in terms of the variables which describe the system dynamics, namely the measurement equations. The measurement equations are a set of m coupled algebraic equations, where m is the number of the output variables of interest. The measurement equations express the vector of output measurements as a linear combination of the system generalized displacement, velocity and acceleration vectors. These equations can be written in matrix notation as:

$$\mathbf{y}(t)=\mathbf{C}_d\mathbf{x}(t)+\mathbf{C}_v\dot{\mathbf{x}}(t)+\mathbf{C}_a\ddot{\mathbf{x}}(t) \tag{3.3}$$

Where $\mathbf{y}(t)$ is a $\mathbb{R}^m$ vector containing the measured output quantities and $\mathbf{C}_d$, $\mathbf{C}_v$ and $\mathbf{C}_a$ are $\mathbb{R}^{m \times n_2}$ matrices representing respectively the output influence matrices for displacement, velocity and acceleration. Note that in practical applications the number of output quantities of interest m is typically lower than the numbers of system degrees of freedom n_2 because it is impractical, or even impossible, to instrument each system degree of freedom with a sensor. The sets of equations of motion and measurement equations describe respectively the system dynamics and the measurement evolution in time by using n_2 configuration variables such as physical coordinate vectors. On the other hand, these sets of equations can also be equivalently represented in different forms by using $n = 2n_2$ state variables defined as follows:

$$\begin{aligned} \mathbf{z}(t) &= \begin{bmatrix} \mathbf{z}_1(t) \\ \mathbf{z}_2(t) \end{bmatrix} = \\ &= \begin{bmatrix} \mathbf{x}(t) \\ \dot{\mathbf{x}}(t) \end{bmatrix} \end{aligned} \tag{3.4}$$

Where $\mathbf{z}(t)$ is a $\mathbb{R}^n$ state vector composed of system generalized displacement and velocity vectors. Indeed, assuming that the mass matrix $\mathbf{M}$ is a non-singular invertible matrix, the equations of motion can be rewritten in terms of the state vector $\mathbf{z}(t)$ as follows:

$$\begin{cases} \dot{\mathbf{z}}_1(t) = \mathbf{z}_2(t) \\ \dot{\mathbf{z}}_2(t) = -\mathbf{M}^{-1}\mathbf{R}\mathbf{z}_2(t) - \mathbf{M}^{-1}\mathbf{K}\mathbf{z}_1(t) + \mathbf{M}^{-1}\mathbf{B}_2\mathbf{u}(t) \end{cases} \tag{3.5}$$

Where an identity equation deriving from the definition of the state vector has been adjoined as first vector equation. Consequently, the original second-order equations of motion can now be rewritten in first-order form as:

$$\dot{\mathbf{z}}(t) = \mathbf{A}_c\mathbf{z}(t) + \mathbf{B}_c\mathbf{u}(t) \tag{3.6}$$

Where $\mathbf{A}_c$ is a $\mathbb{R}^{n\times n}$ matrix representing the continuous-time system state matrix and $\mathbf{B}_c$ is a $\mathbb{R}^{n\times r}$ matrix representing the continuous-time system state influence matrix. These matrices are respectively defined as:

$$\mathbf{A}_c = \begin{bmatrix} \mathbf{O} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{R} \end{bmatrix} \tag{3.7}$$

$$\mathbf{B}_c = \begin{bmatrix} \mathbf{O} \\ \mathbf{M}^{-1}\mathbf{B}_2 \end{bmatrix} \tag{3.8}$$

In addition, the output equations can be expressed in terms of the state vector $\mathbf{z}(t)$ as:

$$\mathbf{y}(t) = \mathbf{C}\mathbf{z}(t) + \mathbf{D}\mathbf{u}(t) \tag{3.9}$$

Where $\mathbf{C}$ is a $\mathbb{R}^{m\times n}$ matrix representing the output influence matrix and $\mathbf{D}$ is a $\mathbb{R}^{m\times r}$ matrix called direct transmission matrix. These matrices can be respectively computed as:

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_d - \mathbf{C}_a\mathbf{M}^{-1}\mathbf{K} & \mathbf{C}_v - \mathbf{C}_a\mathbf{M}^{-1}\mathbf{R} \end{bmatrix} \tag{3.10}$$

$$\mathbf{D} = \mathbf{C}_a\mathbf{M}^{-1}\mathbf{B}_2 \tag{3.11}$$

The sets of equations of motion and measurement equations constitute a continuous-time state-space model of the dynamical system. The state-space model describes the relationship between the inputs and the outputs of a system between an intermediate variable named the state vector $\mathbf{z}(t)$. It is worth to point out that the state-space model is coordinate dependent. Indeed, let the state vector be transformed by a new set of coordinates:

$$\mathbf{z}(t) = \mathbf{T}\overline{\mathbf{z}}(t) \tag{3.12}$$

Where $\mathbf{T}$ is a $\mathbb{R}^{n\times n}$ matrix representing an invertible coordinate transformation. According to this coordinate transformation, the state-space model become:

$$\dot{\bar{\mathbf{z}}}(t) = \bar{\mathbf{A}}_c \bar{\mathbf{z}}(t) + \bar{\mathbf{B}}_c \mathbf{u}(t) \tag{3.13}$$

$$\mathbf{y}(t) = \bar{\mathbf{C}} \bar{\mathbf{z}}(t) + \mathbf{D}\mathbf{u}(t) \tag{3.14}$$

Where $\bar{\mathbf{A}}_c$, $\bar{\mathbf{B}}_c$ and $\bar{\mathbf{C}}$ are respectively $\mathbb{R}^{n\times n}$, $\mathbb{R}^{n\times r}$ and $\mathbb{R}^{m\times n}$ matrices representing the state matrix, the state influence matrix and the output influence matrix referred to the transformed state $\bar{\mathbf{z}}(t)$. These matrices can be computed as:

$$\bar{\mathbf{A}}_c = \mathbf{T}\mathbf{A}_c\mathbf{T}^{-1} \tag{3.15}$$

$$\bar{\mathbf{B}}_c = \mathbf{T}^{-1}\mathbf{B}_c \tag{3.16}$$

$$\bar{\mathbf{C}} = \mathbf{C}\mathbf{T} \tag{3.17}$$

This transformed state-space model is related to the original one by a similarity transformation in the sense that the transformation $\mathbf{T}$ preserves the eigenvalues of the state space matrix $\mathbf{A}_c$. In addition, the transformed state-space model describes the same input-output relationship as the original state space model. Note that the direct transmission matrix $\mathbf{D}$ is coordinate independent. The state-space representations of system equations of motion can be reformulated in a symmetric form. The symmetric reformulation of system state-space model can be achieved in at least two ways. The first method considers the following formulation of system equations of motion in terms of the state vector $\mathbf{z}(t)$:

$$\begin{cases} \mathbf{R}\dot{\mathbf{z}}_1(t) + \mathbf{M}\dot{\mathbf{z}}_2(t) = -\mathbf{K}\mathbf{z}_1(t) + \mathbf{B}_2\mathbf{u}(t) \\ \mathbf{M}\dot{\mathbf{z}}_1(t) = \mathbf{M}\mathbf{z}_2(t) \end{cases} \tag{3.18}$$

Where an identity equation has been adjoined as second matrix equation. Consequently, the second-order equations of motion can be rewritten in first-order form as:

$$\mathbf{V}_c\dot{\mathbf{z}}(t) = \mathbf{S}_c\mathbf{z}(t) + \mathbf{N}_c\mathbf{u}(t) \tag{3.19}$$

Where $\mathbf{V}_c$ and $\mathbf{S}_c$ are $\mathbb{R}^{n\times n}$ symmetric matrices and $\mathbf{N}_c$ is a $\mathbb{R}^{n\times r}$ matrix respectively defined as:

$$\mathbf{V}_c = \begin{bmatrix} \mathbf{R} & \mathbf{M} \\ \mathbf{M} & \mathbf{O} \end{bmatrix} \tag{3.20}$$

$$\mathbf{S}_c = \begin{bmatrix} -\mathbf{K} & \mathbf{O} \\ \mathbf{O} & \mathbf{M} \end{bmatrix} \tag{3.21}$$

$$\mathbf{N}_c = \begin{bmatrix} \mathbf{B}_2 \\ \mathbf{O} \end{bmatrix} \tag{3.22}$$

This is the first method to obtain a symmetric formulation of the system state-space model. The second method to derive a symmetric representation of the system state-space model is based on the following formulation of system equations of motion in terms of the state vector $\mathbf{z}(t)$:

$$\begin{cases} -\mathbf{K}\dot{\mathbf{z}}_1(t) = -\mathbf{K}\mathbf{z}_2(t) \\ \mathbf{M}\dot{\mathbf{z}}_2(t) = -\mathbf{K}\mathbf{z}_1(t) - \mathbf{R}\mathbf{z}_2(t) + \mathbf{B}_2\mathbf{u}(t) \end{cases} \tag{3.23}$$

Where an identity equation has been adjoined as first matrix equation. Similarly to the previous case, the second-order equations of motion can be expressed in first-order form as follows:

$$\mathbf{V}_c \dot{\mathbf{z}}(t) = \mathbf{S}_c \mathbf{z}(t) + \mathbf{N}_c \mathbf{u}(t) \tag{3.24}$$

Where $\mathbf{V}_c$ and $\mathbf{S}_c$ are $\mathbb{R}^{n \times n}$ symmetric matrices and $\mathbf{N}_c$ is a $\mathbb{R}^{n \times r}$ matrix respectively defined as:

$$\mathbf{V}_c = \begin{bmatrix} -\mathbf{K} & \mathbf{O} \\ \mathbf{O} & \mathbf{M} \end{bmatrix} \tag{3.25}$$

$$\mathbf{S}_c = \begin{bmatrix} \mathbf{O} & -\mathbf{K} \\ -\mathbf{K} & -\mathbf{R} \end{bmatrix} \tag{3.26}$$

$$\mathbf{N}_c = \begin{bmatrix} \mathbf{O} \\ \mathbf{B}_2 \end{bmatrix} \tag{3.27}$$

Note that the symmetric formulations of the state-space representation of the system equations of motion are both ascribable to the standard one observing that the state transmission matrix $\mathbf{A}_c$ and the state influence matrix $\mathbf{B}_c$ can be expressed using the matrices $\mathbf{V}_c$, $\mathbf{S}_c$ and $\mathbf{N}_c$. Indeed:

$$\mathbf{A}_c = \mathbf{V}_c^{-1} \mathbf{S}_c \tag{3.28}$$

$$\mathbf{B}_c = \mathbf{V}_c^{-1} \mathbf{N}_c \tag{3.29}$$

One of the major advantages of representing the equations of motion in a state-space formulation is that the now the equations assume the form of a system of first-order matrix differential equations and therefore they can be solved in a straightforward manner by using Duhamel principle to yield:

$$\mathbf{z}(t) = e^{\mathbf{A}_c(t-t_0)}\mathbf{z}_0 + \int_{t_0}^{t} e^{\mathbf{A}_c(t-\tau)}\mathbf{B}_c\mathbf{u}(\tau)d\tau \tag{3.30}$$

Where $\mathbf{z}_0$ is a $\mathbb{R}^n$ vector containing the initial conditions. The output vector can be directly computed by using this expression of state vector to yield:

$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}_c(t-t_0)}\mathbf{z}_0 + \mathbf{C}\int_{t_0}^{t} e^{\mathbf{A}_c(t-\tau)}\mathbf{B}_c\mathbf{u}(\tau)d\tau + \mathbf{D}\mathbf{u}(t) \tag{3.31}$$

The solution of system first-order differential equations of motion can be used to convert the continuous-time state-space model to a discrete-time representation for digital control considering the zero-order hold mechanism. A zero-order hold device takes a continuous signal and turns it in a stepwise one in which the signal is sampled and held for a certain interval of time. In practice, when a control system is implemented by a computer, the inclusion of a sample and hold device is routine. If the sampling interval is Δt, the sampling frequency f_c is equal the inverse of the sampling interval Δt and the Nyquist frequency f_N, that is the maximum frequency captured by the sampling process or, in other words, the frequency at which the aliasing phenomenon starts occurring, is equal to one half of the sampling frequency f_c. Indeed:

$$f_c = \frac{1}{\Delta t} \tag{3.32}$$

$$\begin{aligned} f_N &= \frac{1}{2} f_c = \\ &= \frac{1}{2\Delta t} \end{aligned} \tag{3.33}$$

Consider the following discrete sampling interval:

$$t = 0, \Delta t, 2\Delta t, \ldots, k\Delta t, (k+1)\Delta t, \ldots \tag{3.34}$$

Assume that the input vector is held constant and equal to $\mathbf{u}(k\Delta t)$ over the time interval from $t_0 = k\Delta t$ to $t = (k+1)\Delta t$ by a zero-order hold device:

$$\mathbf{u}(t) = \mathbf{u}(k\Delta t) \quad , \quad k\Delta t \leq t < \left(k+1\right)\Delta t \quad , \quad k = 1, 2, 3, \ldots \tag{3.35}$$

The solution of the continuous-time state-space model can be rewritten by using zero-order hold assumption as:

$$\begin{aligned} \mathbf{z}(\left(k+1\right)\Delta t) &= e^{\mathbf{A}_c \Delta t}\mathbf{z}(k\Delta t) + \int_{k\Delta t}^{(k+1)\Delta t} e^{\mathbf{A}_c\left(\left(k+1\right)\Delta t - \tau\right)}\mathbf{B}_c\mathbf{u}(k\Delta t)d\tau = \\ &= e^{\mathbf{A}_c \Delta t}\mathbf{z}(k\Delta t) + \int_0^{\Delta t} e^{\mathbf{A}_c \tau'} d\tau' \mathbf{B}_c \mathbf{u}(k\Delta t) \end{aligned} \tag{3.36}$$

Where the following change of variable has been used:

$$\tau' = \left(k+1\right)\Delta t - \tau \tag{3.37}$$

Using the simplified notation k for the time argument $k\Delta t$, a discrete-time representation of system equations of motion can be obtained from the previous equations:

$$\mathbf{z}(k+1) = \mathbf{A}\mathbf{z}(k) + \mathbf{B}\mathbf{u}(k) \tag{3.38}$$

Where $\mathbf{A}$ is a $\mathbb{R}^{n \times n}$ matrix representing the discrete-time system state matrix and $\mathbf{B}$ is a $\mathbb{R}^{n \times r}$ matrix representing the discrete-time system state influence matrix. These matrices are respectively defined as:

$$\mathbf{A} = e^{\mathbf{A}_c \Delta t} \tag{3.39}$$

$$\mathbf{B} = \int_0^{\Delta t} e^{\mathbf{A}_c \tau'} d\tau' \mathbf{B}_c \tag{3.40}$$

The discrete-time state matrix $\mathbf{A}$ and the discrete-time state influence matrix $\mathbf{B}$ can be explicitly computed from their continuous-time counterparts directly utilizing their definitions:

$$\begin{aligned}\mathbf{A} = e^{\mathbf{A}_c \Delta t} &= \\ &= \mathbf{I} + \mathbf{A}_c \Delta t + \frac{1}{2}\mathbf{A}_c^2 \Delta t^2 + \frac{1}{6}\mathbf{A}_c^3 \Delta t^3 + \ldots = \\ &= \mathbf{I} + \sum_{k=1}^{\infty}\left(\frac{1}{k!}\mathbf{A}_c^k \Delta t^k\right) = \\ &= \sum_{k=0}^{\infty}\left(\frac{1}{k!}\mathbf{A}_c^k \Delta t^k\right)\end{aligned} \tag{3.41}$$

$$\begin{aligned}\mathbf{B} = \int_0^{\Delta t} e^{\mathbf{A}_c \tau'} d\tau' \mathbf{B}_c &= \\ &= \int_0^{\Delta t}\left(\mathbf{I} + \mathbf{A}_c \tau' + \frac{1}{2}\mathbf{A}_c^2 \tau'^2 + \frac{1}{6}\mathbf{A}_c^3 \tau'^3 + \ldots\right) d\tau' \mathbf{B}_c = \\ &= \left[\mathbf{I}\tau' + \frac{1}{2}\mathbf{A}_c \tau'^2 + \frac{1}{6}\mathbf{A}_c^2 \tau'^3 + \frac{1}{24}\mathbf{A}_c^3 \tau'^4 + \ldots\right]_0^{\Delta t} \mathbf{B}_c = \\ &= \left(\mathbf{I}\Delta t + \frac{1}{2}\mathbf{A}_c \Delta t^2 + \frac{1}{6}\mathbf{A}_c^2 \Delta t^3 + \frac{1}{24}\mathbf{A}_c^3 \Delta t^4 + \ldots\right)\mathbf{B}_c = \\ &= \left(\sum_{k=1}^{\infty}\left(\frac{1}{k!}\mathbf{A}_c^{k-1} \Delta t^k\right)\right)\mathbf{B}_c\end{aligned} \tag{3.42}$$

If none of the eigenvalues of $\mathbf{A}_c$ are zero, then $\mathbf{A}^{-1}$ exists and the expression for $\mathbf{B}$ can be further simplified to yield:

$$\mathbf{B} = \mathbf{A}_c^{-1}\left(\mathbf{A} - \mathbf{I}\right)\mathbf{B}_c \tag{3.43}$$

On the other hand, by using zero-order hold assumption the output equations can be sampled at each instant in an analogous fashion to yield:

$$\mathbf{y}(k) = \mathbf{C}\mathbf{z}(k) + \mathbf{D}\mathbf{u}(k) \tag{3.44}$$

Note that output influence matrix $\mathbf{C}$ and the direct transmission matrix $\mathbf{D}$ do not change in the discrete-time representation. The sets of discretized equations of motion and discretized measurement equations constitute a discrete-time state-space model of the dynamical system. Note that a similarity transformation of the discrete-time state space model produces similar effects as in the case of continuous-time state space model, namely the eigenvalues of the discrete state matrix are unchanged as well as the input-output relationship. Because experimental data are always discrete in practice, these sets of equations form the basis for applied system identification of linear, time-invariant, dynamical systems. It is worth to notice that a continuous-time system can be represented by a discrete-time one which exactly describe its time evolution in the sampling instants. This model is very different from a discretized model which can be obtained by the numerical approximation of the time derivatives with a finite difference scheme. Indeed, the discrete-time model has been obtained by actually integrating the state equations over each successive time interval and therefore the system response derived from the discrete-time model is correct in the sampling instants. On the other hand, unlike the discrete time model, a finite difference scheme is not able to represent exactly the system response at the sampling instants no matter how small the approximation error is.

3.3. MODAL ANALYSIS OF STATE SPACE MODEL

Consider the continuous-time state-space representation of the equations of motion of a linear time-invariant dynamical system and assume that there are not external inputs. The equations of state are:

$$\dot{\mathbf{z}}(t) = \mathbf{A}_c \mathbf{z}(t) \tag{3.45}$$

The solution of these differential equations can be found supposing that the state vector assumes this form:

$$\mathbf{z}(t) = \boldsymbol{\psi} e^{\lambda_c t} \tag{3.46}$$

Where $\boldsymbol{\psi}$ is an $\mathbb{R}^n$ unknown vector and λ_c is an unknown scalar. The assumed solution must satisfy the state equations and therefore to impose it the supposed solution can be put into the state equations in order to get:

$$\left(\mathbf{A}_c - \lambda_c \mathbf{I}\right)\boldsymbol{\psi} = \mathbf{0} \tag{3.47}$$

This is an eigenvalue problem for the state matrix $\mathbf{A}_c$ that can be restated in the standard form as follows:

$$\mathbf{A}_c \boldsymbol{\psi} = \lambda_c \boldsymbol{\psi} \tag{3.48}$$

The results of this problem is a set of n complex conjugate eigenvalues:

$$\begin{cases} \lambda_{c,2j-1} = -\xi_j \omega_{n,j} - \mathbf{i}\omega_{n,j}\sqrt{1-\xi_j^2} \\ \lambda_{c,2j-1}^* = -\xi_j \omega_{n,j} + \mathbf{i}\omega_{n,j}\sqrt{1-\xi_j^2} \end{cases} \quad , \quad j = 1, 2, \ldots, n \tag{3.49}$$

These complex conjugate eigenvalues $\lambda_{c,j}$ correspond to a set of n natural frequencies $\omega_{n,j}$, a set of n damping ratios ξ_j and to a set of n complex conjugate eigenvectors $\boldsymbol{\psi}_j$ which represent the system mode shapes. The sets of eigenvalues and eigenvectors can be grouped in a matrix form as:

$$\mathbf{\Lambda}_c = diag(\lambda_{c,1}, \lambda_{c,2}, \ldots, \lambda_{c,n-1}, \lambda_{c,n}) \tag{3.50}$$

$$\mathbf{\Psi} = \begin{bmatrix} \boldsymbol{\psi}_1 & \boldsymbol{\psi}_2 & \cdots & \boldsymbol{\psi}_{n-1} & \boldsymbol{\psi}_n \end{bmatrix} \tag{3.51}$$

Where $\mathbf{\Lambda}_c$ is a $\mathbb{R}^{n\times n}$ diagonal matrix containing the system eigenvalues and $\mathbf{\Psi}$ is a $\mathbb{R}^{n\times n}$ matrix containing the system eigenvectors stacked by column. By using these definitions the eigenvalues problem of state matrix $\mathbf{A}_c$ can be restated in matrix form as:

$$\mathbf{A}_c\mathbf{\Psi} = \mathbf{\Psi}\mathbf{\Lambda}_c \tag{3.52}$$

Assume that $\boldsymbol{\varphi}_j$ are the n_2 eigenvectors of system equations of motion written in physical coordinates. This set of eigenvectors can be put in a matrix form as:

$$\mathbf{W} = \begin{bmatrix} \boldsymbol{\varphi}_1 & \boldsymbol{\varphi}_1^* & \cdots & \boldsymbol{\varphi}_{n_2} & \boldsymbol{\varphi}_{n_2}^* \end{bmatrix} \tag{3.53}$$

Where $\mathbf{W}$ is a $\mathbb{R}^{n_2\times n}$ eigenvector matrix. The state-space eigenvector matrix $\mathbf{\Psi}$ can be expressed using the physical coordinate eigenvector matrix $\mathbf{W}$ and the eigenvalues matrix $\mathbf{\Lambda}_c$ as:

$$\mathbf{\Psi} = \begin{bmatrix} \mathbf{W} \\ \mathbf{W}\mathbf{\Lambda}_c \end{bmatrix} \tag{3.54}$$

Note that the matrix exponential $e^{\mathbf{A}_c t}$ can be computed by using the spectral decomposition of the state matrix $\mathbf{A}_c$ as follows:

$$e^{\mathbf{A}_c t} = \mathbf{\Psi} e^{\mathbf{\Lambda}_c t} \mathbf{\Psi}^{-1} \tag{3.55}$$

On the other hand, consider the eigenvalue problem of the discrete-time system state matrix $\mathbf{A}$:

$$\mathbf{A\Psi} = \mathbf{\Psi\Lambda} \tag{3.56}$$

Where $\mathbf{\Lambda}$ is a $\mathbb{R}^{n \times n}$ diagonal matrix containing the discrete-time system eigenvalues and $\mathbf{\Psi}$ is a $\mathbb{R}^{n \times n}$ matrix containing the system eigenvectors stacked by column. The discrete-time system eigenvalues are related to the continuous-time system eigenvalues by the following matrix equation:

$$\mathbf{\Lambda} = e^{\mathbf{\Lambda}_c \Delta t} \tag{3.57}$$

Where Δt is the time interval of the digital sampling. Note that the continuous-time system eigenvectors and the discrete-time eigenvectors are represented by the same matrix $\mathbf{\Psi}$ because they are identical. On the other hand, the converse transformation from discrete-time eigenvalues to continuous-time eigenvalues can be written as:

$$\mathbf{\Lambda}_c = \frac{1}{\Delta t} \ln(\mathbf{\Lambda}) \tag{3.58}$$

It is important to note that the transformation from the discrete-time model to the continuous-time model is not unique. Indeed, the imaginary part of a natural logarithm of a complex number can be adjusted by the addition of a multiple of 2π which allows the reconstructed continuous-time eigenvalues to take on different values. For instance, for the generic eigenvalue λ_j:

$$\begin{aligned} \lambda_j &= e^{\lambda_{c,j}\Delta t} = \\ &= e^{\lambda_{c,j}\Delta t + \mathbf{i}2k\pi} \quad , \quad \forall k \in \mathbb{Z} \end{aligned} \tag{3.59}$$

$$\lambda_{c,j} + \mathbf{i}\frac{2k\pi}{\Delta t} = \frac{\ln(\lambda_j)}{\Delta t} \quad , \quad \forall k \in \mathbb{Z} \tag{3.60}$$

This correspond to the fact that any two frequencies which differs by a multiple of $\frac{2\pi}{\Delta t}$ are actually indistinguishable when observed at the sampling frequency $f_c = \frac{1}{\Delta t}$. Therefore, in practical applications to correctly interpret natural frequencies of physical system either the sampling interval Δt must be sufficiently short or a filter must be added to prevent that frequencies beyond the Nyquist frequency are interpreted as real frequencies. Consider now the following modal transformation of coordinates for the discrete-time state-space model:

$$\mathbf{z}(k) = \mathbf{\Psi}\mathbf{p}(k) \tag{3.61}$$

Where $\mathbf{p}(k)$ is a $\mathbb{R}^n$ discrete-time modal state vector. By using this coordinate transformation a modal model of system discrete-time state-space representation can be obtained as:

$$\mathbf{p}(k+1) = \mathbf{A}_m\mathbf{p}(k) + \mathbf{B}_m\mathbf{u}(k) \tag{3.62}$$

$$\mathbf{y}(k) = \mathbf{C}_m\mathbf{p}(k) + \mathbf{D}\mathbf{u}(k) \tag{3.63}$$

Where $\mathbf{A}_m$, $\mathbf{B}_m$ and $\mathbf{C}_m$ are respectively $\mathbb{R}^{n\times n}$, $\mathbb{R}^{n\times r}$ and $\mathbb{R}^{m\times n}$ matrices representing the modal state matrix, the modal state influence matrix and the modal output influence matrix. These matrices can be computed as:

$$\mathbf{A}_m = \mathbf{\Lambda} \tag{3.64}$$

$$\mathbf{B}_m = \mathbf{\Psi}^{-1}\mathbf{B} \tag{3.65}$$

$$\mathbf{C}_m = \mathbf{C}\mathbf{\Psi} \tag{3.66}$$

Indeed, the discrete-time modal state matrix $\mathbf{A}_m$ is exactly equal to the discrete-time eigenvalue matrix $\mathbf{\Lambda}$. The modal state matrix $\mathbf{A}_m$ contains the information of system natural frequencies and damping ratios whereas the modal state influence matrix $\mathbf{B}_m$ define the initial mode amplitudes and the modal output influence matrix $\mathbf{C}_m$ represent the mode shapes at the sensor points. All the modal parameters of a dynamic system can thus be identified by the triplet of matrices $\mathbf{A}_m$, $\mathbf{B}_m$ and $\mathbf{C}_m$. It is important to realize that system modal parameters are unique for a given state-space model and therefore the triplet of modal matrices $\mathbf{A}_m$, $\mathbf{B}_m$ and $\mathbf{C}_m$ are coordinate independent as well as the direct transmission term $\mathbf{D}$.

3.4. MARKOV PARAMETERS

Consider a discrete-time state-space model described by the following set of equations:

$$\mathbf{z}(k+1) = \mathbf{A}\mathbf{z}(k) + \mathbf{B}\mathbf{u}(k) \tag{3.67}$$

$$\mathbf{y}(k) = \mathbf{C}\mathbf{z}(k) + \mathbf{D}\mathbf{u}(k) \tag{3.68}$$

The computation of system response to a general input vector $\mathbf{u}(k)$ can be easily performed because the integration action is already built into the model. Indeed:

$$\mathbf{z}(k) = \mathbf{A}^k\mathbf{z}(0) + \sum_{j=1}^{k}\left(\mathbf{A}^{j-1}\mathbf{B}\mathbf{u}(k-j)\right) \tag{3.69}$$

$$\mathbf{y}(k) = \mathbf{C}\mathbf{A}^k\mathbf{z}(0) + \mathbf{C}\sum_{j=1}^{k}\left(\mathbf{A}^{j-1}\mathbf{B}\mathbf{u}(k-j)\right) + \mathbf{D}\mathbf{u}(k) \tag{3.70}$$

Where $\mathbf{z}(0)$ is a $\mathbb{R}^n$ vector containing the initial state. Note that the response of the discrete-time model differs from the response of the continuous-time model because in the discrete-time case the input functions are discretized with a zero-order hold device. Consider now a series of pulse functions applied at initial instant for each input:

$$\mathbf{u}(k=0)=\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix} \quad , \quad \mathbf{u}(k>0)=\begin{bmatrix}0\\0\\\vdots\\0\end{bmatrix}$$
$$\mathbf{u}(k=0)=\begin{bmatrix}0\\1\\\vdots\\0\end{bmatrix} \quad , \quad \mathbf{u}(k>0)=\begin{bmatrix}0\\0\\\vdots\\0\end{bmatrix} \tag{3.71}$$
$$\vdots$$
$$\mathbf{u}(k=0)=\begin{bmatrix}0\\0\\\vdots\\1\end{bmatrix} \quad , \quad \mathbf{u}(k>0)=\begin{bmatrix}0\\0\\\vdots\\0\end{bmatrix}$$

When the substitution of this input series is performed in the system response assuming zero initial conditions, the results can be assembled into a sequence of pulse-response matrices:

$$\mathbf{H}_0=\mathbf{O},\quad \mathbf{H}_1=\mathbf{B},\quad \mathbf{H}_2=\mathbf{AB},\quad \ldots,\quad \mathbf{H}_k=\mathbf{A}^{k-1}\mathbf{B} \tag{3.72}$$

$$\mathbf{Y}_0=\mathbf{D},\quad \mathbf{Y}_1=\mathbf{CB},\quad \mathbf{Y}_2=\mathbf{CAB},\quad \ldots,\quad \mathbf{Y}_k=\mathbf{CA}^{k-1}\mathbf{B} \tag{3.73}$$

Where $\mathbf{H}_k$ and $\mathbf{Y}_k$ are respectively $\mathbb{R}^{n\times r}$ and $\mathbb{R}^{m\times r}$ matrices which are known as system Markov parameters. Note that these parameters are related by the following equations:

$$\mathbf{Y}_k=\mathbf{CH}_k \quad , \quad k=1,2,\ldots \tag{3.74}$$

System Markov parameters can be obtained from experimental data and are typically used as the basis for system identification algorithms. Indeed, it straightforward to realize that the discrete-time state-space model is embedded in the Markov parameters sequence. Since the Markov parameters sequence is simply the pulse response of the system, they must be unique for a given system. Therefore any coordinate transformation of the state vector yields the same system Markov parameters. Using the definitions of system Markov parameters, the system response to a general input vector assuming zero initial conditions can be rewritten as:

$$\mathbf{z}(k) = \sum_{j=0}^{k} \left(\mathbf{H}_j \mathbf{u}(k-j) \right) \tag{3.75}$$

$$\mathbf{y}(k) = \sum_{j=0}^{k} \left(\mathbf{Y}_j \mathbf{u}(k-j) \right) \tag{3.76}$$

These equations shows that the contributions to the state $\mathbf{z}(k)$ and to the output $\mathbf{y}(k)$ at time step k given by the input $\mathbf{u}(k)$ and by the input $\mathbf{u}(k-j)$ applied at the previous time steps are weighted by the Markov parameters. Therefore the pulse response sequence is also known as the weighting sequence and the input-output description is called weighting sequence description. The weighting sequence description uses the pulse response sequence to characterize the input-output relationship instead of using the state description. The advantage of this description is that the dimension of the matrix sequence needed is determined by the number of inputs r and outputs m only, regardless the order of system state n. On the other hand, the disadvantage of this formulation is that for lightly damped systems a large number of terms must be retained in the summation of the weighting sequence description to obtain a satisfactory approximation. To overcome this problem, the discrete-time state-space model can be slightly modified introducing an observer which provide an estimate of system state from inputs and outputs measurements. The discrete-time state-space model with the introduction of the state estimator becomes:

$$\hat{\mathbf{z}}(k+1) = \mathbf{A}\hat{\mathbf{z}}(k) + \mathbf{B}\mathbf{u}(k) - \mathbf{G}(\mathbf{y}(k) - \hat{\mathbf{y}}(k)) \tag{3.77}$$

$$\hat{\mathbf{y}}(k) = \mathbf{C}\hat{\mathbf{z}}(k) + \mathbf{D}\mathbf{u}(k) \tag{3.78}$$

Where $\hat{\mathbf{z}}(k)$ is an $\mathbb{R}^n$ estimated state vector, $\hat{\mathbf{y}}(k)$ is an $\mathbb{R}^m$ estimated output vector and $\mathbf{G}$ is an $\mathbb{R}^{n\times m}$ observer matrix. These equations forms a discrete-time state-space observer model of a dynamical system. The discrete-time state-space observer state equations can be rewritten in a compact form as follows:

$$\hat{\mathbf{z}}(k+1) = \bar{\mathbf{A}}\hat{\mathbf{z}}(k) + \bar{\mathbf{B}}\mathbf{v}(k) \tag{3.79}$$

Where $\bar{\mathbf{A}}$ is a $\mathbb{R}^{n\times n}$ discrete-time observer state matrix, $\bar{\mathbf{B}}$ is a $\mathbb{R}^{n\times(r+m)}$ discrete-time observer state influence matrix and $\mathbf{v}(k)$ is a $\mathbb{R}^{r+m}$ generalized input vector respectively defined as:

$$\bar{\mathbf{A}} = \mathbf{A} + \mathbf{G}\mathbf{C} \tag{3.80}$$

$$\bar{\mathbf{B}} = \begin{bmatrix} \mathbf{B} + \mathbf{G}\mathbf{D} & -\mathbf{G} \end{bmatrix} \tag{3.81}$$

$$\mathbf{v}(k) = \begin{bmatrix} \mathbf{u}(k) \\ \mathbf{y}(k) \end{bmatrix} \tag{3.82}$$

Note that by using the previous definitions the discrete-time state-space observer model appears identical in form respect to discrete-time state-space model. However, the eigenvalues of observer state matrix $\bar{\mathbf{A}}$ are moved from the eigenvalues of the state matrix $\mathbf{A}$ as a consequence of the introduction of the observer matrix $\mathbf{G}$. Therefore, since the observer matrix $\mathbf{G}$ can be arbitrary chosen, the observer state matrix $\bar{\mathbf{A}}$ can be made as asymptotically stable as desired. In practical applications the presence of process and measurement noise suggests to choose the Kalman filter as gain matrix $\mathbf{G}$. Since the discrete-time state-space observer model is analogous in form to the continuous-time state-space model, a set of Markov parameters can be defined in a similar way:

$$\bar{\mathbf{H}}_0 = \mathbf{O}, \quad \bar{\mathbf{H}}_1 = \bar{\mathbf{B}}, \quad \bar{\mathbf{H}}_2 = \bar{\mathbf{A}}\bar{\mathbf{B}}, \quad \ldots, \quad \bar{\mathbf{H}}_k = \bar{\mathbf{A}}^{k-1}\bar{\mathbf{B}} \tag{3.83}$$

$$\bar{\mathbf{Y}}_0 = \mathbf{D}, \quad \bar{\mathbf{Y}}_1 = \mathbf{C}\bar{\mathbf{B}}, \quad \bar{\mathbf{Y}}_2 = \mathbf{C}\bar{\mathbf{A}}\bar{\mathbf{B}}, \quad \ldots, \quad \bar{\mathbf{Y}}_k = \mathbf{C}\bar{\mathbf{A}}^{k-1}\bar{\mathbf{B}} \tag{3.84}$$

Where $\bar{\mathbf{H}}_k$ and $\bar{\mathbf{Y}}_k$ are respectively $\mathbb{R}^{n\times(r+m)}$ and $\mathbb{R}^{m\times(r+m)}$ matrices which are known as observer Markov parameters. Similarly to system Markov parameters, observer Markov parameters are related by the following equations:

$$\bar{\mathbf{Y}}_k = \mathbf{C}\bar{\mathbf{H}}_k \quad , \quad k = 1, 2, \ldots \tag{3.85}$$

Developing the definitions of observer Markov parameters, these matrices can be expressed in a slightly different form as:

$$\bar{\mathbf{H}}_k = \begin{bmatrix} \bar{\mathbf{H}}_k^{(1)} & -\bar{\mathbf{H}}_k^{(2)} \end{bmatrix} \tag{3.86}$$

$$\bar{\mathbf{Y}}_k = \begin{bmatrix} \bar{\mathbf{Y}}_k^{(1)} & -\bar{\mathbf{Y}}_k^{(2)} \end{bmatrix} \tag{3.87}$$

Where $\bar{\mathbf{H}}_k^{(1)}$, $\bar{\mathbf{H}}_k^{(2)}$, $\bar{\mathbf{Y}}_k^{(1)}$ and $\bar{\mathbf{Y}}_k^{(2)}$ are respectively $\mathbb{R}^{n\times r}$, $\mathbb{R}^{n\times m}$, $\mathbb{R}^{m\times r}$ and $\mathbb{R}^{m\times m}$ matrices defined as:

$$\bar{\mathbf{H}}_k^{(1)} = (\mathbf{A} + \mathbf{G}\mathbf{C})^{k-1}(\mathbf{B} + \mathbf{G}\mathbf{D}) \tag{3.88}$$

$$\bar{\mathbf{H}}_k^{(2)} = (\mathbf{A} + \mathbf{G}\mathbf{C})^{k-1}\mathbf{G} \tag{3.89}$$

$$\bar{\mathbf{Y}}_k^{(1)} = \mathbf{C}(\mathbf{A} + \mathbf{G}\mathbf{C})^{k-1}(\mathbf{B} + \mathbf{G}\mathbf{D}) \tag{3.90}$$

$$\bar{\mathbf{Y}}_k^{(2)} = \mathbf{C}(\mathbf{A} + \mathbf{G}\mathbf{C})^{k-1}\mathbf{G} \tag{3.91}$$

The observer Markov parameters can be obtained from experimental data and they can be used to as a basis to compute system Markov parameters. Therefore, the observer Markov parameters can be used rather than identifying the systems Markov parameters, which can exhibit very slow decay for lightly damped systems. Indeed, the primary purpose of introducing an observer matrix $\mathbf{G}$ is as an artifice to compress the data and to improve the identification results at the same time. The matrix $\mathbf{G}$ can thus be chosen in an optimal way in the sense that the number of computed parameters is the minimum number needed to describe the system input-output relationship. This means that in the case of lightly damped structures, the system can be described by a relatively small number of observer Markov parameters instead of an otherwise large number of system Markov parameters. Consider now the response of discrete-time state-space observer model to a generalized input vector $\mathbf{v}(k)$. This response can be easily computed assuming zero initial conditions as follows:

$$\hat{\mathbf{z}}(k)=\sum_{j=1}^{k}\left(\bar{\mathbf{A}}^{j-1}\bar{\mathbf{B}}\mathbf{v}(k-j)\right) \tag{3.92}$$

$$\hat{\mathbf{y}}(k)=\mathbf{C}\sum_{j=1}^{k}\left(\bar{\mathbf{A}}^{j-1}\bar{\mathbf{B}}\mathbf{v}(k-j)\right)+\mathbf{D}\mathbf{u}(k) \tag{3.93}$$

Using the definitions of observer Markov parameters, the observer system response to a general input vector can be rewritten as:

$$\hat{\mathbf{z}}(k)=\sum_{j=0}^{k}\left(\bar{\mathbf{H}}_j\mathbf{v}(k-j)\right) \tag{3.94}$$

$$\hat{\mathbf{y}}(k)=\sum_{j=0}^{k}\left(\bar{\mathbf{Y}}_j\mathbf{v}(k-j)\right) \tag{3.95}$$

The estimated output vector $\hat{\mathbf{y}}(k)$ can be rewritten by using the matrix partition of observer Markov parameters $\bar{\mathbf{Y}}_k$ and the definition of the generalized input vector $\mathbf{v}(k)$ as:

$$\hat{\mathbf{y}}(k)+\sum_{j=1}^{k}\left(\bar{\mathbf{Y}}_j^{(2)}\mathbf{y}(k-j)\right)=\sum_{j=1}^{k}\left(\bar{\mathbf{Y}}_j^{(1)}\mathbf{u}(k-j)\right)+\mathbf{D}\mathbf{u}(k) \qquad (3.96)$$

The observer state matrix $\bar{\mathbf{A}}$ can be made sufficiently stable with a proper choice of the observer matrix $\mathbf{G}$ and consequently $\bar{\mathbf{A}}^p$ can be neglected, where p is a relatively small integer. Therefore, for a time step k greater than p, the estimated output $\hat{\mathbf{y}}(k)$ closely approaches the measured output $\mathbf{y}(k)$ because the estimation error is related to the power of observer state matrix $\bar{\mathbf{A}}$ which approaches zero. Indeed, for a sufficiently large k:

$$\mathbf{y}(k)+\sum_{j=1}^{p}\left(\bar{\mathbf{Y}}_j^{(2)}\mathbf{y}(k-j)\right)=\sum_{j=1}^{p}\left(\bar{\mathbf{Y}}_j^{(1)}\mathbf{u}(k-j)\right)+\mathbf{D}\mathbf{u}(k) \qquad (3.97)$$

This matrix equation is called the linear difference model for multiple input and multiple output linear time-invariant dynamical systems alias Autoregressive model with Exogeneous input or ARX model. The ARX model represents the input-output description of discrete-time state-space observer systems similar to the weighting sequence description of discrete-time state-space systems. Note that this description is based on the assumption of zero initial conditions or that the system is in the condition of a steady state. The coefficients of the finite difference model can be experimentally computed from input and output data together with the observer matrix. Indeed, define the following sequence of Markov parameters:

$$\mathbf{H}_1^0=\mathbf{G}, \quad \mathbf{H}_2^0=\mathbf{A}\mathbf{G}, \quad \ldots, \quad \mathbf{H}_k^0=\mathbf{A}^{k-1}\mathbf{G} \qquad (3.98)$$

$$\mathbf{Y}_1^0=\mathbf{C}\mathbf{G}, \quad \mathbf{Y}_2^0=\mathbf{C}\mathbf{A}\mathbf{G}, \quad \ldots, \quad \mathbf{Y}_k^0=\mathbf{C}\mathbf{A}^{k-1}\mathbf{G} \qquad (3.99)$$

Where $\mathbf{H}_k^0$ and $\mathbf{Y}_k^0$ are respectively $\mathbb{R}^{n\times m}$ and $\mathbb{R}^{m\times m}$ matrices which are known as observer gain Markov parameters. Similarly to system Markov parameters and to observer Markov parameters, observer gain Markov parameters are related by the following equations:

$$\mathbf{Y}_k^0 = \mathbf{C}\mathbf{H}_k^0 \quad , \quad k = 1, 2, \ldots \tag{3.100}$$

In addition, the ARX model can be expressed in a compact form grouping together the observer Markov parameters:

$$\begin{aligned} \mathbf{y}(k) &= \sum_{j=1}^{p}\left(\overline{\mathbf{Y}}_j \mathbf{v}(k-j)\right) + \mathbf{D}\mathbf{u}(k) = \\ &= \sum_{j=0}^{p}\left(\overline{\mathbf{Y}}_j \mathbf{v}(k-j)\right) \end{aligned} \tag{3.101}$$

The coefficients of the finite difference model can be computed from input and output data.

3.5. OBSERVER/KALMAN FILTER IDENTIFICATION METHOD (OKID)

The Observer/Kalman Filter Identification Method is an identification algorithm which allows to compute Markov parameters from a given set of experimental input and output data. Consider a set of input and output data record of length l. The ARX representation of input and output data can be formulated for each time step and grouped in a matrix form to yield:

$$\mathbf{Y} = \bar{\mathbf{L}}_p \mathbf{V}_p \tag{3.102}$$

Where $\mathbf{Y}$, $\bar{\mathbf{L}}_p$ and $\mathbf{V}_p$ are respectively $\mathbb{R}^{m \times l}$, $\mathbb{R}^{m \times (r+(r+m)\,p)}$ and $\mathbb{R}^{(r+(r+m)\,p) \times l}$ matrices defined as:

$$\mathbf{Y} = \begin{bmatrix} \mathbf{y}(0) & \mathbf{y}(1) & \mathbf{y}(2) & \dots & \mathbf{y}(l-1) \end{bmatrix} \tag{3.103}$$

$$\bar{\mathbf{L}}_p = \begin{bmatrix} \bar{\mathbf{Y}}_0 & \bar{\mathbf{Y}}_1 & \bar{\mathbf{Y}}_2 & \dots & \bar{\mathbf{Y}}_p \end{bmatrix} \tag{3.104}$$

$$\mathbf{V}_p = \begin{bmatrix} \mathbf{u}(0) & \mathbf{u}(1) & \dots & \mathbf{u}(p) & \dots & \mathbf{u}(l-1) \\ \mathbf{0} & \mathbf{v}(0) & \dots & \mathbf{v}(p-1) & \dots & \mathbf{v}(l-2) \\ \vdots & \vdots & \ddots & \vdots & \dots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{v}(0) & \dots & \mathbf{v}(l-p-1) \end{bmatrix} \tag{3.105}$$

The block matrix $\bar{\mathbf{L}}_p$ contains the sequence of first p observer Markov parameters which are necessary for the ARX input/output description of the system. These parameters can be recovered from experimental input and output data by least-squares method yielding:

$$\bar{\mathbf{L}}_p = \mathbf{Y}\mathbf{V}_p^{+} \tag{3.106}$$

Where $\mathbf{V}_p^+$ is a $\mathbb{R}^{m\times(r+(r+m)p)}$ matrix which represents the Moore-Penrose pseudoinverse of matrix $\mathbf{V}_p$. Once that observer Markov parameters has been computed from input/output data, the system Markov parameters and the observer gain Markov parameters can be experimentally computed in the following way:

$$\begin{cases} \mathbf{D} = \mathbf{Y}_0 = \overline{\mathbf{Y}}_0 \\ \mathbf{Y}_k = \overline{\mathbf{Y}}_k^{(1)} - \sum_{j=1}^{k} \overline{\mathbf{Y}}_j^{(2)} \mathbf{Y}_{k-j} \quad , \quad k = 1, 2, \ldots, p \\ \mathbf{Y}_k = -\sum_{j=1}^{p} \overline{\mathbf{Y}}_j^{(2)} \mathbf{Y}_{k-j} \quad , \quad k = p+1, p+2, \ldots \end{cases} \tag{3.107}$$

$$\begin{cases} \mathbf{Y}_1^0 = \mathbf{CG} = \overline{\mathbf{Y}}_1^{(2)} \\ \mathbf{Y}_k^0 = \overline{\mathbf{Y}}_k^{(2)} - \sum_{j=1}^{k-1} \overline{\mathbf{Y}}_j^{(2)} \mathbf{Y}_{k-j}^0 \quad , \quad k = 2, 3, \ldots, p \\ \mathbf{Y}_k^0 = -\sum_{j=1}^{p} \overline{\mathbf{Y}}_j^{(2)} \mathbf{Y}_{k-j}^0 \quad , \quad k = p+1, p+2, \ldots \end{cases} \tag{3.108}$$

The previous two sets of equations show that from time step $p+1$ the system Markov parameter and the observer gain Markov parameters become a linear combination of the past Markov parameters. Consequently, there are only p independent system Markov parameters and observer gain Markov parameters. It can be proved $[5]$ that the number of observer Markov parameters p must be chosen such that $mp \geq n$, where m is the number of outputs and n is the order of system. The number p determine thus the maximum number of independent system Markov parameters and therefore the product mp represents the upper bound on the order of the identified system model. On the other hand, consider the case in which a system discrete-time state-space model is available from a theoretical investigation or from an experimental identification. In this case, the identified observer matrix $\mathbf{G}$ can be computed by the recovered sequence of observer gain Markov parameters $\mathbf{Y}_k^0$ and exploiting the knowledge of system state matrix $\mathbf{A}$ and output influence matrix $\mathbf{C}$. Indeed, consider the following matrix equation derived from the definition of observer gain Markov parameters:

$$\mathbf{P}_p \mathbf{G} = \mathbf{Y}_p^0 \tag{3.109}$$

Where $\mathbf{P}_p$ and $\mathbf{Y}_p^0$ are respectively $\mathbb{R}^{mp \times n}$ and $\mathbb{R}^{mp \times m}$ matrices defined as:

$$\mathbf{P}_p = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{p-1} \end{bmatrix} \tag{3.110}$$

$$\mathbf{Y}_p^0 = \begin{bmatrix} \mathbf{Y}_1^0 \\ \mathbf{Y}_2^0 \\ \vdots \\ \mathbf{Y}_p^0 \end{bmatrix} \tag{3.111}$$

Therefore, the observer gain matrix $\mathbf{G}$ can be computed by using least-squares method as:

$$\mathbf{G} = \mathbf{P}_p^+ \mathbf{Y}_p^0 \tag{3.112}$$

Where $\mathbf{P}_p^+$ is a $\mathbb{R}^{m \times (r+(r+m)p)}$ matrix which represents the Moore-Penrose pseudoinverse of matrix $\mathbf{P}_p$. Finally, it can be proved that if the data length l is sufficiently long and if the order of the observer p is sufficiently large, then the identified observer matrix $\mathbf{G}$ computed from the combined Markov parameters coincides with the steady-state Kalman filter gain $\mathbf{K}$ which produces the same input and output map. Indeed:

$$\mathbf{G} = -\mathbf{K} \tag{3.113}$$

In practical applications the identified filter matrix $\mathbf{G}$ is not a steady-state Kalman filter gain because of the presence of disturbances, nonlinearities, non-whiteness of the process and measurement noises, etcetera. In this case, the identified filter is simply an observer that is computed from input and output data which minimizes the filter residual in a least-squares sense.

3.6. EIGENSYSTEM REALIZATION ALGORITHM WITH DATA CORRELATIONS (ERA/DC) USING OBSERVER/KALMAN FILTER IDENTIFICATION METHOD (OKID)

Eigensystem Realization Algorithm (ERA) is a numerical method which is able to derive a state-space realization of a dynamical system starting from system Markov parameters [5], [6]. A realization is a triplet of state-space matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ representing the state-space model of a dynamical system which can be extracted from a given set of system Markov parameters. Any dynamical system has an infinite number of realization which reproduces the same input-output mapping. Minimum realization means a model with the smallest state-space dimensions among all realizable systems that have identical input-output relationship and all minimum realizations have the same set of modal parameters. The basic development of the state-space realization methods is attributed to Ho and Kalman, who introduced the principles of minimum realization theory for first [14]. The Ho-Kalman method uses the generalized Hankel matrix to derive a state-space representation of a linear dynamical system starting from noise-free data. This method has been modified and substantially extended by Juang to develop the Eigensystem Realization Algorithm (ERA) and subsequently the Eigensystem Realization Algorithm with Data Correlation (ERA/DC) in order to identify a state-space model from system Markov parameters obtained from noisy measurement data [5], [6]. Afterwards, Juang developed a method named Eigensystem Realization Algorithm with Data Correlation (ERA/DC) using Observer/Kalman Filter Identification Method (OKID) which is able to compute simultaneously a state-space realization and an observer gain matrix of a dynamical system starting directly from noisy input-output data [5], [6]. The Eigensystem Realization Algorithm (ERA) begins by forming the generalized Hankel matrix composed of system Markov parameters, which is defined as:

$$\mathbf{H}(k-1)=\begin{bmatrix}\mathbf{Y}_k & \mathbf{Y}_{k+1} & \cdots & \mathbf{Y}_{k+\beta-1}\\ \mathbf{Y}_{k+1} & \mathbf{Y}_{k+2} & \cdots & \mathbf{Y}_{k+\beta}\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{Y}_{k+\alpha-1} & \mathbf{Y}_{k+\alpha} & \cdots & \mathbf{Y}_{k+\alpha+\beta-2}\end{bmatrix} \tag{3.114}$$

Where $\mathbf{H}(k-1)$ is a $\mathbb{R}^{\alpha m\times\beta r}$ block data matrix and α, β are two integer assumed larger than system order n. Usually, for a data record of length l, α is set equal to p and β is set equal to $l-p$. Using the definition of system Markov parameters $\mathbf{Y}_k$ the generalized Hankel matrix $\mathbf{H}(k-1)$ can be decomposed as:

$$\mathbf{H}(k-1)=\mathbf{P}_\alpha\mathbf{A}^{k-1}\mathbf{Q}_\beta \tag{3.115}$$

Where $\mathbf{P}_\alpha$ and $\mathbf{Q}_\beta$ are respectively $\mathbb{R}^{\alpha m\times n}$ and $\mathbb{R}^{n\times\beta r}$ matrices representing the observability matrix and the controllability matrix defined as:

$$\mathbf{P}_\alpha=\begin{bmatrix}\mathbf{C}\\ \mathbf{CA}\\ \mathbf{CA}^2\\ \vdots\\ \mathbf{CA}^{\alpha-1}\end{bmatrix} \tag{3.116}$$

$$\mathbf{Q}_\beta=\begin{bmatrix}\mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \cdots & \mathbf{A}^{\beta-1}\mathbf{B}\end{bmatrix} \tag{3.117}$$

In general, a linear time-invariant dynamical system of order n is observable if and only if its observability matrix $\mathbf{P}_\alpha$ has rank n. An observable system is a dynamical system whose state at a generic time step α can be reconstructed knowing the input and output sequences over the finite time interval $0<k\le\alpha$. On the other hand, a linear time-invariant dynamical system

of order n is controllable if and only if its controllability matrix $\mathbf{Q}_\beta$ has rank n. A controllable system is a dynamical system whose state at a generic time step β can be reached from any initial state by some control input acting on the system over the finite time interval $0 < k \le \beta$. If the system is controllable and observable, then the block matrices $\mathbf{P}_\alpha$ and $\mathbf{Q}_\beta$ are both of rank n. For $k=1$ and for $k=2$ the generalized Hankel matrix $\mathbf{H}(k-1)$ becomes:

$$\mathbf{H}(0) = \begin{bmatrix} \mathbf{Y}_1 & \mathbf{Y}_2 & \cdots & \mathbf{Y}_\beta \\ \mathbf{Y}_2 & \mathbf{Y}_3 & \cdots & \mathbf{Y}_{\beta+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{Y}_\alpha & \mathbf{Y}_{\alpha+1} & \cdots & \mathbf{Y}_{\alpha+\beta-1} \end{bmatrix} \tag{3.118}$$

$$\mathbf{H}(1) = \begin{bmatrix} \mathbf{Y}_2 & \mathbf{Y}_3 & \cdots & \mathbf{Y}_{\beta+1} \\ \mathbf{Y}_3 & \mathbf{Y}_4 & \cdots & \mathbf{Y}_{\beta+2} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{Y}_{\alpha+1} & \mathbf{Y}_{\alpha+2} & \cdots & \mathbf{Y}_{\alpha+\beta} \end{bmatrix} \tag{3.119}$$

Note that $\mathbf{Y}_0 = \mathbf{D}$ is not included in $\mathbf{H}(0)$. These matrices can be respectively decomposed as follows:

$$\mathbf{H}(0) = \mathbf{P}_\alpha \mathbf{Q}_\beta \tag{3.120}$$

$$\mathbf{H}(1) = \mathbf{P}_\alpha \mathbf{A} \mathbf{Q}_\beta \tag{3.121}$$

If the system is controllable and observable, the Hankel matrix $\mathbf{H}(0)$ is rank n and the maximum order of the identified system is equal to αm. The next step of Eigensystem Realization Algorithm (ERA) is the factorization of the Hankel matrix $\mathbf{H}(0)$ by using the Singular Value Decomposition method (SVD) $[15]$ to yield:

$$\mathbf{H}(0) = \mathbf{R}\mathbf{\Sigma}\mathbf{S}^T \tag{3.122}$$

Where $\mathbf{\Sigma}$ is a $\mathbb{R}^{\alpha m \times \beta r}$ diagonal matrix containing the singular values of matrix $\mathbf{H}(0)$ whereas $\mathbf{R}$ and $\mathbf{S}$ are respectively $\mathbb{R}^{\alpha m \times \alpha m}$ and $\mathbb{R}^{\beta r \times \beta r}$ orthonormal matrices containing the left singular vectors and the right singular vectors of matrix $\mathbf{H}(0)$. These matrices can be respectively partitioned as follows:

$$\mathbf{\Sigma} = \begin{bmatrix} \mathbf{\Sigma}_n & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix} \tag{3.123}$$

$$\mathbf{R} = \begin{bmatrix} \mathbf{R}_n & \mathbf{R}_{\alpha m - n} \end{bmatrix} \tag{3.124}$$

$$\mathbf{S} = \begin{bmatrix} \mathbf{S}_n & \mathbf{S}_{\beta r - n} \end{bmatrix} \tag{3.125}$$

Where $\mathbf{\Sigma}_n$, $\mathbf{R}_n$, $\mathbf{R}_{\alpha m - n}$, $\mathbf{S}_n$ and $\mathbf{S}_{\beta r - n}$ are respectively $\mathbb{R}^{n \times n}$, $\mathbb{R}^{\alpha m \times n}$, $\mathbb{R}^{\alpha m \times (\alpha m - n)}$, $\mathbb{R}^{\beta r \times n}$ and $\mathbb{R}^{\beta r \times (\beta r - n)}$ matrices. The matrix $\mathbf{\Sigma}_n$ is a diagonal matrix containing the significant singular values of the system. Indeed:

$$\mathbf{\Sigma}_n = diag(\sigma_1, \sigma_2, \ldots, \sigma_n) \tag{3.126}$$

Because of measurement noise, nonlinearity and round-off errors, the Hankel matrix $\mathbf{H}(0)$ is typically of full rank which generally is not equal to the true order of the system under test. Therefore, in order to do not reproduce exactly the noise sequence of data, or rather to get a realization which reproduces a smoothed version of input-output data and that closely represents the underlying linear dynamics of the system, the Hankel matrix $\mathbf{H}(0)$ can be approximated as:

$$\mathbf{H}(0) = \mathbf{R}_n \mathbf{\Sigma}_n \mathbf{S}_n^T \tag{3.127}$$

One interpretation of this factorization is that the observability matrix $\mathbf{P}_\alpha$ is related to the left singular vector matrix $\mathbf{R}_n$ and the controllability matrix $\mathbf{Q}_\beta$ is related to the right singular vector matrix $\mathbf{S}_n$. Indeed, it can be proved [5], [6] that the identified observability and controllability matrices can be computed as:

$$\hat{\mathbf{P}}_\alpha = \mathbf{R}_n \mathbf{\Sigma}_n^{1/2} \tag{3.128}$$

$$\hat{\mathbf{Q}}_\beta = \mathbf{\Sigma}_n^{1/2} \mathbf{S}_n^T \tag{3.129}$$

This choice of observability and controllability matrices $\hat{\mathbf{P}}_\alpha$ and $\hat{\mathbf{Q}}_\beta$ appear to be balanced in the sense the observability and controllability grammians are equal and diagonal. Indeed:

$$\hat{\mathbf{P}}_\alpha^T \hat{\mathbf{P}}_\alpha = \mathbf{\Sigma}_n \tag{3.130}$$

$$\hat{\mathbf{Q}}_\beta \hat{\mathbf{Q}}_\beta^T = \mathbf{\Sigma}_n \tag{3.131}$$

The fact that the observability and controllability grammians are equal and diagonal implies that the identified state-space model is as observable as it is controllable. This means that the identified state-space model is an internally balanced realization in the sense that the signal transfer from the input to the state and from the state to the output are similar and balanced. Once that the observability matrix $\hat{\mathbf{P}}_\alpha$ and the controllability matrix $\hat{\mathbf{Q}}_\beta$ have been identified, the output influence matrix $\hat{\mathbf{C}}$ and the state influence matrix $\hat{\mathbf{B}}$ can be respectively identified from the first m rows of the observability matrix $\hat{\mathbf{P}}_\alpha$ and from the first r columns of the controllability matrix $\hat{\mathbf{Q}}_\beta$. Indeed:

$$\begin{aligned}\hat{\mathbf{C}} &= \mathbf{E}_{\alpha m}^{T}\hat{\mathbf{P}}_{\alpha} = \\ &= \mathbf{E}_{\alpha m}^{T}\mathbf{R}_{n}\mathbf{\Sigma}_{n}^{1/2}\end{aligned} \tag{3.132}$$

$$\begin{aligned}\hat{\mathbf{B}} &= \hat{\mathbf{Q}}_{\beta}\mathbf{E}_{\beta r} = \\ &= \mathbf{\Sigma}_{n}^{1/2}\mathbf{S}_{n}^{T}\mathbf{E}_{\beta r}\end{aligned} \tag{3.133}$$

Where $\mathbf{E}_{\alpha m}$ and $\mathbf{E}_{\beta m}$ are respectively $\mathbb{R}^{\alpha m\times m}$ and $\mathbb{R}^{\beta r\times r}$ Boolean matrices defined as:

$$\mathbf{E}_{\alpha m}^{T} = \begin{bmatrix}\mathbf{I}_{m,m} & \mathbf{O}_{m,m} & \ldots & \mathbf{O}_{m,m}\end{bmatrix} \tag{3.134}$$

$$\mathbf{E}_{\beta r}^{T} = \begin{bmatrix}\mathbf{I}_{r,r} & \mathbf{O}_{r,r} & \ldots & \mathbf{O}_{r,r}\end{bmatrix} \tag{3.135}$$

On the other hand, using the factorization of Hankel matrix $\mathbf{H}(1)$ and the identified observability and controllability matrices $\hat{\mathbf{P}}_{\alpha}$ and $\hat{\mathbf{Q}}_{\beta}$, it can be proved $[5]$, $[6]$ that the identified state matrix $\hat{\mathbf{A}}$ can be computed as:

$$\hat{\mathbf{A}} = \mathbf{\Sigma}_{n}^{-1/2}\mathbf{R}_{n}^{T}\mathbf{H}(1)\mathbf{S}_{n}\overline{\mathbf{\Sigma}}_{n}^{-1/2} \tag{3.136}$$

In brief, the Eigensystem Realization Algorithm (ERA) leads to the following identified state-space model:

$$\begin{cases}\hat{\mathbf{A}} = \mathbf{\Sigma}_{n}^{-1/2}\mathbf{R}_{n}^{T}\mathbf{H}(1)\mathbf{S}_{n}\overline{\mathbf{\Sigma}}_{n}^{-1/2} \\ \hat{\mathbf{B}} = \mathbf{\Sigma}_{n}^{1/2}\mathbf{S}_{n}^{T}\mathbf{E}_{\beta r} \\ \hat{\mathbf{C}} = \mathbf{E}_{\alpha m}^{T}\mathbf{R}_{n}\mathbf{\Sigma}_{n}^{1/2} \\ \hat{\mathbf{D}} = \mathbf{Y}_{0} = \overline{\mathbf{Y}}_{0}\end{cases} \tag{3.137}$$

It is worth noting that the identified state-space model is not unique in the sense that it is coordinate dependent. Nevertheless, the state-space realization obtained by Eigensystem Realization Algorithm (ERA) is a minimum order, controllable and observable realization whose modal parameters are identical to the modal parameters of the true system. Now consider the Eigensystem Realization Algorithm with Data Correlations (ERA/DC). This method utilizes a set of correlation matrices derived from Hankel matrices. Indeed, define the following correlation matrix:

$$\begin{aligned}
&\mathbf{\Delta}_{HH}(k) = \mathbf{H}(k)\mathbf{H}^T(0) = \\
&= \begin{bmatrix} \mathbf{Y}_{k+1} & \mathbf{Y}_{k+2} & \cdots & \mathbf{Y}_{k+\beta} \\ \mathbf{Y}_{k+2} & \mathbf{Y}_{k+3} & \cdots & \mathbf{Y}_{k+\beta+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{Y}_{k+\alpha} & \mathbf{Y}_{k+\alpha+1} & \cdots & \mathbf{Y}_{k+\alpha+\beta-1} \end{bmatrix} \begin{bmatrix} \mathbf{Y}_{1} & \mathbf{Y}_{2} & \cdots & \mathbf{Y}_{\beta} \\ \mathbf{Y}_{2} & \mathbf{Y}_{3} & \cdots & \mathbf{Y}_{\beta+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{Y}_{\alpha} & \mathbf{Y}_{\alpha+1} & \cdots & \mathbf{Y}_{\alpha+\beta-1} \end{bmatrix}^T = \\
&= \begin{bmatrix} \sum_{j=1}^{\beta}\left(\mathbf{Y}_{k+j}\mathbf{Y}_j^T\right) & \sum_{j=1}^{\beta}\left(\mathbf{Y}_{k+j}\mathbf{Y}_{j+1}^T\right) & \cdots & \sum_{j=1}^{\beta}\left(\mathbf{Y}_{k+j}\mathbf{Y}_{\alpha+j-1}^T\right) \\ \sum_{j=1}^{\beta}\left(\mathbf{Y}_{k+j+1}\mathbf{Y}_j^T\right) & \sum_{j=1}^{\beta}\left(\mathbf{Y}_{k+j+1}\mathbf{Y}_{j+1}^T\right) & \cdots & \sum_{j=1}^{\beta}\left(\mathbf{Y}_{k+j+1}\mathbf{Y}_{\alpha+j-1}^T\right) \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=1}^{\beta}\left(\mathbf{Y}_{k+\alpha+j-1}\mathbf{Y}_j^T\right) & \sum_{j=1}^{\beta}\left(\mathbf{Y}_{k+\alpha+j-1}\mathbf{Y}_{j+1}^T\right) & \cdots & \sum_{j=1}^{\beta}\left(\mathbf{Y}_{k+\alpha+j-1}\mathbf{Y}_{\alpha+j-1}^T\right) \end{bmatrix}
\end{aligned} \tag{3.138}$$

Where $\mathbf{\Delta}_{HH}(k)$ is a $\mathbb{R}^{\alpha m\times\alpha m}$ square matrix obtained from the correlation between the Hankel matrix evaluated at the generic time step k and the Hankel matrix at the initial time step. Indeed, the correlation matrix $\mathbf{\Delta}_{HH}(k)$ consists of auto-correlations and cross-correlation of system Markov parameters at lag time of values from k to $k+\alpha$. Therefore, if the noises in system Markov parameters are not correlated, then the correlation matrix $\mathbf{\Delta}_{HH}(k)$ contain less noise than the Hankel matrix $\mathbf{H}(k)$. The block data correlation matrix $\mathbf{\Delta}_{HH}(k)$

can be factorized by using the factorization of Hankel matrices $\mathbf{H}(0)$ and $\mathbf{H}(k)$ in terms of the observability and controllability matrices $\mathbf{P}_\alpha$ and $\mathbf{Q}_\beta$ as follows:

$$\begin{aligned}\mathbf{\Delta}_{HH}(k) = \mathbf{H}(k)\mathbf{H}^T(0) = \\ = \mathbf{P}_\alpha \mathbf{A}^k \mathbf{Q}_\beta \mathbf{Q}_\beta^T \mathbf{P}_\alpha^T = \\ = \mathbf{P}_\alpha \mathbf{A}^k \mathbf{Q}_c\end{aligned} \tag{3.139}$$

Where $\mathbf{Q}_c$ is a $\mathbb{R}^{n \times \alpha m}$ matrix representing a mixed controllability-observability matrix defined as:

$$\mathbf{Q}_c = \mathbf{Q}_\beta \mathbf{Q}_\beta^T \mathbf{P}_\alpha^T \tag{3.140}$$

The next step of the Eigensystem Realization Algorithm with Data Correlations (ERA/DC) is the definition of the block correlation Hankel matrix $\mathbf{H}_\Delta(k)$ whose block elements are the data correlation matrices $\mathbf{\Delta}_{HH}(k)$ shifted in time with multiple of time lag τ. Indeed:

$$\mathbf{H}_\Delta(k) = \begin{bmatrix} \mathbf{\Delta}_{HH}(k) & \mathbf{\Delta}_{HH}(k+\tau) & \dots & \mathbf{\Delta}_{HH}(k+\delta\tau) \\ \mathbf{\Delta}_{HH}(k+\tau) & \mathbf{\Delta}_{HH}(k+2\tau) & \dots & \mathbf{\Delta}_{HH}(k+(\delta+1)\tau) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{\Delta}_{HH}(k+\gamma\tau) & \mathbf{\Delta}_{HH}(k+(\gamma+1)\tau) & \dots & \mathbf{\Delta}_{HH}(k+(\gamma+\delta)\tau) \end{bmatrix} \tag{3.141}$$

Where $\mathbf{H}_\Delta(k)$ is a $\mathbb{R}^{(\gamma+1)\alpha m \times (\delta+1)\alpha m}$ matrix. The integers γ and δ define how many correlation lags are included in the analysis. Exploiting the factorization of correlation matrix $\mathbf{\Delta}_{HH}(k)$, the block correlation Hankel matrix $\mathbf{H}_\Delta(k)$ can be decomposed as:

$$\mathbf{H}_\Delta(k) = \mathbf{P}_\gamma \mathbf{A}^k \mathbf{Q}_\delta \tag{3.142}$$

Where $\mathbf{P}_\gamma$ and $\mathbf{Q}_\delta$ are respectively $\mathbb{R}^{(\gamma+1)\alpha m\times n}$ and $\mathbb{R}^{n\times(\delta+1)\alpha m}$ matrices representing the block correlation observability matrix and the block correlation mixed controllability-observability matrix. These matrices are defined in terms of the observability matrix $\mathbf{P}_\alpha$ and mixed controllability-observability matrix $\mathbf{Q}_c$ as:

$$\mathbf{P}_\gamma = \begin{bmatrix} \mathbf{P}_\alpha \\ \mathbf{P}_\alpha \mathbf{A}^\tau \\ \mathbf{P}_\alpha \mathbf{A}^{2\tau} \\ \vdots \\ \mathbf{P}_\alpha \mathbf{A}^{\gamma\tau} \end{bmatrix} \tag{3.143}$$

$$\mathbf{Q}_\delta = \begin{bmatrix} \mathbf{Q}_c & \mathbf{A}^\tau \mathbf{Q}_c & \mathbf{A}^{2\tau}\mathbf{Q}_c & \dots & \mathbf{A}^{\delta\tau}\mathbf{Q}_c \end{bmatrix} \tag{3.144}$$

For $k=0$ and for $k=1$ the block correlation Hankel matrix $\mathbf{H}_\Delta(k)$ becomes:

$$\mathbf{H}_\Delta(0) = \begin{bmatrix} \mathbf{\Delta}_{HH}(0) & \mathbf{\Delta}_{HH}(\tau) & \dots & \mathbf{\Delta}_{HH}(\delta\tau) \\ \mathbf{\Delta}_{HH}(\tau) & \mathbf{\Delta}_{HH}(2\tau) & \dots & \mathbf{\Delta}_{HH}((\delta+1)\tau) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{\Delta}_{HH}(\gamma\tau) & \mathbf{\Delta}_{HH}((\gamma+1)\tau) & \dots & \mathbf{\Delta}_{HH}((\gamma+\delta)\tau) \end{bmatrix} \tag{3.145}$$

$$\mathbf{H}_\Delta(1) = \begin{bmatrix} \mathbf{\Delta}_{HH}(1) & \mathbf{\Delta}_{HH}(1+\tau) & \dots & \mathbf{\Delta}_{HH}(1+\delta\tau) \\ \mathbf{\Delta}_{HH}(1+\tau) & \mathbf{\Delta}_{HH}(1+2\tau) & \dots & \mathbf{\Delta}_{HH}(1+(\delta+1)\tau) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{\Delta}_{HH}(1+\gamma\tau) & \mathbf{\Delta}_{HH}(1+(\gamma+1)\tau) & \dots & \mathbf{\Delta}_{HH}(1+(\gamma+\delta)\tau) \end{bmatrix} \tag{3.146}$$

These matrices can be respectively decomposed in terms of the block correlation observability matrix $\mathbf{P}_\gamma$ and of the block correlation mixed controllability-observability matrix $\mathbf{Q}_\delta$ as follows:

$$\mathbf{H}_\Delta(0) = \mathbf{P}_\gamma \mathbf{Q}_\delta \tag{3.147}$$

$$\mathbf{H}_\Delta(1) = \mathbf{P}_\gamma \mathbf{A} \mathbf{Q}_\delta \tag{3.148}$$

Similarly to the Eigensystem Realization Algorithm (ERA), the next step is the factorization of the block correlation Hankel matrix $\mathbf{H}_\Delta(0)$ by using the Singular Value Decomposition method (SVD) $[15]$ to yield:

$$\mathbf{H}_\Delta(0) = \mathbf{R}_\Delta \mathbf{\Sigma}_\Delta \mathbf{S}_\Delta^T \tag{3.149}$$

Where $\mathbf{\Sigma}_\Delta$ is a $\mathbb{R}^{(\gamma+1)\alpha m \times (\delta+1)\alpha m}$ diagonal matrix containing the singular values of matrix $\mathbf{H}_\Delta(0)$ whereas $\mathbf{R}_\Delta$ and $\mathbf{S}_\Delta$ are respectively $\mathbb{R}^{(\gamma+1)\alpha m \times (\gamma+1)\alpha m}$ and $\mathbb{R}^{(\delta+1)\alpha m \times (\delta+1)\alpha m}$ orthonormal matrices containing the left singular vectors and the right singular vectors of matrix $\mathbf{H}_\Delta(0)$. These matrices can be respectively partitioned as follows:

$$\mathbf{\Sigma}_\Delta = \begin{bmatrix} \mathbf{\Sigma}_{\Delta,n} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix} \tag{3.150}$$

$$\mathbf{R}_\Delta = \begin{bmatrix} \mathbf{R}_{\Delta,n} & \mathbf{R}_{\Delta,(\gamma+1)\alpha m-n} \end{bmatrix} \tag{3.151}$$

$$\mathbf{S}_\Delta = \begin{bmatrix} \mathbf{S}_{\Delta,n} & \mathbf{S}_{\Delta,(\delta+1)\alpha m-n} \end{bmatrix} \tag{3.152}$$

Where $\mathbf{\Sigma}_{\Delta,n}$, $\mathbf{R}_{\Delta,n}$, $\mathbf{R}_{\Delta,(\gamma+1)\alpha m-n}$, $\mathbf{S}_{\Delta,n}$ and $\mathbf{S}_{\Delta,(\delta+1)\alpha m-n}$ are respectively $\mathbb{R}^{n\times n}$, $\mathbb{R}^{(\gamma+1)\alpha m\times n}$, $\mathbb{R}^{(\gamma+1)\alpha m\times((\gamma+1)\alpha m-n)}$, $\mathbb{R}^{(\delta+1)\alpha m\times n}$ and $\mathbb{R}^{(\delta+1)\alpha m\times((\delta+1)\alpha m-n)}$

matrices. The matrix $\mathbf{\Sigma}_{\Delta,n}$ is a diagonal matrix containing the significant singular values of the system. Indeed:

$$\mathbf{\Sigma}_{\Delta,n} = diag(\sigma_{\Delta,1}, \sigma_{\Delta,2}, \ldots, \sigma_{\Delta,n}) \tag{3.153}$$

Even in this case, because of measurement noise the block correlation Hankel matrix $\mathbf{H}_{\Delta}(0)$ is typically of full rank which generally is not equal to the true order of the system. Therefore to get a realization which closely represents the underlying linear dynamics of the system, the block correlation Hankel matrix $\mathbf{H}_{\Delta}(0)$ can be approximated as:

$$\mathbf{H}_{\Delta}(0) = \mathbf{R}_{\Delta,n}\mathbf{\Sigma}_{\Delta,n}\mathbf{S}_{\Delta,n}^{T} \tag{3.154}$$

Similarly to the previous method, one interpretation of this factorization is that the block correlation observability matrix $\mathbf{P}_{\gamma}$ is related to the left singular vector matrix $\mathbf{R}_{\Delta,n}$ and the block correlation controllability-observability matrix $\mathbf{Q}_{\delta}$ is related to the right singular vector matrix $\mathbf{S}_{\Delta,n}$. Indeed, it can be proved $[5]$, $[6]$ that these matrices can be computed as:

$$\hat{\mathbf{P}}_{\gamma} = \mathbf{R}_{\Delta,n}\mathbf{\Sigma}_{\Delta,n}^{1/2} \tag{3.155}$$

$$\hat{\mathbf{Q}}_{\delta} = \mathbf{\Sigma}_{\Delta,n}^{1/2}\mathbf{S}_{\Delta,n}^{T} \tag{3.156}$$

Once that the block correlation observability matrix $\hat{\mathbf{P}}_{\gamma}$ and the block correlation mixed controllability-observability matrix $\hat{\mathbf{Q}}_{\delta}$ have been identified, the observability matrix $\hat{\mathbf{P}}_{\alpha}$ and the mixed controllability-observability matrix $\hat{\mathbf{Q}}_{c}$ can be identified from the first αm rows of the block correlation observability matrix $\hat{\mathbf{P}}_{\gamma}$ and from the first αm columns of the block correlation mixed controllability-observability matrix $\hat{\mathbf{Q}}_{\delta}$. Indeed:

$$\begin{aligned}\hat{\mathbf{P}}_{\alpha} &= \mathbf{E}_{(\gamma+1)\alpha m}^{T}\hat{\mathbf{P}}_{\gamma} = \\ &= \mathbf{E}_{(\gamma+1)\alpha m}^{T}\mathbf{R}_{\Delta,n}\mathbf{\Sigma}_{\Delta,n}^{1/2}\end{aligned} \tag{3.157}$$

$$\begin{aligned}\hat{\mathbf{Q}}_{c} &= \hat{\mathbf{Q}}_{\delta}\mathbf{E}_{(\gamma+1)\alpha m} = \\ &= \mathbf{\Sigma}_{\Delta,n}^{1/2}\mathbf{S}_{\Delta,n}^{T}\mathbf{E}_{(\gamma+1)\alpha m}\end{aligned} \tag{3.158}$$

Where $\mathbf{E}_{(\gamma+1)\alpha m}$ is a $\mathbb{R}^{(\gamma+1)\alpha m\times\alpha m}$ Boolean matrix defined as:

$$\mathbf{E}_{(\gamma+1)\alpha m}^{T} = \begin{bmatrix}\mathbf{I}_{\alpha m,\alpha m} & \mathbf{O}_{\alpha m,\alpha m} & \dots & \mathbf{O}_{\alpha m,\alpha m}\end{bmatrix} \tag{3.159}$$

In addition, once that observability matrix $\hat{\mathbf{P}}_{\alpha}$ has been identified, the controllability matrix $\hat{\mathbf{Q}}_{\beta}$ can be computed from the factorization of Hankel matrix $\mathbf{H}(0)$ using least-squares method. Indeed:

$$\begin{aligned}\hat{\mathbf{Q}}_{\beta} &= \hat{\mathbf{P}}_{\alpha}^{+}\mathbf{H}(0) = \\ &= \left(\mathbf{E}_{(\gamma+1)\alpha m}^{T}\mathbf{R}_{\Delta,n}\mathbf{\Sigma}_{\Delta,n}^{1/2}\right)^{+}\mathbf{H}(0)\end{aligned} \tag{3.160}$$

Analogously to Eigensystem Realization Algorithm (ERA), the output influence matrix $\hat{\mathbf{C}}$ and the state influence matrix $\hat{\mathbf{B}}$ can be identified respectively from the first m rows of the observability matrix $\hat{\mathbf{P}}_{\alpha}$ and from the first r columns of the controllability matrix $\hat{\mathbf{Q}}_{\beta}$. Indeed:

$$\begin{aligned}\hat{\mathbf{C}} &= \mathbf{E}_{\alpha m}^{T}\hat{\mathbf{P}}_{\alpha} = \\ &= \mathbf{E}_{\alpha m}^{T}\mathbf{E}_{(\gamma+1)\alpha m}^{T}\mathbf{R}_{\Delta,n}\mathbf{\Sigma}_{\Delta,n}^{1/2}\end{aligned} \tag{3.161}$$

$$\begin{aligned}\hat{\mathbf{B}} &= \hat{\mathbf{Q}}_{\beta}\mathbf{E}_{\beta r} = \\ &= \left(\mathbf{E}_{(\gamma+1)\alpha m}^{T}\mathbf{R}_{\Delta,n}\mathbf{\Sigma}_{\Delta,n}^{1/2}\right)^{+}\mathbf{H}(0)\mathbf{E}_{\beta r}\end{aligned} \tag{3.162}$$

On the other hand, using the factorization of block correlation Hankel matrix $\mathbf{H}_{\Delta}(1)$ and the identified observability and controllability matrices $\hat{\mathbf{P}}_{\alpha}$ and $\hat{\mathbf{Q}}_{\beta}$, it can be proved $[5]$, $[6]$ that the identified state matrix $\hat{\mathbf{A}}$ can be computed as:

$$\hat{\mathbf{A}} = \mathbf{\Sigma}_{\Delta,n}^{-1/2}\mathbf{R}_{\Delta,n}^{T}\mathbf{H}_{\Delta}(1)\mathbf{S}_{\Delta,n}\mathbf{\Sigma}_{\Delta,n}^{-1/2} \tag{3.163}$$

In brief, the Eigensystem Realization Algorithm with Data Correlations (ERA/DC) leads to the following identified state-space model:

$$\begin{cases}\hat{\mathbf{A}} = \mathbf{\Sigma}_{\Delta,n}^{-1/2}\mathbf{R}_{\Delta,n}^{T}\mathbf{H}_{\Delta}(1)\mathbf{S}_{\Delta,n}\mathbf{\Sigma}_{\Delta,n}^{-1/2} \\ \hat{\mathbf{B}} = \left(\mathbf{E}_{(\gamma+1)\alpha m}^{T}\mathbf{R}_{\Delta,n}\mathbf{\Sigma}_{\Delta,n}^{1/2}\right)^{+}\mathbf{H}(0)\mathbf{E}_{\beta r} \\ \hat{\mathbf{C}} = \mathbf{E}_{\alpha m}^{T}\mathbf{E}_{(\gamma+1)\alpha m}^{T}\mathbf{R}_{\Delta,n}\mathbf{\Sigma}_{\Delta,n}^{1/2} \\ \hat{\mathbf{D}} = \mathbf{Y}_{0} = \overline{\mathbf{Y}}_{0}\end{cases} \tag{3.164}$$

Finally, consider the Eigensystem Realization Algorithm with Data Correlation (ERA/DC) using Observer/Kalman Filter Identification Method (OKID). Basically, this method is an extension of the two previous algorithms. Indeed, this algorithm utilizes simultaneously the combined set of system and observer gain Markov parameters $\mathbf{Y}_{k}$ and $\mathbf{Y}_{k}^{0}$, which are obtained directly from input-output measurements by using Observer/Kalman Filter Identification Method (OKID), to identify at the same time a state-space model and an observer matrix of the system under test. As starting point, define the matrix of combined system and observer gain Markov parameters as follows:

$$\mathbf{\Gamma}_k = \begin{bmatrix} \mathbf{Y}_k & \mathbf{Y}_k^0 \end{bmatrix} \tag{3.165}$$

Where $\mathbf{\Gamma}_k$ is a $\mathbb{R}^{m\times(r+m)}$ block matrix containing the combined system and observer gain Markov parameters. This matrix can be used to construct a generalized block Hankel matrix defined as:

$$\bar{\mathbf{H}}(k-1) = \begin{bmatrix} \mathbf{\Gamma}_k & \mathbf{\Gamma}_{k+1} & \cdots & \mathbf{\Gamma}_{k+\beta-1} \\ \mathbf{\Gamma}_{k+1} & \mathbf{\Gamma}_{k+2} & \cdots & \mathbf{\Gamma}_{k+\beta} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{\Gamma}_{k+\alpha-1} & \mathbf{\Gamma}_{k+\alpha} & \cdots & \mathbf{\Gamma}_{k+\alpha+\beta-2} \end{bmatrix} \tag{3.166}$$

Where $\bar{\mathbf{H}}(k-1)$ is a $\mathbb{R}^{\alpha m\times\beta(r+m)}$ block data matrix containing the set of combined Markov parameters and α, β are two integer assumed larger than system order n. Analogously to Eigensystem Realization Algorithm (ERA), for a data record of length l, α is set equal to p and β is set equal to $l-p$. The combined Hankel matrix $\bar{\mathbf{H}}(k-1)$ can be factorized as:

$$\bar{\mathbf{H}}(k-1) = \mathbf{P}_\alpha \mathbf{A}^{k-1} \tilde{\mathbf{Q}}_\beta \tag{3.167}$$

Where $\mathbf{P}_\alpha$ and $\tilde{\mathbf{Q}}_\beta$ are respectively $\mathbb{R}^{\alpha m\times n}$ and $\mathbb{R}^{\beta(r+m)}$ matrices representing the observability matrix and the combined controllability matrix. Indeed, the matrix $\tilde{\mathbf{Q}}_\beta$ is defined as follows:

$$\tilde{\mathbf{Q}}_\beta = \begin{bmatrix} \tilde{\mathbf{B}} & \mathbf{A}\tilde{\mathbf{B}} & \mathbf{A}^2\tilde{\mathbf{B}} & \ldots & \mathbf{A}^{\beta-1}\tilde{\mathbf{B}} \end{bmatrix} \tag{3.168}$$

Where $\tilde{\mathbf{B}}$ is a $\mathbb{R}^{n\times(r+m)}$ matrix representing the combined state influence matrix $\mathbf{B}$ and the observer matrix $\mathbf{G}$. Indeed, this matrix is define as follows:

$$\tilde{\mathbf{B}} = \begin{bmatrix} \mathbf{B} & \mathbf{G} \end{bmatrix} \tag{3.169}$$

In addition, for $k=1$ and for $k=2$ the combined Hankel matrix $\bar{\mathbf{H}}(k-1)$ becomes:

$$\bar{\mathbf{H}}(0) = \begin{bmatrix} \mathbf{\Gamma}_1 & \mathbf{\Gamma}_2 & \cdots & \mathbf{\Gamma}_\beta \\ \mathbf{\Gamma}_2 & \mathbf{\Gamma}_3 & \cdots & \mathbf{\Gamma}_{\beta+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{\Gamma}_\alpha & \mathbf{\Gamma}_{\alpha+1} & \cdots & \mathbf{\Gamma}_{\alpha+\beta-1} \end{bmatrix} \tag{3.170}$$

$$\bar{\mathbf{H}}(1) = \begin{bmatrix} \mathbf{\Gamma}_2 & \mathbf{\Gamma}_3 & \cdots & \mathbf{\Gamma}_{\beta+1} \\ \mathbf{\Gamma}_3 & \mathbf{\Gamma}_4 & \cdots & \mathbf{\Gamma}_{\beta+2} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{\Gamma}_{\alpha+1} & \mathbf{\Gamma}_{\alpha+2} & \cdots & \mathbf{\Gamma}_{\alpha+\beta} \end{bmatrix} \tag{3.171}$$

These matrices can be factorized by using the observability matrix and the combined controllability matrix as:

$$\bar{\mathbf{H}}(0) = \mathbf{P}_\alpha \tilde{\mathbf{Q}}_\beta \tag{3.172}$$

$$\bar{\mathbf{H}}(1) = \mathbf{P}_\alpha \mathbf{A} \tilde{\mathbf{Q}}_\beta \tag{3.173}$$

Now consider a correlation matrix constructed using the combined Hankel matrices to yield:

$$\bar{\mathbf{\Delta}}_{HH}(k) = \bar{\mathbf{H}}(k)\bar{\mathbf{H}}^T(0) =$$
$$= \begin{bmatrix} \mathbf{\Gamma}_{k+1} & \mathbf{\Gamma}_{k+2} & \cdots & \mathbf{\Gamma}_{k+\beta} \\ \mathbf{\Gamma}_{k+2} & \mathbf{\Gamma}_{k+3} & \cdots & \mathbf{\Gamma}_{k+\beta+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{\Gamma}_{k+\alpha} & \mathbf{\Gamma}_{k+\alpha+1} & \cdots & \mathbf{\Gamma}_{k+\alpha+\beta-1} \end{bmatrix} \begin{bmatrix} \mathbf{\Gamma}_{1} & \mathbf{\Gamma}_{2} & \cdots & \mathbf{\Gamma}_{\beta} \\ \mathbf{\Gamma}_{2} & \mathbf{\Gamma}_{3} & \cdots & \mathbf{\Gamma}_{\beta+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{\Gamma}_{\alpha} & \mathbf{\Gamma}_{\alpha+1} & \cdots & \mathbf{\Gamma}_{\alpha+\beta-1} \end{bmatrix}^T =$$
$$= \begin{bmatrix} \sum_{j=1}^{\beta}\left(\mathbf{\Gamma}_{k+j}\mathbf{\Gamma}_{j}^T\right) & \sum_{j=1}^{\beta}\left(\mathbf{\Gamma}_{k+j}\mathbf{\Gamma}_{j+1}^T\right) & \cdots & \sum_{j=1}^{\beta}\left(\mathbf{\Gamma}_{k+j}\mathbf{\Gamma}_{\alpha+j-1}^T\right) \\ \sum_{j=1}^{\beta}\left(\mathbf{\Gamma}_{k+j+1}\mathbf{\Gamma}_{j}^T\right) & \sum_{j=1}^{\beta}\left(\mathbf{\Gamma}_{k+j+1}\mathbf{\Gamma}_{j+1}^T\right) & \cdots & \sum_{j=1}^{\beta}\left(\mathbf{\Gamma}_{k+j+1}\mathbf{\Gamma}_{\alpha+j-1}^T\right) \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=1}^{\beta}\left(\mathbf{\Gamma}_{k+\alpha+j-1}\mathbf{\Gamma}_{j}^T\right) & \sum_{j=1}^{\beta}\left(\mathbf{\Gamma}_{k+\alpha+j-1}\mathbf{\Gamma}_{j+1}^T\right) & \cdots & \sum_{j=1}^{\beta}\left(\mathbf{\Gamma}_{k+\alpha+j-1}\mathbf{\Gamma}_{\alpha+j-1}^T\right) \end{bmatrix}$$
$$(3.174)$$

Where $\bar{\mathbf{\Delta}}_{HH}(k)$ is a $\mathbb{R}^{\alpha m\times\alpha m}$ square matrix obtained from the correlation between the combined Hankel matrix evaluated at the generic time step k and the combined Hankel matrix at the initial time step. The block data correlation matrix $\bar{\mathbf{\Delta}}_{HH}(k)$ can be factorized by using the factorization of Hankel matrices $\bar{\mathbf{H}}(0)$ and $\bar{\mathbf{H}}(k)$ in terms of the observability matrix $\mathbf{P}_\alpha$ and of the combined controllability matrix $\tilde{\mathbf{Q}}_\beta$ as follows:

$$\begin{aligned} \bar{\mathbf{\Delta}}_{HH}(k) &= \bar{\mathbf{H}}(k)\bar{\mathbf{H}}^T(0) = \\ &= \mathbf{P}_\alpha\mathbf{A}^k\tilde{\mathbf{Q}}_\beta\tilde{\mathbf{Q}}_\beta^T\mathbf{P}_\alpha^T = \\ &= \mathbf{P}_\alpha\mathbf{A}^k\tilde{\mathbf{Q}}_c \end{aligned} \tag{3.175}$$

Where $\tilde{\mathbf{Q}}_c$ is a $\mathbb{R}^{n\times\alpha m}$ matrix representing a mixed controllability-observability matrix obtained from combined Markov parameters and it is defined as:

$$\tilde{\mathbf{Q}}_c = \tilde{\mathbf{Q}}_\beta \tilde{\mathbf{Q}}_\beta^T \mathbf{P}_\alpha^T \tag{3.176}$$

Similarly to Eigensystem Realization Algorithm with Data Correlations (ERA/DC), the next step is the definition of the block correlation Hankel matrix $\bar{\mathbf{H}}_\Delta(k)$ obtained from combined system and observer gain Markov parameters whose block elements are the data correlation matrices $\bar{\mathbf{\Delta}}_{HH}(k)$ shifted in time with multiple of time lag τ. Indeed:

$$\bar{\mathbf{H}}_\Delta(k) = \begin{bmatrix} \bar{\mathbf{\Delta}}_{HH}(k) & \bar{\mathbf{\Delta}}_{HH}(k+\tau) & \dots & \bar{\mathbf{\Delta}}_{HH}(k+\delta\tau) \\ \bar{\mathbf{\Delta}}_{HH}(k+\tau) & \bar{\mathbf{\Delta}}_{HH}(k+2\tau) & \dots & \bar{\mathbf{\Delta}}_{HH}(k+(\delta+1)\tau) \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\mathbf{\Delta}}_{HH}(k+\gamma\tau) & \bar{\mathbf{\Delta}}_{HH}(k+(\gamma+1)\tau) & \dots & \bar{\mathbf{\Delta}}_{HH}(k+(\gamma+\delta)\tau) \end{bmatrix} \tag{3.177}$$

Where $\bar{\mathbf{H}}_\Delta(k)$ is a $\mathbb{R}^{(\gamma+1)\alpha m\times(\delta+1)\alpha m}$ matrix. The integers γ and δ define how many correlation lags are included in the analysis. Using the factorization of correlation matrix $\bar{\mathbf{\Delta}}_{HH}(k)$ the block correlation Hankel matrix $\bar{\mathbf{H}}_\Delta(k)$ obtained from combined Markov parameter can be decomposed as:

$$\bar{\mathbf{H}}_\Delta(k) = \mathbf{P}_\gamma \mathbf{A}^k \tilde{\mathbf{Q}}_\delta \tag{3.178}$$

Where $\mathbf{P}_\gamma$ and $\tilde{\mathbf{Q}}_\delta$ are respectively $\mathbb{R}^{(\gamma+1)\alpha m\times n}$ and $\mathbb{R}^{n\times(\delta+1)\alpha m}$ matrices representing the block correlation observability matrix and the block correlation mixed controllability-observability matrix obtained from combined Markov parameters. The matrix $\tilde{\mathbf{Q}}_\delta$ is defined in terms of mixed controllability-observability matrix $\tilde{\mathbf{Q}}_c$ obtained from combined Markov parameters as follows:

$$\tilde{\mathbf{Q}}_\delta = \begin{bmatrix} \tilde{\mathbf{Q}}_c & \mathbf{A}^\tau \tilde{\mathbf{Q}}_c & \mathbf{A}^{2\tau} \tilde{\mathbf{Q}}_c & \dots & \mathbf{A}^{\delta\tau} \tilde{\mathbf{Q}}_c \end{bmatrix} \tag{3.179}$$

For $k=0$ and for $k=1$ the block correlation Hankel matrix $\bar{\mathbf{H}}_{\Delta}(k)$ obtained from combined Markov parameters becomes:

$$\bar{\mathbf{H}}_{\Delta}(0)=\begin{bmatrix} \bar{\mathbf{\Delta}}_{HH}(0) & \bar{\mathbf{\Delta}}_{HH}(\tau) & \dots & \bar{\mathbf{\Delta}}_{HH}(\delta\tau) \\ \bar{\mathbf{\Delta}}_{HH}(\tau) & \bar{\mathbf{\Delta}}_{HH}(2\tau) & \dots & \bar{\mathbf{\Delta}}_{HH}((\delta+1)\tau) \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\mathbf{\Delta}}_{HH}(\gamma\tau) & \bar{\mathbf{\Delta}}_{HH}((\gamma+1)\tau) & \dots & \bar{\mathbf{\Delta}}_{HH}((\gamma+\delta)\tau) \end{bmatrix} \tag{3.180}$$

$$\bar{\mathbf{H}}_{\Delta}(1)=\begin{bmatrix} \bar{\mathbf{\Delta}}_{HH}(1) & \bar{\mathbf{\Delta}}_{HH}(1+\tau) & \dots & \bar{\mathbf{\Delta}}_{HH}(1+\delta\tau) \\ \bar{\mathbf{\Delta}}_{HH}(1+\tau) & \bar{\mathbf{\Delta}}_{HH}(1+2\tau) & \dots & \bar{\mathbf{\Delta}}_{HH}(1+(\delta+1)\tau) \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\mathbf{\Delta}}_{HH}(1+\gamma\tau) & \bar{\mathbf{\Delta}}_{HH}(1+(\gamma+1)\tau) & \dots & \bar{\mathbf{\Delta}}_{HH}(1+(\gamma+\delta)\tau) \end{bmatrix} \tag{3.181}$$

These matrices can be respectively decomposed using the block correlation observability matrix $\mathbf{P}_{\gamma}$ and the block correlation mixed controllability-observability matrix obtained from combined Markov parameters $\tilde{\mathbf{Q}}_{\delta}$ as follows:

$$\bar{\mathbf{H}}_{\Delta}(0)=\mathbf{P}_{\gamma}\tilde{\mathbf{Q}}_{\delta} \tag{3.182}$$

$$\bar{\mathbf{H}}_{\Delta}(1)=\mathbf{P}_{\gamma}\mathbf{A}\tilde{\mathbf{Q}}_{\delta} \tag{3.183}$$

Similarly to the Eigensystem Realization Algorithm with Data Correlations (ERA/DC), the next step is the factorization of the block correlation Hankel matrix $\bar{\mathbf{H}}_{\Delta}(0)$ obtained from combined Markov parameters by using the Singular Value Decomposition method (SVD) $[15]$ to yield:

$$\bar{\mathbf{H}}_{\Delta}(0)=\bar{\mathbf{R}}_{\Delta}\bar{\mathbf{\Sigma}}_{\Delta}\bar{\mathbf{S}}_{\Delta}^{T} \tag{3.184}$$

Where $\bar{\boldsymbol{\Sigma}}_\Delta$ is a $\mathbb{R}^{(\gamma+1)\alpha m\times(\delta+1)\alpha m}$ diagonal matrix containing the singular values of matrix $\bar{\mathbf{H}}_\Delta(0)$ whereas $\bar{\mathbf{R}}_\Delta$ and $\bar{\mathbf{S}}_\Delta$ are respectively $\mathbb{R}^{(\gamma+1)\alpha m\times(\gamma+1)\alpha m}$ and $\mathbb{R}^{(\delta+1)\alpha m\times(\delta+1)\alpha m}$ orthonormal matrices containing the left singular vectors and the right singular vectors of matrix $\bar{\mathbf{H}}_\Delta(0)$. These matrices can be respectively partitioned as follows:

$$\bar{\boldsymbol{\Sigma}}_\Delta = \begin{bmatrix} \bar{\boldsymbol{\Sigma}}_{\Delta,n} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix} \tag{3.185}$$

$$\bar{\mathbf{R}}_\Delta = \begin{bmatrix} \bar{\mathbf{R}}_{\Delta,n} & \bar{\mathbf{R}}_{\Delta,(\gamma+1)\alpha m-n} \end{bmatrix} \tag{3.186}$$

$$\bar{\mathbf{S}}_\Delta = \begin{bmatrix} \bar{\mathbf{S}}_{\Delta,n} & \bar{\mathbf{S}}_{\Delta,(\delta+1)\alpha m-n} \end{bmatrix} \tag{3.187}$$

Where $\bar{\boldsymbol{\Sigma}}_{\Delta,n}$, $\bar{\mathbf{R}}_{\Delta,n}$, $\bar{\mathbf{R}}_{\Delta,(\gamma+1)\alpha m-n}$, $\bar{\mathbf{S}}_{\Delta,n}$ and $\bar{\mathbf{S}}_{\Delta,(\delta+1)\alpha m-n}$ are respectively $\mathbb{R}^{n\times n}$, $\mathbb{R}^{(\gamma+1)\alpha m\times n}$, $\mathbb{R}^{(\gamma+1)\alpha m\times((\gamma+1)\alpha m-n)}$, $\mathbb{R}^{(\delta+1)\alpha m\times n}$ and $\mathbb{R}^{(\delta+1)\alpha m\times((\delta+1)\alpha m-n)}$ matrices. The matrix $\bar{\boldsymbol{\Sigma}}_{\Delta,n}$ is a diagonal matrix containing the significant singular values of the system. Indeed:

$$\bar{\boldsymbol{\Sigma}}_{\Delta,n} = diag(\bar{\sigma}_{\Delta,1}, \bar{\sigma}_{\Delta,2}, \ldots, \bar{\sigma}_{\Delta,n}) \tag{3.188}$$

Even in this case, the block correlation Hankel matrix $\bar{\mathbf{H}}_\Delta(0)$ obtained from combined Markov parameters is typically of full rank which generally is not equal to the true order of the system. Therefore this matrix can be approximated as:

$$\bar{\mathbf{H}}_\Delta(0) = \bar{\mathbf{R}}_{\Delta,n}\bar{\boldsymbol{\Sigma}}_{\Delta,n}\bar{\mathbf{S}}_{\Delta,n}^T \tag{3.189}$$

Similarly to Eigensystem Realization Algorithm with Data Correlations (ERA/DC), the block correlation observability matrix $\mathbf{P}_\gamma$ can be related to the left singular vector matrix $\bar{\mathbf{R}}_{\Delta,n}$ and the block correlation controllability-observability matrix $\tilde{\mathbf{Q}}_{\delta}$ obtained from combined Markov parameters can be related to the right singular vector matrix $\bar{\mathbf{S}}_{\Delta,n}$. Indeed, it can be proved $[5]$, $[6]$ that these matrices can be computed as:

$$\hat{\mathbf{P}}_{\gamma} = \bar{\mathbf{R}}_{\Delta,n}\bar{\mathbf{\Sigma}}_{\Delta,n}^{1/2} \tag{3.190}$$

$$\hat{\hat{\mathbf{Q}}}_{\delta} = \bar{\mathbf{\Sigma}}_{\Delta,n}^{1/2}\bar{\mathbf{S}}_{\Delta,n}^{T} \tag{3.191}$$

Once that the block correlation observability matrix $\hat{\mathbf{P}}_{\gamma}$ and the block correlation mixed controllability-observability matrix $\hat{\hat{\mathbf{Q}}}_{\delta}$ obtained from combined Markov parameters have been identified, the observability matrix $\hat{\mathbf{P}}_{\alpha}$ and the mixed controllability-observability matrix $\hat{\hat{\mathbf{Q}}}_{c}$ obtained from combined Markov parameters can be identified from the first αm rows of the block correlation observability matrix $\hat{\mathbf{P}}_{\gamma}$ and from the first αm columns of the block correlation mixed controllability-observability matrix $\hat{\hat{\mathbf{Q}}}_{\delta}$ obtained from combined Markov parameters. Indeed:

$$\begin{aligned}\hat{\mathbf{P}}_{\alpha} &= \mathbf{E}_{(\gamma+1)\alpha m}^{T}\hat{\mathbf{P}}_{\gamma} = \\ &= \mathbf{E}_{(\gamma+1)\alpha m}^{T}\bar{\mathbf{R}}_{\Delta,n}\bar{\mathbf{\Sigma}}_{\Delta,n}^{1/2}\end{aligned} \tag{3.192}$$

$$\begin{aligned}\hat{\hat{\mathbf{Q}}}_{c} &= \hat{\hat{\mathbf{Q}}}_{\delta}\mathbf{E}_{(\gamma+1)\alpha m} = \\ &= \bar{\mathbf{\Sigma}}_{\Delta,n}^{1/2}\bar{\mathbf{S}}_{\Delta,n}^{T}\mathbf{E}_{(\gamma+1)\alpha m}\end{aligned} \tag{3.193}$$

In addition, once that observability matrix $\hat{\mathbf{P}}_\alpha$ has been identified, the combined controllability matrix $\hat{\hat{\mathbf{Q}}}_\beta$ can be computed from the factorization of generalized Hankel matrix $\bar{\mathbf{H}}(0)$ obtained from combined Markov parameters using least-squares method. Indeed:

$$\begin{aligned}\hat{\hat{\mathbf{Q}}}_\beta &= \hat{\mathbf{P}}_\alpha^+ \bar{\mathbf{H}}(0) = \\ &= \left(\mathbf{E}^T_{(\gamma+1)\alpha m} \bar{\mathbf{R}}_{\Delta,n} \bar{\mathbf{\Sigma}}^{1/2}_{\Delta,n}\right)^+ \bar{\mathbf{H}}(0)\end{aligned} \tag{3.194}$$

Analogously to Eigensystem Realization Algorithm with Data Correlations (ERA/DC), the output influence matrix $\hat{\mathbf{C}}$ and the combined state influence matrix $\hat{\hat{\mathbf{B}}}$ can be identified from the first m rows of the observability matrix $\hat{\mathbf{P}}_\alpha$ and from the first $r+m$ columns of the combined controllability matrix $\hat{\hat{\mathbf{Q}}}_\beta$. Indeed:

$$\begin{aligned}\hat{\mathbf{C}} &= \mathbf{E}^T_{\alpha m} \hat{\mathbf{P}}_\alpha = \\ &= \mathbf{E}^T_{\alpha m} \mathbf{E}^T_{(\gamma+1)\alpha m} \bar{\mathbf{R}}_{\Delta,n} \bar{\mathbf{\Sigma}}^{1/2}_{\Delta,n}\end{aligned} \tag{3.195}$$

$$\begin{aligned}\hat{\hat{\mathbf{B}}} &= \hat{\hat{\mathbf{Q}}}_\beta \mathbf{E}_{\beta(r+m)} = \\ &= \left(\mathbf{E}^T_{(\gamma+1)\alpha m} \bar{\mathbf{R}}_{\Delta,n} \bar{\mathbf{\Sigma}}^{1/2}_{\Delta,n}\right)^+ \bar{\mathbf{H}}(0) \mathbf{E}_{\beta(r+m)}\end{aligned} \tag{3.196}$$

Where $\mathbf{E}_{\beta(r+m)}$ is a $\mathbb{R}^{\beta(r+m)\times(r+m)}$ Boolean matrix defined as:

$$\mathbf{E}^T_{\beta(r+m)} = \begin{bmatrix} \mathbf{I}_{r+m,r+m} & \mathbf{O}_{r+m,r+m} & \dots & \mathbf{O}_{r+m,r+m} \end{bmatrix} \tag{3.197}$$

Moreover, the state influence matrix $\hat{\mathbf{B}}$ and the observer matrix $\hat{\mathbf{G}}$ can be obtained respectively as the first r columns and as the last m columns of the combined state influence matrix $\hat{\hat{\mathbf{B}}}$. Indeed:

$$\begin{aligned}\hat{\mathbf{B}} &= \hat{\hat{\mathbf{B}}}\mathbf{E}_r = \\ &= \left(\mathbf{E}^T_{(\gamma+1)\alpha m}\bar{\mathbf{R}}_{\Delta,n}\bar{\boldsymbol{\Sigma}}^{1/2}_{\Delta,n}\right)^+ \bar{\mathbf{H}}(0)\mathbf{E}_{\beta(r+m)}\mathbf{E}_r\end{aligned} \tag{3.198}$$

$$\begin{aligned}\hat{\mathbf{G}} &= \hat{\hat{\mathbf{B}}}\mathbf{F}_m = \\ &= \left(\mathbf{E}^T_{(\gamma+1)\alpha m}\bar{\mathbf{R}}_{\Delta,n}\bar{\boldsymbol{\Sigma}}^{1/2}_{\Delta,n}\right)^+ \bar{\mathbf{H}}(0)\mathbf{E}_{\beta(r+m)}\mathbf{F}_m\end{aligned} \tag{3.199}$$

Were $\mathbf{E}_r$ and $\mathbf{F}_m$ are respectively $\mathbb{R}^{(r+m)\times r}$ and $\mathbb{R}^{(r+m)\times m}$ Boolean matrices defined as:

$$\mathbf{E}_r = \begin{bmatrix} \mathbf{I}_{r,r} \\ \mathbf{O}_{m,r} \end{bmatrix} \tag{3.200}$$

$$\mathbf{F}_m = \begin{bmatrix} \mathbf{O}_{r,m} \\ \mathbf{I}_{m,m} \end{bmatrix} \tag{3.201}$$

On the other hand, using the factorization of block correlation Hankel matrix $\bar{\mathbf{H}}_\Delta(1)$ obtained from combined Markov parameters and by using the identified observability matrix $\hat{\mathbf{P}}_\alpha$ and the identified combined controllability matrices $\hat{\hat{\mathbf{Q}}}_\beta$, it can be proved $[5]$, $[6]$ that the identified state matrix $\hat{\mathbf{A}}$ can be computed as:

$$\hat{\mathbf{A}} = \bar{\boldsymbol{\Sigma}}^{-1/2}_{\Delta,n}\bar{\mathbf{R}}^T_{\Delta,n}\bar{\mathbf{H}}_\Delta(1)\bar{\mathbf{S}}_{\Delta,n}\bar{\boldsymbol{\Sigma}}^{-1/2}_{\Delta,n} \tag{3.202}$$

Consequently, the Eigensystem Realization Algorithm with Data Correlations (ERA/DC) using Observer/Kalman Filter Identification Method (OKID) can be summarized as follows:

$$\begin{cases} \hat{\mathbf{A}} = \bar{\mathbf{\Sigma}}_{\Delta,n}^{-1/2} \bar{\mathbf{R}}_{\Delta,n}^{T} \bar{\mathbf{H}}_{\Delta}(1) \bar{\mathbf{S}}_{\Delta,n} \bar{\mathbf{\Sigma}}_{\Delta,n}^{-1/2} \\ \hat{\mathbf{B}} = \left(\mathbf{E}_{(\gamma+1)\alpha m}^{T} \bar{\mathbf{R}}_{\Delta,n} \bar{\mathbf{\Sigma}}_{\Delta,n}^{1/2} \right)^{+} \bar{\mathbf{H}}(0) \mathbf{E}_{\beta(r+m)} \mathbf{E}_{r} \\ \hat{\mathbf{G}} = \left(\mathbf{E}_{(\gamma+1)\alpha m}^{T} \bar{\mathbf{R}}_{\Delta,n} \bar{\mathbf{\Sigma}}_{\Delta,n}^{1/2} \right)^{+} \bar{\mathbf{H}}(0) \mathbf{E}_{\beta(r+m)} \mathbf{F}_{m} \\ \hat{\mathbf{C}} = \mathbf{E}_{\alpha m}^{T} \mathbf{E}_{(\gamma+1)\alpha m}^{T} \bar{\mathbf{R}}_{\Delta,n} \bar{\mathbf{\Sigma}}_{\Delta,n}^{1/2} \\ \hat{\mathbf{D}} = \mathbf{Y}_0 = \bar{\mathbf{Y}}_0 \end{cases} \tag{3.203}$$

At this stage, regardless of the method which has been used, the system modal parameters can be extracted from the identified state-space realization $\hat{\mathbf{A}}$, $\hat{\mathbf{B}}$ and $\hat{\mathbf{C}}$. Indeed, the spectral decomposition of identified state matrix $\hat{\mathbf{A}}$ yields:

$$\hat{\mathbf{A}}\hat{\mathbf{\Psi}} = \hat{\mathbf{\Psi}}\hat{\mathbf{\Lambda}} \tag{3.204}$$

Where $\hat{\mathbf{\Lambda}}$ is a $\mathbb{R}^{n \times n}$ diagonal matrix containing the identified system eigenvalues and $\hat{\mathbf{\Psi}}$ is a $\mathbb{R}^{n \times n}$ matrix containing the identified eigenvectors stacked by column. The identified modal state matrix $\hat{\mathbf{A}}_m$, the identified modal state influence matrix $\hat{\mathbf{B}}_m$ and the identified modal output influence matrix $\hat{\mathbf{C}}_m$ can be computed using the spectral decomposition of the identified state matrix $\hat{\mathbf{A}}$ as follows:

$$\hat{\mathbf{A}}_m = \hat{\mathbf{\Lambda}} \tag{3.205}$$

$$\hat{\mathbf{B}}_m = \hat{\mathbf{\Psi}}^{-1}\hat{\mathbf{B}} \tag{3.206}$$

$$\hat{\mathbf{C}}_m = \hat{\mathbf{C}}\hat{\mathbf{\Psi}} \tag{3.207}$$

The identified modal state matrix $\hat{\mathbf{A}}_m$ contains the information of system natural frequencies and damping ratios whereas the identified modal state influence matrix $\hat{\mathbf{B}}_m$ define the identified initial mode amplitudes and the identified modal output influence matrix $\hat{\mathbf{C}}_m$ represent the identified mode shapes at the sensor points. Therefore, all the identified modal parameters of a dynamic system are represented by the triplet of matrices $\hat{\mathbf{A}}_m$, $\hat{\mathbf{B}}_m$ and $\hat{\mathbf{C}}_m$. In conclusion, supposing that all the identified modes are underdamped, in many practical applications the hypothesis of proportional damping can be assumed as satisfied, especially in the case of structural systems in which damping is small and no a priori information about its nature are available. The proportional damping assumption implies that the modal damping ratios ξ_j are related to the natural frequencies $\omega_{n,j}$ according to the following equations:

$$\xi_j = \frac{\alpha}{2\omega_{n,j}} + \frac{\beta\omega_{n,j}}{2} \quad , \quad j = 1, 2, \ldots, n_2 \tag{3.208}$$

Where α and β are the proportional damping coefficients. This coefficient can be estimated in a simple and effective way leveraging on the identified natural frequencies $\hat{\omega}_{n,j}$ and on the identified damping ratios $\hat{\xi}_j$ $[12]$, $[13]$. Indeed, reformulating the previous equations in according to a matrix notation yields:

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{3.209}$$

Where $\mathbf{A}$ is a $\mathbb{R}^{n_2 \times 2}$ rectangular matrix assembled using the identified natural frequencies $\hat{\omega}_{n,j}$ whereas $\mathbf{x}$ is a $\mathbb{R}^2$ vector containing the unknown proportional coefficients $\hat{\alpha}$, $\hat{\beta}$ and $\mathbf{b}$ is a $\mathbb{R}^{n_2}$ vector containing the identified damping ratios $\hat{\xi}_j$. These elements are respectively defined as:

$$\mathbf{A} = \begin{bmatrix} \dfrac{1}{2\hat{\omega}_{n,1}} & \dfrac{\hat{\omega}_{n,1}}{2} \\ \dfrac{1}{2\hat{\omega}_{n,2}} & \dfrac{\hat{\omega}_{n,2}}{2} \\ \vdots & \vdots \\ \dfrac{1}{2\hat{\omega}_{n,n_2}} & \dfrac{\hat{\omega}_{n,n_2}}{2} \end{bmatrix} \tag{3.210}$$

$$\mathbf{x} = \begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \end{bmatrix} \tag{3.211}$$

$$\mathbf{b} = \begin{bmatrix} \hat{\xi}_1 \\ \hat{\xi}_2 \\ \vdots \\ \hat{\xi}_{n_2} \end{bmatrix} \tag{3.212}$$

Therefore, the proportional damping coefficients $\hat{\alpha}$, $\hat{\beta}$ can be approximately computed by using the least-squares method to yield:

$$\mathbf{x} = \mathbf{A}^{+}\mathbf{b} \tag{3.213}$$

Where $\mathbf{A}^{+}$ is a $\mathbb{R}^{2\times n_2}$ matrix which represents the Moore-Penrose pseudoinverse of matrix $\mathbf{A}$. This method represent an useful mathematical tool to deal with realistic experimental data.

3.7. METHOD FOR CONSTRUCTING PHYSICAL MODELS FROM IDENTIFIED STATE SPACE REPRESENTATIONS (MKR)

In this section it is showed a method to derive a second-order physical model of a mechanical system starting from an identified first-order state-space representation of the same system (MKR) $[8]$, $[9]$, $[10]$. This method represents a solution for the general problem known as linear inverse vibration problem $[8]$, $[9]$, $[10]$. Indeed, it is well-known that a physical model of a linear mechanical system is completely described by the triplet of mass matrix $\mathbf{M}$, stiffness matrix $\mathbf{K}$ and damping matrix $\mathbf{R}$. This second-order physical model can be easily converted into a first-order state-space model represented by the triplet of state matrix $\mathbf{A}_c$, state influence matrix $\mathbf{B}_c$ and output influence matrix $\mathbf{C}$. This problem is sometimes referred as the forward problem. On the other hand, the inverse problem is more complex. Indeed, there are several algorithms which allows to experimentally determine from input and output measurements a first-order state-space model represented by the triplet of the identified state matrix $\hat{\mathbf{A}}_c$, the identified state influence matrix $\hat{\mathbf{B}}_c$ and the identified output influence matrix $\hat{\mathbf{C}}$. The transformation of the identified state-space model into a triplet of identified mass matrix $\hat{\mathbf{M}}$, identified stiffness matrix $\hat{\mathbf{K}}$ and identified damping matrix $\hat{\mathbf{R}}$ is not trivial and it can performed using different methods according to the state-space coordinates chosen to represent the system and according to the location of sensors and actuators on each system degree of freedom. Using the method showed here (MKR) the basic requirement is that all system degrees of freedom must be instrumented with a sensor or an actuator, with at least one co-located sensor-actuator pair. In addition, the state-space representation of the system is formulated in a symmetric fashion to yield:

$$\mathbf{V}_c\dot{\mathbf{z}}(t) = \mathbf{S}_c\mathbf{z}(t) + \mathbf{N}_c\mathbf{u}(t) \tag{3.214}$$

Where $\mathbf{V}_c$ and $\mathbf{S}_c$ are $\mathbb{R}^{n\times n}$ symmetric matrices and $\mathbf{N}_c$ is a $\mathbb{R}^{n\times r}$ matrix respectively defined as:

$$\mathbf{V}_c = \begin{bmatrix} \mathbf{R} & \mathbf{M} \\ \mathbf{M} & \mathbf{O} \end{bmatrix} \tag{3.215}$$

$$\mathbf{S}_c = \begin{bmatrix} -\mathbf{K} & \mathbf{O} \\ \mathbf{O} & \mathbf{M} \end{bmatrix} \tag{3.216}$$

$$\mathbf{N}_c = \begin{bmatrix} \mathbf{B}_2 \\ \mathbf{O} \end{bmatrix} \tag{3.217}$$

The peculiarity of this formulation is that the associated eigenvalue problem results to be symmetric and it can be written in a matrix form as:

$$\mathbf{S}_c\mathbf{\Psi} = \mathbf{V}_c\mathbf{\Psi}\mathbf{\Lambda}_c \tag{3.218}$$

Where $\mathbf{\Lambda}_c$ is a $\mathbb{R}^{n\times n}$ diagonal matrix containing the system eigenvalues and $\mathbf{\Psi}$ is a $\mathbb{R}^{n\times n}$ matrix containing the system eigenvectors stacked by column. In particular, even in this case the eigenvector matrix $\mathbf{\Psi}$ can be partitioned as:

$$\mathbf{\Psi} = \begin{bmatrix} \mathbf{W} \\ \mathbf{W}\mathbf{\Lambda}_c \end{bmatrix} \tag{3.219}$$

Where $\mathbf{W}$ is a $\mathbb{R}^{n_2\times n}$ eigenvector matrix representing the physical coordinate eigenvector matrix. Assume that all modes of the underlying dynamical systems are underdamped and therefore the eigenvalues are supposed to appear in complex conjugate pairs. Since the eigenvectors scaling is arbitrary, assume that the eigenvector matrix $\mathbf{\Psi}$ is scaled such that:

$$\mathbf{\Psi}^T\mathbf{V}_c\mathbf{\Psi} = \mathbf{I} \tag{3.220}$$

$$\mathbf{\Psi}^T \mathbf{S}_c \mathbf{\Psi} = \mathbf{\Lambda}_c \tag{3.221}$$

This assumption can be explicitly restated as follows:

$$\begin{bmatrix} \mathbf{W} \\ \mathbf{W}\mathbf{\Lambda}_c \end{bmatrix}^T \begin{bmatrix} \mathbf{R} & \mathbf{M} \\ \mathbf{M} & \mathbf{O} \end{bmatrix} \begin{bmatrix} \mathbf{W} \\ \mathbf{W}\mathbf{\Lambda}_c \end{bmatrix} = \mathbf{I} \tag{3.222}$$

$$\begin{bmatrix} \mathbf{W} \\ \mathbf{W}\mathbf{\Lambda}_c \end{bmatrix}^T \begin{bmatrix} -\mathbf{K} & \mathbf{O} \\ \mathbf{O} & \mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{W} \\ \mathbf{W}\mathbf{\Lambda}_c \end{bmatrix} = \mathbf{\Lambda}_c \tag{3.223}$$

This assumption is a key-step whose consequences are twofold. The first immediate consequence is that the modal state-space model assumes the following particular form:

$$\mathbf{p}(t) = \mathbf{A}_{c,m}\mathbf{p}(t) + \mathbf{B}_{c,m}\mathbf{u}(t) \tag{3.224}$$

$$\mathbf{y}(t) = \mathbf{C}_m\mathbf{p}(t) + \mathbf{D}\mathbf{u}(t) \tag{3.225}$$

Where $\mathbf{A}_{c,m}$, $\mathbf{B}_{c,m}$ and $\mathbf{C}_m$ are respectively $\mathbb{R}^{n\times n}$, $\mathbb{R}^{n\times r}$ and $\mathbb{R}^{m\times n}$ matrices representing the modal state matrix, the modal state influence matrix and the modal output influence matrix. It can be easily proved that these matrices can be computed as:

$$\mathbf{A}_{c,m} = \mathbf{\Lambda}_c \tag{3.226}$$

$$\mathbf{B}_{c,m} = \mathbf{W}^T\mathbf{B}_2 \tag{3.227}$$

$$\mathbf{C}_m = \mathbf{C}_s\mathbf{W}\mathbf{\Lambda}_c^b \tag{3.228}$$

Where $\mathbf{C}_s$ is a $\mathbb{R}^{m\times n}$ matrix and b is a scalar which characterize the type of sensing. Indeed, for displacement sensing one has $\mathbf{C}_s = \mathbf{C}_d$ and $b = 0$, for velocity sensing one has $\mathbf{C}_s = \mathbf{C}_v$ and $b = 1$ whereas for acceleration sensing one has $\mathbf{C}_s = \mathbf{C}_a$ and $b = 2$. Moreover, note that the modal state influence matrix $\mathbf{B}_{c,m}$ is computed by using the transpose of the eigenvector matrix $\mathbf{W}$ instead of using the inverse of the eigenvector matrix $\boldsymbol{\Psi}$. The second consequence of the selected eigenvector scaling is that the triplet of mass matrix $\mathbf{M}$, stiffness matrix $\mathbf{K}$ and damping matrix $\mathbf{R}$ can be directly computed from the eigenvalue matrix $\boldsymbol{\Lambda}_c$ and from the eigenvector matrix $\mathbf{W}$. Indeed, it can be proved $[8]$, $[9]$, $[10]$ that these matrices can be computed as:

$$\mathbf{M} = \left(\mathbf{W}\boldsymbol{\Lambda}_c\mathbf{W}^T\right)^{-1} \tag{3.229}$$

$$\mathbf{K} = -\left(\mathbf{W}\boldsymbol{\Lambda}_c^{-1}\mathbf{W}^T\right)^{-1} \tag{3.230}$$

$$\mathbf{R} = -\mathbf{M}\mathbf{W}\boldsymbol{\Lambda}_c^2\mathbf{W}^T\mathbf{M} \tag{3.231}$$

Therefore, the problem that arises at this point is how to extract the eigenvector matrix $\mathbf{W}$ from an identified state-space representation. Since the system modal parameters must be the same regardless the type of state-space formulation used, the problem is to find a transformation $\mathbf{T}$ which convert the identified modal parameters, characterized by the triplet of matrices $\hat{\boldsymbol{\Lambda}}_c$, $\hat{\boldsymbol{\Psi}}^{-1}\hat{\mathbf{B}}_c$ and $\hat{\mathbf{C}}\hat{\boldsymbol{\Psi}}$, into the symmetric representation modal parameters, characterized by the triplet of matrices $\boldsymbol{\Lambda}_c$, $\mathbf{W}^T\mathbf{B}_2$ and $\mathbf{C}_s\mathbf{W}\boldsymbol{\Lambda}_c^b$. This problem can be mathematically stated as:

$$\mathbf{T}^{-1}\hat{\boldsymbol{\Lambda}}_c\mathbf{T} = \boldsymbol{\Lambda}_c \tag{3.232}$$

$$\mathbf{T}^{-1}\hat{\mathbf{\Psi}}^{-1}\hat{\mathbf{B}}_c = \mathbf{W}^T\mathbf{B}_2 \tag{3.233}$$

$$\hat{\mathbf{C}}\hat{\mathbf{\Psi}}\mathbf{T} = \mathbf{C}_s\mathbf{W}\mathbf{\Lambda}_c^b \tag{3.234}$$

Since the eigenvalues are equal in both the representation, it is straightforward to understand that the transformation matrix $\mathbf{T}$ is a diagonal matrix composed of complex conjugate elements. Moreover, the transformation matrix $\mathbf{T}$ has two effects: it transforms the eigenvectors from those of an asymmetric eigenvalue problem into those of a symmetric problem and it properly scales such eigenvectors. The basic observation necessary to compute the transformation matrix $\mathbf{T}$ is that for a co-located sensor-actuator pair the following matrix equation holds:

$$\mathbf{C}_s(i,:)\mathbf{W} = \left(\mathbf{W}^T\mathbf{B}_2(:,i)\right)^T \tag{3.235}$$

Where $\mathbf{C}_s(i,:)$ indicates the row i of matrix $\mathbf{C}_s$ and $\mathbf{B}_2(:,i)$ indicates the column i of matrix $\mathbf{B}_2$. Note that the previous matrix equation holds because the matrices $\mathbf{C}_s$ and $\mathbf{B}_2$ are simply Boolean matrices. Indeed, it can be proved [8], [9], [10] that leveraging on this observation the matrix transformation $\mathbf{T}$ can be computed by using the identified realization and the identified modal parameters as follows:

$$\hat{\mathbf{C}}^E(i,:)\hat{\mathbf{\Psi}}\hat{\mathbf{\Lambda}}_c^{-b}\mathbf{T}^2 = \left(\hat{\mathbf{\Psi}}^{-1}\hat{\mathbf{B}}_c^E(:,i)\right)^T \tag{3.236}$$

Where $\hat{\mathbf{B}}_c^E$ and $\hat{\mathbf{C}}^E$ are $\mathbb{R}^{n\times n}$ matrices denoting respectively the expanded version of identified state influence matrix $\hat{\mathbf{B}}_c$ and output influence matrix $\hat{\mathbf{C}}$ which include rows and columns of zeros in order to match the dimension n. Once that the transformation matrix $\mathbf{T}$ has been computed, the rows of the eigenvector matrix $\hat{\mathbf{W}}$ can be identified from each degree of freedom which is instrumented with a sensor or with an actuator. Indeed, it can be proved [8], [9], [10] that:

$$\hat{\mathbf{W}}(j,:) = \left(\mathbf{T}^{-1}\hat{\mathbf{\Psi}}^{-1}\hat{\mathbf{B}}_c^E(:,j)\right)^T \tag{3.237}$$

$$\hat{\mathbf{W}}(k,:) = \hat{\mathbf{C}}^E(k,:)\hat{\mathbf{\Psi}}\mathbf{\Lambda}_c^{-b}\mathbf{T} \tag{3.238}$$

Where j is a generic degree of freedom instrumented with an actuator and k is a generic degree of freedom instrumented with a sensor. Finally, using the identified eigenvector matrix $\hat{\mathbf{W}}$ a second-order model of the mechanical system can be identified as:

$$\hat{\mathbf{M}} = \left(\hat{\mathbf{W}}\hat{\mathbf{\Lambda}}_c\hat{\mathbf{W}}^T\right)^{-1} \tag{3.239}$$

$$\hat{\mathbf{K}} = -\left(\hat{\mathbf{W}}\hat{\mathbf{\Lambda}}_c^{-1}\hat{\mathbf{W}}^T\right)^{-1} \tag{3.240}$$

$$\hat{\mathbf{R}} = -\hat{\mathbf{M}}\hat{\mathbf{W}}\hat{\mathbf{\Lambda}}_c^2\hat{\mathbf{W}}^T\hat{\mathbf{M}} \tag{3.241}$$

Where $\hat{\mathbf{M}}$, $\hat{\mathbf{K}}$ and $\hat{\mathbf{R}}$ are matrices denoting respectively the identified mass, stiffness and damping matrices. These matrices can be used to design a controller directly from the system second-order mechanical model. In particular, for lightly damped system it can be proved [12], [13] experimentally that a better estimation of damping matrix $\hat{\mathbf{R}}$ can be obtained from identified mass and stiffness matrices $\hat{\mathbf{M}}$ and $\hat{\mathbf{K}}$ by using an identified set of proportional damping coefficients as follows:

$$\hat{\mathbf{R}} = \hat{\alpha}\hat{\mathbf{M}} + \hat{\beta}\hat{\mathbf{K}} \tag{3.242}$$

Where the coefficients $\hat{\alpha}$ and $\hat{\beta}$ can be computed from identified natural frequencies and damping ratios via least-squares method $[12]$, $[13]$.

4. CONTROL THEORY

4.1. INTRODUCTION

The raison d'etre of a control system is to influence the dynamic of a mechanical system in order to make it behave in a desirable manner [1], [2]. Indeed, the two typical objectives of a control system are regulation and tracking. In a regulation problem, the system is controlled so that its output is maintained at a certain set point [3], [4]. In tracking problem, the system is controlled so that its output follows a particular desired trajectory [5], [6]. A special case of the regulation problem is the stabilization problem in which a control system is designed to bring the system to rest from any nonzero initial conditions and therefore the desirable set point is zero. For a flexible structure that may be subjected to unwanted vibrations, this is usually the most important goal of a control system [7], [8]. Stabilization is the focus of the following sections where a special class of control system is considered, namely the state-feedback controller in which the control input is a function of the system state. In particular, the Linear Quadratic Regulator algorithm (LQR) [9], [10] is derived for both continuous-time and discrete-time systems. In addition, if the state of the system cannot be measured directly, then a state observer is needed to estimate the system state from the measurements. In particular, the Kalman Filter algorithm (KF) [11], [12] is derived for both continuous-time and discrete-time systems. Finally, the system state is used in a state-feedback controller according to the Linear Quadratic Gaussian control method (LQG) [13], [14].

4.2. REGULATION PROBLEM

Consider a linear-dynamic time-invariant mechanical system. From a physical point of view, the regulation problem consists in finding a control action such that the system does not deviate from a given set point, which can be supposed to be in the origin of the configuration space without loss of generality $[9]$, $[10]$. From a mathematical viewpoint, this problem can be formulated for the continuous-time state-space representation of the mechanical system as well as for its discrete-time state space representation. As starting point, consider the system continuous-time state-space formulation:

$$\begin{cases} \dot{\mathbf{z}}(t) = \mathbf{A}_c \mathbf{z}(t) + \mathbf{B}_c \mathbf{u}(t) \\ \mathbf{z}(0) = \mathbf{z}_0 \end{cases} \tag{4.1}$$

Where $\mathbf{z}_0$ is the vector of initial conditions. Assume that there are enough sensors to completely measure the state vector $\mathbf{z}(t)$. Therefore, the output equations is simply:

$$\mathbf{y}(t) = \mathbf{z}(t) \tag{4.2}$$

One method to solve the regulation problem is to construct the control vector $\mathbf{u}(t)$ as a linear combination of the state vector $\mathbf{z}(t)$. Indeed:

$$\mathbf{u}(t) = \mathbf{F}_c \mathbf{z}(t) \tag{4.3}$$

Where $\mathbf{F}_c$ is a $\mathbb{R}^{m\times n}$ matrix which represent the controller gain matrix. Consequently, the regulation problem reduces to properly compute the feedback matrix $\mathbf{F}_c$ in order to control the system. The question which spontaneously arises is if the introduction of the controller destabilizes the system or not. To answer this question, substitute the feedback control in the state equation:

$$\begin{aligned}\dot{\mathbf{z}}(t) &= \mathbf{A}_c\mathbf{z}(t)+\mathbf{B}_c\mathbf{u}(t) = \\ &= \mathbf{A}_c\mathbf{z}(t)+\mathbf{B}_c\mathbf{F}_c\mathbf{z}(t) = \\ &= \left(\mathbf{A}_c+\mathbf{B}_c\mathbf{F}_c\right)\mathbf{z}(t) = \\ &= \mathbf{A}_{F,c}\mathbf{z}(t)\end{aligned} \tag{4.4}$$

Where $\mathbf{A}_{F,c}$ is a $\mathbb{R}^{n\times n}$ matrix which represents the closed-loop state matrix. This matrix is defined as:

$$\mathbf{A}_{F,c} = \mathbf{A}_c + \mathbf{B}_c\mathbf{F}_c \tag{4.5}$$

In order to obtain an asymptotically stable system the feedback matrix $\mathbf{F}_c$ must be chosen such that the eigenvalues of the closed-loop state matrix $\mathbf{A}_{F,c}$ have negative real parts. Therefore, a physically intuitive method to find the controller gain matrix $\mathbf{F}_c$ is to force the eigenvalues of the closed-loop state matrix $\mathbf{A}_{F,c}$ to assume a prescribed set of values. The basic requirement to place the closed-loop poles of matrix $\mathbf{A}_{F,c}$ in a specific location of the complex plane is that the system must be controllable. A linear time-invariant dynamical system of order n is controllable if and only if its controllability matrix $\mathbf{Q}_{F,c}$ has rank n. The controllability matrix $\mathbf{Q}_{F,c}$ is a $\mathbb{R}^{n\times nr}$ matrix defined as:

$$\mathbf{Q}_{F,c} = \begin{bmatrix}\mathbf{B}_c & \mathbf{A}_c\mathbf{B}_c & \mathbf{A}_c^2\mathbf{B}_c & \dots & \mathbf{A}_c^{n-1}\mathbf{B}_c\end{bmatrix} \tag{4.6}$$

Consider now the eigenvalue problem of the closed-loop state matrix $\mathbf{A}_{F,c}$:

$$\mathbf{A}_{F,c}\boldsymbol{\psi}_{F,c} = \lambda_{F,c}\boldsymbol{\psi}_{F,c} \tag{4.7}$$

Where $\lambda_{F,c}$ is a generic eigenvalue of matrix $\mathbf{A}_{F,c}$ and $\boldsymbol{\psi}_{F,c}$ is a $\mathbb{R}^n$ vector representing the eigenvector of the closed-loop state matrix $\mathbf{A}_{F,c}$ corresponding to the eigenvalue $\lambda_{F,c}$. The basic assumption of this method is that the system is nondefective, namely that exist a full set of eigenvectors corresponding to the eigenvalues to be assigned $[3]$. The eigenvalue problem of matrix $\mathbf{A}_{F,c}$ can be explicitly expressed as:

$$\left(\mathbf{A}_c + \mathbf{B}_c \mathbf{F}_c\right) \boldsymbol{\psi}_{F,c} = \lambda_{F,c} \boldsymbol{\psi}_{F,c} \tag{4.8}$$

This eigenvalue problem can be restated as follows:

$$\begin{aligned} \left[\mathbf{A}_c - \lambda_{F,c}\mathbf{I} \quad \mathbf{B}_c\right] \begin{bmatrix} \boldsymbol{\psi}_{F,c} \\ \mathbf{F}_c \boldsymbol{\psi}_{F,c} \end{bmatrix} = \boldsymbol{\Gamma}_{F,c} \begin{bmatrix} \boldsymbol{\psi}_{F,c} \\ \mathbf{F}_c \boldsymbol{\psi}_{F,c} \end{bmatrix} = \\ = \mathbf{0} \end{aligned} \tag{4.9}$$

Where $\boldsymbol{\Gamma}_{F,c}$ is a $\mathbb{R}^{n\times(n+r)}$ matrix defined as:

$$\boldsymbol{\Gamma}_{F,c} = \left[\mathbf{A}_c - \lambda_{F,c}\mathbf{I} \quad \mathbf{B}_c\right] \tag{4.10}$$

Therefore, the matrix $\boldsymbol{\Gamma}_{F,c}$ can be actually computed once that the eigenvalue $\lambda_{F,c}$ has been assigned for the system represented by the state matrix $\mathbf{A}_c$ and the state influence matrix $\mathbf{B}_c$. This matrix can be factorized by using the Singular Value Decomposition method (SVD) $[15]$ to yield:

$$\boldsymbol{\Gamma}_{F,c} = \mathbf{U}_{F,c} \boldsymbol{\Sigma}_{F,c} \mathbf{V}^*_{F,c} \tag{4.11}$$

Where $\mathbf{\Sigma}_{F,c}$ is a $\mathbb{R}^{n\times(n+r)}$ diagonal matrix containing the complex conjugate singular values of matrix $\mathbf{\Gamma}_{F,c}$ whereas $\mathbf{U}_{F,c}$ and $\mathbf{V}_{F,c}$ are respectively $\mathbb{R}^{n\times n}$ and $\mathbb{R}^{(r+n)\times(r+n)}$ orthonormal matrices containing the left singular vectors and the right singular vectors of matrix $\mathbf{\Gamma}_{F,c}$. These matrices can be respectively partitioned as follows:

$$\mathbf{\Sigma}_{F,c} = \begin{bmatrix} \mathbf{S}_{F,c} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix} \tag{4.12}$$

$$\mathbf{U}_{F,c} = \begin{bmatrix} \mathbf{U}_{F,c}^{S} & \mathbf{U}_{F,c}^{O} \end{bmatrix} \tag{4.13}$$

$$\mathbf{V}_{F,c} = \begin{bmatrix} \mathbf{V}_{F,c}^{S} & \mathbf{V}_{F,c}^{O} \end{bmatrix} \tag{4.14}$$

Where $\mathbf{S}_{F,c}$, $\mathbf{U}_{F,c}^{S}$, $\mathbf{U}_{F,c}^{O}$, $\mathbf{V}_{F,c}^{S}$ and $\mathbf{V}_{F,c}^{O}$ are respectively $\mathbb{R}^{q_{F,c}\times q_{F,c}}$, $\mathbb{R}^{n\times q_{F,c}}$, $\mathbb{R}^{n\times\left((n+r)-q_{F,c}\right)}$, $\mathbb{R}^{(n+r)\times q_{F,c}}$ and $\mathbb{R}^{(n+r)\times\left((n+r)-q_{F,c}\right)}$ matrices. The matrix $\mathbf{S}_{F,c}$ is a diagonal matrix containing the significant singular values of the matrix $\mathbf{\Gamma}_{F,c}$. Indeed:

$$\mathbf{S}_{F,c} = diag(\sigma_{F,c}^{1}, \sigma_{F,c}^{2}, \ldots, \sigma_{F,c}^{q_{F,c}}) \tag{4.15}$$

Consequently, multiplying the matrix $\mathbf{\Gamma}_{F,c}$ times $\mathbf{V}_{F,c}$ yields:

$$\begin{gathered} \mathbf{\Gamma}_{F,c}\mathbf{V}_{F,c} = \mathbf{U}_{F,c}\mathbf{\Sigma}_{F,c}\mathbf{V}_{F,c}^{*}\mathbf{V}_{F,c} = \\ = \mathbf{U}_{F,c}\mathbf{\Sigma}_{F,c} \end{gathered} \tag{4.16}$$

This equation can be explicitly restated as:

$$\begin{aligned}\mathbf{\Gamma}_{F,c}\begin{bmatrix}\mathbf{V}_{F,c}^{S} & \mathbf{V}_{F,c}^{O}\end{bmatrix} &= \mathbf{U}_{F,c}\begin{bmatrix}\mathbf{S}_{F,c} & \mathbf{O}\\ \mathbf{O} & \mathbf{O}\end{bmatrix} = \\ &= \begin{bmatrix}\mathbf{U}_{F,c}\mathbf{S}_{F,c} & \mathbf{O}\end{bmatrix}\end{aligned} \tag{4.17}$$

The second matrix equality yields:

$$\mathbf{\Gamma}_{F,c}\mathbf{V}_{F,c}^{O} = \mathbf{O} \tag{4.18}$$

Therefore the matrix $\mathbf{V}_{F,c}^{O}$ represents a set of orthogonal basis vectors spanning the null space of the matrix $\mathbf{\Gamma}_{F,c}$ so that:

$$\begin{aligned}\mathbf{\Gamma}_{F,c}\mathbf{V}_{F,c}^{O}\mathbf{c}_{F,c} &= \mathbf{\Gamma}_{F,c}\boldsymbol{\varphi}_{F,c} = \\ &= \mathbf{0}\end{aligned} \tag{4.19}$$

Where $\mathbf{c}_{F,c}$ is an $\mathbb{R}^{(n+r)-q_{F,c}}$ arbitrary nonzero vector and $\boldsymbol{\varphi}_{F,c}$ is a $\mathbb{R}^{n+r}$ vector defined as:

$$\boldsymbol{\varphi}_{F,c} = \mathbf{V}_{F,c}^{O}\mathbf{c}_{F,c} \tag{4.20}$$

This vector can be partitioned as follows:

$$\boldsymbol{\varphi}_{F,c} = \begin{bmatrix}\overline{\boldsymbol{\varphi}}_{F,c}\\ \overline{\overline{\boldsymbol{\varphi}}}_{F,c}\end{bmatrix} \tag{4.21}$$

Where $\overline{\boldsymbol{\varphi}}_{F,c}$ and $\overline{\overline{\boldsymbol{\varphi}}}_{F,c}$ are respectively $\mathbb{R}^{n}$ and $\mathbb{R}^{r}$ vectors. Observing the matrix reformulation of the eigenvalue problem of matrix $\mathbf{A}_{F,c}$, the following matrix equations can be deduced:

$$\boldsymbol{\psi}_{F,c} = \overline{\boldsymbol{\varphi}}_{F,c} \tag{4.22}$$

$$\mathbf{F}_c \boldsymbol{\psi}_{F,c} = \bar{\bar{\boldsymbol{\varphi}}}_{F,c} \tag{4.23}$$

Note that from the first matrix equation is straightforward to deduce that the vector $\bar{\boldsymbol{\varphi}}_{F,c}$ coincides with the eigenvector of the closed-loop state matrix $\mathbf{A}_{F,c}$ corresponding to the assigned eigenvalue $\lambda_{F,c}$. Consequently:

$$\mathbf{F}_c \bar{\boldsymbol{\varphi}}_{F,c} = \bar{\bar{\boldsymbol{\varphi}}}_{F,c} \tag{4.24}$$

This procedure can be repeated for each prescribed eigenvalue $\lambda_{F,c}^h$ to yield the following generic matrix equations:

$$\mathbf{F}_c \bar{\boldsymbol{\varphi}}_{F,c}^h = \bar{\bar{\boldsymbol{\varphi}}}_{F,c}^h \quad , \quad h = 1, 2, \ldots, n \tag{4.25}$$

Where $\bar{\boldsymbol{\varphi}}_{F,c}^h$ and $\bar{\bar{\boldsymbol{\varphi}}}_{F,c}^h$ are respectively $\mathbb{R}^n$ and $\mathbb{R}^r$ generic vectors corresponding to the assigned eigenvalue $\lambda_{F,c}^h$. These equations can be restated in a compact matrix form as follows:

$$\mathbf{F}_c \bar{\boldsymbol{\Phi}}_{F,c} = \bar{\bar{\boldsymbol{\Phi}}}_{F,c} \tag{4.26}$$

Where $\bar{\boldsymbol{\Phi}}_{F,c}$ and $\bar{\bar{\boldsymbol{\Phi}}}_{F,c}$ are respectively $\mathbb{R}^{n\times n}$ and $\mathbb{R}^{r\times n}$ matrices defined as:

$$\begin{aligned} \bar{\boldsymbol{\Phi}}_{F,c} &= \begin{bmatrix} \bar{\boldsymbol{\varphi}}_{F,c}^1 & \bar{\boldsymbol{\varphi}}_{F,c}^2 & \cdots & \bar{\boldsymbol{\varphi}}_{F,c}^{n-1} & \bar{\boldsymbol{\varphi}}_{F,c}^n \end{bmatrix} = \\ &= \begin{bmatrix} \bar{\boldsymbol{\varphi}}_{F,c}^1 & \bar{\boldsymbol{\varphi}}_{F,c}^{1*} & \cdots & \bar{\boldsymbol{\varphi}}_{F,c}^{n-1} & \bar{\boldsymbol{\varphi}}_{F,c}^{n-1*} \end{bmatrix} \end{aligned} \tag{4.27}$$

$$\begin{aligned} \bar{\bar{\boldsymbol{\Phi}}}_{F,c} &= \begin{bmatrix} \bar{\bar{\boldsymbol{\varphi}}}_{F,c}^1 & \bar{\bar{\boldsymbol{\varphi}}}_{F,c}^2 & \cdots & \bar{\bar{\boldsymbol{\varphi}}}_{F,c}^{n-1} & \bar{\bar{\boldsymbol{\varphi}}}_{F,c}^n \end{bmatrix} = \\ &= \begin{bmatrix} \bar{\bar{\boldsymbol{\varphi}}}_{F,c}^1 & \bar{\bar{\boldsymbol{\varphi}}}_{F,c}^{1*} & \cdots & \bar{\bar{\boldsymbol{\varphi}}}_{F,c}^{n-1} & \bar{\bar{\boldsymbol{\varphi}}}_{F,c}^{n-1*} \end{bmatrix} \end{aligned} \tag{4.28}$$

Note that complex conjugate eigenvalue pairs $\lambda_{F,c}^{h-1}$ and $\lambda_{F,c}^{h}$ corresponds to complex conjugate vector pairs $\bar{\boldsymbol{\varphi}}_{F,c}^{h-1}$, $\bar{\boldsymbol{\varphi}}_{F,c}^{h}$ and $\bar{\bar{\boldsymbol{\varphi}}}_{F,c}^{h-1}$, $\bar{\bar{\boldsymbol{\varphi}}}_{F,c}^{h}$. Finally, the feedback matrix $\mathbf{F}_c$ can be computed as:

$$\mathbf{F}_c = \bar{\bar{\boldsymbol{\Phi}}}_{F,c} \bar{\boldsymbol{\Phi}}_{F,c}^{-1} \tag{4.29}$$

This method to compute the controller gain matrix is sometimes referred as null-space technique for poles placement [3]. This method can be extended to system discrete-time state-space representation in a straightforward manner replacing the continuous-time state matrix $\mathbf{A}_c$ and the continuous-time state influence matrix $\mathbf{B}_c$ respectively with the discrete-time state matrix $\mathbf{A}$ and the discrete-time state influence matrix $\mathbf{B}$ to yield a discrete-time feedback matrix $\mathbf{F}$ instead of the continuous-time controller gain $\mathbf{F}_c$. Another important method to solve the regulation problem for both continuous-time and discrete-time linear state-space systems comes from the optimal control theory and is the Linear Quadratic Regulator algorithm (LQR) [9], [10]. Indeed, consider a continuous-time state-space system:

$$\begin{cases} \dot{\mathbf{z}}(t) = \mathbf{A}_c \mathbf{z}(t) + \mathbf{B}_c \mathbf{u}(t) \\ \mathbf{z}(0) = \mathbf{z}_0 \end{cases} \tag{4.30}$$

Where $\mathbf{z}_0$ is a $\mathbb{R}^n$ vector corresponding to the initial conditions. This method is able to compute the feedback matrix in such a way to minimize a quadratic cost index. The cost index is a performance index which accounts for the actuator power available and at the same time with the deviation of the state from the reference configuration. In the continuous-time case and without considering constraints on the terminal state, for a finite-horizon of time $0 \le t \le T$ the quadratic cost index can be defined as:

$$J_c = \frac{1}{2}\mathbf{z}^T(T)\mathbf{Q}_{c,T}\mathbf{z}(T) + \frac{1}{2}\int_0^T \mathbf{z}^T(t)\mathbf{Q}_{c,z}\mathbf{z}(t) + \mathbf{u}^T(t)\mathbf{Q}_{c,u}\mathbf{u}(t)dt \quad (4.31)$$

Where $\mathbf{Q}_{c,T}$ and $\mathbf{Q}_{c,z}$ are $\mathbb{R}^{n\times n}$ matrices which represent the terminal cost matrix and the weight of the state vector whereas $\mathbf{Q}_{c,u}$ is a $\mathbb{R}^{r\times r}$ matrix representing the weight of the input vector. Note that the matrix $\mathbf{Q}_{c,T}$ is a positive semidefinite matrix which penalizes the deviation of the final state from the desired set point whereas the matrices $\mathbf{Q}_{c,z}$ and $\mathbf{Q}_{c,u}$ are respectively a positive semidefinite matrix and a positive definite matrix which penalize respectively the instantaneous deviation of the state form the reference configuration and the instantaneous control effort. Note that since an initial values problem is considered, the state vector at the final time T is unknown and therefore the terminal cost in the performance index J_c is expressed in terms of unknown quantities. On the other hand, the terminal cost can be expressed in terms of the initial conditions as:

$$\begin{aligned} \frac{1}{2}\mathbf{z}^T(T)\mathbf{Q}_{c,T}\mathbf{z}(T) &= \frac{1}{2}\mathbf{z}^T(T)\mathbf{Q}_{c,T}\mathbf{z}(T) - \frac{1}{2}\mathbf{z}^T(0)\mathbf{Q}_{c,T}\mathbf{z}(0) + \frac{1}{2}\mathbf{z}^T(0)\mathbf{Q}_{c,T}\mathbf{z}(0) = \\ &= \int_0^T \frac{d}{dt}\left(\frac{1}{2}\mathbf{z}^T(t)\mathbf{Q}_{c,T}\mathbf{z}(t)\right)dt + \frac{1}{2}\mathbf{z}_0^T\mathbf{Q}_{c,T}\mathbf{z}_0 = \\ &= \int_0^T \mathbf{z}^T(t)\mathbf{Q}_{c,T}\dot{\mathbf{z}}(t)dt + \frac{1}{2}\mathbf{z}_0^T\mathbf{Q}_{c,T}\mathbf{z}_0 \end{aligned} \quad (4.32)$$

Consequently, the performance index can be reformulated as:

$$J_c = \frac{1}{2}\mathbf{z}^T(T)\mathbf{Q}_{c,T}\mathbf{z}(T) + \frac{1}{2}\int_0^T \mathbf{z}^T(t)\mathbf{Q}_{c,z}\mathbf{z}(t) + \mathbf{u}^T(t)\mathbf{Q}_{c,u}\mathbf{u}(t)dt =$$
$$= \int_0^T \mathbf{z}^T(t)\mathbf{Q}_{c,T}\dot{\mathbf{z}}(t)dt + \frac{1}{2}\mathbf{z}_0^T\mathbf{Q}_{c,T}\mathbf{z}_0 + \frac{1}{2}\int_0^T \mathbf{z}^T(t)\mathbf{Q}_{c,z}\mathbf{z}(t) + \mathbf{u}^T(t)\mathbf{Q}_{c,u}\mathbf{u}(t)dt =$$
$$= \frac{1}{2}\mathbf{z}_0^T\mathbf{Q}_{c,T}\mathbf{z}_0 + \int_0^T \mathbf{z}^T(t)\mathbf{Q}_{c,T}\dot{\mathbf{z}}(t) + \frac{1}{2}\mathbf{z}^T(t)\mathbf{Q}_{c,z}\mathbf{z}(t) + \frac{1}{2}\mathbf{u}^T(t)\mathbf{Q}_{c,u}\mathbf{u}(t)dt$$
(4.33)

In order to find the control input $\mathbf{u}(t)$ as a linear function of the state $\mathbf{z}(t)$, the cost index J_c must be minimized but simultaneously the system state equation must be satisfied. Therefore, the state equation represents a constraint equation for the minimization problem. One way to solve this problem is the method of Lagrange multipliers which consists in adjoining the state equation to the performance index and subsequently minimize this adjoint cost index J_c^* using variational calculus technique $[9]$, $[10]$. Indeed:

$$J_c^* = J_c + \int_0^T \boldsymbol{\lambda}^T(t)\left(\mathbf{A}_c\mathbf{z}(t) + \mathbf{B}_c\mathbf{u}(t) - \dot{\mathbf{z}}(t)\right)dt =$$
$$= \frac{1}{2}\mathbf{z}_0^T\mathbf{Q}_{c,T}\mathbf{z}_0 + \int_0^T \mathbf{z}^T(t)\mathbf{Q}_{c,T}\dot{\mathbf{z}}(t) + \frac{1}{2}\mathbf{z}^T(t)\mathbf{Q}_{c,z}\mathbf{z}(t) + \frac{1}{2}\mathbf{u}^T(t)\mathbf{Q}_{c,u}\mathbf{u}(t)dt +$$
$$+ \int_0^T \boldsymbol{\lambda}^T(t)\left(\mathbf{A}_c\mathbf{z}(t) + \mathbf{B}_c\mathbf{u}(t) - \dot{\mathbf{z}}(t)\right)dt$$
(4.34)

Where $\boldsymbol{\lambda}(t)$ is a $\mathbb{R}^n$ vector containing the Lagrange multipliers. Since the optimal control input minimizes the adjoint performance index J_c^*, it is necessary to compute the first variation of this functional and set it equal to zero. Indeed, taking the first variation of the augmented cost function J_c^* yields:

$$\delta J_c^* = \delta\left(\frac{1}{2}\mathbf{z}_0^T\mathbf{Q}_{c,T}\mathbf{z}_0\right) + \delta\int_0^T \mathbf{z}^T(t)\mathbf{Q}_{c,T}\dot{\mathbf{z}}(t) + \frac{1}{2}\mathbf{z}^T(t)\mathbf{Q}_{c,z}\mathbf{z}(t) + \frac{1}{2}\mathbf{u}^T(t)\mathbf{Q}_{c,u}\mathbf{u}(t)dt +$$
$$+\delta\int_0^T \boldsymbol{\lambda}^T(t)\left(\mathbf{A}_c\mathbf{z}(t) + \mathbf{B}_c\mathbf{u}(t) - \dot{\mathbf{z}}(t)\right)dt =$$
$$= \int_0^T \delta\left(\mathbf{z}^T(t)\mathbf{Q}_{c,T}\dot{\mathbf{z}}(t) + \frac{1}{2}\mathbf{z}^T(t)\mathbf{Q}_{c,z}\mathbf{z}(t) + \frac{1}{2}\mathbf{u}^T(t)\mathbf{Q}_{c,u}\mathbf{u}(t)\right)dt +$$
$$+\int_0^T \delta\left(\boldsymbol{\lambda}^T(t)\left(\mathbf{A}_c\mathbf{z}(t) + \mathbf{B}_c\mathbf{u}(t) - \dot{\mathbf{z}}(t)\right)\right)dt =$$
$$= \int_0^T \delta\mathbf{z}^T(t)\mathbf{Q}_{c,T}\dot{\mathbf{z}}(t) + \mathbf{z}^T(t)\mathbf{Q}_{c,T}\delta\dot{\mathbf{z}}(t) + \mathbf{z}^T(t)\mathbf{Q}_{c,z}\delta\mathbf{z}(t) + \mathbf{u}^T(t)\mathbf{Q}_{c,u}\delta\mathbf{u}(t)dt +$$
$$+\int_0^T +\delta\boldsymbol{\lambda}^T(t)\left(\mathbf{A}_c\mathbf{z}(t) + \mathbf{B}_c\mathbf{u}(t) - \dot{\mathbf{z}}(t)\right) + \boldsymbol{\lambda}^T(t)\left(\mathbf{A}_c\delta\mathbf{z}(t) + \mathbf{B}_c\delta\mathbf{u}(t) - \delta\dot{\mathbf{z}}(t)\right)dt =$$
$$= \int_0^T \dot{\mathbf{z}}^T(t)\mathbf{Q}_{c,T}\delta\mathbf{z}(t) + \mathbf{z}^T(t)\mathbf{Q}_{c,T}\delta\dot{\mathbf{z}}(t) + \mathbf{z}^T(t)\mathbf{Q}_{c,z}\delta\mathbf{z}(t) + \mathbf{u}^T(t)\mathbf{Q}_{c,u}\delta\mathbf{u}(t)dt +$$
$$+\int_0^T \left(\mathbf{A}_c\mathbf{z}(t) + \mathbf{B}_c\mathbf{u}(t) - \dot{\mathbf{z}}(t)\right)^T\delta\boldsymbol{\lambda}(t) + \boldsymbol{\lambda}^T(t)\mathbf{A}_c\delta\mathbf{z}(t) + \boldsymbol{\lambda}^T(t)\mathbf{B}_c\delta\mathbf{u}(t) - \boldsymbol{\lambda}^T(t)\delta\dot{\mathbf{z}}(t)dt =$$
$$= \int_0^T \left(\mathbf{z}^T(t)\mathbf{Q}_{c,T} - \boldsymbol{\lambda}^T(t)\right)\delta\dot{\mathbf{z}}(t) + \left(\dot{\mathbf{z}}^T(t)\mathbf{Q}_{c,T} + \mathbf{z}^T(t)\mathbf{Q}_{c,z} + \boldsymbol{\lambda}^T(t)\mathbf{A}_c\right)\delta\mathbf{z}(t)dt +$$
$$+\int_0^T \left(\mathbf{u}^T(t)\mathbf{Q}_{c,u} + \boldsymbol{\lambda}^T(t)\mathbf{B}_c\right)\delta\mathbf{u}(t) + \left(\mathbf{A}_c\mathbf{z}(t) + \mathbf{B}_c\mathbf{u}(t) - \dot{\mathbf{z}}(t)\right)^T\delta\boldsymbol{\lambda}(t)dt =$$
$$= \int_0^T \left(\mathbf{Q}_{c,T}\mathbf{z}(t) - \boldsymbol{\lambda}(t)\right)^T\delta\dot{\mathbf{z}}(t) + \left(\mathbf{Q}_{c,T}\dot{\mathbf{z}}(t) + \mathbf{Q}_{c,z}\mathbf{z}(t) + \mathbf{A}_c^T\boldsymbol{\lambda}(t)\right)^T\delta\mathbf{z}(t)dt +$$
$$+\int_0^T \left(\mathbf{Q}_{c,u}\mathbf{u}(t) + \mathbf{B}_c^T\boldsymbol{\lambda}(t)\right)^T\delta\mathbf{u}(t) + \left(\mathbf{A}_c\mathbf{z}(t) + \mathbf{B}_c\mathbf{u}(t) - \dot{\mathbf{z}}(t)\right)^T\delta\boldsymbol{\lambda}(t)dt =$$
$$= \left[\left(\mathbf{Q}_{c,T}\mathbf{z}(t) - \boldsymbol{\lambda}(t)\right)^T\delta\mathbf{z}(t)\right]_0^T +$$
$$+\int_0^T \left(-\mathbf{Q}_{c,T}\dot{\mathbf{z}}(t) + \dot{\boldsymbol{\lambda}}(t) + \mathbf{Q}_{c,T}\dot{\mathbf{z}}(t) + \mathbf{Q}_{c,z}\mathbf{z}(t) + \mathbf{A}_c^T\boldsymbol{\lambda}(t)\right)^T\delta\mathbf{z}(t)dt +$$
$$+\int_0^T \left(\mathbf{Q}_{c,u}\mathbf{u}(t) + \mathbf{B}_c^T\boldsymbol{\lambda}(t)\right)^T\delta\mathbf{u}(t) + \left(\mathbf{A}_c\mathbf{z}(t) + \mathbf{B}_c\mathbf{u}(t) - \dot{\mathbf{z}}(t)\right)^T\delta\boldsymbol{\lambda}(t)dt =$$
$$= \left[\left(\mathbf{Q}_{c,T}\mathbf{z}(t) - \boldsymbol{\lambda}(t)\right)^T\delta\mathbf{z}(t)\right]_0^T + \int_0^T \left(\dot{\boldsymbol{\lambda}}(t) + \mathbf{Q}_{c,z}\mathbf{z}(t) + \mathbf{A}_c^T\boldsymbol{\lambda}(t)\right)^T\delta\mathbf{z}(t)dt +$$
$$+\int_0^T \left(\mathbf{Q}_{c,u}\mathbf{u}(t) + \mathbf{B}_c^T\boldsymbol{\lambda}(t)\right)^T\delta\mathbf{u}(t) + \left(\mathbf{A}_c\mathbf{z}(t) + \mathbf{B}_c\mathbf{u}(t) - \dot{\mathbf{z}}(t)\right)^T\delta\boldsymbol{\lambda}(t)dt =$$
$$= 0 \quad , \quad \forall\delta\mathbf{z}(t) \quad , \quad \forall\delta\boldsymbol{\lambda}(t) \quad , \quad \forall\delta\mathbf{u}(t)$$

(4.35)

Observing that the variation of the state vector $\delta\mathbf{z}(t)$, the variation of the co-state vector $\delta\boldsymbol{\lambda}(t)$ and the variation of the input vector $\delta\mathbf{u}(t)$ are all independent, each quantity in the time integral can be independently taken equal to zero:

$$\begin{cases} \delta\mathbf{z}(0)=\mathbf{0} \\ \mathbf{Q}_{c,T}\mathbf{z}(T)-\boldsymbol{\lambda}(T)=\mathbf{0} \\ \dot{\boldsymbol{\lambda}}(t)+\mathbf{Q}_{c,z}\mathbf{z}(t)+\mathbf{A}_c^T\boldsymbol{\lambda}(t)=\mathbf{0} \\ \mathbf{Q}_{c,u}\mathbf{u}(t)+\mathbf{B}_c^T\boldsymbol{\lambda}(t)=\mathbf{0} \\ \mathbf{A}_c\mathbf{z}(t)+\mathbf{B}_c\mathbf{u}(t)-\dot{\mathbf{z}}(t)=\mathbf{0} \end{cases} \tag{4.36}$$

Note that the variation of the initial state is set equal to zero because it is assumed to be known whereas a condition which links the state vector to the adjoint vector at the final state T arises from the minimization procedure. Therefore, the minimization of the adjoint cost index J_c^* yields a set of two differential equations and one algebraic equation:

$$\begin{cases} \dot{\mathbf{z}}(t)=\mathbf{A}_c\mathbf{z}(t)+\mathbf{B}_c\mathbf{u}(t) \\ \mathbf{z}(0)=\mathbf{z}_0 \end{cases} \tag{4.37}$$

$$\begin{cases} \dot{\boldsymbol{\lambda}}(t)=-\mathbf{Q}_{c,z}\mathbf{z}(t)-\mathbf{A}_c^T\boldsymbol{\lambda}(t) \\ \boldsymbol{\lambda}(T)=\mathbf{Q}_{c,T}\mathbf{z}(T) \end{cases} \tag{4.38}$$

$$\mathbf{u}(t)=-\mathbf{Q}_{c,u}^{-1}\mathbf{B}_c^T\boldsymbol{\lambda}(t) \tag{4.39}$$

Where the first differential equation is the state equation with its the initial conditions, the second differential equation is the adjoint equation with its boundary conditions and the last algebraic equation is the stationarity equation. There is a method originated from the optimal control theory to obtain these sets of equations directly defining a so-called Hamiltonian function which depends on the state vector $\mathbf{z}(t)$, the co-state vector $\boldsymbol{\lambda}(t)$ and the control vector $\mathbf{u}(t)$ [9], [10]. Indeed:

$$\begin{aligned} H_c(\mathbf{z}(t),\mathbf{u}(t),\boldsymbol{\lambda}(t)) = {} & \frac{1}{2}\mathbf{z}^T(t)\mathbf{Q}_{c,z}\mathbf{z}(t) + \frac{1}{2}\mathbf{u}^T(t)\mathbf{Q}_{c,u}\mathbf{u}(t) + \\ & + \boldsymbol{\lambda}^T(t)\left(\mathbf{A}_c\mathbf{z}(t) + \mathbf{B}_c\mathbf{u}(t)\right) \end{aligned} \tag{4.40}$$

The state equation, the co-state equation and the stationarity equation can be obtained from the Hamiltonian function as follows:

$$\begin{cases} \dot{\mathbf{z}}(t) = \left(\dfrac{\partial H_c(\mathbf{z}(t),\mathbf{u}(t),\boldsymbol{\lambda}(t))}{\partial \boldsymbol{\lambda}(t)}\right)^T \\ \mathbf{z}(0) = \mathbf{z}_0 \end{cases} \tag{4.41}$$

$$\begin{cases} \dot{\boldsymbol{\lambda}}(t) = -\left(\dfrac{\partial H_c(\mathbf{z}(t),\mathbf{u}(t),\boldsymbol{\lambda}(t))}{\partial \mathbf{z}(t)}\right)^T \\ \boldsymbol{\lambda}(T) = \mathbf{Q}_{c,T}\mathbf{z}(T) \end{cases} \tag{4.42}$$

$$\left(\frac{\partial H_c(\mathbf{z}(t),\mathbf{u}(t),\boldsymbol{\lambda}(t))}{\partial \mathbf{u}(t)}\right)^T = \mathbf{0} \tag{4.43}$$

It is noteworthy to realize that the adjoint equation is a linear differential equation coupled with the state equation which has a boundary condition at the final instant of time T. On the other hand, the stationarity equation relates the optimal control vector with the vector of Lagrange multipliers which derives from the adjoint equation. Moreover, the state equation depends on the optimal input vector. Consequently, the whole problem is coupled and it is sometimes referred as two-point boundary value problem [9], [10]. This problem can be numerically solved by using iterative minimization techniques combined with methods to integrate ordinary differential equations. This method derives from Computational Fluid Dynamics and it is referred to as adjoint method [16], [17], [18], [19], [20]. In practice, the serious drawback of this solution procedure is that the optimal input vector is computed as an explicit function of time instead of a linear function of the state vector making the solution found extremely sensitive to some external disturbances and unfeasible for real-time application. Nevertheless, this algorithm represents and useful method to perform motion planning [21]. On the other hand, the classical method to solve this problem consist in reducing it to the solution of a continuous-time differential Riccati equation which allows to express the same optimal control as a linear function of the state [9], [10]. Indeed, observing that at the final time T the adjoint vector is a linear function of the state vector, assume that this relation holds for each instant of time:

$$\begin{cases} \boldsymbol{\lambda}(t) = \mathbf{S}(t)\mathbf{z}(t) \\ \mathbf{S}(T) = \mathbf{Q}_{c,T} \end{cases} \tag{4.44}$$

Where $\mathbf{S}(t)$ is a $\mathbb{R}^{n\times n}$ symmetric matrix to be computed. Note that this solution method is referred as Sweep Method [13]. Therefore, the input vector can be computed as follows:

$$\begin{aligned} \mathbf{u}(t) &= -\mathbf{Q}_{c,u}^{-1}\mathbf{B}_c^T\boldsymbol{\lambda}(t) = \\ &= -\mathbf{Q}_{c,u}^{-1}\mathbf{B}_c^T\mathbf{S}(t)\mathbf{z}(t) = \\ &= \mathbf{F}_c(t)\mathbf{z}(t) \end{aligned} \tag{4.45}$$

Where the continuous-time feedback matrix $\mathbf{F}_c(t)$ is a $\mathbb{R}^{r\times n}$ matrix function of time defined as:

$$\mathbf{F}_c(t) = -\mathbf{Q}_{c,u}^{-1}\mathbf{B}_c^T\mathbf{S}(t) \tag{4.46}$$

Taking the time derivative of the adjoint vector yields:

$$\begin{aligned}\dot{\boldsymbol{\lambda}}(t) &= \dot{\mathbf{S}}(t)\mathbf{z}(t) + \mathbf{S}(t)\dot{\mathbf{z}}(t) = \\ &= \dot{\mathbf{S}}(t)\mathbf{z}(t) + \mathbf{S}(t)\mathbf{A}_c\mathbf{z}(t) + \mathbf{S}(t)\mathbf{B}_c\mathbf{u}(t) = \\ &= \dot{\mathbf{S}}(t)\mathbf{z}(t) + \mathbf{S}(t)\mathbf{A}_c\mathbf{z}(t) - \mathbf{S}(t)\mathbf{B}_c\mathbf{Q}_{c,u}^{-1}\mathbf{B}_c^T\mathbf{S}(t)\mathbf{z}(t) = \\ &= \left(\dot{\mathbf{S}}(t) + \mathbf{S}(t)\mathbf{A}_c - \mathbf{S}(t)\mathbf{B}_c\mathbf{Q}_{c,u}^{-1}\mathbf{B}_c^T\mathbf{S}(t)\right)\mathbf{z}(t)\end{aligned} \tag{4.47}$$

Where the state equation has been used. On the other hand, from the adjoint equation the time derivative of the adjoint vector can be computed as:

$$\begin{aligned}\dot{\boldsymbol{\lambda}}(t) &= -\mathbf{A}_c^T\boldsymbol{\lambda}(t) - \mathbf{Q}_{c,z}\mathbf{z}(t) = \\ &= -\mathbf{A}_c^T\mathbf{S}(t)\mathbf{z}(t) - \mathbf{Q}_{c,z}\mathbf{z}(t) = \\ &= -\left(\mathbf{A}_c^T\mathbf{S}(t) + \mathbf{Q}_{c,z}\right)\mathbf{z}(t)\end{aligned} \tag{4.48}$$

Equating the two previous equations yields:

$$\left(\dot{\mathbf{S}}(t) + \mathbf{S}(t)\mathbf{A}_c - \mathbf{S}(t)\mathbf{B}_c\mathbf{Q}_{c,u}^{-1}\mathbf{B}_c^T\mathbf{S}(t) + \mathbf{A}_c^T\mathbf{S}(t) + \mathbf{Q}_{c,z}\right)\mathbf{z}(t) = \mathbf{0} \tag{4.49}$$

Setting the terms between the brackets equal to zero gives:

$$\begin{cases}\dot{\mathbf{S}}(t) + \mathbf{S}(t)\mathbf{A}_c + \mathbf{A}_c^T\mathbf{S}(t) - \mathbf{S}(t)\mathbf{B}_c\mathbf{Q}_{c,u}^{-1}\mathbf{B}_c^T\mathbf{S}(t) + \mathbf{Q}_{c,z} = \mathbf{O} \\ \mathbf{S}(T) = \mathbf{Q}_{c,T}\end{cases} \tag{4.50}$$

This is a first-order matrix differential equation named continuous-time differential Riccati equation. The solution of this differential equation can be found numerically with the standard methods and it provides the evolution in time of the symmetric matrix $\mathbf{S}(t)$ necessary to compute the optimal control input. It can be proved [9], [10] that this equation reaches quickly an asymptotic solution $\mathbf{S}_\infty$ which can be used to compute a steady-state continuous-time feedback matrix $\mathbf{F}_{c,\infty}$ as:

$$\mathbf{F}_{c,\infty} = -\mathbf{Q}_{c,u}^{-1}\mathbf{B}_c^T\mathbf{S}_\infty \tag{4.51}$$

In practice the steady-state feedback matrix is preferred especially for real-time applications. This is equivalent to minimize an infinite-horizon continuous-time quadratic cost index defined as:

$$J_{c,\infty} = \frac{1}{2}\int_0^\infty \mathbf{z}^T(t)\mathbf{Q}_{c,z}\mathbf{z}(t) + \mathbf{u}^T(t)\mathbf{Q}_{c,u}\mathbf{u}(t)dt \tag{4.52}$$

Consequently, the control input can be computed as a linear combination of the state as follows:

$$\mathbf{u}(t) = \mathbf{F}_{c,\infty}\mathbf{z}(t) \tag{4.53}$$

Consider now a discrete-time state-state space system:

$$\begin{cases} \mathbf{z}(k+1) = \mathbf{A}\mathbf{z}(k) + \mathbf{B}\mathbf{u}(k) \\ \mathbf{z}(0) = \mathbf{z}_0 \end{cases} \tag{4.54}$$

Where $\mathbf{z}_0$ is a $\mathbb{R}^n$ vector representing the initial conditions. The Linear Quadratic Regulator method (LQR) can be applied even in this case with some slight modifications $[9]$, $[10]$. Indeed, assuming no constraints on the terminal state, consider a discrete-time quadratic cost index J defined as:

$$J = \frac{1}{2}\mathbf{z}^T(N)\mathbf{Q}_T\mathbf{z}(N) + \frac{1}{2}\sum_{k=0}^{N-1}\left(\mathbf{z}^T(k)\mathbf{Q}_z\mathbf{z}(k) + \mathbf{u}^T(k)\mathbf{Q}_u\mathbf{u}(k)\right) \tag{4.55}$$

Where $\mathbf{Q}_T$ and $\mathbf{Q}_z$ are $\mathbb{R}^{n \times n}$ matrices which represent the terminal cost matrix and the weight of the state vector whereas $\mathbf{Q}_u$ is a $\mathbb{R}^{r \times r}$ matrix representing the weight of the input vector. Note that the matrix $\mathbf{Q}_T$ is a positive semidefinite matrix which penalizes the deviation of the final state from the desired set point whereas the matrices $\mathbf{Q}_z$ and $\mathbf{Q}_u$ are respectively a positive semidefinite matrix and a positive definite matrix which penalize respectively the instantaneous deviation of the state form the reference configuration and the instantaneous control effort. Even in this case, in order to find the control input $\mathbf{u}(k)$ as a linear function of the state $\mathbf{z}(k)$, the cost index J must be minimized but simultaneously the system state equation must be satisfied. Therefore, the state equation represents a constraint equation for the minimization problem. One method to solve this problem is the Lagrange multipliers technique which consists in adjoining the state equation to the performance index and subsequently minimize this adjoint cost index J^* using variational calculus methodology $[9]$, $[10]$. Indeed:

$$
\begin{aligned}
J^* &= J + \sum_{k=0}^{N-1} \boldsymbol{\lambda}^T(k+1)\big(\mathbf{A}\mathbf{z}(k) + \mathbf{B}\mathbf{u}(k) - \mathbf{z}(k+1)\big) = \\
&= \frac{1}{2}\mathbf{z}^T(N)\mathbf{Q}_T\mathbf{z}(N) + \frac{1}{2}\sum_{k=0}^{N-1}\Big(\mathbf{z}^T(k)\mathbf{Q}_z\mathbf{z}(k) + \mathbf{u}^T(k)\mathbf{Q}_u\mathbf{u}(k)\Big) + \\
&\quad + \sum_{k=0}^{N-1} \boldsymbol{\lambda}^T(k+1)\big(\mathbf{A}\mathbf{z}(k) + \mathbf{B}\mathbf{u}(k) - \mathbf{z}(k+1)\big)
\end{aligned}
\tag{4.56}
$$

Where $\boldsymbol{\lambda}(k)$ is a $\mathbb{R}^n$ vector containing the Lagrange multipliers. Since the optimal control input minimizes the adjoint performance index J^*, it is necessary to compute the first variation of this functional and set it equal to zero. Indeed, taking the first variation of the augmented cost function J^* yields:

$$
\begin{aligned}
&\delta J^* = \delta\left(\frac{1}{2}\mathbf{z}^T(N)\mathbf{Q}_T\mathbf{z}(N)\right) + \delta\sum_{k=0}^{N-1}\left(\frac{1}{2}\mathbf{z}^T(k)\mathbf{Q}_z\mathbf{z}(k) + \frac{1}{2}\mathbf{u}^T(k)\mathbf{Q}_u\mathbf{u}(k)\right) + \\
&+\delta\sum_{k=0}^{N-1}\left(\boldsymbol{\lambda}^T(k+1)\left(\mathbf{Az}(k) + \mathbf{Bu}(k) - \mathbf{z}(k+1)\right)\right) = \\
&= \mathbf{z}^T(N)\mathbf{Q}_T\delta\mathbf{z}(N) + \\
&+\sum_{k=0}^{N-1}\left(\mathbf{z}^T(k)\mathbf{Q}_z\delta\mathbf{z}(k) + \mathbf{u}^T(k)\mathbf{Q}_u\delta\mathbf{u}(k) + \delta\boldsymbol{\lambda}^T(k+1)\left(\mathbf{Az}(k) + \mathbf{Bu}(k) - \mathbf{z}(k+1)\right)\right) + \\
&+\sum_{k=0}^{N-1}\left(\boldsymbol{\lambda}^T(k+1)\left(\mathbf{A}\delta\mathbf{z}(k) + \mathbf{B}\delta\mathbf{u}(k) - \delta\mathbf{z}(k+1)\right)\right) = \\
&= \mathbf{z}^T(N)\mathbf{Q}_T\delta\mathbf{z}(N) + \\
&+\sum_{k=0}^{N-1}\left(\mathbf{z}^T(k)\mathbf{Q}_z\delta\mathbf{z}(k) + \mathbf{u}^T(k)\mathbf{Q}_u\delta\mathbf{u}(k) + \left(\mathbf{Az}(k) + \mathbf{Bu}(k) - \mathbf{z}(k+1)\right)^T\delta\boldsymbol{\lambda}(k+1)\right) + \\
&+\sum_{k=0}^{N-1}\left(\boldsymbol{\lambda}^T(k+1)\mathbf{A}\delta\mathbf{z}(k) + \boldsymbol{\lambda}^T(k+1)\mathbf{B}\delta\mathbf{u}(k) - \boldsymbol{\lambda}^T(k+1)\delta\mathbf{z}(k+1)\right) = \\
&= \mathbf{z}^T(N)\mathbf{Q}_T\delta\mathbf{z}(N) + \sum_{k=0}^{N-1}\left(\left(\mathbf{z}^T(k)\mathbf{Q}_z + \boldsymbol{\lambda}^T(k+1)\mathbf{A}\right)\delta\mathbf{z}(k)\right) + \\
&+\sum_{k=0}^{N-1}\left(\left(\mathbf{u}^T(k)\mathbf{Q}_u + \boldsymbol{\lambda}^T(k+1)\mathbf{B}\right)\delta\mathbf{u}(k) + \left(\mathbf{Az}(k) + \mathbf{Bu}(k) - \mathbf{z}(k+1)\right)^T\delta\boldsymbol{\lambda}(k+1)\right) + \\
&-\boldsymbol{\lambda}^T(N)\delta\mathbf{z}(N) + \boldsymbol{\lambda}^T(0)\delta\mathbf{z}(0) - \sum_{k=0}^{N-1}\left(\boldsymbol{\lambda}^T(k)\delta\mathbf{z}(k)\right) = \\
&= \boldsymbol{\lambda}^T(0)\delta\mathbf{z}(0) + \left(\mathbf{Q}_T\mathbf{z}(N) - \boldsymbol{\lambda}(N)\right)^T\delta\mathbf{z}(N) + \\
&+\sum_{k=0}^{N-1}\left(\left(\mathbf{Q}_z\mathbf{z}(k) + \mathbf{A}^T\boldsymbol{\lambda}(k+1) - \boldsymbol{\lambda}(k)\right)^T\delta\mathbf{z}(k)\right) + \\
&+\sum_{k=0}^{N-1}\left(\left(\mathbf{Q}_u\mathbf{u}(k) + \mathbf{B}^T\boldsymbol{\lambda}(k+1)\right)^T\delta\mathbf{u}(k) + \left(\mathbf{Az}(k) + \mathbf{Bu}(k) - \mathbf{z}(k+1)\right)^T\delta\boldsymbol{\lambda}(k+1)\right) = \\
&= 0 \quad , \quad \forall\delta\mathbf{z}(k) \quad , \quad \forall\delta\boldsymbol{\lambda}(k) \quad , \quad \forall\delta\mathbf{u}(k)
\end{aligned}
\tag{4.57}
$$

Observing that the variation of the state vector $\delta\mathbf{z}(k)$, the variation of the co-state vector $\delta\boldsymbol{\lambda}(k)$ and the variation of the input vector $\delta\mathbf{u}(k)$ are all independent, each quantity in the time integral can be independently taken equal to zero yielding to the following equations:

$$\begin{cases} \delta\mathbf{z}(0) = \mathbf{0} \\ \mathbf{Q}_T\mathbf{z}(N) - \boldsymbol{\lambda}(N) = \mathbf{0} \\ \mathbf{Q}_z\mathbf{z}(k) + \mathbf{A}^T\boldsymbol{\lambda}(k+1) - \boldsymbol{\lambda}(k) = \mathbf{0} \\ \mathbf{Q}_u\mathbf{u}(k) + \mathbf{B}^T\boldsymbol{\lambda}(k+1) = \mathbf{0} \\ \mathbf{A}\mathbf{z}(k) + \mathbf{B}\mathbf{u}(k) - \mathbf{z}(k+1) = \mathbf{0} \end{cases} \tag{4.58}$$

Even in this dual case, the variation of the initial state is set equal to zero because it is assumed to be known whereas a condition which links the state vector to the adjoint vector at the final state N arises from the minimization procedure. Therefore, the minimization of the adjoint cost index J^* yields a set of two difference equations and one algebraic equation:

$$\begin{cases} \mathbf{z}(k+1) = \mathbf{A}\mathbf{z}(k) + \mathbf{B}\mathbf{u}(k) \\ \mathbf{z}(0) = \mathbf{z}_0 \end{cases} \tag{4.59}$$

$$\begin{cases} \boldsymbol{\lambda}(k) = \mathbf{Q}_z\mathbf{z}(k) + \mathbf{A}^T\boldsymbol{\lambda}(k+1) \\ \boldsymbol{\lambda}(N) = \mathbf{Q}_T\mathbf{z}(N) \end{cases} \tag{4.60}$$

$$\mathbf{u}(k) = -\mathbf{Q}_u^{-1}\mathbf{B}^T\boldsymbol{\lambda}(k+1) \tag{4.61}$$

Where the first difference equation is the state equation with its the initial conditions whereas the second difference equation is the adjoint equation with its boundary conditions and the last algebraic equation is the stationarity equation which relates the optimal control vector with the vector of Lagrange multipliers. There is a method originated from the optimal control theory to obtain these sets of equations directly defining a so-called Hamiltonian function which depends on the state vector $\mathbf{z}(k)$, the co-state vector $\boldsymbol{\lambda}(k)$ and the control vector $\mathbf{u}(k)$ $[9]$, $[10]$:

$$\begin{aligned} H(\mathbf{z}(k),\mathbf{u}(k),\boldsymbol{\lambda}(k)) = \frac{1}{2}\mathbf{z}^T(k)\mathbf{Q}_z\mathbf{z}(k) + \frac{1}{2}\mathbf{u}^T(k)\mathbf{Q}_u\mathbf{u}(k) + \\ + \boldsymbol{\lambda}^T(k+1)\left(\mathbf{A}\mathbf{z}(k) + \mathbf{B}\mathbf{u}(k)\right) \end{aligned} \tag{4.62}$$

The state equation, the co-state equation and the stationarity equation can be obtained from the Hamiltonian function as follows:

$$\begin{cases} \mathbf{z}(k+1) = \left(\dfrac{\partial H(\mathbf{z}(k),\mathbf{u}(k),\boldsymbol{\lambda}(k))}{\partial \boldsymbol{\lambda}(k+1)}\right)^T \\ \mathbf{z}(0) = \mathbf{z}_0 \end{cases} \tag{4.63}$$

$$\begin{cases} \boldsymbol{\lambda}(k) = -\left(\dfrac{\partial H(\mathbf{z}(k),\mathbf{u}(k),\boldsymbol{\lambda}(k))}{\partial \mathbf{z}(k)}\right)^T \\ \boldsymbol{\lambda}(N) = \mathbf{Q}_T\mathbf{z}(N) \end{cases} \tag{4.64}$$

$$\left(\frac{\partial H(\mathbf{z}(k),\mathbf{u}(k),\boldsymbol{\lambda}(k))}{\partial \mathbf{u}(k)}\right)^T = \mathbf{0} \tag{4.65}$$

Note that even in this case the whole problem is coupled and it is sometimes referred as two-point boundary value problem $[9]$, $[10]$. The classical method to solve this problem consist in reducing it to the solution of a discrete-time difference Riccati equation which allows to express the optimal control vector as a linear function of the state $[9]$, $[10]$. Indeed, observing that at the final time N the adjoint vector is a linear function of the state vector, assume that this relation holds for each instant of time:

$$\begin{cases} \boldsymbol{\lambda}(k) = \mathbf{S}(k)\mathbf{z}(k) \\ \mathbf{S}(N) = \mathbf{Q}_T \end{cases} \tag{4.66}$$

Where $\mathbf{S}(k)$ is a $\mathbb{R}^{n\times n}$ symmetric matrix to be computed. Note that this solution method is referred as Sweep Method $[13]$. Consequently, the input vector can be computed as follows:

$$\begin{aligned} \mathbf{u}(k) &= -\mathbf{Q}_u^{-1}\mathbf{B}^T\boldsymbol{\lambda}(k+1) = \\ &= -\mathbf{Q}_u^{-1}\mathbf{B}^T\mathbf{S}(k+1)\mathbf{z}(k+1) = \\ &= -\mathbf{Q}_u^{-1}\mathbf{B}^T\mathbf{S}(k+1)\left(\mathbf{Az}(k) + \mathbf{Bu}(k)\right) = \\ &= -\mathbf{Q}_u^{-1}\mathbf{B}^T\mathbf{S}(k+1)\mathbf{Az}(k) - \mathbf{Q}_u^{-1}\mathbf{B}^T\mathbf{S}(k+1)\mathbf{Bu}(k) \end{aligned} \tag{4.67}$$

Rearranging the common factors yields:

$$\begin{aligned} \mathbf{u}(k) &= -\left(\mathbf{Q}_u + \mathbf{B}^T\mathbf{S}(k+1)\mathbf{B}\right)^{-1}\mathbf{B}^T\mathbf{S}(k+1)\mathbf{Az}(k) = \\ &= \mathbf{F}(k)\mathbf{z}(k) \end{aligned} \tag{4.68}$$

Where the discrete-time feedback matrix $\mathbf{F}(k)$ is a $\mathbb{R}^{r\times n}$ matrix function of time defined as:

$$\mathbf{F}(k) = -\left(\mathbf{Q}_u + \mathbf{B}^T\mathbf{S}(k+1)\mathbf{B}\right)^{-1}\mathbf{B}^T\mathbf{S}(k+1)\mathbf{A} \tag{4.69}$$

On the other hand, substituting the assumed functional form for the adjoint vector in the adjoint difference equation yields:

$$\begin{aligned} \mathbf{S}(k)\mathbf{z}(k) &= \mathbf{Q}_z\mathbf{z}(k) + \mathbf{A}^T\mathbf{S}(k+1)\mathbf{z}(k+1) = \\ &= \mathbf{Q}_z\mathbf{z}(k) + \mathbf{A}^T\mathbf{S}(k+1)\left(\mathbf{Az}(k) + \mathbf{Bu}(k)\right) = \\ &= \mathbf{Q}_z\mathbf{z}(k) + \mathbf{A}^T\mathbf{S}(k+1)\mathbf{Az}(k) + \mathbf{A}^T\mathbf{S}(k+1)\mathbf{Bu}(k) = \\ &= \mathbf{Q}_z\mathbf{z}(k) + \mathbf{A}^T\mathbf{S}(k+1)\mathbf{Az}(k) - \mathbf{A}^T\mathbf{S}(k+1)\mathbf{BF}(k)\mathbf{z}(k) \end{aligned} \tag{4.70}$$

Rearranging the common factors leads to:

$$\left(\mathbf{S}(k)-\mathbf{Q}_z-\mathbf{A}^T\mathbf{S}(k+1)\mathbf{A}-\mathbf{A}^T\mathbf{S}(k+1)\mathbf{B}\mathbf{F}(k)\right)\mathbf{z}(k)=\mathbf{0} \qquad (4.71)$$

Setting the terms between the brackets equal to zero gives:

$$\begin{cases}\mathbf{S}(k)=\mathbf{A}^T\mathbf{S}(k+1)\mathbf{B}\mathbf{F}(k)+\mathbf{A}^T\mathbf{S}(k+1)\mathbf{A}+\mathbf{Q}_z \\ \mathbf{S}(N)=\mathbf{Q}_T\end{cases} \qquad (4.72)$$

This is a first-order matrix difference equation named discrete-time difference Riccati equation. The solution of this difference equation can be found with a marching backward in time. Substituting the definition of the discrete-time feedback matrix yields a more explicit form of this matrix equation:

$$\begin{aligned}
\mathbf{S}(k) &= -\mathbf{A}^T\mathbf{S}(k+1)\mathbf{B}\left(\mathbf{Q}_u+\mathbf{B}^T\mathbf{S}(k+1)\mathbf{B}\right)^{-1}\mathbf{B}^T\mathbf{S}(k+1)\mathbf{A}+\mathbf{A}^T\mathbf{S}(k+1)\mathbf{A}+\mathbf{Q}_z = \\
&= \mathbf{A}^T\left(\mathbf{S}(k+1)-\mathbf{S}(k+1)\mathbf{B}\left(\mathbf{Q}_u+\mathbf{B}^T\mathbf{S}(k+1)\mathbf{B}\right)^{-1}\mathbf{B}^T\mathbf{S}(k+1)\right)\mathbf{A}+\mathbf{Q}_z = \\
&= \mathbf{A}^T\left(\mathbf{S}(k+1)-\mathbf{S}(k+1)\mathbf{B}\left(\mathbf{Q}_u^{-1}-\mathbf{Q}_u^{-1}\mathbf{B}^T\left(\mathbf{S}(k+1)^{-1}+\mathbf{B}\mathbf{Q}_u^{-1}\mathbf{B}^T\right)^{-1}\mathbf{B}\mathbf{Q}_u^{-1}\right)\mathbf{B}^T\mathbf{S}(k+1)\right)\mathbf{A}+ \\
&+\mathbf{Q}_z = \\
&= \mathbf{A}^T\left(\mathbf{S}(k+1)-\mathbf{S}(k+1)\mathbf{B}\mathbf{Q}_u^{-1}\mathbf{B}^T\mathbf{S}(k+1)\right)\mathbf{A}+ \\
&+\mathbf{A}^T\left(\mathbf{S}(k+1)\mathbf{B}\mathbf{Q}_u^{-1}\mathbf{B}^T\left(\mathbf{S}(k+1)^{-1}+\mathbf{B}\mathbf{Q}_u^{-1}\mathbf{B}^T\right)^{-1}\mathbf{B}\mathbf{Q}_u^{-1}\mathbf{B}^T\mathbf{S}(k+1)\right)\mathbf{A}+\mathbf{Q}_z = \\
&= \mathbf{A}^T\left(\mathbf{S}(k+1)-\mathbf{S}(k+1)+\left(\mathbf{S}(k+1)^{-1}+\mathbf{B}\mathbf{Q}_u^{-1}\mathbf{B}^T\right)^{-1}\right)\mathbf{A}+\mathbf{Q}_z = \\
&= \mathbf{A}^T\left(\mathbf{S}(k+1)^{-1}+\mathbf{B}\mathbf{Q}_u^{-1}\mathbf{B}^T\right)^{-1}\mathbf{A}+\mathbf{Q}_z = \\
&= \mathbf{A}^T\mathbf{S}'(k+1)\mathbf{A}+\mathbf{Q}_z
\end{aligned} \qquad (4.73)$$

Where $\mathbf{S}'(k+1)$ is $\mathbb{R}^{n\times n}$ a symmetric matrix defined as:

$$\mathbf{S}'(k+1) = \left(\mathbf{S}(k+1)^{-1} + \mathbf{B}\mathbf{Q}_u^{-1}\mathbf{B}^T\right)^{-1} \tag{4.74}$$

Consequently, the unknown matrix $\mathbf{S}(k)$ can be computed by the following set of matrix difference equation:

$$\begin{cases} \mathbf{S}(k) = \mathbf{A}^T\mathbf{S}'(k+1)\mathbf{A} + \mathbf{Q}_z \\ \mathbf{S}(N) = \mathbf{Q}_T \end{cases} \tag{4.75}$$

By using the definition of matrix $\mathbf{S}'(k+1)$ the discrete-time feedback matrix $\mathbf{F}(k)$ can be expressed as:

$$\begin{aligned} \mathbf{F}(k) &= -\mathbf{Q}_u^{-1}\mathbf{B}^T\mathbf{S}'(k+1)\mathbf{A} = \\ &= -\left(\mathbf{Q}_u + \mathbf{B}^T\mathbf{S}(k+1)\mathbf{B}\right)^{-1}\mathbf{B}^T\mathbf{S}(k+1)\mathbf{A} \end{aligned} \tag{4.76}$$

It can be proved [9], [10] that the discrete-time difference Riccati equation reaches quickly an asymptotic solution $\mathbf{S}_\infty$ which can be used to compute a steady-state discrete-time feedback matrix $\mathbf{F}_\infty$ as:

$$\begin{aligned} \mathbf{F}_\infty &= -\mathbf{Q}_u^{-1}\mathbf{B}^T\mathbf{S}'_\infty\mathbf{A} = \\ &= -\left(\mathbf{Q}_u + \mathbf{B}^T\mathbf{S}_\infty\mathbf{B}\right)^{-1}\mathbf{B}^T\mathbf{S}_\infty\mathbf{A} \end{aligned} \tag{4.77}$$

This is equivalent to minimize an infinite-horizon discrete-time quadratic cost index defined as:

$$J_\infty = \frac{1}{2}\sum_{k=0}^{\infty}\left(\mathbf{z}^T(k)\mathbf{Q}_z\mathbf{z}(k) + \mathbf{u}^T(k)\mathbf{Q}_u\mathbf{u}(k)\right) \tag{4.78}$$

Consequently, the control input can be computed as a linear combination of the state as follows:

$$\mathbf{u}(k) = \mathbf{F}_{\infty}\mathbf{z}(k) \tag{4.79}$$

In practice, this simple form of the control input is widely used for real-time applications.

4.3. STATE ESTIMATION PROBLEM

Consider a linear-dynamic time-invariant mechanical system:

$$\begin{cases} \dot{\mathbf{z}}(t) = \mathbf{A}_c \mathbf{z}(t) + \mathbf{B}_c \mathbf{u}(t) \\ \mathbf{z}(0) = \mathbf{z}_0 \end{cases} \tag{4.80}$$

In practical application, it is common that there are not enough sensors to completely measure the state vector $\mathbf{z}(t)$ and even the system initial state $\mathbf{z}_0$ is unknown. Therefore, the output equations is:

$$\mathbf{y}(t) = \mathbf{C}\mathbf{z}(t) + \mathbf{D}\mathbf{u}(t) \tag{4.81}$$

The state estimation problem consists in finding an estimation of the evolution of system state $\hat{\mathbf{z}}(t)$ using the available input and output measurements represented by the vectors $\mathbf{u}(t)$ and $\mathbf{y}(t)$ $[9]$, $[10]$. Clearly, since the estimated state $\hat{\mathbf{z}}(t)$ is a function of time, to compute it a differential equation is required. The mathematical device that allows to compute an estimation of the state from input and output measurements is known as an observer $[11]$, $[12]$. The simplest state estimator device is represented by a linear differential equation similar to the state equation which have an additional driving input proportional to the difference from the actual measurement vector $\mathbf{y}(t)$ and the reconstructed output vector $\hat{\mathbf{y}}(t)$ in order to ensure that the estimated state does not deviate too much from the actual state. Indeed:

$$\dot{\hat{\mathbf{z}}}(t) = \mathbf{A}_c \hat{\mathbf{z}}(t) + \mathbf{B}_c \mathbf{u}(t) - \mathbf{G}_c \left(\mathbf{y}(t) - \hat{\mathbf{y}}(t) \right) \tag{4.82}$$

$$\hat{\mathbf{y}}(t) = \mathbf{C}\hat{\mathbf{z}}(t) + \mathbf{D}\mathbf{u}(t) \tag{4.83}$$

Where $\mathbf{G}_c$ is a $\mathbb{R}^{n \times m}$ matrix which represent the observer gain matrix. Consequently, the state estimation problem reduces to properly compute the observer matrix $\mathbf{G}_c$ in order to obtain a satisfying estimation of system state. Using the definition of the estimated measurement vector $\hat{\mathbf{y}}(t)$, the observer equation can be expressed in a compact form as:

$$\dot{\hat{\mathbf{z}}}(t) = \left(\mathbf{A}_c + \mathbf{G}_c\mathbf{C}\right)\hat{\mathbf{z}}(t) + \left(\mathbf{B}_c + \mathbf{G}_c\mathbf{D}\right)\mathbf{u}(t) - \mathbf{G}_c\mathbf{y}(t) \qquad (4.84)$$

This equation shows that the evolution of the estimated state vector $\hat{\mathbf{z}}(t)$ is driven from both the input vector $\mathbf{u}(t)$ and the output vector $\mathbf{y}(t)$. The question which spontaneously arises is if the estimated state computed by the observer converges to the actual state or not. To answer this question, define the state estimation error as:

$$\mathbf{e}(t) = \mathbf{z}(t) - \hat{\mathbf{z}}(t) \qquad (4.85)$$

The evolution in time of the state estimation error can be obtained from the following differential equation:

$$\begin{aligned}
&\dot{\mathbf{e}}(t) = \dot{\mathbf{z}}(t) - \dot{\hat{\mathbf{z}}}(t) = \\
&= \mathbf{A}_c\mathbf{z}(t) + \mathbf{B}_c\mathbf{u}(t) - \left(\mathbf{A}_c + \mathbf{G}_c\mathbf{C}\right)\hat{\mathbf{z}}(t) - \left(\mathbf{B}_c + \mathbf{G}_c\mathbf{D}\right)\mathbf{u}(t) + \mathbf{G}_c\mathbf{y}(t) = \\
&= \mathbf{A}_c\mathbf{z}(t) + \mathbf{B}_c\mathbf{u}(t) - \mathbf{A}_c\hat{\mathbf{z}}(t) - \mathbf{G}_c\mathbf{C}\hat{\mathbf{z}}(t) - \mathbf{B}_c\mathbf{u}(t) - \mathbf{G}_c\mathbf{D}\mathbf{u}(t) + \\
&+ \mathbf{G}_c\left(\mathbf{C}\mathbf{z}(t) + \mathbf{D}\mathbf{u}(t)\right) = \\
&= \left(\mathbf{A}_c + \mathbf{G}_c\mathbf{C}\right)\mathbf{z}(t) - \left(\mathbf{A}_c + \mathbf{G}_c\mathbf{C}\right)\hat{\mathbf{z}}(t) = \\
&= \left(\mathbf{A}_c + \mathbf{G}_c\mathbf{C}\right)\left(\mathbf{z}(t) - \hat{\mathbf{z}}(t)\right) = \\
&= \left(\mathbf{A}_c + \mathbf{G}_c\mathbf{C}\right)\mathbf{e}(t) = \\
&= \mathbf{A}_{G,c}\mathbf{e}(t)
\end{aligned} \qquad (4.86)$$

Where $\mathbf{A}_{G,c}$ is a $\mathbb{R}^{n\times n}$ matrix which represents the closed-loop state estimation error matrix. This matrix is defined as:

$$\mathbf{A}_{G,c} = \mathbf{A}_c + \mathbf{G}_c\mathbf{C} \tag{4.87}$$

In order to obtain a state estimation error which converges to zero the observer matrix $\mathbf{G}_c$ must be constructed such that the eigenvalues of the closed-loop state estimation error matrix $\mathbf{A}_{G,c}$ have negative real parts. Therefore, similarly to the regulation problem, a physically intuitive method to find the controller gain matrix $\mathbf{G}_c$ is to force the eigenvalues of the closed-loop state estimation error matrix $\mathbf{A}_{G,c}$ to assume a prescribed set of values $[3]$. The basic requirement to place the closed-loop poles of matrix $\mathbf{A}_{G,c}$ in a specific location of the complex plane is that the system must be observable. A linear time-invariant dynamical system of order n is observable if and only if its observability matrix $\mathbf{Q}_{G,c}$ has rank n. The observability matrix $\mathbf{Q}_{G,c}$ is a $\mathbb{R}^{nm\times n}$ matrix defined as:

$$\mathbf{Q}_{G,c} = \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A}_c \\ \mathbf{C}\mathbf{A}_c^2 \\ \vdots \\ \mathbf{C}\mathbf{A}_c^{n-1} \end{bmatrix} \tag{4.88}$$

Consider now the left eigenvalue problem of the closed-loop state estimation error matrix $\mathbf{A}_{G,c}$:

$$\mathbf{A}_{G,c}^T \boldsymbol{\psi}_{G,c} = \lambda_{G,c} \boldsymbol{\psi}_{G,c} \tag{4.89}$$

Where $\lambda_{G,c}$ is a generic eigenvalue of matrix $\mathbf{A}_{G,c}$ and $\boldsymbol{\psi}_{G,c}$ is a $\mathbb{R}^n$ vector representing the left eigenvector of the closed-loop state estimation error matrix $\mathbf{A}_{G,c}$ corresponding to the eigenvalue $\lambda_{G,c}$. The formulation of the left eigenvalue problem of matrix $\mathbf{A}_{G,c}$ obviously leads to the same right eigenvalues but is necessary in order to use the null space technique in a form similar to the case of the pole placement of closed-loop state matrix $\mathbf{A}_{F,c}$ $[3]$. The left eigenvalue problem of matrix $\mathbf{A}_{G,c}$ can be explicitly expressed as:

$$\left(\mathbf{A}_c + \mathbf{G}_c\mathbf{C}\right)^T \boldsymbol{\psi}_{G,c} = \lambda_{G,c}\boldsymbol{\psi}_{G,c} \tag{4.90}$$

This eigenvalue problem can be restated as follows:

$$\begin{aligned}\begin{bmatrix}\mathbf{A}_c^T - \lambda_{G,c}\mathbf{I} & \mathbf{C}^T\end{bmatrix}\begin{bmatrix}\boldsymbol{\psi}_{G,c} \\ \mathbf{G}_c^T\boldsymbol{\psi}_{G,c}\end{bmatrix} = \boldsymbol{\Gamma}_{G,c}\begin{bmatrix}\boldsymbol{\psi}_{G,c} \\ \mathbf{G}_c^T\boldsymbol{\psi}_{G,c}\end{bmatrix} = \\ = \mathbf{0}\end{aligned} \tag{4.91}$$

Where $\boldsymbol{\Gamma}_{G,c}$ is a $\mathbb{R}^{n\times(n+m)}$ matrix defined as:

$$\boldsymbol{\Gamma}_{G,c} = \begin{bmatrix}\mathbf{A}_c^T - \lambda_{G,c}\mathbf{I} & \mathbf{C}^T\end{bmatrix} \tag{4.92}$$

Therefore, even in this case the matrix $\boldsymbol{\Gamma}_{G,c}$ can be actually computed once that the eigenvalue $\lambda_{G,c}$ has been assigned for the system represented by the state matrix $\mathbf{A}_c$ and the output influence matrix $\mathbf{C}$. This matrix can be factorized by using the Singular Value Decomposition method (SVD) $[15]$ to yield:

$$\boldsymbol{\Gamma}_{G,c} = \mathbf{U}_{G,c}\boldsymbol{\Sigma}_{G,c}\mathbf{V}_{G,c}^* \tag{4.93}$$

Where $\boldsymbol{\Sigma}_{G,c}$ is a $\mathbb{R}^{n\times(n+m)}$ diagonal matrix containing the complex conjugate singular values of matrix $\boldsymbol{\Gamma}_{G,c}$ whereas $\mathbf{U}_{G,c}$ and $\mathbf{V}_{G,c}$ are respectively $\mathbb{R}^{n\times n}$ and $\mathbb{R}^{(m+n)\times(m+n)}$ orthonormal matrices containing the left singular vectors and the right singular vectors of matrix $\mathbf{\Gamma}_{G,c}$. These matrices can be respectively partitioned as follows:

$$\mathbf{\Sigma}_{G,c} = \begin{bmatrix} \mathbf{S}_{G,c} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix} \tag{4.94}$$

$$\mathbf{U}_{G,c} = \begin{bmatrix} \mathbf{U}_{G,c}^{S} & \mathbf{U}_{G,c}^{O} \end{bmatrix} \tag{4.95}$$

$$\mathbf{V}_{G,c} = \begin{bmatrix} \mathbf{V}_{G,c}^{S} & \mathbf{V}_{G,c}^{O} \end{bmatrix} \tag{4.96}$$

Where $\mathbf{S}_{G,c}$, $\mathbf{U}_{G,c}^{S}$, $\mathbf{U}_{G,c}^{O}$, $\mathbf{V}_{G,c}^{S}$ and $\mathbf{V}_{G,c}^{O}$ are respectively $\mathbb{R}^{q_{G,c} \times q_{G,c}}$, $\mathbb{R}^{n \times q_{G,c}}$, $\mathbb{R}^{n \times ((n+m)-q_{G,c})}$, $\mathbb{R}^{(n+m) \times q_{G,c}}$ and $\mathbb{R}^{(n+m) \times ((n+m)-q_{G,c})}$ matrices. The matrix $\mathbf{S}_{G,c}$ is a diagonal matrix containing the significant singular values of the matrix $\mathbf{\Gamma}_{G,c}$. Indeed:

$$\mathbf{S}_{G,c} = diag(\sigma_{G,c}^{1}, \sigma_{G,c}^{2}, \ldots, \sigma_{G,c}^{q_{G,c}}) \tag{4.97}$$

Consequently, multiplying the matrix $\mathbf{\Gamma}_{G,c}$ times $\mathbf{V}_{G,c}$ yields:

$$\begin{aligned} \mathbf{\Gamma}_{G,c}\mathbf{V}_{G,c} &= \mathbf{U}_{G,c}\mathbf{\Sigma}_{G,c}\mathbf{V}_{G,c}^{*}\mathbf{V}_{G,c} = \\ &= \mathbf{U}_{G,c}\mathbf{\Sigma}_{G,c} \end{aligned} \tag{4.98}$$

This equation can be explicitly restated as:

$$\begin{aligned} \mathbf{\Gamma}_{G,c}\begin{bmatrix} \mathbf{V}_{G,c}^{S} & \mathbf{V}_{G,c}^{O} \end{bmatrix} &= \mathbf{U}_{G,c}\begin{bmatrix} \mathbf{S}_{G,c} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix} = \\ &= \begin{bmatrix} \mathbf{U}_{G,c}\mathbf{S}_{G,c} & \mathbf{O} \end{bmatrix} \end{aligned} \tag{4.99}$$

The second matrix equality yields:

$$\mathbf{\Gamma}_{G,c}\mathbf{V}_{G,c}^{O} = \mathbf{O} \tag{4.100}$$

Therefore the matrix $\mathbf{V}_{G,c}^{O}$ represents a set of orthogonal basis vectors spanning the null space of the matrix $\mathbf{\Gamma}_{G,c}$ so that:

$$\begin{aligned}\mathbf{\Gamma}_{G,c}\mathbf{V}_{G,c}^{O}\mathbf{c}_{G,c} = \mathbf{\Gamma}_{G,c}\boldsymbol{\varphi}_{G,c} = \\ = \mathbf{0}\end{aligned} \tag{4.101}$$

Where $\mathbf{c}_{G,c}$ is an $\mathbb{R}^{(n+m)-q_{G,c}}$ arbitrary nonzero vector and $\boldsymbol{\varphi}_{G,c}$ is a $\mathbb{R}^{n+m}$ vector defined as:

$$\boldsymbol{\varphi}_{G,c} = \mathbf{V}_{G,c}^{O}\mathbf{c}_{G,c} \tag{4.102}$$

This vector can be partitioned as follows:

$$\boldsymbol{\varphi}_{G,c} = \begin{bmatrix} \overline{\boldsymbol{\varphi}}_{G,c} \\ \overline{\overline{\boldsymbol{\varphi}}}_{G,c} \end{bmatrix} \tag{4.103}$$

Where $\overline{\boldsymbol{\varphi}}_{G,c}$ and $\overline{\overline{\boldsymbol{\varphi}}}_{G,c}$ are respectively $\mathbb{R}^{n}$ and $\mathbb{R}^{m}$ vectors. Observing the matrix reformulation of the left eigenvalue problem of matrix $\mathbf{A}_{G,c}$, the following matrix equations can be deduced:

$$\boldsymbol{\psi}_{G,c} = \overline{\boldsymbol{\varphi}}_{G,c} \tag{4.104}$$

$$\mathbf{G}_{c}^{T}\boldsymbol{\psi}_{G,c} = \overline{\overline{\boldsymbol{\varphi}}}_{G,c} \tag{4.105}$$

Note that from the first matrix equation is straightforward to deduce that the vector $\bar{\boldsymbol{\varphi}}_{G,c}$ coincides with the left eigenvector of the closed-loop state estimation error matrix $\mathbf{A}_{G,c}$ corresponding to the assigned eigenvalue $\lambda_{G,c}$. Consequently:

$$\mathbf{G}_c^T \bar{\boldsymbol{\varphi}}_{G,c} = \bar{\bar{\boldsymbol{\varphi}}}_{G,c} \tag{4.106}$$

This procedure can be repeated for each prescribed eigenvalue $\lambda_{G,c}^h$ to yield the following generic matrix equations:

$$\mathbf{G}_c^T \bar{\boldsymbol{\varphi}}_{G,c}^h = \bar{\bar{\boldsymbol{\varphi}}}_{G,c}^h \quad , \quad h = 1, 2, \ldots, n \tag{4.107}$$

Where $\bar{\boldsymbol{\varphi}}_{G,c}^h$ and $\bar{\bar{\boldsymbol{\varphi}}}_{G,c}^h$ are respectively $\mathbb{R}^n$ and $\mathbb{R}^m$ generic vectors corresponding to the assigned eigenvalue $\lambda_{G,c}^h$. These equations can be restated in a compact matrix form as follows:

$$\mathbf{G}_c^T \bar{\boldsymbol{\Phi}}_{G,c} = \bar{\bar{\boldsymbol{\Phi}}}_{G,c} \tag{4.108}$$

Where $\bar{\boldsymbol{\Phi}}_{G,c}$ and $\bar{\bar{\boldsymbol{\Phi}}}_{G,c}$ are respectively $\mathbb{R}^{n \times n}$ and $\mathbb{R}^{m \times n}$ matrices defined as:

$$\begin{aligned} \bar{\boldsymbol{\Phi}}_{G,c} &= \begin{bmatrix} \bar{\boldsymbol{\varphi}}_{G,c}^1 & \bar{\boldsymbol{\varphi}}_{G,c}^2 & \ldots & \bar{\boldsymbol{\varphi}}_{G,c}^{n-1} & \bar{\boldsymbol{\varphi}}_{G,c}^n \end{bmatrix} = \\ &= \begin{bmatrix} \bar{\boldsymbol{\varphi}}_{G,c}^1 & \bar{\boldsymbol{\varphi}}_{G,c}^{1*} & \ldots & \bar{\boldsymbol{\varphi}}_{G,c}^{n-1} & \bar{\boldsymbol{\varphi}}_{G,c}^{n-1*} \end{bmatrix} \end{aligned} \tag{4.109}$$

$$\begin{aligned} \bar{\bar{\boldsymbol{\Phi}}}_{G,c} &= \begin{bmatrix} \bar{\bar{\boldsymbol{\varphi}}}_{G,c}^1 & \bar{\bar{\boldsymbol{\varphi}}}_{G,c}^2 & \ldots & \bar{\bar{\boldsymbol{\varphi}}}_{G,c}^{n-1} & \bar{\bar{\boldsymbol{\varphi}}}_{G,c}^n \end{bmatrix} = \\ &= \begin{bmatrix} \bar{\bar{\boldsymbol{\varphi}}}_{G,c}^1 & \bar{\bar{\boldsymbol{\varphi}}}_{G,c}^{1*} & \ldots & \bar{\bar{\boldsymbol{\varphi}}}_{G,c}^{n-1} & \bar{\bar{\boldsymbol{\varphi}}}_{G,c}^{n-1*} \end{bmatrix} \end{aligned} \tag{4.110}$$

Note that complex conjugate eigenvalue pairs $\lambda_{G,c}^{h-1}$ and $\lambda_{G,c}^{h}$ corresponds to complex conjugate vector pairs $\bar{\boldsymbol{\varphi}}_{G,c}^{h-1}$, $\bar{\boldsymbol{\varphi}}_{G,c}^{h}$ and $\bar{\bar{\boldsymbol{\varphi}}}_{G,c}^{h-1}$, $\bar{\bar{\boldsymbol{\varphi}}}_{G,c}^{h}$. Finally, the observer matrix $\mathbf{G}_c$ can be computed as:

$$\begin{aligned}\mathbf{G}_c &= \left(\bar{\bar{\boldsymbol{\Phi}}}_{G,c}\bar{\boldsymbol{\Phi}}_{G,c}^{-1}\right)^T = \\ &= \bar{\boldsymbol{\Phi}}_{G,c}^{-T}\bar{\bar{\boldsymbol{\Phi}}}_{G,c}^{T}\end{aligned} \tag{4.111}$$

This method to compute the observer gain matrix is sometimes referred as null-space technique for poles placement [3]. Since the output influence matrix $\mathbf{C}$ is the same for both discrete-time and continuous-time state-space representations, this method can be extended to system discrete-time state-space representation in a straightforward manner replacing the continuous-time state matrix $\mathbf{A}_c$ with the discrete-time state matrix $\mathbf{A}$ to yield a discrete-time observer matrix $\mathbf{G}$ instead of the continuous-time observer gain $\mathbf{G}_c$. Now consider the more realistic case in which the system analytical model exhibits some inaccuracies and the output measurements are corrupted by noise. In this situation, an important method to solve the state estimation problem for both continuous-time and discrete-time linear state-space systems comes from the optimal estimation theory and is the Kalman Filter algorithm (KF) [11], [12]. Similarly to the pole placement technique, the basic requirement to apply the Kalman Filter algorithm (KF) is that the system must be observable. Indeed, consider a continuous-time state-space system affected by disturbances:

$$\dot{\mathbf{z}}(t) = \mathbf{A}_c\mathbf{z}(t) + \mathbf{B}_c\mathbf{u}(t) + \mathbf{w}(t) \tag{4.112}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{z}(t) + \mathbf{D}\mathbf{u}(t) + \mathbf{v}(t) \tag{4.113}$$

Where $\mathbf{w}(t)$ is a $\mathbb{R}^n$ vector representing the process noise and $\mathbf{v}(t)$ is a $\mathbb{R}^m$ vector representing the measurement noise. The random disturbances $\mathbf{w}(t)$

and $\mathbf{v}(t)$ are not measurable and are assumed zero mean Gaussian white noises whose stochastic characteristics are:

$$E[\mathbf{w}(t)] = \mathbf{0} \quad , \quad \forall t \geq 0 \tag{4.114}$$

$$E[\mathbf{v}(t)] = \mathbf{0} \quad , \quad \forall t \geq 0 \tag{4.115}$$

$$E[\mathbf{w}(t)\mathbf{w}^T(\tau)] = \mathbf{R}_{c,w}\delta(t-\tau) \quad , \quad \forall t,\tau \geq 0 \tag{4.116}$$

$$E[\mathbf{v}(t)\mathbf{v}^T(\tau)] = \mathbf{R}_{c,v}\delta(t-\tau) \quad , \quad \forall t,\tau \geq 0 \tag{4.117}$$

Where $\mathbf{R}_{c,w}$ is a $\mathbb{R}^{n\times n}$ symmetric positive definite matrix defining the process noise covariance matrix and $\mathbf{R}_{c,v}$ is a $\mathbb{R}^{m\times m}$ symmetric positive definite matrix defining the measurement noise covariance matrix. In addition, the process noise and the measurement noise are assumed mutually uncorrelated:

$$E[\mathbf{w}(t)\mathbf{v}^T(\tau)] = \mathbf{O} \quad , \quad \forall t,\tau \geq 0 \tag{4.118}$$

On the other hand, even the initial state $\mathbf{z}_0$ is assumed unknown and it is modelled as a Gaussian distributed random vector whose stochastic characteristics are:

$$E[\mathbf{z}_0] = \overline{\mathbf{z}}_0 \tag{4.119}$$

$$E[\left(\mathbf{z}_0 - \overline{\mathbf{z}}_0\right)\left(\mathbf{z}_0 - \overline{\mathbf{z}}_0\right)^T] = \mathbf{R}_{c,0} \tag{4.120}$$

Where $\overline{\mathbf{z}}_0$ is a $\mathbb{R}^n$ vector representing the expected value of initial state and $\mathbf{R}_{c,0}$ is a $\mathbb{R}^{n\times n}$ symmetric positive definite matrix representing the covariance matrix of the initial state. The initial state vector is modelled as a random process uncorrelated to the stochastic disturbances:

$$E[\mathbf{z}_0\mathbf{w}^T(t)] = \mathbf{O} \quad , \quad \forall t \geq 0 \tag{4.121}$$

$$E[\mathbf{z}_0\mathbf{v}^T(t)] = \mathbf{O} \quad , \quad \forall t \geq 0 \tag{4.122}$$

The Continuous Kalman Filter algorithm (CKF) is capable to derive a continuous-time observer matrix which minimizes a quadratic performance index. The cost index is a quadratic functional which depends on process noise, measurement noise and on the error of the initial state estimation $[11]$, $[12]$. In the continuous-time case, for a finite-horizon of time $0 \leq t \leq T$ the quadratic cost index can be defined as:

$$\begin{aligned} J_c = {} & \frac{1}{2}\left(\mathbf{z}_0 - \overline{\mathbf{z}}_0\right)^T \mathbf{R}_{c,0}^{-1}\left(\mathbf{z}_0 - \overline{\mathbf{z}}_0\right) + \\ & + \frac{1}{2}\int_0^T \mathbf{w}^T(t)\mathbf{R}_{c,w}^{-1}\mathbf{w}(t) + \mathbf{v}^T(t)\mathbf{R}_{c,v}^{-1}\mathbf{v}(t)dt \end{aligned} \tag{4.123}$$

Where the weighting matrices used in the cost function for the process noise and the measurement noise are the inverse of their respective covariance matrices whereas the weighting matrix used for the estimation error of the initial state is the inverse of the covariance matrix of the initial state. Note that the performance index can be seen as an energy index of the disturbances or as an error index $[12]$. This cost index can be reformulated replacing the measurement noise by using the output equation to yield:

$$
\begin{aligned}
J_c &= \frac{1}{2}\left(\mathbf{z}_0 - \overline{\mathbf{z}}_0\right)^T \mathbf{R}_{c,0}^{-1}\left(\mathbf{z}_0 - \overline{\mathbf{z}}_0\right) + \\
&+ \frac{1}{2}\int_0^T \mathbf{w}^T(t)\mathbf{R}_{c,w}^{-1}\mathbf{w}(t) + \mathbf{v}^T(t)\mathbf{R}_{c,v}^{-1}\mathbf{v}(t)dt = \\
&= \frac{1}{2}\left(\mathbf{z}_0 - \overline{\mathbf{z}}_0\right)^T \mathbf{R}_{c,0}^{-1}\left(\mathbf{z}_0 - \overline{\mathbf{z}}_0\right) + \\
&+ \frac{1}{2}\int_0^T \mathbf{w}^T(t)\mathbf{R}_{c,w}^{-1}\mathbf{w}(t) + \left(\mathbf{y}(t) - \mathbf{C}\mathbf{z}(t) - \mathbf{D}\mathbf{u}(t)\right)^T \mathbf{R}_{c,v}^{-1}\left(\mathbf{y}(t) - \mathbf{C}\mathbf{z}(t) - \mathbf{D}\mathbf{u}(t)\right)dt
\end{aligned}
\tag{4.124}
$$

In order to find the Kalman state estimator, the cost index J_c must be minimized and simultaneously the state equation must be satisfied. Therefore, the state equation represents a constraint equation for the minimization problem. To solve this problem the method of Lagrange multipliers can be used $[11]$, $[12]$. This method consists in adjoining the state equation to the performance index and subsequently minimize this adjoint cost index J_c^* using variational calculus technique. Indeed:

$$
\begin{aligned}
J_c^* &= J_c + \int_0^T \boldsymbol{\lambda}^T(t)\left(\dot{\mathbf{z}}(t) - \mathbf{A}_c\mathbf{z}(t) - \mathbf{B}_c\mathbf{u}(t) - \mathbf{w}(t)\right)dt = \\
&= \frac{1}{2}\left(\mathbf{z}_0 - \overline{\mathbf{z}}_0\right)^T \mathbf{R}_{c,0}^{-1}\left(\mathbf{z}_0 - \overline{\mathbf{z}}_0\right) + \\
&+ \frac{1}{2}\int_0^T \mathbf{w}^T(t)\mathbf{R}_{c,w}^{-1}\mathbf{w}(t) + \left(\mathbf{y}(t) - \mathbf{C}\mathbf{z}(t) - \mathbf{D}\mathbf{u}(t)\right)^T \mathbf{R}_{c,v}^{-1}\left(\mathbf{y}(t) - \mathbf{C}\mathbf{z}(t) - \mathbf{D}\mathbf{u}(t)\right)dt + \\
&+ \int_0^T \boldsymbol{\lambda}^T(t)\left(\dot{\mathbf{z}}(t) - \mathbf{A}_c\mathbf{z}(t) - \mathbf{B}_c\mathbf{u}(t) - \mathbf{w}(t)\right)dt
\end{aligned}
\tag{4.125}
$$

Where $\boldsymbol{\lambda}(t)$ is a $\mathbb{R}^n$ vector containing the Lagrange multipliers. Since the optimal estimator minimizes the adjoint performance index J_c^*, it is necessary to compute the first variation of this functional and set it equal to zero. Indeed, taking the first variation of the augmented cost function J_c^* yields:

$$
\begin{aligned}
\delta J_c^* &= \delta\left(\frac{1}{2}\left(\mathbf{z}_0-\overline{\mathbf{z}}_0\right)^T \mathbf{R}_{c,0}^{-1}\left(\mathbf{z}_0-\overline{\mathbf{z}}_0\right)\right)+\delta\int_0^T \frac{1}{2}\mathbf{w}^T(t)\mathbf{R}_{c,w}^{-1}\mathbf{w}(t)dt+\\
&+\delta\int_0^T \frac{1}{2}\left(\mathbf{y}(t)-\mathbf{C}\mathbf{z}(t)-\mathbf{D}\mathbf{u}(t)\right)^T \mathbf{R}_{c,v}^{-1}\left(\mathbf{y}(t)-\mathbf{C}\mathbf{z}(t)-\mathbf{D}\mathbf{u}(t)\right)dt+\\
&+\delta\int_0^T \boldsymbol{\lambda}^T(t)\left(\dot{\mathbf{z}}(t)-\mathbf{A}_c\mathbf{z}(t)-\mathbf{B}_c\mathbf{u}(t)-\mathbf{w}(t)\right)dt=\\
&=\left(\mathbf{z}_0-\overline{\mathbf{z}}_0\right)^T \mathbf{R}_{c,0}^{-1}\delta\left(\mathbf{z}_0-\overline{\mathbf{z}}_0\right)+\int_0^T \delta\left(\frac{1}{2}\mathbf{w}^T(t)\mathbf{R}_{c,w}^{-1}\mathbf{w}(t)\right)dt+\\
&+\int_0^T \delta\left(\frac{1}{2}\left(\mathbf{y}(t)-\mathbf{C}\mathbf{z}(t)-\mathbf{D}\mathbf{u}(t)\right)^T \mathbf{R}_{c,v}^{-1}\left(\mathbf{y}(t)-\mathbf{C}\mathbf{z}(t)-\mathbf{D}\mathbf{u}(t)\right)\right)dt+\\
&+\int_0^T \delta\left(\boldsymbol{\lambda}^T(t)\left(\dot{\mathbf{z}}(t)-\mathbf{A}_c\mathbf{z}(t)-\mathbf{B}_c\mathbf{u}(t)-\mathbf{w}(t)\right)\right)dt=\\
&=\left(\mathbf{z}_0-\overline{\mathbf{z}}_0\right)^T \mathbf{R}_{c,0}^{-1}\left(\delta\mathbf{z}_0-\delta\overline{\mathbf{z}}_0\right)+\int_0^T \mathbf{w}^T(t)\mathbf{R}_{c,w}^{-1}\delta\mathbf{w}(t)dt+\\
&+\int_0^T \left(\mathbf{y}(t)-\mathbf{C}\mathbf{z}(t)-\mathbf{D}\mathbf{u}(t)\right)^T \mathbf{R}_{c,v}^{-1}\delta\left(\mathbf{y}(t)-\mathbf{C}\mathbf{z}(t)-\mathbf{D}\mathbf{u}(t)\right)dt+\\
&+\int_0^T \delta\boldsymbol{\lambda}^T(t)\left(\dot{\mathbf{z}}(t)-\mathbf{A}_c\mathbf{z}(t)-\mathbf{B}_c\mathbf{u}(t)-\mathbf{w}(t)\right)dt+\\
&+\int_0^T \boldsymbol{\lambda}^T(t)\delta\left(\dot{\mathbf{z}}(t)-\mathbf{A}_c\mathbf{z}(t)-\mathbf{B}_c\mathbf{u}(t)-\mathbf{w}(t)\right)dt=\\
&=\left(\mathbf{z}_0-\overline{\mathbf{z}}_0\right)^T \mathbf{R}_{c,0}^{-1}\delta\mathbf{z}_0+\\
&+\int_0^T \mathbf{w}^T(t)\mathbf{R}_{c,w}^{-1}\delta\mathbf{w}(t)+\left(\mathbf{y}(t)-\mathbf{C}\mathbf{z}(t)-\mathbf{D}\mathbf{u}(t)\right)^T \mathbf{R}_{c,v}^{-1}\delta\mathbf{y}(t)dt+\\
&+\int_0^T -\left(\mathbf{y}(t)-\mathbf{C}\mathbf{z}(t)-\mathbf{D}\mathbf{u}(t)\right)^T \mathbf{R}_{c,v}^{-1}\mathbf{C}\delta\mathbf{z}(t)dt+\\
&-\int_0^T +\left(\mathbf{y}(t)-\mathbf{C}\mathbf{z}(t)-\mathbf{D}\mathbf{u}(t)\right)^T \mathbf{R}_{c,v}^{-1}\mathbf{D}\delta\mathbf{u}(t)dt+\\
&+\int_0^T \left(\dot{\mathbf{z}}(t)-\mathbf{A}_c\mathbf{z}(t)-\mathbf{B}_c\mathbf{u}(t)-\mathbf{w}(t)\right)^T \delta\boldsymbol{\lambda}(t)+\boldsymbol{\lambda}^T(t)\delta\dot{\mathbf{z}}(t)dt+\\
&+\int_0^T -\boldsymbol{\lambda}^T(t)\mathbf{A}_c\delta\mathbf{z}(t)-\boldsymbol{\lambda}^T(t)\mathbf{B}_c\delta\mathbf{u}(t)-\boldsymbol{\lambda}^T(t)\delta\mathbf{w}(t)dt=0
\end{aligned}
\tag{4.126}
$$

This formula can be further simplified yielding to:

$$
\begin{aligned}
&\left(\mathbf{z}_0-\overline{\mathbf{z}}_0\right)^T \mathbf{R}_{c,0}^{-1}\delta\mathbf{z}_0 + \\
&+\int_0^T \mathbf{w}^T(t)\mathbf{R}_{c,w}^{-1}\delta\mathbf{w}(t)+\left(\mathbf{y}(t)-\mathbf{C}\mathbf{z}(t)-\mathbf{D}\mathbf{u}(t)\right)^T \mathbf{R}_{c,v}^{-1}\delta\mathbf{y}(t)dt + \\
&+\int_0^T -\left(\mathbf{y}(t)-\mathbf{C}\mathbf{z}(t)-\mathbf{D}\mathbf{u}(t)\right)^T \mathbf{R}_{c,v}^{-1}\mathbf{C}\,\delta\mathbf{z}(t)dt + \\
&-\int_0^T +\left(\mathbf{y}(t)-\mathbf{C}\mathbf{z}(t)-\mathbf{D}\mathbf{u}(t)\right)^T \mathbf{R}_{c,v}^{-1}\mathbf{D}\delta\mathbf{u}(t)dt + \\
&+\int_0^T \left(\dot{\mathbf{z}}(t)-\mathbf{A}_c\mathbf{z}(t)-\mathbf{B}_c\mathbf{u}(t)-\mathbf{w}(t)\right)^T \delta\boldsymbol{\lambda}(t)+\boldsymbol{\lambda}^T(t)\delta\dot{\mathbf{z}}(t)dt + \\
&+\int_0^T -\boldsymbol{\lambda}^T(t)\mathbf{A}_c\delta\mathbf{z}(t)-\boldsymbol{\lambda}^T(t)\mathbf{B}_c\delta\mathbf{u}(t)-\boldsymbol{\lambda}^T(t)\delta\mathbf{w}(t)dt = \\
&=\left(\mathbf{z}_0-\overline{\mathbf{z}}_0\right)^T \mathbf{R}_{c,0}^{-1}\delta\mathbf{z}_0 + \\
&+\int_0^T \mathbf{w}^T(t)\mathbf{R}_{c,w}^{-1}\delta\mathbf{w}(t)-\left(\mathbf{y}(t)-\mathbf{C}\mathbf{z}(t)-\mathbf{D}\mathbf{u}(t)\right)^T \mathbf{R}_{c,v}^{-1}\mathbf{C}\,\delta\mathbf{z}(t)dt + \\
&+\int_0^T \left(\dot{\mathbf{z}}(t)-\mathbf{A}_c\mathbf{z}(t)-\mathbf{B}_c\mathbf{u}(t)-\mathbf{w}(t)\right)^T \delta\boldsymbol{\lambda}(t)+\boldsymbol{\lambda}^T(t)\delta\dot{\mathbf{z}}(t)dt + \\
&+\int_0^T -\boldsymbol{\lambda}^T(t)\mathbf{A}_c\delta\mathbf{z}(t)-\boldsymbol{\lambda}^T(t)\delta\mathbf{w}(t)dt = 0
\end{aligned}
\tag{4.127}
$$

This expression can be further simplified to yield:

$$\left(\mathbf{z}_0-\overline{\mathbf{z}}_0\right)^T\mathbf{R}_{c,0}^{-1}\delta\mathbf{z}_0+$$
$$+\int_0^T\mathbf{w}^T(t)\mathbf{R}_{c,w}^{-1}\delta\mathbf{w}(t)-\left(\mathbf{y}(t)-\mathbf{C}\mathbf{z}(t)-\mathbf{D}\mathbf{u}(t)\right)^T\mathbf{R}_{c,v}^{-1}\mathbf{C}\,\delta\mathbf{z}(t)dt+$$
$$+\int_0^T\left(\dot{\mathbf{z}}(t)-\mathbf{A}_c\mathbf{z}(t)-\mathbf{B}_c\mathbf{u}(t)-\mathbf{w}(t)\right)^T\delta\boldsymbol{\lambda}(t)+\boldsymbol{\lambda}^T(t)\delta\dot{\mathbf{z}}(t)dt+$$
$$+\int_0^T-\boldsymbol{\lambda}^T(t)\mathbf{A}_c\delta\mathbf{z}(t)-\boldsymbol{\lambda}^T(t)\delta\mathbf{w}(t)dt=$$
$$=\left(\mathbf{z}_0-\overline{\mathbf{z}}_0\right)^T\mathbf{R}_{c,0}^{-1}\delta\mathbf{z}_0+$$
$$+\int_0^T\boldsymbol{\lambda}^T(t)\delta\dot{\mathbf{z}}(t)+\left(-\left(\mathbf{y}(t)-\mathbf{C}\mathbf{z}(t)-\mathbf{D}\mathbf{u}(t)\right)^T\mathbf{R}_{c,v}^{-1}\mathbf{C}-\boldsymbol{\lambda}^T(t)\mathbf{A}_c\right)\delta\mathbf{z}(t)dt+$$
$$+\int_0^T\left(-\boldsymbol{\lambda}^T(t)+\mathbf{w}^T(t)\mathbf{R}_{c,w}^{-1}\right)\delta\mathbf{w}(t)+\left(\dot{\mathbf{z}}(t)-\mathbf{A}_c\mathbf{z}(t)-\mathbf{B}_c\mathbf{u}(t)-\mathbf{w}(t)\right)^T\delta\boldsymbol{\lambda}(t)dt=$$
$$=\left(\mathbf{z}_0-\overline{\mathbf{z}}_0\right)^T\mathbf{R}_{c,0}^{-1}\delta\mathbf{z}_0+\left[\boldsymbol{\lambda}^T(t)\delta\mathbf{z}(t)\right]_0^T+$$
$$+\int_0^T\left(-\dot{\boldsymbol{\lambda}}^T(t)-\left(\mathbf{y}(t)-\mathbf{C}\mathbf{z}(t)-\mathbf{D}\mathbf{u}(t)\right)^T\mathbf{R}_{c,v}^{-1}\mathbf{C}-\boldsymbol{\lambda}^T(t)\mathbf{A}_c\right)\delta\mathbf{z}(t)dt+$$
$$+\int_0^T\left(-\boldsymbol{\lambda}^T(t)+\mathbf{w}^T(t)\mathbf{R}_{c,w}^{-1}\right)\delta\mathbf{w}(t)+\left(\dot{\mathbf{z}}(t)-\mathbf{A}_c\mathbf{z}(t)-\mathbf{B}_c\mathbf{u}(t)-\mathbf{w}(t)\right)^T\delta\boldsymbol{\lambda}(t)dt=$$
$$=\left(\left(\mathbf{z}_0-\overline{\mathbf{z}}_0\right)^T\mathbf{R}_{c,0}^{-1}-\boldsymbol{\lambda}^T(0)\right)\delta\mathbf{z}_0+\boldsymbol{\lambda}^T(T)\delta\mathbf{z}(T)+$$
$$+\int_0^T\left(-\dot{\boldsymbol{\lambda}}^T(t)-\left(\mathbf{y}(t)-\mathbf{C}\mathbf{z}(t)-\mathbf{D}\mathbf{u}(t)\right)^T\mathbf{R}_{c,v}^{-1}\mathbf{C}-\boldsymbol{\lambda}^T(t)\mathbf{A}_c\right)\delta\mathbf{z}(t)dt+$$
$$+\int_0^T\left(-\boldsymbol{\lambda}^T(t)+\mathbf{w}^T(t)\mathbf{R}_{c,w}^{-1}\right)\delta\mathbf{w}(t)+\left(\dot{\mathbf{z}}(t)-\mathbf{A}_c\mathbf{z}(t)-\mathbf{B}_c\mathbf{u}(t)-\mathbf{w}(t)\right)^T\delta\boldsymbol{\lambda}(t)dt=$$
$$=\left(\mathbf{R}_{c,0}^{-1}\left(\mathbf{z}_0-\overline{\mathbf{z}}_0\right)-\boldsymbol{\lambda}(0)\right)^T\delta\mathbf{z}_0+\boldsymbol{\lambda}^T(T)\delta\mathbf{z}(T)+$$
$$+\int_0^T\left(-\dot{\boldsymbol{\lambda}}(t)-\mathbf{C}^T\mathbf{R}_{c,v}^{-1}\left(\mathbf{y}(t)-\mathbf{C}\mathbf{z}(t)-\mathbf{D}\mathbf{u}(t)\right)-\mathbf{A}_c^T\boldsymbol{\lambda}(t)\right)^T\delta\mathbf{z}(t)dt+$$
$$+\int_0^T\left(-\boldsymbol{\lambda}(t)+\mathbf{R}_{c,w}^{-1}\mathbf{w}(t)\right)^T\delta\mathbf{w}(t)+\left(\dot{\mathbf{z}}(t)-\mathbf{A}_c\mathbf{z}(t)-\mathbf{B}_c\mathbf{u}(t)-\mathbf{w}(t)\right)^T\delta\boldsymbol{\lambda}(t)dt=$$
$$=0\quad,\quad\forall\delta\mathbf{z}(t)\quad,\quad\forall\delta\boldsymbol{\lambda}(t)\quad,\quad\forall\delta\mathbf{w}(t)$$

(4.128)

Observing that the variation of the state vector $\delta\mathbf{z}(t)$, the variation of the co-state vector $\delta\boldsymbol{\lambda}(t)$ and the variation of the process noise vector $\delta\mathbf{w}(t)$ are all independent, each quantity in the time integral can be independently taken equal to zero:

$$\begin{cases} \mathbf{R}_{c,0}^{-1}\left(\mathbf{z}_0 - \overline{\mathbf{z}}_0\right) - \boldsymbol{\lambda}(0) = \mathbf{0} \\ \delta \mathbf{z}(T) = \mathbf{0} \\ -\dot{\boldsymbol{\lambda}}(t) - \mathbf{C}^T \mathbf{R}_{c,v}^{-1}\left(\mathbf{y}(t) - \mathbf{C}\mathbf{z}(t) - \mathbf{D}\mathbf{u}(t)\right) - \mathbf{A}_c^T \boldsymbol{\lambda}(t) = \mathbf{0} \\ -\boldsymbol{\lambda}(t) + \mathbf{R}_{c,w}^{-1}\mathbf{w}(t) = \mathbf{0} \\ \dot{\mathbf{z}}(t) - \mathbf{A}_c \mathbf{z}(t) - \mathbf{B}_c \mathbf{u}(t) - \mathbf{E}_c \mathbf{w}(t) = \mathbf{0} \end{cases} \tag{4.129}$$

Therefore, the minimization of the adjoint cost index J_c^* yields a set of two differential equations and one algebraic equation:

$$\begin{cases} \dot{\mathbf{z}}(t) = \mathbf{A}_c \mathbf{z}(t) + \mathbf{B}_c \mathbf{u}(t) + \mathbf{w}(t) \\ \mathbf{z}(0) = \mathbf{z}_0 \end{cases} \tag{4.130}$$

$$\begin{cases} \dot{\boldsymbol{\lambda}}(t) = -\mathbf{C}^T \mathbf{R}_{c,v}^{-1}\left(\mathbf{y}(t) - \mathbf{C}\mathbf{z}(t) - \mathbf{D}\mathbf{u}(t)\right) - \mathbf{A}_c^T \boldsymbol{\lambda}(t) \\ \boldsymbol{\lambda}(0) = \mathbf{R}_{c,0}^{-1}\left(\mathbf{z}_0 - \overline{\mathbf{z}}_0\right) \end{cases} \tag{4.131}$$

$$\mathbf{w}(t) = \mathbf{R}_{c,w} \boldsymbol{\lambda}(t) \tag{4.132}$$

Where the first differential equation is the state equation, the second differential equation is the adjoint equation and the last algebraic equation is the stationarity equation. There is a method derived from the optimal estimation theory to obtain these sets of equations directly defining the Hamiltonian function which depends on the state vector $\mathbf{z}(t)$, the co-state vector $\boldsymbol{\lambda}(t)$ and the process noise vector $\mathbf{w}(t)$ $[11]$, $[12]$. Indeed:

$$H_c(\mathbf{z}(t),\mathbf{w}(t),\boldsymbol{\lambda}(t)) = \mathbf{w}^T(t)\mathbf{R}_{c,w}^{-1}\mathbf{w}(t) + \\ +\left(\mathbf{y}(t)-\mathbf{C}\mathbf{z}(t)-\mathbf{D}\mathbf{u}(t)\right)^T \mathbf{R}_{c,v}^{-1}\left(\mathbf{y}(t)-\mathbf{C}\mathbf{z}(t)-\mathbf{D}\mathbf{u}(t)\right)+ \\ -\boldsymbol{\lambda}^T(t)\left(\mathbf{A}_c\mathbf{z}(t)+\mathbf{B}_c\mathbf{u}(t)+\mathbf{w}(t)\right) \tag{4.133}$$

The state equation, the co-state equation and the stationarity equation can be obtained from the Hamiltonian function as follows:

$$\begin{cases} \dot{\mathbf{z}}(t) = -\left(\dfrac{\partial H_c(\mathbf{z}(t),\mathbf{w}(t),\boldsymbol{\lambda}(t))}{\partial \boldsymbol{\lambda}(t)}\right)^T \\ \mathbf{z}(0) = \mathbf{z}_0 \end{cases} \tag{4.134}$$

$$\begin{cases} \dot{\boldsymbol{\lambda}}(t) = \left(\dfrac{\partial H_c(\mathbf{z}(t),\mathbf{w}(t),\boldsymbol{\lambda}(t))}{\partial \mathbf{z}(t)}\right)^T \\ \boldsymbol{\lambda}(0) = \mathbf{R}_{c,0}^{-1}\left(\mathbf{z}_0 - \overline{\mathbf{z}}_0\right) \end{cases} \tag{4.135}$$

$$\left(\frac{\partial H_c(\mathbf{z}(t),\mathbf{w}(t),\boldsymbol{\lambda}(t))}{\partial \mathbf{w}(t)}\right)^T = \mathbf{0} \tag{4.136}$$

The classical method to solve this problem consist in reducing it to the solution of a continuous-time differential Riccati equation which can be used to compute an observer matrix $[11]$, $[12]$. Indeed, since the initial state vector $\mathbf{z}_0$ is a random Gaussian process with mean vector $\overline{\mathbf{z}}_0$ and covariance matrix $\mathbf{R}_{c,0}$, the system state vector $\mathbf{z}(t)$ turns out to be a Gaussian stochastic process which can be expressed as the sum of a mean value function $\hat{\mathbf{z}}(t)$ and a zero mean Gaussian stochastic process whose covariance matrix $\mathbf{P}(t)$ must be determined. Indeed:

$$\begin{cases} \mathbf{z}(t) = \hat{\mathbf{z}}(t) + \mathbf{P}(t)\boldsymbol{\lambda}(t) \\ \hat{\mathbf{z}}(0) = \overline{\mathbf{z}}_0 \\ \mathbf{P}(0) = \mathbf{R}_{c,0} \end{cases} \tag{4.137}$$

Where $\hat{\mathbf{z}}(t)$ is a $\mathbb{R}^n$ vector representing the mean value function of the state vector Gaussian stochastic process and $\mathbf{P}(t)$ is a $\mathbb{R}^{n\times n}$ symmetric matrix representing the covariance matrix of a zero mean Gaussian stochastic process. Since the mean state vector $\hat{\mathbf{z}}(t)$ and the covariance matrix $\mathbf{P}(t)$ are unknown function of time, to compute them two matrix differential equations are required. Taking the time derivative of the state vector and using the adjoint equation yields to:

$$\begin{aligned}
\dot{\mathbf{z}}(t) &= \dot{\hat{\mathbf{z}}}(t) + \dot{\mathbf{P}}(t)\boldsymbol{\lambda}(t) + \mathbf{P}(t)\dot{\boldsymbol{\lambda}}(t) = \\
&= \dot{\hat{\mathbf{z}}}(t) + \dot{\mathbf{P}}(t)\boldsymbol{\lambda}(t) + \mathbf{P}(t)\left(-\mathbf{C}^T\mathbf{R}_{c,v}^{-1}\left(\mathbf{y}(t) - \mathbf{C}\mathbf{z}(t) - \mathbf{D}\mathbf{u}(t)\right) - \mathbf{A}_c^T\boldsymbol{\lambda}(t)\right) = \\
&= \dot{\hat{\mathbf{z}}}(t) + \dot{\mathbf{P}}(t)\boldsymbol{\lambda}(t) + \\
&\quad + \mathbf{P}(t)\left(-\mathbf{C}^T\mathbf{R}_{c,v}^{-1}\left(\mathbf{y}(t) - \mathbf{C}\left(\hat{\mathbf{z}}(t) + \mathbf{P}(t)\boldsymbol{\lambda}(t)\right) - \mathbf{D}\mathbf{u}(t)\right) - \mathbf{A}_c^T\boldsymbol{\lambda}(t)\right) = \\
&= \dot{\hat{\mathbf{z}}}(t) + \dot{\mathbf{P}}(t)\boldsymbol{\lambda}(t) + \\
&\quad + \mathbf{P}(t)\left(-\mathbf{C}^T\mathbf{R}_{c,v}^{-1}\left(\mathbf{y}(t) - \mathbf{C}\hat{\mathbf{z}}(t) - \mathbf{C}\mathbf{P}(t)\boldsymbol{\lambda}(t) - \mathbf{D}\mathbf{u}(t)\right) - \mathbf{A}_c^T\boldsymbol{\lambda}(t)\right) = \\
&= \dot{\hat{\mathbf{z}}}(t) + \dot{\mathbf{P}}(t)\boldsymbol{\lambda}(t) + \\
&\quad + \mathbf{P}(t)\left(-\mathbf{C}^T\mathbf{R}_{c,v}^{-1}\left(\mathbf{y}(t) - \mathbf{C}\hat{\mathbf{z}}(t) - \mathbf{D}\mathbf{u}(t)\right) + \mathbf{C}^T\mathbf{R}_{c,v}^{-1}\mathbf{C}\mathbf{P}(t)\boldsymbol{\lambda}(t) - \mathbf{A}_c^T\boldsymbol{\lambda}(t)\right) = \\
&= \dot{\hat{\mathbf{z}}}(t) + \dot{\mathbf{P}}(t)\boldsymbol{\lambda}(t) - \mathbf{P}(t)\mathbf{C}^T\mathbf{R}_{c,v}^{-1}\left(\mathbf{y}(t) - \mathbf{C}\hat{\mathbf{z}}(t) - \mathbf{D}\mathbf{u}(t)\right) + \\
&\quad + \mathbf{P}(t)\mathbf{C}^T\mathbf{R}_{c,v}^{-1}\mathbf{C}\mathbf{P}(t)\boldsymbol{\lambda}(t) - \mathbf{P}(t)\mathbf{A}_c^T\boldsymbol{\lambda}(t)
\end{aligned}$$

$$(4.138)$$

On the other hand, using the state equation the time derivative of the state vector can be computed as follows:

$$\begin{aligned}\dot{\mathbf{z}}(t) &= \mathbf{A}_c\mathbf{z}(t)+\mathbf{B}_c\mathbf{u}(t)+\mathbf{w}(t)= \\ &= \mathbf{A}_c\left(\hat{\mathbf{z}}(t)+\mathbf{P}(t)\boldsymbol{\lambda}(t)\right)+\mathbf{B}_c\mathbf{u}(t)+\mathbf{R}_{c,w}\boldsymbol{\lambda}(t)= \\ &= \mathbf{A}_c\hat{\mathbf{z}}(t)+\mathbf{A}_c\mathbf{P}(t)\boldsymbol{\lambda}(t)+\mathbf{B}_c\mathbf{u}(t)+\mathbf{R}_{c,w}\boldsymbol{\lambda}(t)\end{aligned} \tag{4.139}$$

Equating the two previous equations yields:

$$\begin{aligned}&\dot{\hat{\mathbf{z}}}(t)-\mathbf{A}_c\hat{\mathbf{z}}(t)-\mathbf{B}_c\mathbf{u}(t)-\mathbf{P}(t)\mathbf{C}^T\mathbf{R}_{c,v}^{-1}\left(\mathbf{y}(t)-\mathbf{C}\hat{\mathbf{z}}(t)-\mathbf{D}\mathbf{u}(t)\right)+ \\ &+\left(\dot{\mathbf{P}}(t)-\mathbf{A}_c\mathbf{P}(t)-\mathbf{P}(t)\mathbf{A}_c^T+\mathbf{P}(t)\mathbf{C}^T\mathbf{R}_{c,v}^{-1}\mathbf{C}\mathbf{P}(t)-\mathbf{R}_{c,w}\right)\boldsymbol{\lambda}(t)=\mathbf{0}\end{aligned} \tag{4.140}$$

Setting the terms between the brackets and the remaining terms independently equal to zero gives:

$$\begin{cases}\dot{\hat{\mathbf{z}}}(t)=\mathbf{A}_c\hat{\mathbf{z}}(t)+\mathbf{B}_c\mathbf{u}(t)+\mathbf{K}_c(t)\left(\mathbf{y}(t)-\hat{\mathbf{y}}(t)\right) \\ \hat{\mathbf{z}}(0)=\overline{\mathbf{z}}_0\end{cases} \tag{4.141}$$

$$\hat{\mathbf{y}}(t)=\mathbf{C}\hat{\mathbf{z}}(t)+\mathbf{D}\mathbf{u}(t) \tag{4.142}$$

$$\begin{cases}\dot{\mathbf{P}}(t)-\mathbf{A}_c\mathbf{P}(t)-\mathbf{P}(t)\mathbf{A}_c^T+\mathbf{P}(t)\mathbf{C}^T\mathbf{R}_{c,v}^{-1}\mathbf{C}\mathbf{P}(t)-\mathbf{R}_{c,w}=\mathbf{O} \\ \mathbf{P}(0)=\mathbf{R}_{c,0}\end{cases} \tag{4.143}$$

Where $\hat{\mathbf{y}}(t)$ is a $\mathbb{R}^m$ vector defining the estimated output measurement vector and $\mathbf{K}_c(t)$ is a $\mathbb{R}^{n\times m}$ matrix representing a continuous-time Kalman gain matrix which is defined as:

$$\mathbf{K}_c(t)=\mathbf{P}(t)\mathbf{C}^T\mathbf{R}_{c,v}^{-1} \tag{4.144}$$

The continuous-time Kalman gain $\mathbf{K}_c(t)$ can be computed once that the covariance matrix $\mathbf{P}(t)$ has been determined from the first-order matrix differential equation which is a continuous-time differential Riccati equation.

Indeed, the Kalman gain matrix $\mathbf{K}_c(t)$ works like a continuous-time observer in the differential equation that describe the evolution of the mean state vector $\hat{\mathbf{z}}(t)$ which can be assumed as an estimation of the state vector $\mathbf{z}(t)$. Moreover, it can be proved $[11]$, $[12]$ that the continuous-time differential Riccati equation reaches quickly an asymptotic solution $\mathbf{P}_\infty$ which can be used to compute a steady-state continuous-time Kalman gain matrix $\mathbf{K}_{c,\infty}$ as:

$$\mathbf{K}_{c,\infty} = \mathbf{P}_\infty \mathbf{C}^T \mathbf{R}_{c,v}^{-1} \tag{4.145}$$

In practice the steady-state Kalman estimator is preferred especially for real-time applications. This is equivalent to minimize an infinite-horizon continuous-time quadratic cost index defined as:

$$J_{c,\infty} = \frac{1}{2}\int_0^\infty \mathbf{w}^T(t)\mathbf{R}_{c,w}^{-1}\mathbf{w}(t) + \mathbf{v}^T(t)\mathbf{R}_{c,v}^{-1}\mathbf{v}(t)dt \tag{4.146}$$

Consider now a discrete-time state-space system affected by disturbances:

$$\mathbf{z}(k+1) = \mathbf{A}\mathbf{z}(k) + \mathbf{B}\mathbf{u}(k) + \mathbf{w}(k) \tag{4.147}$$

$$\mathbf{y}(k) = \mathbf{C}\mathbf{z}(k) + \mathbf{D}\mathbf{u}(k) + \mathbf{v}(k) \tag{4.148}$$

Where $\mathbf{w}(k)$ is a $\mathbb{R}^n$ vector representing the process noise and $\mathbf{v}(k)$ is a $\mathbb{R}^m$ vector representing the measurement noise. Similarly to the continuous-time case, the random disturbances $\mathbf{w}(k)$ and $\mathbf{v}(k)$ are not measurable and are assumed zero mean Gaussian white noises whose stochastic characteristics can be expressed as:

$$E[\mathbf{w}(k)] = \mathbf{0} \quad , \quad \forall k \geq 0 \tag{4.149}$$

$$E[\mathbf{v}(k)] = \mathbf{0} \quad , \quad \forall k \geq 0 \tag{4.150}$$

$$E[\mathbf{w}(h)\mathbf{w}^T(k)] = \mathbf{R}_w\delta_{h,k} \quad , \quad \forall h,k > 0 \tag{4.151}$$

$$E[\mathbf{v}(h)\mathbf{v}^T(k)] = \mathbf{R}_v\delta_{h,k} \quad , \quad \forall h,k \geq 0 \tag{4.152}$$

Where $\mathbf{R}_w$ is a $\mathbb{R}^{n\times n}$ symmetric positive definite matrix defining the process noise covariance matrix and $\mathbf{R}_v$ is a $\mathbb{R}^{m\times m}$ symmetric positive definite matrix defining the measurement noise covariance matrix. The process noise and the measurement noise are assumed mutually uncorrelated:

$$E[\mathbf{w}(h)\mathbf{v}^T(k)] = \mathbf{O} \quad , \quad \forall h,k \geq 0 \tag{4.153}$$

On the other hand, even the initial state $\mathbf{z}_0$ is assumed unknown and it is modelled as a Gaussian distributed random vector whose stochastic characteristics can be expressed as:

$$E[\mathbf{z}_0] = \overline{\mathbf{z}}_0 \tag{4.154}$$

$$E[\left(\mathbf{z}_0 - \overline{\mathbf{z}}_0\right)\left(\mathbf{z}_0 - \overline{\mathbf{z}}_0\right)^T] = \mathbf{R}_0 \tag{4.155}$$

Where $\overline{\mathbf{z}}_0$ is a $\mathbb{R}^n$ vector representing the expected value of initial state and $\mathbf{R}_0$ is a $\mathbb{R}^{n\times n}$ symmetric positive definite matrix representing the covariance matrix of the initial state. The initial state vector is modelled as a random process uncorrelated to the stochastic disturbances:

$$E[\mathbf{z}_0\mathbf{w}^T(k)] = \mathbf{O} \quad , \quad \forall k \geq 0 \tag{4.156}$$

$$E[\mathbf{z}_0\mathbf{v}^T(k)] = \mathbf{O} \quad , \quad \forall k \geq 0 \tag{4.157}$$

The Discrete Kalman Filter algorithm (DKF) is capable to derive a discrete-time observer matrix which minimizes a quadratic performance index $[11]$,

$[12]$. The cost index is a quadratic functional which depends on process noise, measurement noise and on the estimation error of the initial state $[12]$. In the discrete-time case, for a finite-horizon of time $0 \le t \le T$ the quadratic cost index can be defined as:

$$J = \frac{1}{2}\left(\mathbf{z}_0 - \overline{\mathbf{z}}_0\right)^T \mathbf{R}_0^{-1}\left(\mathbf{z}_0 - \overline{\mathbf{z}}_0\right) + \\ + \frac{1}{2}\sum_{k=0}^{N-1}\left(\mathbf{w}^T(k)\mathbf{R}_w^{-1}\mathbf{w}(k) + \mathbf{v}^T(k)\mathbf{R}_v^{-1}\mathbf{v}(k)\right) \tag{4.158}$$

Where the weighting matrices used in the cost function for the process noise and the measurement noise are the inverse of their respective covariance matrices whereas the weighting matrix used for the estimation error of the initial state is the inverse of the covariance matrix of the initial state. Note that the performance index can be seen as an energy index of the disturbances or as an error index. This cost index can be reformulated replacing the measurement noise by using the output equation to yield:

$$\begin{aligned} J &= \frac{1}{2}\left(\mathbf{z}_0 - \overline{\mathbf{z}}_0\right)^T \mathbf{R}_0^{-1}\left(\mathbf{z}_0 - \overline{\mathbf{z}}_0\right) + \\ &+ \frac{1}{2}\sum_{k=0}^{N-1}\left(\mathbf{w}^T(k)\mathbf{R}_w^{-1}\mathbf{w}(k) + \mathbf{v}^T(k)\mathbf{R}_v^{-1}\mathbf{v}(k)\right) = \\ &= \frac{1}{2}\left(\mathbf{z}_0 - \overline{\mathbf{z}}_0\right)^T \mathbf{R}_0^{-1}\left(\mathbf{z}_0 - \overline{\mathbf{z}}_0\right) + \\ &+ \frac{1}{2}\sum_{k=0}^{N-1}\left(\mathbf{w}^T(k)\mathbf{R}_w^{-1}\mathbf{w}(k)\right) + \\ &+ \frac{1}{2}\sum_{k=0}^{N-1}\left(\left(\mathbf{y}(k) - \mathbf{C}\mathbf{z}(k) - \mathbf{D}\mathbf{u}(k)\right)^T \mathbf{R}_v^{-1}\left(\mathbf{y}(k) - \mathbf{C}\mathbf{z}(k) - \mathbf{D}\mathbf{u}(k)\right)\right) \end{aligned} \tag{4.159}$$

In order to find the Kalman state estimator, the cost index J must be minimized and simultaneously the system state equation must be satisfied.

Therefore, the state equation represents a constraint equation for the minimization problem. To solve this problem the method of Lagrange multipliers can be used $[11]$, $[12]$. This method consists in adjoining the state equation to the performance index and subsequently minimize this adjoint cost index J^* using variational calculus technique. Indeed:

$$
\begin{aligned}
&J^* = J + \sum_{k=0}^{N-1}\left(\boldsymbol{\lambda}^T(k+1)\left(\mathbf{z}(k+1) - \mathbf{A}\mathbf{z}(k) - \mathbf{B}\mathbf{u}(k) - \mathbf{w}(k)\right)\right) = \\
&= \frac{1}{2}\left(\mathbf{z}_0 - \overline{\mathbf{z}}_0\right)^T \mathbf{R}_0^{-1}\left(\mathbf{z}_0 - \overline{\mathbf{z}}_0\right) + \\
&+ \frac{1}{2}\sum_{k=0}^{N-1}\left(\mathbf{w}^T(k)\mathbf{R}_w^{-1}\mathbf{w}(k) + \left(\mathbf{y}(k) - \mathbf{C}\mathbf{z}(k) - \mathbf{D}\mathbf{u}(k)\right)^T \mathbf{R}_v^{-1}\left(\mathbf{y}(k) - \mathbf{C}\mathbf{z}(k) - \mathbf{D}\mathbf{u}(k)\right)\right) + \\
&+ \sum_{k=0}^{N-1}\left(\boldsymbol{\lambda}^T(k+1)\left(\mathbf{z}(k+1) - \mathbf{A}\mathbf{z}(k) - \mathbf{B}\mathbf{u}(k) - \mathbf{w}(k)\right)\right)
\end{aligned}
\tag{4.160}
$$

Where $\boldsymbol{\lambda}(k)$ is a $\mathbb{R}^n$ vector containing the Lagrange multipliers. Since the optimal estimator minimizes the adjoint performance index J^*, it is necessary to compute the first variation of this functional and set it equal to zero. Indeed, taking the first variation of the augmented cost function J^* yields:

$$
\begin{aligned}
&\delta J^* = \delta\left(\frac{1}{2}\left(\mathbf{z}_0 - \overline{\mathbf{z}}_0\right)^T \mathbf{R}_0^{-1}\left(\mathbf{z}_0 - \overline{\mathbf{z}}_0\right)\right) + \\
&+\delta\sum_{k=0}^{N-1}\left(\frac{1}{2}\mathbf{w}^T(k)\mathbf{R}_w^{-1}\mathbf{w}(k) + \frac{1}{2}\left(\mathbf{y}(k) - \mathbf{C}\mathbf{z}(k) - \mathbf{D}\mathbf{u}(k)\right)^T \mathbf{R}_v^{-1}\left(\mathbf{y}(k) - \mathbf{C}\mathbf{z}(k) - \mathbf{D}\mathbf{u}(k)\right)\right) + \\
&+\delta\sum_{k=0}^{N-1}\left(\boldsymbol{\lambda}^T(k+1)\left(\mathbf{z}(k+1) - \mathbf{A}\mathbf{z}(k) - \mathbf{B}\mathbf{u}(k) - \mathbf{w}(k)\right)\right) = \\
&= \left(\mathbf{z}_0 - \overline{\mathbf{z}}_0\right)^T \mathbf{R}_0^{-1}\delta\left(\mathbf{z}_0 - \overline{\mathbf{z}}_0\right) + \\
&+\sum_{k=0}^{N-1}\delta\left(\frac{1}{2}\mathbf{w}^T(k)\mathbf{R}_w^{-1}\mathbf{w}(k) + \frac{1}{2}\left(\mathbf{y}(k) - \mathbf{C}\mathbf{z}(k) - \mathbf{D}\mathbf{u}(k)\right)^T \mathbf{R}_v^{-1}\left(\mathbf{y}(k) - \mathbf{C}\mathbf{z}(k) - \mathbf{D}\mathbf{u}(k)\right)\right) + \\
&+\sum_{k=0}^{N-1}\delta\left(\boldsymbol{\lambda}^T(k+1)\left(\mathbf{z}(k+1) - \mathbf{A}\mathbf{z}(k) - \mathbf{B}\mathbf{u}(k) - \mathbf{w}(k)\right)\right) = \\
&= \left(\mathbf{z}_0 - \overline{\mathbf{z}}_0\right)^T \mathbf{R}_0^{-1}\left(\delta\mathbf{z}_0 - \delta\overline{\mathbf{z}}_0\right) + \\
&+\sum_{k=0}^{N-1}\left(\mathbf{w}^T(k)\mathbf{R}_w^{-1}\delta\mathbf{w}(k) + \left(\mathbf{y}(k) - \mathbf{C}\mathbf{z}(k) - \mathbf{D}\mathbf{u}(k)\right)^T \mathbf{R}_v^{-1}\delta\left(\mathbf{y}(k) - \mathbf{C}\mathbf{z}(k) - \mathbf{D}\mathbf{u}(k)\right)\right) + \\
&+\sum_{k=0}^{N-1}\left(\delta\boldsymbol{\lambda}^T(k+1)\left(\mathbf{z}(k+1) - \mathbf{A}\mathbf{z}(k) - \mathbf{B}\mathbf{u}(k) - \mathbf{w}(k)\right)\right) + \\
&+\sum_{k=0}^{N-1}\left(\boldsymbol{\lambda}^T(k+1)\delta\left(\mathbf{z}(k+1) - \mathbf{A}\mathbf{z}(k) - \mathbf{B}\mathbf{u}(k) - \mathbf{w}(k)\right)\right) = \\
&= \left(\mathbf{z}_0 - \overline{\mathbf{z}}_0\right)^T \mathbf{R}_0^{-1}\delta\mathbf{z}_0 + \sum_{k=0}^{N-1}\left(\mathbf{w}^T(k)\mathbf{R}_w^{-1}\delta\mathbf{w}(k) + \left(\mathbf{y}(k) - \mathbf{C}\mathbf{z}(k) - \mathbf{D}\mathbf{u}(k)\right)^T \mathbf{R}_v^{-1}\delta\mathbf{y}(k)\right) + \\
&+\sum_{k=0}^{N-1}\left(-\left(\mathbf{y}(k) - \mathbf{C}\mathbf{z}(k) - \mathbf{D}\mathbf{u}(k)\right)^T \mathbf{R}_v^{-1}\mathbf{C}\delta\mathbf{z}(k)\right) + \\
&+\sum_{k=0}^{N-1}\left(-\left(\mathbf{y}(k) - \mathbf{C}\mathbf{z}(k) - \mathbf{D}\mathbf{u}(k)\right)^T \mathbf{R}_v^{-1}\mathbf{D}\delta\mathbf{u}(k)\right) + \\
&+\sum_{k=0}^{N-1}\left(\left(\mathbf{z}(k+1) - \mathbf{A}\mathbf{z}(k) - \mathbf{B}\mathbf{u}(k) - \mathbf{w}(k)\right)^T \delta\boldsymbol{\lambda}(k+1)\right) + \\
&+\sum_{k=0}^{N-1}\left(\boldsymbol{\lambda}^T(k+1)\delta\mathbf{z}(k+1) - \boldsymbol{\lambda}^T(k+1)\mathbf{A}\delta\mathbf{z}(k) - \boldsymbol{\lambda}^T(k+1)\mathbf{B}\delta\mathbf{u}(k) - \boldsymbol{\lambda}^T(k+1)\delta\mathbf{w}(k)\right)
\end{aligned}
$$

(4.161)

This expression can be further simplified yielding to:

$$
\begin{aligned}
&\left(\mathbf{z}_0-\overline{\mathbf{z}}_0\right)^T \mathbf{R}_0^{-1}\delta\mathbf{z}_0+\sum_{k=0}^{N-1}\left(\mathbf{w}^T(k)\mathbf{R}_w^{-1}\delta\mathbf{w}(k)+\left(\mathbf{y}(k)-\mathbf{C}\mathbf{z}(k)-\mathbf{D}\mathbf{u}(k)\right)^T \mathbf{R}_v^{-1}\delta\mathbf{y}(k)\right)+\\
&+\sum_{k=0}^{N-1}\left(-\left(\mathbf{y}(k)-\mathbf{C}\mathbf{z}(k)-\mathbf{D}\mathbf{u}(k)\right)^T \mathbf{R}_v^{-1}\mathbf{C}\delta\mathbf{z}(k)\right)+\\
&+\sum_{k=0}^{N-1}\left(-\left(\mathbf{y}(k)-\mathbf{C}\mathbf{z}(k)-\mathbf{D}\mathbf{u}(k)\right)^T \mathbf{R}_v^{-1}\mathbf{D}\delta\mathbf{u}(k)\right)+\\
&+\sum_{k=0}^{N-1}\left(\left(\mathbf{z}(k+1)-\mathbf{A}\mathbf{z}(k)-\mathbf{B}\mathbf{u}(k)-\mathbf{w}(k)\right)^T \delta\boldsymbol{\lambda}(k+1)\right)+\\
&+\sum_{k=0}^{N-1}\left(\boldsymbol{\lambda}^T(k+1)\delta\mathbf{z}(k+1)-\boldsymbol{\lambda}^T(k+1)\mathbf{A}\delta\mathbf{z}(k)\right)+\\
&+\sum_{k=0}^{N-1}\left(-\boldsymbol{\lambda}^T(k+1)\mathbf{B}\delta\mathbf{u}(k)-\boldsymbol{\lambda}^T(k+1)\delta\mathbf{w}(k)\right)=\\
&=\left(\mathbf{z}_0-\overline{\mathbf{z}}_0\right)^T \mathbf{R}_0^{-1}\delta\mathbf{z}_0+\sum_{k=0}^{N-1}\left(\mathbf{w}^T(k)\mathbf{R}_w^{-1}\delta\mathbf{w}(k)-\left(\mathbf{y}(k)-\mathbf{C}\mathbf{z}(k)-\mathbf{D}\mathbf{u}(k)\right)^T \mathbf{R}_v^{-1}\mathbf{C}\delta\mathbf{z}(k)\right)+\\
&+\sum_{k=0}^{N-1}\left(\left(\mathbf{z}(k+1)-\mathbf{A}\mathbf{z}(k)-\mathbf{B}\mathbf{u}(k)-\mathbf{w}(k)\right)^T \delta\boldsymbol{\lambda}(k+1)\right)+\\
&+\sum_{k=0}^{N-1}\left(-\boldsymbol{\lambda}^T(k+1)\mathbf{A}\delta\mathbf{z}(k)-\boldsymbol{\lambda}^T(k+1)\delta\mathbf{w}(k)\right)+\boldsymbol{\lambda}^T(N)\delta\mathbf{z}(N)-\boldsymbol{\lambda}^T(0)\delta\mathbf{z}(0)+\\
&+\sum_{k=0}^{N-1}\left(\boldsymbol{\lambda}^T(k)\delta\mathbf{z}(k)\right)=\\
&=\left(\left(\mathbf{z}_0-\overline{\mathbf{z}}_0\right)^T \mathbf{R}_0^{-1}-\boldsymbol{\lambda}^T(0)\right)\delta\mathbf{z}_0+\boldsymbol{\lambda}^T(N)\delta\mathbf{z}(N)+\\
&+\sum_{k=0}^{N-1}\left(\left(-\left(\mathbf{y}(k)-\mathbf{C}\mathbf{z}(k)-\mathbf{D}\mathbf{u}(k)\right)^T \mathbf{R}_v^{-1}\mathbf{C}-\boldsymbol{\lambda}^T(k+1)\mathbf{A}+\boldsymbol{\lambda}^T(k)\right)\delta\mathbf{z}(k)\right)+\\
&+\sum_{k=0}^{N-1}\left(\left(\mathbf{w}^T(k)\mathbf{R}_w^{-1}-\boldsymbol{\lambda}^T(k+1)\right)\delta\mathbf{w}(k)\right)+\\
&+\sum_{k=0}^{N-1}\left(\left(\mathbf{z}(k+1)-\mathbf{A}\mathbf{z}(k)-\mathbf{B}\mathbf{u}(k)-\mathbf{w}(k)\right)^T \delta\boldsymbol{\lambda}(k+1)\right)
\end{aligned}
\tag{4.162}
$$

This formula can be simplified to yield:

$$
\begin{aligned}
&\left(\left(\mathbf{z}_0-\overline{\mathbf{z}}_0\right)^T\mathbf{R}_0^{-1}-\boldsymbol{\lambda}^T(0)\right)\delta\mathbf{z}_0+\boldsymbol{\lambda}^T(N)\delta\mathbf{z}(N)+\\
&+\sum_{k=0}^{N-1}\left(\left(-\left(\mathbf{y}(k)-\mathbf{Cz}(k)-\mathbf{Du}(k)\right)^T\mathbf{R}_v^{-1}\mathbf{C}-\boldsymbol{\lambda}^T(k+1)\mathbf{A}+\boldsymbol{\lambda}^T(k)\right)\delta\mathbf{z}(k)\right)+\\
&+\sum_{k=0}^{N-1}\left(\left(\mathbf{w}^T(k)\mathbf{R}_w^{-1}-\boldsymbol{\lambda}^T(k+1)\right)\delta\mathbf{w}(k)\right)+\\
&+\sum_{k=0}^{N-1}\left(\left(\mathbf{z}(k+1)-\mathbf{Az}(k)-\mathbf{Bu}(k)-\mathbf{w}(k)\right)^T\delta\boldsymbol{\lambda}(k+1)\right)=\\
&=\left(\mathbf{R}_0^{-1}\left(\mathbf{z}_0-\overline{\mathbf{z}}_0\right)-\boldsymbol{\lambda}(0)\right)^T\delta\mathbf{z}_0+\boldsymbol{\lambda}^T(N)\delta\mathbf{z}(N)+\\
&+\sum_{k=0}^{N-1}\left(\left(-\mathbf{C}^T\mathbf{R}_v^{-1}\left(\mathbf{y}(k)-\mathbf{Cz}(k)-\mathbf{Du}(k)\right)-\mathbf{A}^T\boldsymbol{\lambda}(k+1)+\boldsymbol{\lambda}(k)\right)^T\delta\mathbf{z}(k)\right)+\\
&+\sum_{k=0}^{N-1}\left(\left(\mathbf{R}_w^{-1}\mathbf{w}(k)-\boldsymbol{\lambda}(k+1)\right)\delta\mathbf{w}(k)\right)+\\
&+\sum_{k=0}^{N-1}\left(\left(\mathbf{z}(k+1)-\mathbf{Az}(k)-\mathbf{Bu}(k)-\mathbf{w}(k)\right)^T\delta\boldsymbol{\lambda}(k+1)\right)=\\
&=0\quad,\quad\forall\delta\mathbf{z}(k)\quad,\quad\forall\delta\boldsymbol{\lambda}(k)\quad,\quad\forall\delta\mathbf{w}(k)
\end{aligned}
\tag{4.163}
$$

Observing that the variation of the state vector $\delta\mathbf{z}(k)$, the variation of the co-state vector $\delta\boldsymbol{\lambda}(k)$ and the variation of the process noise vector $\delta\mathbf{w}(k)$ are all independent, each quantity in the time integral can be independently taken equal to zero:

$$
\begin{cases}
\mathbf{R}_0^{-1}\left(\mathbf{z}_0-\overline{\mathbf{z}}_0\right)-\boldsymbol{\lambda}(0)=\mathbf{0}\\
\delta\mathbf{z}(N)=\mathbf{0}\\
-\mathbf{C}^T\mathbf{R}_v^{-1}\left(\mathbf{y}(k)-\mathbf{Cz}(k)-\mathbf{Du}(k)\right)-\mathbf{A}^T\boldsymbol{\lambda}(k+1)+\boldsymbol{\lambda}(k)=\mathbf{0}\\
\mathbf{R}_w^{-1}\mathbf{w}(k)-\boldsymbol{\lambda}(k+1)=\mathbf{0}\\
\mathbf{z}(k+1)-\mathbf{Az}(k)-\mathbf{Bu}(k)-\mathbf{w}(k)=\mathbf{0}
\end{cases}
\tag{4.164}
$$

Therefore, the minimization of the adjoint cost index J^* yields a set of two differential equations and one algebraic equation:

$$\begin{cases} \mathbf{z}(k+1) = \mathbf{A}\mathbf{z}(k) + \mathbf{B}\mathbf{u}(k) + \mathbf{w}(k) \\ \mathbf{z}(0) = \mathbf{z}_0 \end{cases} \tag{4.165}$$

$$\begin{cases} \boldsymbol{\lambda}(k) = \mathbf{C}^T\mathbf{R}_v^{-1}\left(\mathbf{y}(k) - \mathbf{C}\mathbf{z}(k) - \mathbf{D}\mathbf{u}(k)\right) + \mathbf{A}^T\boldsymbol{\lambda}(k+1) \\ \boldsymbol{\lambda}(0) = \mathbf{R}_0^{-1}\left(\mathbf{z}_0 - \overline{\mathbf{z}}_0\right) \end{cases} \tag{4.166}$$

$$\mathbf{w}(k) = \mathbf{R}_w\boldsymbol{\lambda}(k+1) \tag{4.167}$$

Where the first difference equation is the state equation, the second difference equation is the adjoint equation and the last algebraic equation is the stationarity equation. Even in this case, there is a method deriving from optimal estimation theory to obtain these set of equations directly defining the Hamiltonian function which depends on the state vector $\mathbf{z}(k)$, the co-state vector $\boldsymbol{\lambda}(k)$ and the process noise vector $\mathbf{w}(k)$ $[11]$, $[12]$. Indeed:

$$\begin{aligned} H(\mathbf{z}(k), \mathbf{w}(k), \boldsymbol{\lambda}(k)) = {} & \frac{1}{2}\mathbf{w}^T(k)\mathbf{R}_w^{-1}\mathbf{w}(k) + \\ & + \frac{1}{2}\left(\mathbf{y}(k) - \mathbf{C}\mathbf{z}(k) - \mathbf{D}\mathbf{u}(k)\right)^T \mathbf{R}_v^{-1}\left(\mathbf{y}(k) - \mathbf{C}\mathbf{z}(k) - \mathbf{D}\mathbf{u}(k)\right) + \\ & - \boldsymbol{\lambda}^T(k+1)\left(\mathbf{A}\mathbf{z}(k) + \mathbf{B}\mathbf{u}(k) + \mathbf{w}(k)\right) \end{aligned} \tag{4.168}$$

The state equation, the co-state equation and the stationarity equation can be obtained from the Hamiltonian function as follows:

$$\begin{cases} \mathbf{z}(k+1) = -\left(\dfrac{\partial H(\mathbf{z}(k), \mathbf{w}(k), \boldsymbol{\lambda}(k))}{\partial \boldsymbol{\lambda}(k+1)}\right)^T \\ \mathbf{z}(0) = \mathbf{z}_0 \end{cases} \tag{4.169}$$

$$\begin{cases} \boldsymbol{\lambda}(k) = -\left(\dfrac{\partial H(\mathbf{z}(k), \mathbf{w}(k), \boldsymbol{\lambda}(k))}{\partial \mathbf{z}(k)} \right)^T \\ \boldsymbol{\lambda}(0) = \mathbf{R}_{c,0}^{-1} \left(\mathbf{z}_0 - \overline{\mathbf{z}}_0 \right) \end{cases} \tag{4.170}$$

$$\left(\frac{\partial H(\mathbf{z}(k), \mathbf{w}(k), \boldsymbol{\lambda}(k))}{\partial \mathbf{w}(k)} \right)^T = \mathbf{0} \tag{4.171}$$

The classical method to solve this problem consist in reducing it to the solution of a discrete-time differential Riccati equation which can be used to compute an observer matrix. Indeed, since the initial state vector $\mathbf{z}_0$ is a random Gaussian process with mean vector $\overline{\mathbf{z}}_0$ and covariance matrix $\mathbf{R}_0$, the system state vector $\mathbf{z}(k)$ turns out to be a Gaussian stochastic process which can be expressed as the sum of a mean value function $\hat{\mathbf{z}}(k)$ and a zero mean Gaussian stochastic process whose covariance matrix $\mathbf{P}(k)$ must be determined. Indeed:

$$\begin{cases} \mathbf{z}(k) = \hat{\mathbf{z}}(k) + \mathbf{P}(k)\boldsymbol{\lambda}(k) \\ \hat{\mathbf{z}}(0) = \overline{\mathbf{z}}_0 \\ \mathbf{P}(0) = \mathbf{R}_0 \end{cases} \tag{4.172}$$

Where $\hat{\mathbf{z}}(k)$ is a $\mathbb{R}^n$ vector representing the mean value function of the state vector Gaussian stochastic process and $\mathbf{P}(k)$ is a $\mathbb{R}^{n \times n}$ symmetric matrix representing the covariance matrix of a zero mean Gaussian stochastic process. Since the mean state vector $\hat{\mathbf{z}}(k)$ and the covariance matrix $\mathbf{P}(k)$ are unknown function of time, to compute them are necessary two matrix difference equations. To derive a discrete-time differential Riccati equation to compute the covariance matrix $\mathbf{P}(k)$, substitute the formulation of the state vector $\mathbf{z}(k)$ in the adjoint equation:

$$\begin{aligned}
\boldsymbol{\lambda}(k) &= \mathbf{C}^T\mathbf{R}_v^{-1}\left(\mathbf{y}(k)-\mathbf{C}\mathbf{z}(k)-\mathbf{D}\mathbf{u}(k)\right)+\mathbf{A}^T\boldsymbol{\lambda}(k+1)= \\
&= \mathbf{C}^T\mathbf{R}_v^{-1}\left(\mathbf{y}(k)-\mathbf{C}\left(\hat{\mathbf{z}}(k)+\mathbf{P}(k)\boldsymbol{\lambda}(k)\right)-\mathbf{D}\mathbf{u}(k)\right)+\mathbf{A}^T\boldsymbol{\lambda}(k+1)= \\
&= \mathbf{C}^T\mathbf{R}_v^{-1}\left(\mathbf{y}(k)-\mathbf{C}\hat{\mathbf{z}}(k)-\mathbf{D}\mathbf{u}(k)\right)-\mathbf{C}^T\mathbf{R}_v^{-1}\mathbf{C}\mathbf{P}(k)\boldsymbol{\lambda}(k)+\mathbf{A}^T\boldsymbol{\lambda}(k+1)
\end{aligned} \tag{4.173}$$

Rearranging the common factors yields:

$$\left(\mathbf{P}^{-1}(k)+\mathbf{C}^T\mathbf{R}_v^{-1}\mathbf{C}\right)\mathbf{P}(k)\boldsymbol{\lambda}(k)=\mathbf{C}^T\mathbf{R}_v^{-1}\left(\mathbf{y}(k)-\mathbf{C}\hat{\mathbf{z}}(k)-\mathbf{D}\mathbf{u}(k)\right)+\mathbf{A}^T\boldsymbol{\lambda}(k+1) \tag{4.174}$$

Consequently, the product $\mathbf{P}(k)\boldsymbol{\lambda}(k)$ can be computed as:

$$\begin{aligned}
\mathbf{P}(k)\boldsymbol{\lambda}(k) &= \left(\mathbf{P}^{-1}(k)+\mathbf{C}^T\mathbf{R}_v^{-1}\mathbf{C}\right)^{-1}\mathbf{C}^T\mathbf{R}_v^{-1}\left(\mathbf{y}(k)-\mathbf{C}\hat{\mathbf{z}}(k)-\mathbf{D}\mathbf{u}(k)\right) \\
&\quad +\left(\mathbf{P}^{-1}(k)+\mathbf{C}^T\mathbf{R}_v^{-1}\mathbf{C}\right)^{-1}\mathbf{A}^T\boldsymbol{\lambda}(k+1)= \\
&= \mathbf{P}'(k)\mathbf{C}^T\mathbf{R}_v^{-1}\left(\mathbf{y}(k)-\mathbf{C}\hat{\mathbf{z}}(k)-\mathbf{D}\mathbf{u}(k)\right)+\mathbf{P}'(k)\mathbf{A}^T\boldsymbol{\lambda}(k+1)
\end{aligned} \tag{4.175}$$

Where $\mathbf{P}'(k)$ is a $\mathbb{R}^{n\times n}$ symmetric matrix defined as:

$$\mathbf{P}'(k)=\left(\mathbf{P}^{-1}(k)+\mathbf{C}^T\mathbf{R}_v^{-1}\mathbf{C}\right)^{-1} \tag{4.176}$$

On the other hand, the state equation can be reformulated by using the previous equation to yield:

$$\begin{aligned}
\mathbf{z}(k+1) &= \mathbf{Az}(k) + \mathbf{Bu}(k) + \mathbf{w}(k) = \\
&= \mathbf{A}\left(\hat{\mathbf{z}}(k) + \mathbf{P}(k)\boldsymbol{\lambda}(k)\right) + \mathbf{Bu}(k) + \mathbf{R}_w\boldsymbol{\lambda}(k+1) = \\
&= \mathbf{A}\left(\hat{\mathbf{z}}(k) + \mathbf{P}'(k)\mathbf{C}^T\mathbf{R}_v^{-1}\left(\mathbf{y}(k) - \mathbf{C}\hat{\mathbf{z}}(k) - \mathbf{Du}(k)\right) + \mathbf{P}'(k)\mathbf{A}^T\boldsymbol{\lambda}(k+1)\right) \\
&\quad + \mathbf{Bu}(k) + \mathbf{R}_w\boldsymbol{\lambda}(k+1) = \\
&= \mathbf{A}\hat{\mathbf{z}}(k) + \mathbf{AP}'(k)\mathbf{C}^T\mathbf{R}_v^{-1}\left(\mathbf{y}(k) - \mathbf{C}\hat{\mathbf{z}}(k) - \mathbf{Du}(k)\right) + \mathbf{AP}'(k)\mathbf{A}^T\boldsymbol{\lambda}(k+1) + \\
&\quad + \mathbf{Bu}(k) + \mathbf{R}_w\boldsymbol{\lambda}(k+1)
\end{aligned} \tag{4.177}$$

The state vector can be also expressed as the sum of the following terms:

$$\mathbf{z}(k+1) = \hat{\mathbf{z}}(k+1) + \mathbf{P}(k+1)\boldsymbol{\lambda}(k+1) \tag{4.178}$$

Equating the last two equations yields:

$$\begin{aligned}
&\hat{\mathbf{z}}(k+1) - \mathbf{A}\hat{\mathbf{z}}(k) - \mathbf{Bu}(k) - \mathbf{AP}'(k)\mathbf{C}^T\mathbf{R}_v^{-1}\left(\mathbf{y}(k) - \mathbf{C}\hat{\mathbf{z}}(k) - \mathbf{Du}(k)\right) + \\
&+ \left(\mathbf{P}(k+1) - \mathbf{AP}'(k)\mathbf{A}^T - \mathbf{R}_w\right)\boldsymbol{\lambda}(k+1) = \mathbf{0}
\end{aligned} \tag{4.179}$$

Setting the terms between the brackets and the remaining terms independently equal to zero gives:

$$\begin{cases} \hat{\mathbf{z}}(k+1) = \mathbf{A}\hat{\mathbf{z}}(k) + \mathbf{Bu}(k) + \mathbf{K}(k)\left(\mathbf{y}(k) - \hat{\mathbf{y}}(k)\right) \\ \hat{\mathbf{z}}(0) = \overline{\mathbf{z}}_0 \end{cases} \tag{4.180}$$

$$\hat{\mathbf{y}}(k) = \mathbf{C}\hat{\mathbf{z}}(k) + \mathbf{Du}(k) \tag{4.181}$$

$$\begin{cases} \mathbf{P}(k+1) = \mathbf{AP}'(k)\mathbf{A}^T + \mathbf{R}_w \\ \mathbf{P}(0) = \mathbf{R}_0 \end{cases} \tag{4.182}$$

Where $\hat{\mathbf{y}}(k)$ is a $\mathbb{R}^m$ vector defining the estimated output measurement vector and $\mathbf{K}(k)$ is a $\mathbb{R}^{n\times m}$ matrix representing a discrete-time Kalman gain matrix which is defined as:

$$\begin{aligned}\mathbf{K}(k) &= \mathbf{A}\mathbf{P}'(k)\mathbf{C}^T\mathbf{R}_v^{-1} = \\ &= \mathbf{A}\mathbf{P}(k)\mathbf{C}^T\left(\mathbf{R}_v + \mathbf{C}\mathbf{P}(k)\mathbf{C}^T\right)^{-1}\end{aligned} \tag{4.183}$$

The discrete-time Kalman gain $\mathbf{K}(k)$ can be computed once that the covariance matrix $\mathbf{P}(k)$ has been determined from the first-order matrix difference equation which is a discrete-time difference Riccati equation. Indeed, the Kalman gain matrix $\mathbf{K}(k)$ works like a discrete-time observer in the difference equation that describe the evolution of the mean state vector $\hat{\mathbf{z}}(k)$ which can be assumed as an estimation of the state vector $\mathbf{z}(k)$. Moreover, it can be proved [11], [12] that the discrete-time difference Riccati equation reaches quickly an asymptotic solution $\mathbf{P}_\infty$ which can be used to compute a steady-state discrete-time Kalman gain matrix $\mathbf{K}_\infty$ as follows:

$$\begin{aligned}\mathbf{K}_\infty &= \mathbf{A}\mathbf{P}'_\infty\mathbf{C}^T\mathbf{R}_v^{-1} = \\ &= \mathbf{A}\mathbf{P}_\infty\mathbf{C}^T\left(\mathbf{R}_v + \mathbf{C}\mathbf{P}_\infty\mathbf{C}^T\right)^{-1}\end{aligned} \tag{4.184}$$

This is equivalent to minimize an infinite-horizon discrete-time quadratic cost index defined as:

$$J_\infty = \frac{1}{2}\sum_{k=0}^{\infty}\left(\mathbf{w}^T(k)\mathbf{R}_w^{-1}\mathbf{w}(k) + \mathbf{v}^T(k)\mathbf{R}_v^{-1}\mathbf{v}(k)\right) \tag{4.185}$$

In practice the steady-state Kalman estimator is preferred especially for real-time applications.

4.4. LINEAR QUADRATIC GAUSSIAN CONTROLLER (LQG)

Consider the regulation problem for a linear dynamical system disturbed by white Gaussian noise in the presence of incomplete state measurements. This problem can be solved using the Linear Quadratic Gaussian controller algorithm (LQG) for both continuous-time and discrete-time state-space systems $[9]$, $[10]$. This method combines the two logical structures of the optimal deterministic state regulator with the optimal stochastic state estimator $[11]$, $[12]$. Indeed, it can be proved that a Linear Quadratic Regulator (LQR) and a Kalman Filter (KF) can be designed independently and then combined together to derive a Linear Quadratic Gaussinan controller (LQG) $[13]$, $[14]$. This important result is known as separation theorem or certainty-equivalence principle. Hence, consider a continuous-time state-space system affected by disturbances:

$$\dot{\mathbf{z}}(t) = \mathbf{A}_c\mathbf{z}(t) + \mathbf{B}_c\mathbf{u}(t) + \mathbf{w}(t) \tag{4.186}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{z}(t) + \mathbf{D}\mathbf{u}(t) + \mathbf{v}(t) \tag{4.187}$$

Where $\mathbf{w}(t)$ is a $\mathbb{R}^n$ vector representing the process noise and $\mathbf{v}(t)$ is a $\mathbb{R}^m$ vector representing the measurement noise. The random disturbances $\mathbf{w}(t)$ and $\mathbf{v}(t)$ are not measurable and are assumed zero mean Gaussian white noises whose stochastic characteristics are:

$$E[\mathbf{w}(t)] = \mathbf{0} \quad , \quad \forall t \geq 0 \tag{4.188}$$

$$E[\mathbf{v}(t)] = \mathbf{0} \quad , \quad \forall t \geq 0 \tag{4.189}$$

$$E[\mathbf{w}(t)\mathbf{w}^T(\tau)] = \mathbf{R}_{c,w}\delta(t-\tau) \quad , \quad \forall t, \tau \geq 0 \tag{4.190}$$

$$E[\mathbf{v}(t)\mathbf{v}^T(\tau)] = \mathbf{R}_{c,v}\delta(t-\tau) \quad , \quad \forall t,\tau \geq 0 \tag{4.191}$$

Where $\mathbf{R}_{c,w}$ is a $\mathbb{R}^{n\times n}$ symmetric positive definite matrix defining the process noise covariance matrix and $\mathbf{R}_{c,v}$ is a $\mathbb{R}^{m\times m}$ symmetric positive definite matrix defining the measurement noise covariance matrix. In addition, the process noise and the measurement noise are assumed mutually uncorrelated:

$$E[\mathbf{w}(t)\mathbf{v}^T(\tau)] = \mathbf{O} \quad , \quad \forall t,\tau \geq 0 \tag{4.192}$$

On the other hand, even the initial state $\mathbf{z}_0$ is assumed unknown and it is modelled as a Gaussian distributed random vector whose stochastic characteristics are:

$$E[\mathbf{z}_0] = \overline{\mathbf{z}}_0 \tag{4.193}$$

$$E[\left(\mathbf{z}_0 - \overline{\mathbf{z}}_0\right)\left(\mathbf{z}_0 - \overline{\mathbf{z}}_0\right)^T] = \mathbf{R}_{c,0} \tag{4.194}$$

Where $\overline{\mathbf{z}}_0$ is a $\mathbb{R}^n$ vector representing the expected value of initial state and $\mathbf{R}_{c,0}$ is a $\mathbb{R}^{n\times n}$ symmetric positive definite matrix representing the covariance matrix of the initial state. The initial state vector is modelled as a random process uncorrelated to the stochastic disturbances:

$$E[\mathbf{z}_0\mathbf{w}^T(t)] = \mathbf{O} \quad , \quad \forall t \geq 0 \tag{4.195}$$

$$E[\mathbf{z}_0\mathbf{v}^T(t)] = \mathbf{O} \quad , \quad \forall t \geq 0 \tag{4.196}$$

For the continuous-time representation, the Linear Quadratic Gaussian regulator method (LQG) is able to compute a feedback matrix in such a way to minimize a quadratic cost index [13], [14]. In the continuous-time case and without considering constraints on the terminal state, for a finite-horizon of time $0 \le t \le T$ the quadratic cost index can be defined as:

$$J_c = E[\frac{1}{2}\mathbf{z}^T(T)\mathbf{Q}_{c,T}\mathbf{z}(T) + \frac{1}{2}\int_0^T \mathbf{z}^T(t)\mathbf{Q}_{c,z}\mathbf{z}(t) + \mathbf{u}^T(t)\mathbf{Q}_{c,u}\mathbf{u}(t)dt] \tag{4.197}$$

Where $\mathbf{Q}_{c,T}$ and $\mathbf{Q}_{c,z}$ are $\mathbb{R}^{n\times n}$ matrices which represent the terminal cost matrix and the weight of the state vector whereas $\mathbf{Q}_{c,u}$ is a $\mathbb{R}^{r\times r}$ matrix representing the weight of the input vector. Note that the matrix $\mathbf{Q}_{c,T}$ is a positive semidefinite matrix which penalizes the deviation of the final state from the desired set point whereas the matrices $\mathbf{Q}_{c,z}$ and $\mathbf{Q}_{c,u}$ are respectively a positive semidefinite matrix and a positive definite matrix which penalize respectively the instantaneous deviation of the state form the reference configuration and the instantaneous control effort. Therefore, performing the minimization procedure the control input can be expressed as:

$$\mathbf{u}(t) = \mathbf{F}_c(t)\hat{\mathbf{z}}(t) \tag{4.198}$$

Where $\mathbf{F}_c(t)$ is a $\mathbb{R}^{r\times n}$ feedback matrix function which derive from the computation of a deterministic state regulator and $\hat{\mathbf{z}}(t)$ is an $\mathbb{R}^n$ observed vector which derive from the computation of a stochastic state estimator [13], [14]. Indeed, the feedback matrix $\mathbf{F}_c(t)$ can be computed minimizing the following deterministic continuous-time cost function:

$$J_c = \frac{1}{2}\mathbf{z}^T(T)\mathbf{Q}_{c,T}\mathbf{z}(T) + \frac{1}{2}\int_0^T \mathbf{z}^T(t)\mathbf{Q}_{c,z}\mathbf{z}(t) + \mathbf{u}^T(t)\mathbf{Q}_{c,u}\mathbf{u}(t)dt \tag{4.199}$$

The minimization of the cost function yields to the following continuous-time differential Riccati equation:

$$\begin{cases} \dot{\mathbf{S}}(t) + \mathbf{S}(t)\mathbf{A}_c + \mathbf{A}_c^T\mathbf{S}(t) - \mathbf{S}(t)\mathbf{B}_c\mathbf{Q}_{c,u}^{-1}\mathbf{B}_c^T\mathbf{S}(t) + \mathbf{Q}_{c,z} = \mathbf{O} \\ \mathbf{S}(T) = \mathbf{Q}_{c,T} \end{cases} \quad (4.200)$$

Where $\mathbf{S}(t)$ is a $\mathbb{R}^{n\times n}$ symmetric matrix function which is necessary to compute the continuous-time feedback matrix function $\mathbf{F}_c(t)$ as:

$$\mathbf{F}_c(t) = -\mathbf{Q}_{c,u}^{-1}\mathbf{B}_c^T\mathbf{S}(t) \quad (4.201)$$

It can be proved [9], [10] that this continuous-time differential Riccati equation reaches quickly an asymptotic solution $\mathbf{S}_\infty$ which can be used to compute a steady-state continuous-time feedback matrix $\mathbf{F}_{c,\infty}$ as:

$$\mathbf{F}_{c,\infty} = -\mathbf{Q}_{c,u}^{-1}\mathbf{B}_c^T\mathbf{S}_\infty \quad (4.202)$$

On the other hand, the estimated state $\hat{\mathbf{z}}(t)$ can be computed from the following observer equations:

$$\begin{cases} \dot{\hat{\mathbf{z}}}(t) = \mathbf{A}_c\hat{\mathbf{z}}(t) + \mathbf{B}_c\mathbf{u}(t) + \mathbf{K}_c(t)\left(\mathbf{y}(t) - \hat{\mathbf{y}}(t)\right) \\ \hat{\mathbf{z}}(0) = \overline{\mathbf{z}}_0 \end{cases} \quad (4.203)$$

$$\hat{\mathbf{y}}(t) = \mathbf{C}\hat{\mathbf{z}}(t) + \mathbf{D}\mathbf{u}(t) \quad (4.204)$$

Where the Kalman filter $\mathbf{K}_c(t)$ can be computed minimizing the following stochastic continuous-time error function:

$$\begin{aligned} J_c = {} & \frac{1}{2}\left(\mathbf{z}_0 - \overline{\mathbf{z}}_0\right)^T \mathbf{R}_{c,0}^{-1}\left(\mathbf{z}_0 - \overline{\mathbf{z}}_0\right) + \\ & + \frac{1}{2}\int_0^T \mathbf{w}^T(t)\mathbf{R}_{c,w}^{-1}\mathbf{w}(t) + \mathbf{v}^T(t)\mathbf{R}_{c,v}^{-1}\mathbf{v}(t)dt \end{aligned} \quad (4.205)$$

The minimization of the error function yields to the following continuous-time differential Riccati equation:

$$\begin{cases} \dot{\mathbf{P}}(t) - \mathbf{A}_c \mathbf{P}(t) - \mathbf{P}(t)\mathbf{A}_c^T + \mathbf{P}(t)\mathbf{C}^T \mathbf{R}_{c,v}^{-1} \mathbf{C}\mathbf{P}(t) - \mathbf{R}_{c,w} = \mathbf{O} \\ \mathbf{P}(0) = \mathbf{R}_{c,0} \end{cases} \tag{4.206}$$

Where $\mathbf{P}(t)$ is a $\mathbb{R}^{n\times n}$ symmetric matrix function which is necessary to compute the continuous-time Kalman matrix function $\mathbf{K}_c(t)$ as:

$$\mathbf{K}_c(t) = \mathbf{P}(t)\mathbf{C}^T \mathbf{R}_{c,v}^{-1} \tag{4.207}$$

It can be proved $[11]$, $[12]$ that this continuous-time differential Riccati equation reaches quickly an asymptotic solution $\mathbf{P}_\infty$ which can be used to compute a steady-state continuous-time Kalman gain matrix $\mathbf{K}_{c,\infty}$ as:

$$\mathbf{K}_{c,\infty} = \mathbf{P}_\infty \mathbf{C}^T \mathbf{R}_{c,v}^{-1} \tag{4.208}$$

Consequently, the overall model of the controlled system combined with the state observer can be expressed as follows:

$$\dot{\bar{\mathbf{z}}}(t) = \bar{\mathbf{A}}_c(t)\bar{\mathbf{z}}(t) \tag{4.209}$$

Where $\bar{\mathbf{z}}(t)$ is a $\mathbb{R}^{2n}$ vector representing the global state vector and $\bar{\mathbf{A}}_c(t)$ is a $\mathbb{R}^{2n\times 2n}$ matrix representing the global state matrix which are respectively defined as:

$$\bar{\mathbf{z}}(t) = \begin{bmatrix} \mathbf{z}(t) \\ \hat{\mathbf{z}}(t) \end{bmatrix} \tag{4.210}$$

$$\bar{\mathbf{A}}_c(t) = \begin{bmatrix} \mathbf{A}_c & \mathbf{B}_c\mathbf{F}_c(t) \\ \mathbf{K}_c(t)\mathbf{C} & \mathbf{A}_c - \mathbf{K}_c(t)\mathbf{C} + \mathbf{B}_c\mathbf{F}_c(t) \end{bmatrix} \tag{4.211}$$

The separation theorem or certainty-equivalence principle states that the eigenvalue set of the global state matrix $\bar{\mathbf{A}}_c(t)$ is the union of the eigenvalue set of the closed-loop control state matrix $\mathbf{A}_c + \mathbf{B}_c\mathbf{F}_c(t)$ and the eigenvalue set of the closed-loop observer state matrix $\mathbf{A}_c - \mathbf{K}_c(t)\mathbf{C}$ $[13]$, $[14]$. Therefore, the state controller and the state observer can be designed independently making stable the closed-loop control state matrix $\mathbf{A}_c + \mathbf{B}_c\mathbf{F}_c(t)$ and the closed-loop observer state matrix $\mathbf{A}_c - \mathbf{K}_c(t)\mathbf{C}$ and consequently the global state matrix $\bar{\mathbf{A}}_c(t)$ is stable. This result can be developed in a straightforward manner considering a slightly different global state vector defined as $[3]$:

$$\tilde{\mathbf{z}}(t) = \begin{bmatrix} \mathbf{z}(t) \\ \mathbf{e}(t) \end{bmatrix} \tag{4.212}$$

Where $\mathbf{e}(t)$ is the error between the system state vector $\mathbf{z}(t)$ and the estimated state vector $\hat{\mathbf{z}}(t)$. Indeed, in terms of this global state vector $\tilde{\mathbf{z}}(t)$ the global state matrix $\tilde{\mathbf{A}}_c(t)$ becomes:

$$\tilde{\mathbf{A}}_c(t) = \begin{bmatrix} \mathbf{A}_c + \mathbf{B}_c\mathbf{F}_c(t) & -\mathbf{B}_c\mathbf{F}_c(t) \\ \mathbf{O} & \mathbf{A}_c - \mathbf{K}_c(t)\mathbf{C} \end{bmatrix} \tag{4.213}$$

Indeed, the eigenvalues of the block triangular state matrix $\tilde{\mathbf{A}}_c(t)$ are union of the eigenvalues of its diagonal block matrices $\mathbf{A}_c + \mathbf{B}_c\mathbf{F}_c(t)$ and $\mathbf{A}_c - \mathbf{K}_c(t)\mathbf{C}$. On the other hand, Consider now a discrete-time state-space system affected by disturbances:

$$\mathbf{z}(k+1) = \mathbf{A}\mathbf{z}(k) + \mathbf{B}\mathbf{u}(k) + \mathbf{w}(k) \tag{4.214}$$

$$\mathbf{y}(k) = \mathbf{C}\mathbf{z}(k) + \mathbf{D}\mathbf{u}(k) + \mathbf{v}(k) \tag{4.215}$$

Where $\mathbf{w}(k)$ is a $\mathbb{R}^n$ vector representing the process noise and $\mathbf{v}(k)$ is a $\mathbb{R}^m$ vector representing the measurement noise. Similarly to the continuous-time case, the random disturbances $\mathbf{w}(k)$ and $\mathbf{v}(k)$ are not measurable and are assumed zero mean Gaussian white noises whose stochastic characteristics can be expressed as:

$$E[\mathbf{w}(k)] = \mathbf{0} \quad , \quad \forall k \geq 0 \tag{4.216}$$

$$E[\mathbf{v}(k)] = \mathbf{0} \quad , \quad \forall k \geq 0 \tag{4.217}$$

$$E[\mathbf{w}(h)\mathbf{w}^T(k)] = \mathbf{R}_w \delta_{h,k} \quad , \quad \forall h, k > 0 \tag{4.218}$$

$$E[\mathbf{v}(h)\mathbf{v}^T(k)] = \mathbf{R}_v \delta_{h,k} \quad , \quad \forall h, k \geq 0 \tag{4.219}$$

Where $\mathbf{R}_w$ is a $\mathbb{R}^{n\times n}$ symmetric positive definite matrix defining the process noise covariance matrix and $\mathbf{R}_v$ is a $\mathbb{R}^{m\times m}$ symmetric positive definite matrix defining the measurement noise covariance matrix. The process noise and the measurement noise are assumed mutually uncorrelated:

$$E[\mathbf{w}(h)\mathbf{v}^T(k)] = \mathbf{O} \quad , \quad \forall h, k \geq 0 \tag{4.220}$$

On the other hand, even the initial state $\mathbf{z}_0$ is assumed unknown and it is modelled as a Gaussian distributed random vector whose stochastic characteristics can be expressed as:

$$E[\mathbf{z}_0] = \overline{\mathbf{z}}_0 \tag{4.221}$$

$$E[(\mathbf{z}_0 - \bar{\mathbf{z}}_0)(\mathbf{z}_0 - \bar{\mathbf{z}}_0)^T] = \mathbf{R}_0 \tag{4.222}$$

Where $\bar{\mathbf{z}}_0$ is a $\mathbb{R}^n$ vector representing the expected value of initial state and $\mathbf{R}_0$ is a $\mathbb{R}^{n\times n}$ symmetric positive definite matrix representing the covariance matrix of the initial state. The initial state vector is modelled as a random process uncorrelated to the stochastic disturbances:

$$E[\mathbf{z}_0 \mathbf{w}^T(k)] = \mathbf{O} \quad , \quad \forall k \geq 0 \tag{4.223}$$

$$E[\mathbf{z}_0 \mathbf{v}^T(k)] = \mathbf{O} \quad , \quad \forall k \geq 0 \tag{4.224}$$

For the continuous-time representation, the Linear Quadratic Gaussian regulator method (LQG) is able to compute a feedback matrix in such a way to minimize a quadratic cost index $[13]$, $[14]$. In the discrete-time case and without considering constraints on the terminal state, for a finite-horizon of time $0 \leq t \leq T$ the quadratic cost index can be defined as:

$$J = E[\frac{1}{2}\mathbf{z}^T(N)\mathbf{Q}_T\mathbf{z}(N) + \frac{1}{2}\sum_{k=0}^{N-1}\left(\mathbf{z}^T(k)\mathbf{Q}_z\mathbf{z}(k) + \mathbf{u}^T(k)\mathbf{Q}_u\mathbf{u}(k)\right)] \tag{4.225}$$

Where $\mathbf{Q}_T$ and $\mathbf{Q}_z$ are $\mathbb{R}^{n\times n}$ matrices which represent the terminal cost matrix and the weight of the state vector whereas $\mathbf{Q}_u$ is a $\mathbb{R}^{r\times r}$ matrix representing the weight of the input vector. Note that the matrix $\mathbf{Q}_T$ is a positive semidefinite matrix which penalizes the deviation of the final state from the desired set point whereas the matrices $\mathbf{Q}_z$ and $\mathbf{Q}_u$ are respectively a positive semidefinite matrix and a positive definite matrix which penalize respectively the instantaneous deviation of the state form the reference configuration and the instantaneous control effort. Therefore, performing the minimization procedure the control input can be expressed as:

$$\mathbf{u}(k) = \mathbf{F}(k)\hat{\mathbf{z}}(k) \tag{4.226}$$

Where $\mathbf{F}(k)$ is a $\mathbb{R}^{r \times n}$ feedback matrix function which derive from the computation of a deterministic state regulator and $\hat{\mathbf{z}}(k)$ is an $\mathbb{R}^{n}$ observed vector which derive from the computation of a stochastic state estimator. Indeed, the feedback matrix $\mathbf{F}(k)$ can be computed minimizing the following deterministic discrete-time cost function:

$$J = \frac{1}{2}\mathbf{z}^T(N)\mathbf{Q}_T\mathbf{z}(N) + \frac{1}{2}\sum_{k=0}^{N-1}\left(\mathbf{z}^T(k)\mathbf{Q}_z\mathbf{z}(k) + \mathbf{u}^T(k)\mathbf{Q}_u\mathbf{u}(k)\right) \tag{4.227}$$

The minimization of the cost function yields to the following discete-time difference Riccati equation:

$$\begin{cases} \mathbf{S}(k) = \mathbf{A}^T\mathbf{S}'(k+1)\mathbf{A} + \mathbf{Q}_z \\ \mathbf{S}(N) = \mathbf{Q}_T \end{cases} \tag{4.228}$$

Where $\mathbf{S}(k)$ is a $\mathbb{R}^{n \times n}$ symmetric matrix function which is necessary to compute the discrete-time feedback matrix function $\mathbf{F}(k)$ as:

$$\begin{aligned} \mathbf{F}(k) &= -\mathbf{Q}_u^{-1}\mathbf{B}^T\mathbf{S}'(k+1)\mathbf{A} = \\ &= -\left(\mathbf{Q}_u + \mathbf{B}^T\mathbf{S}(k+1)\mathbf{B}\right)^{-1}\mathbf{B}^T\mathbf{S}(k+1)\mathbf{A} \end{aligned} \tag{4.229}$$

It can be proved $[9]$, $[10]$ that this discrete-time difference Riccati equation reaches quickly an asymptotic solution $\mathbf{S}_\infty$ which can be used to compute a steady-state discrete-time feedback matrix $\mathbf{F}_\infty$ as:

$$\begin{aligned} \mathbf{F}_\infty &= -\mathbf{Q}_u^{-1}\mathbf{B}^T\mathbf{S}'_\infty\mathbf{A} = \\ &= -\left(\mathbf{Q}_u + \mathbf{B}^T\mathbf{S}_\infty\mathbf{B}\right)^{-1}\mathbf{B}^T\mathbf{S}_\infty\mathbf{A} \end{aligned} \tag{4.230}$$

On the other hand, the estimated state $\hat{\mathbf{z}}(k)$ can be computed from the following observer equations:

$$\begin{cases} \hat{\mathbf{z}}(k+1) = \mathbf{A}\hat{\mathbf{z}}(k) + \mathbf{Bu}(k) + \mathbf{K}(k)\left(\mathbf{y}(k) - \hat{\mathbf{y}}(k)\right) \\ \hat{\mathbf{z}}(0) = \overline{\mathbf{z}}_0 \end{cases} \tag{4.231}$$

$$\hat{\mathbf{y}}(k) = \mathbf{C}\hat{\mathbf{z}}(k) + \mathbf{Du}(k) \tag{4.232}$$

Where the Kalman filter $\mathbf{K}(k)$ can be computed minimizing the following stochastic discrete-time error function:

$$\begin{aligned} J = {} & \frac{1}{2}\left(\mathbf{z}_0 - \overline{\mathbf{z}}_0\right)^T \mathbf{R}_0^{-1}\left(\mathbf{z}_0 - \overline{\mathbf{z}}_0\right) + \\ & + \frac{1}{2}\sum_{k=0}^{N-1}\left(\mathbf{w}^T(k)\mathbf{R}_w^{-1}\mathbf{w}(k) + \mathbf{v}^T(k)\mathbf{R}_v^{-1}\mathbf{v}(k)\right) \end{aligned} \tag{4.233}$$

The minimization of the error function yields to the following discrete-time difference Riccati equation:

$$\begin{cases} \mathbf{P}(k+1) = \mathbf{AP}'(k)\mathbf{A}^T + \mathbf{R}_w \\ \mathbf{P}(0) = \mathbf{R}_0 \end{cases} \tag{4.234}$$

Where $\mathbf{P}(k)$ is a $\mathbb{R}^{n\times n}$ symmetric matrix function which is necessary to compute the discrete-time Kalman matrix function $\mathbf{K}(k)$ as:

$$\begin{aligned} \mathbf{K}(k) &= \mathbf{AP}'(k)\mathbf{C}^T\mathbf{R}_v^{-1} = \\ &= \mathbf{AP}(k)\mathbf{C}^T\left(\mathbf{R}_v + \mathbf{CP}(k)\mathbf{C}^T\right)^{-1} \end{aligned} \tag{4.235}$$

It can be proved [11], [12] that this discrete-time difference Riccati equation reaches quickly an asymptotic solution $\mathbf{P}_\infty$ which can be used to compute a steady-state discrete-time Kalman gain matrix $\mathbf{K}_\infty$ as:

$$\begin{aligned}\mathbf{K}_\infty &= \mathbf{A}\mathbf{P}'_\infty\mathbf{C}^T\mathbf{R}_v^{-1} = \\ &= \mathbf{A}\mathbf{P}_\infty\mathbf{C}^T\left(\mathbf{R}_v + \mathbf{C}\mathbf{P}_\infty\mathbf{C}^T\right)^{-1}\end{aligned} \tag{4.236}$$

Consequently, the overall model of the controlled system combined with the state observer can be expressed as follows:

$$\bar{\mathbf{z}}(k+1) = \bar{\mathbf{A}}(k)\bar{\mathbf{z}}(k) \tag{4.237}$$

Where $\bar{\mathbf{z}}(k)$ is a $\mathbb{R}^{2n}$ vector representing the global state vector and $\bar{\mathbf{A}}(k)$ is a $\mathbb{R}^{2n\times 2n}$ matrix representing the global state matrix which are respectively defined as:

$$\bar{\mathbf{z}}(k) = \begin{bmatrix}\mathbf{z}(k) \\ \hat{\mathbf{z}}(k)\end{bmatrix} \tag{4.238}$$

$$\bar{\mathbf{A}}(k) = \begin{bmatrix}\mathbf{A} & \mathbf{B}\mathbf{F}(k) \\ \mathbf{K}(k)\mathbf{C} & \mathbf{A} - \mathbf{K}(k)\mathbf{C} + \mathbf{B}\mathbf{F}(k)\end{bmatrix} \tag{4.239}$$

The separation theorem or certainty-equivalence principle states that the eigenvalue set of the global state matrix $\bar{\mathbf{A}}(k)$ is the union of the eigenvalue set of the closed-loop control state matrix $\mathbf{A} + \mathbf{B}\mathbf{F}(k)$ and the eigenvalue set of the closed-loop observer state matrix $\mathbf{A} - \mathbf{K}(k)\mathbf{C}$ [13], [14]. Therefore, the state controller and the state observer can be designed independently making stable the closed-loop control state matrix $\mathbf{A} + \mathbf{B}\mathbf{F}(k)$ and the closed-loop observer state matrix $\mathbf{A} - \mathbf{K}(k)\mathbf{C}$ and consequently the global state matrix $\bar{\mathbf{A}}(k)$ is stable. This result can be developed in a straightforward manner considering a slightly different global state vector defined as [3]:

$$\tilde{\mathbf{z}}(k) = \begin{bmatrix} \mathbf{z}(k) \\ \mathbf{e}(k) \end{bmatrix} \tag{4.240}$$

Where $\mathbf{e}(k)$ is the error between the system state vector $\mathbf{z}(k)$ and the estimated state vector $\hat{\mathbf{z}}(k)$. Indeed, in terms of this global state vector $\tilde{\mathbf{z}}(k)$ the global state matrix $\tilde{\mathbf{A}}(k)$ becomes:

$$\tilde{\mathbf{A}}(k) = \begin{bmatrix} \mathbf{A} + \mathbf{BF}(k) & -\mathbf{BF}(k) \\ \mathbf{O} & \mathbf{A} - \mathbf{K}(k)\mathbf{C} \end{bmatrix} \tag{4.241}$$

Indeed, the eigenvalues of the block triangular state matrix $\tilde{\mathbf{A}}(k)$ are union of the eigenvalues of its diagonal block matrices $\mathbf{A} + \mathbf{BF}(k)$ and $\mathbf{A} - \mathbf{K}(k)\mathbf{C}$.

5. CASE STUDY: ACTIVE CONTROL OF A THREE-STORY BUILDING MODEL

5.1. INTRODUCTION

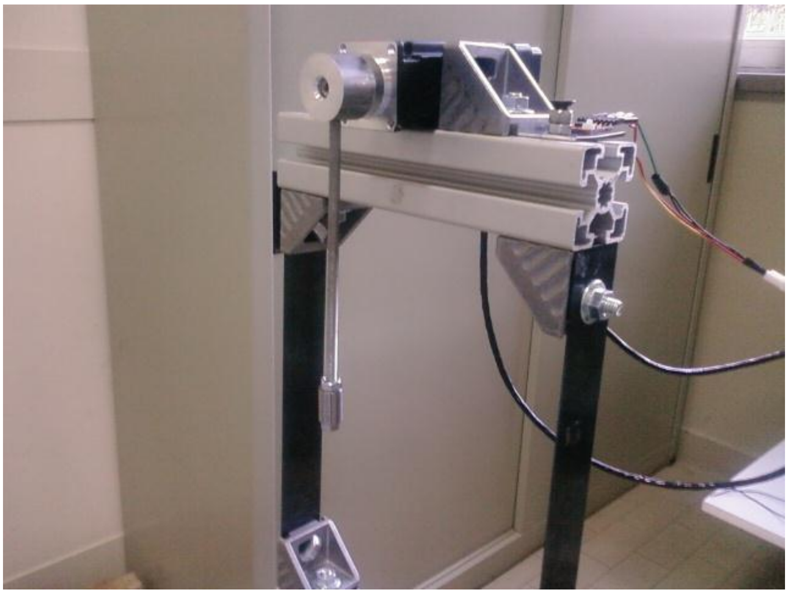
In this chapter the analysis of a case study is presented. The case study examined is a three-story building model with a pendulum hinged on the third floor. The motivations of this choice can be summarized in two points. First, the three-story frame, in spite of its simplicity, is a mechanical system whose dynamical behaviour is qualitatively similar to complex flexible structures. Therefore, all methods able to derive the equations of motion of multibody systems [1], [2], [3], all algorithms capable to identify the modal parameters of structural systems [4], [5], [6], [7], [8], [9], [10], [11], [12], [13] and all strategies adequate to perform active vibration control of mechanical systems [14], [15], [16], [17], [18], [19], [20], [21] can be identically used in order to obtain qualitatively similar results. Second, the three-story building model, by virtue of its simplicity, is a mechanical system which can be quite simply assembled in laboratory making relatively little effort in order to perform a quick and easy-to-test experimental analysis [22], [23]. The following sections contain an accurate description of the three-story frame, of the control system and of all the tools which compose the test rig [24], [25]. Then, the developments of a lumped parameter model and of a finite element model of the three-story frame are presented. Indeed, the system equations of motion has been derived using the finite element formulation of flexible multibody Dynamics

[1]. Subsequently, the development of a data-driven model relative to the system under test is described in the experimental identification section [5]. In particular, the Eigensystem Realization Algorithm with Data Correlation using Observer/Kalman Filter Identification method (ERA/DC OKID) [4] and the Numerical Algorithm for Subspace Identification (N4SID) [6] have been used to determine two different state-space models of the structural system using experimental input and output measurements. In addition, the algorithm to determine a physical model from the identified sate-space representation (MKR) [7], [8], [9] has been used to obtain two different second-order mechanical models of the three-story frame. Subsequently, the design of a Linear Quadratic Gaussian regulator (LQG) [14], [15] has been performed using the previously identified physical model of the system under test. The effectiveness of this controller has been tested in the worst-case scenario in which the system is excited by an external force whose harmonic content is close to the first three system natural frequencies. Finally, a new control algorithm for nonlinear underactuated mechanical systems affected by uncertainties (EUK-EKF) is proposed. The control problem of nonlinear underactuated mechanical system forced with nonholonomic constraints is the main object of many recent researches [26], [27], [28]. In analogy with the Linear Quadratic Gaussian regulation method (LQG), the proposed algorithm represents the extension of the Udwadia-Kalaba control method (UK) [29] to underactuated mechanical systems disturbed by noise. This extension is performed combining the extended Udwadia-Kalaba control method (EUK) [30], [31] which is the extension of the Udwadia-Kalaba control method (UK) [29] to underactuated mechanical systems, with the well-known extended Kalman filter estimation method (EKF) [15]. Even in this case, the effectiveness of the combined algorithms (EUK-EKF) has been tested in the worst-case scenario in which the system is excited by an external force whose harmonic content is close to the first three system natural frequencies.

5.2. TEST RIG DESCRIPTION

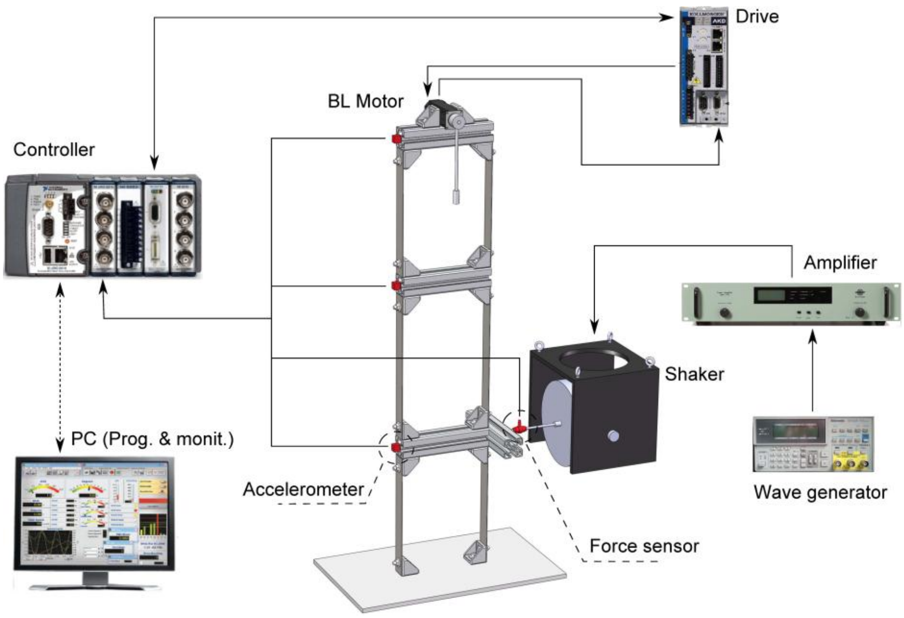
The experimental apparatus is a flexible structure composed of six vertical harmonic steel beams and three aluminium Bosch profiles, which serves as horizontal connecting rods.

The frequency range of interest encompasses all the frequencies below $15\left[Hz\right]$. In the frequency range of interest, the steel beams behave like flexible elements whereas the aluminium rods behave like rigid elements. Since the structure is developed in a plane, its transversal stiffness and its torsional stiffness are much larger than the structure stiffness along the plane. Therefore, in the frequency range of interest, the structure behaves like a plane system. The structural elements are assembled to form a three-story frame which represents a simplified model for a three-story building. The flexible frame is excited in correspondence of the first floor by a Bruel & Kjaer shaker. The shaker is suspended through a steel cable which is fixed on an external support structure. The shaker is connected with the three-story frame by a stinger and a PCB load cell is interposed between the structure and the stinger in order to measure the force transferred to the frame by the shaker. The shaker is fed by a Bruel & Kjaer power amplifiers which is controlled by a Textronics arbitrary function generator. On the other hand, on each floor of the structure there is a Bruel & Kjaer piezoelectric transducer which sense the structure acceleration. In addition, on the third floor there is the control system.



The control actuator is realized by a simple pendulum which can oscillate along the plane of the structure. The pendulum is driven by a Kollmorgen AKM brushless motor, equipped with an encoder, which provides the control torque.

The control torque follows a feedback control law which has been implemented using a National Instruments PAC system, namely the CompactRIO system. The interface between the motor servodrive and the PAC system is achieved by a drive interface module lodged in the chassis of the CompactRIO system, which enables an efficient integration of the two systems. Indeed, the CompactRIO system can read the output signals of the transducers by using the input module and, at the same time, it can accomplish the feedback control for the motor torque by using the drive interface module.



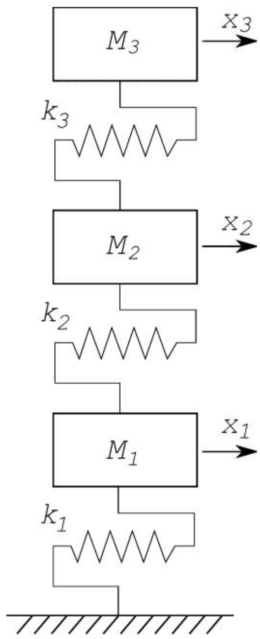
On the other hand, in order to perform the experimental modal analysis of the structure, the structure has been excited by a Bruel & Kjaer impact hammer instrumented with a load cell connected to a Bruel & Kjaer spectrum analyzer whereas, at the same time, the acceleration signals of the system response were recorded by using the spectrum analyzer. Consider now the following data.

The length of the structural elements are equal respectively to $L_1 = 23 \cdot 10^{-2}\,[m]$, $L_2 = 28 \cdot 10^{-2}\,[m]$ and $L_3 = 23 \cdot 10^{-2}\,[m]$ whereas the half-length of the pendulum is equal to $L_4 = 8.25 \cdot 10^{-2}\,[m]$. The dimensions of the cross sections relative to the structural elements are $b_1 = b_2 = b_3 = 3.5 \cdot 10^{-2}\,[m]$

and $h_1 = h_2 = h_3 = 1 \cdot 10^{-3} \left[m\right]$. The areas of the cross sections are equal to $A_1 = A_2 = A_3 = b_1 h_1 = 3.5 \cdot 10^{-5} \left[m^2\right]$. The second moments of area corresponding to each beam cross section can be computed as $J_1 = J_2 = J_3 = \frac{b_1 h_1^3}{12} = 2.917 \cdot 10^{-12} \left[m^4\right]$. The system structural components are made of harmonic steel. The mass densities of the harmonic steel elements are equal to $\rho_1 = \rho_2 = \rho_3 = 7860 \left[\frac{kg}{m^3}\right]$ whereas the elastic moduli are equal to $E_1 = E_2 = E_3 = 207 \cdot 10^9 \left[\frac{N}{m^2}\right]$. The masses of the floors are respectively equal to $m_1 = 1.281\left[kg\right]$, $m_2 = 0.814\left[kg\right]$ and $m_3 = 1.380\left[kg\right]$ whereas the mass of the pendulum is equal to $m_4 = 0.083\left[kg\right]$. The mass moment of inertia relative to the centre of mass of the pendulum is equal to $I_{zz,4} = 8.32 \cdot 10^{-4} \left[kg\ m^2\right]$. The structural damping is assumed proportional with coefficients $\alpha = 0.9751\left[s\right]$ and $\beta = -2.8815 \cdot 10^{-4} \left[\frac{1}{s}\right]$ whereas the pendulum angular damping is assumed equal to $r_4 = 2.432 \cdot 10^{-3} \left[\frac{N \cdot m \cdot s}{rad}\right]$. These data will be used in the following sections to derive different types of models and different types of controller for the three-story frame.

5.3. LUMPED PARAMETER MODEL

Consider the following schematization of the test rig:



According to the lumped parameter description, the system is modelled as a set of three rigid bodies representing the floors of the building connected by three linear springs corresponding to the system structural elements. In addition, a pendulum hinged on the third floor is considered. Therefore, the number of system degrees of freedom is $n_2 = 4$ and the vector of system lagrangian coordinates is selected as follows:

$$\mathbf{q}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \theta(t) \end{bmatrix} \tag{5.1}$$

Where $x_1(t)$, $x_2(t)$ and $x_3(t)$ are the horizontal displacements of the three floors whereas $\theta(t)$ is the pendulum angle. The position vectors relative to each floor centre of mass can be expressed in terms of system configuration coordinates as:

$$\mathbf{R}^1(t) = \begin{bmatrix} x_1(t) \\ L_1 \\ 0 \end{bmatrix} \tag{5.2}$$

$$\mathbf{R}^2(t) = \begin{bmatrix} x_2(t) \\ L_1 + L_2 \\ 0 \end{bmatrix} \tag{5.3}$$

$$\mathbf{R}^3(t) = \begin{bmatrix} x_3(t) \\ L_1 + L_2 + L_3 \\ 0 \end{bmatrix} \tag{5.4}$$

Where L_1, L_2 and L_3 are the dimensions of the structural elements. On the other hand, the position vector of the pendulum centre of mass can be expressed in terms of lagrangian coordinates as follows:

$$\mathbf{R}^4(t) = \begin{bmatrix} x_3(t) + L_4\cos(\theta(t)) \\ L_1 + L_2 + L_3 + L_4\sin(\theta(t)) \\ 0 \end{bmatrix} \tag{5.5}$$

Where L_4 is equal to half length of the pendulum. Indeed, the position vector of a generic point P^4 on the pendulum can be computed as:

$$\mathbf{r}^4(P^4, t) = \mathbf{R}^4(t) + \mathbf{A}^4(t)\bar{\mathbf{u}}^4(P^4) \tag{5.6}$$

Where $\mathbf{A}^4(t)$ is a rotation matrix defined as:

$$\mathbf{A}^4(t) = \begin{bmatrix} \cos(\theta(t)) & -\sin(\theta(t)) & 0 \\ \sin(\theta(t)) & \cos(\theta(t)) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5.7}$$

And $\overline{\mathbf{u}}^4(P^4)$ is the position of the point P^4 referred to the pendulum frame of reference defined as:

$$\overline{\mathbf{u}}^4(P^4) = \begin{bmatrix} \overline{x}^4 \\ 0 \\ 0 \end{bmatrix} \tag{5.8}$$

Consequently, the time derivative of each centre of mass position vector can be computed as:

$$\begin{aligned} \dot{\mathbf{R}}^1(t) &= \begin{bmatrix} \dot{x}_1(t) \\ 0 \\ 0 \end{bmatrix} = \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{\theta}(t) \end{bmatrix} = \\ &= \mathbf{J}_R^1(t)\dot{\mathbf{q}}(t) \end{aligned} \tag{5.9}$$

$$\dot{\mathbf{R}}^2(t) = \begin{bmatrix} \dot{x}_2(t) \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{\theta}(t) \end{bmatrix} = \mathbf{J}_R^2(t)\dot{\mathbf{q}}(t) \tag{5.10}$$

$$\dot{\mathbf{R}}^3(t) = \begin{bmatrix} \dot{x}_3(t) \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{\theta}(t) \end{bmatrix} = \mathbf{J}_R^3(t)\dot{\mathbf{q}}(t) \tag{5.11}$$

$$\dot{\mathbf{R}}^4(t) = \begin{bmatrix} \dot{x}_3(t) - L_4 \sin(\theta(t))\dot{\theta}(t) \\ L_4 \cos(\theta(t))\dot{\theta}(t) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & -L_4 \sin(\theta(t)) \\ 0 & 0 & 0 & L_4 \cos(\theta(t)) \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{\theta}(t) \end{bmatrix} = \mathbf{J}_R^4(t)\dot{\mathbf{q}}(t) \tag{5.12}$$

Where the jacobian transformation matrices $\mathbf{J}_R^1(t)$, $\mathbf{J}_R^2(t)$, $\mathbf{J}_R^3(t)$ and $\mathbf{J}_R^4(t)$ are respectively defined as follows:

$$\mathbf{J}_R^1(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.13}$$

$$\mathbf{J}_R^2(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.14}$$

$$\mathbf{J}_R^3(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.15}$$

$$\mathbf{J}_R^4(t) = \begin{bmatrix} 0 & 0 & 1 & -L_4 \sin(\theta(t)) \\ 0 & 0 & 0 & L_4 \cos(\theta(t)) \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.16}$$

On the other hand, the angular velocity of the pendulum can be expressed in terms of the independent coordinates as follows:

$$\begin{aligned}\boldsymbol{\omega}^4(t) &= \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}(t) \end{bmatrix} = \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{\theta}(t) \end{bmatrix} = \\ &= \mathbf{J}_{\omega}^4(t)\dot{\mathbf{q}}(t)\end{aligned} \tag{5.17}$$

Where the jacobian transformation matrix $\mathbf{J}_{\omega}^4(t)$ is defined as:

$$\mathbf{J}_{\omega}^4(t) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5.18}$$

Using these expressions, the kinetic energy relative to each system element can be computed as:

$$\begin{aligned}T^1(t) &= \frac{1}{2} m_1 \dot{\mathbf{R}}^{1T}(t)\dot{\mathbf{R}}^1(t) = \\ &= \frac{1}{2} m_1 \dot{\mathbf{q}}^T(t)\mathbf{J}_R^{1T}(t)\mathbf{J}_R^1(t)\dot{\mathbf{q}}(t) = \\ &= \frac{1}{2}\dot{\mathbf{q}}^T(t)\mathbf{M}^1(t)\dot{\mathbf{q}}(t)\end{aligned} \tag{5.19}$$

$$\begin{aligned} T^2(t) &= \frac{1}{2} m_2 \dot{\mathbf{R}}^{2T}(t) \dot{\mathbf{R}}^2(t) = \\ &= \frac{1}{2} m_2 \dot{\mathbf{q}}^T(t) \mathbf{J}_R^{2T}(t) \mathbf{J}_R^2(t) \dot{\mathbf{q}}(t) = \\ &= \frac{1}{2} \dot{\mathbf{q}}^T(t) \mathbf{M}^2(t) \dot{\mathbf{q}}(t) \end{aligned} \tag{5.20}$$

$$\begin{aligned} T^3(t) &= \frac{1}{2} m_3 \dot{\mathbf{R}}^{3T}(t) \dot{\mathbf{R}}^3(t) = \\ &= \frac{1}{2} m_3 \dot{\mathbf{q}}^T(t) \mathbf{J}_R^{3T}(t) \mathbf{J}_R^3(t) \dot{\mathbf{q}}(t) = \\ &= \frac{1}{2} \dot{\mathbf{q}}^T(t) \mathbf{M}^3(t) \dot{\mathbf{q}}(t) \end{aligned} \tag{5.21}$$

$$\begin{aligned} T^4(t) &= \frac{1}{2} m_4 \dot{\mathbf{R}}^{4T}(t) \dot{\mathbf{R}}^4(t) + \frac{1}{2} I_{zz,4} \boldsymbol{\omega}^{4T}(t) \boldsymbol{\omega}^4(t) = \\ &= \frac{1}{2} m_4 \dot{\mathbf{q}}^T(t) \mathbf{J}_R^{4T}(t) \mathbf{J}_R^4(t) \dot{\mathbf{q}}(t) + \frac{1}{2} I_{zz,4} \dot{\mathbf{q}}^T(t) \mathbf{J}_\omega^{4T}(t) \mathbf{J}_\omega^4(t) \dot{\mathbf{q}}(t) = \\ &= \frac{1}{2} \dot{\mathbf{q}}^T(t) \left(m_4 \mathbf{J}_R^{4T}(t) \mathbf{J}_R^4(t) + I_{zz,4} \mathbf{J}_\omega^{4T}(t) \mathbf{J}_\omega^4(t) \right) \dot{\mathbf{q}}(t) = \\ &= \frac{1}{2} \dot{\mathbf{q}}^T(t) \mathbf{M}^4(t) \dot{\mathbf{q}}(t) \end{aligned} \tag{5.22}$$

Where m_1, m_2, m_3 and m_4 are respectively the masses of each floor and the pendulum mass whereas $I_{zz,4}$ is the mass moment of inertia relative to the centre of mass of the pendulum. The mass matrices $\mathbf{M}^1(t)$, $\mathbf{M}^2(t)$, $\mathbf{M}^3(t)$ and $\mathbf{M}^4(t)$ can be respectively computed as:

$$\mathbf{M}^1(t) = m_1 \mathbf{J}_R^{1T}(t)\mathbf{J}_R^1(t) =$$
$$= m_1 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \tag{5.23}$$
$$= \begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{M}^2(t) = m_2 \mathbf{J}_R^{2T}(t)\mathbf{J}_R^2(t) =$$
$$= m_2 \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}^T \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \tag{5.24}$$
$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{M}^3(t) = m_3 \mathbf{J}_R^{3T}(t)\mathbf{J}_R^3(t) =$$
$$= m_3 \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \tag{5.25}$$
$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{M}^4(t) = m_4 \mathbf{J}_R^{4T}(t)\mathbf{J}_R^4(t) + I_{zz,4}\mathbf{J}_\omega^{4T}(t)\mathbf{J}_\omega^4(t) =$$
$$= m_4 \begin{bmatrix} 0 & 0 & 1 & -L_4\sin(\theta(t)) \\ 0 & 0 & 0 & L_4\cos(\theta(t)) \\ 0 & 0 & 0 & 0 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 1 & -L_4\sin(\theta(t)) \\ 0 & 0 & 0 & L_4\cos(\theta(t)) \\ 0 & 0 & 0 & 0 \end{bmatrix} +$$
$$+ I_{zz,4}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$
$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & m_4 & -m_4L_4\sin(\theta(t)) \\ 0 & 0 & -m_4L_4\sin(\theta(t)) & m_4L_4^2 + I_{zz,4} \end{bmatrix}$$
(5.26)

Hence, the kinetic energy corresponding to the whole system can be computed as:

$$T(t) = T^1(t) + T^2(t) + T^3(t) + T^4(t) =$$
$$= \frac{1}{2}\dot{\mathbf{q}}^T(t)\mathbf{M}^1(t)\dot{\mathbf{q}}(t) + \frac{1}{2}\dot{\mathbf{q}}^T(t)\mathbf{M}^2(t)\dot{\mathbf{q}}(t) + \frac{1}{2}\dot{\mathbf{q}}^T(t)\mathbf{M}^3(t)\dot{\mathbf{q}}(t) +$$
$$+ \frac{1}{2}\dot{\mathbf{q}}^T(t)\mathbf{M}^4(t)\dot{\mathbf{q}}(t) =$$
$$= \frac{1}{2}\dot{\mathbf{q}}^T(t)\left(\mathbf{M}^1(t) + \mathbf{M}^2(t) + \mathbf{M}^3(t) + \mathbf{M}^4(t)\right)\dot{\mathbf{q}}(t) =$$
$$= \frac{1}{2}\dot{\mathbf{q}}^T(t)\mathbf{M}(t)\dot{\mathbf{q}}(t)$$
(5.27)

Therefore, the mass matrix of the whole system can be obtained summing each element mass matrix yielding:

$$\mathbf{M}(t) = \mathbf{M}^1(t) + \mathbf{M}^2(t) + \mathbf{M}^3(t) + \mathbf{M}^4(t) =$$

$$= \begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 + m_4 & -m_4 L_4 \sin(\theta(t)) \\ 0 & 0 & -m_4 L_4 \sin(\theta(t)) & m_4 L_4^2 + I_{zz,4} \end{bmatrix} \tag{5.28}$$

Once that the mass matrix and the kinetic energy of the whole system have been determined, the two terms which form the quadratic velocity vector can be computed as:

$$-\dot{\mathbf{M}}(t)\dot{\mathbf{q}}(t) = -\left(\frac{d}{dt} \begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 + m_4 & -m_4 L_4 \sin(\theta(t)) \\ 0 & 0 & -m_4 L_4 \sin(\theta(t)) & m_4 L_4^2 + I_{zz,4} \end{bmatrix} \right) \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{\theta}(t) \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_4 L_4 \cos(\theta(t))\dot{\theta}(t) \\ 0 & 0 & m_4 L_4 \cos(\theta(t))\dot{\theta}(t) & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{\theta}(t) \end{bmatrix} =$$

$$= \begin{bmatrix} 0 \\ 0 \\ m_4 L_4 \cos(\theta(t))\dot{\theta}^2(t) \\ m_4 L_4 \cos(\theta(t))\dot{\theta}(t)\dot{x}_3(t) \end{bmatrix} \tag{5.29}$$

$$\left(\frac{\partial T(t)}{\partial \mathbf{q}(t)}\right)^T = \begin{bmatrix} \dfrac{\partial T(t)}{\partial x_1(t)} \\ \dfrac{\partial T(t)}{\partial x_2(t)} \\ \dfrac{\partial T(t)}{\partial x_3(t)} \\ \dfrac{\partial T(t)}{\partial \theta(t)} \end{bmatrix} =$$

$$= \begin{bmatrix} \dfrac{1}{2}\dot{\mathbf{q}}^T(t)\dfrac{\partial \mathbf{M}(t)}{\partial x_1(t)}\dot{\mathbf{q}}(t) \\ \dfrac{1}{2}\dot{\mathbf{q}}^T(t)\dfrac{\partial \mathbf{M}(t)}{\partial x_2(t)}\dot{\mathbf{q}}(t) \\ \dfrac{1}{2}\dot{\mathbf{q}}^T(t)\dfrac{\partial \mathbf{M}(t)}{\partial x_3(t)}\dot{\mathbf{q}}(t) \\ \dfrac{1}{2}\dot{\mathbf{q}}^T(t)\dfrac{\partial \mathbf{M}(t)}{\partial \theta(t)}\dot{\mathbf{q}}(t) \end{bmatrix} =$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ -m_4 L_4 \cos(\theta(t))\dot{\theta}(t)\dot{x}_3(t) \end{bmatrix} \tag{5.30}$$

Consequently, the quadratic velocity vector can be computed as follows:

$$\mathbf{Q}_v(t) = -\dot{\mathbf{M}}(t)\dot{\mathbf{q}}(t) + \left(\frac{\partial T(t)}{\partial \mathbf{q}(t)}\right)^T =$$

$$= \begin{bmatrix} 0 \\ 0 \\ m_4 L_4 \cos(\theta(t))\dot{\theta}^2(t) \\ 0 \end{bmatrix} \tag{5.31}$$

On the other hand, suppose that the elastic springs have zero length in the undeformed state. The deformation vectors of each spring can be expressed as follows:

$$\begin{aligned}\mathbf{l}^1(t) &= \begin{bmatrix} x_1(t) \\ 0 \\ 0 \end{bmatrix} = \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \theta(t) \end{bmatrix} = \\ &= \mathbf{J}_l^1(t)\mathbf{q}(t)\end{aligned} \tag{5.32}$$

$$\begin{aligned}\mathbf{l}^2(t) &= \begin{bmatrix} x_2(t) - x_1(t) \\ 0 \\ 0 \end{bmatrix} = \\ &= \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \theta(t) \end{bmatrix} = \\ &= \mathbf{J}_l^2(t)\mathbf{q}(t)\end{aligned} \tag{5.33}$$

$$\mathbf{l}^3(t) = \begin{bmatrix} x_3(t) - x_2(t) \\ 0 \\ 0 \end{bmatrix} = \\ = \begin{bmatrix} 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \theta(t) \end{bmatrix} = \\ = \mathbf{J}_l^3(t)\mathbf{q}(t) \tag{5.34}$$

Where the jacobian transformation matrices $\mathbf{J}_l^1(t)$, $\mathbf{J}_l^2(t)$ and $\mathbf{J}_l^3(t)$ can be respectively expressed as:

$$\mathbf{J}_l^1(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.35}$$

$$\mathbf{J}_l^2(t) = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.36}$$

$$\mathbf{J}_l^3(t) = \begin{bmatrix} 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.37}$$

Consequently, the strain potential energy relative to each spring of the system can be computed as:

$$\begin{aligned} U_l^1(t) &= \frac{1}{2}k_1\mathbf{l}^{1T}(t)\mathbf{l}^1(t) = \\ &= \frac{1}{2}k_1\mathbf{q}^T(t)\mathbf{J}_l^{1T}(t)\mathbf{J}_l^1(t)\mathbf{q}(t) = \\ &= \frac{1}{2}\mathbf{q}^T(t)\mathbf{K}_l^1(t)\mathbf{q}(t) \end{aligned} \tag{5.38}$$

$$\begin{aligned} U_l^2(t) &= \frac{1}{2}k_2\mathbf{l}^{2T}(t)\mathbf{l}^2(t) = \\ &= \frac{1}{2}k_2\mathbf{q}^T(t)\mathbf{J}_l^{2T}(t)\mathbf{J}_l^2(t)\mathbf{q}(t) = \\ &= \frac{1}{2}\mathbf{q}^T(t)\mathbf{K}_l^2(t)\mathbf{q}(t) \end{aligned} \tag{5.39}$$

$$\begin{aligned} U_l^3(t) &= \frac{1}{2}k_3\mathbf{l}^{3T}(t)\mathbf{l}^3(t) = \\ &= \frac{1}{2}k_3\mathbf{q}^T(t)\mathbf{J}_l^{3T}(t)\mathbf{J}_l^3(t)\mathbf{q}(t) = \\ &= \frac{1}{2}\mathbf{q}^T(t)\mathbf{K}_l^3(t)\mathbf{q}(t) \end{aligned} \tag{5.40}$$

Where k_1, k_2 and k_3 are the elastic constants relative to each spring. These constants can be computed modelling the structural elements as fixed end beams:

$$k_1 = 24\frac{E_1J_1}{L_1^3} \tag{5.41}$$

$$k_2 = 24\frac{E_2J_2}{L_2^3} \tag{5.42}$$

$$k_3 = 24\frac{E_3 J_3}{L_3^3} \tag{5.43}$$

Where E_1, E_2 and E_3 are respectively the Young elastic moduli relative to each beam whereas J_1, J_2 and J_3 are the second moments of area corresponding to each beam cross section. The spring stiffness matrices $\mathbf{K}^1(t)$, $\mathbf{K}^2(t)$ and $\mathbf{K}^3(t)$ can be respectively computed as:

$$\begin{aligned}\mathbf{K}_l^1(t) &= k_1 \mathbf{J}_l^{1T}(t)\mathbf{J}_l^1(t) = \\ &= k_1 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \\ &= \begin{bmatrix} k_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\end{aligned} \tag{5.44}$$

$$\begin{aligned}\mathbf{K}_l^2(t) &= k_2 \mathbf{J}_l^{2T}(t)\mathbf{J}_l^2(t) = \\ &= k_2 \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}^T \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \\ &= \begin{bmatrix} k_2 & -k_2 & 0 & 0 \\ -k_2 & k_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\end{aligned} \tag{5.45}$$

$$\mathbf{K}_l^3(t) = k_3 \mathbf{J}_l^{3T}(t)\mathbf{J}_l^3(t) =$$

$$= k_3 \begin{bmatrix} 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}^T \begin{bmatrix} 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \tag{5.46}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & k_3 & -k_3 & 0 \\ 0 & -k_3 & k_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, the strain potential energy corresponding to the whole system can be computed as:

$$\begin{aligned} U_l(t) &= U_l^1(t) + U_l^2(t) + U_l^3(t) = \\ &= \frac{1}{2}\mathbf{q}^T(t)\mathbf{K}_l^1(t)\mathbf{q}(t) + \frac{1}{2}\mathbf{q}^T(t)\mathbf{K}_l^2(t)\mathbf{q}(t) + \frac{1}{2}\mathbf{q}^T(t)\mathbf{K}_l^3(t)\mathbf{q}(t) = \\ &= \frac{1}{2}\mathbf{q}^T(t)\left(\mathbf{K}_l^1(t) + \mathbf{K}_l^2(t) + \mathbf{K}_l^3(t)\right)\mathbf{q}(t) = \\ &= \frac{1}{2}\mathbf{q}^T(t)\mathbf{K}_l(t)\mathbf{q}(t) \end{aligned} \tag{5.47}$$

Therefore, the spring stiffness matrix of the whole system can be obtained summing each spring stiffness matrix to yield:

$$\mathbf{K}_l(t) = \mathbf{K}_l^1(t) + \mathbf{K}_l^2(t) + \mathbf{K}_l^3(t) =$$

$$= \begin{bmatrix} k_1 + k_2 & -k_2 & 0 & 0 \\ -k_2 & k_2 + k_3 & -k_3 & 0 \\ 0 & -k_3 & k_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.48}$$

Moreover, the pendulum potential energy relative to gravitational force can be simply computed as:

$$\begin{aligned} U_g(t) &= -m_4\mathbf{g}^T\mathbf{R}_4(t) = \\ &= -m_4\begin{bmatrix}0 & -g & 0\end{bmatrix}\begin{bmatrix} x_3(t)+L_4\cos(\theta(t)) \\ L_1+L_2+L_3+L_4\sin(\theta(t)) \\ 0 \end{bmatrix} = \\ &= m_4 g\left(L_1+L_2+L_3+L_4\sin(\theta(t))\right) \end{aligned} \tag{5.49}$$

Consequently, the total potential energy of the system can be computed summing the strain potential energy of the springs and the gravitational potential energy of the pendulum yielding to:

$$\begin{aligned} U(t) &= U_l(t)+U_g(t) = \\ &= \frac{1}{2}\mathbf{q}^T(t)\mathbf{K}_l(t)\mathbf{q}(t)+m_4 g\left(L_1+L_2+L_3+L_4\sin(\theta(t))\right) \end{aligned} \tag{5.50}$$

Therefore, the lagrangian component of the conservative external forces acting on the system can be determined as follows:

$$
\begin{aligned}
\mathbf{Q}_{e,c}(t) &= -\left(\frac{\partial U(t)}{\partial \mathbf{q}(t)}\right)^T = \\
&= -\left(\frac{\partial}{\partial \mathbf{q}(t)}\left(\frac{1}{2}\mathbf{q}^T(t)\mathbf{K}_l(t)\mathbf{q}(t) + m_4 g\left(L_1 + L_2 + L_3 + L_4 \sin(\theta(t))\right)\right)\right)^T = \\
&= -\mathbf{K}_l(t)\mathbf{q}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -m_4 g L_4 \cos(\theta(t)) \end{bmatrix} = \\
&= -\begin{bmatrix} k_1 + k_2 & -k_2 & 0 & 0 \\ -k_2 & k_2 + k_3 & -k_3 & 0 \\ 0 & -k_3 & k_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \theta(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -m_4 g L_4 \cos(\theta(t)) \end{bmatrix} = \\
&= \begin{bmatrix} -\left(k_1 + k_2\right) x_1(t) + k_2 x_2(t) \\ k_2 x_1(t) - \left(k_2 + k_3\right) x_2(t) + k_3 x_3(t) \\ k_3 x_2(t) - k_3 x_3(t) \\ -m_4 g L_4 \cos(\theta(t)) \end{bmatrix}
\end{aligned}
\tag{5.51}
$$

In addition, the effect of the nonconservative external force acting on the first floors can be accounted for computing its virtual work. Indeed:

$$
\begin{aligned}
\delta W_{e,nc}(t) &= \mathbf{F}^T(t)\delta \mathbf{R}_1(t) = \\
&= \mathbf{F}^T(t)\mathbf{J}_R^1(t)\delta \mathbf{q}(t) = \\
&= \mathbf{Q}_{e,nc}^T(t)\delta \mathbf{q}(t)
\end{aligned}
\tag{5.52}
$$

Thence, the lagrangian component of the nonconservative external forces can be determined as:

$$
\begin{aligned}
\mathbf{Q}_{e,nc}(t) &= \mathbf{J}_R^{1T}(t)\mathbf{F}(t) = \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}^T \begin{bmatrix} F(t) \\ 0 \\ 0 \end{bmatrix} = \\
&= \begin{bmatrix} F(t) \\ 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
\tag{5.53}
$$

Consequently, the total lagrangian component of all forces acting on the system can be determined as follows:

$$
\begin{aligned}
\mathbf{Q}(t) &= \mathbf{Q}_v(t) + \mathbf{Q}_{e,c}(t) + \mathbf{Q}_{e,nc}(t) = \\
&= \begin{bmatrix} -\left(k_1 + k_2\right)x_1(t) + k_2 x_2(t) + F(t) \\ k_2 x_1(t) - \left(k_2 + k_3\right)x_2(t) + k_3 x_3(t) \\ m_4 L_4 \cos(\theta(t))\dot{\theta}^2(t) + k_3 x_2(t) - k_3 x_3(t) \\ -m_4 g L_4 \cos(\theta(t)) \end{bmatrix}
\end{aligned}
\tag{5.54}
$$

On the other hand, the electric motor exerts a control torque on the pendulum whose virtual work is:

$$
\begin{aligned}
\delta W_c(t) &= C(t)\delta\theta(t) = \\
&= \mathbf{C}^T(t)\mathbf{J}_\omega^4(t)\delta\mathbf{q}(t) = \\
&= \mathbf{Q}_c^T(t)\delta\mathbf{q}(t)
\end{aligned}
\tag{5.55}
$$

Hence, the lagrangian component of the control torque can be determined as:

$$\mathbf{Q}_c(t) = \mathbf{J}_\omega^{4T}(t)\mathbf{C}(t) =$$
$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \\ C(t) \end{bmatrix} = \tag{5.56}$$
$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ C(t) \end{bmatrix}$$

Finally, the system equations of motion can be expressed in matrix notation using Lagrange equations as:

$$\begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 + m_4 & -m_4 L_4 \sin(\theta(t)) \\ 0 & 0 & -m_4 L_4 \sin(\theta(t)) & m_4 L_4^2 + I_{zz,4} \end{bmatrix} \begin{bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \\ \ddot{x}_3(t) \\ \ddot{\theta}(t) \end{bmatrix} =$$
$$= \begin{bmatrix} -(k_1 + k_2) x_1(t) + k_2 x_2(t) + F(t) \\ k_2 x_1(t) - (k_2 + k_3) x_2(t) + k_3 x_3(t) \\ m_4 L_4 \cos(\theta(t)) \dot{\theta}^2(t) + k_3 x_2(t) - k_3 x_3(t) \\ -m_4 g L_4 \cos(\theta(t)) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ C(t) \end{bmatrix} \tag{5.57}$$

The compact form of these equations is the following:

$$\mathbf{M}(t)\ddot{\mathbf{q}}(t) = \mathbf{Q}(t) + \mathbf{Q}_c(t) \tag{5.58}$$

This set of motion equations represents the system lumped parameter model. Linearizing the lumped parameter model of the system around the stable equilibrium position where $\theta_0 = \frac{3}{2}\pi$ yields:

$$\begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3+m_4 & m_4 L_4 \\ 0 & 0 & m_4 L_4 & m_4 L_4^2 + I_{zz,4} \end{bmatrix} \begin{bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \\ \ddot{x}_3(t) \\ \ddot{\varphi}(t) \end{bmatrix} + \\ + \begin{bmatrix} k_1+k_2 & -k_2 & 0 & 0 \\ -k_2 & k_2+k_3 & -k_3 & 0 \\ 0 & -k_3 & k_3 & 0 \\ 0 & 0 & 0 & m_4 g L_4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \varphi(t) \end{bmatrix} = \\ = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} F(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} C(t) \tag{5.59}$$

Where the following change of variables has been performed:

$$\begin{aligned} \theta(t) &= \varphi(t) + \theta_0 = \\ &= \varphi(t) + \frac{3}{2}\pi \end{aligned} \tag{5.60}$$

The compact form of these equations is the following:

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{B}_{2,e}\mathbf{u}_e(t) + \mathbf{B}_{2,c}\mathbf{u}_c(t) \tag{5.61}$$

Where $\mathbf{x}(t)$ is the vector containing the independent coordinates, $\mathbf{M}$ is the linearized mass matrix and $\mathbf{K}$ is the linearized stiffness matrix whereas $\mathbf{B}_{2,e}$ and $\mathbf{B}_{2,c}$ are the Boolean matrices characterizing the location of the external uncontrolled and controlled inputs $\mathbf{u}_e(t)$ and $\mathbf{u}_c(t)$ acting on the system which correspond to the external uncontrolled force $F(t)$ and to the controlled torque $C(t)$. These quantities are defined as:

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \varphi(t) \end{bmatrix} \tag{5.62}$$

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 + m_4 & m_4 L_4 \\ 0 & 0 & m_4 L_4 & m_4 L_4^2 + I_{zz,4} \end{bmatrix} \tag{5.63}$$

$$\mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 & 0 \\ -k_2 & k_2 + k_3 & -k_3 & 0 \\ 0 & -k_3 & k_3 & 0 \\ 0 & 0 & 0 & m_4 g L_4 \end{bmatrix} \tag{5.64}$$

$$\mathbf{B}_{2,e} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{5.65}$$

$$\mathbf{B}_{2,c} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{5.66}$$

This set of motion equations represents the system linearized lumped parameter model. Finally, using the data reported in the test rig description, the system modal parameters can be determined yielding the system natural frequencies $f_{n,j}$ and mode shapes $\boldsymbol{\varphi}_j = e^{\mathrm{i}\Theta_j}\boldsymbol{\rho}_j$ for $j = 1, 2, \ldots, n_2 = 4$. Indeed:

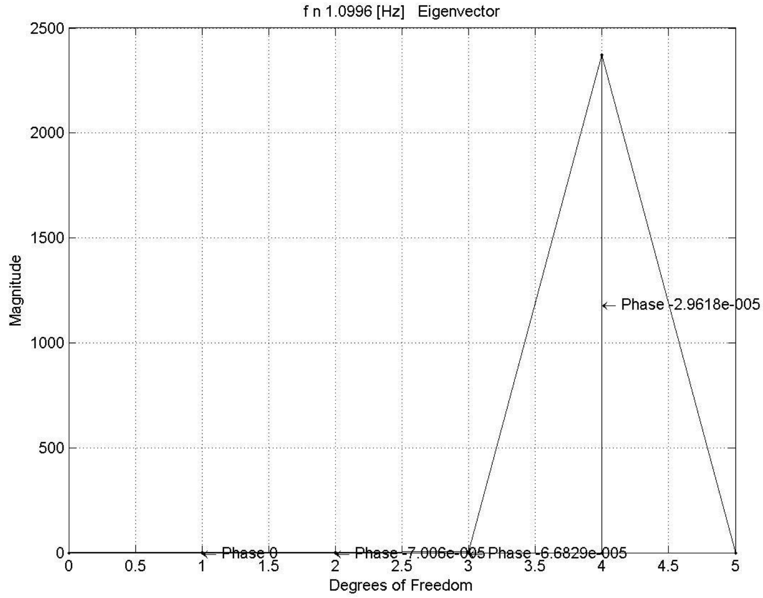
$$f_{n,1} = 1.099 \quad , \quad \boldsymbol{\rho}_1 = \begin{bmatrix} 1 \\ 2.712 \\ 3.572 \\ 237.095 \end{bmatrix} \quad , \quad \boldsymbol{\Theta}_1 = diag(0,0,0,0) \tag{5.67}$$

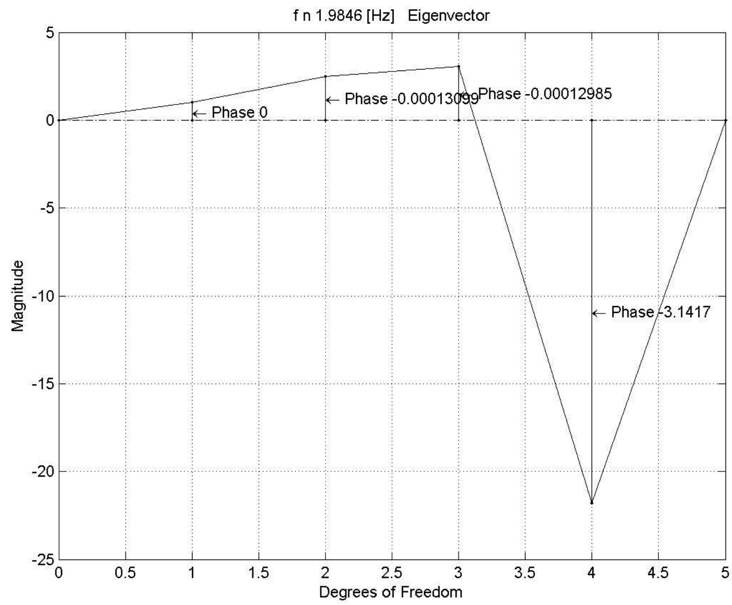
$$f_{n,2} = 1.985 \quad , \quad \boldsymbol{\rho}_2 = \begin{bmatrix} 1 \\ 2.502 \\ 3.069 \\ 21.782 \end{bmatrix} \quad , \quad \boldsymbol{\Theta}_2 = diag(0,0,0,-3.142) \tag{5.68}$$

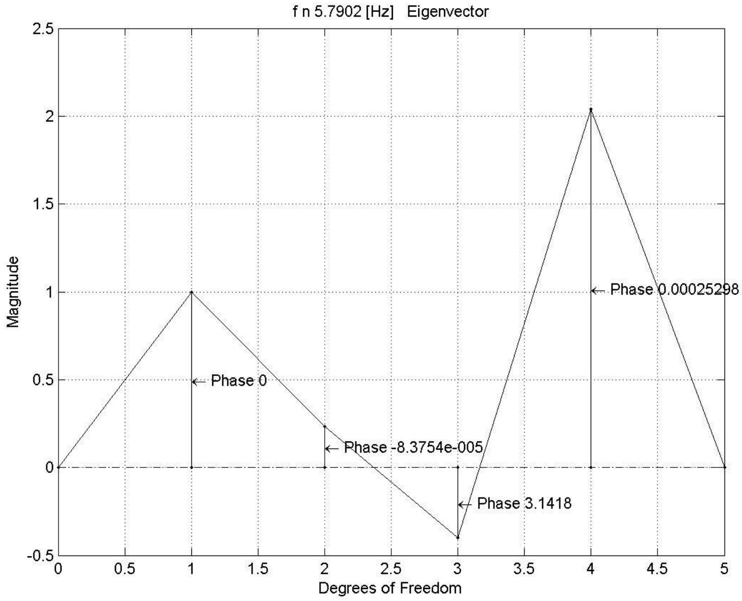
$$f_{n,3} = 5.790 \quad , \quad \boldsymbol{\rho}_3 = \begin{bmatrix} 1 \\ 0.237 \\ 0.401 \\ 2.041 \end{bmatrix} \quad , \quad \boldsymbol{\Theta}_3 = diag(0,0,-3.142,0) \tag{5.69}$$

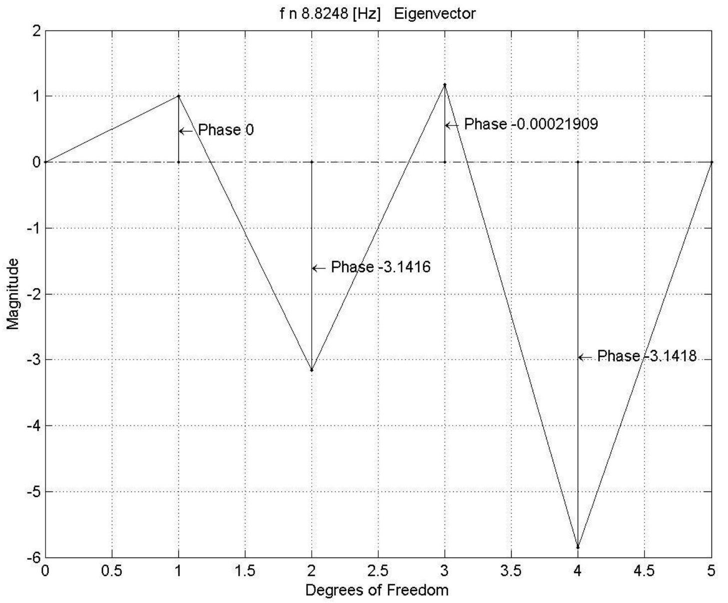
$$f_{n,4} = 8.825 \quad , \quad \boldsymbol{\rho}_4 = \begin{bmatrix} 1 \\ 3.162 \\ 1.176 \\ 5.857 \end{bmatrix} \quad , \quad \boldsymbol{\Theta}_4 = diag(0,-3.142,0,-3.142) \tag{5.70}$$

The system mode shapes can be represented graphically as follows:


f n 1.0996 [Hz] Eigenvector
Magnitude
Degrees of Freedom
Phase -2.9618e-005
Phase 0
Phase -7.006e-005
Phase -6.6829e-005
0
500
1000
1500
2000
2500
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5


f n 1.9846 [Hz] Eigenvector
Magnitude
Degrees of Freedom
Phase 0
Phase -0.00013099
Phase -0.00012985
Phase -3.1417
5
0
-5
-10
-15
-20
-25
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

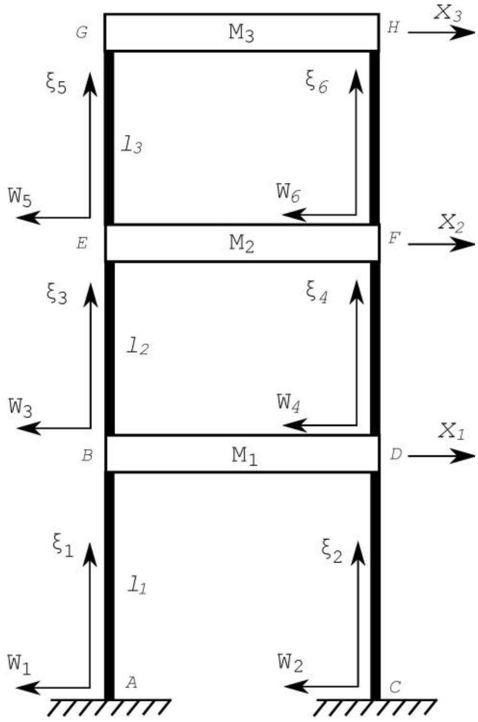

f n 5.7902 [Hz] Eigenvector
Magnitude
Degrees of Freedom
Phase 0
Phase -8.3754e-005
Phase 3.1418
Phase 0.00025298
2.5
2
1.5
1
0.5
0
-0.5
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5


f n 8.8248 [Hz] Eigenvector
Magnitude
Degrees of Freedom
Phase 0
Phase -0.00021909
Phase -3.1416
Phase -3.1418
2
1
0
-1
-2
-3
-4
-5
-6
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

These graphics represent the system mode shapes obtained from the linearized lumped parameter model.

5.4. FINITE ELEMENT MODEL

Consider the following schematization of the test rig:



According to the finite element discretization, the system is modelled as a set of three rigid bodies representing the floors of the building connected by six fixed end beams representing the system structural elements. For the sake of simplicity, each beam is foremost discretized in three elements of equal lengths. Subsequently, a more complex finite element model composed of an arbitrary number of elements can be derived from this preliminary model in a systematic fashion. In addition, a pendulum hinged on the third floor is considered. The preliminary finite element model consider a set of $N_b = 6$ elastic bodies which are all discretized in $N_e^i = 3$ elements for $i = 1, 2, \dots, N_b$. The elastic deformation of each element is modelled assuming a beam shape function $\mathbf{S}^{i,j}(P^{i,j})$ for $i = 1, 2, \dots, N_b$ and $j = 1, 2, \dots, N_e^i$. Indeed:

$$\mathbf{S}^{i,j}(P^{i,j}) = \begin{bmatrix} \mathbf{S}_1^{i,j}(P^{i,j}) \\ \mathbf{S}_2^{i,j}(P^{i,j}) \end{bmatrix} \tag{5.71}$$

Where $\mathbf{S}_1^{i,j}(P^{i,j})$ and $\mathbf{S}_2^{i,j}(P^{i,j})$ are:

$$\mathbf{S}_1^{i,j}(P^{i,j}) = \begin{bmatrix} 1-\xi^{i,j} & 0 & 0 & \xi^{i,j} & 0 & 0 \end{bmatrix} \tag{5.72}$$

$$\mathbf{S}_2^{i,jT}(P^{i,j}) = \\ = \begin{bmatrix} 0 \\ 1-3\xi^{i,j\,2}+2\xi^{i,j\,3} \\ L^{i,j}\left(\xi^{i,j}-2\xi^{i,j\,2}+\xi^{i,j\,3}\right) \\ 0 \\ 3\xi^{i,j\,2}-2\xi^{i,j\,3} \\ L^{i,j}\left(-\xi^{i,j\,2}+\xi^{i,j\,3}\right) \end{bmatrix}^T \tag{5.73}$$

Where $\xi^{i,j}$ is a dimensionless spatial coordinate defined as:

$$\xi^{i,j} = \frac{x^{i,j}}{L^{i,j}} \tag{5.74}$$

Where $L^{i,j}$ is the length relative to the element j of body i. Therefore, the element displacement field $\bar{\mathbf{u}}_f^{i,j}(P^{i,j},t)$ is represented using a set of $n_e^{i,j}=6$ nodal coordinates for $i=1,2,\ldots,N_b$ and $j=1,2,\ldots,N_e^i$. Indeed:

$$\bar{\mathbf{u}}_f^{i,j}(P^{i,j},t) = \mathbf{S}_g^{i,j}(P^{i,j})\mathbf{q}_g^{i,j}(t) \tag{5.75}$$

Where $\mathbf{S}_g^{i,j}(P^{i,j})$ is a matrix function representing the global shape function. The global shape function $\mathbf{S}_g^{i,j}(P^{i,j})$ can be computed by using the local shape function $\mathbf{S}^{i,j}(P^{i,j})$ as:

$$\mathbf{S}_g^{i,j}(P^{i,j}) = \mathbf{C}^{i,j}\mathbf{S}^{i,j}(P^{i,j})\bar{\mathbf{C}}^{i,j} \tag{5.76}$$

Where $\mathbf{C}^{i,j}$ and $\bar{\mathbf{C}}^{i,j}$ are rotation matrices relative to the local element frame of reference which can be computed as:

$$\begin{aligned}\mathbf{C}^{i,j} &= \begin{bmatrix} \cos(\alpha^{i,j}) & -\sin(\alpha^{i,j}) & 0 \\ \sin(\alpha^{i,j}) & \cos(\alpha^{i,j}) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \\ &= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\end{aligned} \tag{5.77}$$

$$\bar{\mathbf{C}}^{i,j} = \begin{bmatrix} \mathbf{C}^{i,j\,T} & \mathbf{O}_{3,3} \\ \mathbf{O}_{3,3} & \mathbf{C}^{i,j\,T} \end{bmatrix} \tag{5.78}$$

Where $\alpha^{i,j} = \dfrac{\pi}{2}$ for $i = 1, 2, \ldots, N_b$ and $j = 1, 2, \ldots, N_e^i$ is the rotation angle between the local reference frame of element j of body i and the inertial reference system. In addition, $\mathbf{q}_g^{i,j}(t)$ for $i = 1, 2, \ldots, N_b$ and $j = 1, 2, \ldots, N_e^i$ is a $\mathbb{R}^{n_e^{i,j}} = \mathbb{R}^6$ vector function expressed in the inertial reference frame corresponding to the vector of nodal coordinates. This vector is defined as:

$$\mathbf{q}_g^{i,j}(t) = \begin{bmatrix} u_{g,1}^{i,j}(t) \\ v_{g,1}^{i,j}(t) \\ \theta_{g,1}^{i,j}(t) \\ u_{g,2}^{i,j}(t) \\ v_{g,2}^{i,j}(t) \\ \theta_{g,2}^{i,j}(t) \end{bmatrix} \tag{5.79}$$

The total number of nodal coordinates relative to each elastic body is $n_b^i = \sum_{j=1}^{N_e^i} n_e^{i,j} = 18$ for $i = 1, 2, \ldots, N_b$ and $j = 1, 2, \ldots, N_e^i$. Since the internal constraints of each elastic body are $n_{ic}^i = 6$ for $i = 1, 2, \ldots, N_b$, the global number of body nodal coordinates is $n_g^i = n_b^i - n_{ic}^i = 18 - 6 = 12$ for $i = 1, 2, \ldots, N_b$. These nodal coordinates can be groped in a $\mathbb{R}^{n_g^i} = \mathbb{R}^{12}$ vector $\mathbf{q}_g^i(t)$ for $i = 1, 2, \ldots, N_b$ corresponding to each beam as follows:

$$\mathbf{q}_g^{i}(t) = \begin{bmatrix} u_{g,1}^{i}(t) \\ v_{g,1}^{i}(t) \\ \theta_{g,1}^{i}(t) \\ u_{g,2}^{i}(t) \\ v_{g,2}^{i}(t) \\ \theta_{g,2}^{i}(t) \\ u_{g,3}^{i}(t) \\ v_{g,3}^{i}(t) \\ \theta_{g,3}^{i}(t) \end{bmatrix} \tag{5.80}$$

The vector of element coordinates $\mathbf{q}_g^{i,j}(t)$ for $i = 1, 2, \ldots, N_b$ and $j = 1, 2, \ldots, N_e^i$ can be expressed in terms of the vector of body coordinates

$\mathbf{q}_g^i(t)$ for each body $i = 1, 2, \ldots, N_b$ by using a set of $\mathbb{R}^{n_e^{i,j} \times n_g^i} = \mathbb{R}^{6\times 12}$ Boolean matrices $\mathbf{B}_c^{i,j}$ for $i = 1, 2, \ldots, N_b$ and $j = 1, 2, \ldots, N_e^i$ relative to internal constraints. These matrices are defined as:

$$\mathbf{B}_c^{i,1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.81}$$

$$\mathbf{B}_c^{i,2} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{5.82}$$

$$\mathbf{B}_c^{i,3} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{5.83}$$

Indeed:

$$\mathbf{q}_g^{i,j}(t) = \mathbf{B}_c^{i,j}\mathbf{q}_g^i(t) \tag{5.84}$$

On the other hand, the first two bodies have a number of external constraints equal to $n_{ec}^1 = n_{ec}^2 = 7$ whereas the external constraints of the last four bodies are $n_{ec}^3 = n_{ec}^4 = n_{ec}^5 = n_{ec}^6 = 6$. Hence, the free nodal coordinates relative to the first two beams are equal to $n_f^1 = n_g^1 - n_{ec}^1 = 12 - 7 = 5 = n_f^2$ whereas the free nodal coordinates relative to the last four beams are $n_f^3 = n_g^3 - n_{ec}^3 = 12 - 6 = 6 = n_f^4 = n_f^5 = n_f^6$. These free nodal coordinates can be grouped in a set of $\mathbb{R}^{n_f^i}$ vector functions $\mathbf{q}_f^i(t)$ for $i = 1, 2, \ldots, N_b$ defined as:

$$\mathbf{q}_f^1(t) = \begin{bmatrix} u_{f,1}^1(t) \\ \theta_{f,1}^1(t) \\ u_{f,2}^1(t) \\ \theta_{f,2}^1(t) \\ u_{f,3}^1(t) \end{bmatrix} \tag{5.85}$$

$$\mathbf{q}_f^2(t) = \begin{bmatrix} u_{f,1}^2(t) \\ \theta_{f,1}^2(t) \\ u_{f,2}^2(t) \\ \theta_{f,2}^2(t) \\ u_{f,3}^2(t) \end{bmatrix} \tag{5.86}$$

$$\mathbf{q}_f^3(t) = \begin{bmatrix} u_{f,1}^3(t) \\ u_{f,2}^3(t) \\ \theta_{f,1}^3(t) \\ u_{f,3}^3(t) \\ \theta_{f,2}^3(t) \\ u_{f,4}^3(t) \end{bmatrix} \tag{5.87}$$

$$\mathbf{q}_f^4(t) = \begin{bmatrix} u_{f,1}^4(t) \\ u_{f,2}^4(t) \\ \theta_{f,1}^4(t) \\ u_{f,3}^4(t) \\ \theta_{f,2}^4(t) \\ u_{f,4}^4(t) \end{bmatrix} \tag{5.88}$$

$$\mathbf{q}_f^5(t) = \begin{bmatrix} u_{f,1}^5(t) \\ u_{f,2}^5(t) \\ \theta_{f,1}^5(t) \\ u_{f,3}^5(t) \\ \theta_{f,2}^5(t) \\ u_{f,4}^5(t) \end{bmatrix} \tag{5.89}$$

$$\mathbf{q}_f^6(t) = \begin{bmatrix} u_{f,1}^6(t) \\ u_{f,2}^6(t) \\ \theta_{f,1}^6(t) \\ u_{f,3}^6(t) \\ \theta_{f,2}^6(t) \\ u_{f,4}^6(t) \end{bmatrix} \tag{5.90}$$

The global vector of body coordinates $\mathbf{q}_g^i(t)$ for $i = 1, 2, \ldots, N_b$ can be expressed in terms of the vector of body free coordinates $\mathbf{q}_f^i(t)$ for $i = 1, 2, \ldots, N_b$ by using a set of $\mathbb{R}^{n_g^i \times n_f^i}$ Boolean matrices $\mathbf{B}_e^i$ for $i = 1, 2, \ldots, N_b$ relative to the external constraints. These matrices are defined as:

$$\mathbf{B}_e^1 = \mathbf{B}_e^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.91}$$

$$\mathbf{B}_e^3 = \mathbf{B}_e^4 = \mathbf{B}_e^5 = \mathbf{B}_e^6 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.92}$$

Indeed:

$$\mathbf{q}_g^i(t) = \mathbf{B}_e^i \mathbf{q}_f^i(t) \tag{5.93}$$

Finally, subtracting the number of mutual constraints between the bodies $n_c = 7$ from the global number of bodies degrees of freedom $n_g = \sum_{i=1}^{N_b} n_f^i = 34$ the system elastic degrees of freedom $n_f = n_g - n_c = 34 - 7 = 27$ can be computed. These elastic degrees of freedom can be grouped in a $\mathbb{R}^{n_f} = \mathbb{R}^{27}$ vector function $\mathbf{q}_f(t)$ defined as:

$$\mathbf{q}_f(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ u_4(t) \\ \theta_4(t) \\ u_5(t) \\ \theta_5(t) \\ u_6(t) \\ \theta_6(t) \\ u_7(t) \\ \theta_7(t) \\ u_8(t) \\ \theta_8(t) \\ u_9(t) \\ \theta_9(t) \\ u_{10}(t) \\ \theta_{10}(t) \\ u_{11}(t) \\ \theta_{11}(t) \\ u_{12}(t) \\ \theta_{12}(t) \\ u_{13}(t) \\ \theta_{13}(t) \\ u_{14}(t) \\ \theta_{14}(t) \\ u_{15}(t) \\ \theta_{15}(t) \end{bmatrix} \tag{5.94}$$

The vector of body free coordinates $\mathbf{q}_f^i(t)$ for $i = 1, 2, \ldots, N_b$ can be expressed in terms of the total vector of free coordinates $\mathbf{q}_f(t)$ by using a set of $\mathbb{R}^{n_f \times n_f^i}$ Boolean matrices $\mathbf{B}_f^i$ for $i = 1, 2, \ldots, N_b$ relative to the mutual constraints. These matrices are defined as:

$$\mathbf{B}_f^1 = \begin{bmatrix} \mathbf{0}_4 & \mathbf{O}_{4,2} & \mathbf{I}_{4,4} & \mathbf{O}_{4,20} \\ 1 & \mathbf{0}_2^T & \mathbf{0}_4^T & \mathbf{0}_{20}^T \end{bmatrix} \tag{5.95}$$

$$\mathbf{B}_f^2 = \begin{bmatrix} \mathbf{0}_4 & \mathbf{O}_{4,6} & \mathbf{I}_{4,4} & \mathbf{O}_{4,16} \\ 1 & \mathbf{0}_6^T & \mathbf{0}_4^T & \mathbf{0}_{16}^T \end{bmatrix} \tag{5.96}$$

$$\mathbf{B}_f^3 = \begin{bmatrix} 1 & 0 & \mathbf{0}_9^T & \mathbf{0}_4^T & \mathbf{0}_{12}^T \\ \mathbf{0}_4 & \mathbf{0}_4 & \mathbf{O}_{4,9} & \mathbf{I}_{4,4} & \mathbf{O}_{4,12} \\ 0 & 1 & \mathbf{0}_9^T & \mathbf{0}_4^T & \mathbf{0}_{12}^T \end{bmatrix} \tag{5.97}$$

$$\mathbf{B}_f^4 = \begin{bmatrix} 1 & 0 & \mathbf{0}_{13}^T & \mathbf{0}_4^T & \mathbf{0}_8^T \\ \mathbf{0}_4 & \mathbf{0}_4 & \mathbf{O}_{4,13} & \mathbf{I}_{4,4} & \mathbf{O}_{4,8} \\ 0 & 1 & \mathbf{0}_{13}^T & \mathbf{0}_4^T & \mathbf{0}_8^T \end{bmatrix} \tag{5.98}$$

$$\mathbf{B}_f^5 = \begin{bmatrix} 0 & 1 & 0 & \mathbf{0}_{16}^T & \mathbf{0}_4^T & \mathbf{0}_4^T \\ \mathbf{0}_4 & \mathbf{0}_4 & \mathbf{0}_4 & \mathbf{O}_{4,16} & \mathbf{I}_{4,4} & \mathbf{O}_{4,4} \\ 0 & 0 & 1 & \mathbf{0}_{16}^T & \mathbf{0}_4^T & \mathbf{0}_4^T \end{bmatrix} \tag{5.99}$$

$$\mathbf{B}_f^6 = \begin{bmatrix} 0 & 1 & 0 & \mathbf{0}_{20}^T & \mathbf{0}_4^T \\ \mathbf{0}_4 & \mathbf{0}_4 & \mathbf{0}_4 & \mathbf{O}_{4,20} & \mathbf{I}_{4,4} \\ 0 & 0 & 1 & \mathbf{0}_{20}^T & \mathbf{0}_4^T \end{bmatrix} \tag{5.100}$$

Indeed:

$$\mathbf{q}_f^i(t) = \mathbf{B}_f^i \mathbf{q}_f(t) \tag{5.101}$$

Consequently, the displacement vector of the generic point $P^{i,j}$ on element j of body i can be expressed as follows:

$$\begin{aligned}\bar{\mathbf{u}}_f^{i,j}(P^{i,j},t) &= \mathbf{S}_g^{i,j}(P^{i,j})\mathbf{q}_g^{i,j}(t) = \\ &= \mathbf{S}_g^{i,j}(P^{i,j})\mathbf{B}_c^{i,j}\mathbf{q}_g^i(t) = \\ &= \mathbf{S}_g^{i,j}(P^{i,j})\mathbf{B}_c^{i,j}\mathbf{B}_e^i\mathbf{q}_f^i(t) = \\ &= \mathbf{S}_g^{i,j}(P^{i,j})\mathbf{B}_c^{i,j}\mathbf{B}_e^i\mathbf{B}_f^i\mathbf{q}_f(t) = \\ &= \mathbf{N}^{i,j}(P^{i,j})\mathbf{B}_e^i\mathbf{B}_f^i\mathbf{q}_f(t)\end{aligned} \tag{5.102}$$

Using the expression of the kinetic energy of the element j of body i the corresponding mass matrix can be computed as:

$$\mathbf{M}_{f,f}^{i,j} = \mathbf{B}_f^{iT}\mathbf{B}_e^{iT}\bar{\mathbf{J}}_{f,f}^{i,j}\mathbf{B}_e^i\mathbf{B}_f^i \tag{5.103}$$

Where $\bar{\mathbf{J}}_{f,f}^{i,j}$ is a symmetric matrix defined as:

$$\bar{\mathbf{J}}_{f,f}^{i,j} = \mathbf{B}_c^{i,jT}\bar{\mathbf{C}}^{i,jT}\mathbf{S}_{f,f}^{i,j}\bar{\mathbf{C}}^{i,j}\mathbf{B}_c^{i,j} \tag{5.104}$$

Where $\mathbf{S}_{f,f}^{i,j}$ is a symmetric matrix defined as follows:

$$\mathbf{S}_{f,f}^{i,j} = \bar{\mathbf{S}}_{1,1}^{i,j} + \bar{\mathbf{S}}_{2,2}^{i,j} \tag{5.105}$$

Where the symmetric matrices $\bar{\mathbf{S}}_{1,1}^{i,j}$ and $\bar{\mathbf{S}}_{2,2}^{i,j}$ come from the integration of the shape function and are defined as:

$$\bar{\mathbf{S}}_{1,1}^{i,j} = \int_{\Omega^{i,j}} \rho^{i,j}\mathbf{S}_1^{i,jT}(P^{i,j})\mathbf{S}_1^{i,j}(P^{i,j})dV^{i,j} \tag{5.106}$$

$$\overline{\mathbf{S}}_{2,2}^{i,j} = \int_{\Omega^{i,j}} \rho^{i,j} \mathbf{S}_2^{i,j\,T}(P^{i,j}) \mathbf{S}_2^{i,j}(P^{i,j}) dV^{i,j} \tag{5.107}$$

Where $\rho^{i,j}$ is the mass density relative to the element j of body i. The spatial integration can be performed using the beam shape function yielding to the following matrix:

$$\mathbf{S}_{f,f}^{i,j} = m^{i,j} \begin{bmatrix} \frac{1}{3} & 0 & 0 & \frac{1}{6} & 0 & 0 \\ 0 & \frac{13}{35} & \frac{11L^{i,j}}{210} & 0 & \frac{9}{70} & -\frac{13L^{i,j}}{420} \\ 0 & \frac{11L^{i,j}}{210} & \frac{L^{i,j\,2}}{105} & 0 & \frac{13L^{i,j}}{420} & -\frac{L^{i,j\,2}}{140} \\ \frac{1}{6} & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{9}{70} & \frac{13L^{i,j}}{420} & 0 & \frac{13}{35} & -\frac{11L^{i,j}}{210} \\ 0 & -\frac{13L^{i,j}}{420} & -\frac{L^{i,j\,2}}{140} & 0 & -\frac{11L^{i,j}}{210} & \frac{L^{i,j\,2}}{105} \end{bmatrix} \tag{5.108}$$

Where $m^{i,j} = \rho^{i,j} A^{i,j} L^{i,j}$ is the mass relative to the element j of body i and $A^{i,j}$ is the cross section area of the element j of body i. On the other hand, using the expression of the elastic strain energy of the element j of body i the corresponding stiffness matrix can be computed as:

$$\mathbf{K}_{f,f}^{i,j} = \mathbf{B}_f^{i\,T} \mathbf{B}_e^{i\,T} \overline{\overline{\mathbf{J}}}_{f,f}^{i,j} \mathbf{B}_e^{i} \mathbf{B}_f^{i} \tag{5.109}$$

Where $\overline{\overline{\mathbf{J}}}_{f,f}^{i,j}$ is a symmetric matrix defined as:

$$\overline{\overline{\mathbf{J}}}_{f,f}^{i,j} = \mathbf{B}_c^{i,j\,T} \overline{\mathbf{C}}^{i,j\,T} \overline{\overline{\mathbf{S}}}_{f,f}^{i,j} \overline{\mathbf{C}}^{i,j} \mathbf{B}_c^{i,j} \tag{5.110}$$

Where $\bar{\bar{\mathbf{S}}}^{i,j}_{f,f}$ is a symmetric matrix which in the case of beam structural element can be computed as follows:

$$\bar{\bar{\mathbf{S}}}^{i,j}_{f,f} = \int_{\Omega^{i,j}} E^{i,j} A^{i,j} \mathbf{S}^{i,j\,T}_{1,x}(P^{i,j}) \mathbf{S}^{i,j}_{1,x}(P^{i,j}) + E^{i,j} J^{i,j} \mathbf{S}^{i,j\,T}_{2,xx}(P^{i,j}) \mathbf{S}^{i,j}_{2,xx}(P^{i,j}) dV^{i,j} \tag{5.111}$$

Where $E^{i,j}$ is the Young elastic modulus relative to the element j of body i whereas $J^{i,j}$ is the second moments of area corresponding to the cross section relative to element j of body i. The spatial integration can be performed using the beam shape function yielding to the following matrix:

$$\bar{\bar{\mathbf{S}}}^{i,j}_{f,f} = \frac{E^{i,j} J^{i,j}}{L^{i,j}} \begin{bmatrix} \frac{A^{i,j}}{J^{i,j}} & 0 & 0 & -\frac{A^{i,j}}{J^{i,j}} & 0 & 0 \\ 0 & \frac{12}{L^{i,j\,2}} & \frac{6}{L^{i,j}} & 0 & -\frac{12}{L^{i,j\,2}} & \frac{6}{L^{i,j}} \\ 0 & \frac{6}{L^{i,j}} & 4 & 0 & -\frac{6}{L^{i,j}} & 2 \\ -\frac{A^{i,j}}{J^{i,j}} & 0 & 0 & \frac{A^{i,j}}{J^{i,j}} & 0 & 0 \\ 0 & -\frac{12}{L^{i,j\,2}} & -\frac{6}{L^{i,j}} & 0 & \frac{12}{L^{i,j\,2}} & -\frac{6}{L^{i,j}} \\ 0 & \frac{6}{L^{i,j}} & 2 & 0 & -\frac{6}{L^{i,j}} & 4 \end{bmatrix} \tag{5.112}$$

Once that the mass matrix and the stiffness matrix relative to the element j of body i have been computed, the respective matrices corresponding to the whole structural system can be obtained by a summation over all elements of all bodies. Indeed:

$$\mathbf{M}_{f,f} = \sum_{i=1}^{N_b} \sum_{j=1}^{N_e^i} \mathbf{M}_{f,f}^{i,j} \tag{5.113}$$

$$\mathbf{K}_{f,f} = \sum_{i=1}^{N_b} \sum_{j=1}^{N_e^i} \mathbf{K}_{f,f}^{i,j} \tag{5.114}$$

To complete the derivation of the system structural model the effect of the mass relative to each floor must be considered. First note that the displacement relative to each floor can be recovered from the global vector of free elastic coordinates $\mathbf{q}_f(t)$ using an appropriate set of Boolean matrices. Indeed:

$$u_1(t) = \mathbf{B}_{m_1} \mathbf{q}_f(t) \tag{5.115}$$

$$u_2(t) = \mathbf{B}_{m_2} \mathbf{q}_f(t) \tag{5.116}$$

$$u_3(t) = \mathbf{B}_{m_3} \mathbf{q}_f(t) \tag{5.117}$$

Where $\mathbf{B}_{m_1}$, $\mathbf{B}_{m_2}$ and $\mathbf{B}_{m_3}$ are $\mathbb{R}^{n_f} = \mathbb{R}^{27}$ Boolean row vectors defined as:

$$\mathbf{B}_{m_1} = \begin{bmatrix} 1 & \mathbf{0}_{26}^T \end{bmatrix} \tag{5.118}$$

$$\mathbf{B}_{m_2} = \begin{bmatrix} 0 & 1 & \mathbf{0}_{25}^T \end{bmatrix} \tag{5.119}$$

$$\mathbf{B}_{m_3} = \begin{bmatrix} 0 & 0 & 1 & \mathbf{0}_{24}^T \end{bmatrix} \tag{5.120}$$

In is straightforward to deduce from the kinetic energy definition that the expressions of the mass matrices relative to each floor are the following:

$$\mathbf{M}_{m_1} = m_1 \mathbf{B}_{m_1}^T \mathbf{B}_{m_1} \tag{5.121}$$

$$\mathbf{M}_{m_2} = m_2\mathbf{B}_{m_2}^T\mathbf{B}_{m_2} \tag{5.122}$$

$$\mathbf{M}_{m_3} = m_3\mathbf{B}_{m_3}^T\mathbf{B}_{m_3} \tag{5.123}$$

Where m_1, m_2 and m_3 are the floors masses. Consider now the modelling of the pendulum. The pendulum angle $\theta(t)$ is an additional degree of freedom of the whole system. Indeed, the total number of system degrees of freedom is $n = n_f + 1 = 27 + 1 = 28$. Consequently, the overall vector of lagrangian coordinates becomes:

$$\mathbf{q}(t) = \begin{bmatrix} \theta(t) \\ \mathbf{q}_f(t) \end{bmatrix} \tag{5.124}$$

Where $\mathbf{q}(t)$ is a $\mathbb{R}^n = \mathbb{R}^{28}$ vector. The pendulum angle $\theta(t)$ and the vector of elastic nodal coordinates $\mathbf{q}_f(t)$ can be recovered from the global vector of lagrangian coordinates $\mathbf{q}(t)$ as follows:

$$\theta(t) = \mathbf{B}_\theta\mathbf{q}(t) \tag{5.125}$$

$$\mathbf{q}_f(t) = \mathbf{B}_{q_f}\mathbf{q}(t) \tag{5.126}$$

Where $\mathbf{B}_\theta$ and $\mathbf{B}_{q_f}$ are two Boolean matrices respectively of dimensions $\mathbb{R}^n = \mathbb{R}^{28}$ and $\mathbb{R}^{n_f \times n} = \mathbb{R}^{27\times 28}$ which are defined as:

$$\mathbf{B}_\theta = \begin{bmatrix} 1 & \mathbf{0}_{27}^T \end{bmatrix} \tag{5.127}$$

$$\mathbf{B}_{q_f} = \begin{bmatrix} \mathbf{0}_{27} & \mathbf{I}_{27,27} \end{bmatrix} \tag{5.128}$$

Consequently, the mass and stiffness matrices relative to the structural system referred to the vector of the overall lagrangian coordinates $\mathbf{q}(t)$ becomes:

$$\mathbf{M}^{q}_{f,f} = \mathbf{B}^{T}_{q_f} \mathbf{M}_{f,f} \mathbf{B}_{q_f} \tag{5.129}$$

$$\mathbf{K}^{q}_{f,f} = \mathbf{B}^{T}_{q_f} \mathbf{K}_{f,f} \mathbf{B}_{q_f} \tag{5.130}$$

In addition, the mass matrices relative to the floors referred to the vector of system degrees of freedom $\mathbf{q}(t)$ becomes:

$$\begin{aligned} \mathbf{M}^{q}_{m_1} &= \mathbf{B}^{T}_{q_f} \mathbf{M}_{m_1} \mathbf{B}_{q_f} = \\ &= m_1 \mathbf{B}^{T}_{q_f} \mathbf{B}^{T}_{m_1} \mathbf{B}_{m_1} \mathbf{B}_{q_f} \end{aligned} \tag{5.131}$$

$$\begin{aligned} \mathbf{M}^{q}_{m_2} &= \mathbf{B}^{T}_{q_f} \mathbf{M}_{m_2} \mathbf{B}_{q_f} = \\ &= m_2 \mathbf{B}^{T}_{q_f} \mathbf{B}^{T}_{m_2} \mathbf{B}_{m_2} \mathbf{B}_{q_f} \end{aligned} \tag{5.132}$$

$$\begin{aligned} \mathbf{M}^{q}_{m_3} &= \mathbf{B}^{T}_{q_f} \mathbf{M}_{m_3} \mathbf{B}_{q_f} = \\ &= m_3 \mathbf{B}^{T}_{q_f} \mathbf{B}^{T}_{m_3} \mathbf{B}_{m_3} \mathbf{B}_{q_f} \end{aligned} \tag{5.133}$$

The position vector relative to pendulum centre of mass can be expressed as a function of system lagrangian coordinates to yield:

$$\mathbf{R}^{4}(t) = \begin{bmatrix} u_3(t) + L_4 \cos(\theta(t)) \\ L_1 + H_1 + L_2 + H_2 + L_3 + H_3 + L_4 \sin(\theta(t)) \\ 0 \end{bmatrix} \tag{5.134}$$

Where L_1, L_2 and L_3 are the length of the system structural elements, L_4 is equal to half length of the pendulum whereas H_1, H_2 and H_3 are the dimensions of the floors. The time derivative of the pendulum centre of mass position vector can be computed as:

$$\begin{aligned}\dot{\mathbf{R}}^4(t) &= \begin{bmatrix} \dot{u}_3(t) - L_4 \sin(\theta(t))\dot{\theta}(t) \\ L_4 \cos(\theta(t))\dot{\theta}(t) \\ 0 \end{bmatrix} = \\ &= \begin{bmatrix} -L_4 \sin(\theta(t)) & 1 \\ L_4 \cos(\theta(t)) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}(t) \\ \dot{u}_3(t) \end{bmatrix} = \\ &= \begin{bmatrix} -L_4 \sin(\theta(t)) & 1 \\ L_4 \cos(\theta(t)) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{B}_\theta \\ \mathbf{B}_{m_3}\mathbf{B}_{q_f} \end{bmatrix} \dot{\mathbf{q}}(t) = \\ &= \begin{bmatrix} -L_4 \sin(\theta(t)) & 1 \\ L_4 \cos(\theta(t)) & 0 \\ 0 & 0 \end{bmatrix} \mathbf{B}^4 \dot{\mathbf{q}}(t) = \\ &= \mathbf{J}_R^4(t)\dot{\mathbf{q}}(t)\end{aligned} \tag{5.135}$$

Where $\mathbf{B}^4$ is a $\mathbb{R}^{2\times n} = \mathbb{R}^{2\times 28}$ Boolean matrix defined as:

$$\mathbf{B}^4 = \begin{bmatrix} \mathbf{B}_\theta \\ \mathbf{B}_{m_3}\mathbf{B}_{q_f} \end{bmatrix} \tag{5.136}$$

The jacobian transformation matrix $\mathbf{J}_R^4(t)$ is defined as follows:

$$\mathbf{J}_R^4(t) = \begin{bmatrix} -L_4 \sin(\theta(t)) & 1 \\ L_4 \cos(\theta(t)) & 0 \\ 0 & 0 \end{bmatrix} \mathbf{B}^4 \tag{5.137}$$

On the other hand, the angular velocity of the pendulum can be expressed in terms of the independent coordinates as follows:

$$\begin{aligned}\boldsymbol{\omega}^4(t) &= \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}(t) \end{bmatrix} = \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}(t) \\ \dot{u}_3(t) \end{bmatrix} = \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{B}_\theta \\ \mathbf{B}_{m_3}\mathbf{B}_{q_f} \end{bmatrix} \dot{\mathbf{q}}(t) = \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \mathbf{B}^4 \dot{\mathbf{q}}(t) = \\ &= \mathbf{J}_\omega^4(t)\dot{\mathbf{q}}(t)\end{aligned} \tag{5.138}$$

Where the jacobian transformation matrix $\mathbf{J}_\omega^4(t)$ is defined as:

$$\mathbf{J}_\omega^4(t) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \mathbf{B}^4 \tag{5.139}$$

Consequently, the mass matrix relative to the pendulum can be easily derived from the pendulum kinetic energy as follows:

$$
\begin{aligned}
\mathbf{M}^4(t) &= m_4 \mathbf{J}_R^{4T}(t)\mathbf{J}_R^4(t) + I_{zz,4}\mathbf{J}_\omega^{4T}(t)\mathbf{J}_\omega^4(t) = \\
&= m_4 \mathbf{B}^{4T} \begin{bmatrix} -L_4\sin(\theta(t)) & 1 \\ L_4\cos(\theta(t)) & 0 \\ 0 & 0 \end{bmatrix}^T \begin{bmatrix} -L_4\sin(\theta(t)) & 1 \\ L_4\cos(\theta(t)) & 0 \\ 0 & 0 \end{bmatrix} \mathbf{B}^4 + \\
&+ I_{zz,4}\mathbf{B}^{4T} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}^T \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \mathbf{B}^4 = \\
&= m_4 \mathbf{B}^{4T} \begin{bmatrix} L_4^2 & -L_4\sin(\theta(t)) \\ -L_4\sin(\theta(t)) & 1 \end{bmatrix} \mathbf{B}^4 + \\
&+ I_{zz,4}\mathbf{B}^{4T} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{B}^4 = \\
&= \mathbf{B}^{4T} \begin{bmatrix} m_4 L_4^2 + I_{zz,4} & -m_4 L_4\sin(\theta(t)) \\ -m_4 L_4 \sin(\theta(t)) & m_4 \end{bmatrix} \mathbf{B}^4
\end{aligned} \tag{5.140}
$$

The mass matrix relative to the whole system can be derived again from the expression of the kinetic energy thus obtaining the summation of all the components mass matrices. Indeed:

$$
\begin{aligned}
T(t) &= T_{f,f}(t) + T_{m_1}(t) + T_{m_2}(t) + T_{m_3}(t) + T^4(t) = \\
&= \frac{1}{2}\dot{\mathbf{q}}^T(t)\mathbf{M}_{f,f}^q\dot{\mathbf{q}}(t) + \frac{1}{2}\dot{\mathbf{q}}^T(t)\mathbf{M}_{m_1}^q\dot{\mathbf{q}}(t) + \frac{1}{2}\dot{\mathbf{q}}^T(t)\mathbf{M}_{m_2}^q\dot{\mathbf{q}}(t) + \\
&+ \frac{1}{2}\dot{\mathbf{q}}^T(t)\mathbf{M}_{m_3}^q\dot{\mathbf{q}}(t) + \frac{1}{2}\dot{\mathbf{q}}^T(t)\mathbf{M}^4(t)\dot{\mathbf{q}}(t) = \\
&= \frac{1}{2}\dot{\mathbf{q}}^T(t)\left(\mathbf{M}_{f,f}^q + \mathbf{M}_{m_1}^q + \mathbf{M}_{m_2}^q + \mathbf{M}_{m_3}^q + \mathbf{M}^4(t)\right)\dot{\mathbf{q}}(t) = \\
&= \frac{1}{2}\dot{\mathbf{q}}^T(t)\mathbf{M}(t)\dot{\mathbf{q}}(t)
\end{aligned} \tag{5.141}
$$

Where the global mass matrix $\mathbf{M}(t)$ is defined as:

$$\mathbf{M}(t) = \mathbf{M}^{q}_{f,f} + \mathbf{M}^{q}_{m_1} + \mathbf{M}^{q}_{m_2} + \mathbf{M}^{q}_{m_3} + \mathbf{M}^{4}(t) \tag{5.142}$$

Once that the mass matrix and the kinetic energy of the whole system have been obtained, the two terms which form the quadratic velocity vector can be computed as:

$$\begin{aligned}
&-\dot{\mathbf{M}}(t)\dot{\mathbf{q}}(t) = -\dot{\mathbf{M}}^{4}(t)\dot{\mathbf{q}}(t) = \\
&= -\mathbf{B}^{4T}\left(\frac{d}{dt}\begin{bmatrix} m_4 L_4^2 + I_{zz,4} & -m_4 L_4 \sin(\theta(t)) \\ -m_4 L_4 \sin(\theta(t)) & m_4 \end{bmatrix}\right)\mathbf{B}^{4}\dot{\mathbf{q}}(t) = \\
&= -\mathbf{B}^{4T}\begin{bmatrix} 0 & -m_4 L_4 \cos(\theta(t))\dot{\theta}(t) \\ -m_4 L_4 \cos(\theta(t))\dot{\theta}(t) & 0 \end{bmatrix}\begin{bmatrix} \dot{\theta}(t) \\ \dot{u}_3(t) \end{bmatrix} = \\
&= \mathbf{B}^{4T}\begin{bmatrix} m_4 L_4 \cos(\theta(t))\dot{\theta}(t)\dot{u}_3(t) \\ m_4 L_4 \cos(\theta(t))\dot{\theta}^2(t) \end{bmatrix} = \\
&= \begin{bmatrix} m_4 L_4 \cos(\theta(t))\dot{\theta}(t)\dot{u}_3(t) \\ 0 \\ 0 \\ m_4 L_4 \cos(\theta(t))\dot{\theta}^2(t) \\ \mathbf{0}_{24} \end{bmatrix} = \\
&= \mathbf{B}_{\theta}^{T} m_4 L_4 \cos(\theta(t))\dot{\theta}(t)\dot{u}_3(t) + \mathbf{B}_{q_f}^{T}\mathbf{B}_{m_3}^{T} m_4 L_4 \cos(\theta(t))\dot{\theta}^2(t)
\end{aligned} \tag{5.143}$$

$$
\begin{aligned}
\left(\frac{\partial T(t)}{\partial \mathbf{q}(t)}\right)^T &= \begin{bmatrix} \dfrac{\partial T(t)}{\partial \theta(t)} \\ \left(\dfrac{\partial T(t)}{\partial \mathbf{q}_f(t)}\right)^T \end{bmatrix} = \\
&= \begin{bmatrix} \dfrac{1}{2}\dot{\mathbf{q}}^T(t)\dfrac{\partial \mathbf{M}(t)}{\partial \theta(t)}\dot{\mathbf{q}}(t) \\ \mathbf{0}_{27} \end{bmatrix} = \\
&= \begin{bmatrix} \dfrac{1}{2}\dot{\mathbf{q}}^T(t)\dfrac{\partial \mathbf{M}^4(t)}{\partial \theta(t)}\dot{\mathbf{q}}(t) \\ \mathbf{0}_{27} \end{bmatrix} = \\
&= \begin{bmatrix} \dfrac{1}{2}\dot{\mathbf{q}}^T(t)\mathbf{B}^{4T}\dfrac{\partial}{\partial \theta(t)}\begin{bmatrix} m_4 L_4^2 + I_{zz,4} & -m_4 L_4 \sin(\theta(t)) \\ -m_4 L_4 \sin(\theta(t)) & m_4 \end{bmatrix}\mathbf{B}^4\dot{\mathbf{q}}(t) \\ \mathbf{0}_{27} \end{bmatrix} = \\
&= \begin{bmatrix} \dfrac{1}{2}\begin{bmatrix}\dot{\theta}(t) & \dot{u}_3(t)\end{bmatrix}\begin{bmatrix} 0 & -m_4 L_4 \cos(\theta(t)) \\ -m_4 L_4 \cos(\theta(t)) & 0 \end{bmatrix}\begin{bmatrix}\dot{\theta}(t) \\ \dot{u}_3(t)\end{bmatrix} \\ \mathbf{0}_{27} \end{bmatrix} = \\
&= \begin{bmatrix} \dfrac{1}{2}\begin{bmatrix}\dot{\theta}(t) & \dot{u}_3(t)\end{bmatrix}\begin{bmatrix} -m_4 L_4 \cos(\theta(t))\dot{u}_3(t) \\ -m_4 L_4 \cos(\theta(t))\dot{\theta}(t) \end{bmatrix} \\ \mathbf{0}_{27} \end{bmatrix} = \\
&= \begin{bmatrix} -m_4 L_4 \cos(\theta(t))\dot{\theta}(t)\dot{u}_3(t) \\ \mathbf{0}_{27} \end{bmatrix} = \\
&= -\mathbf{B}_\theta^T m_4 L_4 \cos(\theta(t))\dot{\theta}(t)\dot{u}_3(t)
\end{aligned} \tag{5.144}
$$

Consequently, the system quadratic velocity vector can be computed as follows:

$$\mathbf{Q}_v(t) = -\dot{\mathbf{M}}(t)\dot{\mathbf{q}}(t) + \left(\frac{\partial T(t)}{\partial \mathbf{q}(t)}\right)^T =$$
$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ m_4 L_4 \cos(\theta(t))\dot{\theta}^2(t) \\ \mathbf{0}_{24} \end{bmatrix} = \tag{5.145}$$
$$= \mathbf{B}_{q_f}^T \mathbf{B}_{m_3}^T m_4 L_4 \cos(\theta(t))\dot{\theta}^2(t)$$

Moreover, the pendulum potential energy relative to gravitational force can be simply computed as:

$$U_g(t) = -m_4 \mathbf{g}^T \mathbf{R}_4(t) =$$
$$= -m_4 \begin{bmatrix} 0 & -g & 0 \end{bmatrix} \begin{bmatrix} u_3(t) + L_4 \cos(\theta(t)) \\ L_1 + H_1 + L_2 + H_2 + L_3 + H_3 + L_4 \sin(\theta(t)) \\ 0 \end{bmatrix} =$$
$$= m_4 g \left(L_1 + H_1 + L_2 + H_2 + L_3 + H_3 + L_4 \sin(\theta(t))\right) \tag{5.146}$$

Consequently, the total potential energy of the system can be computed summing the strain potential energy of the beams and the gravitational potential energy of the pendulum yielding to:

$$U(t) = U_f(t) + U_g(t) =$$
$$= \frac{1}{2}\mathbf{q}^T(t)\mathbf{K}_{f,f}^q \mathbf{q}(t) + m_4 g \left(L_1 + H_1 + L_2 + H_2 + L_3 + H_3 + L_4 \sin(\theta(t))\right) \tag{5.147}$$

Therefore, the lagrangian component of the conservative external forces acting on the system can be determined as follows:

$$
\begin{aligned}
&\mathbf{Q}_{e,c}(t) = -\left(\frac{\partial U(t)}{\partial \mathbf{q}(t)}\right)^T = \\
&= -\left(\frac{\partial}{\partial \mathbf{q}(t)}\left(\frac{1}{2}\mathbf{q}^T(t)\mathbf{K}^q_{f,f}\mathbf{q}(t) + m_4 g\left(L_1 + H_1 + L_2 + H_2 + L_3 + H_3 + L_4\sin(\theta(t))\right)\right)\right)^T = \\
&= -\mathbf{K}^q_{f,f}(t)\mathbf{q}(t) + \begin{bmatrix} -m_4 g L_4 \cos(\theta(t)) \\ \mathbf{0}_{27} \end{bmatrix} = \\
&= -\mathbf{K}^q_{f,f}(t)\mathbf{q}(t) - \mathbf{B}^T_{\theta} m_4 g L_4 \cos(\theta(t))
\end{aligned}
\tag{5.148}
$$

In addition, the effect of the non-conservative external force acting on the first floors can be accounted for computing its virtual work. Indeed:

$$
\begin{aligned}
\delta W_{e,nc}(t) &= F(t)\delta u_1(t) = \\
&= F(t)\mathbf{B}_{m_1}\mathbf{B}_{q_f}\delta\mathbf{q}(t) = \\
&= \mathbf{Q}^T_{e,nc}(t)\delta\mathbf{q}(t)
\end{aligned}
\tag{5.149}
$$

Thence, the lagrangian component of the non-conservative external forces can be determined as:

$$
\begin{aligned}
\mathbf{Q}_{e,nc}(t) &= \mathbf{B}^T_{q_f}\mathbf{B}^T_{m_1}F(t) = \\
&= \begin{bmatrix} 0 \\ F(t) \\ \mathbf{0}_{26} \end{bmatrix}
\end{aligned}
\tag{5.150}
$$

Consequently, the total lagrangian component of all forces acting on the system can be determined as follows:

$$\mathbf{Q}(t) = \mathbf{Q}_v(t) + \mathbf{Q}_{e,c}(t) + \mathbf{Q}_{e,nc}(t) =$$
$$= -\mathbf{K}^q_{f,f}\mathbf{q}(t) + \begin{bmatrix} -m_4 g L_4 \cos(\theta(t)) \\ F(t) \\ 0 \\ m_4 L_4 \cos(\theta(t))\dot{\theta}^2(t) \\ \mathbf{0}_{24} \end{bmatrix} =$$
$$= -\mathbf{K}^q_{f,f}\mathbf{q}(t) - \mathbf{B}^T_{\theta} m_4 g L_4 \cos(\theta(t)) + \mathbf{B}^T_{q_f}\mathbf{B}^T_{m_1} F(t) + \mathbf{B}^T_{q_f}\mathbf{B}^T_{m_3} m_4 L_4 \cos(\theta(t))\dot{\theta}^2(t) \tag{5.151}$$

On the other hand, the electric motor exerts a control torque on the pendulum whose virtual work is:

$$\begin{aligned} \delta W_c(t) &= C(t)\delta\theta(t) = \\ &= C(t)\mathbf{B}_{\theta}\delta\mathbf{q}(t) = \\ &= \mathbf{Q}^T_c(t)\delta\mathbf{q}(t) \end{aligned} \tag{5.152}$$

Hence, the lagrangian component of the control torque can be determined as:

$$\begin{aligned} \mathbf{Q}_c(t) &= \mathbf{B}^T_{\theta} C(t) = \\ &= \begin{bmatrix} C(t) \\ \mathbf{0}_{27} \end{bmatrix} \end{aligned} \tag{5.153}$$

Finally, the system equations of motion can be expressed in matrix notation using Lagrange equations as:

$$\left(\mathbf{M}^q_{f,f} + \mathbf{M}^q_{m_1} + \mathbf{M}^q_{m_2} + \mathbf{M}^q_{m_3} + \mathbf{B}^{4T}\begin{bmatrix} m_4 L_4^2 + I_{zz,4} & -m_4 L_4 \sin(\theta(t)) \\ -m_4 L_4 \sin(\theta(t)) & m_4 \end{bmatrix}\mathbf{B}^4\right)\ddot{\mathbf{q}}(t) =$$
$$= -\mathbf{K}^q_{f,f}\mathbf{q}(t) - \mathbf{B}^T_\theta m_4 g L_4 \cos(\theta(t)) + \mathbf{B}^T_{q_f}\mathbf{B}^T_{m_1}F(t) + \mathbf{B}^T_{q_f}\mathbf{B}^T_{m_3}m_4 L_4 \cos(\theta(t))\dot{\theta}^2(t) +$$
$$+\mathbf{B}^T_\theta C(t) \tag{5.154}$$

The compact form of these equations is the following:

$$\mathbf{M}(t)\ddot{\mathbf{q}}(t) = \mathbf{Q}(t) + \mathbf{Q}_c(t) \tag{5.155}$$

This set of motion equations represents the system flexible multibody model obtained using the finite element method. Linearizing the flexible multibody model of the system around the stable equilibrium position where $\theta_0 = \frac{3}{2}\pi$ yields:

$$\left(\mathbf{M}^q_{f,f} + \mathbf{M}^q_{m_1} + \mathbf{M}^q_{m_2} + \mathbf{M}^q_{m_3} + \mathbf{B}^{4T}\begin{bmatrix} m_4 L_4^2 + I_{zz,4} & m_4 L_4 \\ m_4 L_4 & m_4 \end{bmatrix}\mathbf{B}^4\right)\ddot{\mathbf{x}}(t) +$$
$$+\left(\mathbf{K}^q_{f,f} + m_4 g L_4 \mathbf{B}^T_\theta \mathbf{B}_\theta\right)\mathbf{x}(t) = \mathbf{B}^T_{q_f}\mathbf{B}^T_{m_1}F(t) + \mathbf{B}^T_\theta C(t) \tag{5.156}$$

Where the following change of variables has been performed:

$$\begin{aligned}\theta(t) &= \varphi(t) + \theta_0 = \\ &= \varphi(t) + \frac{3}{2}\pi\end{aligned} \tag{5.157}$$

The compact form of these equations is the following:

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{B}_{2,e}\mathbf{u}_e(t) + \mathbf{B}_{2,c}\mathbf{u}_c(t) \tag{5.158}$$

Where $\mathbf{x}(t)$ is the vector containing the independent coordinates, $\mathbf{M}$ is the linearized mass matrix and $\mathbf{K}$ is the linearized stiffness matrix whereas $\mathbf{B}_{2,e}$ and $\mathbf{B}_{2,c}$ are the Boolean matrices characterizing the location of the external uncontrolled and controlled inputs $\mathbf{u}_e(t)$ and $\mathbf{u}_c(t)$ acting on the system which correspond to the external uncontrolled force $F(t)$ and to the controlled torque $C(t)$. These quantities are defined as:

$$\mathbf{x}(t) = \begin{bmatrix} \varphi(t) \\ \mathbf{q}_f(t) \end{bmatrix} \tag{5.159}$$

$$\mathbf{M} = \mathbf{M}_{f,f}^{q} + \mathbf{M}_{m_1}^{q} + \mathbf{M}_{m_2}^{q} + \mathbf{M}_{m_3}^{q} + \mathbf{B}^{4T} \begin{bmatrix} m_4 L_4^2 + I_{zz,4} & m_4 L_4 \\ m_4 L_4 & m_4 \end{bmatrix} \mathbf{B}^4 \tag{5.160}$$

$$\mathbf{K} = \mathbf{K}_{f,f}^{q} + m_4 g L_4 \mathbf{B}_\theta^T \mathbf{B}_\theta \tag{5.161}$$

$$\mathbf{B}_{2,e} = \mathbf{B}_{q_f}^T \mathbf{B}_{m_1}^T \tag{5.162}$$

$$\mathbf{B}_{2,c} = \mathbf{B}_\theta^T \tag{5.163}$$

This set of motion equations represents the system linearized flexible multibody model obtained using the finite element method. Finally, using the data reported in the test rig description and considering a discretization of $N_e = 40$ for all the elastic bodies, the system modal parameters can be determined yielding the system natural frequencies $f_{n,j}$ and mode shapes $\boldsymbol{\varphi}_j = e^{\mathbf{i}\boldsymbol{\Theta}_j}\boldsymbol{\rho}_j$ for $j = 1, 2, \ldots, n_{2,t} = 4$. Indeed, the first four modal parameters are the following:

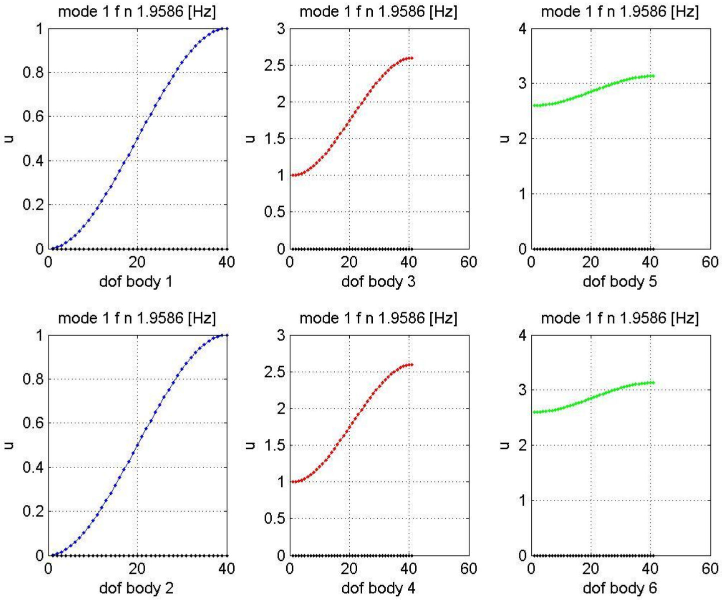
$$f_{n,1} = 1.097 \quad , \quad \boldsymbol{\rho}_1 = \begin{bmatrix} 1 \\ 2.823 \\ 3.667 \\ 2481.958 \end{bmatrix} \quad , \quad \boldsymbol{\Theta}_1 = diag(0,0,0,0) \quad (5.164)$$

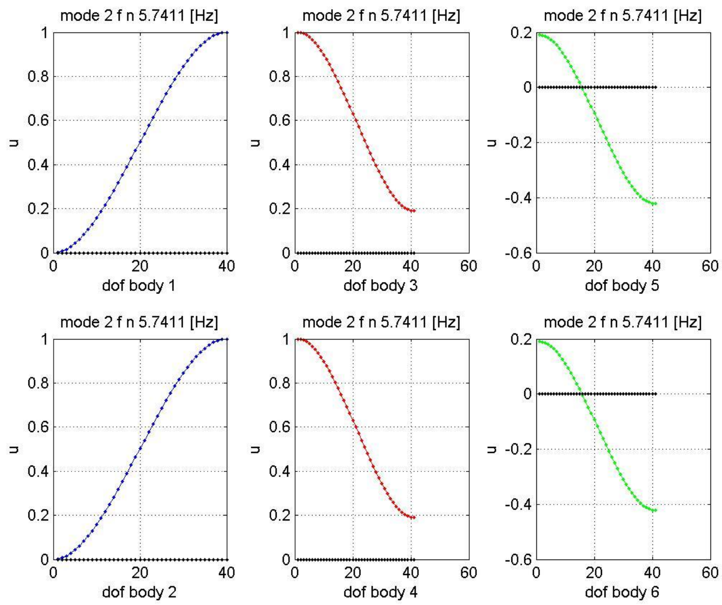
$$f_{n,2} = 1.945 \quad , \quad \boldsymbol{\rho}_2 = \begin{bmatrix} 1 \\ 2.602 \\ 3.145 \\ 26.991 \end{bmatrix} \quad , \quad \boldsymbol{\Theta}_2 = diag(3.142,0,0,0) \quad (5.165)$$

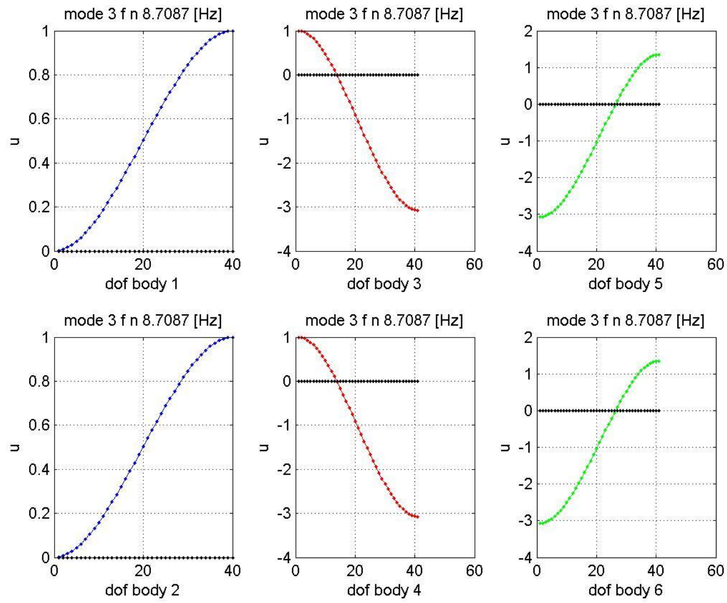
$$f_{n,3} = 5.727 \quad , \quad \boldsymbol{\rho}_3 = \begin{bmatrix} 1 \\ 0.204 \\ 0.413 \\ 2.147 \end{bmatrix} \quad , \quad \boldsymbol{\Theta}_3 = diag(0,0,0,-3.142) \quad (5.166)$$

$$f_{n,4} = 8.678 \quad , \quad \boldsymbol{\rho}_4 = \begin{bmatrix} 1 \\ 3.034 \\ 1.286 \\ 6.473 \end{bmatrix} \quad , \quad \boldsymbol{\Theta}_4 = diag(-3.142,0,-3.142,0)$$
(5.167)

The first three mode shapes relative only to the structural components of the system can be represented graphically as follows:


mode 1 f n 1.9586 [Hz]
mode 1 f n 1.9586 [Hz]
mode 1 f n 1.9586 [Hz]
u
dof body 1
dof body 3
dof body 5
mode 1 f n 1.9586 [Hz]
mode 1 f n 1.9586 [Hz]
mode 1 f n 1.9586 [Hz]
dof body 2
dof body 4
dof body 6

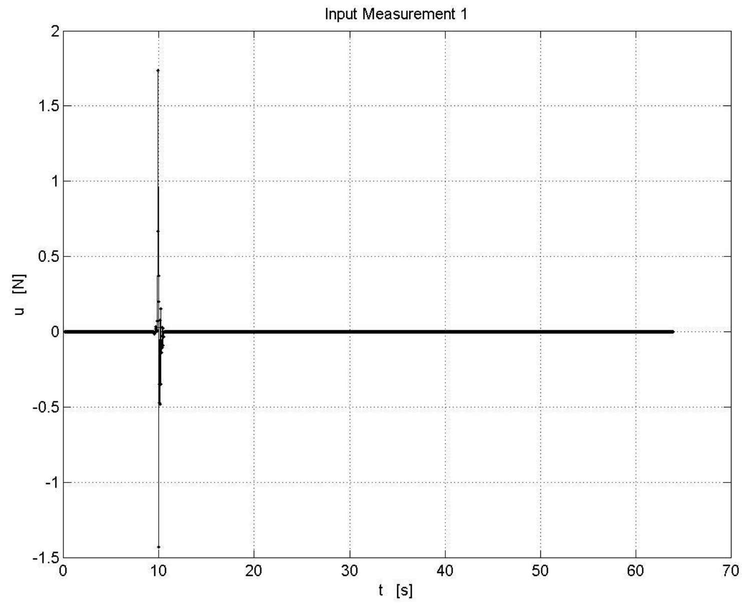

mode 2 f n 5.7411 [Hz]
mode 2 f n 5.7411 [Hz]
mode 2 f n 5.7411 [Hz]
u
dof body 1
dof body 3
dof body 5
mode 2 f n 5.7411 [Hz]
mode 2 f n 5.7411 [Hz]
mode 2 f n 5.7411 [Hz]
dof body 2
dof body 4
dof body 6



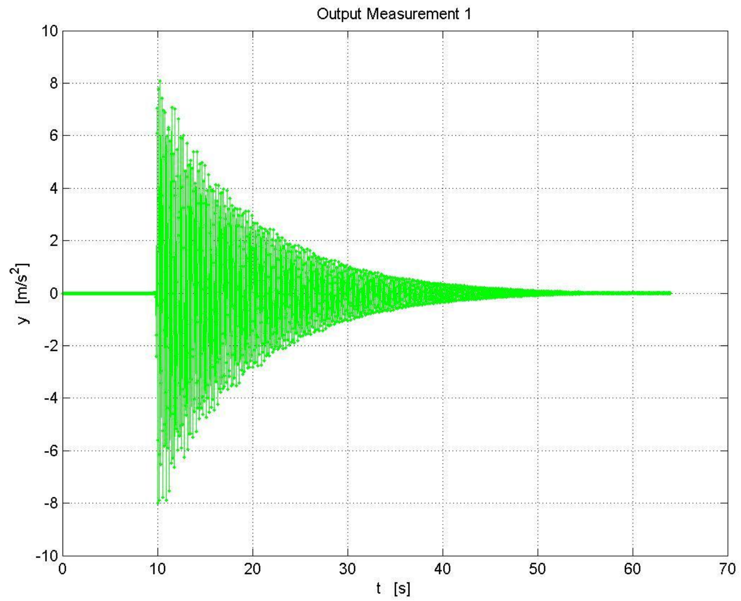
These graphics represent the system mode shapes derived from the linearized flexible multibody model obtained using the finite element method.

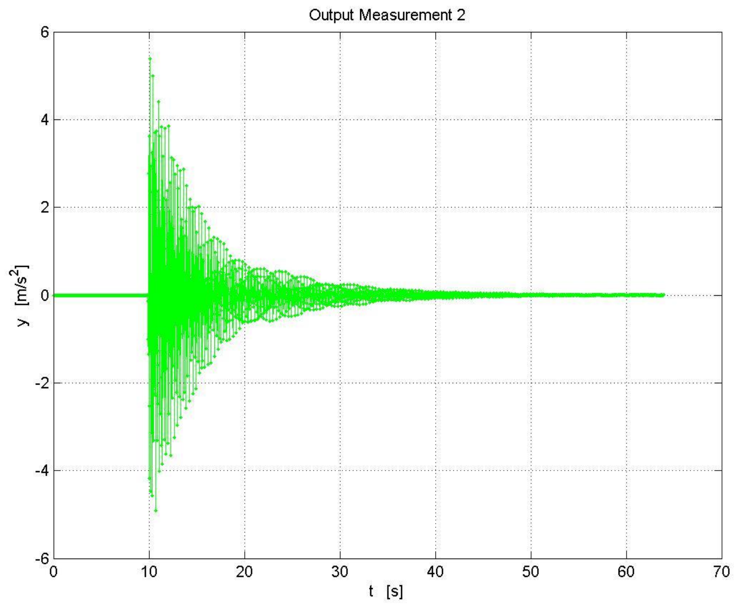
5.5. EXPERIMENTAL IDENTIFICATION

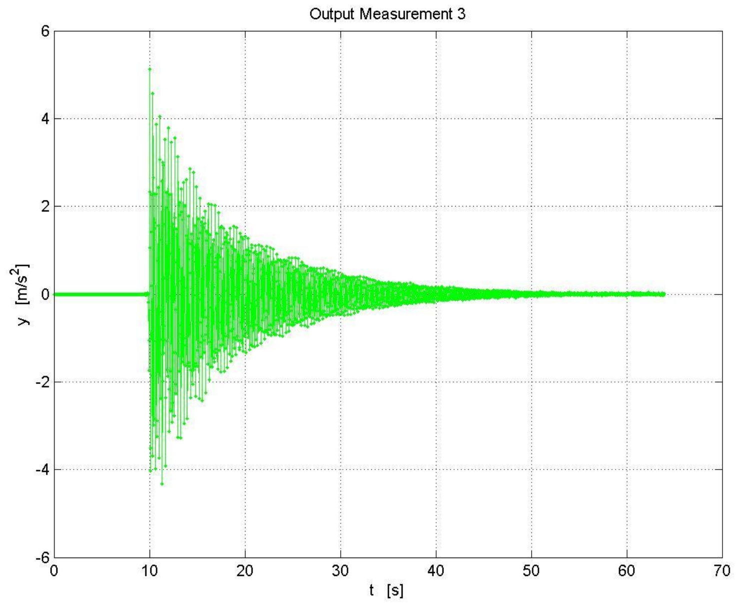
An experimental model of the three-story building system has been obtained experimentally by the numerical technique named Eigensystem Realization Algorithm with Data Correlations (ERA/DC) using Observer/Kalman Filter Method (OKID). The test campaign included $N_c = 4$ test configurations and $N_t = 10$ experiments has been performed in each test configuration. In all the test configurations the accelerations of the floors has been recorded as output data by using three piezoelectric accelerometers placed on each floor. Therefore, the test configurations differ for the type and for the location of the input signal transferred to the system. In the first three test configurations an impulsive input has been delivered to the system floors by using an impact hammer instrumented with a load cell whereas in the last test configuration the impulsive input has been provided by a shaker instrumented with a load cell. Hence, in the first test configuration the input is located on the first floor, in the second test configuration the input is located on the second floor and in the third test configuration the input is located on the third floor whereas in the fourth test configuration the input is located on the first floor. In all test configuration the sampling frequency used is $f_s = 32\left[Hz\right]$ which corresponds to a sampling time equal to $\Delta t = 31.25 \cdot 10^{-3}\left[s\right]$. Note that the Nyquist frequency corresponding to the sampling frequency used is equal to $f_N = 16\left[Hz\right]$. The record length used for the measurements is composed of $l = 2048$ points and consequently the time span during which the input and output measurements has been recorded is equal to $T_s = 64\left[s\right]$. Since the results of all the four test configurations are comparable, here are presented only the results of the first configuration. The input measurement of this test configuration is the force signal transferred on the first floor and it can be graphically represented as follows:



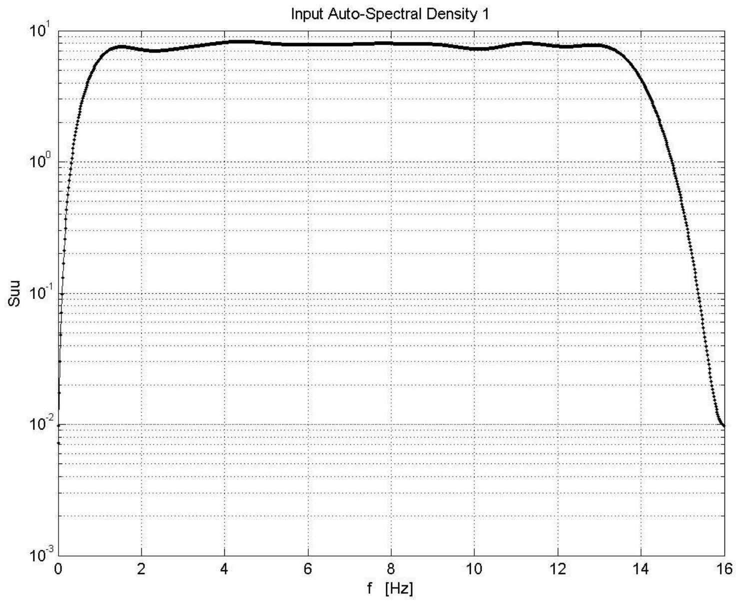
On the other hand, the output measurements are the accelerations of the system floors corresponding to the force transferred to the system. These acceleration signals can be represented graphically as follows:


Output Measurement 1
y [m/s^2]
t [s]
10
8
6
4
2
0
-2
-4
-6
-8
-10
0
10
20
30
40
50
60
70

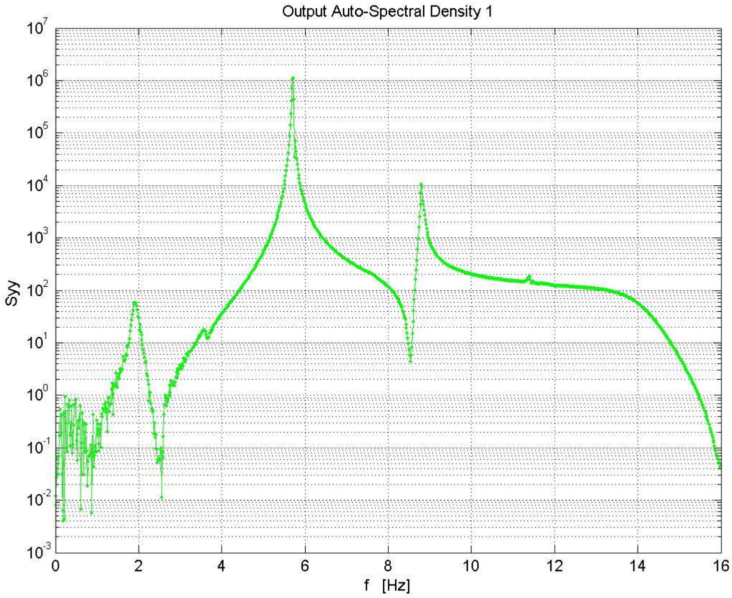

Output Measurement 2
y [m/s^2]
t [s]
6
4
2
0
-2
-4
-6
0
10
20
30
40
50
60
70

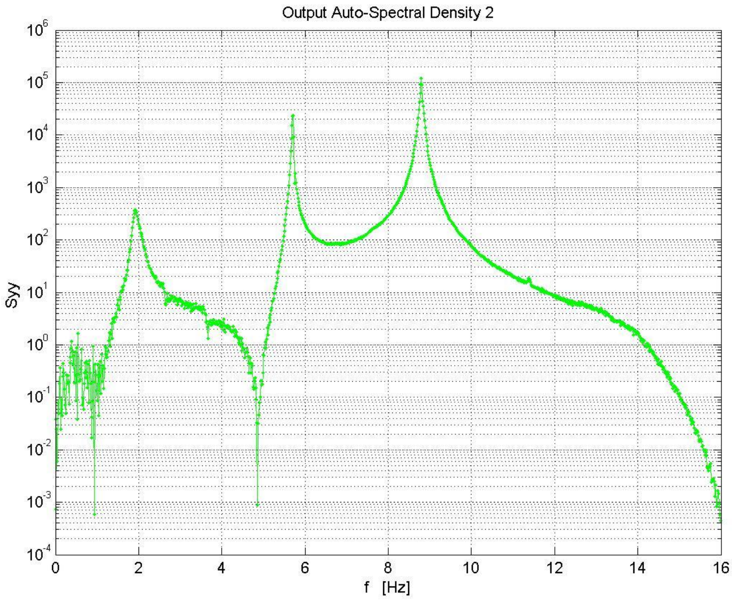


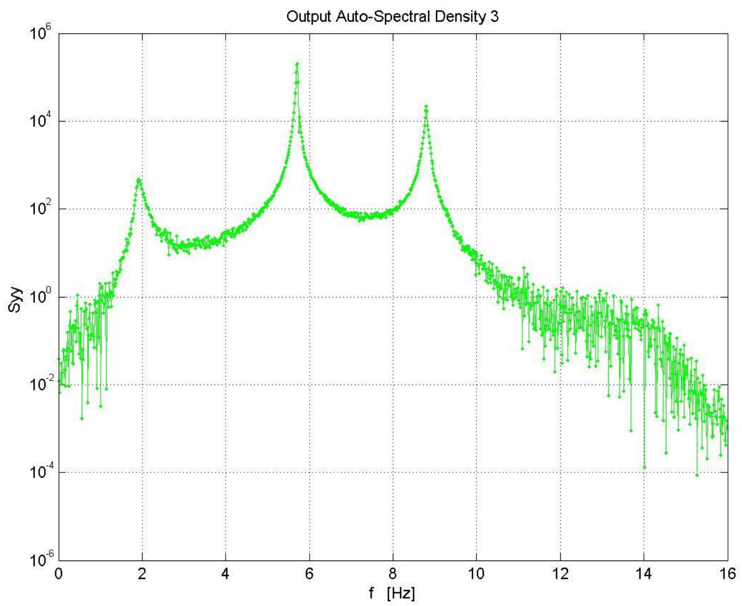
Note that the input and output measurements were filtered in both time domain and frequency domain. In the frequency domain, a low-pass filter with a cut-off frequency of $f_c = 12.5\left[Hz\right]$ has been used whereas in the time domain the inconsistent parts of the input and output measurements were deleted. The auto spectral density relative to the input signal can be computed by using the Fast Fourier Transform algorithm (FFT) to yield:



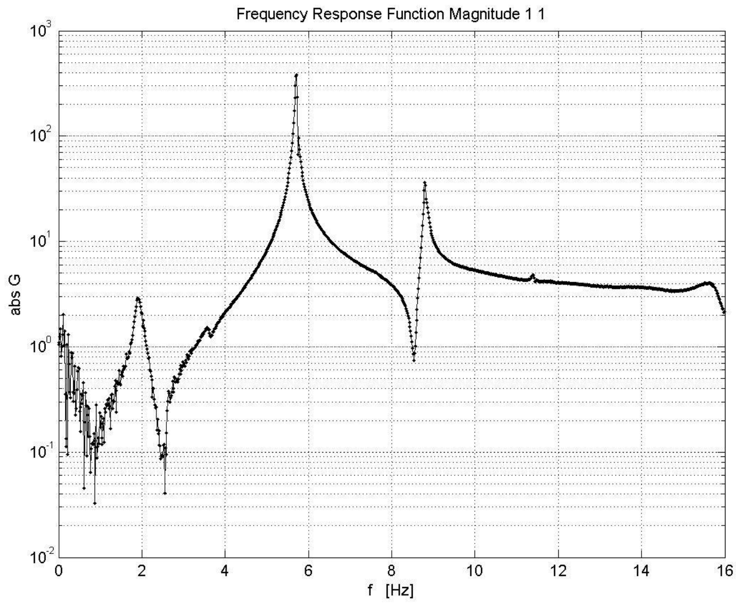
Similarly, the auto spectral density relative to the output signals can be computed yielding to the following plots:

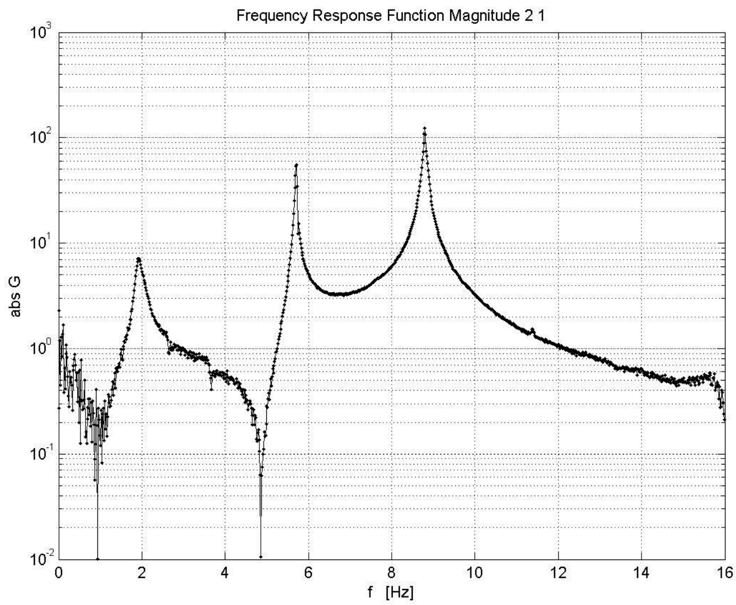

Output Auto-Spectral Density 1
Syy
f [Hz]
0
2
4
6
8
10
12
14
16
10^{7}
10^{6}
10^{5}
10^{4}
10^{3}
10^{2}
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}

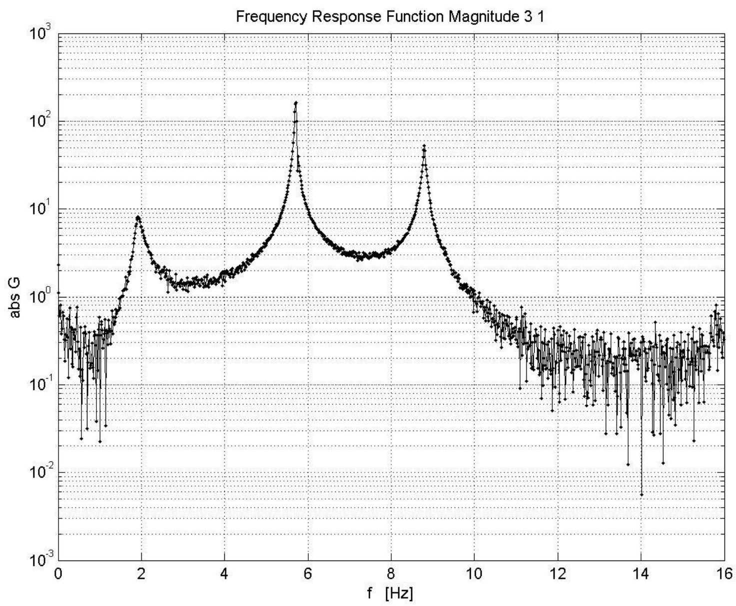

Output Auto-Spectral Density 2
Syy
f [Hz]
0
2
4
6
8
10
12
14
16
10^{6}
10^{5}
10^{4}
10^{3}
10^{2}
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}



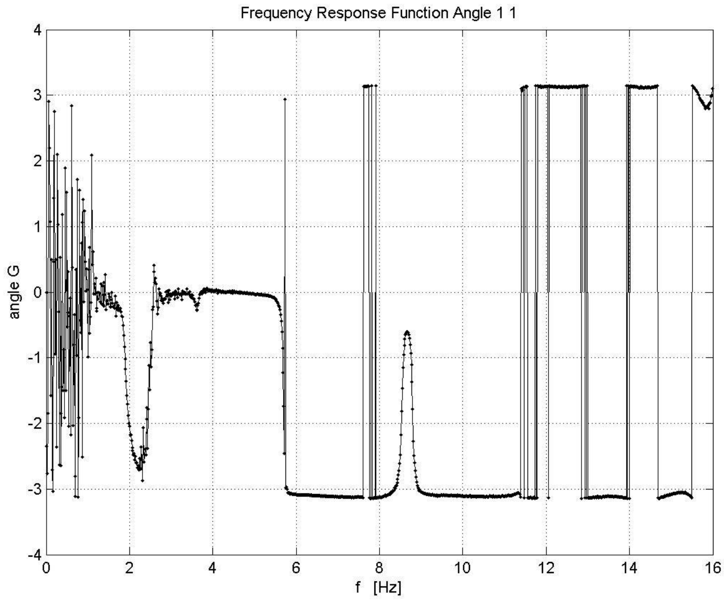
It is worth noting that the first three system natural frequencies are already recognizable from these plots. Once that the auto and cross spectral densities have both been computed for the input and output measurements, the system frequency response function can be derived. The magnitude of the frequency response function referred to each input-output combination can be represented as follows:

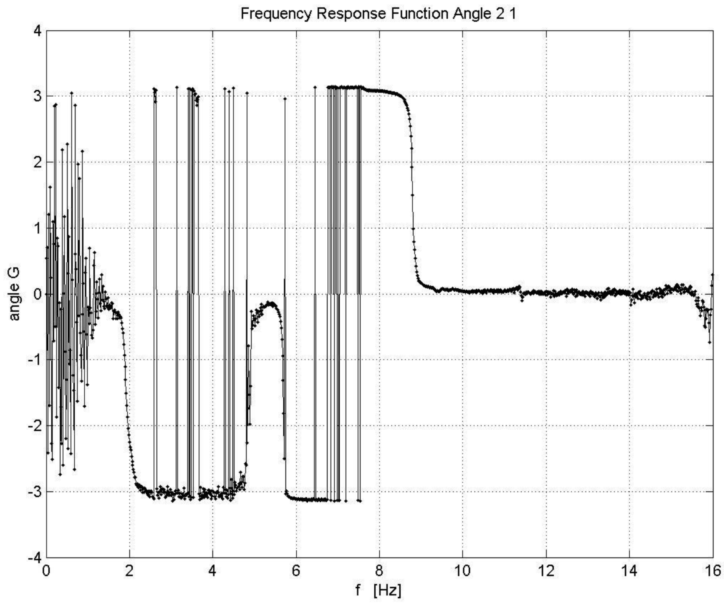

Frequency Response Function Magnitude 1 1
abs G
f [Hz]

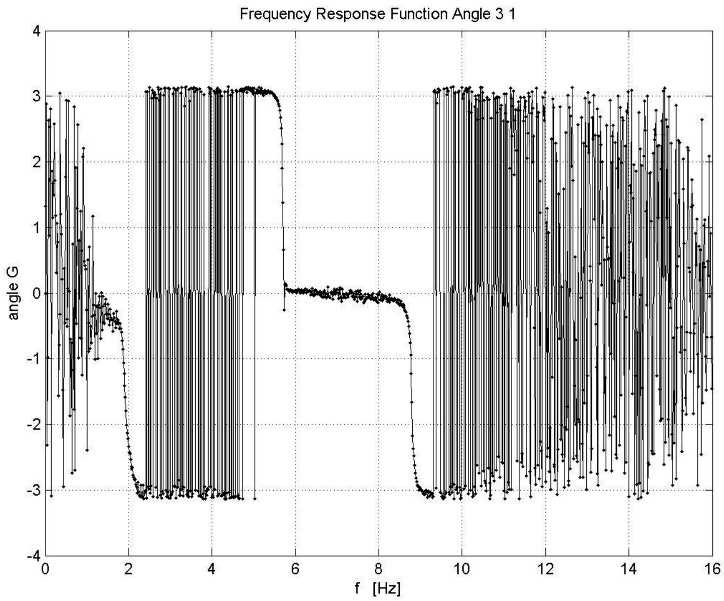

Frequency Response Function Magnitude 2 1
abs G
f [Hz]



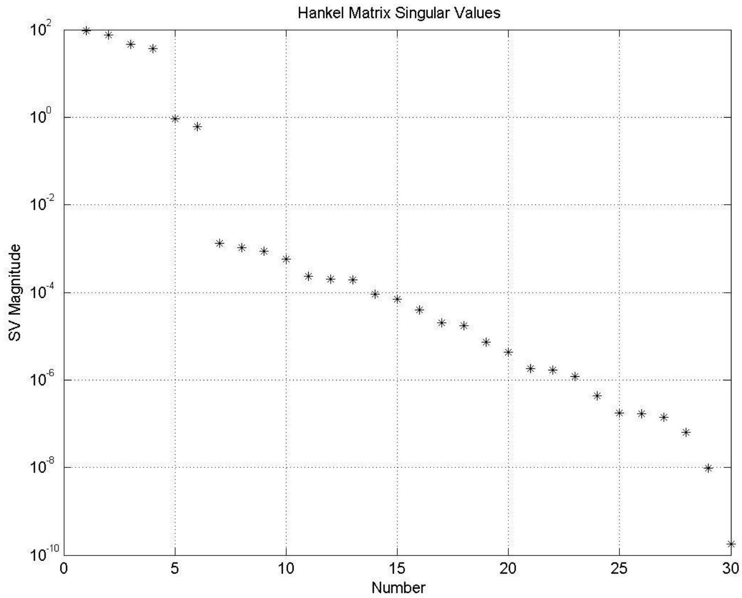
On the other hand, the angles relative to the frequency response function can be represented as:


Frequency Response Function Angle 1 1
angle G
f [Hz]
4
3
2
1
0
-1
-2
-3
-4
0
2
4
6
8
10
12
14
16


Frequency Response Function Angle 2 1
angle G
f [Hz]
4
3
2
1
0
-1
-2
-3
-4
0
2
4
6
8
10
12
14
16



Using this set of input and output data, the Eigensystem Realization Algorithm with Data Correlations (ERA/DC) using Observer/Kalman Filter Method (OKID) has been applied. Indeed, the system Markov parameters and the observer gain Markov parameters have been computed from the identified observer Markov parameters and the Hankel matrix composed of the combined Markov parameters was constructed. The singular values of the system Hankel matrix are showed graphically in the following plot:



Examining the singular values of the Hankel matrix it is possible to determine the order of the identified system model. Indeed, it is clear from the plot that there are only six singular values whose magnitude is not negligible and therefore the order of the identified state-space model is equal to $\hat{n} = 6$. In fact, the singular values successive to the sixth does not correspond to actual system modes but represents noise modes induced by external disturbances which affected the measurements. The discrete-time state-space realization resulting from the identified Markov parameters is represented by the following set of matrices:

$$\hat{\mathbf{A}} = \begin{bmatrix} 0.4374 & 0.8852 & -0.0220 & -0.0415 & -0.0017 & -0.0002 \\ -0.9029 & 0.4310 & -0.0875 & 0.0191 & -0.0040 & 0.0010 \\ 0.0500 & -0.0111 & -0.1490 & 0.9742 & -0.0041 & -0.0021 \\ 0.0621 & -0.0617 & -0.9841 & -0.1538 & -0.0067 & 0.0009 \\ -0.0146 & -0.0241 & -0.0015 & -0.0312 & 0.9230 & 0.3530 \\ -0.0148 & -0.0340 & 0.0100 & -0.0425 & -0.3777 & 0.9070 \end{bmatrix} \tag{5.168}$$

$$\hat{\mathbf{B}} = \begin{bmatrix} -0.0946 \\ -0.0203 \\ -0.1133 \\ 0.0030 \\ -0.0885 \\ -0.1079 \end{bmatrix} \tag{5.169}$$

$$\hat{\mathbf{C}} = \begin{bmatrix} -26.355 & 7.736 & -5.170 & -1.132 & -0.677 & 0.162 \\ -4.217 & 2.626 & 14.217 & 1.220 & -1.340 & 0.307 \\ 11.219 & -4.228 & -5.732 & -0.055 & -1.549 & 0.470 \end{bmatrix} \tag{5.170}$$

$$\hat{\mathbf{D}} = \begin{bmatrix} -2.574 \\ -0.208 \\ 0.003 \end{bmatrix} \tag{5.171}$$

$$\hat{\mathbf{G}} = \begin{bmatrix} 0.0103 & 0.0189 & 0.0016 \\ -0.0234 & 0.0041 & 0.0039 \\ -0.0156 & -0.0257 & -0.0001 \\ 0.0018 & 0.0141 & -0.0059 \\ 0.0581 & 0.1584 & 0.0272 \\ 0.0165 & 0.0006 & -0.0855 \end{bmatrix} \tag{5.172}$$

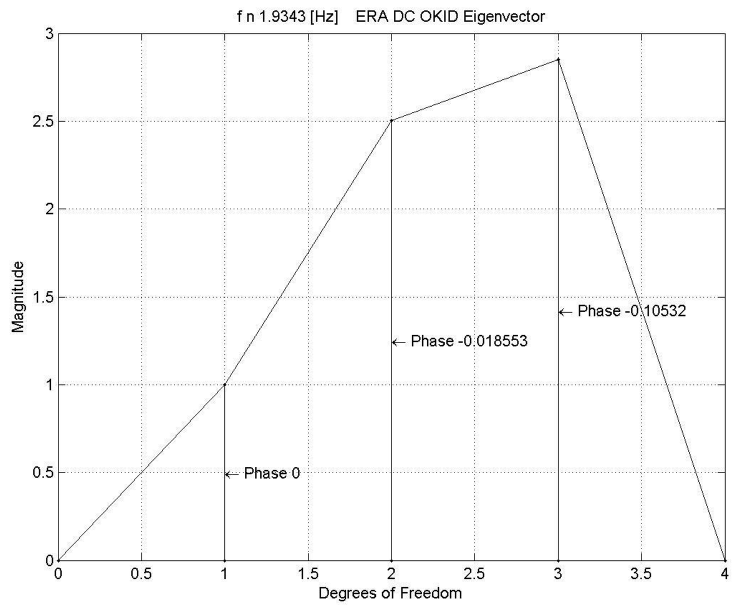
Where $\hat{\mathbf{A}}$ is the identified state matrix, $\hat{\mathbf{B}}$ is the identified state influence matrix, $\hat{\mathbf{C}}$ is the identified output influence matrix, $\hat{\mathbf{D}}$ is the identified direct transmission matrix and $\hat{\mathbf{G}}$ is the identified observer matrix. The system discrete-time state-space realization can be transformed into its continuous-time counterpart by using the zero-order hold assumption. Consequently, the system modal parameters can be determined yielding the identified natural frequencies $\hat{f}_{n,j}$, damping ratios $\hat{\xi}_j$ and mode shapes $\hat{\boldsymbol{\varphi}}_j = e^{\mathbf{i}\hat{\boldsymbol{\Theta}}_j}\hat{\boldsymbol{\rho}}_j$ for $j = 1, 2, \ldots, \hat{n}_2 = 3$. Indeed:

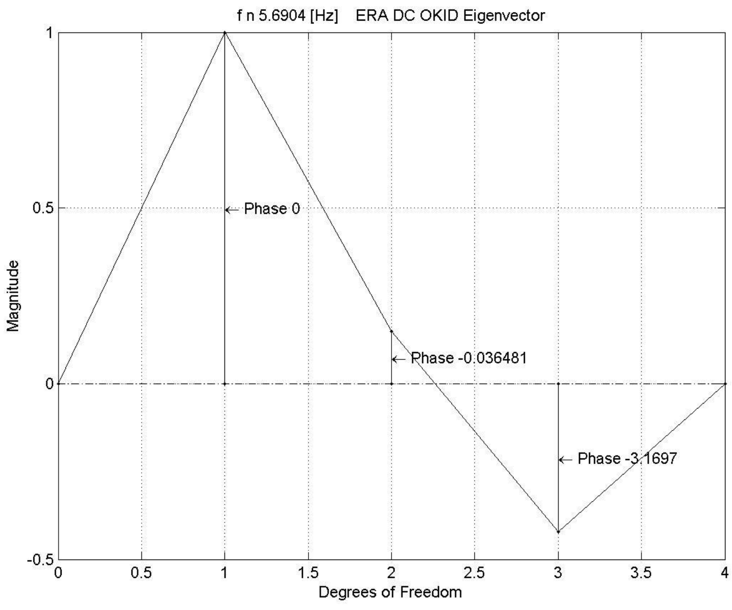
$$\hat{f}_{n,1} = 1.934\,[Hz] \quad , \quad \hat{\xi}_1 = 0.0395\,[\backslash]$$
$$\hat{\boldsymbol{\rho}}_1 = \begin{bmatrix} 1 \\ 2.503 \\ 2.851 \end{bmatrix} \quad , \quad \hat{\boldsymbol{\Theta}}_1 = diag(0, \text{-}0.0186, \text{-}0.1053) \tag{5.173}$$

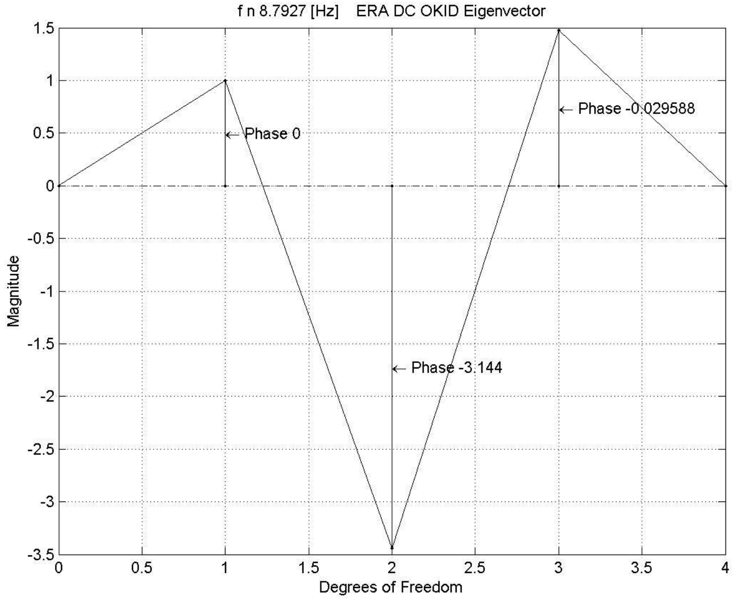
$$\hat{f}_{n,2} = 5.690\,[Hz] \quad , \quad \hat{\xi}_2 = 0.0030\,[\backslash]$$
$$\hat{\boldsymbol{\rho}}_2 = \begin{bmatrix} 1 \\ 0.1492 \\ 0.4221 \end{bmatrix} \quad , \quad \hat{\boldsymbol{\Theta}}_2 = diag(0, \text{-}0.0365, \text{-}3.170) \tag{5.174}$$

$$\hat{f}_{n,3} = 8.793\,[Hz] \quad , \quad \hat{\xi}_3 = 0.0042\,[\backslash]$$
$$\hat{\boldsymbol{\rho}}_3 = \begin{bmatrix} 1 \\ 3.443 \\ 1.476 \end{bmatrix} \quad , \quad \hat{\boldsymbol{\Theta}}_3 = diag(0, \text{-}3.144, \text{-}0.0296) \tag{5.175}$$

The identified mode shapes can be represented graphically as follows:


f n 1.9343 [Hz] ERA DC OKID Eigenvector
Magnitude
Degrees of Freedom
Phase 0
Phase -0.018553
Phase -0.10532
0
0.5
1
1.5
2
2.5
3
3.5
4


f n 5.6904 [Hz] ERA DC OKID Eigenvector
Magnitude
Degrees of Freedom
Phase 0
Phase -0.036481
Phase -3.1697
-0.5
0
0.5
1
1.5
2
2.5
3
3.5
4



Note that the identified modal damping is small and the identified mode shapes are all roughly in phase or approximately completely out of phase. Consequently, the system can be assumed proportionally damped and the proportional damping coefficients $\hat{\alpha}$ and $\hat{\beta}$ can be approximately determined from the identified natural frequencies and damping ratios by a least-squares approach yielding to the following results:

$$\hat{\alpha} = 0.9751\left[\frac{1}{s}\right] \quad , \quad \hat{\beta}=-2.8815\cdot 10^{-4}\left[s\right] \tag{5.176}$$

On the other hand, a physical model can be constructed from the identified sate-space representation by using the algorithm showed in the previous chapters (MKR). The result of the implementation of this method with the identified data is the set of physical coordinates mass matrix $\hat{\mathbf{M}}$, stiffness matrix $\hat{\mathbf{K}}$ and damping matrix $\hat{\mathbf{R}}$. Indeed:

$$\hat{\mathbf{M}} = \begin{bmatrix} 0.3871 & -0.0156 & 0.0011 \\ -0.0156 & 0.2621 & -0.0323 \\ 0.0011 & -0.0323 & 0.3790 \end{bmatrix} \tag{5.177}$$

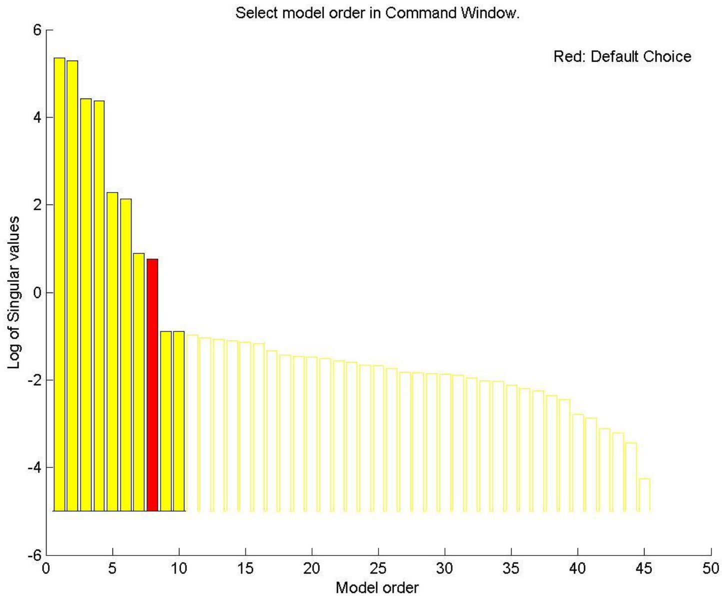
$$\hat{\mathbf{K}} = \begin{bmatrix} 534.061 & -221.931 & 25.696 \\ -221.931 & 614.564 & -432.792 \\ 25.696 & -432.792 & 422.005 \end{bmatrix} \tag{5.178}$$

$$\hat{\mathbf{R}} = \begin{bmatrix} 9.960 & -4.823 & 1.018 \\ -4.823 & 12.480 & -8.366 \\ 1.018 & -8.366 & 7.017 \end{bmatrix} \tag{5.179}$$

While the experimental results of the identification method for mass matrix $\hat{\mathbf{M}}$ and for the identified stiffness matrix $\hat{\mathbf{K}}$ are physically acceptable, the result for the identified damping matrix $\hat{\mathbf{R}}$ appears to be incongruous. On the other hand, a better result for the identified damping matrix $\hat{\mathbf{R}}$ can be obtained from the identified mass matrix $\hat{\mathbf{M}}$ and from the identified stiffness matrix $\hat{\mathbf{K}}$ using the identified proportional damping coefficients $\hat{\alpha}$ and $\hat{\beta}$. Indeed:

$$\begin{aligned} \hat{\mathbf{R}} &= \alpha\hat{\mathbf{M}} + \beta\hat{\mathbf{K}} = \\ &= \begin{bmatrix} 0.22356 & 0.0487 & -0.0063 \\ 0.0487 & 0.0785 & 0.0932 \\ -0.0063 & 0.0932 & 0.2480 \end{bmatrix} \end{aligned} \tag{5.180}$$

This model represents a second-order physical model of the system derived from a set of experimental data. Consequently, this experimental model is the most suitable model to design a real-time controller. Finally, using the same set of input and output data, the Numerical Algorithm for State Space Subspace System Identification (N4SID), which is implemented in MATLAB, has been applied. Similarly, the singular values of the system Hankel matrix are showed graphically in the following plot:



Examining the singular values of the Hankel matrix it is possible to determine the order of the identified system model. Indeed, it is clear from the plot that there are only six singular values whose magnitude is not negligible and therefore the order of the identified state-space model is equal to $\hat{n} = 6$. In fact, the singular values successive to the sixth does not correspond to actual system modes but represents noise modes induced by external disturbances which affected the measurements. The discrete-time state-space realization resulting from the identified Markov parameters is represented by the following set of matrices:

$$\hat{\mathbf{A}} = \begin{bmatrix} 0.4184 & 0.8019 & 0.3995 & 0.1444 & 0.0153 & -0.0073 \\ -0.6853 & 0.2661 & 0.3680 & -0.5466 & -0.0164 & 0.0120 \\ -0.5789 & 0.1414 & 0.0405 & 0.7964 & 0.0423 & -0.0117 \\ -0.1367 & 0.5145 & -0.8260 & -0.1599 & 0.0415 & 0.0040 \\ 0.0117 & -0.0324 & 0.0257 & -0.0387 & 0.9263 & -0.3689 \\ -0.0001 & -0.0059 & 0.0485 & -0.0300 & 0.3585 & 0.8969 \end{bmatrix} \tag{5.181}$$

$$\hat{\mathbf{B}} = \begin{bmatrix} 0.0176 \\ 0.0521 \\ -0.0265 \\ -0.0352 \\ 0.0867 \\ -0.0552 \end{bmatrix} \tag{5.182}$$

$$\hat{\mathbf{C}} = \begin{bmatrix} 56.6682 & 39.7860 & 6.4378 & -6.8943 & 0.7703 & -0.1882 \\ 12.0417 & -4.2486 & 17.2738 & 20.3517 & 1.5212 & -0.4419 \\ -26.3375 & -11.1040 & -11.0976 & -8.1824 & 3.1637 & -0.1651 \end{bmatrix} \tag{5.183}$$

$$\hat{\mathbf{D}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{5.184}$$

$$\hat{\mathbf{G}} = \begin{bmatrix} 0.0020 & 0.0005 & 0.0012 \\ -0.0002 & -0.0011 & 0.0001 \\ -0.0018 & 0.0034 & -0.0008 \\ 0.0019 & -0.0033 & 0.0020 \\ 0.0202 & 0.0218 & 0.0345 \\ 0.0115 & -0.0188 & 0.0153 \end{bmatrix} \tag{5.185}$$

Where $\hat{\mathbf{A}}$ is the identified state matrix, $\hat{\mathbf{B}}$ is the identified state influence matrix, $\hat{\mathbf{C}}$ is the identified output influence matrix, $\hat{\mathbf{D}}$ is the identified direct transmission matrix and $\hat{\mathbf{G}}$ is the identified observer matrix. The system discrete-time state-space realization can be transformed into its continuous-time counterpart by using the zero-order hold assumption. Consequently, the system modal parameters can be determined yielding the identified natural frequencies $\hat{f}_{n,j}$, damping ratios $\hat{\xi}_j$ and mode shapes $\hat{\boldsymbol{\varphi}}_j = e^{\mathbf{i}\hat{\boldsymbol{\Theta}}_j}\hat{\boldsymbol{\rho}}_j$ for $j = 1, 2, \ldots, \hat{n}_2 = 3$. Indeed:

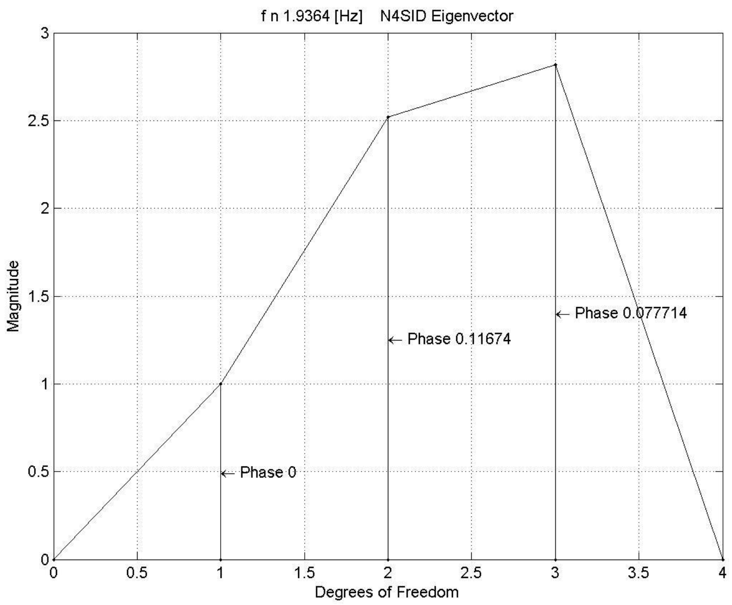
$$\begin{gathered} \hat{f}_{n,1} = 1.936\left[Hz\right] \quad , \quad \hat{\xi}_1 = 0.0464\left[\backslash\right] \\ \hat{\boldsymbol{\rho}}_1 = \begin{bmatrix} 1 \\ 2.519 \\ 2.820 \end{bmatrix} \quad , \quad \hat{\boldsymbol{\Theta}}_1 = diag(0, 0.1167, 0.0777) \end{gathered} \tag{5.186}$$

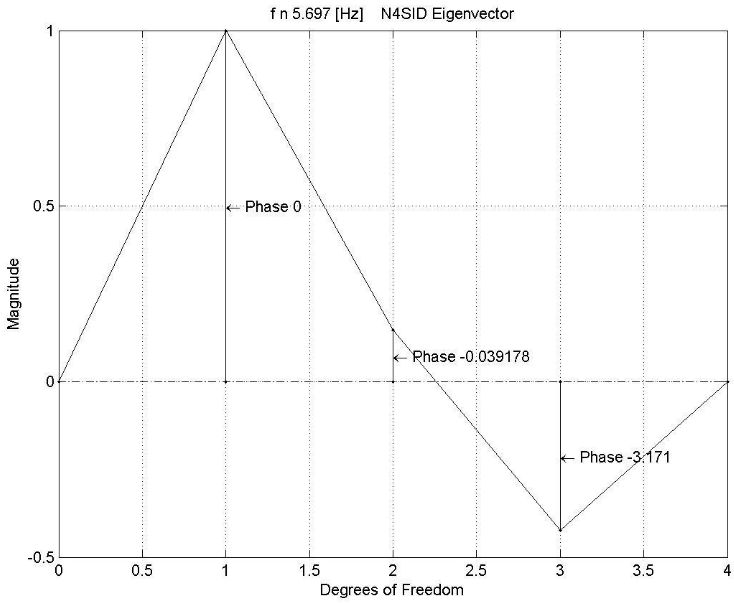
$$\begin{gathered} \hat{f}_{n,2} = 5.697\left[Hz\right] \quad , \quad \hat{\xi}_2 = 0.0028\left[\backslash\right] \\ \hat{\boldsymbol{\rho}}_2 = \begin{bmatrix} 1 \\ 0.1461 \\ 0.4241 \end{bmatrix} \quad , \quad \hat{\boldsymbol{\Theta}}_2 = diag(0, \text{-}0.0392, \text{-}3.1710) \end{gathered} \tag{5.187}$$

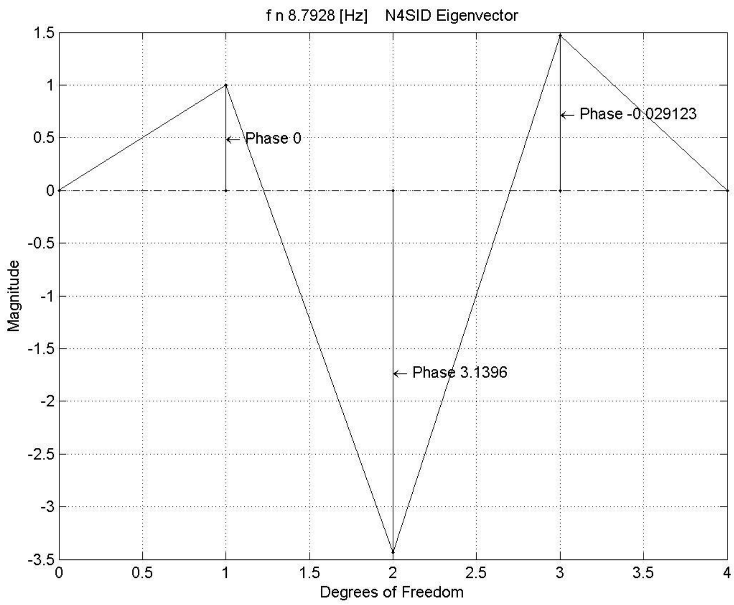
$$\hat{f}_{n,3} = 8.793\left[Hz\right] \quad , \quad \hat{\xi}_3 = 0.0042\left[\backslash\right]$$

$$\hat{\boldsymbol{\rho}}_3 = \begin{bmatrix} 1 \\ 3.436 \\ 1.472 \end{bmatrix} \quad , \quad \hat{\boldsymbol{\Theta}}_3 = diag(0, 3.1396, \text{-}0.0291) \tag{5.188}$$

The identified mode shapes can be represented graphically as follows:




f n 5.697 [Hz] N4SID Eigenvector
Magnitude
Degrees of Freedom
Phase 0
Phase -0.039178
Phase -3.171
1
0.5
0
-0.5
0
0.5
1
1.5
2
2.5
3
3.5
4


f n 8.7928 [Hz] N4SID Eigenvector
Magnitude
Degrees of Freedom
Phase 0
Phase -0.029123
Phase 3.1396
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
-3.5
0
0.5
1
1.5
2
2.5
3
3.5
4

Note that the identified modal damping is small and the identified mode shapes are all roughly in phase or approximately completely out of phase. Consequently, the system can be assumed proportionally damped and the proportional damping coefficients $\hat{\alpha}$ and $\hat{\beta}$ can be approximately determined from the identified natural frequencies and damping ratios by a least-squares approach yielding to the following results:

$$\hat{\alpha} = 1.1494\left[\frac{1}{s}\right] \quad , \quad \hat{\beta} = -3.6937 \cdot 10^{-4}\left[s\right] \tag{5.189}$$

On the other hand, a physical model can be constructed from the identified sate-space representation by using the algorithm showed in the previous chapters (MKR). The result of the implementation of this method with the identified data is the set of physical coordinates mass matrix $\hat{\mathbf{M}}$, stiffness matrix $\hat{\mathbf{K}}$ and damping matrix $\hat{\mathbf{R}}$. Indeed:

$$\hat{\mathbf{M}} = \begin{bmatrix} 0.4089 & -0.0132 & 0.0191 \\ -0.0132 & 0.3110 & -0.0004 \\ 0.0191 & -0.0004 & 0.4664 \end{bmatrix} \tag{5.190}$$

$$\hat{\mathbf{K}} = \begin{bmatrix} 569.1692 & -266.3515 & 33.1538 \\ -266.3515 & 693.3727 & -458.0419 \\ 33.1538 & -458.0419 & 442.9695 \end{bmatrix} \tag{5.191}$$

$$\hat{\mathbf{R}} = \begin{bmatrix} 13.3370 & -6.0425 & 2.1404 \\ -6.0425 & 18.6675 & -8.5170 \\ 2.1404 & -8.5170 & 13.2783 \end{bmatrix} \tag{5.192}$$

While the experimental results of the identification method for mass matrix $\hat{\mathbf{M}}$ and for the identified stiffness matrix $\hat{\mathbf{K}}$ are physically acceptable, the result for the identified damping matrix $\hat{\mathbf{R}}$ appears to be incongruous. On the other hand, a better result for the identified damping matrix $\hat{\mathbf{R}}$ can be obtained from the identified mass matrix $\hat{\mathbf{M}}$ and from the identified stiffness matrix $\hat{\mathbf{K}}$ using the identified proportional damping coefficients $\hat{\alpha}$ and $\hat{\beta}$. Indeed:

$$\hat{\mathbf{R}} = \alpha\hat{\mathbf{M}} + \beta\hat{\mathbf{K}} = \\ = \begin{bmatrix} 0.2597 & 0.0832 & 0.0098 \\ 0.0832 & 0.1013 & 0.1687 \\ 0.0098 & 0.1687 & 0.3724 \end{bmatrix} \tag{5.193}$$

This model represents a second-order physical model of the system derived from a set of experimental data. It is worth to emphasize that the experimental results obtained using the Eigensystem Realization Algorithm with Data Correlations (ERA\DC) using Observer/Kalman Filter Method (OKID) are comparable with the experimental results obtained from the same data set using the Numerical Algorithm for Subspace Identification (N4SID) implemented in MATLAB.

5.6. LINEAR QUADRATIC GUAUSSIAN REGULATOR (LQG) DESIGN

Once that an identified model of the mechanical system has been obtained, a Linear Quadratic Gaussian regulator (LQG) can be designed. The purpose of the regulator is to mitigate the structural vibrations of the system due to an external excitation on the first floor by a control torque acting on a pendulum hinged on the third floor. Thus, the model used to design the Linear Quadratic Gaussian regulator (LQG) is a combination of the identified structural model and of the lumped parameter model. Indeed, the vector of the lagrangian coordinates used in the model is formed by the set of the displacement of the floors and the pendulum angle:

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \varphi(t) \end{bmatrix} \tag{5.194}$$

The floors displacements and the pendulum angle can be recovered from the vector of lagrangian coordinates $\mathbf{x}(t)$ by using the following Boolean matrices:

$$\mathbf{B}_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{5.195}$$

$$\mathbf{B}_\varphi = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \tag{5.196}$$

In particular, the displacement of the third floor can be recovered from the vector of lagrangian coordinates by using the following Boolean matrix:

$$\mathbf{B}_{x_3} = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \tag{5.197}$$

Similarly to the case of the lumped parameter model, the effect of the pendulum on the global mass matrix of the system can be accounted for by using the following Boolean matrix:

$$\mathbf{B}^4 = \begin{bmatrix} \mathbf{B}_{x_3}\mathbf{B}_x \\ \mathbf{B}_\varphi \end{bmatrix} \tag{5.198}$$

The mathematical model of the system can be expressed as:

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{R}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{B}_{2,e}\mathbf{u}_e(t) + \mathbf{B}_{2,c}\mathbf{u}_c(t) \tag{5.199}$$

Where the system mass matrix $\mathbf{M}$, stiffness matrix $\mathbf{K}$ and damping matrix $\mathbf{R}$ can be expressed as follows:

$$\mathbf{M} = \mathbf{B}_x^T\hat{\mathbf{M}}\mathbf{B}_x + \mathbf{B}^{4T}\mathbf{M}^4\mathbf{B}^4 \tag{5.200}$$

$$\mathbf{K} = \mathbf{B}_x^T\hat{\mathbf{K}}\mathbf{B}_x + \mathbf{B}^{4T}\mathbf{K}^4\mathbf{B}^4 \tag{5.201}$$

$$\mathbf{R} = \mathbf{B}_x^T\hat{\mathbf{R}}\mathbf{B}_x + \mathbf{B}^{4T}\mathbf{R}^4\mathbf{B}^4 \tag{5.202}$$

Where $\hat{\mathbf{M}}$, $\hat{\mathbf{K}}$, and $\hat{\mathbf{R}}$ are respectively the identified mass matrix, stiffness matrix and damping matrix whereas the matrices $\mathbf{M}^4$, $\mathbf{K}^4$ and $\mathbf{R}^4$ are respectively the pendulum mass matrix, stiffness matrix and damping matrix which are defined as:

$$\mathbf{M}^4 = \begin{bmatrix} m_4 & m_4L_4 \\ m_4L_4 & m_4L_4^2 + I_{zz,4} \end{bmatrix} \tag{5.203}$$

$$\mathbf{K}^4 = m_4gL_4 \tag{5.204}$$

$$\mathbf{R}^4 = r_4 \tag{5.205}$$

Where m_4 is the pendulum mass, $I_{zz,4}$ is the mass moment of inertia relative to the centre of mass of the pendulum, L_4 is equal to half the length of the pendulum and r_4 is the pendulum angular damping. On the other hand, the matrices $\mathbf{B}_{2,e}$ and $\mathbf{B}_{2,c}$ define the locations of the inputs. In particular, on the first floor there is an external force and on the pendulum there is the control torque:

$$\mathbf{B}_{2,e} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{5.206}$$

$$\mathbf{B}_{2,c} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{5.207}$$

Considering a worst-case scenario, the external force is assumed as a superposition of three harmonic force whose excitation frequencies are close to the first three system natural frequencies. Indeed:

$$F(t) = F_{0,1}\sin(2\pi f_1) + F_{0,2}\sin(2\pi f_2) + F_{0,3}\sin(2\pi f_3) \tag{5.208}$$

Where the force amplitudes are assumed to be $F_{0,1} = F_{0,2} = F_{0,3} = 0.1\left[N\right]$ and the force excitation frequencies are assumed equal to $f_1 = 1.9\left[Hz\right]$, $f_2 = 5.7\left[Hz\right]$ and $f_3 = 8.8\left[Hz\right]$. Note that the excitation force $F(t)$ acting on the first floor can be measured using a load cell and therefore can be used in the control algorithm. On the other hand, it has been assumed that only the acceleration of each floor and the pendulum angle can be measured. Therefore, the system output equations are the following:

$$\mathbf{y}(t) = \mathbf{C}_d \mathbf{x}(t) + \mathbf{C}_v \dot{\mathbf{x}}(t) + \mathbf{C}_a \ddot{\mathbf{x}}(t) \tag{5.209}$$

Where the sensing matrices $\mathbf{C}_d$, $\mathbf{C}_v$ and $\mathbf{C}_a$ can be computed as follows:

$$\mathbf{C}_d = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5.210}$$

$$\mathbf{C}_v = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.211}$$

$$\mathbf{C}_a = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.212}$$

Once that the system model in physical coordinates has been obtained, a discrete-time state-space model can be easily derived using a sampling time equal to $\Delta t = 31.25 \cdot 10^{-3} \left[s\right]$ and the zero order hold assumption. Indeed:

$$\mathbf{z}(k+1) = \mathbf{A}\mathbf{z}(k) + \mathbf{B}_e \mathbf{u}_e(k) + \mathbf{B}_c \mathbf{u}_c(k) \tag{5.213}$$

$$\mathbf{y}(k) = \mathbf{C}\mathbf{z}(k) + \mathbf{D}_e \mathbf{u}_e(k) + \mathbf{D}_c \mathbf{u}_c(k) \tag{5.214}$$

In order to take in account the effects of the uncertainty relative to the system model and relative to the data acquisition system, a process noise vector and a measurement noise vector are considered:

$$\mathbf{z}(k+1) = \mathbf{A}\mathbf{z}(k) + \mathbf{B}_e\mathbf{u}_e(k) + \mathbf{B}_c\mathbf{u}_c(k) + \mathbf{w}(k) \tag{5.215}$$

$$\mathbf{y}(k) = \mathbf{C}\mathbf{z}(k) + \mathbf{D}_e\mathbf{u}_e(k) + \mathbf{D}_c\mathbf{u}_c(k) + \mathbf{v}(k) \tag{5.216}$$

For the simulation purposes, consider a time span equal to $T_s = 50\left[s\right]$. In addition, consider the following initial conditions:

$$\begin{cases} x_1(0) = 10^{-3}\left[m\right] \\ x_2(0) = 10^{-3}\left[m\right] \\ x_3(0) = 10^{-3}\left[m\right] \\ \varphi(0) = 10^{-3}\left[rad\right] \end{cases}, \quad \begin{cases} \dot{x}_1(0) = 10^{-4}\left[{m}/{s}\right] \\ \dot{x}_2(0) = 10^{-4}\left[{m}/{s}\right] \\ \dot{x}_3(0) = 10^{-4}\left[{m}/{s}\right] \\ \dot{\varphi}(0) = 10^{-4}\left[{rad}/{s}\right] \end{cases} \tag{5.217}$$

The initial state $\mathbf{z}_0$ is modelled as Gaussian distributed random vector whose mean value $\bar{\mathbf{z}}_0$ and covariance matrix $\mathbf{R}_0$ are assumed as:

$$\mathbf{Z}_0 = \begin{bmatrix} 10^{-4}\left[m\right] \\ 10^{-4}\left[m\right] \\ 10^{-4}\left[m\right] \\ 10^{-4}\left[rad\right] \\ 10^{-5}\left[{m}/{s}\right] \\ 10^{-5}\left[{m}/{s}\right] \\ 10^{-5}\left[{m}/{s}\right] \\ 10^{-5}\left[{rad}/{s}\right] \end{bmatrix} \tag{5.218}$$

$$\bar{\mathbf{z}}_0 = \mathbf{z}_0 + \mathbf{Z}_0 \tag{5.219}$$

$$\mathbf{R}_0 = diag(10^{-8}\left[m^2\right], 10^{-8}\left[m^2\right], 10^{-8}\left[m^2\right], 10^{-8}\left[rad^2\right],$$
$$10^{-10}\left[{m^2}/{s^2}\right], 10^{-10}\left[{m^2}/{s^2}\right], 10^{-10}\left[{m^2}/{s^2}\right], 10^{-10}\left[{rad^2}/{s^2}\right]) \tag{5.220}$$

The random disturbances $\mathbf{w}(k)$ and $\mathbf{v}(k)$ are assumed zero mean Gaussian white noises whose covariance matrices are assumed equal to:

$$\begin{aligned}\mathbf{W}_0 = diag(&10^{-3}\left[{m}/{s}\right], 10^{-3}\left[{m}/{s}\right], 10^{-3}\left[{m}/{s}\right], 10^{-3}\left[{rad}/{s}\right],\\ &10^{-3}\left[{m}/{s^2}\right], 10^{-3}\left[{m}/{s^2}\right], 10^{-3}\left[{m}/{s^2}\right], 10^{-3}\left[{rad}/{s^2}\right])\end{aligned} \tag{5.221}$$

$$\mathbf{V}_0 = diag(10^{-3}\left[{m}/{s^2}\right], 10^{-3}\left[{m}/{s^2}\right], 10^{-3}\left[{m}/{s^2}\right], 10^{-3}\left[rad\right]) \tag{5.222}$$

$$\mathbf{R}_w = \mathbf{W}_0^2 \tag{5.223}$$

$$\mathbf{R}_v = \mathbf{V}_0^2 \tag{5.224}$$

Where $\mathbf{W}_0$ is the amplitude vector relative to the process noise, $\mathbf{V}_0$ is the amplitude vector relative to the measurement noise, $\mathbf{R}_w$ is the process noise covariance matrix and $\mathbf{R}_v$ is the measurement noise covariance matrix. Assuming these stochastic characteristics relative to the process noise, the measurement noise and to the initial state a discrete-time infinite-horizon Kalman filter gain has been computed to yield:

$$\mathbf{K}_{\infty} = \begin{bmatrix} -0.0008 & -0.0008 & -0.0014 & 0.0023 \\ -0.0013 & -0.0019 & -0.0038 & 0.0069 \\ -0.0013 & -0.0022 & -0.0043 & 0.0084 \\ -0.0000 & -0.0003 & -0.0037 & 0.6372 \\ 0.0246 & 0.0018 & -0.0001 & 0.0444 \\ 0.0029 & 0.0210 & 0.0057 & 0.1229 \\ -0.0001 & 0.0033 & 0.0257 & 0.1433 \\ -0.0019 & -0.0312 & -0.2077 & -0.6372 \end{bmatrix} \quad (5.225)$$

Indeed, an estimation of the system state $\hat{\mathbf{z}}(k)$ can be obtained from the following difference equations:

$$\hat{\mathbf{z}}(k+1) = \mathbf{A}\hat{\mathbf{z}}(k) + \mathbf{B}_e\mathbf{u}_e(k) + \mathbf{B}_c\mathbf{u}_c(k) + \mathbf{K}_{\infty}\left(\mathbf{y}(k) - \hat{\mathbf{y}}(k)\right) \quad (5.226)$$

The estimation equations are initialized setting the esteem of the initial state $\hat{\mathbf{z}}(0)$ equal to the expected value of the real initial state $\overline{\mathbf{z}}_0$. In addition, the estimated output vector $\hat{\mathbf{y}}(k)$ can be computed from the following output equations:

$$\hat{\mathbf{y}}(k) = \mathbf{C}\hat{\mathbf{z}}(k) + \mathbf{D}_e\mathbf{u}_e(k) + \mathbf{D}_c\mathbf{u}_c(k) \quad (5.227)$$

On the other hand, the control action is computed as an optimal feedback control minimizing a discrete-time infinite-horizon quadratic performance index defined as:

$$J_{\infty} = \frac{1}{2}\sum_{k=0}^{\infty}\left(\mathbf{z}^T(k)\mathbf{Q}_z\mathbf{z}(k) + \mathbf{u}^T(k)\mathbf{Q}_u\mathbf{u}(k)\right) \quad (5.228)$$

The weight matrices relative to the state vector $\mathbf{Q}_z$ and to the input vector $\mathbf{Q}_u$ has been chosen as follows:

$$\mathbf{Q}_z = diag(10^2 \left[{}^{1}\!/_{m^2} \right], 10^2 \left[{}^{1}\!/_{m^2} \right], 10^2 \left[{}^{1}\!/_{m^2} \right], 10^{-2} \left[{}^{1}\!/_{rad^2} \right],$$
$$10^2 \left[{}^{s^2}\!/_{m^2} \right], 10^2 \left[{}^{s^2}\!/_{m^2} \right], 10^2 \left[{}^{s^2}\!/_{m^2} \right], 10^{-2} \left[{}^{s^2}\!/_{rad^2} \right])$$
(5.229)

$$\mathbf{Q}_u = 10^{-4} \left[\frac{1}{N^2 \cdot m^2} \right] \tag{5.230}$$

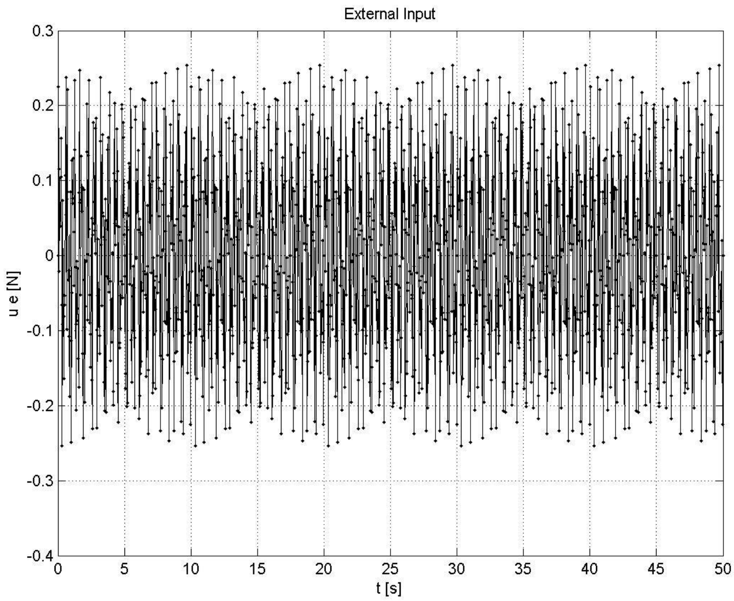
Consequently, an infinite-horizon optimal feedback gain has been computed as:

$$\mathbf{F}_\infty^T = \begin{bmatrix} -3.0331 \\ -0.4309 \\ 7.4299 \\ 0.0584 \\ -0.2140 \\ -0.4318 \\ 1.3853 \\ -0.0068 \end{bmatrix} \tag{5.231}$$

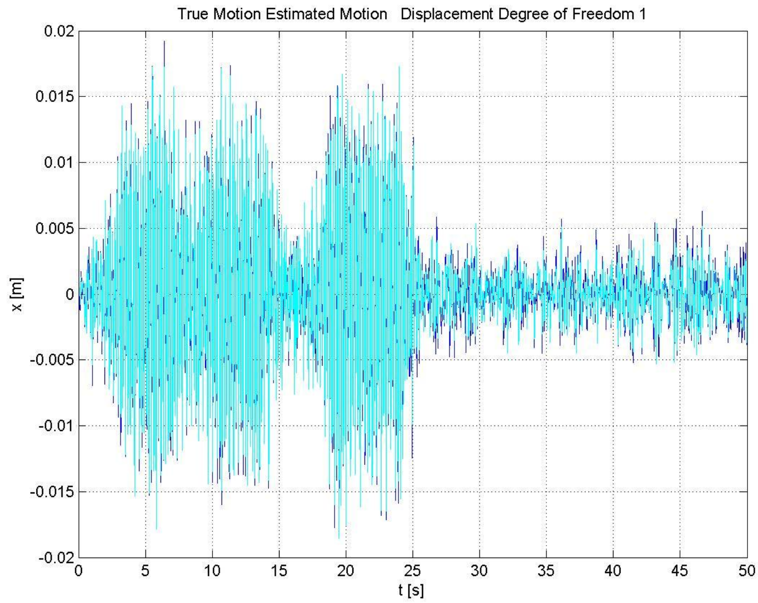
Indeed, the control action $\mathbf{u}_c(k)$ can be expressed using the feedback matrix $\mathbf{F}_\infty$ and the estimated state $\hat{\mathbf{z}}(k)$ as follows:

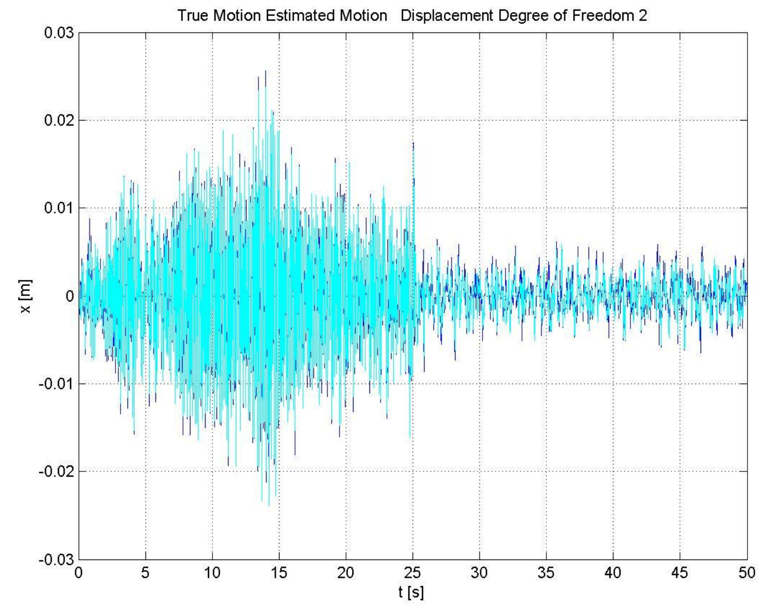
$$\mathbf{u}_c(k) = \mathbf{F}_\infty \hat{\mathbf{z}}(k) \tag{5.232}$$

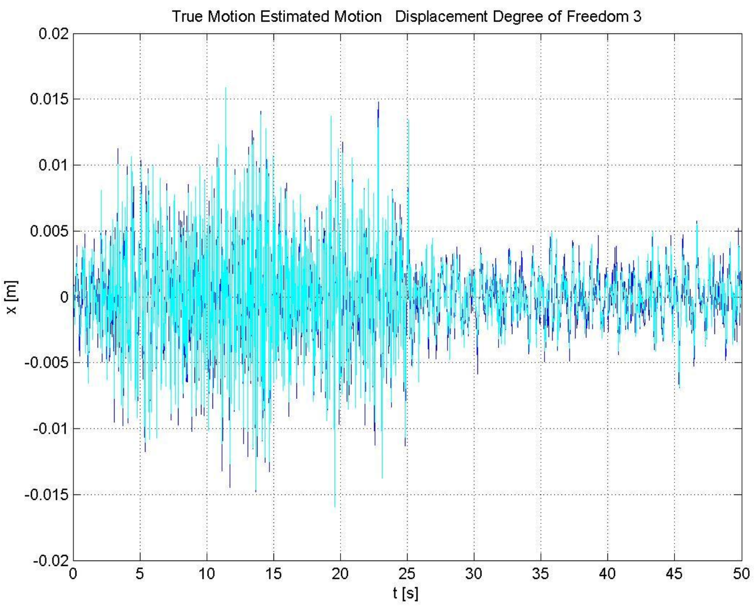
The external input force acting on the first floor is the following:



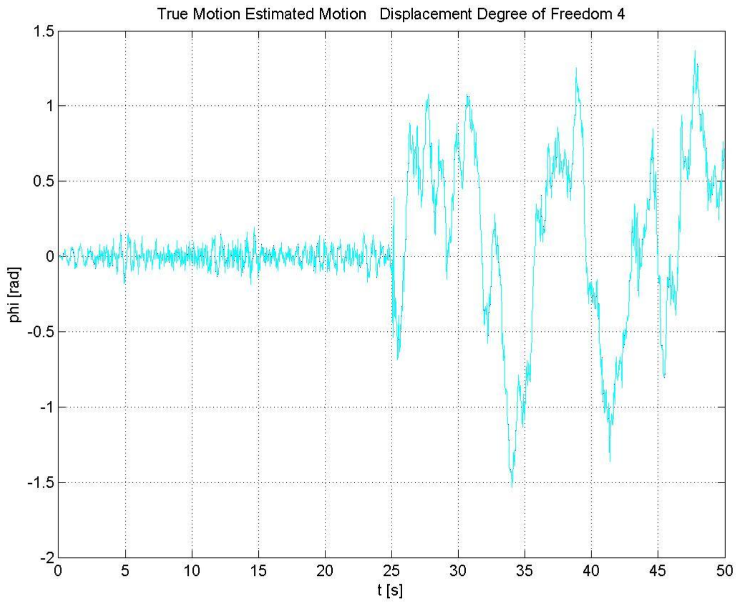
The controller has been designed to start after that half the time span has elapsed. The time evolution of the system displacement and of the estimated displacement relative to each floor are the following:


True Motion Estimated Motion Displacement Degree of Freedom 1
x [m]
t [s]
0.02
0.015
0.01
0.005
0
-0.005
-0.01
-0.015
-0.02
0
5
10
15
20
25
30
35
40
45
50

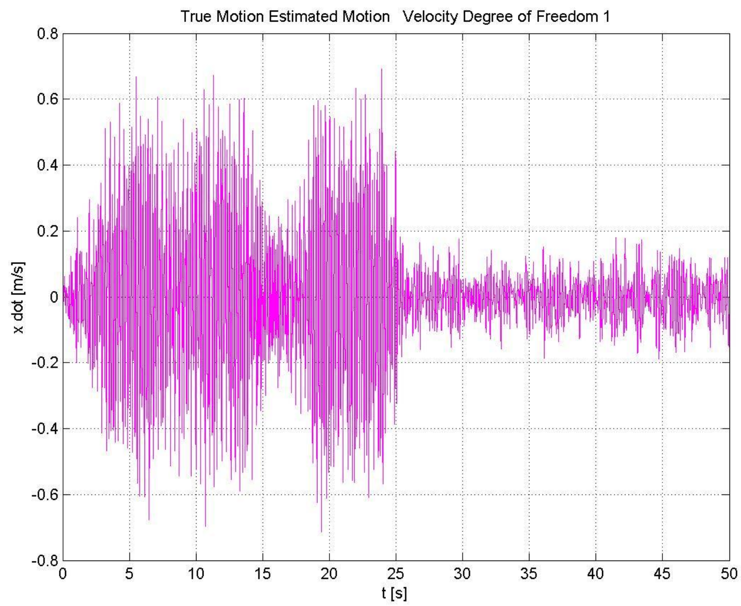

True Motion Estimated Motion Displacement Degree of Freedom 2
x [m]
t [s]
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
0
5
10
15
20
25
30
35
40
45
50

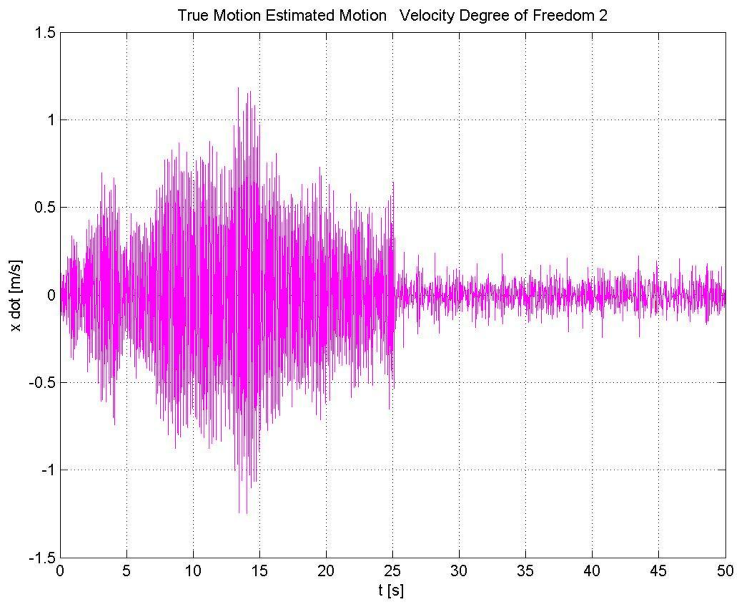


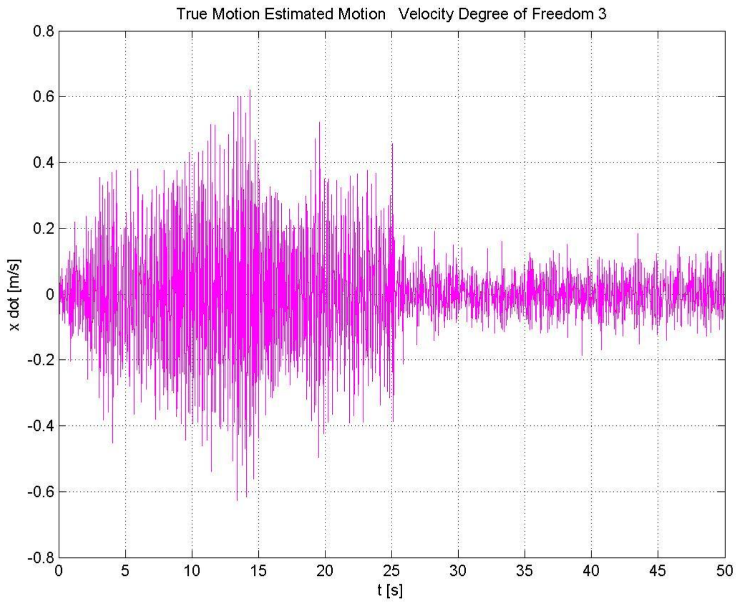
The time evolution of the system angular displacement and of the estimated angular displacement relative to the pendulum are the following:



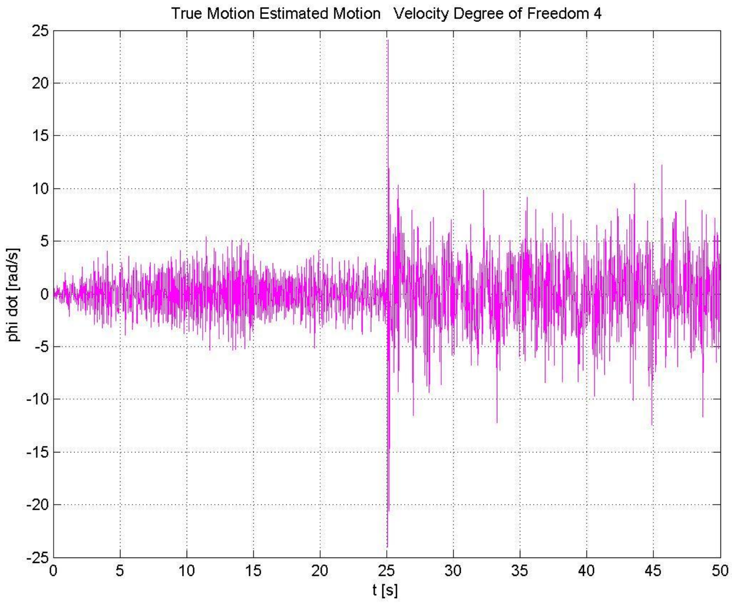
Since the pendulum serves as an actuator, when the controller starts working the amplitude of the pendulum angular displacement increases. On the other hand, the time evolution of the system velocity and of the estimated velocity relative to each floor are the following:


True Motion Estimated Motion Velocity Degree of Freedom 1
x dot [m/s]
t [s]
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
0
5
10
15
20
25
30
35
40
45
50

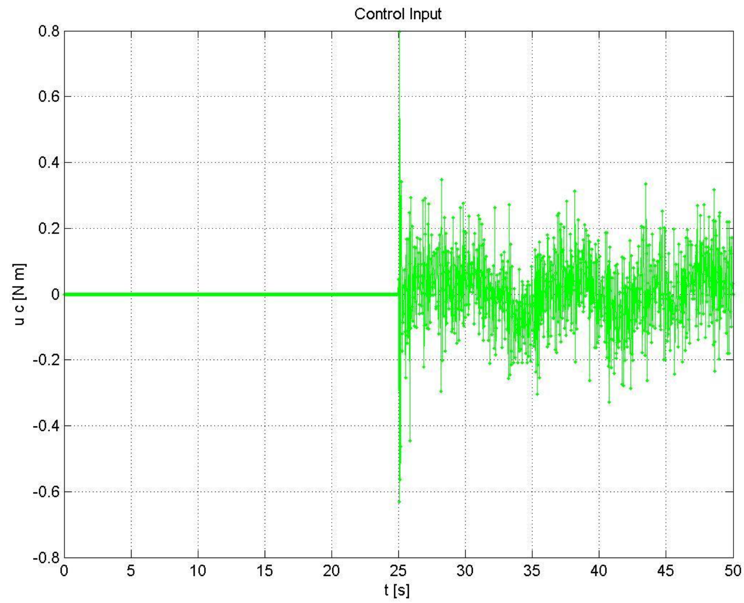

True Motion Estimated Motion Velocity Degree of Freedom 2
x dot [m/s]
t [s]
1.5
1
0.5
0
-0.5
-1
-1.5
0
5
10
15
20
25
30
35
40
45
50



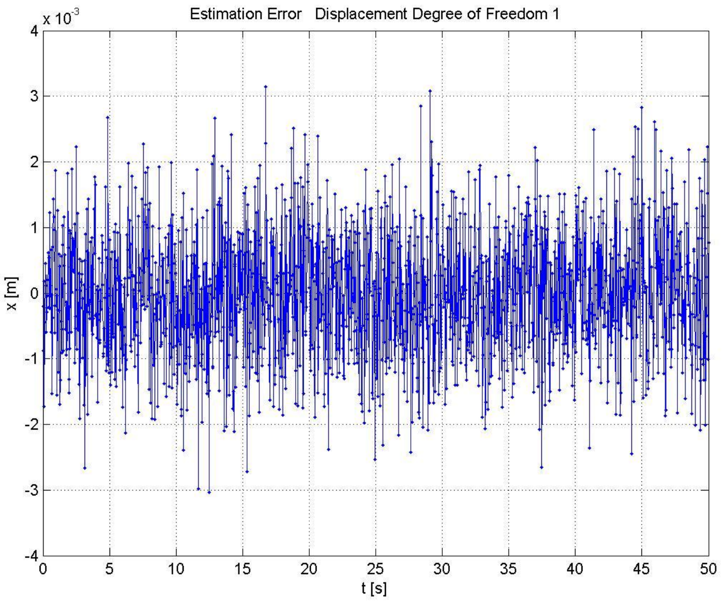
The time evolution of the system angular velocity and of the estimated angular velocity relative to the pendulum are the following:

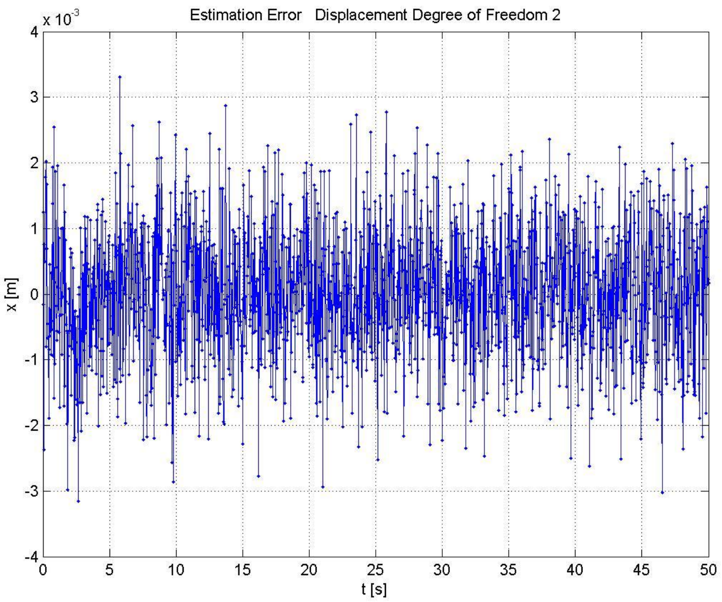


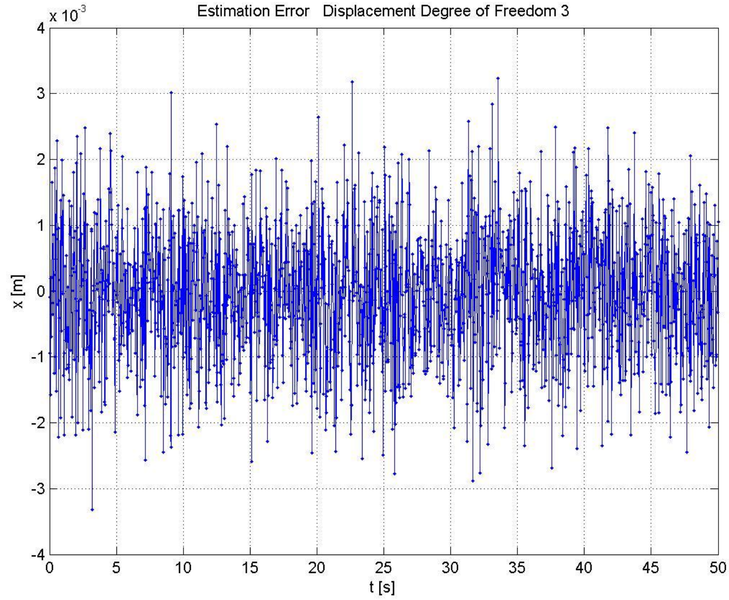
Since the pendulum serves as an actuator, when the controller starts working the amplitude of the pendulum angular velocity increases. Finally, the time evolution of the control torque is the following:



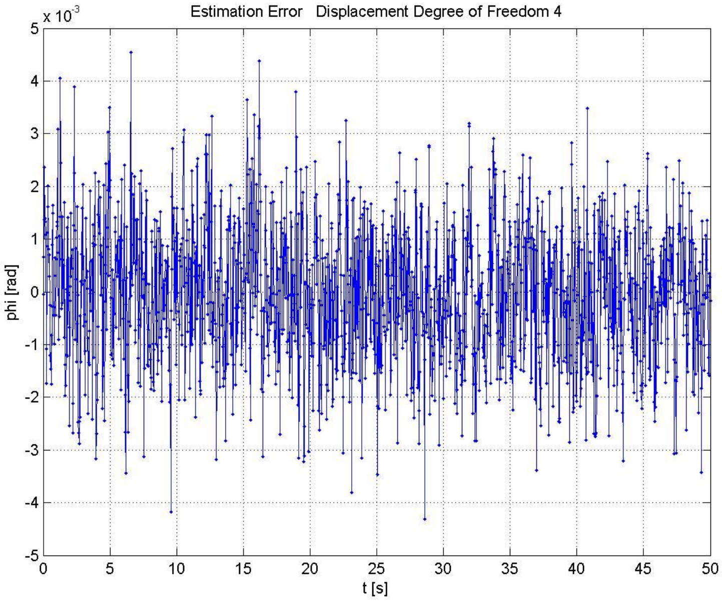
The time evolution of the estimation error relative to the system displacement corresponding to each floor is the following:


Estimation Error Displacement Degree of Freedom 1
x 10^-3
x [m]
t [s]
4
3
2
1
0
-1
-2
-3
-4
0
5
10
15
20
25
30
35
40
45
50

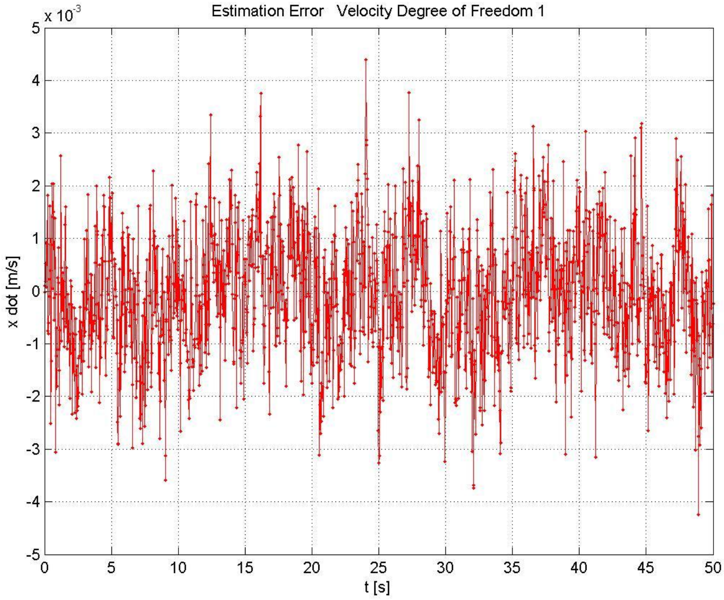

Estimation Error Displacement Degree of Freedom 2
x 10^-3
x [m]
t [s]
4
3
2
1
0
-1
-2
-3
-4
0
5
10
15
20
25
30
35
40
45
50

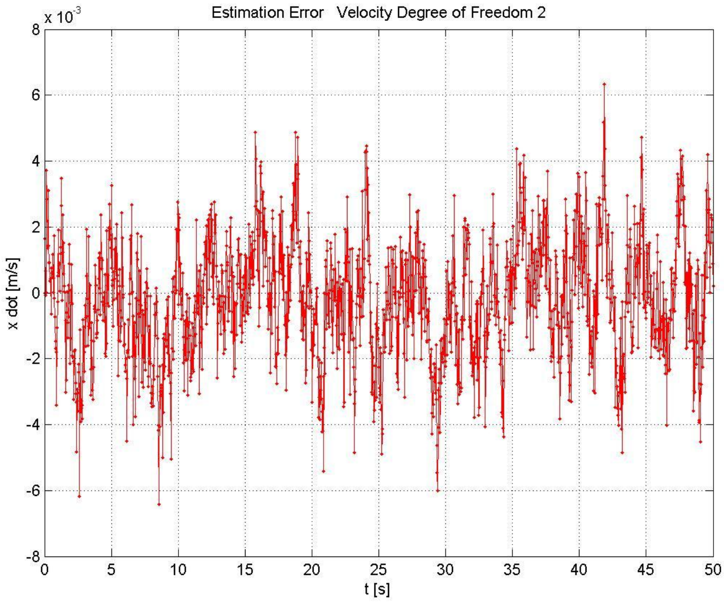


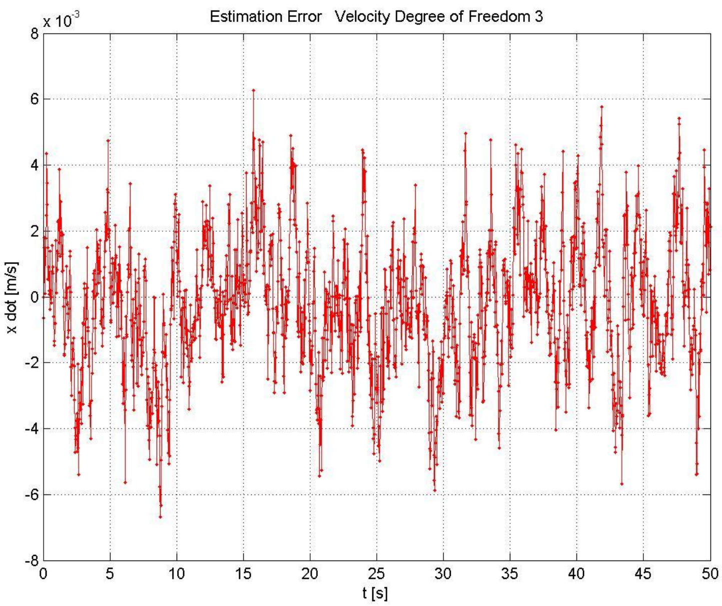
The time evolution of the estimation error relative to the pendulum angular displacement is the following:



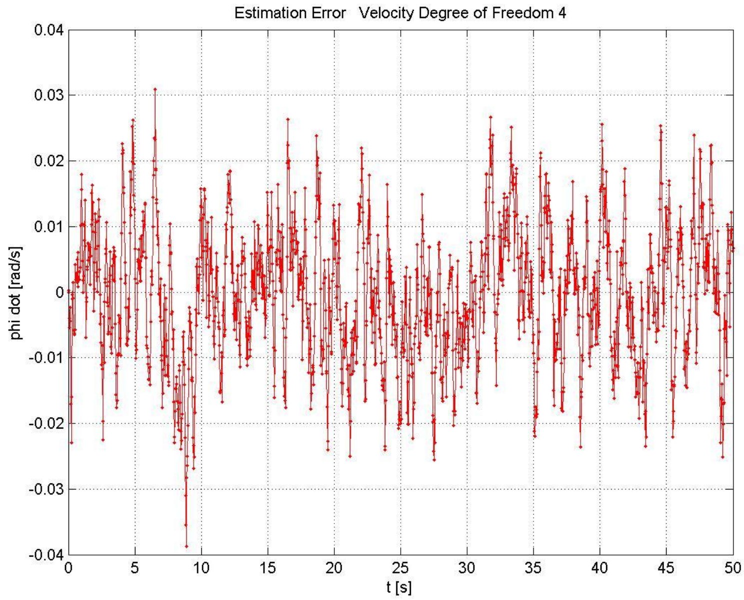
The time evolution of the estimation error relative to the system velocity corresponding to each floor is the following:


Estimation Error Velocity Degree of Freedom 1
x 10^-3
x dot [m/s]
t [s]


Estimation Error Velocity Degree of Freedom 2
x 10^-3
x dot [m/s]
t [s]



The time evolution of the estimation error relative to the pendulum angular velocity is the following:



Note that the estimation error is bounded in a relatively small range for both system generalized displacement and velocity. It is worth to emphasize that the control action is confined in an acceptable working range. Finally, the percentage decrease of the maximum amplitude of the system response at the steady state due to the action of the controller is the following:

$$\begin{cases} \dfrac{x_{1,\max}^{nc} - x_{1,\max}^{c}}{x_{1,\max}^{nc}} = 71.09\,[\%] \\ \dfrac{x_{2,\max}^{nc} - x_{2,\max}^{c}}{x_{2,\max}^{nc}} = 74.95\,[\%] \\ \dfrac{x_{3,\max}^{nc} - x_{3,\max}^{c}}{x_{3,\max}^{nc}} = 56.94\,[\%] \end{cases} \quad , \quad \begin{cases} \dfrac{\dot{x}_{1,\max}^{nc} - \dot{x}_{1,\max}^{c}}{\dot{x}_{1,\max}^{nc}} = 73.59\,[\%] \\ \dfrac{\dot{x}_{2,\max}^{nc} - \dot{x}_{2,\max}^{c}}{\dot{x}_{2,\max}^{nc}} = 80.49\,[\%] \\ \dfrac{\dot{x}_{3,\max}^{nc} - \dot{x}_{3,\max}^{c}}{\dot{x}_{3,\max}^{nc}} = 70.48\,[\%] \end{cases}$$

(5.233)

It is clear that the Linear Quadratic Gaussian controller (LQG) drastically reduces the amplitude of displacement and velocity relative to each system floor even in the worst-case scenario of an external excitation which is close to the first three system natural frequencies.

5.7. EXTENDED UDWADIA-KALABA CONTROLLER (EUK) DESIGN AND EXTENDED KALMAN FILTER (EKF) DESIGN

A new control algorithm for nonlinear underactuated mechanical systems affected by uncertainties (EUK-EKF) has been developed. This algorithm is based on the combination of the extended Udwadia-Kalaba control method (EUK) and the extended Kalman filter estimation method (EKF). The extended Udwadia-Kalaba control method (EUK) is the extension of the Udwadia-Kalaba control algorithm (UK) to underactuated mechanical systems whereas the extended Kalman filter estimation method (EUK) is the well-known extension of Kalman filter estimation algorithm (KF) to nonlinear mechanical systems. The basic idea of the Udwadia-Kalaba control method (UK) consists in setting a virtual set of constraint equations, which represent the desired behaviour for the system, and subsequently use the fundamental equations of constrained Dynamics to derive the corresponding constraints action which satisfy the constraint equations. The constraints action is then used as a feedback control law. This control strategy can be extended to underactuated mechanical system adding an extra set of constraint equations. This set of constraint equations express the requirement that some of system degrees of freedom must be unactuated, namely there must be no actuators on some specified system degrees of freedom. The key idea to translate the underactuation requirement into a set of analytical constraint equations is simply to use the unconstrained system equations of motion as an additional set of constraint equations. Indeed, consider the nonlinear lumped parameter model of the three-story building system:

$$\mathbf{M}(t)\ddot{\mathbf{q}}(t) = \mathbf{Q}(t) \tag{5.234}$$

Where $\mathbf{q}(t)$ is the vector of system lagrangian coordinates, $\mathbf{M}(t)$ is the system mass matrix and $\mathbf{Q}(t)$ is the vector lagrangian components relative to the external forces acting on the system. Considering the presence of an additional viscous damping on each degree of freedom, these quantities can be expressed as:

$$\mathbf{q}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \theta(t) \end{bmatrix} \tag{5.235}$$

$$\mathbf{M}(t) = \begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 + m_4 & -m_4 L_4 \sin(\theta(t)) \\ 0 & 0 & -m_4 L_4 \sin(\theta(t)) & m_4 L_4^2 + I_{zz,4} \end{bmatrix} \tag{5.236}$$

$$\mathbf{Q}(t) = \begin{bmatrix} -\left(k_1 + k_2\right) x_1(t) + k_2 x_2(t) - \left(\sigma_1 + \sigma_2\right) \dot{x}_1(t) + \sigma_2 \dot{x}_2(t) + F(t) \\ k_2 x_1(t) - \left(k_2 + k_3\right) x_2(t) + k_3 x_3(t) + \sigma_2 \dot{x}_1(t) - \left(\sigma_2 + \sigma_3\right) \dot{x}_2(t) + \sigma_3 \dot{x}_3(t) \\ m_4 L_4 \cos(\theta(t)) \dot{\theta}^2(t) + k_3 x_2(t) - k_3 x_3(t) + \sigma_3 \dot{x}_2(t) - \sigma_3 \dot{x}_3(t) \\ -m_4 g L_4 \cos(\theta(t)) - \sigma_4 \dot{\theta}(t) \end{bmatrix} \tag{5.237}$$

Now consider the following virtual equation of constraint:

$$x_3(t) = c \tag{5.238}$$

This virtual equation of constraint express the requirement of maintaining the displacement of the third floor constantly equal to a constant c, that is to block the movement of the third floor even in presence of an external exciting force $F(t)$ acting on the first floor. Deriving twice this equation respect to time, it can be expressed in the standard form which is suitable to use the fundamental equations of constrained Dynamics:

$$\dot{x}_3(t) = 0 \tag{5.239}$$

$$\ddot{x}_3(t) = 0 \tag{5.240}$$

The additional requirement is that the system is underactuated, namely there is only a control torque acting on the pendulum. The underactuation requirement can be accomplished using the system equations of motion relative to the floors degrees of freedom as an additional set of constraint equations:

$$\begin{cases} m_1\ddot{x}_1(t) = -\left(k_1 + k_2\right)x_1(t) + k_2 x_2(t) - \left(\sigma_1 + \sigma_2\right)\dot{x}_1(t) + \\ \quad + \sigma_2\dot{x}_2(t) + F(t) \\ m_2\ddot{x}_2(t) = k_2 x_1(t) - \left(k_2 + k_3\right)x_2(t) + k_3 x_3(t) + \sigma_2\dot{x}_1(t) + \\ \quad - \left(\sigma_2 + \sigma_3\right)\dot{x}_2(t) + \sigma_3\dot{x}_3(t) \\ \left(m_3 + m_4\right)\ddot{x}_3(t) - m_4 L_4 \sin(\theta(t))\ddot{\theta}(t) = m_4 L_4 \cos(\theta(t))\dot{\theta}^2(t) + k_3 x_2(t) + \\ \quad - k_3 x_3(t) + \sigma_3\dot{x}_2(t) - \sigma_3\dot{x}_3(t) \end{cases} \tag{5.241}$$

As a consequence, grouping together the preceding equations, the standard from of the equations of constrains can be written as follows:

$$\mathbf{A}(t)\ddot{\mathbf{q}}(t) = \mathbf{b}(t) \tag{5.242}$$

Where the constraint matrix $\mathbf{A}(t)$ and the constraint vector $\mathbf{b}(t)$ are respectively defined as:

$$\mathbf{A}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 + m_4 & -m_4 L_4 \sin(\theta(t)) \end{bmatrix} \tag{5.243}$$

$$\mathbf{b}(t) = \begin{bmatrix} 0 \\ -\left(k_1 + k_2\right) x_1(t) + k_2 x_2(t) - \left(\sigma_1 + \sigma_2\right) \dot{x}_1(t) + \sigma_2 \dot{x}_2(t) + F(t) \\ k_2 x_1(t) - \left(k_2 + k_3\right) x_2(t) + k_3 x_3(t) + \sigma_2 \dot{x}_1(t) - \left(\sigma_2 + \sigma_3\right) \dot{x}_2(t) + \sigma_3 \dot{x}_3(t) \\ m_4 L_4 \cos(\theta(t)) \dot{\theta}^2(t) + k_3 x_2(t) - k_3 x_3(t) + \sigma_3 \dot{x}_2(t) - \sigma_3 \dot{x}_3(t) \end{bmatrix} \tag{5.244}$$

At this stage it turns to be crucial to check the rank of the generalized controllability matrix which is defined as follows:

$$\mathbf{M}_c(t) = \begin{bmatrix} \mathbf{M}(t) \\ \mathbf{A}(t) \end{bmatrix} \tag{5.245}$$

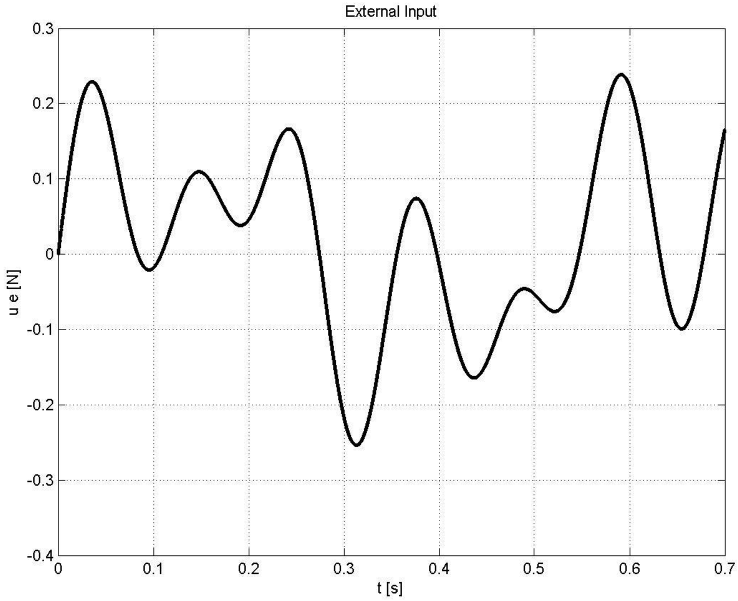
Since the generalized controllability matrix $\mathbf{M}_c(t)$ has full rank, the solution of the fundamental problem of constrained Dynamics exists and it is unique. In particular, since the constraint matrix $\mathbf{A}(t)$ is a square matrix which has full rank, it can be simply proved that the constraint action which satisfy the preceding prescribed constraint equations can be computed as:

$$\mathbf{Q}_c(t) = \mathbf{M}(t)\mathbf{A}^{-1}(t)\mathbf{b}(t) - \mathbf{Q}(t) =$$
$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\dfrac{m_4 L_4^2 + I_{zz,4}}{m_4 L_4 \sin(\theta(t))}\left(m_4 L_4 \cos(\theta(t)) \dot{\theta}^2(t) + k_3 x_2(t) - k_3 x_3(t) + \sigma_3 \dot{x}_2(t) - \sigma_3 \dot{x}_3(t)\right) + \\ + m_4 g L_4 \cos(\theta(t)) + \sigma_4 \dot{\theta}(t) \end{bmatrix} \tag{5.246}$$

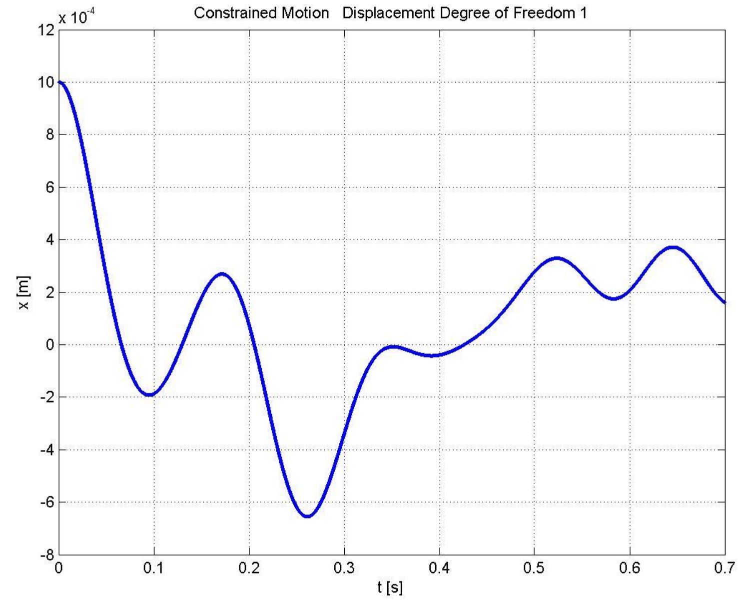
This lagrangian component of constraints action represents a control vector field which forces the system to satisfy the constraint equations. In particular, only the last component of this vector is different from zero as prescribed by the underactuation constraints. Indeed, it represents a nonlinear feedback control law for the control torque acting on the pendulum which is able to maintain the position of the third floor into a fixed value. For the simulation purposes, consider a time span equal to $T_s = 0.7\left[s\right]$ and a sampling time equal to $\Delta t = 1\cdot 10^{-4}\left[s\right]$. Considering a worst-case scenario, the external force is assumed as a superposition of three harmonic force whose excitation frequencies are close to the first three system natural frequencies. In addition, consider the following initial conditions:

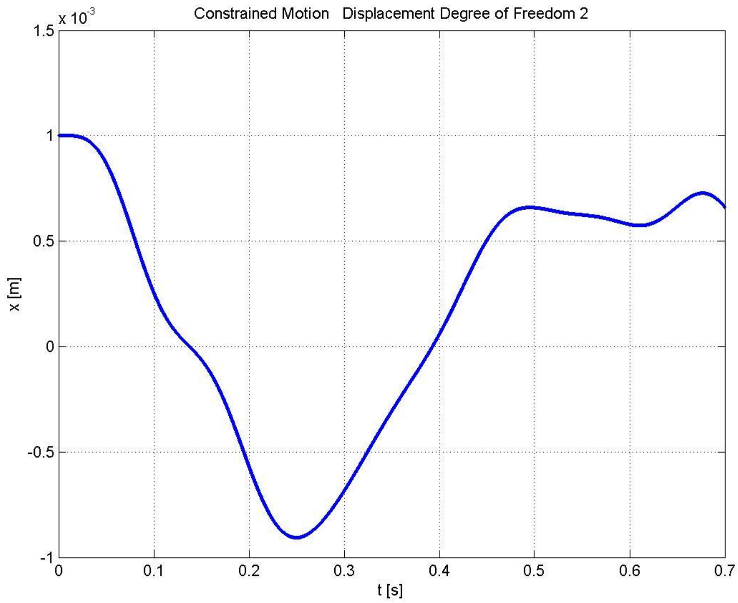
$$\begin{cases} x_1(0) = 10^{-3}\left[m\right] \\ x_2(0) = 10^{-3}\left[m\right] \\ x_3(0) = 10^{-3}\left[m\right] \\ \theta(0) = \dfrac{3}{2}\pi\left[rad\right] \end{cases}, \quad \begin{cases} \dot{x}_1(0) = 10^{-4}\left[{}^{m}\!/\!{}_{s}\right] \\ \dot{x}_2(0) = 10^{-4}\left[{}^{m}\!/\!{}_{s}\right] \\ \dot{x}_3(0) = 10^{-4}\left[{}^{m}\!/\!{}_{s}\right] \\ \dot{\theta}(0) = 10^{-4}\left[{}^{rad}\!/\!{}_{s}\right] \end{cases} \tag{5.247}$$

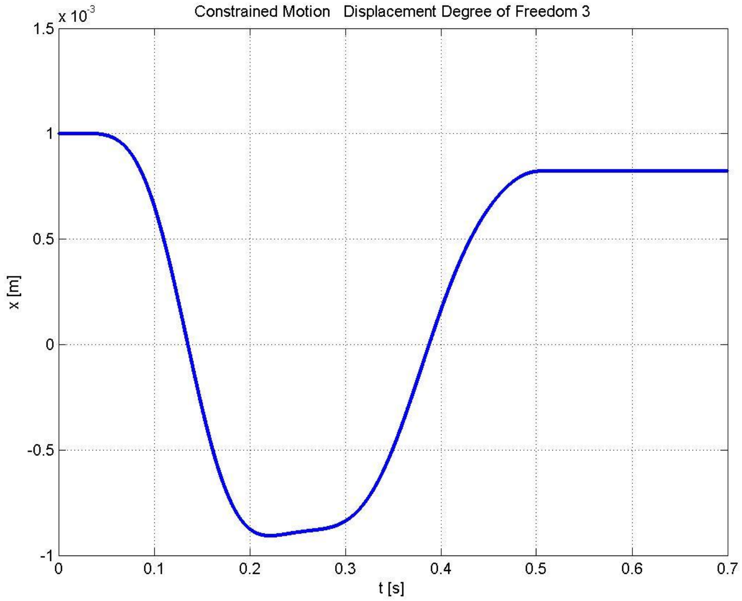
The external input force acting on the first floor is the following:



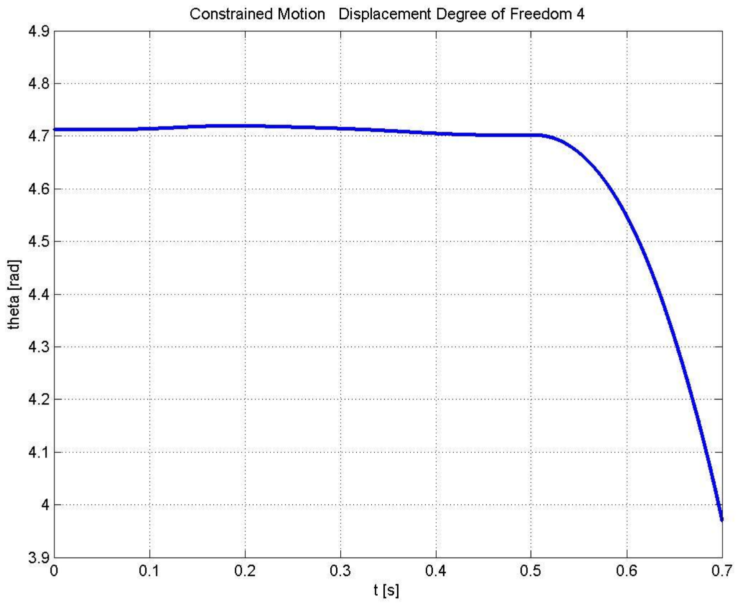
The controller has been designed to start after that half the time span has elapsed and when the velocity of the third floor is relatively small in order to evidence the difference of the system response to the external input with and without the controller. Note that to satisfy the constraint equations the controller must start when, in theory, the velocity of the third floor is zero. The time evolution of the displacement relative to each system floor are the following:


Constrained Motion Displacement Degree of Freedom 1
x 10^-4
12
10
8
6
4
2
0
-2
-4
-6
-8
x [m]
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
t [s]

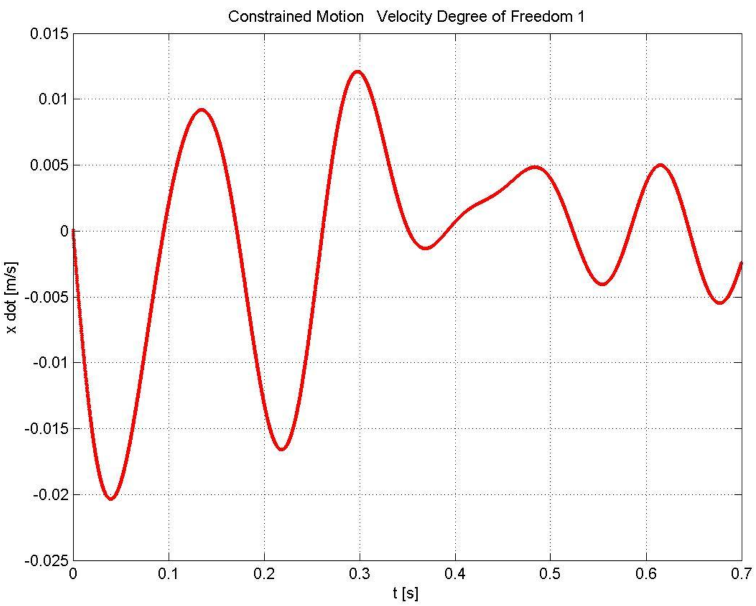

Constrained Motion Displacement Degree of Freedom 2
x 10^-3
1.5
1
0.5
0
-0.5
-1
x [m]
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
t [s]

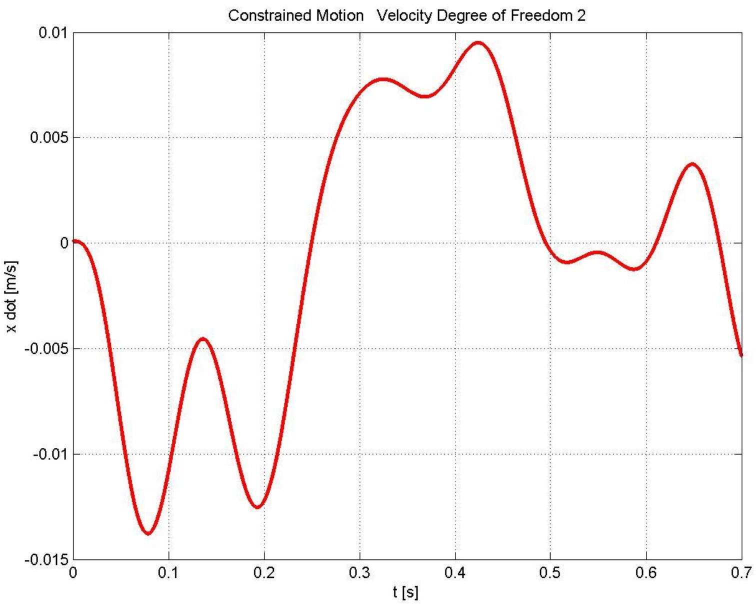


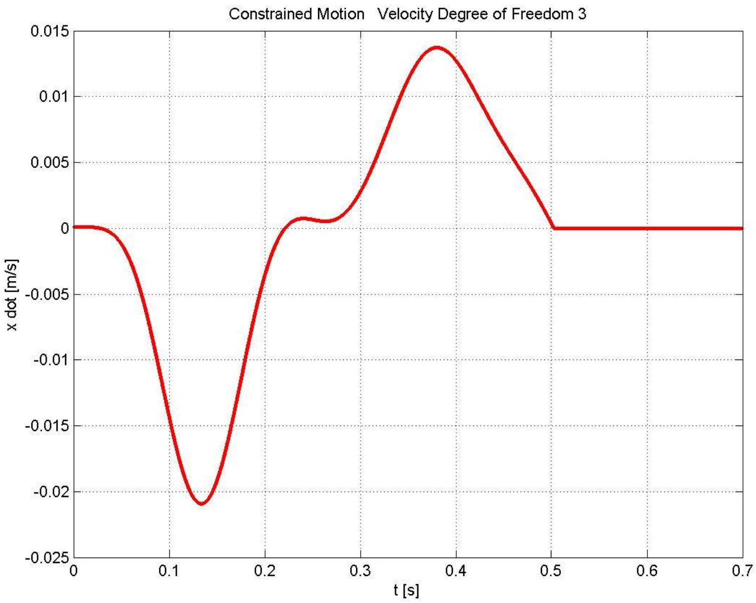
The time evolution of the angular displacement relative to the pendulum is the following:



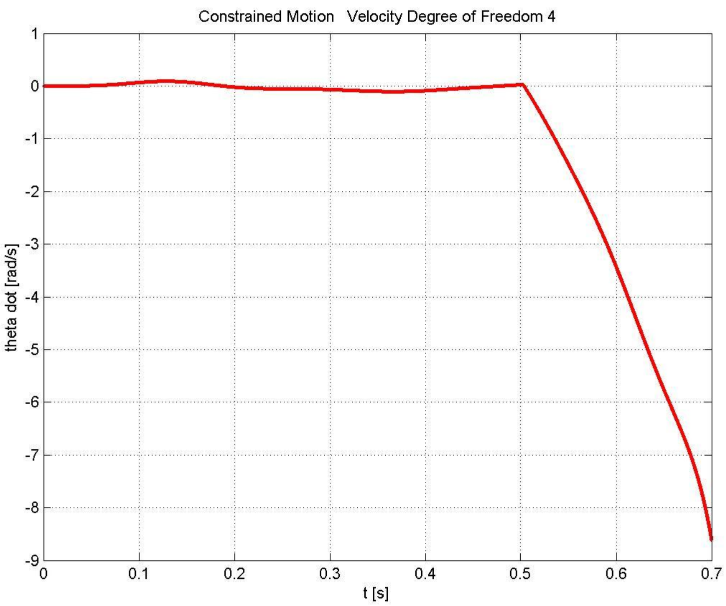
On the other hand, the time evolution of the velocity relative to each floor are the following:


Constrained Motion Velocity Degree of Freedom 1
x dot [m/s]
t [s]
0.015
0.01
0.005
0
-0.005
-0.01
-0.015
-0.02
-0.025
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7

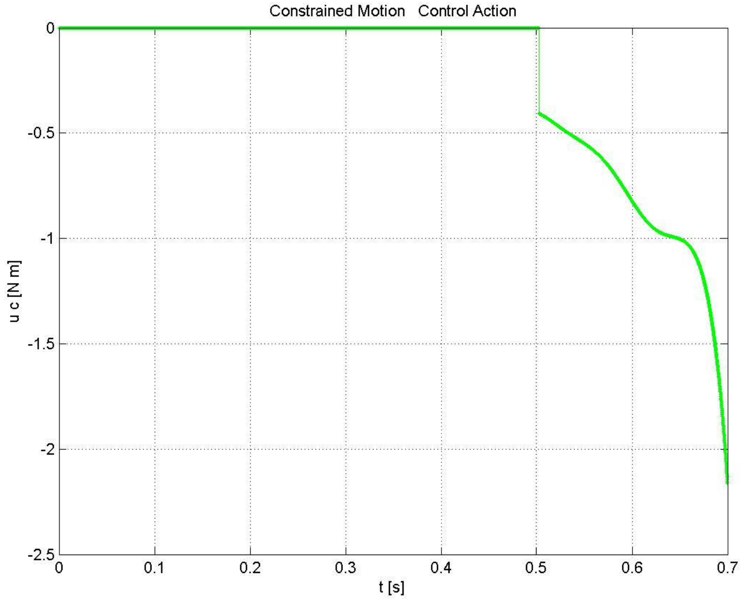

Constrained Motion Velocity Degree of Freedom 2
x dot [m/s]
t [s]
0.01
0.005
0
-0.005
-0.01
-0.015
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7



The time evolution of the angular velocity of the pendulum is the following:



Finally, the time evolution of the control torque is the following:



These plots show that when the controller start working the third floor stops vibrating and its position is hold constant in time. On the other hand, to realize the control action the pendulum must suddenly accelerate and the control torque rapidly increases. The simulation shows the effectiveness of the controller designed using the extended Udwadia-Kalaba control method (EUK). Nevertheless, from the simulation can be deduced that the choice of placing the actuator on the pendulum is not the best one. Indeed, when the pendulum approaches the horizontal position the force transferred from the actuator to the structure tends to zero and therefore the control system degenerates into a singular configuration. In order to avoid this singular configuration for the control system, the location of the actuator must be changed directly from the pendulum to the third floor. Consequently, the underactuation requirement must encompass the first two floors and the pendulum:

$$\begin{cases} m_1\ddot{x}_1(t) = -\left(k_1+k_2\right)x_1(t) + k_2x_2(t) - \left(\sigma_1+\sigma_2\right)\dot{x}_1(t) + \sigma_2\dot{x}_2(t) + F(t) \\ m_2\ddot{x}_2(t) = k_2x_1(t) - \left(k_2+k_3\right)x_2(t) + k_3x_3(t) + \sigma_2\dot{x}_1(t) - \left(\sigma_2+\sigma_3\right)\dot{x}_2(t) + \sigma_3\dot{x}_3(t) \\ -m_4L_4\sin(\theta(t))\ddot{x}_3(t) + \left(m_4L_4^2 + I_{zz,4}\right)\ddot{\theta}(t) = -m_4gL_4\cos(\theta(t)) - \sigma_4\dot{\theta}(t) \end{cases}$$
(5.248)

Therefore, the constraint matrix $\mathbf{A}(t)$ and the constraint vector $\mathbf{b}(t)$ must be redefined respectively as follows:

$$\mathbf{A}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & -m_4L_4\sin(\theta(t)) & m_4L_4^2 + I_{zz,4} \end{bmatrix} \tag{5.249}$$

$$\mathbf{b}(t) = \begin{bmatrix} 0 \\ -\left(k_1+k_2\right)x_1(t) + k_2x_2(t) - \left(\sigma_1+\sigma_2\right)\dot{x}_1(t) + \sigma_2\dot{x}_2(t) + F(t) \\ k_2x_1(t) - \left(k_2+k_3\right)x_2(t) + k_3x_3(t) + \sigma_2\dot{x}_1(t) - \left(\sigma_2+\sigma_3\right)\dot{x}_2(t) + \sigma_3\dot{x}_3(t) \\ -m_4gL_4\cos(\theta(t)) - \sigma_4\dot{\theta}(t) \end{bmatrix}$$
(5.250)

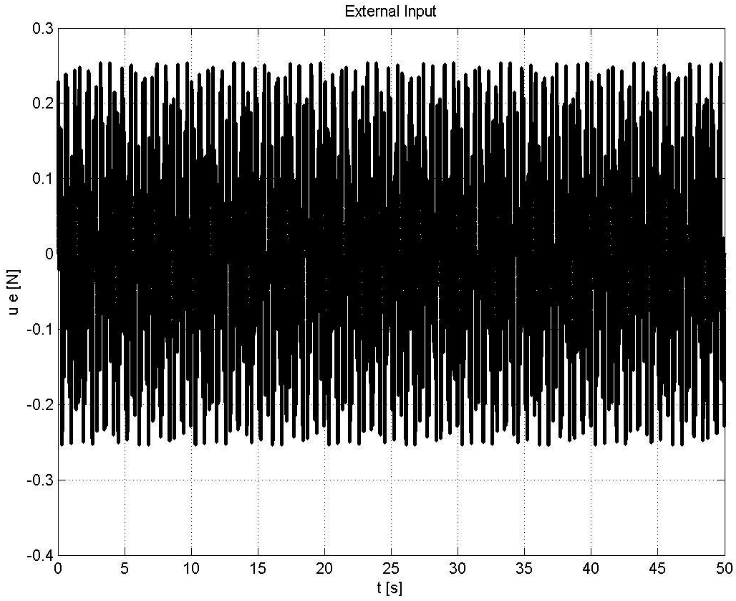
Even in this case the generalized controllability matrix $\mathbf{M}_c(t)$ has full rank. Now using the fundamental equations of constrained Dynamics it can be simply proved that the constraint action which satisfy the preceding prescribed constraint equations is the following:

$$\mathbf{Q}_c(t) = \begin{bmatrix} 0 \\ 0 \\ -\left(m_4 L_4 \cos(\theta(t))\dot{\theta}^2(t) + k_3 x_2(t) - k_3 x_3(t) + \sigma_3 \dot{x}_2(t) - \sigma_3 \dot{x}_3(t)\right) + \\ +m_4 L_4 \sin(\theta(t)) \dfrac{m_4 g L_4 \cos(\theta(t)) + \sigma_4 \dot{\theta}(t)}{m_4 L_4^2 + I_{zz,4}} \\ 0 \end{bmatrix} \tag{5.251}$$

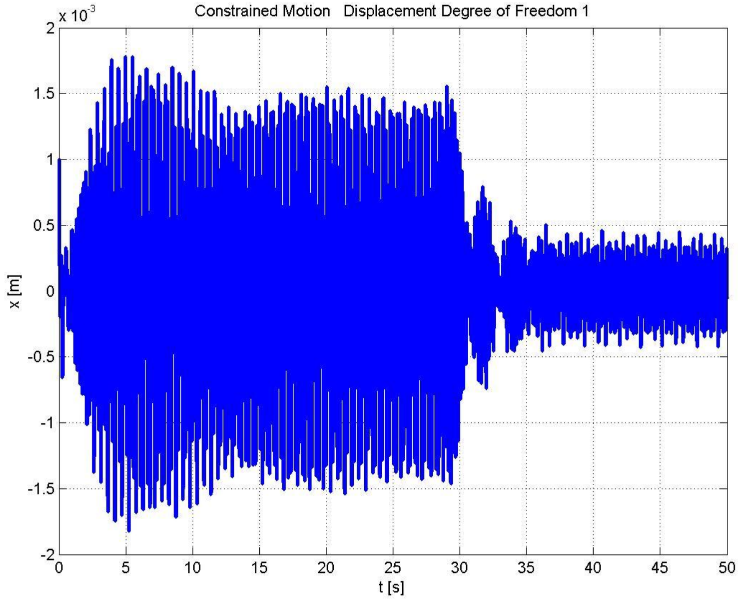
Similarly to the preceding case, the lagrangian component of constraints action represents a control vector field which forces the system to satisfy the constraint equations. In particular, only the third component of this vector is different from zero as prescribed by the underactuation constraints. Indeed, it represents a nonlinear feedback control law for the control force acting on the third floor which is able to maintain this floor into a fixed position. For the simulation purposes, consider a time span equal to $T_s = 50\left[s\right]$ and a sampling time equal to $\Delta t = 1 \cdot 10^{-4}\left[s\right]$. Considering a worst-case scenario, the external force is assumed as a superposition of three harmonic force whose excitation frequencies are close to the first three system natural frequencies. In addition, consider the following initial conditions:

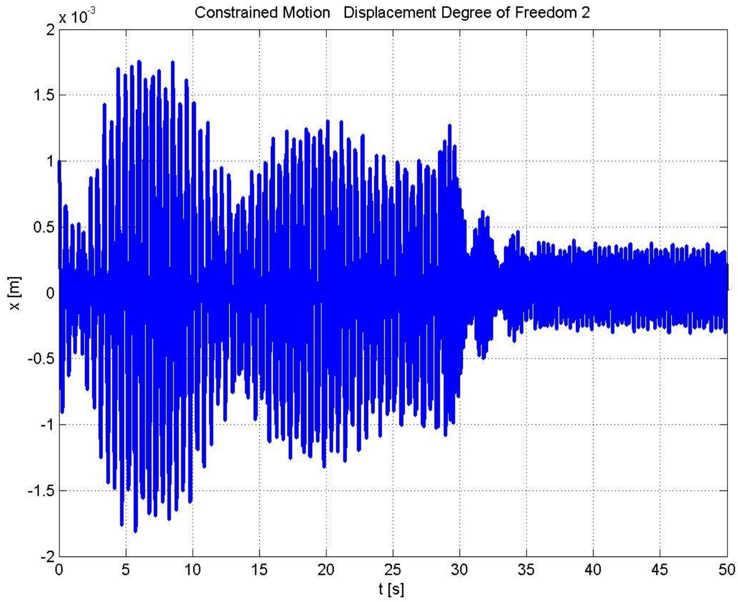
$$\begin{cases} x_1(0) = 10^{-3}\left[m\right] \\ x_2(0) = 10^{-3}\left[m\right] \\ x_3(0) = 10^{-3}\left[m\right] \\ \theta(0) = \dfrac{3}{2}\pi\left[rad\right] \end{cases}, \quad \begin{cases} \dot{x}_1(0) = 10^{-4}\left[{}^{m}\!/\!{}_{s}\right] \\ \dot{x}_2(0) = 10^{-4}\left[{}^{m}\!/\!{}_{s}\right] \\ \dot{x}_3(0) = 10^{-4}\left[{}^{m}\!/\!{}_{s}\right] \\ \dot{\theta}(0) = 10^{-4}\left[{}^{rad}\!/\!{}_{s}\right] \end{cases} \tag{5.252}$$

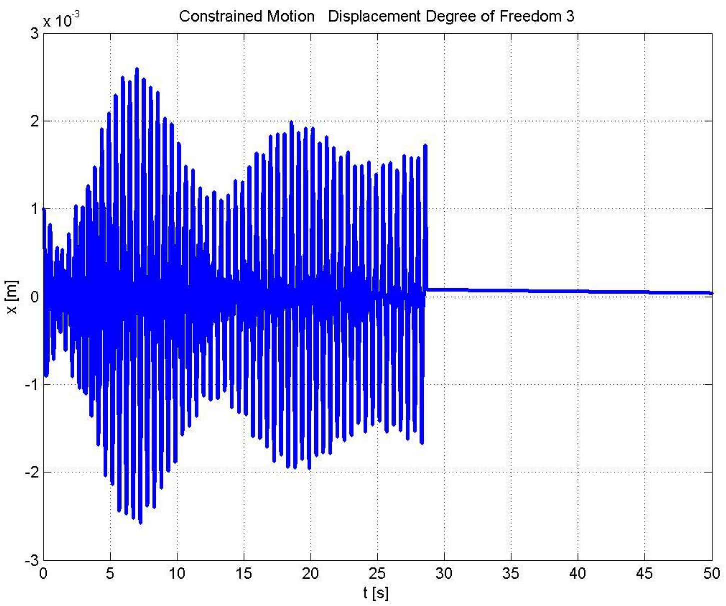
The external input force acting on the first floor is the following:



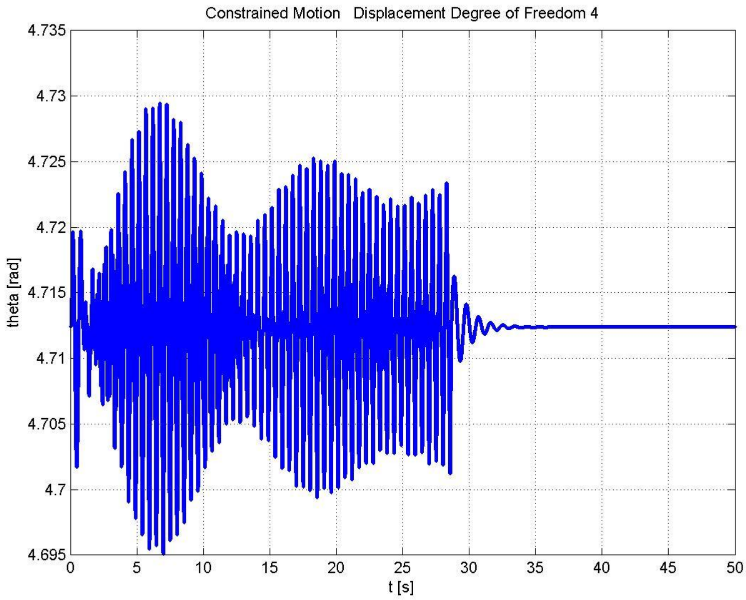
The controller has been designed to start after that half the time span has elapsed and when the velocity of the third floor is relatively small in order to evidence the difference of the system response to the external input with and without the controller. Note that to satisfy the constraint equations the controller must start when, in theory, the velocity of the third floor is zero. The time evolution of the displacement relative to each system floor are the following:


Constrained Motion Displacement Degree of Freedom 1
x 10^-3
x [m]
t [s]
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
5
10
15
20
25
30
35
40
45
50

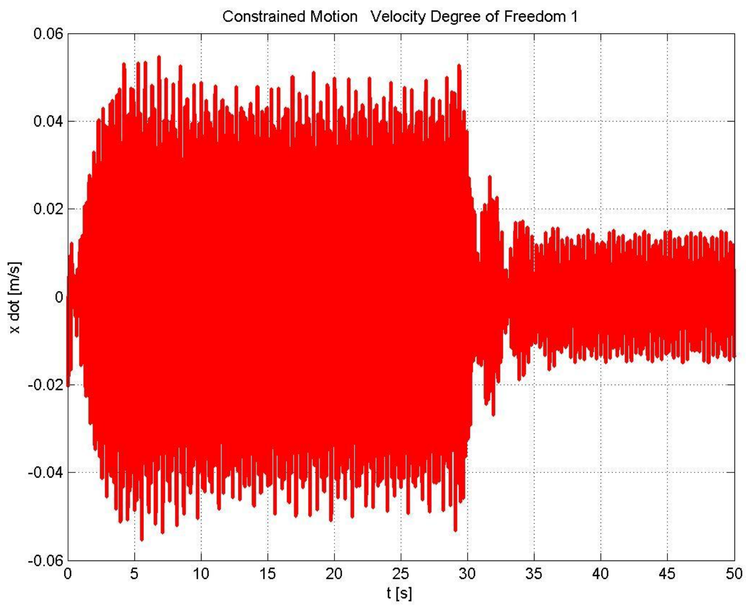

Constrained Motion Displacement Degree of Freedom 2
x 10^-3
x [m]
t [s]
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
5
10
15
20
25
30
35
40
45
50

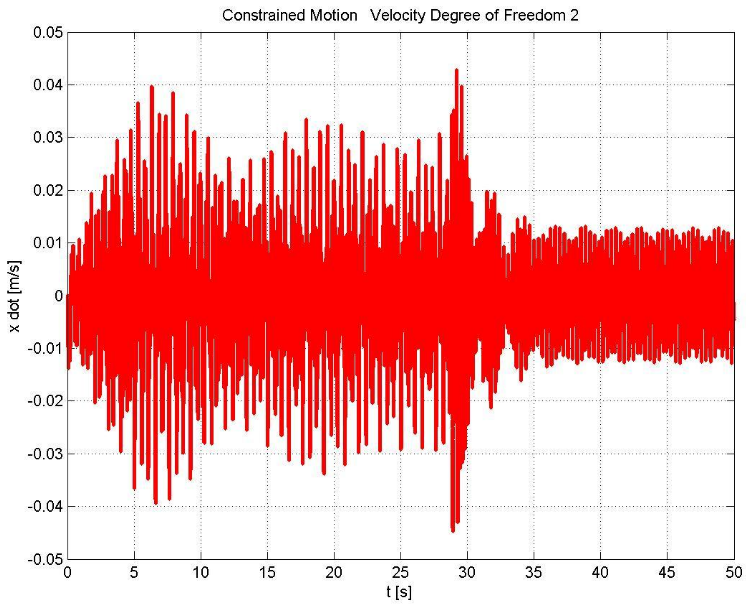


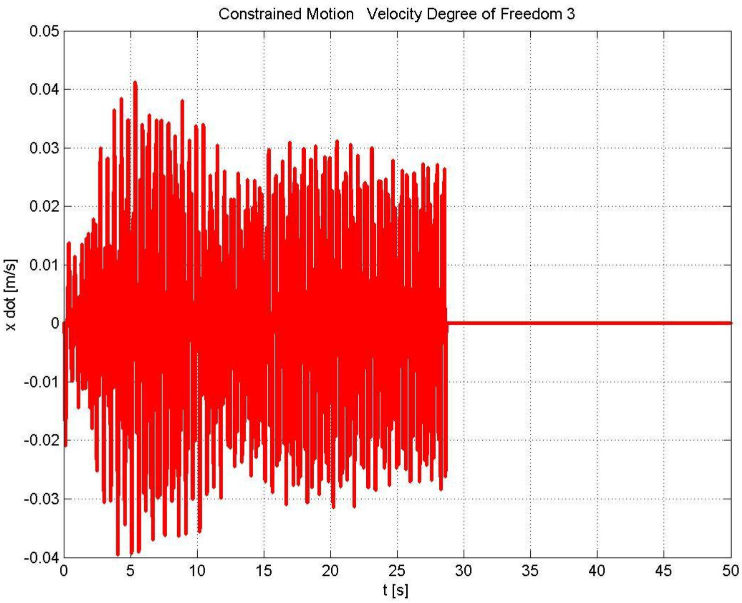
The time evolution of the angular displacement relative to the pendulum is the following:



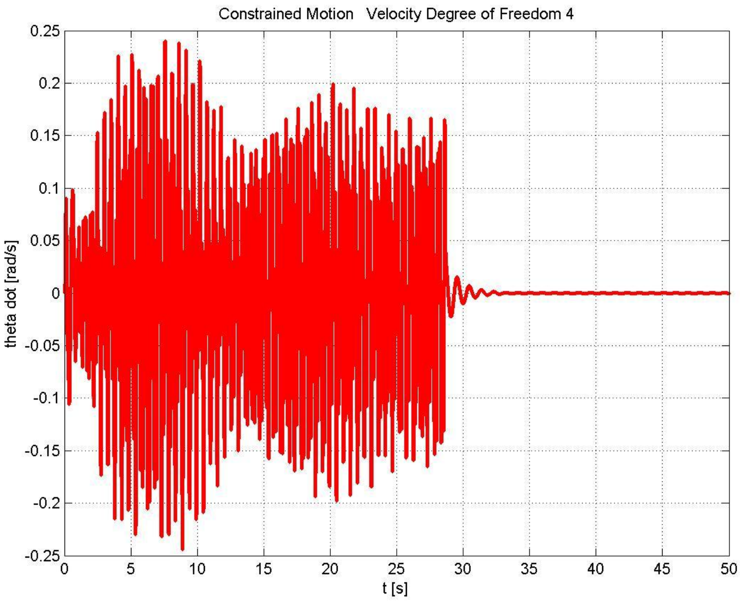
On the other hand, the time evolution of the velocity relative to each floor are the following:


Constrained Motion Velocity Degree of Freedom 1
x dot [m/s]
t [s]
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
0
5
10
15
20
25
30
35
40
45
50

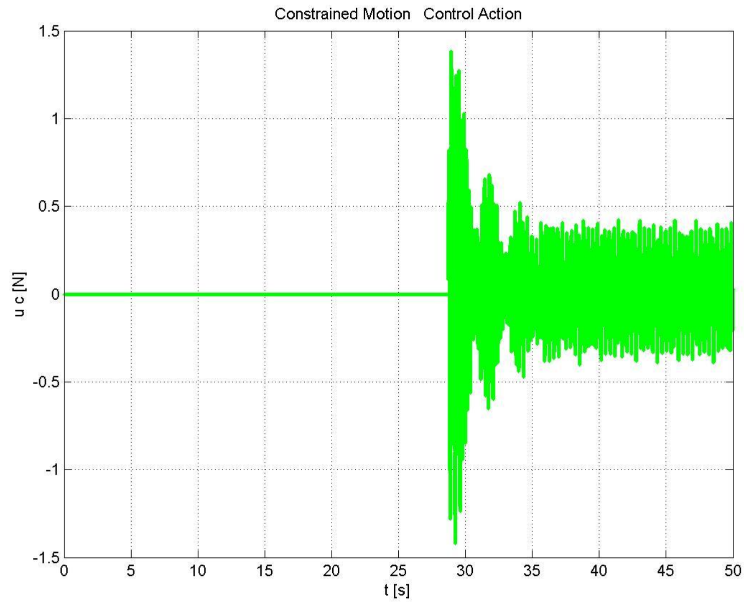

Constrained Motion Velocity Degree of Freedom 2
x dot [m/s]
t [s]
0.05
0.04
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
-0.04
-0.05
0
5
10
15
20
25
30
35
40
45
50



The time evolution of the angular velocity relative to the pendulum is the following:



Finally, the time evolution of the control force is the following:



These plots show that when the controller start working the third floor stops vibrating and its position is hold constant in time. In addition, even the displacement and the velocity relative to the other degrees of freedom are drastically reduced by the indirect action of the controller. It is worth to emphasize that in this configuration the control action is confined in an acceptable working range. Finally, consider the more realistic case in which the system state cannot be measured completely. In this case, the extended Kalman filter method (EKF) can be used to estimate the system state from the available measurements and subsequently the estimated state can be used to evaluate the feedback control law designed using the extended Udwadia-Kalaba control method (EUK). This strategy yields to a robust control algorithm which, in analogy with the Linear Quadratic Gaussian control method (LQG), represents the natural extension of the extended Udwadia-Kalaba control technique (EUK) to nonlinear underactuated mechanical systems affected by uncertainties. According to this algorithm (EUK-EKF), the system state equation can be written as follows:

$$\begin{cases} \dot{\mathbf{z}}(t) = \mathbf{f}(\mathbf{z}(t), \mathbf{u}_e(t), t) + \mathbf{f}_c(\hat{\mathbf{z}}(t), \mathbf{u}_e(t), t) + \mathbf{w}(t) \\ \mathbf{z}(0) = \mathbf{z}_0 \end{cases} \tag{5.253}$$

Where $\mathbf{z}(t)$ is the system state vector, $\hat{\mathbf{z}}(t)$ is the estimated state vector, $\mathbf{f}(t)$ is the system state vector function, $\mathbf{u}_e(t)$ is the vector of external input acting on the system, $\mathbf{f}_c(t)$ is the controller vector function and $\mathbf{w}(t)$ is the process noise vector. The system state is defined as:

$$\mathbf{z}(t) = \begin{bmatrix} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{bmatrix} \tag{5.254}$$

The state function $\mathbf{f}(t)$ is a nonlinear vector function defined as follows:

$$\mathbf{f}(t) = \begin{bmatrix} \dot{\mathbf{q}}(t) \\ \mathbf{a}(\mathbf{q}(t), \dot{\mathbf{q}}(t), t) \end{bmatrix} \tag{5.255}$$

Where $\mathbf{a}(t)$ is the generalized acceleration vector relative to the unconstrained system which can be computed according to the fundamental equations of constrained Dynamics. The controller function $\mathbf{f}_c(t)$ is a nonlinear vector function defined as:

$$\mathbf{f}_c(t) = \begin{bmatrix} \mathbf{0} \\ \mathbf{a}_c(\mathbf{q}(t), \dot{\mathbf{q}}(t), t) \end{bmatrix} \tag{5.256}$$

Where $\mathbf{a}_c(t)$ is the generalized acceleration vector corresponding to the action of the constraints which can be computed according to the fundamental equations of constrained Dynamics. Note that in the state equation the controller vector function $\mathbf{f}_c(t)$ is computed using the estimated state $\hat{\mathbf{z}}(t)$. In addition to the state equation there is the measurement equation which is a nonlinear algebraic equation defined as:

$$\mathbf{y}(t) = \mathbf{h}(\mathbf{z}(t), \mathbf{u}_e(t), t) + \mathbf{C}_a \mathbf{a}_c(\hat{\mathbf{z}}(t), t) + \mathbf{v}(t) \quad (5.257)$$

Where $\mathbf{v}(t)$ is the measurements noise vector and $\mathbf{h}(t)$ is a nonlinear measurement vector function defined in analogy to the linear systems:

$$\mathbf{h}(t) = \mathbf{C}_d \mathbf{q}(t) + \mathbf{C}_v \dot{\mathbf{q}}(t) + \mathbf{C}_a \mathbf{a}(\mathbf{q}(t), \dot{\mathbf{q}}(t), t) \quad (5.258)$$

Where $\mathbf{C}_d$, $\mathbf{C}_v$ and $\mathbf{C}_a$ identifies the output influence matrix referred respectively to system generalized displacement, velocity and acceleration. Indeed, the measurement vector function $\mathbf{h}(t)$ is a linear combination of system generalized displacement $\mathbf{q}(t)$, velocity $\dot{\mathbf{q}}(t)$ and free acceleration vector $\mathbf{a}(t)$. Note that in the measurement equation the generalized acceleration vector due to constraints action $\mathbf{a}_c(t)$ is computed using the estimated state $\hat{\mathbf{z}}(t)$. On the other hand, the evolution of the estimated state $\hat{\mathbf{z}}(t)$ can be computed from the following estimation equation:

$$\begin{cases} \dot{\hat{\mathbf{z}}}(t) = \mathbf{f}(\hat{\mathbf{z}}(t), \mathbf{u}_e(t), t) + \mathbf{f}_c(\hat{\mathbf{z}}(t), \mathbf{u}_e(t), t) + \mathbf{K}_c(\hat{\mathbf{z}}(t), t)\big(\mathbf{y}(t) - \hat{\mathbf{y}}(t)\big) \\ \hat{\mathbf{z}}(0) = \bar{\mathbf{z}}_0 \end{cases} \quad (5.259)$$

Where $\mathbf{K}_c(t)$ is the Kalman gain matrix and $\hat{\mathbf{y}}(t)$ is the measurement vector corresponding to the estimated state $\hat{\mathbf{z}}(t)$. Indeed, the measurement vector $\hat{\mathbf{y}}(t)$ corresponding to the estimated state $\hat{\mathbf{z}}(t)$ can be computed from the following nonlinear algebraic measurement equation:

$$\hat{\mathbf{y}}(t) = \mathbf{h}(\hat{\mathbf{z}}(t), \mathbf{u}_e(t), t) + \mathbf{C}_a \mathbf{a}_c(\hat{\mathbf{z}}(t), t) \quad (5.260)$$

The random disturbances $\mathbf{w}(t)$ and $\mathbf{v}(t)$ are not measurable and are assumed zero mean Gaussian white noise whose stochastic characteristics are the following:

$$E[\mathbf{w}(t)] = \mathbf{0} \quad , \quad \forall t \geq 0 \tag{5.261}$$

$$E[\mathbf{v}(t)] = \mathbf{0} \quad , \quad \forall t \geq 0 \tag{5.262}$$

$$E[\mathbf{w}(t)\mathbf{w}^T(\tau)] = \mathbf{R}_{c,w}\delta(t-\tau) \quad , \quad \forall t,\tau \geq 0 \tag{5.263}$$

$$E[\mathbf{v}(t)\mathbf{v}^T(\tau)] = \mathbf{R}_{c,v}\delta(t-\tau) \quad , \quad \forall t,\tau \geq 0 \tag{5.264}$$

Where $\mathbf{R}_{c,w}$ is the symmetric positive definite matrix defining the process noise covariance matrix and $\mathbf{R}_{c,v}$ is the symmetric positive definite matrix defining the measurement noise covariance matrix. In addition, the process noise and the measurement noise are assumed mutually uncorrelated:

$$E[\mathbf{w}(t)\mathbf{v}^T(\tau)] = \mathbf{O} \quad , \quad \forall t,\tau \geq 0 \tag{5.265}$$

On the other hand, even the initial state $\mathbf{z}_0$ is assumed unknown and it is modelled as a Gaussian distributed random vector whose stochastic characteristics are:

$$E[\mathbf{z}_0] = \overline{\mathbf{z}}_0 \tag{5.266}$$

$$E[(\mathbf{z}_0 - \overline{\mathbf{z}}_0)(\mathbf{z}_0 - \overline{\mathbf{z}}_0)^T] = \mathbf{R}_{c,0} \tag{5.267}$$

Where $\overline{\mathbf{z}}_0$ is the vector representing the expected value of initial state and $\mathbf{R}_{c,0}$ is the symmetric positive definite matrix representing the covariance matrix of the initial state. The initial state vector is modelled as a random process uncorrelated to the stochastic disturbances:

$$E[\mathbf{z}_0\mathbf{w}^T(t)] = \mathbf{O} \quad , \quad \forall t \geq 0 \tag{5.268}$$

$$E[\mathbf{z}_0 \mathbf{v}^T(t)] = \mathbf{O} \quad , \quad \forall t \geq 0 \tag{5.269}$$

In virtue of these assumptions on the stochastic part of the model, a continuous-time Kalman filter can be developed linearizing the system model around the estimated state $\hat{\mathbf{z}}(t)$. Consequently, the following linearized state matrix and output influence matrix can be defined:

$$\mathbf{A}_c(\hat{\mathbf{z}}(t), t) = \left. \frac{\partial \mathbf{f}(t)}{\partial \mathbf{z}(t)} \right|_{\hat{\mathbf{z}}(t)} \tag{5.270}$$

$$\mathbf{C}(\hat{\mathbf{z}}(t), t) = \left. \frac{\partial \mathbf{h}(t)}{\partial \mathbf{z}(t)} \right|_{\hat{\mathbf{z}}(t)} \tag{5.271}$$

Therefore, the Kalman gain matrix $\mathbf{K}_c(t)$ can be computed as:

$$\mathbf{K}_c(\hat{\mathbf{z}}(t), t) = \mathbf{P}(t)\mathbf{C}^T(\hat{\mathbf{z}}(t), t)\mathbf{R}_{c,v}^{-1} \tag{5.272}$$

Where the covariance matrix $\mathbf{P}(t)$ can be determined from the following continuous-time Riccati matrix differential equation:

$$\begin{cases} \dot{\mathbf{P}}(t) = \mathbf{A}_c(\hat{\mathbf{z}}(t), t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{A}_c^T(\hat{\mathbf{z}}(t), t) - \mathbf{P}(t)\mathbf{C}^T(\hat{\mathbf{z}}(t), t)\mathbf{R}_{c,v}^{-1}\mathbf{C}(\hat{\mathbf{z}}(t), t)\mathbf{P}(t) + \mathbf{R}_{c,w} \\ \mathbf{P}(0) = \mathbf{R}_{c,0} \end{cases} \tag{5.273}$$

It is worth noting that the state equation, the estimation equation and the filter equation are coupled and therefore they must be solved at the same time in order to find the evolution of the controlled system. Consider now the three-story building system with the control actuator located on the third floor. Using the fundamental equations of the constrained Dynamics, since the system mass matrix $\mathbf{M}(t)$ is a square matrix which has full rank, the system free acceleration vector $\mathbf{a}(t)$ can be symbolically computed as:

$$\mathbf{a}(t) = \mathbf{M}^{-1}(t)\mathbf{Q}(t) =$$

$$= \begin{bmatrix} \dfrac{-\left(k_1 + k_2\right)x_1(t) + k_2 x_2(t) - \left(\sigma_1 + \sigma_2\right)\dot{x}_1(t) + \sigma_2 \dot{x}_2(t) + F(t)}{m_1} \\ \dfrac{k_2 x_1(t) - \left(k_2 + k_3\right)x_2(t) + k_3 x_3(t) + \sigma_2 \dot{x}_1(t) - \left(\sigma_2 + \sigma_3\right)\dot{x}_2(t) + \sigma_3 \dot{x}_3(t)}{m_2} \\ -\dfrac{m_4 L_4 \sin(\theta(t))\left(m_4 g L_4 \cos(\theta(t)) + \sigma_4 \dot{\theta}(t)\right)}{\left(m_3 + m_4\right)\left(m_4 L_4^2 + I_{zz,4}\right) - \left(m_4 L_4 \sin(\theta(t))\right)^2} + \\ + \dfrac{\left(m_4 L_4^2 + I_{zz,4}\right)\left(m_4 L_4 \cos(\theta(t))\dot{\theta}^2(t) + k_3 x_2(t) - k_3 x_3(t) + \sigma_3 \dot{x}_2(t) - \sigma_3 \dot{x}_3(t)\right)}{\left(m_3 + m_4\right)\left(m_4 L_4^2 + I_{zz,4}\right) - \left(m_4 L_4 \sin(\theta(t))\right)^2} \\ -\dfrac{\left(m_3 + m_4\right)\left(m_4 g L_4 \cos(\theta(t)) + \sigma_4 \dot{\theta}(t)\right)}{\left(m_3 + m_4\right)\left(m_4 L_4^2 + I_{zz,4}\right) - \left(m_4 L_4 \sin(\theta(t))\right)^2} + \\ + \dfrac{m_4 L_4 \sin(\theta(t))\left(m_4 L_4 \cos(\theta(t))\dot{\theta}^2(t) + k_3 x_2(t) - k_3 x_3(t) + \sigma_3 \dot{x}_2(t) - \sigma_3 \dot{x}_3(t)\right)}{\left(m_3 + m_4\right)\left(m_4 L_4^2 + I_{zz,4}\right) - \left(m_4 L_4 \sin(\theta(t))\right)^2} \end{bmatrix} \tag{5.274}$$

In addition, since the constraint matrix $\mathbf{A}(t)$ is a square matrix which has full rank, it can be proved that the acceleration vector induced by the constraints action $\mathbf{a}_c(t)$ can be simply computed as follows:

$$\mathbf{a}_c(t) = \mathbf{A}^{-1}(t)\mathbf{b}(t) - \mathbf{a}(t) =$$

$$= \begin{bmatrix} 0 \\ 0 \\ -\dfrac{m_4 L_4 \sin(\theta(t))\left(m_4 g L_4 \cos(\theta(t)) + \sigma_4 \dot{\theta}(t)\right)}{\left(m_3+m_4\right)\left(m_4 L_4^2 + I_{zz,4}\right) - \left(m_4 L_4 \sin(\theta(t))\right)^2} + \\ -\dfrac{\left(m_4 L_4^2 + I_{zz,4}\right)\left(m_4 L_4 \cos(\theta(t))\dot{\theta}^2(t) + k_3 x_2(t) - k_3 x_3(t) + \sigma_3 \dot{x}_2(t) - \sigma_3 \dot{x}_3(t)\right)}{\left(m_3+m_4\right)\left(m_4 L_4^2 + I_{zz,4}\right) - \left(m_4 L_4 \sin(\theta(t))\right)^2} \\ -\dfrac{m_4 L_4 \cos(\theta(t))\dot{\theta}^2(t) + k_3 x_2(t) - k_3 x_3(t) + \sigma_3 \dot{x}_2(t) - \sigma_3 \dot{x}_3(t)}{m_4 L_4 \sin(\theta(t))} + \\ +\dfrac{\left(m_3+m_4\right)\left(m_4 g L_4 \cos(\theta(t)) + \sigma_4 \dot{\theta}(t)\right)}{\left(m_3+m_4\right)\left(m_4 L_4^2 + I_{zz,4}\right) - \left(m_4 L_4 \sin(\theta(t))\right)^2} + \\ -\dfrac{m_4 L_4 \sin(\theta(t))\left(m_4 L_4 \cos(\theta(t))\dot{\theta}^2(t) + k_3 x_2(t) - k_3 x_3(t) + \sigma_3 \dot{x}_2(t) - \sigma_3 \dot{x}_3(t)\right)}{\left(m_3+m_4\right)\left(m_4 L_4^2 + I_{zz,4}\right) - \left(m_4 L_4 \sin(\theta(t))\right)^2} \end{bmatrix} \quad (5.275)$$

For the simulation purposes, consider a time span equal to $T_s = 50\left[s\right]$ and a sampling time equal to $\Delta t = 1\cdot 10^{-4}\left[s\right]$. Considering a worst-case scenario, the external force is assumed as a superposition of three harmonic force whose excitation frequencies are close to the first three system natural frequencies. In addition, consider the following initial conditions:

$$\begin{cases} x_1(0) = 10^{-3}\left[m\right] \\ x_2(0) = 10^{-3}\left[m\right] \\ x_3(0) = 10^{-3}\left[m\right] \\ \theta(0) = \dfrac{3}{2}\pi\left[rad\right] \end{cases}, \quad \begin{cases} \dot{x}_1(0) = 10^{-4}\left[{}^{m}\!/_{s}\right] \\ \dot{x}_2(0) = 10^{-4}\left[{}^{m}\!/_{s}\right] \\ \dot{x}_3(0) = 10^{-4}\left[{}^{m}\!/_{s}\right] \\ \dot{\theta}(0) = 10^{-4}\left[{}^{rad}\!/_{s}\right] \end{cases} \quad (5.276)$$

Moreover, the initial state is assumed as a stochastic process with the following characteristics:

$$\mathbf{Z}_0 = \begin{bmatrix} 10^{-3}\,[m] \\ 10^{-3}\,[m] \\ 10^{-3}\,[m] \\ 1\,[rad] \\ 10^{-4}\left[{}^{m}\!/_{s}\right] \\ 10^{-4}\left[{}^{m}\!/_{s}\right] \\ 10^{-4}\left[{}^{m}\!/_{s}\right] \\ 10^{-4}\left[{}^{rad}\!/_{s}\right] \end{bmatrix} \tag{5.277}$$

$$\bar{\mathbf{z}}_0 = \mathbf{z}_0 + \mathbf{Z}_0 \tag{5.278}$$

$$\mathbf{R}_{c,0} = diag(10^{-6}\left[m^2\right], 10^{-6}\left[m^2\right], 10^{-6}\left[m^2\right], 1\left[rad^2\right], \\ 10^{-8}\left[{}^{m^2}\!/_{s^2}\right], 10^{-8}\left[{}^{m^2}\!/_{s^2}\right], 10^{-8}\left[{}^{m^2}\!/_{s^2}\right], 10^{-8}\left[{}^{rad^2}\!/_{s^2}\right]) \tag{5.279}$$

The measured output vector contains the acceleration relative to each floor and the angular position of the pendulum. Consequently, the output influence matrices referred respectively to system generalized displacement, velocity and acceleration are assumed as follows:

$$\mathbf{C}_d = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5.280}$$

$$\mathbf{C}_v = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.281}$$

$$\mathbf{C}_a = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.282}$$

The process noise and the measurement noise are assumed zero mean Gaussian white noise whose stochastic characteristics are the following:

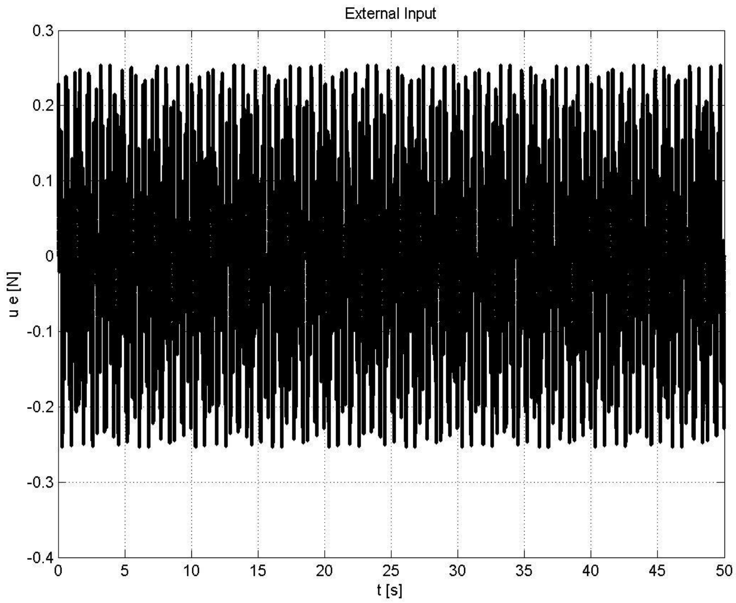
$$\mathbf{W}_0 = diag(10^{-3}\left[{}^{m}\!/\!{}_{s}\right], 10^{-3}\left[{}^{m}\!/\!{}_{s}\right], 10^{-3}\left[{}^{m}\!/\!{}_{s}\right], 10^{-3}\left[{}^{rad}\!/\!{}_{s}\right], \\ 10^{-3}\left[{}^{m}\!/\!{}_{s^2}\right], 10^{-3}\left[{}^{m}\!/\!{}_{s^2}\right], 10^{-3}\left[{}^{m}\!/\!{}_{s^2}\right], 10^{-3}\left[{}^{rad}\!/\!{}_{s^2}\right]) \tag{5.283}$$

$$\mathbf{V}_0 = diag(10^{-3}\left[{}^{m}\!/\!{}_{s^2}\right], 10^{-3}\left[{}^{m}\!/\!{}_{s^2}\right], 10^{-3}\left[{}^{m}\!/\!{}_{s^2}\right], 10^{-3}\left[rad\right]) \tag{5.284}$$

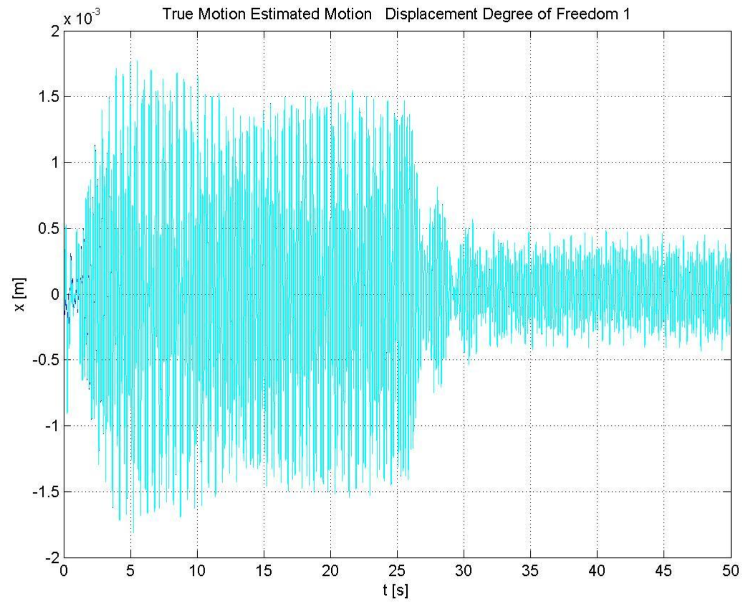
$$\mathbf{R}_{c,w} = \mathbf{W}_0^2 \tag{5.285}$$

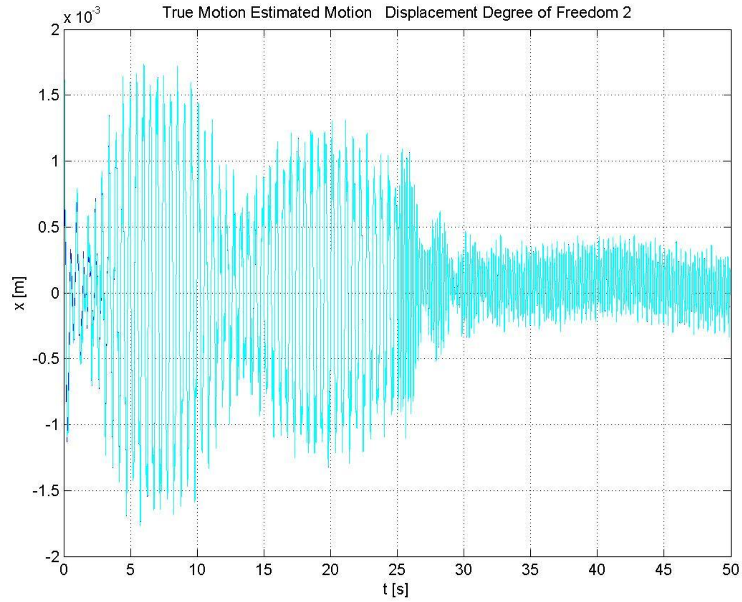
$$\mathbf{R}_{c,v} = \mathbf{V}_0^2 \tag{5.286}$$

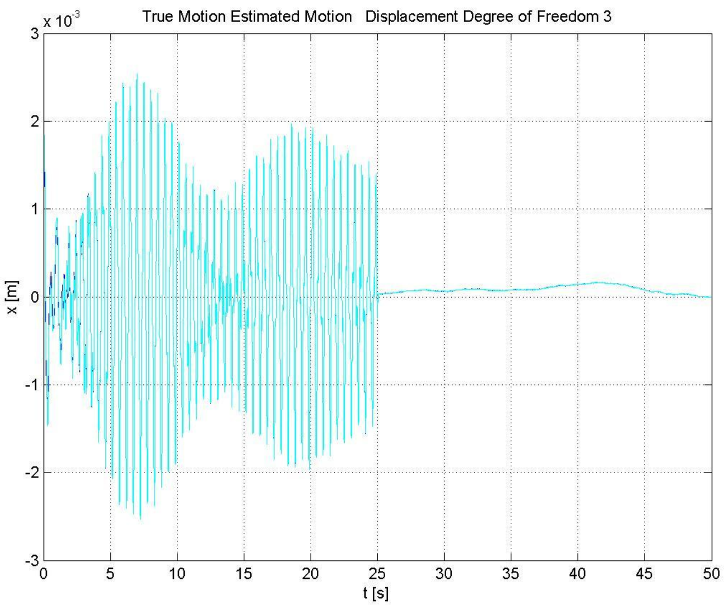
Where $\mathbf{W}_0$ is the amplitude vector relative to the process noise, $\mathbf{V}_0$ is the amplitude vector relative to the measurement noise, $\mathbf{R}_w$ is the process noise covariance matrix and $\mathbf{R}_v$ is the measurement noise covariance matrix. The external input force acting on the first floor is the following:



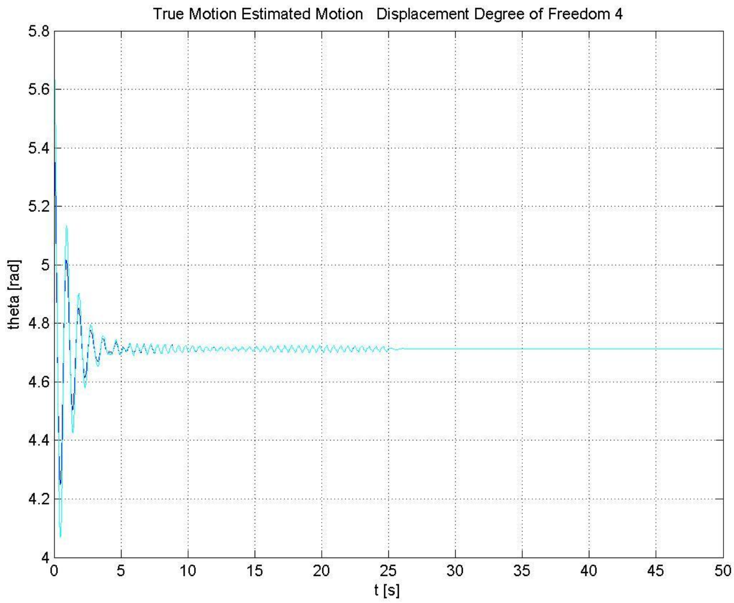
Even in this case, the controller has been designed to start after that half the time span has elapsed and when the velocity of the third floor is relatively small in order to evidence the difference of the system response to the external input with and without the controller. Note that to satisfy the constraint equations the controller must start when, in theory, the velocity of the third floor is zero. The time evolution of the system displacement and of the estimated displacement relative to each floor are the following:


True Motion Estimated Motion Displacement Degree of Freedom 1
x 10^-3
x [m]
t [s]

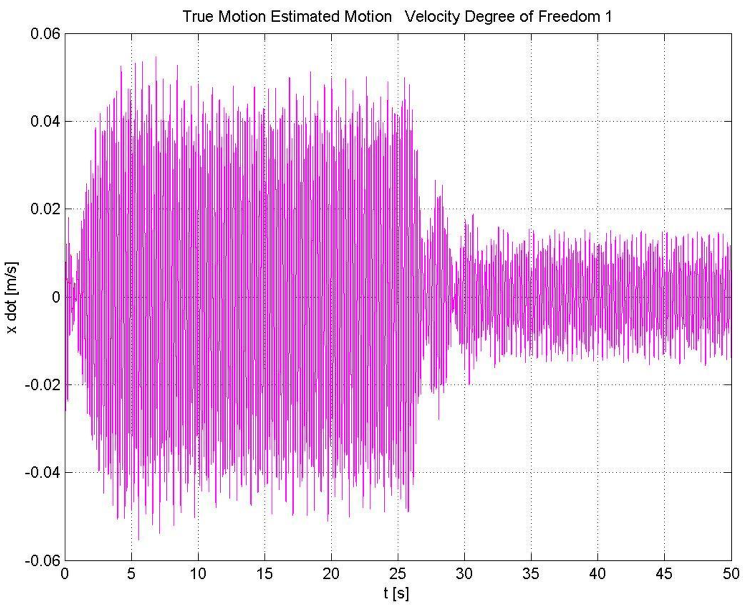

True Motion Estimated Motion Displacement Degree of Freedom 2
x 10^-3
x [m]
t [s]

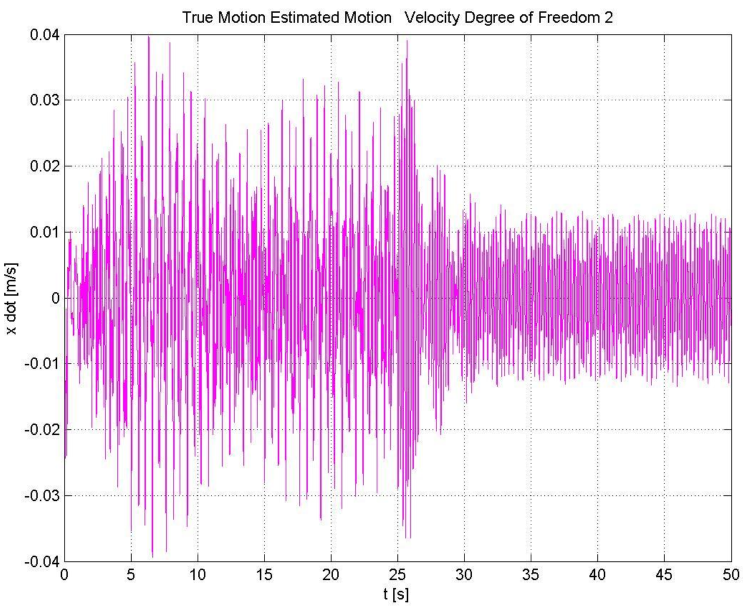


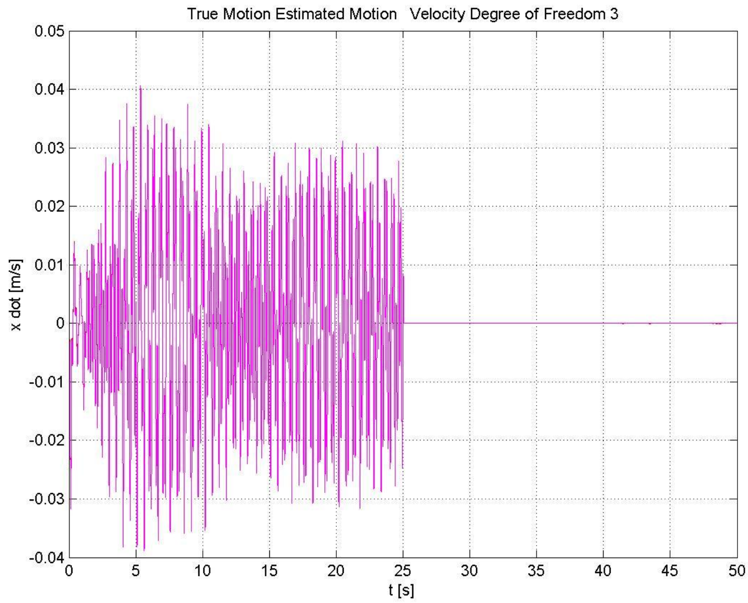
The time evolution of the system angular displacement and of the estimated angular displacement relative to the pendulum are the following:



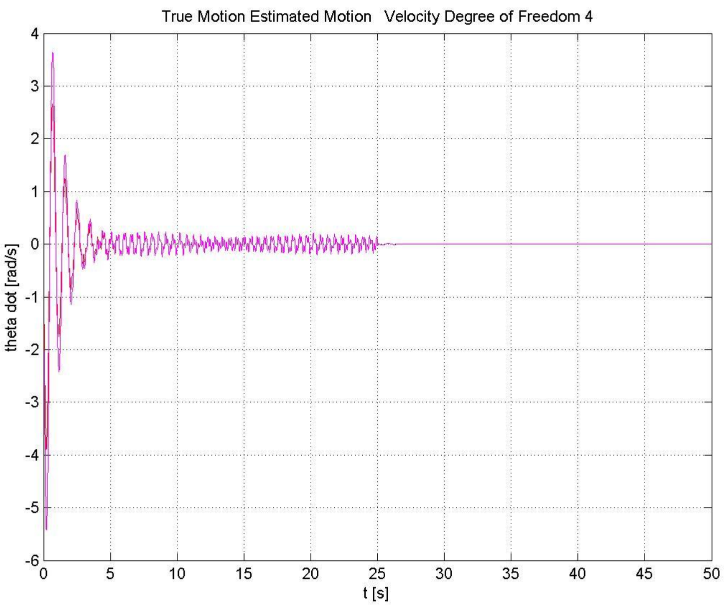
On the other hand, the time evolution of the system velocity and of the estimated velocity relative to each floor are the following:


True Motion Estimated Motion Velocity Degree of Freedom 1
x dot [m/s]
t [s]

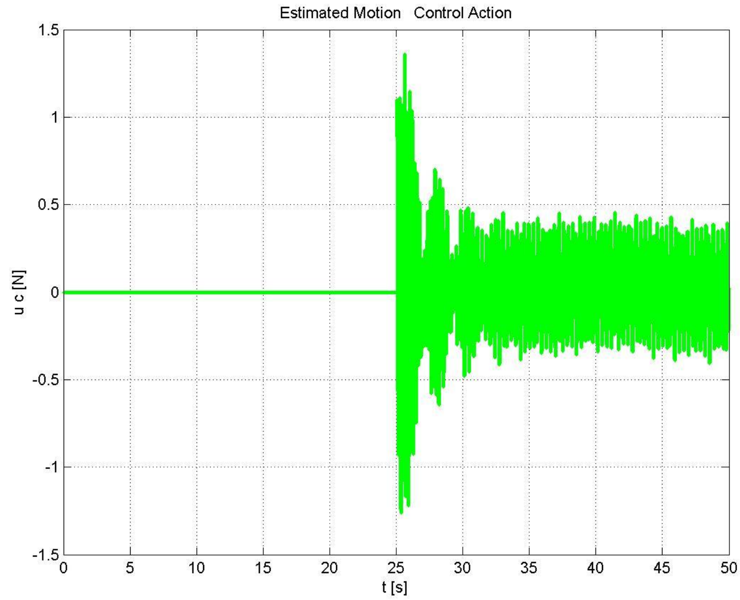

True Motion Estimated Motion Velocity Degree of Freedom 2
x dot [m/s]
t [s]



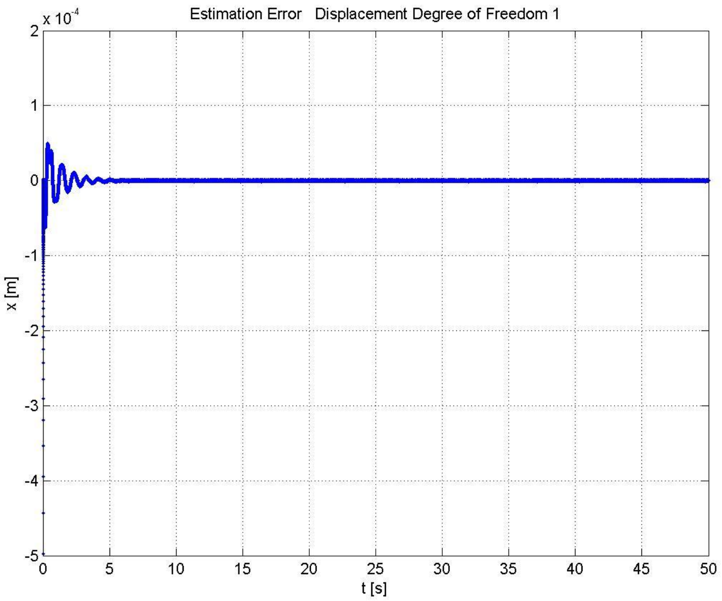
The time evolution of the system angular velocity and of the estimated angular velocity relative to the pendulum are the following:

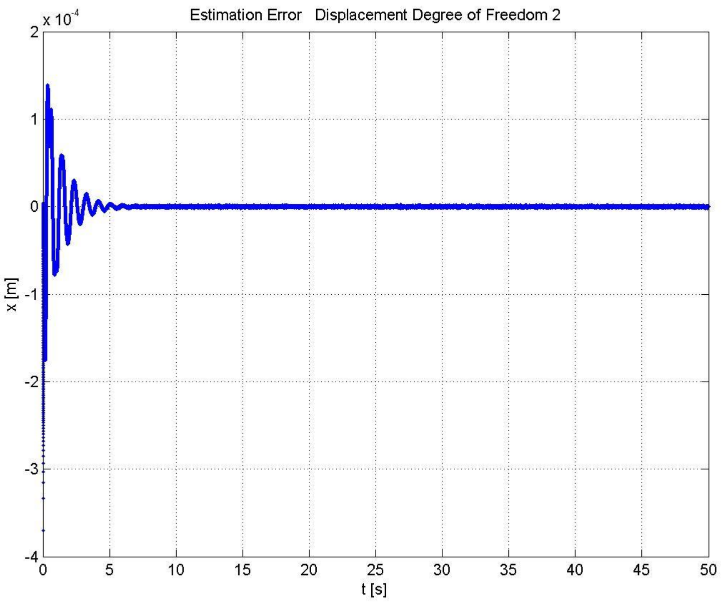


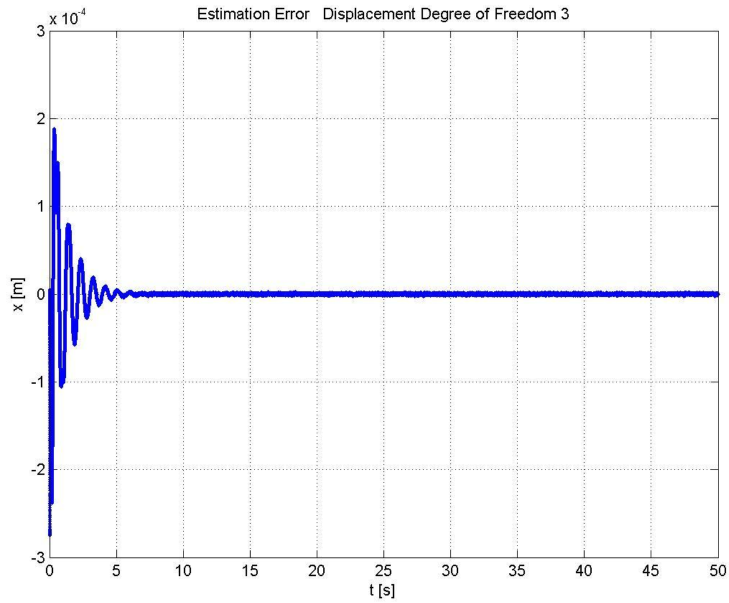
Finally, the time evolution of the control force is the following:



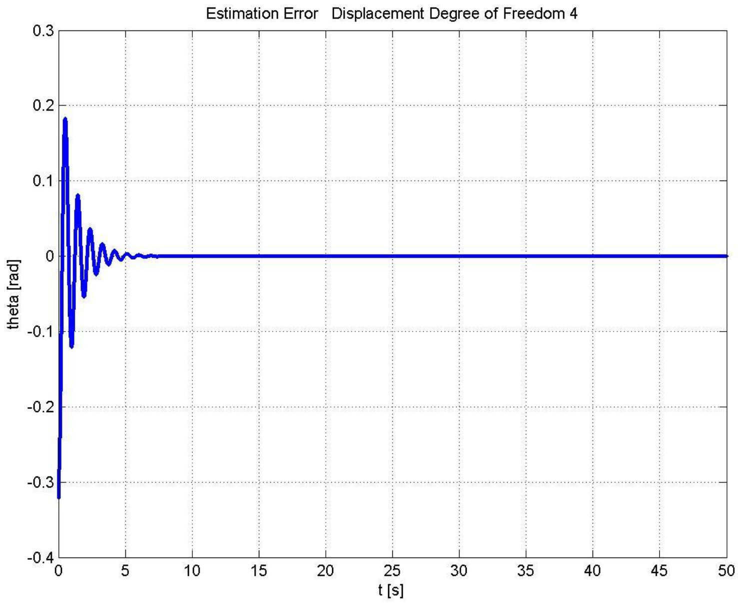
The time evolution of the estimation error relative to the system displacement corresponding to each floor is the following:


Estimation Error Displacement Degree of Freedom 1
x 10^-4
x [m]
t [s]

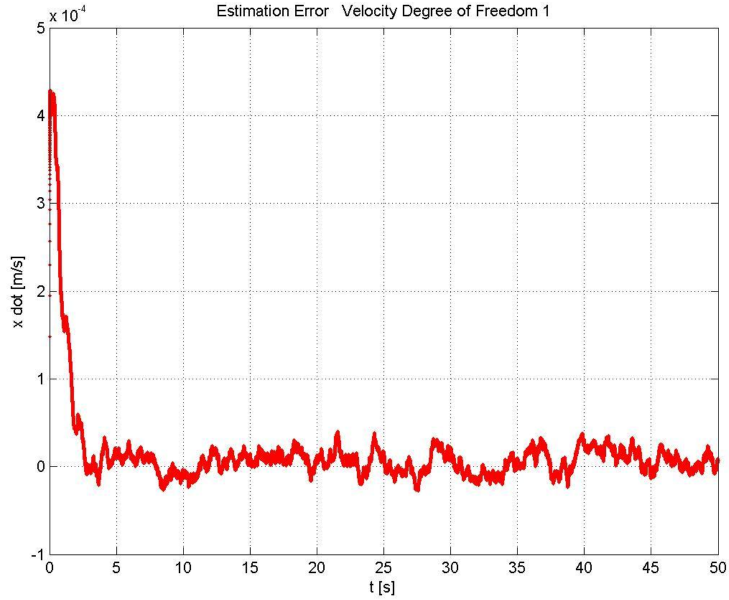

Estimation Error Displacement Degree of Freedom 2
x 10^-4
x [m]
t [s]

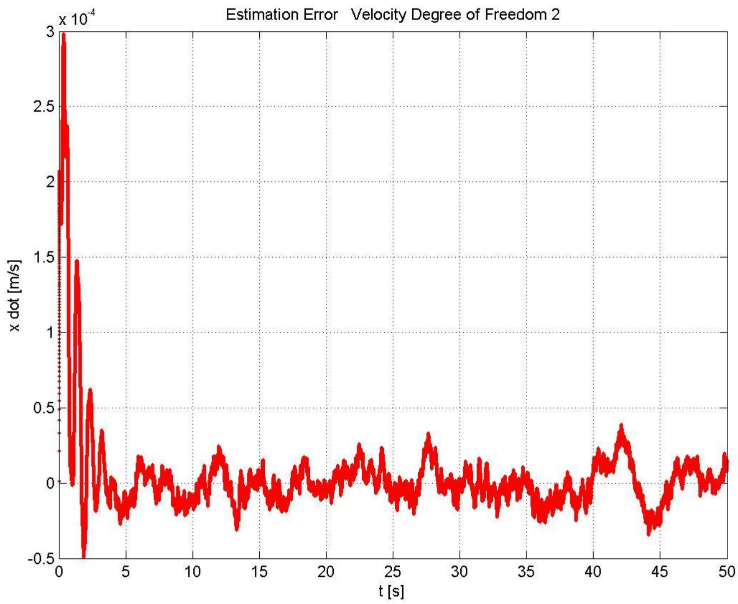


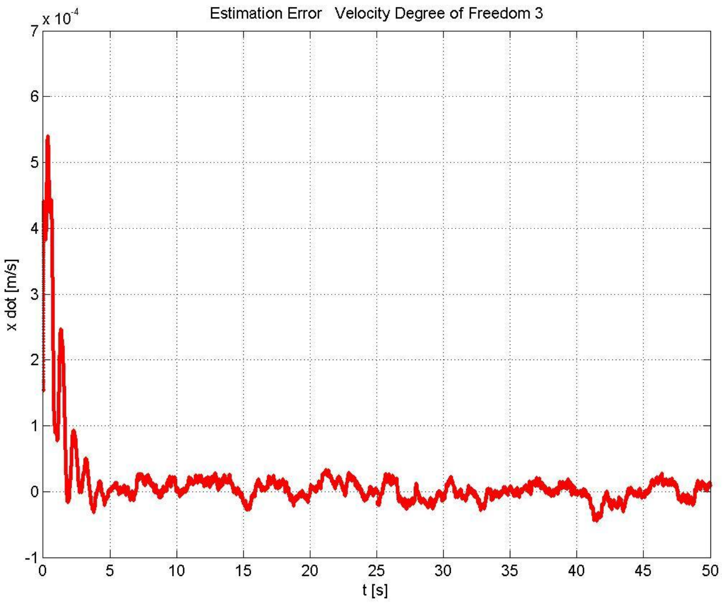
The time evolution of the estimation error relative to the pendulum angular displacement is the following:



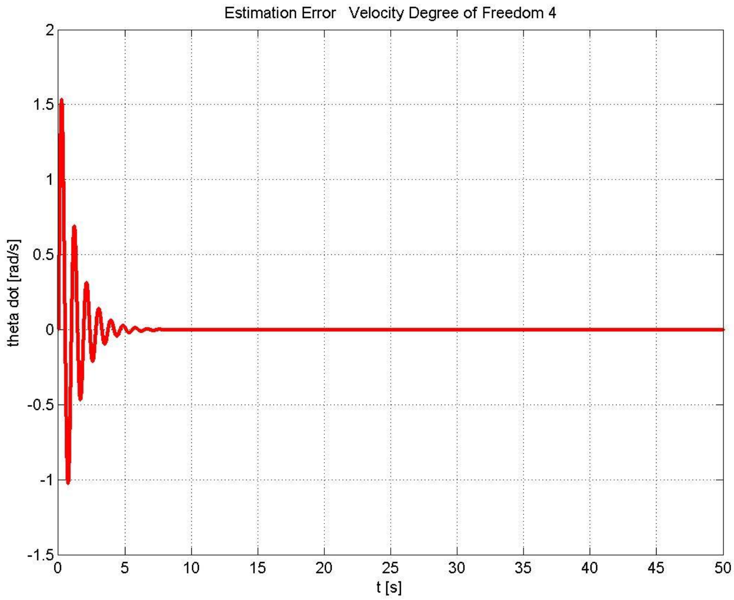
The time evolution of the estimation error relative to the system velocity corresponding to each floor is the following:


Estimation Error Velocity Degree of Freedom 1
x 10^-4
x dot [m/s]
t [s]
5
4
3
2
1
0
-1
0 5 10 15 20 25 30 35 40 45 50


Estimation Error Velocity Degree of Freedom 2
x 10^-4
x dot [m/s]
t [s]
3
2.5
2
1.5
1
0.5
0
-0.5
0 5 10 15 20 25 30 35 40 45 50



The time evolution of the estimation error relative to the pendulum angular velocity is the following:



Note that the estimation rapidly converges in a relatively small range for both system generalized displacement and velocity. Indeed, these plots show that when the controller start working the third floor stops vibrating and its position is hold approximately constant in time. In this case, the small deviation from the reference configuration of the displacement relative to the third floor is due to the presence of process and measurement noise. In addition, the displacement and the velocity relative to the other degrees of freedom are drastically reduced by the indirect action of the controller. It is worth to emphasize that even in this case the control action is confined in an acceptable working range. Finally, the percentage decrease of the maximum amplitude of the system response at the steady state due to the action of the controller is the following:

$$\begin{cases} \dfrac{x_{1,\max}^{nc} - x_{1,\max}^{c}}{x_{1,\max}^{nc}} = 75.90\,[\%] \\ \dfrac{x_{2,\max}^{nc} - x_{2,\max}^{c}}{x_{2,\max}^{nc}} = 78.16\,[\%] \\ \dfrac{x_{3,\max}^{nc} - x_{3,\max}^{c}}{x_{3,\max}^{nc}} = 93.45\,[\%] \end{cases} , \quad \begin{cases} \dfrac{\dot{x}_{1,\max}^{nc} - \dot{x}_{1,\max}^{c}}{\dot{x}_{1,\max}^{nc}} = 71.95\,[\%] \\ \dfrac{\dot{x}_{2,\max}^{nc} - \dot{x}_{2,\max}^{c}}{\dot{x}_{2,\max}^{nc}} = 66.04\,[\%] \\ \dfrac{\dot{x}_{3,\max}^{nc} - \dot{x}_{3,\max}^{c}}{\dot{x}_{3,\max}^{nc}} = 99.94\,[\%] \end{cases} \tag{5.287}$$

It is clear that the controller drastically reduces the amplitude of displacement and velocity relative to each system floor even in the worst-case scenario. Indeed, the extended Udwadia-Kalaba control method (EUK) combined with the extended Kalman filter estimation method (EKF), compared to the Linear Quadratic Gaussian control and estimation method (LQG), presents remarkable performances. On the other hand, the main drawback of this algorithm is that the numerical integration must be performed using a smaller sampling time to get accurate results. Consequently, the performances improvement require a greater computation time.

6. CONCLUSIONS

This thesis represents an effort to demonstrate that Multibody Dynamics, System Identification and Control Theory are actually strongly linked matters. Consequently, the study of one of these subjects cannot be separated from the study of the other two. The structure of this works is an attempt to encompass the essence of Multibody Dynamics, System Identification and Control Theory. In the first chapter a synthesis of the most important principles and techniques to derive the equations of motion of multibody systems is presented. In the second chapter a synthesis of the most important methodologies to obtain modal parameters of a dynamical system using force and vibration measurements is presented. In the third chapter a synthesis of the most important algorithm to design a feedback control system based on an observer is presented. The case study examined is a three-story building model with a pendulum hinged on the third floor [1], [2]. In particular, a lumped parameter model and a finite element model of the three-story frame have been developed. Subsequently, a data-driven model relative to the system under test has been developed. Indeed, the Eigensystem Realization Algorithm with Data Correlation using Observer/Kalman Filter Identification method (ERA/DC OKID) [3] and the Numerical Algorithm for Subspace Identification (N4SID) [4] have been used to determine two different state-space models of the structural system using experimental input and output measurements. Moreover, the algorithm to determine a physical model from the identified sate-space representation (MKR) [5], [6], [7] has been used to obtain two different second-order mechanical models of the three-story frame. Subsequently, the design of a Linear Quadratic

Gaussian regulator (LQG) [8] has been performed using the previously identified physical model of the system under test. The effectiveness of this controller has been tested in the worst-case scenario in which the system is excited by an external force whose harmonic content is close to the first three system natural frequencies. From the simulation results it is clear that the Linear Quadratic Gaussian controller (LQG) [8] drastically reduces the amplitude of displacement and velocity relative to each system floor even in this worst-case scenario. Finally, a new control algorithm for nonlinear underactuated mechanical systems affected by uncertainties (EUK-EKF) is proposed. In analogy with the Linear Quadratic Gaussian regulation method (LQG) [8], this algorithm represents the extension of the Udwadia-Kalaba control method (UK) [9], [10] to underactuated mechanical systems disturbed by noise. This extension is performed combining the extended Udwadia-Kalaba control method (EUK), which is the extension of the Udwadia-Kalaba control method (UK) [9], [10] to underactuated mechanical systems, with the well-known extended Kalman filter estimation method (EKF) [8]. Even in this case, the effectiveness of the combined algorithms (EUK-EKF) has been tested in the worst-case scenario in which the system is excited by an external force whose harmonic content is close to the first three system natural frequencies. From the simulation results it is clear that the controller drastically reduces the amplitude of displacement and velocity relative to each system floor even in the worst-case scenario. Indeed, the extended Udwadia-Kalaba control method (EUK) combined with the extended Kalman filter estimation method (EKF), compared to the Linear Quadratic Gaussian control and estimation method (LQG) [8], presents remarkable performances. On the other hand, the main drawback of this algorithm is that the numerical integration must be performed using a smaller sampling time to get accurate results. Consequently, the performances improvement require a greater computation time.

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