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Noncoercive Nonlinear Elliptic Equations in Unbounded Domains

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A mio fratello Michele, a mia mamma, a mio padre, a Egidio.

Voi la mia roccia, il mio cuore, la mia casa,

l'inscindibile filo d'oro della mia vita.

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Introduction

This thesis is mainly devoted to the study of existence results for noncoercive nonlinear Dirichlet problems in unbounded domains.

Let Ω be an open subset of $\mathbb{R}^N$, $N > 2$. Consider the classical linear Dirichlet problem

$$\begin{cases} -\mathrm{div}(M(x)\nabla u) + \mu u = -\mathrm{div}(u\,E(x)) + f(x) \quad \text{in } \Omega, \\ \\ u \in W_0^{1,2}(\Omega), \end{cases} \tag{1}$$

where $M : \Omega \to \mathbb{R}^{N^2}$ is a measurable matrix field such that there exist α, $\beta \in \mathbb{R}_+$ such that

$$\alpha|\xi|^2 \le M(x)\,\xi\cdot\xi, \quad |M(x)| \le \beta, \quad \text{a.e. } x \in \Omega, \quad \forall\, \xi \in \mathbb{R}^N, \tag{2}$$

$$\mu > 0, \tag{3}$$

$$E : \Omega \to \mathbb{R}^N \text{ is a vector field} \tag{4}$$

and

$$f : \Omega \to \mathbb{R} \text{ is a real function.} \tag{5}$$

In the Sixties, Guido Stampacchia, in [27, 28], studies problem (1), assuming that the set Ω is bounded.

He proves existence, uniqueness and regularity results for (1) considering that the problem is coercive due to some particular assumptions on μ and $|E|$, with $|E|$ belonging to an opportune Sobolev space.

Namely, in [27], he shows, among other important results, that, if $\mu > 0$ is large enough, then problem (1) is coercive and that, if $\|E\|_{L^N(\Omega)}$ is small enough, problem (1) still remains coercive, even if μ is small or null. Under hypotheses (2), (4), (5), with $f \in L^{\frac{2N}{N+2}}(\Omega)$, and if the problem is coercive, he proves that (1) has a unique weak solution u. In particular, in order to obtain these results, he uses the Lax-Milgram Lemma taking μ large enough or measure of Ω small enough.

Successively, in [3], Lucio Boccardo obtaines the same results, also considering the case $\mu = 0$, assuming Ω bounded, $|E| \in L^N(\Omega)$ and $f \in L^m(\Omega)$, $1 \leq m < \dfrac{N}{2}$.

We point out that the main difficulty here is due to the noncoercivity of the operator $-\mathrm{div}(M(x)\nabla u) + \mathrm{div}(u\, E(x))$. Indeed, on $\|E\|_{L^N(\Omega)}$ no smallness assumptions are done, while, as already observed, in order to obtain the coercivity in the case $\mu = 0$, one has to require that $\|E\|_{L^N(\Omega)}$ is small enough. The lack of the coercivity of the operator does not allow to use classical theorems to achieve the existence and uniqueness results.

Thus, in order to prove that there exists a unique solution u, Lucio Boccardo, inspired from the papers [27, 28] by Guido Stampacchia and from [9, 10, 12], follows a nonlinear approach. Namely, he approximates noncoercive linear problem (1) by nonlinear coercive problems. By means of Schauder fixed point Theorem, he shows that, for every fixed n, there exists a weak solution u_n of the approximate problem. Later on, using the classical truncate function introduced by Stampacchia, he obtaines the boundedness of u_n in $W_0^{1,2}(\Omega)$. Exploiting this result and passing to the limit in the variational formulation of the problem, he firstly proves the existence and, later on, the uniqueness result.

Few years later, in [4], Lucio Boccardo considers, always in the case of bounded domains, a nonlinear version of the noncoercive boundary value problem (1) with $\mu = 0$ studied in [3]. He proves existence and uniqueness results assuming $1 < p < N$, $|E| \in L^{\frac{N}{p-1}}(\Omega)$ and $f \in L^m(\Omega)$ with $m \geq (p^*)'$, where by $(\ )^*$ we denote the Sobolev conjugate of $(\ )$ and by $(\ )'$ the Hölder conjugate of $(\ )$.

In order to do this, the key point is to approximate his problem by coercive nonlinear problems, following the same approach of linear case. In particular, he obtaines the existence result for the approximate problems by means of Schauder fixed point Theorem and, then, he proves the boundedness in $W_0^{1,p}(\Omega)$ of these solutions using Stampacchia's truncate functions. Successively, he passes to the limit in the variational formulation proving the existence of a weak solution of the original problem. Finally, he is able to obtain also the uniqueness result for u, but only for $1 < p \leq 2$.

The study of problem (1) is extended to the case of Ω unbounded for the first time in [14] by Gianfranco Bottaro and Maria Erminia Marina. They assume that the problem is coercive and give existence and uniqueness results under opportune hypotheses on the coefficients. Successively, in [29], Maria Transirico and Mario Troisi generalize the results of [14] proving, among other important results, that problem (1) admits a unique weak solution u. This is done always assuming the coercivity of the bilinear form associated to the matrix M, but under assumptions on μ, $|E|$ and f weaker than those of [14].

The main difficulties one has to deal with when working on unbounded sets are the following well known ones:

- there are no natural decreasing inclusions among the $L^p(\Omega)$ spaces;
- there are no compactness results;
- the norm in $W_0^{1,p}(\Omega)$ is not equivalent to the norm of gradient since Poincaré inequality does not hold.

This has lead to consider the $M^p(\Omega)$ spaces, with $p \in [1, +\infty[$, introduced for the firts time in [29] and recently recalled in [1].

We remind that, for $p \in [1, +\infty[$, $M^p(\Omega)$ denotes the space of all the functions f in $L^p_{loc}(\overline{\Omega})$ such that

$$\|f\|_{M^p(\Omega)} = \sup_{x\in\Omega} \|f\|_{L^p(\Omega\cap B(x,1))} < +\infty,$$

endowed with the previous norm.

The importance of these Sobolev spaces derives from their special properties. Indeed, for these spaces the natural decreasing inclusions are valid also in the case of unbounded domains. Moreover, a compactness result holds (see Theorem 1.10, Chapter 1).

The noncoercive linear problem, analysed by Lucio Boccardo in [3], is generalized by Sara Monsurrò and Maria Transirico in [24], to the case when Ω is unbounded, with different hypotheses due to the unboundedness of the domain. The authors suppose $\mu > 0$, $|E| \in L^2(\Omega) \cap M_0^N(\Omega)$ and $f \in L^1(\Omega) \cap L^{\frac{2N}{N+2}}(\Omega)$.

Under these assumptions, in [24], the authors obtain existence and uniqueness results by approximating the noncoercive linear problem via coercive nonlinear problems and, then, passing to the limit in the variational formulation of approximate problems. Namely, by means of Schauder fixed point Theorem, they prove that there exists a solution u_n of the approximate problem, for every fixed n. Later on, using the classical truncate function introduced by Stampacchia, the authors prove that the u_n are bounded in $W_0^{1,2}(\Omega)$, for every fixed n. Thus, they can pass to the limit using the compactness result stated in Theorem 1.10, that applies in view of the assumption on the coefficient appearing in the noncoercive term. This leads to the proof of the existence of a weak solution of their problem. Successively, they also obtain the uniqueness of the solution.

Inspired by the work [4], in [2], we consider a nonlinear version of the noncoercive boundary value problem, in the case of unbounded domains. Aim of the paper [2] is to extend the results of [24] to the nonlinear case, when the set Ω is unbounded. We consider the problem

$$\begin{cases} -\mathrm{div}(b(x)|\nabla u|^{p-2}\nabla u) + \mu|u|^{p-2}u = \\ \\ -\mathrm{div}(|u|^{p-2}u\, E(x)) + f(x) \quad \text{in } \Omega, \\ \\ u \in W_0^{1,p}(\Omega), \end{cases} \tag{6}$$

assuming

$$1 < p < N,$$

$$\alpha \leq b(x) \leq \beta, \quad \text{for some } 0 < \alpha \leq \beta, \text{ a.e. } x \in \Omega,$$

$$\mu > 0,$$

$$|E| \in L^{p'}(\Omega) \cap M_0^{\frac{N}{p-1}}(\Omega)$$

and

$$f \in L^1(\Omega) \cap L^m(\Omega),\ m \geq (p^*)'.$$

We emphasize the presence of the noncoercive operator $-\mathrm{div}(b(x)|\nabla u|^{p-2}\nabla u)+\mathrm{div}(|u|^{p-2}u\, E(x))$, where on the second term no smallness assumptions are done. Due to the unboundedness of the domain, the hypothesis $\mu > 0$ is necessary (see Section 3.1). Despite this, since μ is not required to be large enough, the operator in (6) still remains noncoercive.
The technique used to achieve the existence result follows by the ideas of Lucio Boccardo used in [3] and in [4]. Hence, in order to prove the existence of a solution of (6), our noncoercive nonlinear problem is approximated by

coercive nonlinear problems, depending on n, and, then, we pass to the limit. Differently from [4] and [24] where the existence of the solutions u_n of approximate problems is an immediate consequence of the Schauder fixed point Theorem, here it must be explicitly proved by means of the Surjectivity Theorem. Successively, we prove the boundedness in $W_0^{1,p}(\Omega)$ of these solutions, using Stampacchia's truncate functions. Later on, thanks to the hypothesis on the coefficient of the noncoercive term and in view of the compactness result in $M_0^N(\Omega)$, it is possible to pass to the limit obtaining the existence of the solution of the initial problem (6). The proof of the uniqueness of the solution is quite delicate and will be object of a forthcoming study. Also in the case of bounded domains, in [4], only a partial result ($1 < p \leq 2$) has been achieved.

This thesis is organized as follows.

In Chapter 1 we give an overview about the $M^p(\Omega)$ spaces. In particular, we recall the definitions and the properties of some important subspaces of $M^p(\Omega)$. One of the most relevant results of this Chapter is contained in Theorem 1.10 that deals with some compactness results holding when the coefficients of the operators belong to this class of suitable Sobolev spaces.

Chapter 2 is devoted to existence, uniqueness and regularity results for coercive and noncoercive elliptic Dirichlet problems on bounded domains. Firstly, we recall the papers [27, 28] by Guido Stampacchia about coercive problems in the linear case and, later on, the main techniques introduced by Lucio Boccardo, both in the linear case and in nonlinear one (see [3, 4]).

Chapter 3 opens with the analysis of the main difficulties one has to deal with when working on unbounded sets. Then, preliminary results, that are useful in the linear and nonlinear case, are recalled. Successively, the existence, uniqueness and regularity results for noncoercive elliptic Dirichlet problems on unbounded domains, treated in the paper [24] by Sara Monsurrò and Maria Transirico, are examined. In this thesis, we give the complete proof of a regularity result, contained in Lemma 3.9, that was only outlined by authors in [24].

Finally, in Chapter 4, we study the noncoercive nonlinear elliptic problem in unbounded domains. We consider some approximate problems, that depend on n, and give the existence of these coercive nonlinear problems by means of the Surjectivity Theorem. Later on, thanks to some boundedness results of the solutions of the approximate problems, it is possible to pass to the limit obtaining the existence of the solution of the initial problem. The results of this section are contained in the paper [2] by Emilia Anna Alfano and Sara Monsurrò.

Chapter 1

A class of suitable Sobolev spaces

In this chapter we recall the definition and some properties of a class of functional spaces, suitable for our aim, introduced for the first time in [29].

1.1 Some notations

Let F be a subset of $\mathbb{R}^N$, $N > 2$. We define $F(x,t) = F \cap B(x,t)$, for every $x \in F$ and every $t \in \mathbb{R}_+$, where $B(x,t)$ is the open ball with center x and radius t, and $F(x) = F \cap B(x,1)$. The σ-algebra of all Lebesgue measurable subset of F is denoted by $\Sigma(F)$.

Given $A \in \Sigma(F)$, $|A|$ denotes the Lebesgue measure of A and χ_A denotes its characteristic function. We set by $\mathcal{D}(F)$ the class of restrictions to F of functions $\zeta \in C_0^\infty(\mathbb{R}^N)$ with $\overline{F} \cap \operatorname{supp} \zeta \subseteq F$ and, for $p \in [1, +\infty[$, we denote by $L^p_{loc}(F)$ the class of all functions $g : F \to \mathbb{R}$ such that $\zeta g \in L^p(F)$ for any $\zeta \in \mathcal{D}(F)$.

1.2 The space $M^p(\Omega)$

From now on, Ω is assumed to be an unbounded open subset of $\mathbb{R}^N$, $N > 2$. For $p \in [1, +\infty[$ and for fixed $t \in \mathbb{R}_+$, $M^p(\Omega, t)$ denotes the space of all the functions f in $L^p_{loc}(\overline{\Omega})$ such that

$$\|f\|_{M^p(\Omega,t)} = \sup_{x\in\Omega} \|f\|_{L^p(\Omega(x,t))} < +\infty, \tag{1.1}$$

endowed with the norm defined in (1.1).

The properties of $M^p(\Omega, t)$ spaces and of some of their subspaces, introduced for the first time in [29], are studied in different works (see, for istance, [1, 15, 30, 31]). Here we recall some results on these spaces, useful in the sequel.

In the next proposition we prove that, for every $t \in \mathbb{R}_+$, the $M^p(\Omega, t)$ spaces are isomorphic.

Proposition 1.1. *For every t_1 and $t_2 \in \mathbb{R}_+$:*

i) $f \in M^p(\Omega, t_1)$ *is equivalent to* $f \in M^p(\Omega, t_2)$*;*

ii) *if* $t_2 > t_1$*, we have*

$$\|f\|_{M^p(\Omega,t_1)} \le \|f\|_{M^p(\Omega,t_2)} \le 8^N \left(\frac{t_2}{t_1}\right)^N \|f\|_{M^p(\Omega,t_1)}, \quad \forall f \in M^p(\Omega, t_1).$$

Proof. It is sufficient to prove *ii)*.

Let $t_2 > t_1$. Then

$$\|f\|_{M^p(\Omega,t_1)} = \sup_{x\in\Omega} \|f\|_{L^p(\Omega(x,t_1))} \le \sup_{x\in\Omega} \|f\|_{L^p(\Omega(x,t_2))} = \|f\|_{M^p(\Omega,t_2)}.$$

In order to prove the converse inequality, we observe that, for every $x \in \Omega$,

$$\Omega \cap B(x,t_2) \subset \Omega \cap Q(x,2t_2)$$

where $Q(x,2t_2)$ is the cube with center x, sides parallel to the axes and edges with length $2t_2$.

Since $t_2 > t_1$, there exists $k \in \mathbb{N}$ such that

$$2^{k-1} < \frac{t_2}{t_1} \le 2^k$$

that implies

$$\frac{2t_2}{2^{k+2}} \le \frac{t_1}{2}.$$

Therefore, every cube $\Omega \cap Q(x,2t_2)$ can be diadically decomposed in $2^{N(k+2)}$ cubes with sides $\dfrac{2t_2}{2^{k+2}}$ and center $x_i \in \Omega \cap Q(x,2t_2)$. Furthermore, each cube is contained in the ball with center x_i and radius $\dfrac{t_1}{2}$. Thus

$$Q(x,2t_2) = \bigcup_{i=1}^{2^{N(k+2)}} Q\left(x_i, \frac{2t_2}{2^{k+2}}\right) \subset \bigcup_{i=1}^{2^{N(k+2)}} B\left(x_i, \frac{t_1}{2}\right).$$

Hence, for every fixed $x \in \Omega$, one has

$$\|f\|_{L^p(\Omega(x,t_2))} \le \sum_{i=1}^{2^{N(k+2)}} \|f\|_{L^p(\Omega(x_i,\frac{t_1}{2}))} \le \sum_{i=1}^{2^{N(k+2)}} \|f\|_{L^p(\Omega(x_i,t_1))}$$

$$\le \sum_{i=1}^{2^{N(k+2)}} \|f\|_{M^p(\Omega,t_1)} = 2^{N(k+2)}\|f\|_{M^p(\Omega,t_1)}.$$

In view of the choice of k and thanks to the arbitrariness of $x \in \Omega$, we obtain

$$\|f\|_{M^p(\Omega,t_2)} \le 8^N \left(\frac{t_2}{t_1}\right)^N \|f\|_{M^p(\Omega,t_1)}.$$

□

From now on, we consider the space

$$M^p(\Omega) = M^p(\Omega, 1). \tag{1.2}$$

The following result shows that, also in the case of unbounded domains, the natural inclusions are still valid for $M^p(\Omega)$ spaces, differently from $L^p(\Omega)$ ones.

Proposition 1.2. *For every $p, q \in [1, +\infty[$,*

(i) $L^p(\Omega) \subset M^p(\Omega)$ *and* $L^\infty(\Omega) \subset M^p(\Omega)$*;*

(ii) $M^q(\Omega) \subseteq M^p(\Omega)$ *if* $p \le q$*.*

Proof. We start proving *(i)*.

If $f \in L^p(\Omega)$, then

$$\|f\|_{M^p(\Omega)} = \sup_{x\in\Omega} \|f\|_{L^p(\Omega(x))} \leq \|f\|_{L^p(\Omega)}.$$

If $f \in L^\infty(\Omega)$, since $\Omega(x) \subset B(x,1)$, then

$$\|f\|_{M^p(\Omega)} = \sup_{x\in\Omega} \|f\|_{L^p(\Omega(x))} \leq \|f\|_{L^\infty(\Omega)} \sup_{x\in\Omega} |\Omega(x)|^{\frac{1}{p}} \leq C \ \|f\|_{L^\infty(\Omega)},$$

where $C = C(N,p)$. This gives *(i)*.

Now, let us prove *(ii)*.

Let $p \leq q$. For every fixed $x \in \Omega$, since $\Omega(x) \subset B(x,1)$, by Hölder inequality one gets

$$\|f\|_{L^p(\Omega(x))} \leq \|f\|_{L^q(\Omega(x))} \cdot |\Omega(x)|^{\frac{1}{p}-\frac{1}{q}} \leq C \ \|f\|_{L^q(\Omega(x))} \leq C \ \|f\|_{M^q(\Omega)},$$

where $C = C(N,p,q)$. Thanks to the arbitrariness of $x \in \Omega$, we obtain

$$\|f\|_{M^p(\Omega)} = \sup_{x\in\Omega} \|f\|_{L^p(\Omega(x))} \leq C\|f\|_{M^q(\Omega)}.$$

This gives *(ii)*. $\square$

We observe that the previous inclusions are both algebraic and topological. Moreover, in the case of Ω bounded, we point out that Propositions 1.1 and 1.2 show that $f \in M^p(\Omega)$ is equivalent to $f \in L^p(\Omega)$. This justifies the choice to consider Ω unbounded.

1.3 The space $\widetilde{M}^p(\Omega)$

For $p \in [1, +\infty[$, $\widetilde{M}^p(\Omega)$ is the subspace of $M^p(\Omega)$ made up of the functions $f \in M^p(\Omega)$ such that

$$\lim_{h\to+\infty} \sup_{\substack{E\in\Sigma(\Omega)\\ \sup\limits_{x\in\Omega}|E(x)|\le 1/h}} \|f\chi_E\|_{M^p(\Omega)} = 0. \tag{1.3}$$

This is equivalent to require that

$$\forall\epsilon \in \mathbb{R}_+ \ \exists h_\epsilon \in \mathbb{R}_+ \ : \forall E \in \Sigma(\Omega), \ \sup_{x\in\Omega} |E(x)| \le \frac{1}{h_\epsilon} \Rightarrow \|f\chi_E\|_{M^p(\Omega)} < \epsilon. \tag{1.4}$$

The next two propositions show that $\widetilde{M}^p(\Omega)$ is a closed subspace of $M^p(\Omega)$.

Proposition 1.3. *For every $p \in [1, +\infty[$,*

$$L^\infty(\Omega) \subset \widetilde{M}^p(\Omega).$$

Proof. Proposition 1.2 gives $L^\infty(\Omega) \subset M^p(\Omega)$.
Let $E \in \Sigma(\Omega)$ and $f \in L^\infty(\Omega)$. One has

$$\|f\chi_E\|_{M^p(\Omega)} = \sup_{x\in\Omega} \|f\chi_E\|_{L^p(\Omega(x))} \le \|f\|_{L^\infty(\Omega)} \cdot \sup_{x\in\Omega} |E(x)|^{\frac{1}{p}}. \tag{1.5}$$

Fixed $\epsilon > 0$, let $h_\epsilon > 0$ be such that

$$h_\epsilon = \left(\frac{\|f\|_{L^\infty(\Omega)}}{\epsilon}\right)^p.$$

If we show that $\sup_{x\in\Omega}|E(x)| \leq \dfrac{1}{h_\epsilon}$, we obtain

$$\|f\chi_E\|_{M^p(\Omega)} \leq \|f\|_{L^\infty(\Omega)} \cdot \frac{\epsilon}{\|f\|_{L^\infty(\Omega)}} = \epsilon$$

and hence $f \in \widetilde{M}^p(\Omega)$. $\square$

Proposition 1.4. *For every $p \in [1, +\infty[$,*

$$\widetilde{M}^p(\Omega) \textit{ is the closure of } L^\infty(\Omega) \textit{ in } M^p(\Omega).$$

Proof. Let $f \in \widetilde{M}^p(\Omega)$. For every $k \in \mathbb{N}$, we define

$$F_k = \{x \in \Omega : |f(x)| \geq k\}. \tag{1.6}$$

Now, putting $F_k(x) = F_k \cap B(x,1)$ and since

$$\|f\|_{M^p(\Omega)} = \sup_{x\in\Omega}\|f\|_{L^p(\Omega(x))} \geq \sup_{x\in\Omega}\|f\|_{L^p(F_k(x))} \geq k \sup_{x\in\Omega}|F_k(x)|^{\frac{1}{p}},$$

one has

$$\sup_{x\in\Omega}|F_k(x)| \leq \left(\frac{\|f\|_{M^p(\Omega)}}{k}\right)^p. \tag{1.7}$$

Fixed $\epsilon > 0$, let h_ϵ such that (1.4) is verified, we put

$$k_\epsilon = \|f\|_{M^p(\Omega)} \cdot h_\epsilon^{\frac{1}{p}}.$$

By (1.7), one has

$$\sup_{x\in\Omega} |F_{k_\epsilon}(x)| \leq \frac{1}{h_\epsilon},$$

that implies

$$\|f\chi_{F_{k_\epsilon}}\|_{M^p(\Omega)} < \epsilon. \tag{1.8}$$

We put $f_\epsilon = f - f\chi_{F_{k_\epsilon}}$. Observing that

$$f_\epsilon = \begin{cases} 0 & \text{if } x \in \chi_{F_{k_\epsilon}} \\ f & \text{if } x \in \Omega \setminus \chi_{F_{k_\epsilon}}, \end{cases}$$

by (1.6), we obtain $f_\epsilon \in L^\infty(\Omega)$. Furthermore, by (1.8),

$$\|f - f_\epsilon\|_{M^p(\Omega)} = \|f\chi_{F_{k_\epsilon}}\|_{M^p(\Omega)} < \epsilon,$$

then f belongs to the closure of $L^\infty(\Omega)$ in $M^p(\Omega)$.

Conversely, let $f \in M^p(\Omega)$ the limit of a sequence $(f_h)_{h\in\mathbb{N}}$ of functions of $L^\infty(\Omega)$. Thus, thanks to Proposition 1.3, for every $\epsilon > 0$ there exists $h_\epsilon > 0$ such that, if $E \in \Sigma(\Omega)$ with $\sup_{x\in\Omega} |E(x)| \leq \frac{1}{h_\epsilon}$, we have

$$\|f - f_{h_\epsilon}\|_{M^p(\Omega)} < \frac{\epsilon}{2}$$

and

$$\|f_{h_\epsilon}\chi_E\|_{M^p(\Omega)} < \frac{\epsilon}{2}.$$

Hence,

$$\|f\chi_E\|_{M^p(\Omega)} = \|(f - f_{h_\epsilon})\chi_E\|_{M^p(\Omega)} + \|f_{h_\epsilon}\chi_E\|_{M^p(\Omega)}$$

$$\leq \|f - f_{h_\epsilon}\|_{M^p(\Omega)} + \|f_{h_\epsilon}\chi_E\|_{M^p(\Omega)} < \epsilon,$$

that gives $f \in \widetilde{M}^p(\Omega)$. □

Remark 1. *From Proposition 1.2 it easily follows that, for every* $p, q \in [1, +\infty[$ *with* $p \leq q$,

$$\widetilde{M}^q(\Omega) \subseteq \widetilde{M}^p(\Omega). \tag{1.9}$$

The following result improves (1.9) and *(ii)* of Proposition 1.2.

Proposition 1.5. *For every* $p, q \in [1, +\infty[$ *with* $p < q$,

$$M^q(\Omega) \subset \widetilde{M}^p(\Omega).$$

Proof. Proposition 1.2 gives $M^q(\Omega) \subset M^p(\Omega)$. Now, let $E \in \Sigma(\Omega)$ and $f \in M^q(\Omega)$, by Hölder inequality, one has

$$\|f\chi_E\|_{M^p(\Omega)} = \sup_{x\in\Omega} \|f\chi_E\|_{L^p(\Omega(x))} \leq \sup_{x\in\Omega} \|f\|_{L^q(\Omega(x))} \cdot |\Omega(x) \cap E|^{\frac{1}{p}-\frac{1}{q}}$$

$$\leq \|f\|_{M^q(\Omega)} \cdot \sup_{x\in\Omega} |E(x)|^{\frac{1}{p}-\frac{1}{q}}.$$

For $\epsilon > 0$, let $h_\epsilon > 0$ such that $h_\epsilon = \left(\dfrac{\|f\|_{M^q(\Omega)}}{\epsilon}\right)^{\frac{pq}{q-p}}$.

If $\sup_{x\in\Omega} |E(x)| \leq \dfrac{1}{h_\epsilon}$, we obtain

$$\|f\chi_E\|_{M^p(\Omega)} < \|f\|_{M^q(\Omega)} \cdot \left(\frac{1}{h_\epsilon}\right)^{\frac{q-p}{pq}} = \epsilon$$

that implies $f \in \widetilde{M}^p(\Omega)$. □

1.4 The space $M_0^p(\Omega)$

For $p \in [1, +\infty[$, $M_0^p(\Omega)$ is the subspace of $M^p(\Omega)$ made up of the functions $f \in M^p(\Omega)$ such that

$$\lim_{|x|\to+\infty} \|f\|_{L^p(\Omega(x))} = 0. \tag{1.10}$$

This is equivalent to require that

$$\forall \epsilon \in \mathbb{R}_+ \ \exists k_\epsilon \in \mathbb{R}_+ \text{ s.t. } \forall x \in \Omega, \ |x| > k_\epsilon \Rightarrow \|f\|_{L^p(\Omega(x))} < \epsilon. \tag{1.11}$$

Now, we recall the following important results.

Proposition 1.6. *For every* $p \in [1, +\infty[$,

(i) $f \in L^p_{loc}(\overline{\Omega})$, $\lim_{|x|\to+\infty} f(x) = 0 \quad \Rightarrow \quad f \in M_0^p(\Omega)$;

(ii) $f \in L^\infty(\Omega)$, $\lim_{|x|\to+\infty} \|f\|_{L^1(\Omega(x))} = 0 \quad \Rightarrow \quad f \in M_0^p(\Omega)$.

PROOF. We start proving *(i)*.

Let $f \in L^p_{\mathrm{loc}}(\overline{\Omega})$ and $\lim_{|x| \to +\infty} f(x) = 0$. Then

$$\forall \epsilon \in \mathbb{R}_+ \ \exists k_\epsilon \in \mathbb{R}_+ \text{ s.t. } \forall x \in \Omega, \ |x| > k_\epsilon \Rightarrow |f(x)| < \epsilon.$$

Hence, since $\Omega(x) \subset B(x, 1)$, for $x \in \Omega$ such that $|x| > k_\epsilon$,

$$\|f\|_{L^p(\Omega(x))} \leq \epsilon \, |\Omega(x)|^{\frac{1}{p}} < C \cdot \epsilon,$$

where $C = C(N, p)$.

This implies that $f \in M^p_0(\Omega)$ and, thus, *(i)* holds.

Now, we want to prove *(ii)*.

Let $f \in L^\infty(\Omega)$ and $\lim_{|x| \to +\infty} \|f\|_{L^1(\Omega(x))} = 0$. Then

$$\forall \epsilon \in \mathbb{R}_+ \ \exists k_\epsilon \in \mathbb{R}_+ \text{ s.t. } \forall x \in \Omega, \ |x| > k_\epsilon \Rightarrow \|f\|_{L^1(\Omega(x))} < \epsilon.$$

Hence, for $x \in \Omega$ such that $|x| > k_\epsilon$,

$$\|f\|_{L^p(\Omega(x))} = \left(\int_{\Omega(x)} |f|^{p-1} \cdot |f| dy \right)^{\frac{1}{p}} \leq \|f\|^{\frac{p-1}{p}}_{L^\infty(\Omega)} \cdot \|f\|^{\frac{1}{p}}_{L^1(\Omega)} < \|f\|^{\frac{p-1}{p}}_{L^\infty(\Omega)} \cdot \epsilon^{\frac{1}{p}}.$$

This implies that $f \in M^p_0(\Omega)$ and, thus, *(ii)* holds. $\square$

Now, we introduce a class of functions useful in the sequel.

For $h \in \mathbb{R}_+$, we denote by ζ_h a function of class $C^\infty_0(\mathbb{R}^N)$ such that

$$0 \leq \zeta_h \leq 1, \qquad \zeta_{h|_{\overline{B(0,h)}}} = 1, \qquad \operatorname{supp} \zeta_h \subset B(0, 2h). \tag{1.12}$$

In order to prove that also $M_0^p(\Omega)$ is a closed subspace of $M^p(\Omega)$, we focus on the relationship between $M_0^p(\Omega)$ and $\widetilde{M}^p(\Omega)$.

Proposition 1.7. *For every $p \in [1,+\infty[$, one has that $f \in M_0^p(\Omega)$ if and only if $f \in \widetilde{M}^p(\Omega)$ and*

$$\lim_{h\to+\infty} \|(1-\zeta_h)f\|_{M^p(\Omega)} = 0. \tag{1.13}$$

Proof. We start proving that, if $f \in M_0^p(\Omega)$, then $f \in \widetilde{M}^p(\Omega)$ and $\lim_{h\to+\infty} \|(1-\zeta_h)f\|_{M^p(\Omega)} = 0$.
Observe that, by the properties of ζ_h, one has, for $r \geq 2h$,

$$\|(1-\zeta_r)f\|_{M^p(\Omega)} \leq \|(1-\zeta_{2h})f\|_{M^p(\Omega)} \leq \|f\chi_{\Omega\setminus B(0,2h)}\|_{M^p(\Omega)}. \tag{1.14}$$

If $f \in M_0^p(\Omega)$, then

$$\forall\epsilon \in \mathbb{R}_+ \ \exists h_\epsilon \in \mathbb{R}_+ \text{ s.t. } \forall x \in \Omega,\ |x| > 2h_\epsilon \Rightarrow \|f\|_{L^p(\Omega(x))} < \epsilon. \tag{1.15}$$

Therefore, by (1.14) and (1.15), one has that for every $\epsilon > 0$ there exists $r_\epsilon \geq 2h_\epsilon$ such that

$$\|(1-\zeta_{r_\epsilon})f\|_{M^p(\Omega)} \leq \|f\chi_{\Omega\setminus B(0,2h_\epsilon)}\|_{M^p(\Omega)}$$

$$\leq \sup_{x\in\Omega} \|f\|_{L^p(\Omega(x)\setminus B(0,2h_\epsilon))} \leq \sup_{\substack{x\in\Omega \\ |x|>2h_\epsilon}} \|f\|_{L^p(\Omega(x))} < \epsilon,$$

that is

$$\lim_{h\to+\infty} \|(1-\zeta_h)f\|_{M^p(\Omega)} = 0. \tag{1.16}$$

Let $\epsilon > 0$ and $E \in \Sigma(\Omega)$. By (1.16), there exists $h_\epsilon > 0$ such that

$$\begin{aligned} \|f\chi_E\|_{M^p(\Omega)} &\le \|(1-\zeta_{h_\epsilon})\,\chi_E\, f\|_{M^p(\Omega)} + \|\zeta_{h_\epsilon}\,\chi_E\, f\|_{M^p(\Omega)} \\ &\le \|(1-\zeta_{h_\epsilon})\, f\|_{M^p(\Omega)} + \|\zeta_{h_\epsilon}\,\chi_E\, f\|_{M^p(\Omega)} \qquad (1.17) \\ &< \frac{\epsilon}{2} + \|\zeta_{h_\epsilon}\,\chi_E\, f\|_{M^p(\Omega)}. \end{aligned}$$

Observe that, by the properties of ζ_{h_ϵ}, one has

$$\|\zeta_{h_\epsilon}\,\chi_E\, f\|_{M^p(\Omega)} = \sup_{x\in\Omega} \|\zeta_{h_\epsilon}\,\chi_E\, f\|_{L^p(\Omega(x))} \le \sup_{x\in\Omega} \|f\|_{L^p(E(x)\cap B(0,2h_\epsilon))}. \tag{1.18}$$

On the other hand, there exist $m_\epsilon \in \mathbb{N}$ and $x_1, ..., x_{m_\epsilon} \in \Omega$ such that

$$E(x) \cap B(0, 2h_\epsilon) \;\subset\; \bigcup_{i=1}^{m_\epsilon} E \cap B(x_i, 1)$$

that implies

$$|E(x) \cap B(0, 2h_\epsilon)| \le \sum_{i=1}^{m_\epsilon} |E \cap B(x_i, 1)| \le m_\epsilon \sup_{x\in\Omega} |E(x)|. \tag{1.19}$$

Thanks to the absolute continuity in the spaces $L^p(\Omega)$, one has that there exists $\delta_\epsilon > 0$ such that $\|f\|_{L^p(A)} < \dfrac{\epsilon}{2}$, if $A \in \Sigma(\Omega)$ with $|A| < \delta_\epsilon$.

Therefore, if $\sup_{x\in\Omega}|E(x)| \leq \dfrac{\delta_\epsilon}{m_\epsilon}$, from (1.19), we have

$$\|f\|_{L^p(E(x)\,\cap\, B(0,2h_\epsilon))} < \frac{\epsilon}{2}.$$

Thus, by (1.18),

$$\|\zeta_{h_\epsilon}\,\chi_E\, f\|_{M^p(\Omega)} < \frac{\epsilon}{2}. \tag{1.20}$$

From (1.17) and (1.20), it follows that $f \in \widetilde{M}^p(\Omega)$.

Now we prove that, if $f \in \widetilde{M}^p(\Omega)$ and (1.13) holds, then $f \in M^p_0(\Omega)$.

From the hypotheses and from the properties of the functions ζ_h, we have

$$\forall \epsilon \in \mathbb{R}_+ \; \exists h_\epsilon > 1 \text{ s.t. } \|(1-\zeta_{h_\epsilon})f\|_{M^p(\Omega)} < \epsilon.$$

For $|x| > 3h_\epsilon$, one has that, if $y \in B(x,1)$, then $|y| > 2h_\epsilon$. Therefore, for $x \in \Omega$ such that $|x| > 3h_\epsilon$, one has

$$\|f\|_{L^p(\Omega(x))} = \|(1-\zeta_{h_\epsilon})f\|_{L^p(\Omega(x))} \leq \|(1-\zeta_{h_\epsilon})f\|_{M^p(\Omega)} < \epsilon$$

and, hence, $f \in M^p_0(\Omega)$. □

Remark 2. *The following algebraic and topological inclusions are valid:*

$$M^q_0(\Omega) \subseteq M^p_0(\Omega) \quad \textit{if} \;\; 1 \leq p \leq q \leq +\infty, \tag{1.21}$$

$$L^p(\Omega) \subset M^p_0(\Omega) \quad \textit{if} \;\; 1 \leq p < +\infty. \tag{1.22}$$

Remark 3. *For the readers' convenience, let us recall that, as proved in [30], if* $f \in M^p(\Omega)$ *the following properties are equivalent:*

i) $f \in M_0^p(\Omega)$,

ii) for any $\varepsilon \in \mathbb{R}_+$ *there exist* $\nu_\varepsilon, \sigma_\varepsilon \in \mathbb{R}_+$ *such that*

$$E \in \Sigma(\Omega),\ |E(0,\sigma_\varepsilon)| \le \nu_\varepsilon \Rightarrow ||f\chi_E||_{M^p(\Omega)} \le \varepsilon, \tag{1.23}$$

iii) for any $\varepsilon \in \mathbb{R}_+$ *there exist* $h_\varepsilon, k_\varepsilon \in \mathbb{R}_+$ *such that*

$$\begin{gathered} \|(1-\zeta_{h_\varepsilon})\, f\|_{M^p(\Omega)} \le \varepsilon,\ E \in \Sigma(\Omega),\ \sup_{x\in\Omega} |E(x)| \le \frac{1}{k_\varepsilon} \\ \Rightarrow \|f\,\chi_E\|_{M^p(\Omega)} \le \varepsilon. \end{gathered} \tag{1.24}$$

Now, we are able to prove the following closure result.

Proposition 1.8. *For every* $p \in [1, +\infty[$,

$$M_0^p(\Omega) \textit{ is the closure of } C_0^\infty(\Omega) \textit{ in } M^p(\Omega).$$

Proof. Let $f \in M_0^p(\Omega)$. Fixed $\epsilon > 0$, from Proposition 1.7 there exists $h_\epsilon > 0$ such that

$$\|(1-\zeta_{h_\epsilon})f\|_{M^p(\Omega)} < \frac{\epsilon}{2}. \tag{1.25}$$

Observe that, from the properties of the functions ζ_h and from Proposition 1.2, we have

$$\|\zeta_h\, f\|_{M^p(\Omega)} \le \|f\|_{M^p(\Omega)} \le \|f\|_{L^p(\Omega)} \quad \forall f \in L^p(\Omega) \text{ and } \forall h > 0. \tag{1.26}$$

Moreover, since $f \in L^p_{loc}(\overline{\Omega})$, we know that $\zeta_{2h_\epsilon} f \in L^p(\Omega)$. From the density of $C_0^\infty(\Omega)$ in $L^p(\Omega)$, there exists $\varphi_{k_\epsilon} \in C_0^\infty(\Omega)$ such that

$$\|\zeta_{2h_\epsilon} f - \varphi_{k_\epsilon}\|_{L^p(\Omega)} < \frac{\epsilon}{2}. \tag{1.27}$$

Since $\zeta_{2h}\, \zeta_h = \zeta_h$, by (1.25), (1.26) and (1.27), one obtains

$$\|f - \zeta_{h_\epsilon}\varphi_{k_\epsilon}\|_{M^p(\Omega)} \leq \|f - \zeta_{h_\epsilon} f\|_{M^p(\Omega)} + \|\zeta_{h_\epsilon} f - \zeta_{h_\epsilon}\varphi_{k_\epsilon}\|_{M^p(\Omega)}$$

$$= \|(1 - \zeta_{h_\epsilon}) f\|_{M^p(\Omega)} + \|\zeta_{2h_\epsilon}\zeta_{h_\epsilon} f - \zeta_{h_\epsilon}\varphi_{k_\epsilon}\|_{M^p(\Omega)}$$

$$< \frac{\epsilon}{2} + \|\zeta_{h_\epsilon}(\zeta_{2h_\epsilon} f - \varphi_{k_\epsilon})\|_{M^p(\Omega)} \leq \frac{\epsilon}{2} + \|\zeta_{2h_\epsilon} f - \varphi_{k_\epsilon}\|_{L^p(\Omega)} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Namely, since $\zeta_{h_\epsilon}\varphi_{k_\epsilon} \in C_0^\infty(\Omega)$, f is in the closure of $C_0^\infty(\Omega)$.

Now, we suppose that $f \in M^p(\Omega)$ and that there exists a sequence $\varphi_k \in C_0^\infty(\Omega)$ $(k \in \mathbb{N})$ such that

$$\lim_{k \to +\infty} \|f - \varphi_k\|_{M^p(\Omega)} = 0. \tag{1.28}$$

For every $k \in \mathbb{N}$, let $h_k > 0$ such that $\operatorname{supp} \varphi_k \subset B(0, h_k)$.

From the properties of ζ_{h_k}, we have

$$\|(1 - \zeta_{h_k}) f\|_{M^p(\Omega)} = \|(1 - \zeta_{h_k})(f - \varphi_k)\|_{M^p(\Omega)} \leq \|f - \varphi_k\|_{M^p(\Omega)},$$

from which

$$\lim_{k\to+\infty} \|(1-\zeta_{h_k})f\|_{M^p(\Omega)} \le \lim_{k\to+\infty} \|f-\varphi_k\|_{M^p(\Omega)} = 0. \tag{1.29}$$

Now, let $E \in \Sigma(\Omega)$. One has

$$\begin{aligned} \|f\chi_E\|_{M^p(\Omega)} &\le \|(f-\varphi_k)\chi_E\|_{M^p(\Omega)} + \|\varphi_k\chi_E\|_{M^p(\Omega)} \\ &\le \|f-\varphi_k\|_{M^p(\Omega)} + \|\varphi_k\chi_E\|_{M^p(\Omega)}. \end{aligned} \tag{1.30}$$

Observing that $\varphi_k \in C_0^\infty(\Omega)$ and that $C_0^\infty(\Omega) \subset L^\infty(\Omega) \subset \widetilde{M}^p(\Omega)$, by (1.28) and (1.30) it follows that $f \in \widetilde{M}^p(\Omega)$. Therefore, by (1.29) and Proposition 1.7, we obtain that $f \in M_0^p(\Omega)$. □

1.5 Further results about the $M^p(\Omega)$ spaces

In this section we recall some properties of the $M^p(\Omega)$ spaces, that will be useful in the study of our differential operators.

Firstly, we define the modulus of continuity of a function in $M_0^p(\Omega)$. Namely, if g belongs to $M_0^p(\Omega)$, a modulus of continuity of g in $M_0^p(\Omega)$ is an application $\sigma_o^p[g] : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|(1-\zeta_h)g\|_{M^p(\Omega)} + \sup_{\substack{E\in\Sigma(\Omega)\\ \sup\limits_{x\in\Omega}|E(x)|\leq \frac{1}{h}}} \|g\,\chi_E\|_{M^p(\Omega)} \leq \sigma_o^p[g](h), \tag{1.31}$$

$$\text{with} \lim_{h\to+\infty} \sigma_o^p[g](h) = 0\,.$$

This definition is well posed thanks to Proposition 1.7 and to the definition of $\widetilde{M}^p(\Omega)$.

Let us remind the following result proved in Lemma 3.1 of [30], see also [15], adapted here to our needs, that allow us to approximate functions in $M_0^p(\Omega)$ by means of sequences of functions in $L^1(\Omega)\cap L^p(\Omega)$.

Lemma 1.9. *If $g \in M_0^p(\Omega)$, with $p > 1$, then there exists a sequence g_h, $h \in \mathbb{N}$, with $g_h \in L^1(\Omega)\cap L^p(\Omega)$, such that*

$$g_h \to g \quad \textit{in } M^p(\Omega), \tag{1.32}$$

$$|g_h(x)| \leq |g(x)|, \quad \textit{a.e. in } \Omega, \forall\, h \in \mathbb{N}, \tag{1.33}$$

$$\sigma_o^p[g_h] = \sigma_o^p[g], \forall\, h \in \mathbb{N}. \tag{1.34}$$

Now, we recall the following results concerning the multiplication operator

$$u \in W_0^{1,p}(\Omega) \longrightarrow g\,u \in L^p(\Omega), \tag{1.35}$$

where the function g belongs to $M^N(\Omega)$.

These results have been proved in [31] (see also [15]) in a more general case. Here, we report only a specific one, required in the sequel.

Theorem 1.10. *Let $1 < p < N$. If $g \in M^N(\Omega)$, then the operator in* (1.35) *is bounded and there exists a positive constant c such that*

$$\|g\,u\|_{L^p(\Omega)} \leq c\,\|g\|_{M^N(\Omega)}\,\|u\|_{W^{1,p}(\Omega)} \quad \forall\, u \in W_0^{1,p}(\Omega), \tag{1.36}$$

with $c = c\,(N, p)$.
Moreover, if $g \in M_0^N(\Omega)$, then the operator in (1.35) *is also compact.*

Chapter 2

Elliptic equations in bounded domains

This chapter is dedicated to recall the most important existence, uniqueness and regularity results about coercive and noncoercive elliptic Dirichlet problems on bounded domains.
We start with the milestones results by Guido Stampacchia [27, 28] dealing with coercive problems in the linear case and, later on, we recall those by Lucio Boccardo concerning noncoercive problems in the linear and nonlinear case, contained in his papers [3] and [4].

2.1 Stampacchia's results

From now on, let Ω be a bounded, open subset of $\mathbb{R}^N$, $N > 2$. In the Sixties, Guido Stampacchia studies the following linear Dirichlet problem

$$\begin{cases} -\mathrm{div}(M(x)\nabla u) + \mu u = -\mathrm{div}(u\,E(x)) + f(x) \quad \text{in } \Omega, \\ \\ u \in W_0^{1,2}(\Omega), \end{cases} \tag{2.1}$$

where $M : \Omega \to \mathbb{R}^{N^2}$ is a measurable matrix field such that there exist α, $\beta \in \mathbb{R}_+$ such that

$$\alpha|\xi|^2 \le M(x)\,\xi\cdot\xi, \quad |M(x)| \le \beta, \quad \text{a.e. } x \in \Omega, \quad \forall\, \xi \in \mathbb{R}^N, \tag{2.2}$$

$$E : \Omega \to \mathbb{R}^N \text{ is a vector field} \tag{2.3}$$

and

$$f : \Omega \to \mathbb{R} \text{ is a real function such that } f \in L^{\frac{2N}{N+2}}(\Omega). \tag{2.4}$$

In [27] Stampacchia proves that, if $\mu > 0$ is large enough, then problem (2.1) is coercive. Moreover, he shows that, if $\|E\|_{L^N(\Omega)}$ is small enough, problem (2.1) still remains coercive, even if μ is small or null.

Under hypotheses (2.2), (2.3), (2.4) and if the problem is coercive, in [27, 28], he proves that (2.1) has a unique weak solution u. Furthermore, he also obtaines the following regularity results:

- if $|E| \in L^N(\Omega)$ and $f \in L^m(\Omega)$, $m > \dfrac{N}{2}$, then the solution u of (2.1) is in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$;

- if $|E| \in L^N(\Omega)$ and $f \in L^m(\Omega)$, $\frac{2N}{N+2} \le m < \frac{N}{2}$, then the solution u is in $W_0^{1,2}(\Omega) \cap L^{m^{**}}(\Omega)$, with $m^{**} = \frac{Nm}{N-2m}$.

In [27, 28], Guido Stampacchia also proves the existence of a solution u of (2.1) even if the summability of f is less than $\frac{2N}{N+2}$, and further regularity results, namely:

- if $f \in L^m(\Omega)$, $1 < m < \frac{2N}{N+2}$, then $u \in W_0^{1,m^*}(\Omega)$, $m^* = \frac{Nm}{N-m}$;
- if $f \in L^1(\Omega)$, then $u \in W_0^{1,q}(\Omega)$, $\forall q < \frac{N}{N-1}$.

2.2 Boccardo's results

In this section we report the main results obtained by Lucio Boccardo in [3, 4] for noncoercive problems in the linear and nonlinear cases. We focus on the techniques used to obtain existence and uniqueness of the solution in bounded domains that will be generalized to unbounded ones to get our results.

2.2.1 The noncoercive linear case

In [3], Lucio Boccardo considers the following linear Dirichlet problem

$$\begin{cases} -\mathrm{div}(M(x)\nabla u) = -\mathrm{div}(u\,E(x)) + f(x) \quad \text{in } \Omega, \\ \\ u \in W_0^{1,2}(\Omega), \end{cases} \tag{2.5}$$

where $M : \Omega \to \mathbb{R}^{N^2}$ is a measurable matrix field such that there exist α, $\beta \in \mathbb{R}_+$ such that

$$\alpha|\xi|^2 \le M(x)\,\xi\cdot\xi, \quad |M(x)| \le \beta, \quad \text{a.e. } x \in \Omega, \quad \forall\, \xi \in \mathbb{R}^N, \tag{2.6}$$

$E : \Omega \to \mathbb{R}^N$ is a vector field such that

$$|E| \in L^N(\Omega) \tag{2.7}$$

and $f : \Omega \to \mathbb{R}$ is a real function such that

$$f \in L^m(\Omega),\ 1 \le m < \frac{N}{2}. \tag{2.8}$$

The Dirichlet problem (2.5) is problem (2.1) studied by Stampacchia in the case $\mu = 0$. (The linear case in bounded domains has also been studied in [16, 25], assuming $|E|$ in classes wider than $L^N(\Omega)$).
We observe that the main difficulty here is due to the presence of the differential operator $-\mathrm{div}(M(x)\nabla v) + \mathrm{div}(v\,E(x))$ bacause it is noncoercive, since no smallness assumptions are done on $\|E\|_{L^N(\Omega)}$. While, as already observed, in order to obtain the coercivity in the case $\mu = 0$, one has to require that $\|E\|_{L^N(\Omega)}$ is small enough.

In order to obtain existence and uniqueness results for (2.5), Boccardo follows a nonlinear approch. In particular, he approximates (2.5) by the following nonlinear coercive problems

$$
\begin{cases}
-\operatorname{div}(M(x)\nabla u_n) = -\operatorname{div}\left(\dfrac{u_n}{1+\frac{1}{n}|u_n|}\,\dfrac{E(x)}{1+\frac{1}{n}|E(x)|}\right) + f_n(x), \\
\\
u_n \in W_0^{1,2}(\Omega),
\end{cases}
\tag{2.9}
$$

with $f_n(x) = T_n(f(x))$, where, for $k \in \mathbb{R}_+$,

$$
T_k(t) = \begin{cases}
t, & \text{if } |t| \le k \\
\\
k\frac{t}{|t|}, & \text{if } |t| > k,
\end{cases}
\tag{2.10}
$$

is the classical truncate function introduced by Stampacchia.

Thanks to the Schauder fixed point Theorem, he proves that, for every fixed n, a weak solution u_n of (2.9) exists.

The main idea to obtain this existence result is the following:

fixed $n \in \mathbb{N}$, let $w_n \in W_0^{1,2}(\Omega)$. He considers the problems

$$
\begin{cases}
-\operatorname{div}(M(x)\nabla u_n) = -\operatorname{div}\left(\dfrac{w_n}{1+\frac{1}{n}|w_n|}\,\dfrac{E(x)}{1+\frac{1}{n}|E(x)|}\right) + f_n(x), \\
\\
u_n \in W_0^{1,2}(\Omega)
\end{cases}
\tag{2.11}
$$

and proves that there exists a unique and bounded solution u_n of (2.11). Later on, he considers the operator

$$P : w_n \in W_0^{1,2}(\Omega) \to P(w_n) = u_n \in W_0^{1,2}(\Omega)$$

and, by means of Schauder fixed point Theorem, he shows that P has a fixed point. This gives the existence result for the solution of problem (2.9).

Successively, he shows that, if $m = \dfrac{2N}{N+2}$, the sequence u_n is bounded in $W_0^{1,2}(\Omega)$. Namely, for every $k \in \mathbb{R}_+$, he proves that the sequence $T_k(u_n)$ is bounded in $W_0^{1,2}(\Omega)$ and later, for sufficiently large k, he gets that the sequence $G_k(u_n) := u_n - T_k(u_n)$ is bounded in $W_0^{1,2}(\Omega)$. This leads to the boundedness of the sequence u_n.
Later on, exploiting this result and passing to the limit, he obtains the existence of a weak solution $u \in W_0^{1,2}(\Omega)$ of problem (2.5), with $m = \dfrac{2N}{N+2}$.
Indeed, since $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$, up to a subsequence, u_n converges weakly in $W_0^{1,2}(\Omega)$ to a function u. Furthermore, u_n is a solution of (2.9) and then, passing to the limit as $n \to +\infty$ in the variational formulation and thanks to the linearity of the problem, he obtains that u is a weak solution of (2.5).
Now, by taking $T_\epsilon(u-w)$ as test function in the variational formulation of the problem, where $\epsilon > 0$ and where u and w are weak solutions of (2.5), thanks to the Hölder inequality, Lucio Boccardo also proves that, under hypotheses (2.6), (2.7) and (2.8) with $m = \dfrac{2N}{N+2}$, the solution is unique.

As in Stampacchia's works, also in this case some regularity results are obtained, namely:

- if $|E| \in L^N(\Omega)$ and $f \in L^m(\Omega)$, $\dfrac{2N}{N+2} \leq m < \dfrac{N}{2}$, then there exists a solution u of (2.5), $u \in W_0^{1,2}(\Omega) \cap L^{m^{**}}(\Omega)$, with $m^{**} = \dfrac{Nm}{N-2m}$;
- if $|E| \in L^r(\Omega)$, $r > N$ and $f \in L^m(\Omega)$, $m > \dfrac{N}{2}$, then there exists a solution u of (2.5), $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.

Moreover, he also proves the existence of a distributional solution u of (2.5) and, even if the summability of f in less than $\dfrac{2N}{N+2}$, the following regularity results:

- if $|E| \in L^N(\Omega)$ and $f \in L^m(\Omega)$, $1 < m < \dfrac{2N}{N+2}$, then $u \in W_0^{1,m^*}(\Omega)$, $m^* = \dfrac{Nm}{N-m}$;
- if $|E| \in L^N(\Omega)$ and $f \in L^1(\Omega)$, then $u \in W_0^{1,1^*}(\Omega)$, where $1^* = \dfrac{N}{N-1}$.

2.2.2 The noncoercive nonlinear case

In this section we recall the main results obtained by Lucio Boccardo in [4], where he considers, in the case of bounded domains, a nonlinear version of the noncoercive boundary value problem studied in [3].

Let A be the differential operator defined as

$$A(v) = -\mathrm{div}(a(x, \nabla v)),$$

where $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function such that

$$\begin{cases} a(x,\xi)\xi \geq \alpha|\xi|^p, \\ \\ |a(x,\xi)| \leq \beta|\xi|^{p-1}, \\ \\ \left(a(x,\xi) - a(x,\eta)\right)(\xi - \eta) > 0, \end{cases} \tag{2.12}$$

for a.e. $x \in \Omega$, for every $\xi \in \mathbb{R}^N$ and $\eta \in \mathbb{R}^N$ with $\xi \neq \eta$, and where α, β are strictly positive costants.

In [4], Lucio Boccardo proves the existence and uniqueness of the weak solution of the following nonlinear Dirichlet problem

$$\begin{cases} A(u) = -\mathrm{div}(g(u)\, E(x)) + f(x) \quad \text{in } \Omega, \\ \\ u \in W_0^{1,p}(\Omega), \end{cases} \tag{2.13}$$

where

$$1 < p < N, \tag{2.14}$$

$$|E| \in L^{\frac{N}{p-1}}(\Omega), \tag{2.15}$$

$$f \in L^m(\Omega),\ m \geq (p^*)', \tag{2.16}$$

where p^* denotes the Sobolev conjugate of p and p' the Hölder conjugate of

p,

$$g(s) \text{ is a real continuous function such that } |g(s)| \le \gamma |s|^{p-1}, \tag{2.17}$$

for some $\gamma > 0$.
We observe that, thanks to (2.12), the operator A is monotone and coercive, therefore the Surjectivity Theorem applies and thus A is surjective. As in the linear case studied in [3], the main difficulty here is due to the noncoercivity on $W_0^{1,p}(\Omega)$ of problem (2.13).

Following the same approach of the linear case (cfr. [3]), inspired by the papers of Guido Stampacchia [28, 27], Lucio Boccardo in [4] approximates the noncoercive nonlinear problem by coercive nonlinear problems and then passes to the limit. In particular, he considers the following approximate problems

$$\begin{cases} -\operatorname{div}(a(x, \nabla u_n)) = \\ \\ -\operatorname{div}\left(\dfrac{g(u_n)}{1 + \frac{1}{n}|u_n|^{p-1}} \, \dfrac{E(x)}{1 + \frac{1}{n}|E(x)|} \right) + \dfrac{f(x)}{1 + \frac{1}{n}|f(x)|}, \\ \\ u_n \in W_0^{1,p}(\Omega). \end{cases} \tag{2.18}$$

Thanks to (2.12) and under the hypotheses (2.14), (2.15), (2.16) and (2.17), the Schauder fixed point Theorem immediately applies and then he obtaines the existence of a weak solution u_n of (2.18), for every fixed n.

Exploiting Stampacchia's truncates (2.10) and remembering that $G_k(u_n) := u_n - T_k(u_n)$, by the boundedness of the sequences $T_k(u_n)$ and $G_k(u_n)$ in $W_0^{1,p}(\Omega)$, for sufficiently large k, he obtains that also the sequence u_n is bounded in $W_0^{1,p}(\Omega)$.

Later on, passing to the limit, he obtains the existence of a weak solution $u \in W_0^{1,p}(\Omega)$ of (2.13). Namely, since $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$, up to a subsequence, u_n converges weakly in $W_0^{1,p}(\Omega)$ to a function u. Furthermore, he proves that u_n converges strongly to u in $W_0^{1,p}(\Omega)$. This allows him to pass to the limit, as $n \to +\infty$, in the variational formulation of (2.18), and thus he obtains the existence of a weak solution $u \in W_0^{1,p}(\Omega)$ of (2.13).

To achieve the uniqueness of the solution, under the same hypotheses of (2.13), he considers the problem

$$\begin{cases} -\mathrm{div}(b(x)|\nabla u|^{p-2}\nabla u) = -\mathrm{div}(g(u)\,E(x)) + f(x) & \text{in } \Omega, \\ \\ u \in W_0^{1,p}(\Omega), \end{cases} \tag{2.19}$$

where

$$\alpha \le b(x) \le \beta, \ \text{ for some } 0 < \alpha \le \beta$$

and where the function g is required to be such that $|g'(s)| \le \mu |s|^{p-1} + \mu$ for some $\mu > 0$.

Obviously for this problem, that is a special case of (2.13), all the results obtained previously are valid. Differently from linear case analysed in [3], where the uniqueness result can be achieved for $p = 2$, in the nonlinear case it can be obtained, for problem (2.19), only for $p: \ 1 < p \leq 2$. In particular, Lucio Boccardo proves the uniqueness of the solution taking $T_h(u-w)^+$ as test function in the variational formulation of the problem, where $h > 0$ and where u and w are weak solutions of (2.19) and proving that u is equal to w.

Chapter 3

Noncoercive elliptic equations in unbounded domains: the linear case

In this chapter we consider the Dirichlet problem for noncoercive linear elliptic equations in unbounded domains studied in the paper [24], by Sara Monsurrò and Maria Transirico. First of all, we remark the main difficulties one has to deal with when working on unbounded sets. Later on, we introduce some preliminary tools, useful both in the linear case and in the nonlinear one. Finally, we focus on the existence, uniqueness and regularity results.

3.1 Main difficulties in unbounded domains

The main difficulties one has to afford when working on unbounded sets are the following:

1) There are no natural decreasing inclusions among the $L^p(\Omega)$ spaces;

2) There are no compactness results;

3) The norm in $W_0^{1,p}(\Omega)$ is not equivalent to the norm of gradient (Poincaré inequality does not hold).

How can we overcome these problems?

1) For the first problem, we use the $M^p(\Omega)$ spaces defined in Chapter 1 for which the natural inclusions are valid also in unbounded domains. In particular, we suppose that the coefficients of our problem belong to the intersections of these spaces with $L^t(\Omega)$ ones, for a suitable t.

2) In order to solve the second problem, we exploit a compactenss result on $M_0^p(\Omega)$ spaces, stated in Theorem 1.10 of Chapter 1, proved in [31] by Maria Transirico, Mario Troisi and Antonio Vitolo.

3) Since the norm in $W_0^{1,p}(\Omega)$ is not equivalent to the norm of gradient when Ω is unbounded, it is necessary to take $\mu > 0$ in the equation of problem (2.1). Nevertheless, since μ is not required to be large enough, the problem remains noncoercive.

3.2 Preliminary results

Let us recall some important results useful in the sequel. Let Ω be an unbounded, open subset of $\mathbb{R}^N$, $N > 2$.

Let $k \in \mathbb{R}_+$ and $T_k(t)$, defined in (2.10), be Stampacchia's truncate functions. Put

$$G_k(t) = t - T_k(t). \tag{3.1}$$

Given $u \in W_0^{1,p}(\Omega)$, define

$$A_k = \{x \in \Omega \,:\, |u(x)| > k\}. \tag{3.2}$$

The following lemma contains some useful properties of the composition of the functions T_k and G_k with $u \in W_0^{1,p}(\Omega)$, needed in the sequel.

Lemma 3.1. *Let $p > 1$. For every $u \in W_0^{1,p}(\Omega)$ and $k \in \mathbb{R}_+$ one has*

$$T_k(u) = T_k \circ u \in W_0^{1,p}(\Omega), \tag{3.3}$$

$$|\nabla u|^{p-2}\nabla u \nabla T_k(u) = |\nabla T_k(u)|^p, \;\; a.e. \;\, in\ \Omega, \tag{3.4}$$

$$|u|^{p-2}u \; T_k(u) \geq |T_k(u)|^p, \;\; a.e. \;\, in\ \Omega, \tag{3.5}$$

$$|u|^{p-2}u \; \nabla T_k(u) = |T_k(u)|^{p-1}\nabla T_k(u), \;\; a.e. \;\, in\ \Omega, \tag{3.6}$$

$$G_k(u) = G_k \circ u \in W_0^{1,p}(\Omega), \tag{3.7}$$

$$|G_k(u)| \leq |u|, \;\; a.e. \;\; in \; \Omega, \tag{3.8}$$

$$|u| \leq |G_k(u)| + k, \;\; a.e. \;\; in \; \Omega, \tag{3.9}$$

$$|u|^{p-1} \leq 2^{p-2}(|G_k(u)|^{p-1} + k^{p-1}), \;\; a.e. \;\; in \; \Omega, \tag{3.10}$$

$$|\nabla u|^{p-2}\nabla u \nabla G_k(u) = |\nabla G_k(u)|^p, \;\; a.e. \;\; in \; \Omega, \tag{3.11}$$

$$|u|^{p-2}u \; G_k(u) \geq |G_k(u)|^p, \;\; a.e. \;\; in \; \Omega, \tag{3.12}$$

$$\text{supp } G_k(u) \subseteq \overline{A}_k, \tag{3.13}$$

$$(G_k(u))_{x_i} = \begin{cases} u_{x_i} & a.e. \;\; in \; A_k, \\ 0 & a.e. \;\; in \; \Omega \setminus A_k, \;\; i = 1 \dots n. \end{cases} \tag{3.14}$$

Let us mention a generalization to unbounded sets of a result proved in [28], in the case of bounded domains, and already showed, for the case $p = 2$, in [14].

Lemma 3.2. *Let $p > 1$, G be a uniformly Lipschitz function such that $G(0) = 0$ and $u \in W_0^{1,p}(\Omega)$. Then $G \circ u \in W_0^{1,p}(\Omega)$.*

Proof. The proof is obtained following the same arguments of [14], with opportune modifications. □

Now, we recall Lemma 4.1 of [28] by Stampacchia. This is useful to prove an important summability result that can lead us to the existence of a weak solution for the problem.

Lemma 3.3. *Let $k_0 > 0$ and $\varphi : [k_0, +\infty[\to \mathbb{R}$ be a non negative and non* *increasing function such that*

$$\varphi(h) \leq \frac{C}{(h-k)^{\gamma}}[\varphi(k)]^{\delta} \quad \forall\, h > k \geq k_0, \tag{3.15}$$

where C, γ and δ are positive constants, with $\delta > 1$. Then, for

$$d = 2^{\frac{\delta}{\delta-1}} C^{1/\gamma} \left[\varphi(k_0)\right]^{\frac{\delta-1}{\gamma}}, \tag{3.16}$$

one has

$$\varphi(k_0 + d) = 0. \tag{3.17}$$

Proof. We consider

$$k_s = k_0 + d - \frac{d}{2^s}.$$

By (3.15), we obtain

$$\varphi(k_{s+1}) \leq \frac{C\ 2^{(s+1)\gamma}}{d^{\gamma}}\left(\varphi(k_s)\right)^{\delta} \tag{3.18}$$

because

$$\varphi(k_{s+1}) \leq \frac{C}{(k_{s+1}-k_s)^{\gamma}}\left(\varphi(k_s)\right)^{\delta} = \frac{C}{\left(\frac{d}{2^{s+1}}\right)^{\gamma}}\left(\varphi(k_s)\right)^{\delta} = \frac{C\ 2^{(s+1)\gamma}}{d^{\gamma}}\left(\varphi(k_s)\right)^{\delta}.$$

We want to prove by induction that

$$\varphi(k_s) \leq \frac{\varphi(k_0)}{2^{-s\mu}} \tag{3.19}$$

where $\mu = \dfrac{\gamma}{1-\delta}$.

The case $s = 0$ is trivially true.

Now, we suppose the inequality (3.19) true for s and let us prove it for $s+1$.

By (3.18) and hypothesis of induction,

$$\varphi(k_{s+1}) \leq \frac{C\ 2^{(s+1)\gamma}}{d^\gamma}\,(\varphi(k_s))^\delta \leq \frac{C\ 2^{(s+1)\gamma}}{d^\gamma}\frac{(\varphi(k_0))^\delta}{2^{-s\delta\mu}}.$$

Thanks to hypotesis (3.16), we obtain

$$\varphi(k_{s+1}) < \frac{\varphi(k_0)}{2^{-(s+1)\mu}}$$

and hence (3.19) holds for every s.

Now, passing to the limit, as $s \to +\infty$,

$$\varphi(k_s) = \varphi\left(k_0 + d - \frac{d}{2^s}\right) \to \varphi(k_0 + d)$$

and then the proof is done. $\square$

For sake of completeness, we recall now two important theorems, useful in the sequel: the Schauder fixed point Theorem in its formulation given, for instance, in Theorem 1.11 of [6], and the Vitali Theorem (see, for instance, [26]).

Theorem 3.4 (Schauder)**.** *Let X be a Banach space. If F is a function completely continuous and F admits a bounded and closed invariant convex subset K of X, then F has a fixed point on K.*

Theorem 3.5 (Vitali). *Let $u_n \subset L^p(\Omega)$ be a sequence such that $u_n \to u$ a.e. in Ω. Then $u \in L^p(\Omega)$ and $u_n \to u$ in $L^p(\Omega)$ if and only if*

(i) for each $\varepsilon > 0$ there exists a set $A_\varepsilon \subset \Omega$ such that $|A_\varepsilon| < +\infty$ and

$$\int_{\Omega \setminus A_\varepsilon} \|u_n\|^p < \varepsilon \qquad \forall n \in \mathbb{N};$$

(ii) for each $\varepsilon > 0$ these exists $\delta > 0$ such that

$$\int_A \|u_n\|^p < \varepsilon \qquad \forall n \in \mathbb{N},$$

for every $A \subset \Omega$ with $|A| < \delta$.

3.3 Existence, Uniqueness and Regularity results

Let us now recall the existence, uniqueness and regularity results, obtained in [24], in the case when Ω is unbounded.
Consider the following noncoercive linear Dirichlet problem

$$\begin{cases} -\mathrm{div}(M(x)\nabla u) + \mu u = -\mathrm{div}(u\, E(x)) + f(x) & \text{in } \Omega, \\ \\ u \in W_0^{1,2}(\Omega), \end{cases} \tag{3.20}$$

where $M : \Omega \to \mathbb{R}^{N^2}$ is a measurable matrix field such that there exist α, $\beta \in \mathbb{R}_+$ such that

$$\alpha|\xi|^2 \leq M(x)\,\xi \cdot \xi, \quad |M(x)| \leq \beta, \quad \text{a.e. } x \in \Omega, \quad \forall\, \xi \in \mathbb{R}^N, \tag{3.21}$$

$$\mu > 0, \tag{3.22}$$

$E : \Omega \to \mathbb{R}^N$ is a vector field such that

$$|E| \in L^2(\Omega) \cap M_0^N(\Omega) \tag{3.23}$$

and $f : \Omega \to \mathbb{R}$ is a real function such that

$$f \in L^1(\Omega) \cap L^{\frac{2N}{N+2}}(\Omega), \tag{3.24}$$

where $M_0^N(\Omega)$ is the functional space strictly containing $L^N(\Omega)$, described in Section 1.4 of Chapter 1.

The techniques used to obtain these results issue from an idea of [3], inspired by the papers of Guido Stampacchia [27, 28], and by [9, 10, 12], where nonlinear problems are treated.

In particular, as already mentioned, in [3], the noncoercive problem is approximated by coercive nonlinear problems and then the author passes to the limit. Here, it is possible to pass to the limit, thanks to the compactness result in $M_0^N(\Omega)$ (see Theorem 1.10) that applies in view of the assumption (3.23) on the coefficient appearing in the noncoercive term, as showed in the next section.

3.3.1 Existence and uniqueness results

In order to obtain the existence and uniqueness results, the authors, inspired by the technique of Lucio Boccardo in [3], approximate noncoercive linear problem (3.20) by the following coercive nonlinear problems

$$\begin{cases} -\mathrm{div}(M(x)\nabla u_n) + \mu u_n = \\ \\ -\mathrm{div}\left(\dfrac{u_n}{1+\frac{1}{n}|u_n|}\,\dfrac{E(x)}{1+\frac{1}{n}|E(x)|}\right) + \dfrac{f}{1+\frac{1}{n}|f|}, \\ \\ u_n \in W_0^{1,2}(\Omega). \end{cases} \tag{3.25}$$

To prove that a bounded weak solution of (3.25) exists, for every fixed n, a previous result is needed (proved in Lemma 3.4 of [24]).

Lemma 3.6. *Assume (3.21), (3.22), $|F| \in L^2(\Omega)$ and $f \in L^{\frac{2N}{N+2}}(\Omega)$. Then there exists a unique solution u of the problem*

$$\begin{cases} -\mathrm{div}(M(x)\nabla u) + \mu u = -\mathrm{div}(F(x)) + f(x) \quad \text{in } \Omega, \\ \\ u \in W_0^{1,2}(\Omega). \end{cases} \tag{3.26}$$

If in addition $|F| \in L^p(\Omega)$ and $f \in L^{\frac{p}{2}}(\Omega)$, $p > N$, then the solution u is of *class* $L^\infty(\Omega)$.

Let us give an idea of the proof.

Thanks to the Lax-Milgram Lemma, one obtains the existence and uniqueness of the solution. In order to prove the boundedness, one takes $G_k(u)$ as test function in the variational formulation of (3.26), in view of (3.7) with $p = 2$. Using (3.21), the definition of A_k (3.2), properties (3.11), (3.12), (3.13) in the case $p = 2$ and Hölder and Sobolev inequalities, one gets

$$\|G_k(u)\|_{L^{2^*}(\Omega)} \leq C(|A_k|^{\frac{1}{2}-\frac{1}{p}} + |A_k|^{1-\frac{1}{2^*}-\frac{2}{p}}),$$

with $C = C(\alpha, S, \|F\|_{L^p(\Omega)}, \|f\|_{L^{\frac{p}{2}}(\Omega)})$ and where S is the Sobolev constant (cfr. Theorem 3.17 of [6]).

Now, since $|A_k| \to 0$, as $k \to +\infty$, it is possible to assume that there exists $k_0 \in \mathbb{R}_+$ such that $|A_k| \leq 1$, for $k \geq k_0$. Thanks to (3.2) and (3.9) (where $p = 2$), one gets the following inequality

$$|A_h| \leq C'' \frac{|A_k|^{\frac{2^*}{2}-\frac{2^*}{p}}}{(h-k)^{2^*}}, \quad \forall h > k \geq k_0,$$

with $C'' = C''(\alpha, S, \|F\|_{L^p(\Omega)}, \|f\|_{L^{\frac{p}{2}}(\Omega)})$.

Finally, since $N < p$, one gets that $\frac{2^*}{2} - \frac{2^*}{p} > 1$, hence Lemma 3.3 applies and therefore there exists $d \in \mathbb{R}_+$ such that $|A_{k_0+d}| = 0$, thus $u \in L^\infty(\Omega)$.

We are now in a position to show, by means of the Schauder fixed point Theorem, the existence and boundedness of a solution of approximate problem (3.25). This is done for $n = 1$ and it can be analogously obtained for $n \geq 2$.

Theorem 3.7. *Assume (3.21), (3.22), $|E| \in L^2(\Omega) \cap M_0^N(\Omega)$ and $f \in L^1(\Omega)$. Then there exists a weak solution u of class $L^\infty(\Omega)$ of the following problem*

$$\begin{cases} -\mathrm{div}(M(x)\nabla u) + \mu u = -\mathrm{div}\left(\dfrac{u}{1+|u|}\,\dfrac{E(x)}{1+|E(x)|}\right) + \dfrac{f}{1+|f|}, \\ \\ u \in W_0^{1,2}(\Omega). \end{cases} \tag{3.27}$$

Proof. Let $w \in W_0^{1,2}(\Omega)$. Thanks to Lemma 3.6, there exists a unique and bounded solution u of the problem

$$\begin{cases} -\mathrm{div}(M(x)\nabla u) + \mu u = -\mathrm{div}\left(\dfrac{w}{1+|w|}\,\dfrac{E(x)}{1+|E(x)|}\right) + \dfrac{f}{1+|f|}, \\ \\ u \in W_0^{1,2}(\Omega). \end{cases} \tag{3.28}$$

In order to apply the Schauder fixed point Theorem, one considers the operator

$$P : w \in W_0^{1,2}(\Omega) \to u = Pw \in W_0^{1,2}(\Omega) \tag{3.29}$$

and shows that the following two hypotheses are satisfied:

1. P admits a bounded and closed invariant convex set.
2. P is completely continuous.

In order to prove point 1, one takes u as test function in the variational formulation of (3.28), obtaining

$$\int_\Omega M(x)\nabla u\cdot\nabla u+\int_\Omega \mu u^2=\int_\Omega \frac{w}{1+|w|}\frac{E(x)}{1+|E(x)|}\cdot\nabla u+\int_\Omega \frac{f\,u}{1+|f|}\,.$$

Thus, by hypotheses (3.21), (3.22) and thanks to the Hölder and Sobolev inequalities, one has that there exist two positive constants $C_0=C_0(\alpha,\mu)$ and $C=C(\alpha,\mu,\|E\|_{L^2(\Omega)},\|\frac{f}{1+|f|}\|_{L^{\frac{2N}{N+2}}(\Omega)},S)$ such that

$$\|u\|^2_{W^{1,2}(\Omega)}\le C_0(\|E\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)}+\Big\|\frac{f}{1+|f|}\Big\|_{L^{\frac{2N}{N+2}}(\Omega)}\|u\|_{L^{2^*}(\Omega)})\le C\|u\|_{W^{1,2}(\Omega)}.$$

Hence, if one considers the closed ball $\|w\|_{W^{1,2}(\Omega)}\le C$, one obtains that $\|Pw\|_{W^{1,2}(\Omega)}=\|u\|_{W^{1,2}(\Omega)}\le C$. This concludes the proof of the first point.
In order to prove the point 2, one has to show that if $w_n\rightharpoonup\bar w$ weakly in $W_0^{1,2}(\Omega)$, then $Pw_n\to P\bar w$ in $W_0^{1,2}(\Omega)$.
Let $u_n=Pw_n$ and $\bar u=P\bar w$. One takes $u_n-\bar u$ as test function in the variational formulations of (3.28) written in correspondence of $w=w_n$ and $w=\bar w$, respectively, and subtracts member from member obtaining

$$\int_\Omega M(x)[\nabla(u_n-\bar u)]^2+\int_\Omega \mu(u_n-\bar u)^2$$

$$=\int_\Omega\Big(\frac{w_n}{1+|w_n|}-\frac{\bar w}{1+|\bar w|}\Big)\frac{E(x)}{1+|E(x)|}\cdot\nabla(u_n-\bar u).$$

Thanks to hypotheses (3.21), (3.22) and by the Hölder inequality, one gets the following inequality

$$\|u_n - \bar{u}\|_{W^{1,2}(\Omega)} \leq \left\| \left(\frac{w_n}{1+|w_n|} - \frac{\bar{w}}{1+|\bar{w}|} \right) |E| \right\|_{L^2(\Omega)}.$$

Now, thanks to the compactness result of the operator $u \in W_0^{1,2}(\Omega) \to |E|u \in L^2(\Omega)$, stated in Theorem 1.10 used in the case $p = 2$, since $w_n \rightharpoonup \bar{w}$ weakly in $W_0^{1,p}(\Omega)$, one has $|E|w_n \to |E|\bar{w}$ in $L^2(\Omega)$, and hence, up to a subsequence, w_n converges to $\bar{w}$ a.e. in Ω. Thus, it is possible to apply the Lebesgue dominated convergence Theorem obtaining

$$\left\| \left(\frac{w_n}{1+|w_n|} - \frac{\bar{w}}{1+|\bar{w}|} \right) |E| \right\|_{L^2(\Omega)} \to 0.$$

This concludes the proof. □

Successively, in [24], the authors show that the sequence u_n of the solutions of problems (3.25) is bounded in $W_0^{1,2}(\Omega)$ thanks to some preliminary estimates on the sequence $T_k(u_n)$ and $G_k(u_n)$. Namely, assuming $|E| \in L^2(\Omega) \cap M_0^N(\Omega)$ and $f \in L^1(\Omega) \cap L^{\frac{2N}{N+2}}(\Omega)$, one gets that the sequence $T_k(u_n)$ is bounded in $W_0^{1,2}(\Omega)$, for any $k \in \mathbb{R}_+$. This is done taking $T_k(u_n)$ as test function in the variational formulation of (3.25), that can be done in view of (3.3) for $p = 2$. Then, to obtain that the sequence $G_k(u_n)$ is bounded in $W_0^{1,2}(\Omega)$ too, for sufficiently large k, one uses $G_k(u_n)$ as test function in the variational formulation of (3.25), that can be done in view of (3.7) for $p = 2$. This allows to obtain the boundedness of u_n in $W_0^{1,2}(\Omega)$, fixed k sufficiently large, in view of (3.1). Moreover, by the estimates on

$T_k(u_n)$ and $G_k(u_n)$, one gets the following a priori bound on $\{u_n\}$

$$\|u_n\|^2_{W^{1,2}(\Omega)} \leq C\left(\|E\|^2_{L^2(\Omega)} + \|f\|_{L^1(\Omega)} + \|f\|^2_{L^{\frac{2N}{N+2}}(\Omega)}\right). \tag{3.30}$$

where $C = C\left(N, \alpha, \mu, S, \sigma_o{}^N[E]\right)$.

Finally, in Theorem 3.8 below, by approximation, S. Monsurrò and M. Transirico get the existence result of a weak solution of problem (3.20).

Theorem 3.8. *Assume (3.21), (3.22), $|E| \in L^2(\Omega) \cap M_0^N(\Omega)$ and $f \in L^1(\Omega) \cap L^{\frac{2N}{N+2}}(\Omega)$. Then there exists $u \in W_0^{1,2}(\Omega)$ weak solution of (3.20), that is*

$$\int_\Omega M(x)\nabla u\cdot\nabla v + \mu\int_\Omega uv = \int_\Omega u\, E(x)\cdot\nabla v + \int_\Omega f\, v, \qquad \forall\ v \in W_0^{1,2}(\Omega). \tag{3.31}$$

Moreover, there exists a positive constant $C = C(N, \alpha, \mu, S, \sigma_o{}^N[E])$ such that

$$\|u\|^2_{W^{1,2}(\Omega)} \leq C(\|E\|^2_{L^2(\Omega)} + \|f\|_{L^1(\Omega)} + \|f\|^2_{L^{\frac{2N}{N+2}}(\Omega)}). \tag{3.32}$$

Proof. Since u_n is a solution of (3.25) and the sequence $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$, one has that

$$\begin{aligned}&\int_\Omega M(x)\nabla u_n\cdot\nabla v + \mu\int_\Omega u_n v\\ &= \int_\Omega \frac{u_n}{1+\frac{1}{n}|u_n|}\frac{E(x)}{1+\frac{1}{n}|E(x)|}\cdot\nabla v + \int_\Omega \frac{f}{1+\frac{1}{n}|f|}v\,,\end{aligned} \tag{3.33}$$

for every $v \in W_0^{1,2}(\Omega)$. Now, let us to pass to the limit, as $n \to +\infty$, in (3.33).

Clearly the first, the second and the last integral do not give problems by the weak convergence $u_n \rightharpoonup u$ in $W_0^{1,2}(\Omega)$. Let us now analyse the following integral

$$\int_\Omega \frac{u_n}{1+\frac{1}{n}|u_n|} \frac{E(x)}{1+\frac{1}{n}|E(x)|} \cdot \nabla v.$$

Since u_n converges weakly to u in $W_0^{1,2}(\Omega)$, by Lemma 1.10 one has that $|E|u_n$ converges strongly to $|E|u$ in $L^2(\Omega)$. Hence, it is possible to use the Vitali Theorem (cfr. Theorem 3.5) obtaining that for any $\varepsilon > 0$ there exists $\Omega_\varepsilon \subset \Omega$ with $|\Omega_\varepsilon| < +\infty$ such that

$$\int_{\Omega\setminus\Omega_\varepsilon} |u_n|^2|E|^2 < \varepsilon, \text{ uniformly with respect to } n,$$

and that there exists $\delta > 0$ such that for every $A \subset \Omega$ with $|A| < \delta$, one has

$$\int_A |u_n|^2|E|^2 < \varepsilon, \text{ uniformly with respect to } n.$$

Now,

$$\int_{\Omega\setminus\Omega_\varepsilon} \frac{|u_n|^2}{(1+\frac{1}{n}|u_n|)^2} \frac{|E(x)|^2}{(1+\frac{1}{n}|E(x)|)^2} \le \int_{\Omega\setminus\Omega_\varepsilon} |u_n|^2|E|^2 < \varepsilon,$$

and

$$\int_A \frac{|u_n|^2}{(1+\frac{1}{n}|u_n|)^2} \frac{|E(x)|^2}{(1+\frac{1}{n}|E(x)|)^2} \le \int_A |u_n|^2|E|^2 < \varepsilon,$$

uniformly with respect to n and furthermore, since u_n converges a.e. to u, one gets

$$\frac{u_n}{1+\frac{1}{n}|u_n|}\frac{|E(x)|}{1+\frac{1}{n}|E(x)|} \to u|E| \text{ a.e. in } \Omega.$$

Hence, using again the Vitali Theorem (cfr. Theorem 3.5), in the reverse sense, we obtain that

$$\frac{u_n}{1+\frac{1}{n}|u_n|}\frac{|E(x)|}{1+\frac{1}{n}|E(x)|} \to u\,|E| \text{ in } L^2(\Omega).$$

Passing to the limit, as $n \to +\infty$, in (3.33) one obtains (3.31).
Estimate (3.32) follows then by (3.30). □

In order to achieve the uniqueness result, the authors follow some ideas of [3, 11]. In particular, they prove that, if (3.21) and (3.22) hold, $|E| \in L^2(\Omega) \cap M_0^N(\Omega)$ and $f \in L^1(\Omega) \cap L^{\frac{2N}{N+2}}(\Omega)$, then the weak solution u of (3.20) is unique.
The idea is to consider u and w weak solutions of (3.20) and to obtain u equal to w almost everywhere. This is done assuming $\delta \in \mathbb{R}_+$ and $\epsilon \in]0, \delta[$ and using $T_\epsilon(u-w)$ as test function in the variational formulation of problem (3.20), written in correspondence of the solutions u and w respectively. Then, subtracting, one has

$$\int_\Omega M(x)\nabla(u-w)\nabla T_\epsilon(u-w) + \mu \int_\Omega (u-w)T_\epsilon(u-w)$$

$$= \int_\Omega (u-w)E(x)\nabla T_\epsilon(u-w).$$

Finally, thanks to some inequalities and to hypotheses, it is simple to show that $u(x) = w(x)$ almost everywhere, getting the uniqueness.

3.3.2 Regularity results

This section is devoted to prove two regularity results for the weak solution $u \in W_0^{1,2}(\Omega)$ of problem (3.20). More precisely,

(i) if $|E| \in L^2(\Omega) \cap M_0^N(\Omega)$ and $f \in L^1(\Omega) \cap L^m(\Omega)$, $\dfrac{2N}{N+2} \leq m < \dfrac{N}{2}$, then the solution u of (3.20) is in $L^{m^{**}}(\Omega)$, with $m^{**} = (m^*)^* = \dfrac{Nm}{N-2m}$;

(ii) if one requires stronger assumptions on E and f, namely if $|E| \in L^2(\Omega) \cap L^r(\Omega)$, $r > N$, and $f \in L^1(\Omega) \cap L^m(\Omega)$, $m > \dfrac{N}{2}$, then the solution u of (3.20) is in $L^\infty(\Omega)$.

To show (i), some preliminary results for the sequences $T_k(u_n)$ and $G_k(u_n)$ are needed.

Firstly, assuming (3.21) and (3.22), $|E| \in L^2(\Omega) \cap M_0^N(\Omega)$ and $f \in L^1(\Omega) \cap L^{\frac{2N}{N+2}}(\Omega)$, for any $k \in \mathbb{R}_+$, one gets that the sequence $T_k(u_n)$ is bounded in $L^{m^{**}}(\Omega)$, for every $\dfrac{2N}{N+2} \leq m < \dfrac{N}{2}$. More precisely, there exists a positive constant $C = C(N, m, \alpha, S)$ such that

$$\left[\int_\Omega |T_k(u_n)|^{m^{**}}\right]^{\frac{2}{2^*}} \leq C\Big(k^{\frac{2m^{**}}{2^*}} \int_\Omega |E|^2 + k^{\frac{2m^{**}}{2^*}-1} \int_\Omega |f|\Big). \tag{3.34}$$

To prove (3.34), one takes $\dfrac{|T_k(u_n)|^{2(\lambda-1)}T_k(u_n)}{2\lambda - 1}$, with $\lambda = \dfrac{m^{**}}{2^*}$, as test function in the variational formulation of problem (3.25). This can be done in view of Lemma 3.2, for $p = 2$, and of Theorem 3.7. Thus, estimate (3.34) is obtained using some properties and Young and Sobolev inequalities.

For the sequence $G_k(u_n)$, it is not possible to obtain an analogous result with $|E| \in L^2(\Omega) \cap M_0^N(\Omega)$. But, as we see in Lemma 3.9, under the stronger assumption $|E| \in L^2(\Omega) \cap L^N(\Omega)$ and if $f \in L^1(\Omega) \cap L^m(\Omega)$, with $\frac{2N}{N+2} \leq m < \frac{N}{2}$, it is possible to prove the boundedness of $G_k(u_n)$ in $L^{m^{**}}(\Omega)$, for sufficiently large k. We explicitly give the complete proof of Lemma 3.9, that was only outlined in [24].

Lemma 3.9. *Assume (4.3), (4.4), $|E| \in L^2(\Omega) \cap L^N(\Omega)$ and $f \in L^1(\Omega) \cap L^m(\Omega)$. If $\frac{2N}{N+2} \leq m < \frac{N}{2}$, then there exists a $\tilde{k} \in \mathbb{R}_+$ such that the sequence $\{G_k(u_n)\}$ is bounded in $L^{m^{**}}(\Omega)$, for every $k > \tilde{k}$. More precisely, there exists a positive constant $C = C(N, m, \alpha, S)$ such that*

$$\left[\int_\Omega |G_k(u_n)|^{m^{**}}\right]^{\frac{2}{2^*} - \frac{1}{m'}} \leq C\Big(k^2 + \|f\|_{L^m(\Omega)}\Big). \tag{3.35}$$

Proof. Since the function $|t|^{2(\lambda-1)}t$, with $\lambda > 1$, satisfies the hypotheses of Lemma 3.2, provided that $|t| \leq M$, for some $M > 0$, and since $u_n \in L^\infty(\Omega)$ by Theorem 3.7, we can take $\frac{|G_k(u_n)|^{2(\lambda-1)}G_k(u_n)}{2\lambda - 1}$, with $\lambda = \frac{m^{**}}{2^*}$, as test function in the variational formulation of (3.25).

Observe that

$$\nabla\left(\frac{|G_k(u_n)|^{2(\lambda-1)}G_k(u_n)}{2\lambda-1}\right) = |G_k(u_n)|^{2(\lambda-1)}\nabla G_k(u_n).$$

By (3.11), (3.21), one obtains

$$\alpha \int_{\Omega} |G_k(u_n)|^{2(\lambda-1)} \nabla G_k(u_n)$$

$$\leq \int_{\Omega} |u_n| \left(|G_k(u_n)|^{\lambda-1}|E|\right) \left(|G_k(u_n)|^{\lambda-1}|\nabla G_k(u_n)|\right) + \frac{1}{2\lambda-1} \int_{\Omega} |f|\, |G_k(u_n)|^{2\lambda-1}$$

Now, Young and Hölder inequalities imply that

$$\alpha \int_{\Omega} |G_k(u_n)|^{2(\lambda-1)} \nabla G_k(u_n)$$

$$\leq \epsilon \int_{\Omega} |G_k(u_n)|^{2(\lambda-1)} |\nabla G_k(u_n)|^2 + \frac{1}{4\epsilon} \int_{A_n(k)} |u_n|^2 |G_k(u_n)|^{2(\lambda-1)} |E|^2$$

$$+ \frac{\|f\|_{L^m(\Omega)}}{2\lambda-1} \left(\int_{\Omega} |G_k(u_n)|^{(2\lambda-1)m'} \right)^{\frac{1}{m'}}$$

where, for $k \in \mathbb{R}_+$ and $n \in \mathbb{N}$,

$$A_n(k) = \{x \in \Omega : k < |u_n(x)|\}.$$

Taking $\epsilon = \dfrac{\alpha}{2}$, by (3.9) and thanks to Sobolev inequality, we have

$$C_1 \left(\int_{\Omega} |G_k(u_n)|^{2^*\lambda} \right)^{\frac{2}{2^*}}$$

$$\leq C_2 \int_{A_n(k)} |G_k(u_n)|^{2\lambda} \, |E|^2 + C_2\, k^2 \int_{A_n(k)} |G_k(u_n)|^{2(\lambda-1)} \, |E|^2$$

$$+\frac{\|f\|_{L^m(\Omega)}}{2\lambda-1}\left(\int_\Omega |G_k(u_n)|^{(2\lambda-1)m'}\right)^{\frac{1}{m'}},$$

where C_1 and C_2 are positive constants indipendent on n such that $C_1 = C_1(N, m, \alpha, S)$ and $C_2 = C_2(N, m, \alpha, S)$, where S is the Sobolev constant as in Theorem 3.17 of [6]. Using Hölder inequality again, we obtain

$$C_1\left(\int_\Omega |G_k(u_n)|^{2^*\lambda}\right)^{\frac{2}{2^*}}$$

$$\leq C_2\left(\int_\Omega |G_k(u_n)|^{2^*\lambda}\right)^{\frac{2}{2^*}}\left(\int_{A_n(k)} |E|^N\right)^{\frac{2}{N}}$$

$$+ C_2\, k^2(\text{meas } A_n(k))^{\frac{2}{2^*\lambda}}\left(\int_\Omega |G_k(u_n)|^{2^*\lambda}\right)^{\frac{2(\lambda-1)}{2^*\lambda}}\left(\int_{A_n(k)} |E|^N\right)^{\frac{2}{N}}$$

$$+\frac{\|f\|_{L^m(\Omega)}}{2\lambda-1}\left(\int_\Omega |G_k(u_n)|^{(2\lambda-1)m'}\right)^{\frac{1}{m'}}.$$

Since $|E| \in L^N(\Omega)$, (45) of [24] implies that there exists $\tilde{k}$ such that

$$C_2\left(\int_{A_n(k)} |E|^N\right)^{\frac{2}{N}} \leq \frac{C_1}{2} \quad \forall k > \tilde{k}.$$

Notice, now, that

$$2^*\lambda = (2\lambda-1)m' = m^{**}, \qquad \frac{2(\lambda-1)}{2^*\lambda} < \frac{2}{2^*}$$

and
$$\frac{2}{2^*} > \frac{1}{m'} \text{ if and only if } m < \frac{N}{2} \text{ (as in our case).}$$

Then, we deduce that, for $k > \tilde{k}$,

$$\frac{C_1}{2}\left(\int_\Omega |G_k(u_n)|^{m^{**}}\right)^{\frac{2}{2^*}}$$

$$\leq C_3\, k^2 \left(\int_\Omega |G_k(u_n)|^{m^{**}}\right)^{\frac{2(\lambda-1)}{2^*\lambda}} + \frac{\|f\|_{L^m(\Omega)}}{2\lambda - 1}\left(\int_\Omega |G_k(u_n)|^{m^{**}}\right)^{\frac{1}{m'}},$$

where $C_3 = C_3(N, m, \alpha, S)$.
Thus, for $k > \tilde{k}$, since $\dfrac{2}{2^*} > \dfrac{1}{m'}$, the sequence $\{G_k(u_n)\}$ is bounded in $L^{m^{**}}(\Omega)$.

$\square$

Now, putting together (3.34) and (3.35), in view of (3.1), if $|E| \in L^2(\Omega) \cap L^N(\Omega)$ and $f \in L^1(\Omega) \cap L^m(\Omega)$, with $\dfrac{2N}{N+2} \leq m < \dfrac{N}{2}$, fixed k sufficiently large, one gets the boundedness of u_n in $L^{m^{**}}(\Omega)$, i. e. there exists a positive constant $C = C\left(N, m, \alpha, S, \sigma_o^N[E]\right)$ such that

$$\|u_n\|_{L^{m^{**}}(\Omega)}^{m^{**}} \leq C(\|E\|_{L^2(\Omega)}^{\frac{2}{2^*}} + \|f\|_{L^1(\Omega)}^{\frac{2}{2^*}} + \|f\|_{L^m(\Omega)}^{\frac{2^*}{2-2^*/m'}} + 1). \tag{3.36}$$

This allows to obtain that, under the same hypotheses, the weak solution u of problem (3.20) is in $L^{m^{**}}(\Omega)$. Indeed, there exists a positive constant

$C = C(N, m, \alpha, S, \sigma_o^N[E])$ such that

$$\|u\|_{L^{m^{**}}(\Omega)}^{m^{**}} \leq C(\|E\|_{L^2(\Omega)}^{\frac{2}{2^*}} + \|f\|_{L^1(\Omega)}^{\frac{2}{2^*}} + \|f\|_{L^m(\Omega)}^{\frac{2^*}{2-2^*/m'}} + 1). \tag{3.37}$$

Finally, by approximation, one gets the regularity result (i) for u. Namely, assuming (3.21), (3.22), $|E| \in L^2(\Omega) \cap M_0^N(\Omega)$ and $f \in L^1(\Omega) \cap L^m(\Omega)$, $\dfrac{2N}{N+2} \leq m < \dfrac{N}{2}$, the weak solution u of (3.20) belongs to $W_0^{1,2}(\Omega) \cap L^{m^{**}}(\Omega)$. More precisely, there exists a positive constant $C = C\left(N, m, \alpha, S, \sigma_o^N[E]\right)$ such that

$$\|u\|_{L^{m^{**}}(\Omega)}^{m^{**}} \leq C(\|E\|_{L^2(\Omega)}^{\frac{2}{2^*}} + \|f\|_{L^1(\Omega)}^{\frac{2}{2^*}} + \|f\|_{L^m(\Omega)}^{\frac{2^*}{2-2^*/m'}} + 1). \tag{3.38}$$

To show (ii), the authors follow Stampacchia's method ([27], see also [3]) based on the boundedness of the function $\log(1+|u|)$. In particular, assuming (3.21), (3.22), if $|E| \in L^2(\Omega) \cap L^r(\Omega)$, $r > N$, and $f \in L^1(\Omega) \cap L^m(\Omega)$, $m > \dfrac{N}{2}$, then the weak solution u of (3.20) belongs to $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.
The proof is done defining the function

$$G(t) = \begin{cases} 0, & \text{if } |t| \leq l, \\ \dfrac{t}{1+t} - \dfrac{l}{1+l}, & \text{if } t > l, \\ \dfrac{t}{1-t} + \dfrac{l}{1+l}, & \text{if } t < -l, \end{cases}$$

with $l \in \mathbb{R}_+$. Namely, let $u \in W_0^{1,2}(\Omega)$ be the solution of (3.20). In view of Lemma 3.2 for $p = 2$, one takes $G(u)$ as test function in the variational formulation of (3.20) and gets that

$$\int\limits_{u>l} M(x)\nabla u\cdot\nabla\Big(\frac{u}{1+u}\Big)+\int\limits_{u<-l} M(x)\nabla u\cdot\nabla\Big(\frac{u}{1-u}\Big)$$

$$+\mu\int\limits_{u>l} u\Big(\frac{u}{1+u}-\frac{l}{1+l}\Big)+\mu\int\limits_{u<-l} u\Big(\frac{u}{1-u}+\frac{l}{1+l}\Big)$$

$$=\int\limits_{u>l} uE(x)\;\cdot\nabla\Big(\frac{u}{1+u}\Big)+\int\limits_{u<-l} uE(x)\;\cdot\nabla\Big(\frac{u}{1-u}\Big)$$

$$+\int\limits_{u>l} f\Big(\frac{u}{1+u}-\frac{l}{1+l}\Big)+\int\limits_{u<-l} f\Big(\frac{u}{1-u}+\frac{l}{1+l}\Big).$$

They obtain the result thanks to some known inequalities and to Lemma 3.3.

Chapter 4

Existence results for noncoercive elliptic equations in unbounded domains: the nonlinear case

In this chapter we consider a nonlinear version of the noncoercive boundary value problem analysed by Sara Monsurrò and Maria Transirico in [24], where the domain Ω is still supposed to be unbounded. These results are contained in the recent paper by Emilia Anna Alfano and Sara Monsurrò [2].

Let Ω be an unbounded open subset of $\mathbb{R}^N$, $N > 2$. We consider the following Dirichlet problem

$$\begin{cases} -\mathrm{div}(b(x)|\nabla u|^{p-2}\nabla u) + \mu|u|^{p-2}u = \\ \\ -\mathrm{div}(|u|^{p-2}u\, E(x)) + f(x) \quad \text{in } \Omega, \\ \\ u \in W_0^{1,p}(\Omega), \end{cases} \tag{4.1}$$

where

$$1 < p < N, \tag{4.2}$$

$$\alpha \le b(x) \le \beta, \quad \text{for some } 0 < \alpha \le \beta,\ \text{a.e. } x \in \Omega, \tag{4.3}$$

$$\mu > 0, \tag{4.4}$$

$$|E| \in L^{p'}(\Omega) \cap M_0^{\frac{N}{p-1}}(\Omega) \tag{4.5}$$

and

$$f \in L^1(\Omega) \cap L^m(\Omega),\ m \ge (p^*)', \tag{4.6}$$

where by $(\ )^*$ we denote the Sobolev conjugate of $(\ )$ and by $(\ )'$ the Hölder conjugate of $(\ )$. Therefore, one has:

- $p' = \dfrac{p}{p-1}$;
- $p^* = \dfrac{Np}{N-p}$;
- $(p^*)' = \dfrac{Np}{Np-N+p}$.

We emphasize the presence of the noncoercive operator $-\mathrm{div}(b(x)|\nabla u|^{p-2}\nabla u)+\mathrm{div}(|u|^{p-2}u\,E(x))$ where on the second term no smallness assumptions are done. Due to the unboundedness of the domain, the hypothesis (4.4) is necessary (as we see in Section 3.1). Despite this, since μ is not required to be large enough, the operator in (4.1) still remains noncoercive.
The idea is to extend the results of [24] to the nonlinear case and to generalize the existence result of [4] to the case when Ω is unbounded. In order to obtain the existence of a solution of our problem, inspired by an idea of [3], we approximate noncoercive nonlinear problem (4.1) by coercive nonlinear problems and then we pass to the limit. Differently from [4] where the existence of the solutions of the approximate problems is immediately obtained thanks to the Schauder fixed point Theorem, here it is done by means of the Surjectivity Theorem. Due to the assumption (4.5) on the coefficient appearing in the noncoercive term and thanks to a compactness result in $M_0^N(\Omega)$ proved in [30] (see Theorem 1.10), it is finally possible to pass to the limit.
For related problems on bounded domains we quote here [5, 7, 8, 13, 25, 32] while for linear coercive problems on unbounded domains we refer the reader to [18, 19, 20, 21, 22, 23].

4.1 A coercive approximate problem

Let us start by proving a useful property, needed in the sequel.

Proposition 4.1. *Let $p > 2$. Then*

$$\left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right)(\xi - \eta) > 0 \qquad \forall \xi, \eta \in \mathbb{R}^N \textit{ with } \xi \neq \eta. \tag{4.7}$$

PROOF. Let $\xi \neq \eta$. If $|\xi| = |\eta|$, then

$$\left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right)(\xi - \eta) = |\xi|^{p-2}(\xi - \eta)^2 > 0.$$

Hence (4.7) holds.

Let us now consider the case $|\xi| \neq |\eta|$.

$$\left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right)(\xi - \eta)$$

$$= |\xi|^p - |\xi|^{p-2} < \xi, \eta > -|\eta|^{p-2} < \eta, \xi > +|\eta|^p$$

$$= |\xi|^p + |\eta|^p - \left(|\xi|^{p-2} + |\eta|^{p-2}\right) < \xi, \eta > .$$

Since by Young inequality $| < \xi, \eta > | \leq |\xi| \; |\eta| \leq \frac{1}{2} \; |\xi|^2 + \frac{1}{2} \; |\eta|^2$ and $- < \xi, \eta > \geq -| < \xi, \eta > |$,

$$|\xi|^p + |\eta|^p - \left(|\xi|^{p-2} + |\eta|^{p-2}\right) < \xi, \eta >$$

$$\geq |\xi|^p + |\eta|^p - \left(|\xi|^{p-2} + |\eta|^{p-2}\right)\left(\frac{1}{2} \; |\xi|^2 + \frac{1}{2} \; |\eta|^2\right)$$

$$= \frac{1}{2}\left(|\xi|^2 - |\eta|^2\right)\left(|\xi|^{p-2} - |\eta|^{p-2}\right) > 0.$$

This gives (4.7). $\square$

We will prove the existence of a weak solution of problem (4.1) following an idea of [3] and [4], inspired from the papers [27, 28] by Guido Stampacchia and from [9, 10, 12].

We consider the following class of nonlinear coercive approximate problems

$$\begin{cases} -\mathrm{div}(b(x)|\nabla u_n|^{p-2}\nabla u_n) + \mu|u_n|^{p-2}u_n = \\ \\ -\mathrm{div}\left(\dfrac{|u_n|^{p-2}u_n}{1+\frac{1}{n}|u_n|^{p-1}} \dfrac{E(x)}{1+\frac{1}{n}|E(x)|}\right) + \dfrac{f}{1+\frac{1}{n}|f|}, \\ \\ u_n \in W_0^{1,p}(\Omega). \end{cases} \tag{4.8}$$

We start proving, in Theorem 4.5, that a weak solution u_n of (4.8) exists, for every fixed $n \in \mathbb{N}$. Then, we show, in Theorem 4.6, that this solution of (4.8) is also bounded. The proofs are done for $n = 1$, but they are analogous for $n \geq 2$.

The existence of a solution of approximate problems (4.8) will be proved by means of the following Surjectivity Theorem (see also [6]).

Theorem 4.2 (Surjectivity)**.** *Let V be a reflexive and separable Banach space. Let the operator $A : V \to V'$ be*

1. *coercive, i.e.*
$$\frac{< A(u), u >}{\|u\|} \to +\infty, \quad \|u\| \to +\infty;$$

2. *pseudomonotone, i.e.*

i) A is bounded (it trasforms bounded sets of V in bounded sets of V');

ii) if $u_n \rightharpoonup u$ weakly in V and $\limsup_{n\to+\infty} < A(u_n), u_n - u > \leq 0$, then $\liminf_{n\to+\infty} < A(u_n), u_n - v > \geq < A(u), u - v >$ for all v in V.

Then A is surjective, i.e. for every f in V' there exists u in V such that $A(u) = f$.

To our aim, some further preliminary results are needed. In particular, we recall the next lemma, proved in [6] in the case of bounded domains, that remains valid also in the case of unbounded sets (cfr. Theorem 2.1 of [6]).

Lemma 4.3. *Let $p > 1$, $\{f_n\}$ be a sequence of functions in $L^p(\Omega)$ and f be a function in $L^p(\Omega)$. Assume that*

1. *$\{f_n\}$ is uniformly bounded in $L^p(\Omega)$;*

2. *$f_n \to f$ a.e. in Ω.*

Then $f_n \rightharpoonup f$ weakly in $L^p(\Omega)$.

Now, we prove a preliminary lemma, useful in the sequel.

Lemma 4.4. *Let $p > 1$ and u_n, $u \in W_0^{1,p}(\Omega)$. Under hypotheses (4.3) and (4.4) and if*

$$b(x)\left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)\nabla(u_n - u)$$

$$+\mu\left(|u_n|^{p-2}u_n - |u|^{p-2}u\right)(u_n - u) \to 0 \quad a.e. \text{ in } \Omega,$$

then

$$\nabla u_n \to \nabla u \quad a.e. \;\; in \; \Omega \tag{4.9}$$

and

$$u_n \to u \quad a.e. \;\; in \; \Omega. \tag{4.10}$$

Proof. Let us start observing that (4.7) holds. Now, since

$$b(x)\left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)\nabla(u_n - u)$$

$$+\mu\left(|u_n|^{p-2}u_n - |u|^{p-2}u\right)(u_n - u) \to 0 \;\; \text{a.e. in } \Omega,$$

by (4.3), (4.4) and (4.7), one gets

$$b(x)\left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)\nabla(u_n - u) \to 0 \;\; \text{a.e. in } \Omega, \tag{4.11}$$

$$\mu\left(|u_n|^{p-2}u_n - |u|^{p-2}u\right)(u_n - u) \to 0 \;\; \text{a.e. in } \Omega. \tag{4.12}$$

By (4.11) there exists $c(x)$ such that

$$\left|b(x)\left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)\nabla(u_n - u)\right| \leq c(x), \tag{4.13}$$

up to a set of null measure Z.

We want to show that there exists a function C such that, in $\Omega \setminus Z$, one has

$$|\nabla u_n(x)| \leq C(x). \tag{4.14}$$

Indeed, by (4.3) and (4.7), from (4.13) we get

$$c(x) \geq b(x)\left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)\nabla(u_n - u)$$

$$\geq \alpha|\nabla u_n|^p - \beta|\nabla u_n|^{p-1}|\nabla u| - \beta|\nabla u|^{p-1}|\nabla u_n| + \alpha|\nabla u|^p.$$

Hence, since on the right-hand side we have a polynomial in $|\nabla u_n|$, we get (4.14).

Let us now prove (4.9). By contradiction, assume that there exists $x_0 \in \Omega \setminus Z$ such that $\nabla u_n(x_0)$ does not converge to $\nabla u(x_0)$.

In view of (4.14) and the Bolzano-Weierstrass Theorem, up to a subsequence, one has $\nabla u_{n_k}(x_0) \to \zeta \in \mathbb{R}^N$.

Then, passing to the limit in (4.11), we get

$$b(x_0)\left(|\zeta|^{p-2}\zeta - |\nabla u(x_0)|^{p-2}\nabla u(x_0)\right)(\zeta - \nabla u(x_0)) = 0.$$

Therefore, by (4.3) and (4.7), $\zeta = \nabla u(x_0)$. This gives (4.9).

Following a similar argument, by (4.12) one gets (4.10). □

We are now in a position to prove the existence of a weak solution of the approximate problems. We give the proof just for $n = 1$, the cases $n > 2$ being completely analogous.

Theorem 4.5. *Assume (4.2), (4.3), (4.4), (4.5) and $f \in L^1(\Omega)$. Then there* *exists a weak solution u of the following problem*

$$
\begin{cases}
-\operatorname{div}(b(x)|\nabla u|^{p-2}\nabla u) + \mu|u|^{p-2}u = \\[2ex]
-\operatorname{div}\left(\dfrac{|u|^{p-2}u}{1+|u|^{p-1}}\,\dfrac{E(x)}{1+|E(x)|}\right) + \dfrac{f}{1+|f|}, \\[2ex]
u \in W_0^{1,p}(\Omega).
\end{cases}
\tag{4.15}
$$

Proof. We want to apply Theorem 4.2 to the following operator:

$$
A : u \in W_0^{1,p}(\Omega) \to -\operatorname{div}\left(b(x)|\nabla u|^{p-2}\nabla u\right) + \mu|u|^{p-2}u
$$

$$
+\operatorname{div}\left(\frac{|u|^{p-2}u}{1+|u|^{p-1}}\frac{E(x)}{1+|E(x)|}\right) - \frac{f}{1+|f|} \in W^{-1,p}(\Omega).
$$

Therefore we have to prove that A is coercive and pseudomonotone.

Step 1. A is a coercive opearator. Indeed,

$$
< A(u), u >
$$

$$
\geq \int_\Omega b(x)|\nabla u|^p + \mu\int_\Omega |u|^p - \int_\Omega \frac{|u|^{p-1}}{1+|u|^{p-1}}\frac{|E(x)|}{1+|E(x)|}|\nabla u| - \int_\Omega \frac{|f||u|}{1+|f|}
$$

$$
\geq c\|u\|^p_{W^{1,p}(\Omega)} - \|E\|_{L^{p'}(\Omega)}\|u\|_{W^{1,p}(\Omega)} - \left\|\frac{f}{1+|f|}\right\|_{L^{p'}(\Omega)}\|u\|_{W^{1,p}(\Omega)}
$$

$$= \left(c\|u\|^{p-1}_{W^{1,p}(\Omega)} - \|E\|_{L^{p'}(\Omega)} - \left\| \frac{f}{1+|f|} \right\|_{L^{p'}(\Omega)} \right) \|u\|_{W^{1,p}(\Omega)}$$

where the constant $c = \min\{\alpha, \mu\}$.

Step 2. A is a bounded operator. Indeed,

$$< A(u), v >= \int_\Omega b(x)|\nabla u|^{p-2}\nabla u \nabla v + \mu \int_\Omega |u|^{p-2}uv$$

$$- \int_\Omega \frac{|u|^{p-2}u}{1+|u|^{p-1}} \frac{E(x)}{1+|E(x)|} \nabla v - \int_\Omega \frac{f}{1+|f|} v$$

$$\leq \beta \|\nabla u\|^{p-1}_{L^p(\Omega)} \|\nabla v\|_{L^p(\Omega)} + \mu \|u\|^{p-1}_{L^p(\Omega)} \|v\|_{L^p(\Omega)}$$

$$+ \|E\|_{L^{p'}(\Omega)} \|\nabla v\|_{L^p(\Omega)} + \left\| \frac{f}{1+|f|} \right\|_{L^{p'}(\Omega)} \|v\|_{L^p(\Omega)}$$

$$\leq \left(C\|u\|^{p-1}_{W^{1,p}(\Omega)} + \|E\|_{L^{p'}(\Omega)} + \left\| \frac{f}{1+|f|} \right\|_{L^{p'}(\Omega)} \right) \|v\|_{W^{1,p}(\Omega)}$$

where $C = \max\{\beta, \mu\}$.

Step 3. Let

$$u_n \rightharpoonup u \text{ weakly in } W_0^{1,p}(\Omega) \tag{4.16}$$

and

$$\limsup_{n\to+\infty} < A(u_n), u_n - u > \leq 0, \tag{4.17}$$

we must show that

$$\liminf_{n\to+\infty} < A(u_n), u_n - w > \geq < A(u), u - w > \quad \forall w \in W_0^{1,p}(\Omega). \tag{4.18}$$

To this aim, we start proving that

$$\lim_{n\to+\infty} \left[\int_\Omega b(x) \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u \right) \nabla(u_n - u) \right.$$
$$\left. + \mu \int_\Omega \left(|u_n|^{p-2}u_n - |u|^{p-2}u \right)(u_n - u) \right] = 0. \tag{4.19}$$

Observe that

$$\limsup_{n\to+\infty} < A(u_n), u_n - u >$$

$$= \limsup_{n\to+\infty} \left[\int_\Omega b(x)|\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - u) + \mu \int_\Omega |u_n|^{p-2}u_n(u_n - u) \right.$$

$$\left. - \int_\Omega \frac{|u_n|^{p-2}u_n}{1 + |u_n|^{p-1}} \frac{E(x)}{1 + |E(x)|} \nabla(u_n - u) - \int_\Omega \frac{f}{1 + |f|}(u_n - u) \right].$$

Now, by (4.16)

$$\lim_{n\to+\infty} \left[\int_\Omega b(x)|\nabla u|^{p-2}\nabla u \nabla(u_n - u) + \mu \int_\Omega |u|^{p-2}u(u_n - u) \right] = 0. \tag{4.20}$$

Therefore

$$\limsup_{n\to+\infty} < A(u_n), u_n - u >$$

$$= \limsup_{n\to+\infty} \left[\int_\Omega b(x)(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)\nabla(u_n - u) \right.$$

$$+\mu \int_\Omega (|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u) \tag{4.21}$$

$$\left. - \int_\Omega \frac{|u_n|^{p-2}u_n}{1+|u_n|^{p-1}} \frac{E(x)}{1+|E(x)|} \nabla(u_n - u) - \int_\Omega \frac{f}{1+|f|}(u_n - u) \right].$$

Let us prove that

$$\lim_{n\to+\infty} \int_\Omega \frac{|u_n|^{p-2}u_n}{1+|u_n|^{p-1}} \frac{E(x)}{1+|E(x)|} \nabla(u_n - u) = 0. \tag{4.22}$$

Indeed, arguing as before, by (4.16) one has

$$\lim_{n\to+\infty} \int_\Omega \frac{|u|^{p-2}u}{1+|u|^{p-1}} \frac{E(x)}{1+|E(x)|} \nabla(u_n - u) = 0. \tag{4.23}$$

Thus

$$\lim_{n\to+\infty} \int_\Omega \frac{|u_n|^{p-2}u_n}{1+|u_n|^{p-1}} \frac{E(x)}{1+|E(x)|} \nabla(u_n - u)$$

$$= \lim_{n\to+\infty} \int_\Omega \left(\frac{|u_n|^{p-2}u_n}{1+|u_n|^{p-1}} - \frac{|u|^{p-2}u}{1+|u|^{p-1}} \right) \frac{E(x)}{1+|E(x)|} \nabla(u_n - u). \tag{4.24}$$

Now, by the compactness result stated in Theorem 1.10, $|E(x)|^{\frac{1}{p-1}}u_n \to |E(x)|^{\frac{1}{p-1}}u$ in $L^p(\Omega)$ and $u_n \to u$ a.e., up to a subsequence, hence

$$\frac{|u_n|^{p-2}u_n}{1+|u_n|^{p-1}}\frac{E(x)}{1+|E(x)|} \to \frac{|u|^{p-2}u}{1+|u|^{p-1}}\frac{E(x)}{1+|E(x)|} \quad \text{a.e. in } \Omega.$$

Furthermore

$$\frac{|u_n|^{p-1}}{1+|u_n|^{p-1}}\frac{|E(x)|}{1+|E(x)|} \leq |E(x)| \in L^{p'}(\Omega).$$

Thus the Lebesgue dominated convergence Theorem applies and we get that

$$\frac{|u_n|^{p-2}u_n}{1+|u_n|^{p-1}}\frac{E(x)}{1+|E(x)|} \to \frac{|u|^{p-2}u}{1+|u|^{p-1}}\frac{E(x)}{1+|E(x)|} \text{ in } L^{p'}(\Omega). \tag{4.25}$$

Thus, thanks to (4.16), (4.24) and (4.25), we get (4.22).

By (4.16), (4.17), (4.21) and (4.22) we obtain

$$\limsup_{n\to+\infty}\Bigg(\int_\Omega b(x)(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)\nabla(u_n-u)$$

$$+\mu\int_\Omega(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n-u)\Bigg) \leq 0.$$

Furthermore, by (4.3), (4.4) and (4.7),

$$\int_\Omega b(x)(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)\nabla(u_n-u)$$

$$+\mu\int_\Omega(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n-u) \geq 0$$

and hence (4.19) holds.

By (4.3), (4.4), (4.7) and (4.19), we deduce that

$$\begin{gathered} b(x)\left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)\nabla(u_n - u) \\ +\mu\left(|u_n|^{p-2}u_n - |u|^{p-2}u\right)(u_n - u) \to 0 \ \text{ in } L^1(\Omega). \end{gathered} \tag{4.26}$$

Now, we want to prove (4.18).

Let u_{n_k} be the subsequence of u_n such that

$$\liminf_{n\to+\infty} < A(u_n), u_n - w >= \lim_{k\to+\infty} < A(u_{n_k}), u_{n_k} - w > . \tag{4.27}$$

Let us observe that (4.26) clearly holds with u_{n_k} in place of u_n, hence there exists $u_{n_{k_m}}$ such that

$$b(x)\left(|\nabla u_{n_{k_m}}|^{p-2}\nabla u_{n_{k_m}} - |\nabla u|^{p-2}\nabla u\right)\nabla(u_{n_{k_m}} - u)$$

$$+\mu\left(|u_{n_{k_m}}|^{p-2}u_{n_{k_m}} - |u|^{p-2}u\right)(u_{n_{k_m}} - u) \to 0 \ \text{ a.e. in } \Omega.$$

Hence, by Lemma 4.4

$$\begin{gathered} u_{n_{k_m}} \to u \text{ a.e. in } \Omega \\ \nabla u_{n_{k_m}} \to \nabla u \text{ a.e. in } \Omega. \end{gathered} \tag{4.28}$$

By (4.27)

$$\liminf_{n\to+\infty} < A(u_n), u_n - w >= \lim_{m\to+\infty} < A(u_{n_{k_m}}), u_{n_{k_m}} - w >$$

$$= \lim_{m\to+\infty} \Bigg(\int_\Omega b(x)|\nabla u_{n_{k_m}}|^p + \mu \int_\Omega |u_{n_{k_m}}|^p - \int_\Omega \frac{|u_{n_{k_m}}|^{p-2}u_{n_{k_m}}}{1+|u_{n_{k_m}}|^{p-1}} \frac{E(x)}{1+|E(x)|} \nabla u_{n_{k_m}}$$

$$- \int_\Omega \frac{f}{1+|f|} u_{n_{k_m}} - \int_\Omega b(x)|\nabla u_{n_{k_m}}|^{p-2}\nabla u_{n_{k_m}} \nabla w - \mu \int_\Omega |u_{n_{k_m}}|^{p-2}u_{n_{k_m}} w$$

$$+ \int_\Omega \frac{|u_{n_{k_m}}|^{p-2}u_{n_{k_m}}}{1+|u_{n_{k_m}}|^{p-1}} \frac{E(x)}{1+|E(x)|} \nabla w + \int_\Omega \frac{f}{1+|f|} w \Bigg).$$

Passing to the limit as $m \to +\infty$ in the right-hand side, by (4.16), (4.25), (4.28), the Fatou Lemma, the Lebesgue dominated convergence Theorem and Theorem 4.3 give

$$\liminf_{n\to+\infty} < A(u_n), u_n - w > \geq \int_\Omega b(x)|\nabla u|^p + \mu \int_\Omega |u|^p - \int_\Omega \frac{|u|^{p-2}u}{1+|u|^{p-1}} \frac{E(x)}{1+|E(x)|} \nabla u$$

$$- \int_\Omega \frac{f}{1+|f|} u - \int_\Omega b(x)|\nabla u|^{p-2}\nabla u \nabla w - \mu \int_\Omega |u|^{p-2}uw$$

$$+ \int_\Omega \frac{|u|^{p-2}u}{1+|u|^{p-1}} \frac{E(x)}{1+|E(x)|} \nabla w + \int_\Omega \frac{f}{1+|f|} w =< A(u), u - w >,$$

hence (4.18) holds.

This concludes our proof. □

The last theorem of this section is an essential tool to prove our main result, obtained following some techniques of [17, 27].

Theorem 4.6. *Assume (4.2), (4.3), (4.4), (4.5) and (4.6). Then every solution u of problem (4.15) is of class $L^\infty(\Omega)$.*

PROOF. In order to prove the boundedness of u, take $G_k(u)$ as test function in the variational formulation of (4.15) (this is allowed by (3.7)). Then, by (4.3), (3.2), (3.11), (3.12), (3.13), Hölder and Sobolev inequalities, one gets

$$\alpha \int_\Omega |\nabla G_k(u)|^p + \mu \int_\Omega |G_k(u)|^p$$

$$\leq \int_{A_k} \left| \frac{E(x)}{1+|E(x)|} \right| |\nabla G_k(u)| + \int_{A_k} \left| \frac{f}{1+|f|} \right| |G_k(u)|$$

$$\leq \left\| \frac{E}{1+|E|} \right\|_{L^{p'}(A_k)} \|\nabla G_k(u)\|_{L^p(\Omega)} + \left\| \frac{f}{1+|f|} \right\|_{L^{(p^*)'}(A_k)} \| G_k(u)\|_{L^{p^*}(\Omega)}$$

$$\leq \left(\left\| \frac{E}{1+|E|} \right\|_{L^{p'}(A_k)} + \frac{1}{S} \left\| \frac{f}{1+|f|} \right\|_{L^{(p^*)'}(A_k)} \right) \|\nabla G_k(u)\|_{L^p(\Omega)}$$

$$\leq \left(\left\| \frac{E}{1+|E|} \right\|_{L^{q}(\Omega)} |A_k|^{\frac{1}{p'}-\frac{1}{q}} + \frac{1}{S} \left\| \frac{f}{1+|f|} \right\|_{L^{q}(\Omega)} |A_k|^{\frac{1}{(p^*)'}-\frac{1}{q}} \right) \|\nabla G_k(u)\|_{L^p(\Omega)}$$

and where, to our aim, we take $q > \dfrac{N}{p-1}$ $(> p' > (p^*)')$ and with $S = S(N,p)$ Sobolev constant as in Theorem 3.17 of [6].

Whence, using again Sobolev inequalities and (4.4), one has

$$\|G_k(u)\|^{p-1}_{L^{p^*}(\Omega)} \leq C(|A_k|^{\frac{1}{p'}-\frac{1}{q}} + |A_k|^{\frac{1}{(p^*)'}-\frac{1}{q}}),$$

with $C = C\left(N, p, \alpha, \left\|\frac{E}{1+|E|}\right\|_{L^q(\Omega)}, \left\|\frac{f}{1+|f|}\right\|_{L^q(\Omega)}\right)$.

Observe that since $|A_k| \to 0$, as $k \to +\infty$, we can assume that there exists

$k_0 \in \mathbb{R}_+$ such that $|A_k| \leq 1$, for $k \geq k_0$. Moreover, since $\dfrac{1}{p'} - \dfrac{1}{q} < \dfrac{1}{(p^*)'} - \dfrac{1}{q}$, we get

$$\|G_k(u)\|_{L^{p^*}(\Omega)} \leq C'|A_k|^{\left(\frac{1}{p'}-\frac{1}{q}\right)\frac{1}{p-1}}, \forall k \geq k_0, \tag{4.29}$$

with $C' = C'\left(N, p, \alpha, \left\|\frac{E}{1+|E|}\right\|_{L^q(\Omega)}, \left\|\frac{f}{1+|f|}\right\|_{L^q(\Omega)}\right)$.

Now, by (3.2) and (3.9),

$$h|A_h|^{\frac{1}{p^*}} = \left(\int_{A_h} |h|^{p^*}\right)^{\frac{1}{p^*}} \leq \|u\|_{L^{p^*}(A_h)} \leq \|G_k(u)\|_{L^{p^*}(A_h)} + k|A_h|^{\frac{1}{p^*}}.$$

Thus

$$(h-k)|A_h|^{\frac{1}{p^*}} \leq \|G_k(u)\|_{L^{p^*}(A_h)}. \tag{4.30}$$

Putting together (4.29) and (4.30), we obtain

$$|A_h| \leq C'' \frac{|A_k|^{\left(\frac{1}{p'}-\frac{1}{q}\right)\frac{p^*}{p-1}}}{(h-k)^{p^*}} \quad \forall h > k \geq k_0,$$

with $C'' = C''\left(N, p, \alpha, \left\|\frac{E}{1+|E|}\right\|_{L^q(\Omega)}, \left\|\frac{f}{1+|f|}\right\|_{L^q(\Omega)}\right)$.
Finally, as a consequence of the fact that $q > \dfrac{N}{p-1}$, one has $\left(\dfrac{1}{p'} - \dfrac{1}{q}\right)\dfrac{p^*}{p-1} > 1$, therefore Lemma 3.3 applies and there exists $d \in \mathbb{R}_+$ such that $|A_{k_0+d}| = 0$, thus $u \in L^\infty(\Omega)$. □

4.2 Existence result

In this section we finally achieve the existence of a weak solution of problem (4.1).

Lemma 4.7. *Assume (4.2), (4.3), (4.4), (4.5) and* $f \in L^1(\Omega)$*, and let* u_n *be a solution of (4.8). Then, for any* $k \in \mathbb{R}_+$*, the sequence* $\{T_k(u_n)\}$ *is bounded in* $W_0^{1,p}(\Omega)$*. More precisely we have:*

$$\frac{\alpha}{2}\int_\Omega |\nabla T_k(u_n)|^p + \mu\int_\Omega |T_k(u_n)|^p \le C\ k^p \int_\Omega |E|^{p'} + k\int_\Omega |f|, \tag{4.31}$$

where $C = C(p, \alpha)$.

PROOF. Let us take $T_k(u_n)$ as test function in the variational formulation of (4.8), this can be done in view of (3.3). We have

$$\int_\Omega b(x)|\nabla u_n|^{p-2}\nabla u_n \nabla T_k(u_n) + \mu\int_\Omega |u_n|^{p-2}u_n T_k(u_n)$$

$$= \int_\Omega \frac{|u_n|^{p-2}u_n}{1+\frac{1}{n}|u_n|^{p-1}}\frac{E(x)}{1+\frac{1}{n}|E(x)|}\nabla T_k(u_n) + \int_\Omega \frac{f}{1+\frac{1}{n}|f|}T_k(u_n)\,.$$

In view of (4.3), (3.4), (3.5), (3.6) and by Young inequality we get

$$\alpha \int\limits_{\Omega} |\nabla T_k(u_n)|^p + \mu \int\limits_{\Omega} |T_k(u_n)|^p$$

$$\leq \int\limits_{\Omega} |T_k(u_n)|^{p-1} \, |E(x)| \, |\nabla T_k(u_n)| + \int\limits_{\Omega} |f| \, |T_k(u_n)|$$

$$\leq \frac{\alpha}{2} \int\limits_{\Omega} |\nabla T_k(u_n)|^p + C(p,\alpha) \int\limits_{\Omega} |T_k(u_n)|^p \, |E(x)|^{\frac{p}{p-1}} + \int\limits_{\Omega} |f| \, |T_k(u_n)|.$$

Therefore, in view of (2.10), (4.31) follows. □

Lemma 4.8. *Assume (4.2), (4.3), (4.4), (4.5) and $f \in L^1(\Omega)$. Then every solution u_n of (4.8) satisfies*

$$\left[\int\limits_{\Omega} |\log(1+|u_n|)|^{p^*}\right]^{\frac{p}{p^*}} \leq C\left(\int\limits_{\Omega} |E|^{p'} + \int\limits_{\Omega} |f|\right), \tag{4.32}$$

where $C = C(N,p,\alpha)$.

Proof. In view of Lemma 3.2, we can take $\dfrac{1}{p-1}\left[1 - \dfrac{1}{(1+|u_n|)^{p-1}}\right] \operatorname{sign}(u_n)$ as test function in (4.8).

Now, observe that $\frac{1}{1+\frac{1}{n}|u_n|^{p-1}} \leq 1$, $\frac{|u_n|^{p-1}}{(1+|u_n|)^{p-1}} \leq 1$, $\frac{1}{1+\frac{1}{n}|E(x)|} \leq 1$, $\frac{1}{1+\frac{1}{n}|f|} \leq 1$ and
$\left|\left[1 - \dfrac{1}{(1+|u_n|)^{p-1}}\right] \operatorname{sign}(u_n)\right| \leq 1$, hence, using (4.3) and (4.4), we have

$$\alpha \int\limits_{\Omega} \frac{|\nabla u_n|^p}{(1+|u_n|)^p} \leq \int\limits_{\Omega} \frac{|E(x)||\nabla u_n|}{1+|u_n|} + \frac{1}{p-1} \int\limits_{\Omega} |f| \, .$$

Hence, in view of Young inequality, we get

$$\frac{\alpha}{2}\int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^p} \le C'(p,\alpha)\int_\Omega |E(x)|^{p'} + \frac{1}{p-1}\int_\Omega |f|,$$

which implies, by Sobolev inequality,

$$\frac{S^p\alpha}{2}\left[\int_\Omega |\log(1+|u_n|)|^{p^*}\right]^{\frac{p}{p^*}} \le \frac{\alpha}{2}\int_\Omega |\nabla \log(1+|u_n|)|^p$$

$$= \frac{\alpha}{2}\int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^p} \le C'(p,\alpha)\int_\Omega |E(x)|^{p'} + \frac{1}{p-1}\int_\Omega |f|,$$

which gives (4.32). □

Remark 4. *Remark that, thanks to the estimate (4.32), one has*

$$\text{meas}\left\{x \in \Omega : |u_n(x)| > k\right\}^{p/p^*} \le \frac{C}{|\log(1+k)|^p}\left(\int_\Omega |E|^{p'} + \int_\Omega |f|\right). \quad (4.33)$$

Thus, for any $\epsilon > 0$, it is possible to choose k_ϵ such that

$$\text{meas}\left\{x \in \Omega : |u_n(x)| > k\right\}^{p/p^*} \le \epsilon,\ \forall\, k > k_\epsilon, \forall\, n \in \mathbb{N}. \quad (4.34)$$

□

Lemma 4.9. *Assume (4.2), (4.3), (4.4), (4.5) and (4.6), and let u_n be a solution of (4.8). Then there exists $k^* \in \mathbb{R}_+$, with $k^* = k^*(N, p, \sigma_o{}^{\frac{N}{p-1}}[E])$, such that the sequence $\{G_k(u_n)\}$ is bounded in $W_0^{1,p}(\Omega)$, for every $k > k^*$.*

More precisely we have:

$$\int\limits_{\Omega} |\nabla G_k(u_n)|^p + \int\limits_{\Omega} |G_k(u_n)|^p \le C\left(k^p \|E\|^{p'}_{L^{p'}(\Omega)} + \|f\|^{p'}_{L^{(p^*)'}(\Omega)}\right), \tag{4.35}$$

where $C = C(N, p, \alpha, \mu)$.

Proof. Let $k \in \mathbb{R}_+$ and $n \in \mathbb{N}$, define

$$A_n(k) = \{x \in \Omega : k < |u_n(x)|\}.$$

The use of $G_k(u_n)$ as test function in the variational formulation of (4.8) (that can be done in view of (3.7)), (4.3), (3.10), (3.11) and (3.12) give that

$$\alpha \int\limits_{\Omega} |\nabla G_k(u_n)|^p + \mu \int\limits_{\Omega} |G_k(u_n)|^p$$

$$\le c'\Bigg(\int\limits_{\Omega} |G_k(u_n)|^{p-1}|E(x)||\nabla G_k(u_n)| \tag{4.36}$$

$$+k^{p-1} \int\limits_{\Omega} |E(x)||\nabla G_k(u_n)| + \int\limits_{\Omega} |G_k(u_n)||f|\Bigg),$$

with $c' = c'(p)$.

By (4.5), (3.13), Hölder inequality and (1.36) of Theorem 1.10, we get that

By (4.5), (3.13), Hölder inequality and (1.36) of Lemma 1.10, we get that

$$\int_\Omega |G_k(u_n)|^{p-1}|E(x)||\nabla G_k(u_n)|$$

$$\leq \left(\int_{A_n(k)} |E(x)|^{p'}|G_k(u_n)|^p\right)^{\frac{1}{p'}} \left(\int_\Omega |\nabla G_k(u_n)|^p\right)^{\frac{1}{p}} \tag{4.37}$$

$$\leq c''\|E\|_{M^{\frac{N}{p-1}}(A_n(k))}\|G_k(u_n)\|^{p-1}_{W^{1,p}(\Omega)}\|\nabla G_k(u_n)\|_{L^p(\Omega)}$$

$$\leq c''\|E\|_{M^{\frac{N}{p-1}}(A_n(k))}\|G_k(u_n)\|^{p}_{W^{1,p}(\Omega)},$$

with $c'' = c''(N,p)$.

Therefore, by (4.36), (4.37) and Young, Hölder and Sobolev inequalities, one has that, for $\epsilon > 0$,

$$\alpha\int_\Omega |\nabla G_k(u_n)|^p + \mu\int_\Omega |G_k(u_n)|^p$$

$$\leq c'''\Bigg[\|E\|_{M^{\frac{N}{p-1}}(A_n(k))}\left(\int_\Omega |\nabla G_k(u_n)|^p + \int_\Omega |G_k(u_n)|^p\right)$$

$$+\epsilon\int_\Omega |\nabla G_k(u_n)|^p + \frac{k^p}{(\epsilon p)^{\frac{1}{p-1}}p'}\|E\|^{p'}_{L^{p'}(\Omega)}$$

$$+\epsilon\int_\Omega |\nabla G_k(u_n)|^p + \frac{1}{S^{p'}(\epsilon p)^{\frac{1}{p-1}}p'}\|f\|^{p'}_{L^{(p^*)'}(\Omega)}\Bigg],$$

with $c''' = c'''(N,p)$.

Thus it results

$$
\left[\frac{\alpha}{c'''} - \|E\|_{M^{\frac{N}{p-1}}(A_n(k))} - 2\epsilon\right] \int_\Omega |\nabla G_k(u_n)|^p
$$

$$
+\left[\frac{\mu}{c'''} - \|E\|_{M^{\frac{N}{p-1}}(A_n(k))}\right] \int_\Omega |G_k(u_n)|^p
$$

$$
\leq \frac{k^p}{(\epsilon p)^{\frac{1}{p-1}} p'} \|E\|^{p'}_{L^{p'}(\Omega)} + \frac{1}{S^{p'}(\epsilon p)^{\frac{1}{p-1}} p'} \|f\|^{p'}_{L^{(p^*)'}(\Omega)}.
$$

Fix ϵ so that $2\epsilon = \dfrac{\alpha}{4c'''}$. Then (1.23) and (4.34) imply that there exists $k^* \in \mathbb{R}_+$ such that

$$
\|E\|_{M^{\frac{N}{p-1}}(A_n(k))} \leq \min\left\{\frac{\alpha}{4c'''}, \frac{\mu}{2c'''}\right\}, \quad \forall k > k^*. \tag{4.38}
$$

Let us explicitly observe that, in view of (1.23), (1.24) and by the definition (1.31) of $\sigma_{\mathcal{O}}^{\frac{N}{p-1}}[E]$, one has $k^* = k^*(N, p, \sigma_{\mathcal{O}}^{\frac{N}{p-1}}[E])$. This concludes our proof.

□

Theorem 4.10. *Assume (4.2), (4.3), (4.4), (4.5) and (4.6). Then the sequence* $\{u_n\}$ *of the solutions of problems (4.8) is bounded in* $W_0^{1,p}(\Omega)$.

Proof. Let k^* be given by Lemma 4.9. Definition (3.1) together with the estimates (4.31) and (4.35) imply that for any $k > k^*$ there exists a positive constant $C' = C'(N, p, \alpha, \mu)$ such that

$$\int_\Omega |\nabla u_n|^p + \int_\Omega |u_n|^p$$

$$\le C'\left(k^p \int_\Omega |E(x)|^{p'} + k\int_\Omega |f| + \left(\int_\Omega |f|^{(p^*)'}\right)^{1+\frac{p'}{N}}\right). \tag{4.39}$$

This concludes the proof. □

Finally, let us prove the existence result.

Theorem 4.11. *Assume (4.2), (4.3), (4.4), (4.5) and (4.6). Then there exists $u \in W_0^{1,p}(\Omega)$ weak solution of (4.1), that is*

$$\int_\Omega b(x)|\nabla u|^{p-2}\nabla u \nabla v + \int_\Omega \mu |u|^{p-2}uv$$

$$= \int_\Omega |u|^{p-2}u\, E(x)\nabla v + \int_\Omega fv,\ \forall v \in W_0^{1,p}(\Omega). \tag{4.40}$$

Let us explicitly observe that the right hand side of formula (4.40) makes sense. In particular, for the term

$$\int_\Omega |u|^{p-2}u\, E(x)\nabla v \tag{4.41}$$

one has that, in view of hypothesis (4.5), by Lemma 1.10 it follows that $|E(x)|^{\frac{1}{p-1}}u$ is in $L^p(\Omega)$, thus $|E(x)||u|^{p-2}u$ belongs to $L^{p'}(\Omega)$. Therefore by Hölder inequality the integral in (4.41) is bounded.

Proof of Theorem 4.11. The sequence $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$ by Theorem 4.10. Then, up to a subsequence,

$$u_n \rightharpoonup u \text{ weakly in } W_0^{1,p}(\Omega). \tag{4.42}$$

Let us start proving that if u_n solves (4.8), then, up to a subsequence,

$$\begin{aligned} &b(x)\left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)\nabla(u_n - u) \\ &+\mu\left(|u_n|^{p-2}u_n - |u|^{p-2}u\right)(u_n - u) \to 0 \ \text{ a.e. in } \Omega. \end{aligned} \tag{4.43}$$

To this aim, we firstly show that

$$\begin{aligned} \lim_{n\to+\infty}\Bigg[&\int_\Omega b(x)\left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)\nabla(u_n - u) \\ &+\mu\int_\Omega \left(|u_n|^{p-2}u_n - |u|^{p-2}u\right)(u_n - u)\Bigg] = 0. \end{aligned} \tag{4.44}$$

Indeed, if we take $u_n - u$ as test function in the variational formulation of (4.8) we get

$$\begin{aligned} &\int_\Omega b(x)|\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - u) + \mu\int_\Omega |u_n|^{p-2}u_n(u_n - u) \\ &= \int_\Omega \frac{|u_n|^{p-2}u_n}{1+\frac{1}{n}|u_n|^{p-1}}\frac{E(x)}{1+\frac{1}{n}|E(x)|}\nabla(u_n - u) + \int_\Omega \frac{f}{1+\frac{1}{n}|f|}(u_n - u). \end{aligned} \tag{4.45}$$

Now, let us show that

$$\lim_{n\to+\infty}\int_{\Omega}\frac{|u_n|^{p-2}u_n}{1+\frac{1}{n}|u_n|^{p-1}}\frac{E(x)}{1+\frac{1}{n}|E(x)|}\nabla(u_n-u)=0. \tag{4.46}$$

By (4.42) and by the compactness result in Lemma 1.10, we obtain that, up to subsequences,

$$|E(x)|^{\frac{1}{p-1}}u_n \to |E(x)|^{\frac{1}{p-1}}u \text{ in } L^p(\Omega)$$

and

$$u_n \to u \text{ a.e. in } \Omega.$$

Thus, in view of the Vitali Theorem (cfr., for istance, [26]), we get that for any $\epsilon > 0$ there exists $\Omega_\epsilon \subset \Omega$ with $|\Omega_\epsilon| < +\infty$ such that

$$\int_{\Omega\setminus\Omega_\epsilon}|u_n|^p|E(x)|^{\frac{p}{p-1}} < \epsilon, \text{ uniformly with respect to } n$$

and there exists $\delta > 0$ such that for every $A \subset \Omega$ with $|A| < \delta$, one has

$$\int_A |u_n|^p|E(x)|^{\frac{p}{p-1}} < \epsilon, \text{ uniformly with respect to } n.$$

On the other hand,

$$\int_{\Omega\setminus\Omega_\epsilon}\frac{(|u_n|^{p-2}u_n)^{\frac{p}{p-1}}}{\left(1+\frac{1}{n}|u_n|^{p-1}\right)^{\frac{p}{p-1}}}\frac{|E(x)|^{\frac{p}{p-1}}}{\left(1+\frac{1}{n}|E(x)|\right)^{\frac{p}{p-1}}} \le \int_{\Omega\setminus\Omega_\epsilon}|u_n|^p|E(x)|^{\frac{p}{p-1}} < \epsilon$$

and

$$\int_A \frac{(|u_n|^{p-2}u_n)^{\frac{p}{p-1}}}{\left(1+\frac{1}{n}|u_n|^{p-1}\right)^{\frac{p}{p-1}}} \frac{|E(x)|^{\frac{p}{p-1}}}{\left(1+\frac{1}{n}|E(x)|\right)^{\frac{p}{p-1}}} \leq \int_A |u_n|^p |E(x)|^{\frac{p}{p-1}} < \epsilon,$$

uniformly with respect to n.

Furthermore, since $|u_n|^{p-2}u_n$ converges a.e. to $|u|^{p-2}u$ in Ω, we obtain

$$\frac{|u_n|^{p-2}u_n}{1+\frac{1}{n}|u_n|^{p-1}} \frac{|E(x)|}{1+\frac{1}{n}|E(x)|} \to |u|^{p-2}u|E(x)| \ \text{ a.e. in } \Omega.$$

Hence, by using in the reverse sense the Vitali Theorem, we get

$$\frac{|u_n|^{p-2}u_n}{1+\frac{1}{n}|u_n|^{p-1}} \frac{|E(x)|}{1+\frac{1}{n}|E(x)|} \to |u|^{p-2}u|E(x)| \ \text{ in } L^{p'}(\Omega). \tag{4.47}$$

Putting together (4.42) and (4.47) we obtain (4.46).

Furthermore, by (4.6) and (4.42), one has

$$\lim_{n\to+\infty} \int_\Omega \frac{f}{1+\frac{1}{n}|f|}(u_n - u) = 0. \tag{4.48}$$

Thus, by (4.46) and (4.48), identity (4.45) gives

$$\lim_{n\to+\infty} \left[\int_\Omega b(x)|\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - u) + \mu \int_\Omega |u_n|^{p-2}u_n(u_n - u) \right] = 0. \tag{4.49}$$

Moreover, always in view of (4.42) and of (4.3) and (4.4)

$$\lim_{n\to+\infty}\left[\int\limits_{\Omega} b(x)|\nabla u|^{p-2}\nabla u\nabla(u_n-u)+\mu\int\limits_{\Omega}|u|^{p-2}u(u_n-u)\right]=0. \tag{4.50}$$

Therefore, subtracting (4.50) from (4.49) we obtain (4.44).

Hence, in view of (4.3), (4.4) and (4.7)

$$\begin{gathered} b(x)\left(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u|^{p-2}\nabla u\right)\nabla(u_n-u) \\ +\mu\left(|u_n|^{p-2}u_n-|u|^{p-2}u\right)(u_n-u)\to 0 \ \text{ in } L^1(\Omega) \end{gathered} \tag{4.51}$$

and this gives (4.43) up to a subsequence.

Convergence (4.43) together with Lemma 4.4 yield then

$$\nabla u_n\to\nabla u \ \text{ a.e. in } \Omega \tag{4.52}$$

and

$$u_n\to u \ \text{ a.e. in } \Omega. \tag{4.53}$$

We are now able to pass to the limit, as $n\to+\infty$, in the variational formulation of (4.8)

$$\begin{gathered} \int_\Omega b(x)|\nabla u_n|^{p-2}\nabla u_n\nabla v+\int_\Omega \mu|u_n|^{p-2}u_n v \\ =\int_\Omega \frac{|u_n|^{p-2}u_n}{1+\frac{1}{n}|u_n|^{p-1}}\frac{E(x)}{1+\frac{1}{n}|E(x)|}\nabla v+\int_\Omega\frac{f}{1+\frac{1}{n}|f|}v, \end{gathered} \tag{4.54}$$

$v \in W_0^{1,p}(\Omega)$.

By (4.3), Theorem 4.10, (4.52), (4.53) and the Lebesgue dominated convergence Theorem, we get

$$\int_\Omega b(x)|\nabla u_n|^{p-2}\nabla u_n \nabla v \to \int_\Omega b(x)|\nabla u|^{p-2}\nabla u \nabla v \tag{4.55}$$

and

$$\int_\Omega \mu|u_n|^{p-2}u_n v \to \int_\Omega \mu|u|^{p-2}uv. \tag{4.56}$$

Moreover, by (4.47)

$$\int_\Omega \frac{|u_n|^{p-2}u_n}{1+\frac{1}{n}|u_n|^{p-1}} \frac{E(x)}{1+\frac{1}{n}|E(x)|}\nabla v \to \int_\Omega |u|^{p-2}uE(x)\nabla v. \tag{4.57}$$

On the other hand,

$$\frac{f}{1+\frac{1}{n}|f|}v \to fv \text{ a.e. in } \Omega$$

and

$$\left|\frac{f}{1+\frac{1}{n}|f|}v\right| \leq |fv| \in L^1(\Omega).$$

Thus, again in view of the Lebesgue dominated convergence Theorem, we have

$$\int_\Omega \frac{f}{1+\frac{1}{n}|f|}v \to \int_\Omega fv. \tag{4.58}$$

Taking into account (4.55), (4.56), (4.57) and (4.58) and passing to the limit, as $n \to +\infty$, in (4.54), we conclude the proof.

$\square$

The uniqueness of the solution will be object of a forthcoming study; probably this can be obtained for $1 < p \leq 2$ as in the case of bounded domains (cfr. [4]). In order to prove this, the following last result can be useful.

Corollary 4.12. *If $f(x) \geq 0$ then $u(x) \geq 0$.*

PROOF. We use $T_h(u^-)$ as test function in (4.1). We have

$$\begin{aligned} &\int\limits_\Omega b(x)|\nabla u|^{p-2}\nabla u \nabla T_h(u^-) + \mu \int\limits_\Omega |u|^{p-2}u T_h(u^-) \\ &= \int\limits_\Omega |u|^{p-2}u E(x) \nabla T_h(u^-) + \int\limits_\Omega f T_h(u^-). \end{aligned} \tag{4.59}$$

We obtain that

$$T_h(u^-) = \begin{cases} 0, & \text{if } u \geq 0 \\ -u, & \text{if } -h \leq u < 0 \\ h, & \text{if } u < -h. \end{cases} \tag{4.60}$$

Thus

$$|u|^{p-2}u T_h(u^-) \begin{cases} = 0, & \text{if } u \geq 0 \\ = -|T_h(u^-)|^p, & \text{if } -h \leq u < 0 \\ \leq -|T_h(u^-)|^p, & \text{if } u < -h. \end{cases} \tag{4.61}$$

Hence, by (4.59), (4.60) and (4.61)

$$-\int\limits_\Omega b(x)|\nabla T_h(u^-)|^p - \mu \int\limits_\Omega |T_h(u^-)|^p$$

$$\geq -\int_\Omega b(x)|\nabla T_h(u^-)|^p + \mu \int_\Omega |u|^{p-2} u T_h(u^-)$$

$$= \int_\Omega |u|^{p-2} u E(x) \nabla T_h(u^-) + \int_\Omega f T_h(u^-).$$

Therefore,

$$\alpha \int_\Omega |\nabla T_h(u^-)|^p + \mu \int_\Omega |T_h(u^-)|^p \leq \int_\Omega |u|^{p-1} |E(x)| |\nabla T_h(u^-)| - \int_\Omega f T_h(u^-).$$

Hence

$$\int_\Omega |\nabla T_h(u^-)|^p + \int_\Omega |T_h(u^-)|^p \leq C \int_\Omega |u|^{p-1} |E(x)| |\nabla T_h(u^-)|$$

with $C = C(\alpha, \mu)$.

Let $0 < h < \delta$. Then

$$\int_\Omega |\nabla T_h(u^-)|^p + \int_\Omega |T_h(u^-)|^p \leq C\ h^{p-1} \left(\int_{-h<u<0} |E(x)|^{p'} \right)^{\frac{1}{p'}} \|\nabla T_h(u^-)\|_{L^p(\Omega)}.$$

Thus,

$$\|T_h(u^-)\|^p_{W^{1,p}(\Omega)} \leq C\ h^{p-1} \left(\int_{-h<u<0} |E(x)|^{p'} \right)^{\frac{1}{p'}} \|T_h(u^-)\|_{W^{1,p}(\Omega)},$$

$$\|T_h(u^-)\|^{p-1}_{W^{1,p}(\Omega)} \leq C\ h^{p-1} \left(\int_{-h<u<0} |E(x)|^{p'} \right)^{\frac{1}{p'}}$$

and

$$\|T_h(u^-)\|_{L^p(\Omega)}^{p-1} \le C\ h^{p-1} \left(\int\limits_{-h<u<0} |E(x)|^{p'} \right)^{\frac{1}{p'}} .$$

The previous inequality gives

$$h^{\frac{p}{p'}} \operatorname{meas}\{u < -\delta\} \le C\ h^{p-1} \left(\int\limits_{-h<u<0} |E(x)|^{p'} \right)^{\frac{1}{p'}}$$

namely

$$\operatorname{meas}\{u < -\delta\} \le C \left(\int\limits_{-h<u<0} |E(x)|^{p'} \right)^{\frac{1}{p'}} .$$

Then, in view of (4.5), the right hand side goes to 0, as $h \to 0$. This means that $\operatorname{meas}\{u < -\delta\} = 0$, for every $\delta > 0$, which concludes our proof.

$\square$

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