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On the regularity of solutions of free boundary problems

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Alla mia famiglia
e ai miei nonni.
A Marianna.

Contents

Abstract

Optimal design problems have aroused particular interest in the scientific community over the past thirty years. In physics, for example, they find application in the investigation of the minimal energy configurations of a mixture of two materials in a bounded and connected open set.

The fascination of such problems derives from their variational formulation, which involves not only the state function of a system, but also a *shape*, that is a set. If a penalizing contribution of perimeter form, due to a *surface energy*, is added to the integral *mass energy*, dependent on the configuration state-shape, the problem becomes even more intriguing and inspiring.

It is not straightforward to investigate the regularity of minimizing pairs because the two energies have different dimensions under commong scalings: once a homothety of factor r is applied, the first energy “behaves” as a volume (rescaling with factor r^n), the second as a perimeter (rescaling with factor r^{n-1}). The coexistence of the two types of energies is managed using techniques and tools of both the Calculus of Variations and the Geometric Measure Theory.

In the first part of this thesis we deal with two optimal design problems, in which the integral functions that constitute the mass energy have different growths.

If their growth is at most quadratic, we prove the $C^{1,\mu}$ regularity of the interface of the shape that constitutes the optimal pair, up to a singular set of Hausdorff dimension less than $n-1$. The technique used combines the regularity theories of the Λ-minimizers of the perimeter and the minimizers of the Mumford-Shah functional.

If the integrands have at most a polynomial growth of degree p, the analysis becomes more involved. The $C^{1,\mu}$ regularity of the interface remains an open problem. However, it is proved that the optimal shape of the problem is equivalent to an open set with a topological boundary that differs from its reduced boundary for a set of Hausdorff dimension less than or equal to $n-1$.

In the second part of the thesis we address to a completely different variational problem, involving a frustrated spin system on a (one-dimensional and two-dimensional) lattice confined in two magnetic anisotropy circles.

This topic is of significant scientific interest, as it is useful for understanding the behavior of low-dimensional magnetic structures existing in nature.

The frustration parameter $\alpha > 0$ of the system averages the ferromagnetic and antiferromagnetic interactions that coexist in the energy. The minimal energy state of the system, for $\alpha \leqslant 4$, consists of a spin that "lives" within only one of the two magnetic anisotropy circles and has a positive or negative chirality.

We find the correct rescaling of the functional and prove the energy needed to detect the two phenomena that break the rigid minimal symmetry described. These are chirality transitions and magnetic anisotropy transitions of the spin.

Sommario

I problemi di design ottimale hanno suscitato un particolare interesse nella comunità scientifica negli ultimi trent'anni. In campo fisico, per esempio, trovano immediata applicazione nella ricerca della configurazione di minima energia di una miscela di due materiali in un aperto limitato e connesso.

Il fascino di tali problemi deriva dalla loro formulazione variazionale, la quale coinvolge non soltanto la funzione di stato di un sistema, ma anche una *forma*, un insieme. Se poi alla classica *energia di massa* di forma integrale, dipendente dalla configurazione funzione di stato-forma, si aggiunge un contributo penalizzante di forma perimetrale, dovuto ad un'*energia di superficie*, il problema diventa ancora più intrigante e stimolante.

Non è immediato investigare la regolarità delle coppie minimizzanti perché le due energie hanno dimensioni diverse sotto lo stesso riscalamento: applicata un'omotetia di fattore r, la prima si "comporta" come un volume (riscalando come r^n), la seconda come un perimetro (riscalando come r^{n-1}). La compresenza dei due tipi di energie viene gestita adoperando tecniche e strumenti propri sia del Calcolo delle Variazioni che della Teoria Geometrica della Misura.

Nella prima parte di questa tesi si trattano due problemi di design ottimale, in cui le funzioni integrande che compongono l'energia di massa hanno crescite diverse.

Se la crescita è al più quadratica, si prova la regolarità $C^{1,\mu}$ dell'interfaccia della forma che costituisce la coppia ottimale, a meno di un insieme di singolarità di dimensione di Hausdorff strettamente inferiore a $n-1$. La tecnica adoperata coniuga la teorie di regolarità dei Λ-minimi del perimetro e dei minimi del funzionale di Mumford-Shah.

Qualora l'integranda abbia crescita al più polinomiale di grado p, l'analisi diventa più complessa. La regolarità $C^{1,\mu}$ dell'interfaccia resta un problema aperto. Tuttavia, si prova che la forma ottimale del problema è equivalente ad un aperto con una frontiera topologica che differisce dalla sua frontiera ridotta per un insieme di dimensione di Hausdorff inferiore o uguale a $n-1$.

Nella seconda parte della tesi viene affrontato un problema variazionale completamente diverso, che coinvolge un sistema di spin frustrato su un reticolo (unidimensionale e bidimensionale) confinato in due circonferenze di anisotropia magnetica. L'argomento è di rilevante interesse scientifico, siccome utile a comprendere il comportamento di strutture magnetiche di basse dimensioni esistenti in natura. Il parametro di frustrazione $\alpha > 0$ del sistema media le interazioni ferromagnetiche e antiferromagnetiche che si riflettono nell'energia. Lo stato di minima energia del sistema, per $\alpha \leqslant 4$, è composto da uno spin che "vive" all'interno di una sola delle due circonferenze di anisotropia magnetica e ha una chiralità positiva o negativa.

Si prova quali sono il riscalamento corretto del funzionale e l'energia necessaria per individuare i fenomeni di transizione di chiralità e anisotropia magnetica degli spin, le quali rompono la rigida simmetria minimale descritta.

Introduction

This thesis is structured in two parts: Part I is devoted to the study of regularity properties of solutions of optimal design problems with perimeter penalization, which is the main topic that I studied in my PhD course. Part II is focused on the study of chirality and magnetic anisotropy transitions of a frustrated lattice system and originates from my visit, lasted four months, to Prof. Dr. Marco Cicalese at Technische Universität München.

Introduction to Part I

Free boundary problems involving bulk and interface energy have recently attracted the attention of the scientific community. This interest is justified by the large applications they find in the description of plethora of fenomena such as non linear elasticity, material sciences and image segmentation in the computer vision.

Among free boundary problems, optimal design with perimeter penalization concerns the study of the minimal energy configurations of a mixture of two materials in a bounded connected open set, where the energy is penalized by the area of the interface between the two materials (see for instance [4], [5], [31], [34], [38], [42], [43], [44], [57]).

An optimal design problem is a variational problem whose set of competitors is a family of shapes, i.e. domains of $\mathbb{R}^n$. Its mathematical formulation is the following:

$$\min_{E\in\mathcal{A}} \mathcal{F}(E), \tag{0.1}$$

where $\mathcal{A}$ is the class of all admissible domains and $\mathcal{F}$ is the cost function to be minimized over $\mathcal{A}$ (see [12]). A typical example of this kind of problems is the well-known euclidean isoperimetric problem,

$$\min_{\substack{E\subset\mathbb{R}^n \\ |E|=d}} P(E), \tag{0.2}$$

where $d \in (0, +\infty)$ is a fixed number.

It is worth noticing that the class $\mathcal{A}$ does not have any linear or convex structure, so in optimal design problems it is meaningless to speak of convex functionals and similar notions. Moreover, even if several topologies on families of domains are available, in general there is not an a priori choice of a topology in order to apply the direct methods of the calculus of variations, for obtaining the existence of at least an optimal shape. One would have to exclude that minimizing sequences of problem (0.2) locally converge to a set of locally finite perimeter that have a Lebesgue measure strictly less than d, fact that is not guaranteed by the usual compactness results.

In Part I, we deal with variational cost functions of the type

$$\mathcal{F}(E) = \min_{u \in u_0 + W_0^{1,p}(\Omega)} \int_\Omega H_E(x, u(x), \nabla u(x))\, dx + P(E; \Omega), \tag{0.3}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set, $H_E \colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a discontinuous function on the boundary of E and $u_0 \in W_0^{1,p}(\Omega)$ is a fixed boundary datum. The competitor $E \subset \Omega$ is a set of finite perimeter in Ω. In this case, the minimization problem (0.1) involving the functional $\mathcal{F}$ defined in (0.3), i.e.

$$\min_{E \in \mathcal{A}(\Omega)} \mathcal{F}(E), \tag{0.4}$$

is called an **optimal design problem with perimeter penalization** and we say that the energy $\mathcal{F}$ is made up of the bulk energy and the perimetral energy. In the specific case, if the competitors run over the family

$$\mathcal{A}(\Omega) = \{E \subset \Omega \, : \, P(E; \Omega) < +\infty\},$$

we call problem (0.4) an **unconstrained problem**. If

$$\mathcal{A} = \{E \subset \Omega \, : \, P(E; \Omega) < +\infty, \, |E| = d\},$$

for some fixed number $d \in (0, |\Omega|)$, we call problem (0.4) a **constrained problem**.

We remark that if $\mathcal{F}$ has the form (0.3), the starting problem (0.4) can be written as follows:

$$\min_{(E,u) \in \mathcal{A} \times \left(u_0 + W_0^{1,p}(\Omega)\right)} \int_\Omega H_E(x, u(x), \nabla u(x))\, dx + P(E; \Omega).$$

For example, the problem of finding the minimal energy configuration of a mixture of two conducting materials of permittivities $\alpha, \beta > 0$ in a container Ω can be described by the unconstrained problem

$$\min_{(E,u) \in \mathcal{A} \times \left(u_0 + H_0^1(\Omega)\right)} \int_\Omega [(\alpha \mathbb{1}_E(x) + \beta \mathbb{1}_{\Omega \setminus E}(x)) |\nabla u|^2(x) - 2 f(x) u(x)]\, dx + P(E; \Omega),$$

where f denotes the source density, u the electrostatic potential of the system and $P(E; \Omega)$ stands for the energetic dispersion due to the contact of the interfaces of the two materials (see [5] for more details).

Optimal design problems are strictly linked to other classes of problems.

We discuss in Chapter 2 their substantial connection with the Mumford-Shah functional

$$J(K,u) := \int_{\Omega\setminus K} \left[|\nabla u|^2 + \alpha(u-g)^2\right] dx + \beta\mathcal{H}^{n-1}(K \cap \Omega).$$

Here $g \in L^\infty(\Omega) \cap L^2(\Omega)$ is fixed and α, β are positive parameters. The problem consists in minimizing J among all pairs (K,u), being $K \subset \mathbb{R}^n$ a closed set and $u \in C^1(\Omega\setminus K)$.

Another problem linked to optimal design that we study is the model describing the shape of charged liquid droplets under a suitable free energy composed by an attractive term, coming from surface tension forces, and a repulsive one, due to the electric forces generated by the interaction between charged particles, i.e., for $K > 0$,

$$\min_{(E,u,\rho)\in\mathcal{A}\times W(E)} Q^2\left\{\int_{\mathbb{R}^n}[(\mathbb{1}_E(x)+\beta\mathbb{1}_{\mathbb{R}^n\setminus E}(x))|\nabla u|^2(x)]\,dx + K\int_E \rho^2(x)\,dx\right\} + P(E),$$

(see [23], [49], [50], [57]). Here $E \subset \mathbb{R}^n$ represents the droplet, the constant $Q > 0$ is the total charge enclosed in E, $\beta > 0$ is the permittivity of the liquid, ρ and u represent respectively the normalized density of charge and the electrostatic potential, both belonging to the space

$$W(E) = \left\{(u,\rho) \in \mathcal{D}'(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \;:\; -\mathrm{div}((\mathbb{1}_E + \beta\mathbb{1}_{\mathbb{R}^n\setminus E}\nabla u) = \rho,\right.$$
$$\left.\rho\mathbb{1}_{\mathbb{R}^n\setminus E} = 0,\ \int_E \rho\,dx = 1\right\},$$

where $\mathcal{D}^1(\mathbb{R}^n)$ is the closure of $C_c^\infty(\mathbb{R}^n)$ with respect to the gradient norm of $W^{1,2}(\mathbb{R}^n)$.

In addition to existence issues, regularity properties of the optimal shape can be analyzed, e.g. the regularity of its boundary and the Hausdorff dimension of its singular set. This will be our main issue of concern.

One of the first results concerning the unconstrained problem (0.4) was due by L. Ambrosio and G. Buttazzo in 1993 (see [5]). As mentioned before, they considered

$$H_E(x,s,\xi) = (\alpha\mathbb{1}_E(x) + \beta\mathbb{1}_{\Omega\setminus E}(x))|\xi|^2 + \mathbb{1}_E(x)g(x,s) + \mathbb{1}_{\Omega\setminus E}(x)h(x,s), \quad (0.5)$$

with g and h satisfying

$$g(x,s) \geqslant \gamma(x) - k|s|^2 \quad \text{and} \quad h(x,s) \geqslant \gamma(x) - k|s|^2, \qquad (0.6)$$

for a.e. $x \in \Omega$ and $s \in \mathbb{R}$, where $\gamma \in L^1(\Omega)$ and $k < \alpha\lambda_1$, being λ_1 the first eigenvalue of $-\Delta$ on Ω. The previous conditions on g and h ensure the existence of a minimizing couple of problem (0.3). The authors proved the following result.

Theorem 0.0.1. *Let us assume that g and h satisfy the assumption* (0.6),

$$|g(x,s)| \leqslant C(1+|s|^q) \quad \textit{and} \quad |h(x,s)| \leqslant C(1+|s|^q),$$

for a.e. $(x,s) \in \Omega \times \mathbb{R}$, *where*

$$q \in \begin{cases} [p,+\infty) & \textit{if } n=2, \\ [p,p^*) & \textit{if } n>2. \end{cases}$$

If (E,u) *is a solution of the uncostrained problem* (0.4) *with* H_E *as in* (0.5), *then*

1. *u is locally Hölder continuous;*
2. *E is equivalent to an open set $\tilde{E}$, that is*

$$|E\Delta\tilde{E}| = 0 \quad \textit{and} \quad P(E;\Omega) = P(\tilde{E};\Omega) = \mathcal{H}^{n-1}(\partial\tilde{E} \cap \Omega).$$

In the same volume of the same journal, F.H. Lin proved the regularity of the interface (see [44]) of minimizers of the uncostrained problem (0.4) with

$$H_E(x,\xi) = (1+\mathbb{1}_E(x))|\xi|^2, \quad \forall (x,\xi) \in \Omega \times \mathbb{R}^n. \tag{0.7}$$

The author proved that, for a minimal configuration (E,u), ∂E is regular outside a relatively closed set of vanishing $\mathcal{H}^{n-1}$-measure. To be more precise, we define the set of regular points of ∂E as follows:

$$\begin{aligned} \mathrm{Reg}(E) := \big\{ & x \in \partial E \cap \Omega \,:\, \partial E \text{ is a } C^{1,\gamma} \text{ hypersurface in some } I(x) \\ & \text{and for some } \gamma \in (0,1) \big\}, \end{aligned}$$

where $I(x)$ denotes a neighborhood of x. Accordingly, we define the set of singular points of ∂E

$$\Sigma(E) := (\partial E \cap \Omega) \backslash \mathrm{Reg}(E).$$

The theorem proved in [44] is the following.

Theorem 0.0.2. *There exists a solution* (E,u) *of the uncostrained problem* (0.4) *with* H_E *as in* (0.7). *Furthermore,*

1. $u \in C^{\frac{1}{2}}(\Omega)$*;*
2. *∂E is $(n-1)$-countably rectifiable. More precisely, $(\partial E \cap \Omega)\backslash\Sigma(E)$ is a $C^{1,\alpha}$-hypersurface, for some $\alpha \in (0,1)$, and $\mathcal{H}^{n-1}(\Sigma(E)) = 0$.*

We remark that the Hölder exponent $\frac{1}{2}$ is critical. If one can show that $u \in C^{\frac{1}{2}+\eta}(\Omega)$, for some $\eta \in \big(0,\frac{1}{2}\big]$, then we get

$$\int_{B_r(x_0)} |\nabla u|^2 \, dx \leqslant C\big(n,[u]_{\frac{1}{2}+\eta}\big) r^{n-1+2\eta}, \quad \forall B_r(x_0) \subset\subset \Omega.$$

If (E,u) is a solution of problem (0.4) with H_E as in (0.7) (H_E can be more generally a function with quadratic growth in ξ), then, for any $F \subset \mathbb{R}^n$ with $E\Delta F \subset\subset B_r(x_0)$, by minimality, we infer

$$\begin{aligned} P(E;B_r(x_0)) - P(F;B_r(x_0)) &\leqslant \int_{B_r(x_0)} (\mathbb{1}_E(x) - \mathbb{1}_F(x))|\nabla u|^2\,dx \\ &\leqslant 2\int_{B_r(x_0)} |\nabla u|^2\,dx \leqslant C\big(n,[u]_{\frac{1}{2}+\eta}\big) r^{n-1+2\eta}, \end{aligned}$$

obtaining

$$\begin{aligned} \psi(E;B_r(x_0)) &:= P(E;B_r(x_0)) - \min_{F\Delta E\subset\subset B_r(x_0)} P(F;B_r(x_0)) \\ &\leqslant C\big(n,[u]_{\frac{1}{2}+\eta}\big) r^{n-1+2\eta}. \end{aligned}$$

The previous inequality guaranties that E has the same regularity property of a perimeter minimizer, as proved by I. Tamanini in 1982 (see [56]). For the sake of completeness we recall this result below.

Theorem 0.0.3. *Let $E \subset \mathbb{R}^n$ be a set of finite perimeter satisfying, for some $\eta \in \big(0,\frac{1}{2}\big)$ and two positive constants C, r_0, such that*

$$\psi(E;B_r(x_0)) \leqslant C\big(n,[u]_{\frac{1}{2}+\eta}\big) r^{n-1+2\eta},$$

for any $x_0 \in \Omega$ and $0 < r < r_0$. Then the reduced boundary $\partial^* E$ of E is a $C^{1,\eta}$-hypersurface in Ω and

$$dim_{\mathcal{H}}(\Omega \cap (\partial E \backslash \partial^* E)) \leqslant n-8,$$

Summing up, if (E,u) is a solution of the uncostrained problem (0.4) with H_E belonging to a large class of functions and $u \in C^{\frac{1}{2}+\eta}(\Omega)$, then the boundary ∂E of E contains a regular hypersurface $\partial^* E$ which differs from it in Ω by a singular set of Hausdorff dimension less than $n-8$. This is the best regularity one may expect. Indeed, De Giorgi in [18] showed that the non-existence of a singular minimal cone in $\mathbb{R}^n$ implies non-existence in $\mathbb{R}^{n-1}$ and Simons in [55] showed the non-existence of singular minimal cones in dimensions $2 \leqslant n \leqslant 7$. For $n = 8$, Bombieri, De Giorgi, and Giusti proved in [9] that the **Simons' cone**

$$\{x \in \mathbb{R}^8 \,:\, x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_5^2 + x_6^2 + x_7^2 + x_8^2\}$$

is a singular minimal cone with singular set $\{0\}$. Furthermore, if $n > 8$, there exists a perimeter minimizer $E \subset \mathbb{R}^n$ with $\mathcal{H}^{n-8}(\partial E \backslash \partial^* E) = +\infty$ (see Theorem 2.12.1 and, for further details, [46, Chapter 28]). Federer concluded in [33] by proving the Hausdorff dimension of the singular set is less than or equal to $n-8$.

The optimal regularity of the interface is hard to obtain in most cases. This is due to the fact that the two terms in the functional (0.3) have different dimension under common scalings. In [29] L. Esposito and N. Fusco studied the regularity of the costrained problem (0.4) with

$$H_E(x,\xi) = (\alpha \mathbb{1}_E(x) + \beta \mathbb{1}_{\Omega\backslash E}(x))|\xi|^2, \quad \forall (x,\xi) \in \Omega \times \mathbb{R}^n, \tag{0.8}$$

where $0 < \beta < \alpha$ are two fixed constants. Assuming that α is sufficiently close to β, the authors obtained the full regularity of the free interface by proving the following result.

Theorem 0.0.4. *There exists $\gamma_n > 1$ such that if $\frac{\alpha}{\beta} < \gamma_n$ and (E,u) is a solution of the costrained problem (0.4) with H_E as in (0.8), then $u \in C^{\frac{1}{2}+\eta}(\Omega)$, for some $\eta = \eta(n,\alpha,\beta) \in \left(0, \frac{1}{2}\right)$ and $\partial^* E$ is a $C^{1,\eta}$-hypersurface, with $\mathcal{H}^s(\Omega \cap (\partial E \backslash \partial^* E)) = 0$, for any $s > n-8$.*

The problem of handling with the constraint $|E| = d$ is overtaken ensuring that every minimizer of the constrained problem (0.4) is also a minimizer of a penalized functional of the type

$$\mathcal{F}_\Lambda(E,v;\Omega) = \mathcal{F}(E,v;\Omega) + \Lambda \big| |E| - d \big|,$$

for some suitable $\Lambda > 0$. In Chapter 2 the same idea will be carried out in a more general context (see Theorem 2.2.1).

The optimal regularity of the free interface in the general case is still an open problem. However, partial regularity results are available. In 2015, G. De Philippis and A. Figalli in [21], N. Fusco and V. Julin in [35], independently of each other and by different approaches, improved Lin's result by finding a sharper estimate of the singular set's size.

Theorem 0.0.5. *Let (E,u) be a solution of the costrained or uncostrained problem (0.4) with H_E as in (0.8). Then*

1. *there exists a relatively open set $\Gamma \subset \partial E$ such that Γ is a $C^{1,\mu}$ hypersurface for all $0 < \mu < \frac{1}{2}$;*
2. *there exists $\varepsilon = \varepsilon\left(n, \frac{\alpha}{\beta}\right) > 0$, such that*

$$\mathcal{H}^{n-1-\varepsilon}(\Omega \cap (\partial E \backslash \Gamma)) = 0.$$

The technique used by G. De Philippis and A. Figalli consists in proving that, if (E,u) is a solution of the problem, the singular set of ∂E is σ-porous in ∂E, for some $\sigma > 0$. Using density lower and upper bounds on the perimeter of E, the estimate follows from a classical result of measure theory. However, in Chapter 2 we follow the strategy adopted by N. Fusco and V. Julin.

Some aforementioned results were obtained in literature for more general functions H_E. In 1999, F.H. Lin and R.V. Kohn considered the functions

$$H_E(x,s,\xi) = F(x,s,\xi) + \mathbb{1}_E(x) G(x,s,\xi), \quad \forall (x,s,\xi) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n. \tag{0.9}$$

with $F, G \in C^{\ell,\eta}(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$, for $\ell \geqslant 2$ and $\eta \in (0,1)$, satisfying the following properties:

- F and G are uniformly elliptic and uniformly bounded, i.e. there exist two positive constants ν and N such that

$$\nu|\xi|^2 \leqslant F_{\xi_i\xi_j}(x,s,\xi) \leqslant N|\xi|^2, \quad \nu|\xi|^2 \leqslant G_{\xi_i\xi_j}(x,s,\xi) \leqslant N|\xi|^2, \tag{0.10}$$

for any $(x,s,\xi) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$;

- F and G have a controlled growth, that is there exists a constant $M > 0$ such that

$$|\nabla^2_{\xi s}F| + |\nabla^2_{\xi x}F| \leqslant M(1+|\xi|), \quad |\nabla^2_{ss}F| + |\nabla^2_{sx}F| + |\nabla^2_{xx}F| \leqslant M(1+|\xi|^2), \tag{0.11}$$

for any $\xi \in \mathbb{R}^n$.

They proved the following assertion (see [45]).

Theorem 0.0.6. *Assuming that H_E is as in* (0.9)*, and* (0.10)*,* (0.11) *are in force, then there exists a solution (E,u) of the costrained problem* (0.4)*. Furthermore, $u \in C^{\frac{1}{2}}(\Omega)$ and ∂A is $(n-1)$-countably rectifiable.*

A particular example of functions of the type (0.9) are integrands of the type $H_E = F + \mathbb{1}_E G$, with

$$F(x,s,\xi) = \sum_{i,j=1}^{n} a_{ij}(x,s)\xi_i\xi_j + \sum_{i=1}^{n} a_i(x,s)\xi_i + a(x,s), \tag{0.12}$$

$$G(x,s,\xi) = \sum_{i,j=1}^{n} b_{ij}(x,s)\xi_i\xi_j + \sum_{i=1}^{n} b_i(x,s)\xi_i + b(x,s), \tag{0.13}$$

for any $(x,s,\xi) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$. The same authors proved the result below.

Theorem 0.0.7. *Assuming that H_E is as in* (0.9)*, and* (0.10)-(0.13) *are in force, then, for any solution (E,u) of the costrained problem* (0.4)*, $(\partial E \cap \Omega)\backslash\Sigma(E)$ is a $C^{1,\sigma}$-hypersurface, for some $\sigma \in (0,1)$ and $\mathcal{H}^{n-1}(\Sigma(E)) = 0$.*

Actually, the previous results were proved for a more general problem, an optimal design problem with anisotropic perimeter penalization, i.e.

$$\min_{(E,u)\in\mathcal{A}\times\left(u_0+W^{1,p}_0(\Omega)\right)} \int_\Omega H_E(x,u(x),\nabla u(x))\,dx + \Psi(\partial E),$$

with

$$\Psi(\partial E) = \int_\Omega \psi(x,\nu_E(x))\,d|\mu_E|,$$

where ν_E is the exterior unit normal vector to ∂E and ψ satisfies some additional assumptions.

As explained so far, regularity results are based on the study of the interplay between the perimeter and the bulk energy. For this reason, in Chapter

1 we recall some classical notions and well-known results from the geometric measure theory and the regularity theory of minima of variational functions and solutions of partial differential equations.

In Chapter 2 we address the issue of improving the dimensional estimate for the singular part $\Sigma(E)$ of optimal configurations for the model quadratic functionals treated by F.H. Lin and R.V. Kohn. We prove the same kind of regularity of the interface proved in the model case (0.8) in [21] and [35], namely $\dim_{\mathcal{H}}(\Sigma(E)) \leqslant n-1-\varepsilon$, for some $\varepsilon > 0$ depending on the initial data. We consider the model function

$$H_E(x,s,\xi) = F(x,s,\xi) + \mathbb{1}_E(x)G(x,s,\xi), \quad \forall (x,s,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n.$$

under the structure conditions (0.12) and (0.13). Concerning the coefficients, we assume that they are Lipschitz continuous, i.e.

$$a_{ij}, b_{ij}, a_i, b_i, a, b \in C^{0,1}(\Omega \times \mathbb{R}).$$

Moreover, to ensure the existence of minimizers, we assume the uniform boundedness of the coefficients and the uniform ellipticity of the matrices a_{ij} and b_{ij}, i.e.

$$\nu|\xi|^2 \leqslant a_{ij}(x,s)\xi_i\xi_j \leqslant N|\xi|^2, \quad \nu|\xi|^2 \leqslant b_{ij}(x,s)\xi_i\xi_j \leqslant N|\xi|^2,$$
$$\sum_{i=1}^n |a_i(x,s)| + \sum_{i=1}^n |b_i(x,s)| + |a(x,s)| + |b(x,s)| \leqslant L,$$

for any $(x,s,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$, where ν, N and L are three positive constants.

We remark that our regularity assumptions are weaker than the ones assumed by F.H. Lin and R.V. Kohn in [45] for the same model quadratic functional. The aforementioned results con be found in a joint work with L. Esposito, [30].

The problem discussed so far can be easily generalized. Indeed, one may ask whether the aforementioned results are still true if the functional has a p-polynomial growth in place the quadratic one. While in the quadratic case many regularity results are available in literature, the problem is less studied in the p-polynomial growth case. Actually, it turns out to be more involved (see [13], [14], [28]).

As before, the formulation of the problem is

$$\min_{E \in \mathcal{A}(\Omega)} \mathcal{F}(E), \tag{0.14}$$

with a variational cost functions of the type

$$\mathcal{F}(E) = \min_{u \in u_0 + W_0^{1,p}(\Omega)} \int_\Omega H_E(x,u(x),\nabla u(x))\,dx + P(E;\Omega).$$

In 2014, M. Carozza, I. Fonseca and A. Passarelli di Napoli dealt with the costrained problem (0.14) involving a discontinuous class of integrands H_E of the type

$$H_E(x,\xi) = F(\xi) + \mathbb{1}_E(x)G(\xi), \quad \forall (x,\xi) \in \Omega \times \mathbb{R}^n. \tag{0.15}$$

(see [13]). They assumed that $F, G \in C^1(\mathbb{R}^n)$ and the existence of some positive constants l, L, α, β and $\mu \geqslant 0$ such that

- F and G have p-growth ($p > 1$):

$$0 \leqslant F(\xi) \leqslant L(\mu^2 + |\xi|^2)^{\frac{p}{2}}, \tag{F1}$$

$$0 \leqslant G(\xi) \leqslant \beta L(\mu^2 + |\xi|^2)^{\frac{p}{2}}, \tag{G1}$$

  for all $\xi \in \mathbb{R}^n$;

- F and G are strongly quasi-convex:

$$\int_\Omega F(\xi + \nabla\varphi)\,dx \geqslant \int_\Omega \left[F(\xi) + l(\mu^2 + |\xi|^2 + |\nabla\varphi|^2)^{\frac{p-2}{2}}|\nabla\varphi|^2\right] dx, \tag{F2}$$

$$\int_\Omega G(\xi + \nabla\varphi)\,dx \geqslant \int_\Omega \left[G(\xi) + \alpha l(\mu^2 + |\xi|^2 + |\nabla\varphi|^2)^{\frac{p-2}{2}}|\nabla\varphi|^2\right] dx, \tag{G2}$$

  for all $\xi \in \mathbb{R}^n$ and $\varphi \in C_c^1(\Omega)$.

Following the same argument adopted in [29], the authors proved that the constraint $|E| = d$ can be overtaken ensuring that every minimizer of the constrained problem is also a minimizer of a suitable unconstrained energy functional with a volume penalization, i.e.

$$\mathcal{F}_\Lambda(E, v; \Omega) = \mathcal{F}(E, v; \Omega) + \Lambda\big||E| - d\big|,$$

for some $\Lambda > 0$ sufficiently large. Inspired by the same article, they also obtained the optimal regularity of the interface, under the condition

$$\left(\frac{\beta}{\alpha+1}\right)\left(\frac{\beta+1}{\alpha+1}\right)^{\tilde{\sigma}} \leqslant \gamma,$$

for some $\gamma = \gamma\big(n, p, \frac{l}{L}\big) < 1$ and $\tilde{\sigma}(n,p) > 0$. In the general case, the authors proved the following theorem.

Theorem 0.0.8. *Let* (E, u) *be a solution of the costrained problem* (0.14) *with* H_E *as in* (0.15)*, under the conditions* (F1)*,* (F2)*,* (G1)*,* (G2)*. Then there exists an open set* $\Omega_0 \subset \Omega$ *with full measure such that* $u \in C^{0,\eta}(\Omega_0)$*, for every* $\eta \in (0,1)$*. In addition,* $\partial^* E \cap \Omega_0$ *is a* $C^{1,\tilde{\eta}}$*-hypersuface in* Ω_0*, for every* $\tilde{\eta} \in \big(0, \frac{1}{2}\big)$*, and* $\mathcal{H}^s(\Omega_0 \cap (\partial E \backslash \partial E^*)) = 0$*, for all* $s > n - 8$*.*

The stated regularity is only partial; indeed, the singular set of the optimal interface could lie in $\Omega \backslash \Omega_0$, which could have a positive s-dimensional Hausdorff measure, for some $s \in (n-8, n)$. In 2019, L. Esposito proved in [28] that this possibility does not occur for $s \in (n-1, n)$. Indeed, he proved the following lower bound estimate on the perimeter of the optimal set in Ω.

Theorem 0.0.9. *Let* (E, u) *be a solution of the costrained problem* (0.14) *with* H_E *as in* (0.15)*, under the conditions* (F1), (F2), (G1), (G2)*, and let* $U \subset\subset \Omega$. *Then there exists a constant* $c = c\big(U, \|\nabla u\|_{L^p(\Omega)}\big)$ *such that, for every* $x_0 \in \partial E$ *and* $B_r(x_0) \subset U$,

$$P(E; B_r(x_0)) \geqslant c r^{n-1}.$$

Moreover, $\mathcal{H}^{n-1}(\Omega \cap (\partial E \backslash \partial^* E)) = 0$.

In the general case, the C^1 regularity of the optimal interface is still an open problem.

In Chapter 3 we extend the partial regularity result obtained by [5] when the integrand of $\mathcal{F}$ is of the type

$$H_E(x, s, \xi) = F(\xi) + \mathbb{1}_E(x) G(\xi) + f_E(x, s), \quad \forall (x, s, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n,$$

where the function $f_E = g + \mathbb{1}_E h$ is discontinuous.

Regarding F and G, we assume that $F, G \in C^1(\mathbb{R}^n)$ and that (F1), (F2), (G1), (G2) hold. Furthermore, we impose on F and G some proximity conditions that are trivially satisfied if F and G are positively p-homogeneous. In particular, we assume that there exist two positive constants t_0, a and $0 < m < p$ such that for every $t > t_0$ and $\xi \in \mathbb{R}^n$ with $|\xi| = 1$, it holds

$$\left| F_p(\xi) - \frac{F(t\xi)}{t^p} \right| \leqslant \frac{a}{t^m},$$

$$\left| G_p(\xi) - \frac{G(t\xi)}{t^p} \right| \leqslant \frac{a}{t^m},$$

where F_p and G_p are the p-recession functions of F and G (see Definition 3.1.1).

With regard to g and h, we assume that they are Borel measurable, lower semicontinuous with respect to the real variable and that there exist a function $\gamma \in L^1(\Omega)$ and two constants $C_0 > 0$ and $k \in \mathbb{R}$, with $k < \frac{l}{2^{p-1}\lambda}$, being $\lambda = \lambda(\Omega)$ the first eigenvalue of the p-Laplacian on Ω with boundary datum u_0, such that

- g and h satisfy the following assumptions:

  $$g(x, s) \geqslant \gamma(x) - k|s|^p, \quad h(x, s) \geqslant \gamma(x) - k|s|^p,$$

  for almost all $(x, s) \in \Omega \times \mathbb{R}$;

- g and h satisfy the following growth conditions:

  $$|g(x, s)| \leqslant C_0(1 + |s|^q), \quad |h(x, s)| \leqslant C_0(1 + |s|^q),$$

  for all $(x, s) \in \Omega \times \mathbb{R}$, with the exponent

  $$q \in \begin{cases} [p, +\infty) & \text{if } n = 2, \\ [p, p^*) & \text{if } n > 2 \end{cases}$$

  fixed.

The first of the previous assumptions on g and h is essential to prove the existence of a minimal configuration. The same condition turns out to be crucial in the proof of the regularity result as well. All the results concerning this optimal design problem in the p-polynomial growth case can be found in [41].

Introduction to Part II

Lattice systems are discrete variational models, whose energy depends on a spin function defined in a lattice. A lattice system is said to be frustrated, when a competition between ferromagnetic (F) nearest-neighbor (NN) and antiferromagnetic (AF) next-nearest-neighbor (NNN) interactions occurs (see [24] for a complete discussion).

For example, three-dimensional frustrated magnets generally exist in the magnetic diamond and pyrochlore lattices (see [25]) and edge-sharing chains of cuprates provide a natural example of frustrated lattice systems. Furthermore, jarosites are the prototype for a spin-frustrated magnetic structure, because these materials are composed exclusively of kagomé layers (see [51]).

In 2015, M. Cicalese and F. Solombrino in [17] set the problem in the one-dimensional lattice

$$\mathcal{I}^n(I) := \mathbb{Z}^n(I) \backslash \left\{ \left\lfloor \frac{1}{\lambda_n} \right\rfloor - 1, \left\lfloor \frac{1}{\lambda_n} \right\rfloor \right\},$$

where $\mathbb{Z}_n(I) = \{i \in \mathbb{Z} : \lambda_n i \in \overline{I}\}$ and $I := (0,1)$ and $\{\lambda_n\}_{n \in \mathbb{N}}$ is a vanishing sequence of lattice spacings.

They considered, as spins of the system, functions of the type $u \colon i \in \mathbb{Z}_n(I) \longmapsto u^i \in \mathbb{S}^1$, satisfying the boundary condition

$$u^{\lfloor \frac{1}{\lambda_n} \rfloor - 1} \cdot u^{\lfloor \frac{1}{\lambda_n} \rfloor} = u^0 \cdot u^1, \tag{0.16}$$

where $\mathbb{S}^1$ is the unit circle of $\mathbb{R}^2$ centred in the origin. The energy of the system is a scalar functional E_n defined as

$$E_n(u) = \sum_{i \in \mathcal{I}^n(I)} \lambda_n \left(-\alpha u^i \cdot u^{i+1} + u^i \cdot u^{i+2} \right),$$

where $\alpha > 0$ is the frustation parameter that rules the NN and NNN interactions. As usual in the analysis of discrete systems, the family of energies may be embedded on a common functional space, thus extending

$$E_n(u) = \begin{cases} \sum_{i \in \mathcal{I}^n(I)} \lambda_n \left(-\alpha u^i \cdot u^{i+1} + u^i \cdot u^{i+2} \right) & \text{for } u \in C_n(I; \mathbb{S}^1), \\ +\infty & \text{for } u \in L^\infty(I; \mathbb{S}^1) \backslash C_n(I; \mathbb{S}^1), \end{cases} \tag{0.17}$$

where

$$C_n(I;\mathbb{S}^1) := \{u\colon \mathbb{Z}_n(I) \to \mathbb{S}^1 \,:\, u \text{ satisfies (0.16)}, \\ u \text{ is constant on } \lambda_n(i+[0,1)),\ \forall i \in \mathbb{Z}_n(I)\}.$$

While the first term of the energy E_n is ferromagnetic and favors the alignment of neighboring spins, the second one, being antiferromagnetic, frustrates it as it favors antipodal next-to-nearest neighboring spins. A more refined analysis is contained in the following result (see [17, Proposition 2.2 and Remark 2.3]).

Proposition 0.0.10. *Let* $E_n\colon L^\infty(I;\mathbb{S}^1) \to (-\infty,+\infty]$ *be the functional defined in* (0.17)*. We distinguish two cases.*

- *if* $\alpha \geqslant 4$*, then*

$$\min_{u\in L^\infty(I;\mathbb{S}^1)} E_n(u) = -(\alpha-1)\#\mathcal{I}^n(I).$$

  Furthermore,every minimizer $u_n \in L^\infty(I;\mathbb{S}^1)$ *of* E_n *is constant;*

- *if* $\alpha \in (0,4)$*, then*

$$\min_{u\in L^\infty(I;\mathbb{S}^1)} E_n(u) = -\left(1+\frac{\alpha^2}{8}\right)\#\mathcal{I}^n(I).$$

  Furthermore, a minimizer $u_n \in L^\infty(I;\mathbb{S}^1)$ *of* E_n *satisfies*

$$u_n^i \cdot u_n^{i+1} = \frac{\alpha}{4} \quad \text{and} \quad u_n^i \cdot u_n^{i+2} = \frac{\alpha^2}{8} - 1,$$

  for any $i \in \mathcal{I}^n(I)$*.*

In other words, the ground state of the system for $\alpha \geqslant 4$ is ferromagnetic (the spin is made up of alligned vectors), while for $0 < \alpha \leqslant 4$ it is helimagnetic (the spin consists in rotating vectors with a constant angle $\psi = \pm\arccos(\alpha/4)$). If $0 < \alpha \leqslant 4$, the sign of the angle ψ represents the sense of the spin's rotation. Hence, minimizers can be made up of clockwise or counterclockwise spin. In the first case we say that the spin chain has a **positive chirality**, while in the second case we say that it has a **negative chirality**.

In [17], the authors address to a system, whose interactions are close to the ferromagnet/helimagnet transition point as the number of particles diverges. Examples of edge-sharing cuprates in the vicinity of the ferromagnetic/helimagnetic transition point can be found in [26]. From a mathematical point of view, this means that the parameter α depends on n and tends to 4 from below, as $n \to +\infty$. By means of Γ-convergence's techniques, they provide a careful description of the admissible states and compute their associated energy. In particular, they find the correct scalings to detect chirality transitions, which break the simmetry of minimal configurations.

Setting $\alpha = \alpha_n = 4(1 - \delta_n)$ for some positive vanishing sequence $\{\delta_n\}_{n\in\mathbb{N}}$, the Γ-limit of the energy E_n (with respect to the weak-star convergence in L^∞), as $n \to +\infty$, does not provide a detailed description of the fenomenon. For this reason, M. Cicalese and F. Solombrino need to consider higher order Γ-limits, expanding E_n at the first order, that is

$$E_n = \min E_n + \lambda_n \mu_n H_n,$$

for some infinitesimal $\{\mu_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ to be found. The Γ-convergence of H_n can be better studied by changing the variable. It turns out that the best candidate z for the order chirality parameter is linked to the angular velocity of the spin. Hence, $H_n(z)$ can be redefined in $L^1(I)$, since, up to rotation, z is uniquely associated with a spin u.

For $\mu_n = \sqrt{2}\delta_n$, they proved the following result.

Theorem 0.0.11. *Let $H_n\colon L^1(I) \to [0, +\infty]$ be as above. Assume that there exists $l := \lim_{n\to+\infty} \dfrac{\lambda_n}{\sqrt{2\delta_n}} \in [0, +\infty]$. Then $H := \Gamma\text{-}\lim_n H_n$ with respect to the $L^1(I)$-convergence is given by one of the following formulas:*

i) if $l = 0$

$$H(z) = \begin{cases} \frac{4}{3}|Dz|(I) & \text{if } z \in BV(I; \{-1, 1\}), \\ +\infty & \text{otherwise}; \end{cases}$$

ii) if $l \in (0, +\infty)$

$$H(z) = \begin{cases} \dfrac{1}{l}\displaystyle\int_I (z^2(x) - 1)^2\, dx + l \int_I (z'(x))^2\, dx & \text{if } z \in H^1_{|per|}(I), \\ +\infty & \text{otherwise}, \end{cases}$$

where we set $H^1_{|per|}(I) := \{z \in H^1(I) \;:\; |z(0)| = |z(1)|\}$;

iii) if $l = +\infty$

$$H(z) = \begin{cases} 0 & \text{if } z = \text{constant}, \\ +\infty & \text{otherwise}. \end{cases}$$

Therefore, at scale $\lambda_n \delta_n^{3/2}$ several regimes are possible. Different values of the limit number $l \in [0, +\infty]$ entail different scenarios. If $l = +\infty$, the rigidity of the system does not allow the spin to make a chirality transition. If $l > 0$, the spin system may have diffuse and regular macroscopic chirality transitions whose limit energy is finite on $H^1(I)$ (provided some boundary conditions are taken into account). When $l = 0$, chirality transitions on a scale of order $\lambda_n/\sqrt{\delta_n}$ can occur. In this case, the continuum limit energy is finite on $BV(I)$ and counts the number of jumps of the chirality of the system.

In this case the presence of periodic boundary conditions allowed to turn E_n into a Modica-Mortola type energy, whose Γ-convergence is well known in literature (see [47] and [48]). Indeed, expanding the functional at the first order, under a suitable scaling, the spin system makes a chirality transition on a scale of order $\frac{\lambda_n}{\sqrt{\delta_n}}$, when $\frac{\lambda_n}{\sqrt{\delta_n}}$ approaches to a finite nonnegative value, as $n \to +\infty$ (otherwise no chirality transitions emerge).

In chemical and physical literature, frustrated lattice systems appear also in bidimensional settings. The frustration mechanisms originates form the presence of short-range ferromagnetic (F) and antiferromagnetic (AF) inter-atomic interactions of more complex geometric structures. This type of model is known as the J_1-J_3 F-AF classical spin model on the square lattice (see [53]). Whenever there are no NNN interactions, the energy describes the so-called XY model, whose variational analysis has been carried out in [2], [8], [16].

In 2019, M. Cicalese, M. Forster and G. Orlando in [15] addressed a J_1-J_3 F-AF classical spin model in a two dimensional setting. We give here its mathematical formulation.

Let $\Omega \in \mathfrak{A}_0$, that is an open, bounded, regular domain of $\mathbb{R}^2$ (see (5.2) for the precise definition of $\mathfrak{A}_0$). The spin functions u are parametrized over the points of the discrete set $\Omega \cap \lambda_n\mathbb{Z}^2$, and the energy of the system is

$$H_n(u;\Omega) := \frac{1}{2}\lambda_n^2 \sum_{(i,j)\in\mathcal{I}^n(\Omega)} \left[\left| u^{i+2,j} - \frac{\alpha_n}{2}u^{i+1,j} + u^{i,j} \right|^2 + \left| u^{i,j+2} - \frac{\alpha_n}{2}u^{i,j+1} + u^{i,j} \right|^2 \right], \tag{0.18}$$

where $\mathcal{I}^n(\Omega)$ is the equispaced lattice on Ω defined in (5.3) with spacing λ_n.

The authors proved that the two-dimensional problem can be decoupled in two one-dimensional ones, to which the main result of [17] can be applied. Similarly to the one-dimensional case, they redefined $H_n(z,w)$ in $L^1(\mathbb{R}^2)$, where the couple (w,z) of chirality parameters is related to the oriented horizontal and vertical angles between the adjacent vectors of the spin u. Defining the functional $H\colon L^1_{loc}(\mathbb{R}^2;\mathbb{R}^2) \times \mathfrak{A}_0 \to [0,+\infty]$ by setting

$$H(h;\Omega) := \begin{cases} \frac{4}{3}(|D_1 w|(\Omega) + |D_2 z|(\Omega)) & \text{if } h = (w,z) \in \mathrm{Dom}(H;\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$\mathrm{Dom}(H;\Omega) := \Big\{ (w,z) \in L^1_{loc}(\mathbb{R}^2;\mathbb{R}^2) \;:\; (w,z) \in BV(\Omega;\{-1,1\}^2), \; \mathrm{curl}(w,z) = 0 \text{ in } \mathcal{D}'(\Omega) \Big\},$$

the authors proved the following Γ-convergence result.

Theorem 0.0.12. *Assume that $\Omega \in \mathfrak{A}_0$. Then the following results hold true:*

i) *(Compactness) Let* $(w_n, z_n) \in L^1_{loc}(\mathbb{R}^2;\mathbb{R}^2)$ *be a sequence satisfying*

$$H_n(w_n, z_n; \Omega) \leqslant C,$$

for some positive constant C*. Then there exists* $(w, z) \in \mathrm{Dom}(H;\Omega)$ *such that, up to a subsequence,* $(w_n, z_n) \to (w, z)$ *in* $L^1_{loc}(\Omega;\mathbb{R}^2)$*;*

ii) *(*$\liminf$ *inequality) Let* $(w_n, z_n), (w, z) \in L^1_{loc}(\mathbb{R}^2;\mathbb{R}^2)$*. Assume that* $(w_n, z_n) \to (w, z)$ *in* $L^1_{loc}(\Omega;\mathbb{R}^2)$ *and*

$$H(w, z; \Omega) \leqslant \liminf_{n\to+\infty} H_n(w_n, z_n; \Omega);$$

iii) *(*$\limsup$ *inequality) Assume that* $(w, z) \in L^1_{loc}(\mathbb{R}^2;\mathbb{R}^2)$*. Then there exists a sequence* $(w_n, z_n) \in L^1_{loc}(\mathbb{R}^2;\mathbb{R}^2)$ *such that* $(w_n, z_n) \to (w, z)$ *in* $L^1(\Omega;\mathbb{R}^2)$ *and*

$$\limsup_{n\to+\infty} H_n(w_n, z_n; \Omega) \leqslant H(w, z; \Omega).$$

In Part II we study a frustrated lattice spin system with values on the unit sphere of $\mathbb{R}^3$. We investigate both the one-dimensional and the two-dimensional settings proposed in [17] and [15]. We force the spin of the system u to be confined in the union of two magnetic anisotropy circles, S_1 and S_2, lying on the unit sphere $\mathbb{S}^2$, both having the same radius and identified by two versors, v_1 and v_2 (see Figure 4.1). In the one-dimensional case studied in Chapter 4, we consider the following energy

$$\mathcal{E}_n = E_n + P_n,$$

with

$$P_n(\cdot) = \lambda_n k_n |D\mathcal{A}(\cdot)|(I) \tag{0.19}$$

and E_n defined as in (0.17). Here $\alpha \in (0, +\infty)$ is the frustration parameter of the system, k_n is a divergent sequence of positive numbers and $|D\mathcal{A}(u)|(I)$ counts the magnetic anistropy transitions that the spin u makes "jumping" from one circle S_i to the other one S_j (see (4.1)).

In this case, ground states are confined in one of the two magnetic anisotropy circles and turn out to have a symmetric and rigid structure similar to the one explained before.

We carry out our variational analysis when the system is close to the ferromagnet/helimagnet transition point. We estimate the amount of energy the system spends to break the symmetry and the rigidity of minimal configurations. One one hand, we compute how much energy is spent to allow spins to switch their chiralities (chirality transitions); on the other hand, we calculate the quantity of energy needed to let spins "jump" from one magnetic anisotropy circle to the other one (magnetic anisotropy transitions).

In the two-dimensional setting analyzed in Chapter 5 we deal with the functional

$$\mathcal{H}_n = H_n + P_n,$$

where P_n is defined as in (0.19) and H_n as in (0.18).

Also in this case, we address to a system close to the ferromagnet/helimagnet transition point and we find the correct scalings to detect the spin's chirality transitions.

The aforementioned results can be found in a joint work with A. Kubin, [27] .

Part I

Optimal design problems with perimeter penalization

Chapter 1

Notions and preliminaries

In this chapter we recall some basic notions and well-known properties that will be useful in the following. The chapter is divided in two sections: the first section addresses the topic of sets of finite perimeter and BV functions from geometric measure theory and is taken from Maggi's book [46]. In particular, we focus on the Gauss Green measure associated with a set of locally finite perimeter and we highlight concepts of reduced boundary and essential boundary. A subsection is entirely devoted to the definition of excess and its basic properties. The second section illustrates few basic tools of classical regularity theory, collected in the books [7] and [37]. We emphasize some consequences of Caccioppoli's inequality concerning regularity issues, i.e. Hölder continuity of minimizers, Morrey and Campanato estimates and existence of second derivatives.

1.1 Sets of finite perimeter and BV functions

We start by giving the main definition of this section.

Definition 1.1.1. *Let $E \subset \mathbb{R}^n$ be a Lebesgue measurable set. We say the E is a* ***set of locally finite perimeter*** *(in $\mathbb{R}^n$) if and only if, for every compact set $K \subset \mathbb{R}^n$, we have*

$$\sup\left\{\int_E \operatorname{div} T\,dx \;:\; T \in C_c^1(\mathbb{R}^n;\mathbb{R}^n),\, \operatorname{spt} T \subset K,\, \|T\|_\infty \leqslant 1\right\} < +\infty.$$

If this quantity is bounded independently of K, we say that E is a ***set of finite perimeter*** *(in $\mathbb{R}^n$).*

Sets of finite perimeter naturally induce a vector-valued Radon measure, satisfying a generalized Gauss-Green formula.

Proposition 1.1.2 (Distributional Gauss-Green theorem)**.** *If $E \subset \mathbb{R}^n$ is a Lebesgue measurable set, then E is a set of locally finite perimeter if and only if there exists a $\mathbb{R}^n$-valued Radon measure μ_E on $\mathbb{R}^n$ such that*

$$\int_E \operatorname{div} T\,dx = \int_{\mathbb{R}^n} T \cdot d\mu_E, \quad \forall T \in C_c^1(\mathbb{R}^n;\mathbb{R}^n). \tag{1.1}$$

The measure μ_E is unique. Moreover, E is a set of finite perimeter if and only if $|\mu_E|(\mathbb{R}^n) < +\infty$.

Proof. Let us assume that E is a set of locally finite perimeter . The linear functional $L_E\colon C^1_c(\mathbb{R}^n;\mathbb{R}^n) \to \mathbb{R}$ defined by

$$\langle L_E, T\rangle := \int_E \operatorname{div} T\, dx, \quad \forall T \in C^1_c(\mathbb{R}^n;\mathbb{R}^n),$$

is bounded; indeed, for every compact set $K \subset \mathbb{R}^n$ there exists $C = C(K) > 0$ such that $|\langle L_E, T\rangle| \leqslant C(K)\sup_{\mathbb{R}^n}|T|$. Therefore, L_E can be extended to a bounded linear functional on $C_c(\mathbb{R}^n;\mathbb{R}^n)$. The thesis follows by applying Riesz's theorem (see Theorem A.1.3) and setting $\mu_E := |L_E|$. Clearly, if E is a set of finite perimeter, then $|\mu_E|(\mathbb{R}^n) = |L_E|(\mathbb{R}^n) < +\infty$. The converse implication is fairly trivial. Indeed, let μ_E be a $\mathbb{R}^n$-valued measure on $\mathbb{R}^n$ such that (1.1) holds. Then

$$\begin{aligned}
&\sup\left\{\int_E \operatorname{div} T\, dx \;:\; T \in C^1_c(\mathbb{R}^n;\mathbb{R}^n),\ \operatorname{spt} T \subset K,\ \|T\|_\infty \leqslant 1\right\}\\
&= \sup\left\{\int_{\mathbb{R}^n} T\cdot d\mu_E \;:\; T \in C^1_c(\mathbb{R}^n;\mathbb{R}^n),\ \operatorname{spt} T \subset K,\ \|T\|_\infty \leqslant 1\right\}\\
&\leqslant \int_K |d\mu_E| = |\mu_E|(K) < +\infty,
\end{aligned}$$

which implies the thesis. Moreover, if $|\mu_E|(\mathbb{R}^n) < +\infty$, then

$$\begin{aligned}
&\sup\left\{\int_E \operatorname{div} T\, dx \;:\; T \in C^1_c(\mathbb{R}^n;\mathbb{R}^n),\ \operatorname{spt} T \subset K,\ \|T\|_\infty \leqslant 1\right\}\\
&\leqslant |\mu_E|(K) \leqslant |\mu_E|(\mathbb{R}^n) < +\infty,
\end{aligned}$$

as we wanted to prove. Finally, we show the unicity of μ_E. Let ν be a $\mathbb{R}^n$-valued measure such that

$$\int_E \operatorname{div} T\, dx = \int_{\mathbb{R}^n} T\cdot d\nu, \quad \forall T \in C^1_c(\mathbb{R}^n;\mathbb{R}^n).$$

Let us fix $i \in \{1,\cdots,n\}$. Taking (1.1) into account and choosing $T \in C^1_c(\mathbb{R}^n;\mathbb{R}^n)$ with all the components null beside the i-th one set as $T^{(i)} = \phi$, where $\phi \in C^1_c(\mathbb{R}^n)$, we get

$$\int_{\mathbb{R}^n} \phi\, d\mu_E^{(i)} = \int_{\mathbb{R}^n} \phi\, d\nu^{(i)}.$$

By the same density argument, the previous equality holds also for $\phi \in C_c(\mathbb{R}^n)$. This implies that $\mu_E^{(i)} = \nu^{(i)}$ and, by the arbitrariety of i, the assertions follows. Indeed, if $K \subset \mathbb{R}^n$ is compact and $A \subset \mathbb{R}^n$ is open such that $K \subset A$, then we can find $\phi \in C^0_c(\mathbb{R}^n)$ such that $\mathbb{1}_K \leqslant \phi \leqslant \mathbb{1}_A$. In particular

$$\mu_E^{(i)}(K) = \int_{\mathbb{R}^n} \mathbb{1}_K\, d\mu_E^{(i)} \leqslant \int_{\mathbb{R}^n} \phi\, d\mu_E^{(i)} = \int_{\mathbb{R}^n} \phi\, d\nu^{(i)} \leqslant \nu^{(i)}(A).$$

Since any Borel set $F \subset \mathbb{R}^n$ can be approximated in measure from below by compact sets and from above by open sets (see Proposition A.1.14), passing to the supremum for $K \subset F$ and to the infimum for $A \supset F$ we have $\mu_E^{(i)}(F) \leqslant \nu^{(i)}(F)$. Since $\mu_E^{(i)}$ is Borel regular, we have $\mu_E^{(i)}(F) \leqslant \nu^{(i)}(F)$, for every $F \subset \mathbb{R}^n$. On the other side, with the same argument, $\nu^{(i)}(F) \leqslant \mu_E^{(i)}(F)$, for every $F \subset \mathbb{R}^n$. This shows that $\nu^{(i)} = \mu_E^{(i)}$ on $\mathcal{P}(\mathbb{R}^n)$. □

The measure μ_E that appears in the previous theorem is called the **Gauss-Green measure** associated with E and we define respectively the **relative perimeter** of E in $F \subset \mathbb{R}^n$ and the **perimeter** of E (in $\mathbb{R}^n$) as

$$P(E;F) := |\mu_E|(F) \quad \text{and} \quad P(E) = |\mu_E|(\mathbb{R}^n).$$

Remark 1.1.3. *The equality* (1.1) *is equivalent to*

$$\int_E \nabla\phi \, dx = \int_{\mathbb{R}^n} \phi \, d\mu_E, \quad \forall \phi \in C_c^1(\mathbb{R}^n). \tag{1.2}$$

Indeed, if (1.1) *holds true, we can choose $T \in C_c^1(\mathbb{R}^n;\mathbb{R}^n)$ with all the components null beside the i-th one set as $T^{(i)} = \phi$, where $\phi \in C_c^1(\mathbb{R}^n)$. On the other hand, if we assume that* (1.2) *is true, then we can choose n functions $\phi_i \in C_c^1(\mathbb{R}^n)$ such that $\phi_i = T^{(i)}$ for any $i \in \mathbb{N}$, where $T \in C_c^1(\mathbb{R}^n;\mathbb{R}^n)$.*

Example 1.1.4. *If $E \subset \mathbb{R}^n$ is an open set with C^1-boundary, then E is a set of finite perimeter with $\mu_E = \nu_E \mathcal{H}^{n-1} \llcorner \partial E$ and $P(E;F) = \mathcal{H}^{n-1}(F \cap \partial E)$, for any $F \subset \mathbb{R}^n$.*

We recall here some useful properties of the Gauss-Green measure.

Lemma 1.1.5 (Complement)**.** *If E is a set of locally finite perimeter, then $\mathbb{R}^n \backslash E$ is a set of locally finite perimeter with*

$$\mu_{\mathbb{R}^n \backslash E} = -\mu_E, \quad P(E) = P(\mathbb{R}^n \backslash E).$$

Proof. Let $\phi \in C_c^1(\mathbb{R}^n)$. By Lemma A.3.1 we have

$$\int_{\mathbb{R}^n \backslash E} \nabla\phi \, dx = -\int_E \nabla\phi \, dx = \int_{\mathbb{R}^n} \phi \cdot (-d\mu_E).$$

Since $-\mu_E$ is a Radon measure, by Proposition 1.1.2, we get the thesis. □

Lemma 1.1.6 (Symmetric difference)**.** *If E and F are sets of locally finite perimeter, then $\mu_E = \mu_F$ on the class $\mathcal{B}(\mathbb{R}^n)$ of the Borel sets of $\mathbb{R}^n$ if and only if $|E \Delta F| = 0$.*

Proof. We start assuming that $|E\Delta F| = 0$. By Proposition 1.1.2 we get

$$\int_{\mathbb{R}^n} T \cdot \mu_E = \int_E \operatorname{div} T \, dx = \int_F \operatorname{div} T \, dx = \int_{\mathbb{R}^n} T \cdot \mu_F, \quad \forall T \in C_c^1(\mathbb{R}^n, \mathbb{R}^n).$$

By the unicity of the Gauss-Green measure, we infer that $\mu_E = \mu_F$. On the other hand, if $\mu_E = \mu_F$, the assertion is trivial when $|F| = 0$ or $|\mathbb{R}^n \backslash F| = 0$. We may assume that $|F| \neq 0$ and $|\mathbb{R}^n \backslash F| \neq 0$. In this case, by Remark 1.1.3, we get

$$\int_{\mathbb{R}^n} (\mathbb{1}_E - \mathbb{1}_F)\nabla\phi = 0, \quad \forall \phi \in C_c^1(\mathbb{R}^n),$$

which implies the existence of a constant c such that $\mathbb{1}_E - \mathbb{1}_F = c \in \{0, 1\}$ for a.e. $x \in \mathbb{R}^n$. Assume by contradiction that $c = 1$. Then $|E| = |\mathbb{R}^n \backslash F|$ and, using the implication proved before and Lemma 1.1.5, we infer

$$\mu_F = \mu_E = \mu_{\mathbb{R}^n \backslash F} = -\mu_F,$$

which contradicts that $|F| \neq 0$ and $|\mathbb{R}^n \backslash F| \neq 0$. Hence, $c = 1$ and thus $|E \Delta F| = 0$. □

Lemma 1.1.7 (Gauss-Green measure of blow-ups)**.** *If E is a set of locally finite perimeter, $x \in \mathbb{R}^n$ and $r > 0$, then $E_{x,r} := \frac{E-x}{r}$ is a set of finite perimeter in $\mathbb{R}^n$ with*

$$\mu_{E_{x,r}} = \frac{(\Phi_{x,r})_\sharp \mu_E}{r^{n-1}},$$

where $\Phi_{x,r}(y) := \frac{y-x}{r}$, for $y \in \mathbb{R}^n$ and $(\Phi_{x,r})_\sharp$ is as in Definition A.1.5.

Proof. If $\phi \in C_c^1(\mathbb{R}^n)$ and $\phi_{x,r} = \phi \circ \Phi_{x,r}$, then $\nabla \phi_{x,r} = r^{-1}(\nabla\phi \circ \Phi_{x,r})$ and

$$\int_{E_{x,r}} \nabla\phi \, dy = \frac{1}{r^{n-1}} \int_E \nabla\phi_{x,r} \, dz = \frac{1}{r^{n-1}} \int_{\mathbb{R}^n} \phi_{x,r} \, d\mu_E = \int_{\mathbb{R}^n} \phi \, d\frac{(\Phi_{x,r})_\sharp \mu_E}{r^{n-1}},$$

by Proposition A.1.6. Since $r^{1-n}(\Phi_{x,r})_\sharp \mu_E$ is a Radon measure, $E_{x,r}$ is a set of locally finite perimeter with $\mu_{E_{x,r}} = r^{1-n}(\Phi_{x,r})_\sharp \mu_E$. □

Proposition 1.1.8 (Support of the Gauss-Green measure)**.** *If E is a set of locally finite perimeter, then*

$$\operatorname{spt} \mu_E = \{x \in \mathbb{R}^n \, : \, 0 < |E \cap B_r(x)| < \omega_n r^n, \, \forall r > 0\} \subset \partial E.$$

Moreover, there exists a Borel set F such that

$$|E \Delta F| = 0, \quad \operatorname{spt} \mu_F = \partial F.$$

Proof. **Step 1:** If $x \in \mathbb{R}^n$ is such that $x \notin \{x \in \mathbb{R}^n \, : \, 0 < |E \cap B_r(x)| < \omega_n r^n, \, \forall r > 0\}$, then two alternatives may occur: $|E \cap B_r(x)| = 0$ or $|E \cap B_r(x)| = \omega_n r^n$, for some $r > 0$. If $|E \cap B_r(x)| = 0$, then, by Remark 1.1.3,

$$|\mu_E|(B_r(x)) = \sup_{\substack{\phi \in C_c^1(B_r(x)) \\ |\phi| \leqslant 1}} \int_{\mathbb{R}^n} \phi \, d\mu_E = \sup_{\substack{\phi \in C_c^1(B_r(x)) \\ |\phi| \leqslant 1}} \int_E \nabla\phi \, dx = 0,$$

which implies that $\mu_E(B_r(x)) = 0$ because $\mu_E << |\mu_E|$. Thus $x \notin \operatorname{spt} \mu_E$. On the other hand, if $|E \cap B_r(x)| = \omega_n r^n$, then by Lemma A.3.1 and Remark 1.1.3,

$$|\mu_E|(B_r(x)) = \sup_{\substack{\phi \in C_c^1(B_r(x)) \\ |\phi| \leqslant 1}} \int_{\mathbb{R}^n} \phi \, d\mu_E = \sup_{\substack{\phi \in C_c^1(B_r(x)) \\ |\phi| \leqslant 1}} \int_E \nabla \phi \, dx$$
$$= \sup_{\substack{\phi \in C_c^1(B_r(x)) \\ |\phi| \leqslant 1}} \int_{\mathbb{R}^n} \nabla \phi \, dx = 0,$$

which again implies that $\mu_E(B_r(x)) = 0$. In order to prove the other inclusion, we assume that $x \notin \operatorname{spt} \mu_E$. Then $|\mu_E|(B_r(x)) = 0$ for some $r > 0$ and, with the same argument, we infer

$$0 = \int_{\mathbb{R}^n} \phi \, d\mu_E = \int_E \nabla \phi \, dx = \int_{\mathbb{R}^n} \mathbb{1}_E \nabla \phi \, dx, \quad \forall \phi \in C_c^1(B_r(x)).$$

Consequently, there exists $c \in \mathbb{R}$ such that $\mathbb{1}_E = c$ a.e. on $B_r(x)$. Necessarily $c \in \{0, 1\}$ and, correspondingly, $|E \cap B_r(x)| \in \{0, \omega_n r^n\}$, which completes the first part of the proof.

Step 2: Up to modifying E on a set of measure zero, we may assume that E is a Borel set. We now construct a Borel set F with $|F \Delta E| = 0$ and

$$\partial F = \{x \in \mathbb{R}^n \,:\, 0 < |F \cap B_r(x)| < \omega_n r^n \text{ for every } r > 0\}.$$

To this end, let us define two disjoint open sets by setting

$$A_0 = \{x \in \mathbb{R}^n \,:\, \text{there exists } r > 0 \text{ s.t. } |E \cap B_r(x)| = 0\},$$

$$A_1 = \{x \in \mathbb{R}^n \,:\, \text{there exists } r > 0 \text{ s.t. } |E \cap B_r(x)| = \omega_n r^n\},$$

and consider a sequence $\{x_h\}_{h \in \mathbb{N}} \subset A_0$ such that $A_0 \subset \bigcup_{h \in \mathbb{N}} B_{r_h}(x_h)$, $r_h > 0$ and $|E \cap B_{r_h}(x_h)| = 0$. Hence $|E \cap A_0| = 0$ and, by Lemma 1.1.5, we also have $|A_1 \backslash E| = 0$. Therefore, setting $F := (A_1 \cup E) \backslash A_0$, then F is a Borel set, with

$$|F \backslash E| \leqslant |A_1 \backslash E| = 0 \quad \text{and} \quad |E \backslash F| \leqslant |E \cap A_0| = 0,$$

that is $|E \Delta F| = 0$. By step 1 and Lemma 1.1.6, $\mathbb{R}^n \backslash (A_0 \cup A_1) = \operatorname{spt} \mu_E = \operatorname{spt} \mu_F \subset \partial F$. Finally, at the same time, $\partial F \subset \mathbb{R}^n \backslash (A_0 \cup A_1)$, since, by construction,

$$A_1 \subset \mathring{F} \quad \text{and} \quad \overline{F} \subset \mathbb{R}^n \backslash A_0.$$

□

We now recall two well-known results, i.e. the relative isoperimetric inequality in a ball and an approximation theorem.

Proposition 1.1.9 (Relative isoperimetric inequality). *If $n \geqslant 2$, $t \in (0,1)$, $x \in \mathbb{R}^n$ and $r > 0$, then there exists a positive constant $c = c(n,t)$ such that*

$$P(E; B_r(x)) \geqslant c|E \cap B_r(x)|^{\frac{n-1}{n}},$$

for every set of locally finite perimeter E such that $|E \cap B_r(x)| \leqslant t|B_r(x)|$. In particular, if $E \subset B_r(x)$, then

$$P(E; B_r(x)) \geqslant c \min\{|E \cap B_r(x)|, |B_r(x) \backslash E|\}^{\frac{n-1}{n}},$$

for some positive constant $c = c(n)$.

Theorem 1.1.10 (Approximation by smooth sets). *A Lebesgue measurable set $E \subset \mathbb{R}^n$ is locally of finite perimeter if and only if there exist a sequence $\{E_h\}_{h\in\mathbb{N}} \subset \mathbb{R}^n$ of open sets with smooth boundary and $\varepsilon_h \to 0^+$, such that*

$$E_h \overset{loc}{\to} E, \qquad \sup_{h\in\mathbb{N}} P(E_h; B_R) < +\infty, \quad \forall R > 0,$$

$$|\mu_{E_h}| \overset{\star}{\rightharpoonup} |\mu_E|, \quad \partial E_h \subset I_{\varepsilon_h}(\partial E).$$

In particular, $P(E_h; F) \to P(E; F)$, whenever $P(E; \partial F) = 0$. Moreover,

i) if $|E| < +\infty$, then $E_h \to E$;

ii) if $P(E) < +\infty$, then $P(E_h) \to P(E)$.

1.1.1 The reduced boundary and the essential boundary

The key notion to consider in order to understand the geometric structure of sets of finite perimeter is that of reduced boundary, which provides a general definition of unitary normal vector.

Definition 1.1.11. *The* ***reduced boundary*** *of a set E of locally finite perimeter is defined as*

$$\partial^* E := \left\{ x \in \operatorname{spt} \mu_E \,:\, \exists \lim_{r\to 0^+} \frac{\mu_E(B_r(x))}{|\mu_E|(B_r(x))} =: \nu_E(x) \in \mathbb{S}^{n-1} \right\}.$$

The function $\nu_E \colon \partial^* E \to \mathbb{S}^{n-1}$ is the restriction of the $|\mu_E|$-density of μ_E on $\partial^* E$ and is a Borel function. We call ν_E the **outer unit normal to** E. Since $\mu_E << |\mu_E|$, by the Radon-Nikodym theorem (see Theorem A.1.8), we have

$$\mu_E = \nu_E |\mu_E| \llcorner \partial^* E, \tag{1.3}$$

so that the distributional Gauss-Green theorem (Proposition 1.1.2) takes the form

$$\int_E \nabla\phi \, dx = \int_{\partial^* E} \phi \, \nu_E \, d|\mu_E|, \quad \forall \phi \in C^1_c(\mathbb{R}^n).$$

Remark 1.1.12. *By definition and by Proposition 1.1.8, $\partial^* E \subset \operatorname{spt}\mu_E \subset \partial E$. In fact, by* (1.3)*, the Gauss-Green measure μ_E is concentrated on $\partial^* E$, and hence on $\overline{\partial^* E}$. By definition of support, $\operatorname{spt}\mu_E \subset \overline{\partial^* E}$, and therefore $\operatorname{spt}\mu_E = \overline{\partial^* E}$. By Proposition 1.1.8, up to modifying E on a set of Lebesgue measure zero, we have that $\operatorname{spt}\mu_E = \partial E$. Therefore, up to modification on sets of Lebesgue measure zero, $\overline{\partial^* E} = \partial E$.*

From the following theorem it turns out that the reduced boundary $\partial^* E$ of a set E of locally finite perimeter has the structure of a $(n-1)$-dimensional hypersurface and that ν_E has a precise geometric meaning as the outer unit normal to E.

Theorem 1.1.13 (De Giorgi's structure theorem)**.** *If E is a set of locally finite perimeter, then the Gauss-Green measure μ_E of E satisfies*

$$\mu_E = \nu_E \mathcal{H}^{n-1} \llcorner \partial^* E, \quad |\mu_E| = \mathcal{H}^{n-1} \llcorner \partial^* E,$$

and the generalized Gauss-Green formula holds true:

$$\int_E \nabla\phi\, dx = \int_{\partial^* E} \phi\, \nu_E\, d\mathcal{H}^{n-1}, \quad \forall \phi \in C^1_c(\mathbb{R}^n).$$

Moreover, there exist countably many C^1-hypersurfaces $M_h \subset \mathbb{R}^n$, compact sets $K_h \subset M_h$, and a Borel set F with $\mathcal{H}^{n-1}(F) = 0$, such that

$$\partial^* E = F \cup \bigcup_{h\in\mathbb{N}} K_h,$$

and, for every $x \in K_h$, $\nu_E(x)^\perp = T_x M_h$, the tangent space to M_h at x.

The next theorem establishes that the blow-up around a point of the reduced boundary tends to a half-space. As a corollary, two density results are proved.

Theorem 1.1.14. *If E is a set of locally finite perimeter and $x \in \partial^* E$, then*

$$E_{x,r} \stackrel{\text{loc}}{\to} H_x := \{y \in \mathbb{R}^n \,:\, y \cdot \nu_E(x) \leqslant 0\},$$

as $r \to 0^+$. Similarly, if $\pi_x := \partial H_x = \nu_E(x)^\perp$, then, as $r \to 0^+$,

$$\mu_{E_{x,r}} \stackrel{*}{\rightharpoonup} \nu_E(x)\mathcal{H}^{n-1} \llcorner \pi_x, \quad |\mu_{E_{x,r}}| \stackrel{*}{\rightharpoonup} \mathcal{H}^{n-1} \llcorner \pi_x.$$

Corollary 1.1.15. *If E is a set of locally finite perimeter and $x \in \partial^* E$, then*

$$\lim_{r\to 0^+} \frac{|E \cap B_r(x)|}{\omega_n r^n} = \frac{1}{2},$$

$$\lim_{r\to 0^+} \frac{P(E; B_r(x))}{\omega_{n-1} r^{n-1}} = 1.$$

In particular, $\partial^* E \subset E^{(1/2)}$.

Proof. Let $x \in \partial^* E$, $H_x := \{y \in \mathbb{R}^n : y \cdot \nu_E(x) \leqslant 0\}$ and $\pi_x := \partial H_x$. Since $|H_x \cap B_1| = \frac{\omega_n}{2}$, by the local convergence of $E_{x,r}$ to H_x stated in Theorem 1.1.14, we infer

$$\lim_{r\to 0^+} \frac{|E \cap B_r(x)|}{\omega_n r^n} = \lim_{r\to 0^+} \frac{|E_{x,r} \cap B_1|}{\omega_n} = \frac{H_x \cap B_1}{\omega_n} = \frac{1}{2}.$$

Since $\pi_x \cap \partial B_1$ is an $(n-2)$-dimensional unit sphere, we have $\mathcal{H}^{n-1}(\pi_x \cap \partial B_1) = 0$. Thus, by Lemma 1.1.7, Theorem 1.1.14 and Proposition A.1.14,

$$\lim_{r\to 0^+} \frac{P(E; B_r(x))}{\omega_{n-1} r^{n-1}} = \lim_{r\to 0^+} \frac{|\mu_{E_{x,r}}|(B_1)}{\omega_{n-1}} = \frac{\mathcal{H}^{n-1}(\pi_x \cap B_1)}{\omega_{n-1}} = 1.$$

□

We recall another useful definition in geometric measure theory.

Definition 1.1.16 (Essential boundary)**.** *Let $E \subset \mathbb{R}^n$ be a Lebesgue measurable set. We define the* ***essential boundary*** *$\partial^e E$ of E as*

$$\partial^e E := \mathbb{R}^n \backslash \big(E^{(0)} \cup E^{(1)}\big),$$

where

$$E^{(t)} := \left\{x \in \mathbb{R}^n : \lim_{r\to 0^+} \frac{|E \cap B_r(x)|}{\omega_n r^n} = t\right\}$$

is the ***sets of points of density*** *t* ***of*** *E.*

We prove Federer's theorem, stating the $\mathcal{H}^{n-1}$-equivalence between the reduced boundary of E, the set $E^{(1/2)}$ of its points of density one-half, and the essential boundary of E.

Theorem 1.1.17 (Federer's Theorem)**.** *If E is a set of locally finite perimeter, then $\partial^* E \subset E^{(1/2)} \subset \partial^e E$, with $\mathcal{H}^{n-1}(\partial^e E \backslash \partial^* E) = 0$.*

Proof. Of course $E^{(1/2)} \subset \partial^e E$. By the relative isoperimetric inequality (see Proposition 1.1.9), and since

$$|E \cap B_r(x)| \leqslant |B_r(x)|^{\frac{1}{n}} |E \cap B_r(x)|^{\frac{n-1}{n}} \leqslant \omega_n^{\frac{1}{n}} r |E \cap B_r(x)|^{\frac{n-1}{n}},$$

we find that

$$\frac{P(E; B_r(x))}{r^{n-1}} \geqslant c(n) \min\left\{\frac{|E \cap B_r(x)|}{r^n}, \frac{|B_r(x) \backslash E|}{r^n}\right\}.$$

Thus, passing to the upper limit as $r \to 0^+$ in the last inequality, we infer that if

$\limsup_{r\to 0^+} \dfrac{P(E; B_r(x))}{r^{n-1}} = 0$ implies $x \in E^{(0)} \cup E^{(1)}$. In particular,

$$\partial^e E \subset \left\{x \in \mathbb{R}^n : \limsup_{r\to 0^+} \frac{P(E; B_r(x))}{r^{n-1}} > 0\right\},$$

so that

$$\partial^e E \backslash \partial^* E \subset \left\{ x \in \mathbb{R}^n \backslash \partial^* E \ : \ \limsup_{r \to 0^+} \frac{P(E; B_r(x))}{r^{n-1}} > 0 \right\}.$$

By Proposition A.1.11, this last set is $\mathcal{H}^{n-1}$-negligible. □

We recall a representation formula for Gauss–Green measures of intersections of two sets of locally finite perimeter, which can be derived by Federer's theorem.

Theorem 1.1.18 (Gauss-Green measure of the intersection of sets)**.** *If E and F are subsets of $\mathbb{R}^n$ of locally finite perimeter and we let*

$$\{\nu_E = \nu_F\} := \{x \in \partial^* E \cap \partial^* F \ : \ \nu_E(x) = \nu_F(x)\},$$

then $E \cap F$ is a set of locally finite perimeter, with

$$\mu_{E \cap F} = \mu_E \llcorner F^{(1)} + \mu_F \llcorner E^{(1)} + \nu_E \mathcal{H}^{n-1} \llcorner \{\nu_E = \nu_F\}.$$

Theorem 1.1.19 (Comparison sets by replacement)**.** *If $E, G \subset \mathbb{R}^n$ are sets of locally finite perimeter and $A \subset \mathbb{R}^n$ is an open set of finite perimeter such that*

$$\mathcal{H}^{n-1}(\partial^* A \cap \partial^* E) = \mathcal{H}(\partial^* A \cap \partial^* G) = 0,$$

then $F := (G \cap A) \cup (E \backslash A)$ is a set of locally finite perimeter. Moreover, if $A \subset\subset A'$, with $A' \subset \mathbb{R}^n$ open, then

$$P(F; A') = P(G; A) + P(E; A' \backslash \overline{A}) + \mathcal{H}^{n-1}((E^{(1)} \Delta G^{(1)}) \cap \partial^* A).$$

1.1.2 Excess

We introduce the fundamental notion of excess $\mathbf{e}(E, x, r)$, a key concept in the regularity theory for Λ-minimizers of the perimeter. It is used to measure the integral oscillation of the outer unit normal to E over $B_r(x) \cap \partial^* E$.

Definition 1.1.20. *Let E be a set of locally finite perimeter, $x \in \partial E$, $r > 0$ and $\nu \in \mathbb{S}^{n-1}$. We define:*

- *the **cylindrical excess** of E at the point x, at the scale r and with respect to the direction ν, as*

$$\begin{aligned} \mathbf{e}^C(E, x, r, \nu) &:= \frac{1}{r^{n-1}} \int_{\mathbf{C}_r(x,\nu) \cap \partial^* E} \frac{|\nu_E - \nu|^2}{2} d\,\mathcal{H}^{n-1} \\ &= \frac{1}{r^{n-1}} \int_{\mathbf{C}_r(x,\nu) \cap \partial^* E} (1 - \nu_E \cdot \nu)\, d\mathcal{H}^{n-1}; \end{aligned}$$

- *the **spherical excess** of E at the point x, at the scale r and with respect to the direction ν, as*

$$\mathbf{e}(E, x, r, \nu) := \frac{1}{r^{n-1}} \int_{\partial E \cap B_r(x)} \frac{|\nu_E - \nu|^2}{2} d\mathcal{H}^{n-1};$$

- *the* ***spherical excess*** *of E at the point x and at the scale r, as*

$$\mathbf{e}(E,x,r) := \min_{\nu\in\mathbb{S}^{n-1}} \mathbf{e}(E,x,r,\nu).$$

We do not stress the dependence on E when it is clear from the contest. We will often denote the excess of E at the point x, at the scale r and with respect to the direction e_n by $\mathbf{e}_n(x,r)$. Furthermore, if $x=0$, we will write $\mathbf{e}_n(r)$ and, finally, if also $r=1$, we will simply write $\mathbf{e_n}$.

We recall below some properties of the excess.

Proposition 1.1.21 (Scaling of the excess)**.** *If E is a set of locally finite perimeter, $x\in\partial E$, $r>0$, $\nu\in\mathbb{S}^{n-1}$, then*

$$\mathbf{e}^C(E,x,r,\nu) = \mathbf{e}^C(E_{x,r},0,1,\nu), \quad \mathbf{e}(E,x,r) = \mathbf{e}(E_{x,r},0,1),$$

where, as usual, $E_{x,r} = \frac{E-x}{r}$.

Proof. Since $|\nu-\nu_E|^2 = 2(1-\nu\cdot\nu_E)$, using Lemma 1.1.7, we get

$$\begin{aligned}
\mathbf{e}^C(E,x,r,\nu) &= \frac{|\mu_E|(\mathbf{C}_r(x,\nu)) - \nu\cdot\displaystyle\int_{\mathbf{C}_r(x,\nu)\cap\partial^*E} \nu_E\, d\mathcal{H}^{n-1}}{r^{n-1}} \\
&= \frac{|\mu_E|(\mathbf{C}_r(x,\nu)) - \nu\cdot\mu_E(\mathbf{C}_r(x,\nu))}{r^{n-1}} \\
&= \frac{|\mu_E|\circ\Phi_{x,r}^{-1}(\mathbf{C}_1(0,\nu)) - \nu\cdot(\mu_E\circ\Phi_{x,r}^{-1})(\mathbf{C}_1(0,\nu))}{r^{n-1}} \\
&= \frac{\big((\Phi_{x,r})_\sharp|\mu_E|\big)(\mathbf{C}_1(0,\nu))}{r^{n-1}} - \frac{\nu\cdot\big((\Phi_{x,r})_\sharp\mu_E\big)(\mathbf{C}_1(0,\nu))}{r^{n-1}} \\
&= |\mu_{E_{x,r}}|(\mathbf{C}_1(0,\nu)) - \mu_{E_{x,r}}(\mathbf{C}_1(0,\nu)) = \mathbf{e}^C(E_{x,r},0,1,\nu),
\end{aligned}$$

where $(\Phi_{x,r})_\sharp\mu_E$ is as in Definition A.1.5. Similarly, applying Lemma 1.1.7 again, we have

$$\begin{aligned}
\mathbf{e}(E,x,r) &= \min_{\nu\in\mathbb{S}^{n-1}} \frac{|\mu_E|(B_r(x)) - \nu\cdot\mu_E(B_r(x))}{r^{n-1}} \\
&= \frac{1}{r^{n-1}}\Big(|\mu_E|(B_r(x)) - \max_{\nu\in\mathbb{S}^{n-1}} \nu\cdot\mu_E(B_r(x))\Big) \\
&= \frac{|\mu_E|(B_r(x))}{r^{n-1}}\left(1 - \frac{|\mu_E(B_r(x))|}{|\mu_E|(B_r(x))}\right) \\
&= |\mu_{E_{x,r}}|(B_1(0))\left(1 - \frac{|\mu_{E_{x,r}}(B_1(0))|}{|\mu_{E_{x,r}}|(B_1(0))}\right) = \mathbf{e}(E_{x,r},0,1).
\end{aligned}$$

□

Proposition 1.1.22 (Excess at different scales)**.** *If E is a set of locally finite perimeter, $x\in\partial E$, $0<s<r$, $\nu\in\mathbb{S}^{n.1}$, then*

$$\mathbf{e}^C(E,x,s,\nu) \leqslant \left(\frac{r}{s}\right)^{n-1} \mathbf{e}^C(E,x,r,\nu).$$

Proof. The proof follows directly from the definition of the cylindrical excess, since $B_s(x) \subset B_r(x)$. □

The next proposition states that the sets of locally finite perimeter with null cylindrical excess in some point of their boundary are locally a half-space.

Proposition 1.1.23 (Zero excess implies being a half-space)**.** *If E is a set of locally finite perimeter, with* $\operatorname{spt}\mu_E = \partial E$, $x \in \partial E$, $r > 0$, $\nu \in \mathbb{S}^{n-1}$, *then*

$$\mathbf{e}^C(E, x, r, \nu) = 0$$

if and only if $|E \cap \mathbf{C}_r(x,\nu)| = |\{y \in \mathbf{C}_r(x,\nu) \,:\, (y-x)\cdot\nu \leqslant 0\}|$.

Proof. Let us assume that $|E \cap \mathbf{C}_r(x,\nu)| = |\{y \in \mathbf{C}_r(x,\nu) \,:\, (y-x)\cdot\nu \leqslant 0\}|$. Then, by Lemma 1.1.6, $\mu_E = \mu_{\{y\in\mathbb{R}^n \,:\, (y-x)\cdot\nu\leqslant 0\}}$ on subsets of $\mathbf{C}_r(x,\nu)$ and thus

$$\begin{aligned}\mathbf{e}^C(E,x,r,\nu) &= \frac{|\mu_E|(\mathbf{C}_r(x,\nu)) - \nu\cdot\mu_E(\mathbf{C}_r(x,\nu))}{r^{n-1}}\\ &= \frac{|\mu_{\{y\in\mathbb{R}^n \,:\, (y-x)\cdot\nu\leqslant 0\}}|(\mathbf{C}_r(x,\nu)) - \nu\cdot\mu_{\{y\in\mathbb{R}^n \,:\, (y-x)\cdot\nu\leqslant 0\}}(\mathbf{C}_r(x,\nu))}{r^{n-1}}\\ &= \mathbf{e}^C(\{y\in\mathbb{R}^n \,:\, (y-x)\cdot\nu\leqslant 0\}, x, r, \nu) = 0.\end{aligned}$$

On the other hand, if $\mathbf{e}^C(E,x,r,\nu) = 0$, then, by definition, $\nu_E(y) = \nu$ for $\mathcal{H}^{n-1}$-a.e. $y \in \mathbf{C}_r(x,\nu) \cap \partial E$, which implies the thesis. □

As stated in the assertion of the following result, the reduced boundary is made up of points where the excess is very small.

Proposition 1.1.24 (Vanishing of the excess at the reduced boundary)**.** *If E is a set of locally finite perimeter and $x \in \partial^* E$, then*

$$\lim_{r\to 0^+} \mathbf{e}(E,x,r) = 0.$$

Hence, given $\varepsilon > 0$, there exist $r > 0$ and $\nu \in \mathbb{S}^{n-1}$ with $\mathbf{e}^C(E,x,r,\nu) < \varepsilon$.

Proof. Let us fix $x \in \partial^* E$. By the definition of reduced boundary and by Corollary 1.1.15 we have

$$\lim_{r\to 0^+} \frac{|\mu_E(B_r(x))|}{|\mu_E|(B_r(x))} = |\nu_E(x)| = 1, \quad \text{and} \quad \lim_{r\to 0^+} \frac{|\mu_E|(B_r(x))}{\omega_{n-1} r^{n-1}} = 1.$$

Thus, reasoning as in Proposition 1.1.21,

$$\lim_{r\to 0^+} \mathbf{e}(E,x,r) = \lim_{r\to 0^+} \frac{|\mu_E|(B_r(x))}{r^{n-1}}\left(1 - \frac{|\mu_E(B_r(x))|}{|\mu_E|(B_r(x))}\right) = 0.$$

Finally, since $\mathbf{C}_r(x,\nu) \subset B_{\sqrt{2}r}(x)$ for every $\nu \in \mathbb{S}^{n-1}$, we infer that if $r > 0$ is such that $\mathbf{e}(E,x,r) < \varepsilon$ for some $\varepsilon > 0$, then, by the definition of spherical excess, there exists $\nu \in \mathbb{S}^{n-1}$ such that

$$\begin{aligned}\mathbf{e}^C(E,x,r,\nu) &\leqslant 2^{\frac{1-n}{2}}\mathbf{e}^C\left(E, x, \frac{r}{\sqrt{2}}, \nu\right) \leqslant \frac{1}{r^{n-1}}\int_{\partial^* E\cap B_r(x)} (1 - \nu_E\cdot\nu)\, d\mathcal{H}^{n-1}\\ &= \mathbf{e}(E,x,r) < \varepsilon,\end{aligned}$$

where the first inequality is due to Proposition 1.1.22. □

1.1.3 *BV* functions

The notion of BV functions is a generalization of the notion of sets of finite perimeter. Indeed, a set is of (locally) finite perimeter if and only if its characteristic function is a (locally) BV function.

Definition 1.1.25 (BV functions)**.** *Let $u \in L^1(\Omega;\mathbb{R}^m)$. We say that u is a* ***function of bounded variation*** *(or* ***BV function****) in Ω if and only if the distributional derivative of u is representable by a finite $\mathbb{R}^{nm}$-valued Radon measure $Du = (D_i u^\alpha)_{\substack{i\in\{1,\dots,n\}\\ \alpha\in\{1,\dots,m\}}}$ in Ω, i.e., for any $i \in \{1,\dots,n\}$ and $\alpha \in \{1,\dots,m\}$,*

$$\int_\Omega u^\alpha \nabla_{x_i}\phi \, dx = -\int_\Omega \phi \, dD_i u^\alpha, \quad \forall \phi \in C^1_c(\Omega).$$

The vector space of all functions of bounded variation in Ω is denoted by $BV(\Omega;\mathbb{R}^m)$. Furthermore, we set the space of those BV functions that take values in $S \subset \mathbb{R}^m$ as $BV(\Omega;S)$.

Proposition 1.1.26 (Lower semicontinuity of the total variation of the distributional gradient)**.** *Let $u \in L^1(\Omega;\mathbb{R}^m)$. The map $u \longmapsto |Du|(\Omega)$ is lower semicontinuous in $BV(\Omega;\mathbb{R}^m)$ with respect to the $L^1_{loc}(\Omega;\mathbb{R}^m)$ topology.*

Since the gradient of a BV function is a Radon measure, it is possibile to introduce the notion of weak-star convergence. Actually, we give its characterization as a definition (see Definition A.1.13).

Definition 1.1.27 (Weak-star convergence of BV functions)**.** *Let $\{u_h\}_{h\in\mathbb{N}} \subset BV(\Omega;\mathbb{R}^m)$ and $u \in BV(\Omega;\mathbb{R}^m)$. We say that u_h* ***weakly-star converges*** *in $BV(\Omega;\mathbb{R}^m)$ to u if and only if $u_h \to u$ in $L^1(\Omega;\mathbb{R}^m)$ and*

$$\sup_{h\in\mathbb{N}} |Du_h|(\Omega) < +\infty.$$

1.2 Some tools from the regularity theory

In this section we introduce some useful and classical tools from the regularity theory of minimizers of quadratic functionals or solutions of quadratic partial differential equations in divergence form.

Actually, the two subjects are closely related to each other; indeed, it is well-known that minimizers of functionals, under very general assumptions, are weak solutions of the associated Euler-Lagrange equation. Conversely, under dual assumptions, solutions of a partial differential equations are quasi-minima of a suitable functional.

Neverthless, we recall the results in a convenient formulation that will be suitable for our aims in Part I. Our model functional is

$$\mathcal{F}(u,\Omega) := \int_\Omega F(x,u(x),\nabla u(x)) \, dx, \quad \forall u \in W^{1,p}(\Omega),$$

where $F\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a measurable function. We will often write $\mathcal{F}(u)$ in lieu of $\mathcal{F}(E, u)$ if F is integrated over the whole space Ω.

For interior and boundary regularity results regarding $\mathcal{F}$, we shall make use of local and global minimizers.

Definition 1.2.1 (Local and global minimizers). *Let $u_0 \in W^{1,p}(\Omega)$. We say that*

- *$u \in W^{1,p}_{loc}(\Omega)$ is a local minimizer of the functional $\mathcal{F}$ if for every $\phi \in W^{1,p}(\Omega)$, with $\operatorname{spt}\phi \subset\subset \Omega$ we have*
$$\mathcal{F}(u, \operatorname{spt}\phi) \leqslant \mathcal{F}(u+\phi, \operatorname{spt}\phi);$$
- *$u \in u_0 + W^{1,p}_0(\Omega)$ is a minimizer of the functional $\mathcal{F}$ if for every $v \in u_0 + W^{1,p}_0(\Omega)$ we have*
$$\mathcal{F}(u) \leqslant \mathcal{F}(v).$$

We are interested in integrands with quadratic growth of the type

$$F(x,\xi) = \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j, \quad \forall (x,\xi) \in \Omega \times \mathbb{R}^n.$$

The Euler-Lagrange equation associated with the corrispondent class of functionals is

$$\nabla_j(a_{ij}(x)\nabla_i u) = 0, \quad \forall x \in \Omega,$$

where we have adopted the Einstein's notation for which repeated indices are implicitly summed over.

The first powerful tool of regularity theory is Caccioppoli's inequality. We will see in this section how the it can be employed, following an idea due to L. Nirenberg, to prove existence of higher-order weak derivatives of minimizers of quadratic (and also more general) functionals and suitable integrability results thereof, and how to translate these estimates into actual regularity results by means of the Sobolev embedding theorems. The latter aim will be pursued mostly in Part I.

Theorem 1.2.2 (Caccioppoli's inequality). *Let $u \in H^1_{loc}(\Omega)$ be a local weak solution of*

$$\nabla_j(a_{ij}\nabla_i u) = 0 \quad \text{in } \Omega, \tag{1.4}$$

with measurable coefficients $a_{ij}\colon \Omega \to \mathbb{R}$ satisfying

$$\nu|\xi|^2 \leqslant a_{ij}(x)\xi_i\xi_j \leqslant N|\xi|^2, \quad \forall (x,\xi) \in \Omega \times \mathbb{R}^n,$$

for some positive constants ν and N. Then, for any $B_{2\rho}(x_0) \subset \Omega$, it holds

$$\int_{B_\rho(x_0)} |\nabla u|^2\,dx \leqslant \frac{c(\nu,N)}{\rho^2}\int_{B_{2\rho}(x_0)} (u - u_{x_0,2\rho})^2\,dx.$$

Proof. Let $\phi := (u - u_{x_0,2\rho})\eta^2 \in H^1_0(B_{2\rho}(x_0))$, where $\eta \in C^1_c(B_{2\rho}(x_0))$ is a cut-off function such that $0 \leqslant \eta \leqslant 1$, $\eta = 1$ in $B_\rho(x_0)$ and $|\nabla\eta| \leqslant \frac{c}{\rho}$, for some positive constant c. Then, plugging ϕ as a test function in the weak formulation of (1.4), we get

$$\int_{B_{2\rho}(x_0)} a_{ij}\nabla_i u \nabla_j u \eta^2 \, dx = -2\int_{B_{2\rho}(x_0)} a_{ij}\eta(u - u_{x_0,2\rho})\nabla_i u \nabla_j \eta \, dx.$$

Using the uniform ellipticity and the uniform boundedness condition on a_{ij}, by Young's inequality, we infer

$$\nu \int_{B_{2\rho}(x_0)} \eta^2|\nabla u|^2 \, dx \leqslant N\varepsilon \int_{B_{2\rho}(x_0)} \eta^2|\nabla u|^2 \, dx + \frac{N}{\varepsilon}\int_{B_{2\rho}(x_0)} (u - u_{x_0,2\rho})^2|\nabla\eta|^2 \, dx,$$

for any positive ε. Choosing $\varepsilon = \varepsilon(\nu, N)$ sufficiently small, it holds

$$\int_{B_{2\rho}(x_0)} \eta^2|\nabla u|^2 \, dx \leqslant c(\nu, N)\int_{B_{2\rho}(x_0)} (u - u_{x_0,2\rho})^2|\nabla\eta|^2 \, dx.$$

Finally, the properties of η yield to the thesis. □

Definition 1.2.3 (Superlevel sets)**.** *Let $u \in H^1_{loc}(\Omega)$, $B_R(x_0) \subset \Omega$ and $k \in \mathbb{R}$. We define the superlevel set of u as*

$$A(k, R) := \{x \in B_R(x_0) \, : \, u(x) > k\}.$$

A variant of Caccioppoli's inequality on superlevel sets will be the main tool to prove the Hölder continuity of local minimizers of quadratic functionals.

Remark 1.2.4 (Caccioppoli's inequality on superlevel sets)**.** *We point out that choosing $\phi := (u - k)_+\eta^2$, with $k \geqslant 0$, as a test function in the weak formulation of* (1.5) *and following the same proof, we get*

$$\int_{A(k,\rho)} |\nabla u|^2 \, dx \leqslant \frac{c(\nu, N)}{\rho^2}\int_{A(k,2\rho)} (u - k)^2 \, dx.$$

A rather surprising feature of Caccioppoli's inequality on superlevel sets is that it contain practically all the information deriving from the minimum properties of the function u, at least for what concerns its Hölder continuity. In the next theorem, according to a brilliant idea by E. De Giorgi, we will show a strong maximum principle in a quantitative form (more precisely a L^2 to L^∞ estimate), that is the first step to prove the Hölder continuity of solutions.

We shall need the following iterative lemma.

Lemma 1.2.5. *Let $\alpha > 0$ and let $\{x_i\}_{i\in\mathbb{N}_0} \subset \mathbb{R}^+$ such that*

$$x_{i+1} \leqslant CB^i x_i^{1+\alpha},$$

for some $C > 0$ and $B > 1$. If $x_0 \leqslant C^{-\frac{1}{\alpha}}B^{-\frac{1}{\alpha^2}}$, we have

$$x_i \leqslant B^{-\frac{i}{\alpha}}x_0,$$

and hence in particular $\lim_{i\to+\infty} x_i = 0$.

Proof. We prove the assertion by induction. If $i = 0$, the inequality is trivially true. We assume that it holds for some $i \in \mathbb{N}$. Using our assumptions and the induction hypothesis, we conclude

$$x_{i+1} \leqslant CB^i x_i^{1+\alpha} \leqslant CB^{i\left(1-\frac{1+\alpha}{\alpha}\right)} x_0^{1+\alpha} = \left(CB^{\frac{1}{\alpha}} x_0^\alpha\right) B^{-\frac{i+1}{\alpha}} x_0 \leqslant B^{-\frac{i+1}{\alpha}} x_0.$$

□

The following theorem is the main result proved by E. De Giorgi.

Theorem 1.2.6 (De Giorgi's regularity theorem)**.** *Let $u \in H^1_{loc}(\Omega)$ be a local weak solution of*

$$\nabla_j(a_{ij}\nabla_i u) = 0 \quad \text{in } \Omega, \tag{1.5}$$

with measurable coefficients $a_{ij} \colon \Omega \to \mathbb{R}$ satisfying

$$\nu|\xi|^2 \leqslant a_{ij}(x)\xi_i\xi_j \leqslant N|\xi|^2, \quad \forall (x,\xi) \in \Omega \times \mathbb{R}^n,$$

for some positive constants ν and N. Then there exists two constants $\alpha = \alpha(n,\nu,N) \in (0,1)$ and $c = c(n,\nu,N) > 0$ such that $u \in C^{0,\alpha}_{loc}(\Omega)$ and, for any $B_s(x) \subset \Omega$, the following two estimates hold

$$\sup_{B_{\frac{\rho}{2}}(x)} |u|^2 \leqslant \frac{c(n,\nu,N)}{\rho^n} \int_{B_\rho(x)} |u|^2\, dx, \tag{1.6}$$

$$\int_{B_\rho(x)} |u - u_{x_0,\rho}|^2\, dx \leqslant c(n,\nu,N) \left(\frac{\rho}{s}\right)^{n+2\alpha} \int_{B_s(x)} |u - u_{x,s}|^2\, dx, \tag{1.7}$$

for any $\rho \in \left(0, \frac{s}{2}\right)$.

Proof. We may assume for simplicity of notation that $x = 0$. Let $\frac{\rho}{2} \leqslant \sigma \leqslant \tau \leqslant \rho$, $k \geqslant 0$ and $\zeta := \eta(u-k)_+$, where $\eta \in C^\infty_c(B_{\frac{\sigma+\tau}{2}})$ is a cut-off function such that $\eta = 1$ on B_σ and $|\nabla\eta| \leqslant \frac{c}{\tau-\sigma}$, for some positive constant c. Computing $\nabla\zeta = (u-k)_+\nabla\eta + \eta\nabla u$, by Hölder's and Sobolev's inequalities, Caccioppoli's inequality on superlevel sets, we obtain

$$\begin{aligned}
&\int_{A(k,\sigma)} (u-k)^2\, dx \qquad (1.8)\\
&\leqslant \int_{A(k,\sigma)} \zeta^2\, dx \leqslant \left(\int_{A(k,\sigma)} \zeta^{2^*}\, dx\right)^{\frac{2}{2^*}} |A(k,\tau)|^{1-\frac{2}{2^*}}\\
&\leqslant c(n)|A(k,\tau)|^{\frac{2}{n}} \int_{A(k,\sigma)} |\nabla\zeta|^2\, dx\\
&\leqslant c(n)|A(k,\tau)|^{\frac{2}{n}} \left[\int_{A\left(k,\frac{\sigma+\tau}{2}\right)} |\nabla u|^2\, dx + \frac{1}{(\tau-\sigma)^2} \int_{A\left(k,\frac{\sigma+\tau}{2}\right)} (u-k)^2\, dx\right]\\
&\leqslant c(n,\nu,N) \frac{|A(k,\tau)|^{\frac{2}{n}}}{(\tau-\sigma)^2} \int_{A(k,\tau)} (u-k)^2\, dx.
\end{aligned}$$

For $h < k$, using that $A(k,\tau) \subset A(h,\tau)$, we easily deduce the following two inequalities:

$$\int_{A(k,\tau)} (u-k)^2\,dx \leqslant \int_{A(k,\tau)} (u-h)^2\,dx \leqslant \int_{A(h,\tau)} (u-h)^2\,dx,$$

$$\int_{A(h,\tau)} (u-h)^2\,dx \geqslant \int_{A(k,\tau)} (u-h)^2\,dx \geqslant (k-h)^2|A(k,\tau)|.$$

Inserting them in (1.8), we deduce that

$$\begin{aligned}\int_{A(k,\sigma)} (u-k)^2\,dx &\leqslant c(n,\nu,N)\frac{|A(k,\tau)|^{\frac{2}{n}}}{(\tau-\sigma)^2}\int_{A(h,\tau)} (u-h)^2\,dx \\ &\leqslant \frac{c(n,\nu,N)}{(\tau-\sigma)^2(k-h)^{\frac{4}{n}}}\left(\int_{A(h,\tau)} (u-h)^2\,dx\right)^{1+\frac{2}{n}}.\end{aligned}$$

Let $d \geqslant 0$ a number to be chosen later, $k_i := 2d(1-2^{-i})$ and $\sigma_i := \frac{\rho}{2}(1+2^{-i})$, for $i \in \mathbb{N}_0$. We plug $\sigma = \sigma_{i+1}$, $\tau = \sigma_i$, $k = k_{i+1}$, $h = k_i$ in the previous inequality, getting

$$\int_{A(k_{i+1},\sigma_{i+1})} (u-k_{i+1})^2\,dx \leqslant \frac{c(n,\nu,N)2^{2i\left(1+\frac{2}{n}\right)}}{\rho^2 d^{\frac{4}{n}}}\left(\int_{A(k_i,\tau)} (u-k_i)^2\,dx\right)^{1+\frac{2}{n}}.$$

If we set $\Phi_i := \frac{1}{d^2}\int_{A(k_i,\sigma_i)} (u-k_i)^2\,dx$ and we divide the previous inequality by d^2, we infer

$$\Phi_{i+1} \leqslant \frac{c(n,\nu,N)}{\rho^2} 2^{2i\left(1+\frac{2}{n}\right)} \Phi_i^{1+\frac{2}{n}}.$$

Choosing

$$d = c(n,\nu,N)\left(⨍_{B_\rho} u_+^2\,dx\right)^{\frac{1}{2}},$$

so that $\Phi_0 \leqslant c(n,\nu,N)\rho^n$, we are in position to apply Lemma 1.2.5 and, thus, we obtain that $\lim\limits_{i\to+\infty} \Phi_i = 0$, that is

$$\sup_{Q_{\frac{\rho}{2}}} u \leqslant 2d = c(n,\nu,N)\left(⨍_{B_\rho} u_+^2\,dx\right)^{\frac{1}{2}}.$$

Substituting u with $-u$, we get (1.6). Furthermore, it can be showed that there exists $\alpha = \alpha(n,\nu,N) \in (0,1)$ such that

$$\operatorname{osc}(u;B_\rho(x)) \leqslant c(n,\nu,N)\left(\frac{\rho}{s}\right)^\alpha \operatorname{osc}(u;B_s(x)), \tag{1.9}$$

for every $B_s(x) \subset \Omega$, with $\rho \in \left(0,\frac{s}{2}\right)$. This estimate yields to $u \in C^{0,\alpha}_{loc}(\Omega)$. Now we prove (1.7). Let $x = 0$ again for simplicity. Since $u - u_{\frac{s}{2}}$ and $u_{\frac{s}{2}} - u$ are both weak solutions of (1.5), we apply (1.6) twice,

$$\sup_{B_{\frac{s}{2}}} (u-u_{\frac{s}{2}})^2 \leqslant \frac{c(n,\nu,N)}{s^n}\int_{B_s} (u-u_{\frac{s}{2}})^2\,dx,$$

$$\sup_{B_{\frac{s}{2}}}(u_{\frac{s}{2}} - u)^2 \leqslant \frac{c(n,\nu,N)}{s^n} \int_{B_s} (u_{\frac{s}{2}} - u)^2 \, dx.$$

Combining the previous two inequalities, we infer

$$\begin{aligned}
&\operatorname{osc}(u^2; B_{\frac{s}{2}}) \\
&\leqslant c(n,\nu,N) \fint_{B_s} |u - u_{\frac{s}{2}}|^2 \, dx \leqslant c(n,\nu,N) \left[\fint_{B_s} |u - u_s|^2 \, dx + |u_s - u_{\frac{s}{2}}|^2 \right] \\
&\leqslant c(n,\nu,N) \left[\fint_{B_s} |u - u_s|^2 \, dx + \fint_{B_{\frac{s}{2}}} |u_s - u|^2 \, dx \right] \leqslant c(n,\nu,N) \fint_{B_s} |u_s - u|^2 \, dx.
\end{aligned}$$

Using (1.9), we get

$$\begin{aligned}
\fint_{B_\rho} |u - u_\rho|^2 \, dx \leqslant \operatorname{osc}(u^2; B_\rho) &\leqslant c(n,\nu,N) \left(\frac{\rho}{s}\right)^{2\alpha} \operatorname{osc}(u^2; B_{\frac{s}{2}}) \\
&\leqslant c(n,\nu,N) \left(\frac{\rho}{s}\right)^{2\alpha} \fint_{B_s} |u_s - u|^2 \, dx,
\end{aligned}$$

which leads to (1.7). □

Another useful consequence of Caccioppoli's inequality is the control of the oscillation of minimizers of a "frozen" functional by the oscillation of the boundary datum. This result will come in handy in proofs based on comparison strategies.

We state the next proposition for more general integrands with quadratic growth of the type

$$|\xi|^2 - \bar{c} \leqslant F(x,s,\xi) \leqslant |\xi|^2 + \bar{c}, \tag{1.10}$$

for some positive constant $\bar{c}$. We shall assume that the functional is "freezed" in the first two variables.

Proposition 1.2.7 (Oscillation)**.** *Let $B_R(x_0) \subset \mathbb{R}^n$, F be an integrand satisfying* (1.10) *and $v \in H^1(B_R(x_0))$ be a minimizer of the functional*

$$\mathcal{F}_0(w) = \int_{B_R(x_0)} F(x_0, u(x_0), \nabla w) \, dx, \quad \forall w \in H^1(B_R(x_0)),$$

under the boundary condition $w = u$ on $\partial B_R(x_0)$, for some bounded function $u \in H^1(B_R(x_0))$. Then there exists a constant $C = C(n, \bar{c})$ such that

$$\operatorname{osc}(v; B_R(x_0)) \leqslant \operatorname{osc}(u; B_R(x_0)) + CR.$$

Proof. We define $k \geqslant k_0 := \sup_{\partial B_R(x_0)} u$ and $w := \min\{v, k\}$. By minimality we have

$$\int_{A(k,R)} F(x_0, u(x_0), \nabla v) \, dx \leqslant \int_{A(k,R)} F(x_0, u(x_0), \nabla w) \, dx.$$

Thanks to the growth conditions on F we can write

$$\int_{A(k,R)} |\nabla v|^2\,dx - \overline{c}|A(k,R)| \leqslant \int_{A(k,R)} |\nabla w|^2\,dx + \overline{c}|A(k,R)| = \overline{c}|A(k,R)|,$$

that is

$$\int_{A(k,R)} |\nabla v|^2\,dx \leqslant 2\overline{c}|A(k,R)|.$$

Considering any $h > k$, by means of Hölder's and Poincaré's inequalities and the previous estimate, we get

$$\begin{aligned}(h-k)^2|A(h,R)| &\leqslant \int_{A(h,R)} (v-k)^2\,dx \leqslant \int_{A(k,R)} (v-k)^2\,dx \\ &\leqslant \left(\int_{A(k,R)} (v-k)^{2^\star}\,dx\right)^{\frac{2}{2^\star}} |A(k,R)|^{\frac{2}{n}} \\ &\leqslant c(n)\int_{A(k,R)} |\nabla v|^2\,dx|A(k,R)|^{\frac{2}{n}} \leqslant c(n,\overline{c})|A(k,R)|^{1+\frac{2}{n}}.\end{aligned}$$

Choosing $k_i := k_0 + d(1-2^{-i})$ and $a_i = |A(k_i,R)|$, with $i \in \mathbb{N}_0$, we write the previous estimate for $h = k_{i+1}$ and $h = k_i$, obtaining

$$a_{i+1} \leqslant c(n,\overline{c})d^{-2}2^{2i}a_i^{1+\frac{2}{n}}.$$

Choosing $d = c(n,\overline{c})R$ so that $a_0 \leqslant c(n,\overline{c})d^n$, we are in position to apply Lemma 1.2.5. As a result,

$$0 = \lim_{i\to+\infty} a_i = |A(k_0+d,R)|,$$

that is

$$\sup_{B_R(x_0)} v \leqslant \sup_{\partial B_R(x_0)} u + c(n,\overline{c})R.$$

We obtain the thesis, by applying the same argument to $-v$. □

As mentioned before, some conventional regularity results, i.e. results concerning existence and quantitative bounds for higher derivatives of weak solutions of elliptic equations and systems, can be obtained by Caccioppoli's inequality. We will follow the Nirenberg's method, which gives a uniform bound on the difference quotient of the gradient of minimizers. This will be sufficient thanks to Lemma 1.2.9. Actually, the method is more general and it fits well for problem with non-costant coefficients.

Definition 1.2.8 (Difference quotient function). *Let $u\colon \Omega \to \mathbb{R}$, $h \in \mathbb{R}$ and $s \in \{1,\dots,n\}$. We call the **difference quotient** of u with respect to the direction e_s the function*

$$\Delta_{s,h}f(x) := \frac{f(x+he_s) - f(x)}{h}.$$

The function $\Delta_{s,h}f$ is well-defined in the set

$$\Delta_{s,h}(\Omega) := \{x \in \Omega \,:\, x + he_s \in \Omega\},$$

and hence in the set

$$\Omega_{|h|} := \{x \in \Omega \,:\, \operatorname{dist}(x, \partial\Omega) > |h|\}.$$

The following properties of difference quotients are immediate:

- if $u \in H^1(\Omega)$, then $\Delta_h u \in H^1(\Omega_{|h|})$ and

$$\nabla_i \Delta_{s,h} u = \Delta_{s,h} \nabla_i u, \quad \forall i \in \{1, \ldots, n\};$$

- if at least one of the functions f and g has its support contained in $\Omega_{|h|}$, then an integration by parts formula holds:

$$\int_\Omega f \Delta_{s,h} g \, dx = -\int_\Omega g \Delta_{s,-h} f \, dx;$$

- a Leibniz property holds:

$$\Delta_{s,h}(fg)(\cdot) = f(\cdot + he_s)\Delta_{s,h} g(\cdot) + g(\cdot)\Delta_{s,h} f(\cdot).$$

We also recall a well-known lemma.

Lemma 1.2.9. *Let $u \in L^2(\Omega)$ and assume that there exists a positive constant K such that, for every $h \in \mathbb{R}$ sufficiently small, we have*

$$\|\Delta_{s,h} u\|_{L^2(\Omega_{|h|})} \leqslant K,$$

for some $s \in \{1, \ldots, n\}$. Then, $\nabla_s u \in L^2(\Omega)$ and $\|\nabla_s v\|_{L^2(\Omega)} \leqslant K$. Moreover, for $h \to 0$, $\Delta_{s,h} u \to \nabla_s u$ in $L^2_{loc}(\Omega)$.

Theorem 1.2.10 (Regularity of the second order derivatives)**.** *Let $u \in H^1_{loc}(\Omega)$ a local weak solution of*

$$\nabla_j(a_{ij}\nabla_i u) = 0 \quad \text{in } \Omega, \tag{1.11}$$

with constant coefficients a_{ij} satisfying

$$\nu|\xi|^2 \leqslant a_{ij}\xi_i\xi_j \leqslant N|\xi|^2, \quad \forall \xi \in \mathbb{R}^n,$$

for some positive constants ν and N. Then $u \in H^2_{loc}(\Omega)$.

Proof. Let us assume for simplicity of notation that $x_0 = 0$ and fix $B_\rho \subset \Omega$. For some direction $s \in \{1, \cdots, n\}$ and $|h| < \frac{\operatorname{dist}(B_\rho, \Omega)}{2}$, we set $\phi := \Delta_{s,-h}(\eta^2 \Delta_{s,h} u)$, where $\eta \in C^\infty_c(B_\rho)$ is a cut-off function such that $0 \leqslant \eta \leqslant 1$, $\eta = 1$ in $B_{\frac{\rho}{2}}$ and

$|\nabla\eta| \leqslant \frac{c}{\rho}$, for some positive constant c. Then, plugging ϕ as a test function in the weak formulation of (1.11), we get

$$\int_{B_\rho} a_{ij}\nabla_i u \nabla_j \big(\Delta_{s,-h}(\eta^2\Delta_{s,h}u)\big)\,dx = 0$$

Commuting the derivative and the difference quotient and integrating by parts, we are led to

$$\int_{B_\rho} a_{ij}\Delta_{s,h}\nabla_i u(\eta^2\Delta_{s,h}\nabla_j u + 2\eta\Delta_{s,h}u\nabla_j\eta)\,dx = 0.$$

Using the ellipticity of a_{ij} and applying Young's inequality, we obtain

$$\begin{aligned}\nu\int_{B_\rho}\eta^2|\nabla\Delta_{s,h}u|^2\,dx &\leqslant 2N\int_{B_\rho}\eta|\Delta_{s,h}\nabla u||\nabla\eta||\Delta_{s,h}u|\,dx\\ &\leqslant N\varepsilon\int_{B_\rho}\eta^2|\Delta_{s,h}\nabla u|^2\,dx + \frac{N}{\varepsilon}\int_{B_\rho}|\nabla\eta|^2|\Delta_{s,h}u|^2\,dx,\end{aligned}$$

for any $\varepsilon > 0$. Choosing $\varepsilon = \varepsilon(\nu, N)$ sufficiently small and exploiting the properties of η, we can write

$$\int_{B_{\frac{\rho}{2}}} |\nabla\Delta_{s.h}u|^2\,dx \leqslant \frac{c(\nu,N)}{\rho^2}\int_{B_\rho}|\Delta_{s,h}u|^2\,dx.$$

Since the sequence $\{\nabla\Delta_{s,h}u\}_{h\in\mathbb{N}}$ is bounded in L^2, then, by Lemma 1.2.9, $\nabla\Delta_{s,h}u \to \nabla\nabla_s u$ in $L^2_{loc}(B_{\frac{\rho}{2}})$, which implies that $u \in H^2_{loc}(B_{\frac{\rho}{2}})$ and, summing over s,

$$\int_{B_{\frac{\rho}{2}}} |\Delta^2 u|^2\,dx \leqslant \frac{c(\nu,N)}{r^2}\int_{B_\rho}|\nabla u|^2\,dx.$$

□

The following theorem can be found in a far more general version in [31]. It proves that local minimizers u are Lipschitz continuous by means of the Moser's iteration technique, which provides a L^2 to L^∞ estimate of ∇u.

Theorem 1.2.11. *Let $u \in H^1_{loc}(\Omega)$ a local weak solution of*

$$\nabla_j(a_{ij}\nabla_i u) = 0 \quad \text{in } \Omega, \tag{1.12}$$

with constant coefficients a_{ij} satisfying

$$\nu|\xi|^2 \leqslant a_{ij}\xi_i\xi_j \leqslant N|\xi|^2, \quad \forall\xi\in\mathbb{R}^n,$$

for some positive constants ν and N. Then there exists a positive constant $C = C(n,\nu,N)$ such that, for any $B_\rho(x) \subset \Omega$,

$$\sup_{B_{\frac{\rho}{2}}(x)} |\nabla u|^2 \leqslant C ⨍_{B_\rho(x)} |\nabla u|^2\,dx.$$

Proof. We may assume, up to a rescaling argument, that $x = 0$ and $r = 1$. For $s \in \{1, \dots, n\}$, we set $\phi := \eta^2 \nabla_s \psi$, where the function $\psi \in C^2(B_1)$ is arbitrary and $\eta \in C_c^\infty(B_1)$ is a cut-off function such that $0 \leqslant \eta \leqslant 1$. Then, plugging ϕ as a test function in the weak formulation of (1.12), we get

$$\int_{B_\rho} a_{ij} \nabla_i u \eta^2 \nabla^2_{js} \psi \, dx + 2 \int_{B_\rho} a_{ij} \nabla_i u \nabla_s \psi \eta \nabla_j \eta \, dx = 0.$$

Integrating by parts the first integral we get

$$-\int_{B_\rho} a_{ij} \nabla^2_{is} u \eta^2 \nabla_j \psi \, dx - 2 \int_{B_\rho} a_{ij} \nabla_i u \eta \nabla_s \eta \nabla_j \psi \, dx + 2 \int_{B_\rho} a_{ij} \nabla_i u \nabla_s \psi \eta \nabla_j \eta \, dx = 0,$$

which holds for any $\psi \in H^1(B_1)$. We choose $\psi := |\nabla u|^{2\beta} \nabla_s u$, with $\beta \geqslant 0$, obtaining

$$\begin{aligned}
&\int_{B_\rho} a_{ij} \nabla^2_{is} u \eta^2 \nabla^2_{js} u |\nabla u|^{2\beta} \, dx + \beta \int_{B_\rho} a_{ij} \nabla^2_{is} u \nabla_s u \eta^2 \nabla_j (|\nabla u|^2) |\nabla u|^{2\beta - 2} \, dx \\
&= -2 \int_{B_\rho} a_{ij} \nabla_i u \eta \nabla_s \eta \nabla^2_{js} u |\nabla u|^{2\beta} \, dx \\
&- 2\beta \int_{B_\rho} a_{ij} \nabla_i u \eta \nabla_s \eta \nabla_s u |\nabla u|^{2\beta-2} \nabla_j (|\nabla u|^2) \, dx \\
&+ 2 \int_{B_\rho} a_{ij} \nabla_i u \nabla^2_{ss} u |\nabla u|^{2\beta} \eta \nabla_j \eta \, dx \\
&+ \beta \int_{B_\rho} a_{ij} \nabla_i u |\nabla u|^{2\beta-2} \nabla_s (|\nabla u|^2) \nabla_s u \eta \nabla_j \eta \, dx \\
&\leqslant c(n, N) \int_{B_\rho} |\nabla u| \eta |\nabla \eta| \big(|\nabla u|^{2\beta} |\nabla^2 u| + \beta |\nabla u|^{2\beta-2} |\nabla u| |\nabla (|\nabla u|^2)| \big) \, dx.
\end{aligned}$$

Summing over s, using the ellipticity of a_{ij} and applying Young's inequality, we get

$$\begin{aligned}
&\nu \int_{B_\rho} |\nabla^2 u|^2 |\nabla u|^{2\beta} \eta^2 \, dx + \frac{\beta \nu}{2} \int_{B_\rho} |\nabla u|^{2\beta-2} |\nabla (|\nabla u|^2)|^2 \eta^2 \, dx \\
&\leqslant c(n, N) \bigg[\varepsilon \int_{B_\rho} \eta^2 |\nabla u|^{2\beta} |\nabla^2 u|^2 \, dx + \frac{1}{\varepsilon} \int_{B_\rho} |\nabla u|^{2\beta+2} |\nabla \eta|^2 \, dx \\
&+ \beta \varepsilon \int_{B_\rho} |\nabla u|^{2\beta-2} |\nabla (|\nabla u|^2)|^2 \eta^2 \, dx + \frac{\beta}{\varepsilon} \int_{B_\rho} |\nabla u|^{2\beta+2} |\nabla \eta|^2 \, dx \bigg].
\end{aligned}$$

Choosing $\varepsilon = \varepsilon(n, \nu, N)$ sufficiently small, we get

$$\begin{aligned}
&\frac{\nu}{2} \int_{B_\rho} |\nabla u|^{2\beta} |\nabla^2 u|^2 \eta^2 \, dx + \frac{\beta \nu}{4} \int_{B_\rho} |\nabla u|^{2\beta-2} |\nabla (|\nabla u|^2)|^2 \eta^2 \, dx \\
&\leqslant c(n, \nu, N)(1 + \beta) \int_{B_\rho} |\nabla u|^{2\beta+2} |\nabla \eta|^2 \, dx. \qquad (1.13)
\end{aligned}$$

Since

$$\int_{B_\rho} |\nabla u|^{2\beta-2}|\nabla(|\nabla u|^2)|^2\eta^2\,dx \leqslant c(n)\int_{B_\rho}|\nabla u|^{2\beta}|\nabla^2 u|^2\eta^2\,dx,$$

dividing both sides of (1.13) by $1+\beta$, we obtain

$$\int_{B_\rho} |\nabla u|^{2\beta-2}|\nabla(|\nabla u|^2)|^2\eta^2\,dx \leqslant c(n,\nu,N)\int_{B_\rho}|\nabla u|^{2\beta+2}|\nabla\eta|^2\,dx. \tag{1.14}$$

Let $\gamma := \frac{1}{2}+\frac{\beta}{2} \geqslant \frac{1}{2}$. Computing

$$\frac{|\nabla\big(|\nabla u|^{2\gamma}\eta\big)|^2}{\gamma^2} \leqslant \eta^2|\nabla u|^{4\gamma-4}|\nabla(|\nabla u|^2)|^2 + \frac{|\nabla u|^{4\gamma}|\nabla\eta|^2}{\gamma^2},$$

the combination of Sobolev-Poincaré's inequality and (1.14) yield

$$\begin{aligned}\left(\int_{B_\rho}(|\nabla u|^{2\gamma}\eta)^{2\chi}\,dx\right)^{\frac{1}{2\chi}} &\leqslant \left(\int_{B_\rho}|\nabla(|\nabla u|^{2\gamma}\eta)|^2\,dx\right)^{\frac{1}{2}}\\ &\leqslant c(n,\nu,N)\gamma\left(\int_{B_\rho}|\nabla u|^{4\gamma}|\nabla\eta|^2\,dx\right)^{\frac{1}{2}},\end{aligned}$$

where $\chi := \frac{n}{n-2}$ if $n>2$ and any positive number greater than 1. We apply the previous inequality for $\rho_i := \frac{1}{2}+\frac{1}{2^i}$, $\gamma_i := \frac{\chi^{i-1}}{2}$, $\eta \in C^1_c(B_{\rho_i})$ such that $0 \leqslant \eta \leqslant 1$, $\eta = 1$ on $B_{\rho_{i+1}}$ and $|\nabla\eta| \leqslant c2^i$, for some positive constant c and for any $i \in \mathbb{N}$, getting

$$\left\||\nabla u|^2\right\|^{\gamma_i}_{L^{2\gamma_{i+1}}(B_{\rho_{i+1}})} \leqslant c(n,\nu,N)\gamma_i 2^i \left\||\nabla u|^2\right\|^{\gamma_i}_{L^{2\gamma_{i+1}}(B_{\rho_i})},$$

and so, iterating the previous estimate,

$$\left\||\nabla u|^2\right\|_{L^{2\gamma_{i+1}}(B_{\rho_{i+1}})} \leqslant \prod_{j=1}^{i}\big(c(n,\nu,N)\gamma_j 2^j\big)^{\frac{1}{\gamma_j}}\left\||\nabla u|^2\right\|_{L^1(B_1)}. \tag{1.15}$$

We remark that the product $\prod_{j=1}^{i}\big(c(n,\nu,N)\gamma_j 2^j\big)^{\frac{1}{\gamma_j}}$ is convergent, as $i \to +\infty$, because

$$\log\left(\prod_{j=1}^{i}\big(c(n,\nu,N)\gamma_j 2^j\big)^{\frac{1}{\gamma_j}}\right) = \sum_{j=1}^{i}\chi\frac{2}{\chi^j}\left[\log\left(\frac{c(n,\nu,N)}{2\chi}\right)+j\log(2\chi)\right],$$

and the series in right-hand side is convergent, being $\chi \geqslant 1$. Thus, letting $i \to +\infty$ in (1.15), $\gamma_i \to +\infty$ and we conclude

$$\|\nabla u\|^2_{L^\infty(B_{\frac{1}{2}})} \leqslant c(n,\nu,N)\,\|\nabla u\|^2_{L^2(B_1)}.$$

□

Chapter 2

The quadratic case

In this chapter we deal with energy functionals of the type

$$\mathcal{F}(E,u;\Omega) = \int_\Omega \big[F(x,u,\nabla u) + \mathbb{1}_E G(x,u,\nabla u)\big]\,dx + P(E;\Omega), \tag{2.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded connected open set, $u \in H^1(\Omega)$ and $E \subset \mathbb{R}^n$ is a set of finite perimeter in Ω. We assume that the density energies F and G in (2.1) satisfy the following structural assumptions:

$$F(x,s,\xi) = \sum_{i,j=1}^n a_{ij}(x,s)\xi_i\xi_j + \sum_{i=1}^n a_i(x,s)\xi_i + a(x,s), \tag{2.2}$$

$$G(x,s,z) = \sum_{i,j=1}^n b_{ij}(x,s)\xi_i\xi_j + \sum_{i=1}^n b_i(x,s)\xi_i + b(x,s), \tag{2.3}$$

for any $(x,s,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$. Concerning the coefficients we assume that

$$a_{ij}, b_{ij}, a_i, b_i, a, b \in C^{0,1}(\Omega \times \mathbb{R}).$$

We denote by L_D the greatest Lipschitz constant of the data $a_{ij}, b_{ij}, a_i, b_i, a, b$, that is

$$|\nabla a_{ij}| \leqslant L_D, \quad |\nabla b_{ij}| \leqslant L_D \quad \text{in } \Omega \times \mathbb{R}, \tag{2.4}$$

and the same holds true for a_i, b_i, a, b.
Moreover, to ensure the existence of minimizers we assume the uniform boundedness of the coefficients and the uniform ellipticity of the matrices a_{ij} and b_{ij},

$$\nu|\xi|^2 \leqslant a_{ij}(x,s)\xi_i\xi_j \leqslant N|\xi|^2, \quad \nu|\xi|^2 \leqslant b_{ij}(x,s)\xi_i\xi_j \leqslant N|\xi|^2, \tag{2.5}$$

$$\sum_{i=1}^n |a_i(x,s)| + \sum_{i=1}^n |b_i(x,s)| + |a(x,s)| + |b(x,s)| \leqslant L, \tag{2.6}$$

for any $(x,s,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$, where ν, N and L are three positive constants.

We are interested in the regularity of minimizers of the following constrained problem.

Definition 2.0.1. *We shall denote by* (P_c) *the constrained problem*

$$\min_{\substack{E\in\mathcal{A}(\Omega)\\ v\in u_0+H_0^1(\Omega)}} \left\{\mathcal{F}(E,v;\Omega)\,:\,|E|=d\right\}, \tag{P_c}$$

where $u_0 \in H^1(\Omega)$ *is the boundary datum and* $0 < d < |\Omega|$ *is a fixed number.*

It is clear that any minimizer u of problem (P_c) is a local minimizer of the functional (2.1) and therefore satisfies the Euler-Lagrange equation

$$\frac{\partial}{\partial x_i}\big[F_{z_i}(x,u,\nabla u)+\mathbb{1}_E(x)G_{z_i}(x,u,\nabla u)\big)\big] = F_u(x,u,\nabla u)+\mathbb{1}_E(x)G_u(x,u,\nabla u),$$

where, as usual, we have used Einstein's convention on repeated indices.

The problem of handling with the constraint $|E| = d$ is overtaken using an argument introduced in [29], ensuring that every minimizer of the constrained problem (P_c) is also a minimizer of a penalized functional of the type

$$\mathcal{F}_\Lambda(E,v;\Omega) = \mathcal{F}(E,v;\Omega) + \Lambda\big||E|-d\big|,$$

for some suitable $\Lambda > 0$ (see Theorem 2.2.1 below). Therefore, we give in addition the following definition.

Definition 2.0.2. *We shall denote by* (P) *the penalized problem*

$$\min_{\substack{E\in\mathcal{A}(\Omega)\\ v\in u_0+H_0^1(\Omega)}} \mathcal{F}_\Lambda(E,v;\Omega), \tag{P}$$

where $u_0 \in H^1(\Omega)$ *is fixed and* $\mathcal{A}(\Omega)$ *is the same class defined in Definition 2.0.1.*

From the point of view of regularity, the extra term $\Lambda\big||E|-|F|\big|$ is a higher order negligible perturbation.

Our aim is to prove the reduction of the Hausdorff dimension of the singular set of ∂E for minimizing couples (E,u) of (2.1). Basically, we adopt the same strategy of [35]. The main result of the chapter is stated in the following theorem.

Theorem 2.0.3. *Let* (E,u) *be a minimizer of problem* (P)*, under assumptions* (2.2) – (2.6)*. Then*

a) there exists a relatively open set $\Gamma \subset \partial E$ *such that* Γ *is a* $C^{1,\mu}$*-hypersurface for all* $0 < \mu < \frac{1}{2}$*;*

b) there exists $\varepsilon = \varepsilon(n,\nu,N,L) > 0$ *such that*

$$\mathcal{H}^{n-1-\varepsilon}((\partial E\backslash\Gamma)\cap\Omega) = 0.$$

For reader's convenience the chapter is structured in sections which reflect the proof strategy. Section 2.1 collects some prelimary definitions and two useful and well-known iterative lemmata that will be applied later on. Section 2.2 is devoted to proving that minimizers of the constrained problem (P_c) solve also problem (P). In Section 2.3 some higher integrability results are proved.

As in the case of minimizers of the Mumford-Shah functional, the proof of regularity is based on the study of the interplay between the perimeter and the bulk energy (see [7], [44]). We recall that the Hölder exponent $\frac{1}{2}$ is critical for solutions u of either (P) or (P_c), in the sense that, whenever $u \in C^{0,\frac{1}{2}}$, under appropriate scaling, the bulk term locally has the same dimension $n-1$ as the perimeter term. In this regard, our starting point is to prove suitable energy decay estimates for the bulk energy. These estimates are presented in Section 2.4. The key point of this approach is contained in Lemma 2.4.6, where it is proved that the bulk energy decays faster than ρ^{n-1}, that is, for any $\delta > 0$,

$$\int_{B_\rho(x_0)} |\nabla u|^2 \, dx \leqslant C\rho^{n-\delta}, \tag{2.7}$$

either in the case that

$$\min\{|E \cap B_\rho(x_0)|, |B_\rho(x_0) \backslash E|\} < \varepsilon_0 |B_\rho(x_0)|,$$

or in the case that there exists an half-space H such that

$$|(E \Delta H) \cap B_\rho(x_0)| \leqslant \varepsilon_0 |B_\rho(x_0)|,$$

for some $\varepsilon_0 > 0$. The latter case is the hardest one to handle because it relies on the regularity properties of solutions of a transmission problem. Let us notice that, for any given $E \subset \Omega$, local minimizers u of the functional

$$\int_\Omega \big[F(x,u,\nabla u) + \mathbb{1}_E G(x,u,\nabla u)\big] \, dx \tag{2.8}$$

are Hölder continuous, $u \in C^{0,\alpha}_{loc}(\Omega)$, but the needed bound $\alpha > \frac{1}{2}$ cannot be expected in the general case without any information on the set E. In Section 2.4 we prove that minimizers of the functional (2.8) are in $C^{0,\alpha}$ for every $\alpha \in (0,1)$, in the case E is an half-space. In this context the linearity of the equation strongly comes into play ensuring that the derivatives of the Euler-Lagrange equation are again solutions of the same equation.

For the proof of the regularity results, we readapt a technique depicted in the book [7] in the context of the Mumford-Shah functional and recently used in a paper by E. Mukoseeva and G. Vescovo, [49].

Once obtained the estimates of Section 2.4, in Section 2.5 we are in position to prove that, if in a ball $B_\rho(x_0)$ the perimeter of E is sufficiently small, then the total energy

$$\int_{B_r(x_0)} |\nabla u|^2 \, dx + P(E; B_r(x_0)), \qquad 0 < r < \rho,$$

decays as r^n (see Lemma 2.5.2). Making use of the latter energy density estimate we are in position to deduce in the same section a density lower bound for the perimeter of E as well. In the subsequent sections the proof strategy follows the path traced from the regularity theory for perimeter minimizers.

In particular, in Section 2.6 it is proved the compactness for sequences of minimizers which more or less follows in a standard way from the density lower bound.

Sections 2.7, 2.8 and 2.9 are devoted to proving some additional consequences of the density lower bound which involve the excess

$$\mathbf{e}(x,r) = \inf_{\nu\in\mathbb{S}^{n-1}} \mathbf{e}(x,r,\nu) := \inf_{\nu\in\mathbb{S}^{n-1}} \frac{1}{r^{n-1}} \int_{\partial E\cap B_r(x)} \frac{|\nu_E(y)-\nu|^2}{2} d\mathcal{H}^{n-1}(y),$$

(see Definition 1.1.20). Indeed, we prove the height bound lemma, the Lipschitz approximation theorem and the reverse Poincaré inequality.

In Section 2.10 we compute the Euler-Lagrange equation for $\mathcal{F}(E,u)$ involving the variation of the set E.

Section 2.11 is devoted to proving the excess improvement, which follows from the fact that, whenever the excess $\mathbf{e}(x,r)$ goes to zero, for $r\to 0$, the Dirichlet integral $\int_{B_\rho(x_0)} |\nabla u|^2\,dx$ decays as in (2.7).

Finally, in Section 2.12 we provide the proof of Theorem 2.0.3, which is a consequence of the excess improvement proved before.

2.1 Some definitions and two iterative lemmata

For any $\mu \geqslant 0$, we define the **Morrey space** $L^{2,\mu}(\Omega)$ as

$$L^{2,\mu}(\Omega) := \left\{ u \in L^2(\Omega) \;:\; \sup_{x_0\in\Omega,\, r>0} r^{-\mu} \int_{\Omega\cap B_r(x_0)} |u|^2\,dx < +\infty \right\}. \tag{2.9}$$

We recall a classical result involving Morrey spaces, which can be obtain by the combination of Poincaré's inequality and the characterization of Camapanato's spaces.

Lemma 2.1.1. *Let $\mu \in [0,2)$, $B_r(x_0) \subset \mathbb{R}^n$ and $u \in H^1(B_r(x_0))$. If $|\nabla u| \in L^{2,n-\mu}(B_r(x_0))$, then $u \in C^\alpha(B_r(x_0))$, where $\alpha = 1-\frac{\mu}{2}$.*

The following definition is standard.

Definition 2.1.2. *Let $v \in H^1_{loc}(\Omega)$ and assume that $E \subset \Omega$ is fixed. We define the functional $\mathcal{F}_E$ as*

$$\mathcal{F}_E(w,\Omega) := \mathcal{F}(E,w;\Omega), \quad \forall w \in H^1(\Omega).$$

It is worth pointing out that for a quadratic integrand F of the type given in (2.2) the following growth conditions can be immediately deduced from assumptions (2.5) and (2.6):

$$\frac{\nu}{2}|z|^2 - \frac{L^2}{\nu} \leqslant F(x,s,z) \leqslant (N+1)|z|^2 + L(L+1), \quad \forall (x,s,z) \in \Omega \times \mathbb{R} \times \mathbb{R}^n. \tag{2.10}$$

Here we recall the proofs of two useful iterative lemmata.

Lemma 2.1.3. *Let $Z(t)$ be a bounded non-negative function in the interval $[\rho, R]$ and assume that, for $\rho \leqslant t < s \leqslant R$, we have*

$$Z(t) \leqslant \theta Z(s) + \frac{A}{(s-t)^2} + B, \tag{2.11}$$

with $A, B \geqslant 0$ and $0 \leqslant \theta < 1$. Then

$$Z(\rho) \leqslant c\left[\frac{A}{(R-\rho)^2} + B\right],$$

for some $c = c(\theta) > 0$.

Proof. Although the proof of this lemma is standard and can be found in [37, Lemma 6.1], we show it here for the sake of completeness. Consider the increasing sequence $\{t_i\}_{i\in\mathbb{N}_0} \subset [\rho, R]$ such that

$$t_0 = \rho \quad \text{and} \quad t_{i+1} - t_1 = (1-\lambda)\lambda^i(R-\rho),$$

where $\lambda \in (0,1)$ will be chosen later on. Iterating (2.11) for $t = t_i$ and $s = t_{i+1}$ we infer

$$Z(\rho) \leqslant \theta^k Z(t_k) + \left[\frac{A}{(1-\lambda)^2(R-\rho)^2} + B\right]\sum_{i=0}^{k-1}(\theta\lambda^{-2})^i.$$

Now we choose $\lambda = \lambda(\theta)$ such that $\theta\lambda^{-2} < 1$. Passing to the limit for $k \to +\infty$, the geometric series in the right-hand side converges and we get the conclusion with $c = \frac{1}{(1-\lambda)^2(1-\theta\lambda^{-2})}$. □

The next lemma can be found in [7, Lemma 7.54].

Lemma 2.1.4. *Let $f : (0,a] \to [0,+\infty)$ be an increasing function such that*

$$f(\rho) \leqslant A\left[\left(\frac{\rho}{R}\right)^p + R^s\right]f(R) + BR^q,$$

whenever $0 < \rho < R \leqslant a$, for some constants $A, B \geqslant 0$, $0 < q < p$, $s > 0$. Then there exist two positive constants $R_0(p,q,s,A)$ and $c(p,q,A)$ such that

$$f(\rho) \leqslant c\left(\frac{\rho}{R}\right)^q f(R) + cB\rho^q,$$

whenever $0 < \rho < R \leqslant \min\{R_0, a\}$.

Proof. Let us fix $r = r(p,q,A) \in (p,q)$ and $\tau = \tau(p,q,A) \in (0,1)$ such that $2A\tau^p \leqslant \tau^r$. Let $R_0 = R_0(p,q,s,A) > 0$ such that $R_0^s \leqslant \tau^p$ and $R \leqslant \min\{R_0, a\}$. By assumption, for any $i \in \mathbb{N}$, we easily get

$$f(\tau^i R) \leqslant A\big(\tau^p + \tau^{is}R^s\big) f\big(\tau^i R\big) + B\tau^{iq}R^q \leqslant \tau^r f\big(\tau^i R\big) + B\tau^{iq}R^q.$$

The previous inequality can be iterated obtaining

$$f(\tau^k R) \leqslant \tau^{kq} f(R) + B(\tau^k R)^q \tau^{-q} \sum_{i=0}^{k-1} \tau^{i(r-q)},$$

for some $k \in \mathbb{N}$ to be chosen. We now distinguish two cases: if $\rho \leqslant \tau R$, we choose $k \in \mathbb{N}$ such that $\tau^{(k+1)}R < \rho \leqslant \tau^k R$ and we conclude

$$\begin{aligned} f(\rho) \leqslant f(\tau^k R) &\leqslant \tau^{-q}\tau^{(k+1)q} f(R) + B\tau^{-2q}\big(\tau^{k+1}R\big)^q \sum_{i=0}^{k-1} \tau^{i(r-q)} \\ &\leqslant c(p,q,A)\left(\frac{\rho}{R}\right)^q f(R) + c(p,q,A)B\rho^q. \end{aligned}$$

If $\rho \in (\tau R, R]$, choosing $c = c(p,q,A)$ such that $c\tau^q \geqslant 1$, since f is increasing, we infer

$$f(\rho) \leqslant c(p,q,A)\tau^q f(R) \leqslant c(p,q,A)\left(\frac{\rho}{R}\right)^q f(R) + c(p,q,A)B\rho^q.$$

□

2.2 From constrained to penalized problem

The next theorem allows us to overcome the difficulty of handling with the constraint $|E| = d$. Indeed, it can be proved that every minimizer of the constrained problem (P_c) is also a minimizer of a suitable unconstrained problem with a volume penalization of the type given in (P).

Theorem 2.2.1. *There exists $\Lambda_0 > 0$ such that if (E,u) is a minimizer of the functional*

$$\mathcal{F}_\Lambda(A,w) = \int_\Omega \big[F(x,w,\nabla w) + \mathbb{1}_A G(x,w,\nabla w)\,dx\big]\,dx + P(A;\Omega) + \Lambda\big||A| - d\big|, \tag{2.12}$$

for some $\Lambda \geqslant \Lambda_0$, among all configurations (A,w) such that $w = u_0$ on $\partial\Omega$, then $|E| = d$ and (E,u) is a minimizer of problem (P_c). Conversely, if (E,u) is a minimizer of problem (P_c), then it is a minimizer of (2.12)*, for all $\Lambda \geqslant \Lambda_0$.*

Proof. The proof can be carried out as in [29, Theorem 1]. For reader's convenience we give here its sketch, emphasizing main ideas and minor differences with respect to the case treated in [29].

The first part of the theorem can be proved by contradiction. Assume that there exist a sequence $\{\lambda_h\}_{h\in\mathbb{N}}$ such that $\lambda_h \to +\infty$ as $h \to +\infty$ and a sequence of configurations (E_h, u_h) minimizing $\mathcal{F}_{\lambda_h}$ and such that $u_h = u_0$ on $\partial\Omega$ and $|E_h| \neq d$ for all $h \in \mathbb{N}$. Let us choose now an arbitrary fixed $E_0 \subset \Omega$ with finite perimeter such that $|E_0| = d$. Let us point out that

$$\mathcal{F}_{\lambda_h}(E_h, u_h) \leqslant \mathcal{F}(E_0, u_0) := \Theta. \tag{2.13}$$

Without loss of generality we may assume that $|E_h| < d$. Indeed, the case $|E_h| > d$ can be treated in the same way considering the complement of E_h in Ω. Our aim is to show that for h sufficiently large, there exists a configuration $(\widetilde{E}_h, \tilde{u}_h)$ such that $\mathcal{F}_{\lambda_h}(\widetilde{E}_h, \tilde{u}_h) < \mathcal{F}_{\lambda_h}(E_h, u_h)$, thus proving the result by contradiction.

By the condition (2.13), it follows that the sequence $\{u_h\}_{h\in\mathbb{N}}$ is bounded in $H^1(\Omega)$, the perimeters of the sets E_h in Ω are bounded and $|E_h| \to d$. Therefore, possibly extracting a not relabelled subsequence, we may assume that there exists a configuration (E, u) such that $u_h \to u$ weakly in $H^1(\Omega)$, $\mathbb{1}_{E_h} \to \mathbb{1}_E$ a.e. in Ω, where the set E is of finite perimeter in Ω and $|E| = d$. The couple (E, u) will be used as reference configuration for the definition of $(\widetilde{E}_h, \tilde{u}_h)$.

Step 1. *Construction of* $(\widetilde{E}_h, \tilde{u}_h)$. Proceeding exactly as in [29], we take a point $x \in \partial^* E \cap \Omega$ and observe that the sets $E_r = (E - x)/r$ converge locally in measure to the half-space $H = \{z \cdot \nu_E(x) < 0\}$, i.e., $\mathbb{1}_{E_r} \to \mathbb{1}_H$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Let $y \in B_1(0)\backslash H$ be the point $y = \nu_E(x)/2$. Given ε (that will be chosen in the step 4), since $\mathbb{1}_{E_r} \to \mathbb{1}_H$ in $L^1(B_1(0))$, there exists $0 < r < 1$ such that

$$|E_r \cap B_{1/2}(y)| < \varepsilon, \qquad |E_r \cap B_1(y)| \geqslant |E_r \cap B_{1/2}(0)| > \frac{\omega_n}{2^{n+2}}\,,$$

where ω_n denotes the measure of the unit ball of $\mathbb{R}^n$. Then if we define $x_r = x + ry \in \Omega$, we have that

$$|E \cap B_{r/2}(x_r)| < \varepsilon r^n, \qquad |E \cap B_r(x_r)| > \frac{\omega_n r^n}{2^{n+2}}\,.$$

Let us assume, without loss of generality, that $x_r = 0$. From the convergence of E_h to E we have that for all h sufficiently large

$$|E_h \cap B_{r/2}| < \varepsilon r^n, \qquad |E_h \cap B_r| > \frac{\omega_n r^n}{2^{n+2}}\,. \tag{2.14}$$

Let us now define the following bi-Lipschitz function used in [29] which maps B_r into itself:

$$\Phi(x) = \begin{cases} \big(1 - \sigma_h(2^n - 1)\big)x & \text{if } |x| < \dfrac{r}{2}, \\ x + \sigma_h\Big(1 - \dfrac{r^n}{|x|^n}\Big)x & \text{if } \dfrac{r}{2} \leqslant |x| < r, \\ x & \text{if } |x| \geqslant r\,, \end{cases} \tag{2.15}$$

for some $0 < \sigma_h < 1/2^n$ sufficiently small in such a way that, setting

$$\widetilde{E}_h = \Phi(E_h), \qquad \tilde{u}_h = u_h \circ \Phi^{-1},$$

we have $|\widetilde{E}_h| < d$. We obtain

$$\begin{aligned}
\mathcal{F}_{\lambda_h}(E_h, u_h) - \mathcal{F}_{\lambda_h}(\widetilde{E}_h, \tilde{u}_h) &= \bigg[\int_{B_r} \big[F(x, u_h, \nabla u_h) + \mathbb{1}_{E_h} G(x, u_h, \nabla u_h)\big]\,dx \\
&- \int_{B_r} \big[F(x, \tilde{u}_h, \nabla \tilde{u}_h) + \mathbb{1}_{\widetilde{E}_h} G(x, \tilde{u}_h, \nabla \tilde{u}_h)\big]\,dy\bigg] \\
&+ \big[P(E_h; \overline{B}_r) - P(\widetilde{E}_h; \overline{B}_r)\big] + \lambda_h\big(|\widetilde{E}_h| - |E_h|\big) \\
&= I_{1,h} + I_{2,h} + I_{3,h}\,.
\end{aligned} \tag{2.16}$$

Step 2. *Estimate of* $I_{1,h}$. First observe that, for $|x| < r/2$, Φ is simply a homothety and all the estimates that we need are very easy to obtain. Conversely, for $r/2 < |x| < r$ we have

$$\frac{\partial \Phi_i}{\partial x_j}(x) = \left(1 + \sigma_h - \frac{\sigma_h r^n}{|x|^n}\right)\delta_{ij} + n\sigma_h r^n \frac{x_i x_j}{|x|^{n+2}}, \quad \forall i, j \in \{1, \dots, n\}. \tag{2.17}$$

Hence, if $\eta \in \mathbb{R}^n$,

$$(\nabla\Phi\eta)\cdot\eta = \left(1 + \sigma_h - \frac{\sigma_h r^n}{|x|^n}\right)|\eta|^2 + n\sigma_h r^n \frac{(x\cdot\eta)^2}{|x|^{n+2}},$$

from which it follows that

$$|\nabla\Phi\eta| \geqslant \left(1 + \sigma_h - \frac{\sigma_h r^n}{|x|^n}\right)|\eta|.$$

From this inequality we easily deduce an estimate on the norm of $\nabla\phi^{-1}$, that is

$$\begin{aligned}
\left\|\nabla\Phi^{-1}\Phi\right\|_\infty &= \max_{|\eta|=1}\left|\nabla\Phi^{-1}\frac{\nabla\Phi\eta}{|\nabla\Phi\eta|}\right| \\
&= \max_{|\eta|=1}\frac{1}{|\nabla\Phi\eta|} \leqslant \left(1 + \sigma_h - \frac{\sigma_h r^n}{|x|^n}\right)^{-1} \\
&\leqslant \big(1 - (2^n - 1)\sigma_h\big)^{-1} \leqslant 1 + 2^n n\sigma_h, \quad \forall x \in B_r.
\end{aligned} \tag{2.18}$$

Moreover, it is clear from (2.17) that, since σ_h is small, Φ is a small perturbation of the identity in the sense that

$$|z - z\nabla\Phi(y)| \leqslant C_1(n)\sigma_h|z|, \quad \text{for all } y, z \in \mathbb{R}^n. \tag{2.19}$$

Concerning $J\Phi$, the Jacobian of Φ, from (2.17) we deduce

$$J\Phi(x) = \left(1 + \sigma_h + \frac{(n-1)\sigma_h r^n}{|x|^n}\right)\left(1 + \sigma_h - \frac{\sigma_h r^n}{|x|^n}\right)^{n-1}.$$

Using the fact that $r/2 < |x| < r$, we can estimate

$$\begin{aligned} J\Phi(x) &\geqslant \left(1+\sigma_h+\frac{(n-1)\sigma_h r^n}{|x|^n}\right)(1+\sigma_h)^{n-2}\left(1+\sigma_h-(n-1)\frac{\sigma_h r^n}{|x|^n}\right) \\ &\geqslant \left(1+\sigma_h+\frac{(n-1)\sigma_h r^n}{|x|^n}\right)\left(1+\sigma_h-\frac{(n-1)\sigma_h r^n}{|x|^n}\right) \\ &= (1+\sigma_h)^2-(n-1)^2\frac{\sigma_h^2 r^{2n}}{|x|^{2n}} \geqslant (1+\sigma_h)^2-4^n(n-1)^2\sigma_h^2 \\ &= 1+2\sigma_h-\big(4^n(n-1)^2-1\big)\sigma_h^2 > 1+\sigma_h, \end{aligned}$$

provided that we choose

$$\sigma_h < \frac{1}{4^n(n-1)^2-1}.$$

Summarizing, we gain the following inequalities for the Jacobian of Φ:

$$\begin{aligned} 1+\sigma_h &\leqslant J\Phi(x), \quad \text{for all } x \in B_r \backslash B_{r/2}, \\ J\Phi(x) &\leqslant 1+2^n n\sigma_h, \quad \text{for all } x \in B_r. \end{aligned} \tag{2.20}$$

Now we can perform the change of variables $y = \Phi(x)$ and, observing that $\mathbb{1}_{\tilde{E}_h}(\Phi(x)) = \mathbb{1}_{E_h}(x)$, we get

$$\begin{aligned} I_{1,h} &= \int_{B_r} \big[F(x,u_h,\nabla u_h) - J\Phi(x)F(\Phi(x),u_h(x),\nabla u_h(x)\nabla\Phi^{-1}(\Phi(x)))\big]\,dx \\ &\quad + \int_{B_r\cap E_h} \big[G(x,u_h,\nabla u_h) - J\Phi(x)G(\Phi(x),u_h(x),\nabla u_h(x)\nabla\Phi^{-1}(\Phi(x)))\big]\,dx \\ &=: J_{1,h}+J_{2,h}. \end{aligned}$$

The two terms $J_{1,h}$ and $J_{2,h}$, involving F and G in B_r and $B_r \cap E_h$ respectively, can be treated in the same way. Therefore we just perform the calculation for $J_{1,h}$.

To make the argument more clear, since we shall use the structure conditions (2.2) and (2.3), we introduce the following notation. $A_2(x,s)$ denotes the quadratic form and $A_1(x,s)$ denotes the linear form defined as follows:

$$A_2(x,s)[z] := a_{ij}(x,s)z_iz_j, \quad A_1(x,s)[z] := a_i(x,s)z_i,$$

for any $(x,s,z) \in \Omega\times\mathbb{R}\times\mathbb{R}^n$. Analogously we set $A_0(x,s) = a(x,s)$. Accordingly, we can write down

$$\begin{aligned} &J_{1,h} \\ &= \int_{B_r}\Big\{A_2(x,u_h(x))[\nabla u_h(x)] \\ &- A_2(\Phi(x),u_h(x))[\nabla u_h(x)\nabla\Phi^{-1}(\Phi(x))]J\Phi(x)\Big\}\,dx \end{aligned} \tag{2.21}$$

$$
\begin{aligned}
&+\int_{B_r}\Big\{A_1(x,u_h(x))[\nabla u_h(x)]\\
&-A_1(\Phi(x),u_h(x))[\nabla u_h(x)\nabla\Phi^{-1}(\Phi(x))]J\Phi(x)\Big\}\,dx\\
&+\int_{B_r}\Big\{A_0(x,u_h(x))-A_0(1\Phi(x),u_h(x))J\Phi(x)\Big\}\,dx.
\end{aligned}
$$

We proceed estimating the first difference in the previous equality, being the others similar and indeed easier to handle.

$$
\begin{aligned}
&\int_{B_r}\Big\{A_2(x,u_h(x))[\nabla u_h(x)]-A_2(\Phi(x),u_h(x))[\nabla u_h(x)\nabla\Phi^{-1}(\Phi(x))]J\Phi(x)\Big\}\,dx\\
&=\int_{B_r}\Big\{A_2(\Phi(x),u_h(x))[\nabla u_h(x)]\\
&-A_2(\Phi(x),u_h(x))[\nabla u_h(x)\nabla\Phi^{-1}(\Phi(x))]J\Phi(x)\Big\}\,dx\\
&+\int_{B_r}\Big\{A_2(x,u_h(x))[\nabla u_h(x)]-A_2(\Phi(x),u_h(x))[\nabla u_h(x)]\Big\}\,dx=:H_{1,h}+H_{2,h}.
\end{aligned}
$$

The first term $H_{1,h}$ can be estimated observing that, as a consequence of (2.5), we have:

$$
|A_2[\xi]-A_2[\eta]|\leqslant N|\xi+\eta||\xi-\eta|,\quad \forall\xi,\eta\in\mathbb{R}^n.
$$

If we apply the last inequality to the vectors

$$
\xi:=\nabla u_h(x),\qquad\qquad \eta:=\sqrt{J\Phi(x)}[\nabla u_h(x)\nabla\Phi^{-1}(\Phi(x))],
$$

we are led to estimate $|\xi-\eta|$.
We start observing that, being $J\Phi(x)=\big(1-\sigma_h(2^n-1)\big)^n$ for $|x|<r/2$, by also using (2.20) we deduce

$$
|\sqrt{J\Phi(x)}-1|<C(n)\sigma_h,\qquad \text{for all } x\in\mathbb{R}^n.
$$

Therefore we have

$$
|\sqrt{J\Phi}\xi-\xi|\leqslant C(n)\sigma_h|\xi|.
$$

In addition choosing $z=\xi\nabla\Phi^{-1}(\Phi(x))$ in (2.19) and using also (2.18), we can deduce

$$
\begin{aligned}
|\xi\nabla\Phi^{-1}(\Phi(x))-\xi|&\leqslant\sigma_hC_1(n)|\xi\nabla\Phi^{-1}(\Phi(x))|\\
&\leqslant\sigma_h|\xi|C_1(n)\left\|\nabla\Phi^{-1}\circ\Phi\right\|_\infty\leqslant n2^nC_1(n)\sigma_h|\xi|.
\end{aligned}
$$

Summarizing, we finally get

$$
|\xi-\eta|\leqslant\sigma_hC(n)|\nabla u_h(x)|,\qquad\qquad |\xi+\eta|\leqslant C(n)|\nabla u_h(x)|,
$$

for some constant $C=C(n)>0$. From the previous estimates we deduce that

$$
|H_{1,h}|\leqslant\sigma_hNC^2(n)\int_{B_r}|\nabla u_h(x)|^2\,dx\leqslant\sigma_hNC^2(n)\Theta, \tag{2.22}
$$

where Θ is defined in (2.13). The second term $H_{2,h}$ can be estimated using the Lipschitz assumption of a_{ij} and observing that $|x - \Phi(x)| \leqslant \sigma_h r 2^n$. Therefore, we deduce that

$$|H_{2,h}| \leqslant \sigma_h r 2^n L_D \int_{B_r} |\nabla u_h(x)|^2 \, dx \leqslant \sigma_h C(n, L_D)\Theta. \tag{2.23}$$

In conclusion, since the other terms in (2.21) can be estimated in the same way, collecting estimates (2.22) and (2.23), we get

$$|J_{1,h}| \leqslant \sigma_h C(n, N, L_D)\Theta.$$

Since the same estimate holds true for $J_{2,h}$, we conclude that

$$I_{1,h} \geqslant -\sigma_h C_2(n, N, L_D)\Theta, \tag{2.24}$$

for some constant $C_2 = C_2(n, N, L_D) > 0$.

Step 3. *Estimate of* $I_{2,h}$. In order to estimate $I_{2,h}$, we use the area formula for maps between rectifiable sets. We fix $x \in \partial^* E_h \cap (B_r \backslash B_{\frac{r}{2}})$. We denote by $\{\tau_1, \ldots, \tau_{n-1}\}$ an orthonormal base for $T_x \partial^* E_h$, and by L the $n \times (n-1)$ matrix representing $\nabla^{\partial^* E_h} \Phi(x)$ with respect the previous base and the canonical base $\{e_1, \ldots, e_n\}$ in $\mathbb{R}^n$ (see Definition A.2.3 for the notation). We remark that, for $x \in B_r \backslash B_{\frac{r}{2}}$,

$$\Phi(x) = \phi(|x|)\frac{x}{|x|},$$

where

$$\phi(t) := t\left(1 + \sigma_h - \frac{\sigma_h r^n}{t^n}\right), \quad \forall t \in \left[\frac{r}{2}, r\right].$$

Hence, we have

$$L_{ij} = \nabla \Phi_i \cdot \tau_j = \frac{\phi(|x|)}{|x|} e_i \cdot \tau_j + \left(\phi'(|x|) - \frac{\phi(|x|)}{|x|}\right)\frac{x_i}{|x|}\frac{x \cdot \tau_j}{|x|},$$

for any $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, n-1\}$. Thus, for $j, l \in \{1, \ldots, n-1\}$, we obtain

$$(L^* L)_{jl} = \frac{\phi^2(|x|)}{|x|^2} \sum_{i=1}^{n} (e_i \cdot \tau_j)(e_i \cdot \tau_l) + \left(\phi'^2(|x|) - \frac{\phi^2(|x|)}{|x|^2}\right)\frac{(x \cdot \tau_j)(x \cdot \tau_l)}{|x|^2}.$$

Since $\nabla^{\partial^* E_h} \Phi(x)$ is invariant by rotation, in order to evaluate $\det(L^* L)$, we may assume, without loss of generality, that $\tau_j = e_j$, for any $j \in \{1, \ldots, n-1\}$. We deduce that

$$L^* L = \frac{\phi^2(|x|)}{|x|^2} I^{(n-1)} + \left(\phi'^2(|x|) - \frac{\phi^2(|x|)}{|x|^2}\right)\frac{x' \otimes x'}{|x|^2},$$

where $I^{(n-1)}$ denotes the identity map on $\mathbb{R}^{n-1}$ and $x' = (x_1, \ldots, x_{n-1})$. With a calculation similar to the one performed to obtain $J\Phi$, from the equality above we easily get that

$$\det(L^* L) = \left(\frac{\phi^2(|x|)}{|x|^2}\right)^{n-1}\left[1 + \frac{|x|^2}{\phi^2(|x|)}\left(\phi'^2(|x|) - \frac{\phi^2(|x|)}{|x|^2}\right)\frac{|x'|^2}{|x|^2}\right],$$

and so we can write, for $x \in \partial^* E_h \cap (B_r \backslash B_{\frac{r}{2}})$,

$$
\begin{aligned}
J^{\partial^* E_h}\Phi(x) &= \sqrt{\det(L^* L)} \\
&= \left(\frac{\phi(|x|)}{|x|}\right)^{n-1} \sqrt{1 + \frac{|x|^2}{\phi^2(|x|)}\left(\phi'^2(|x|) - \frac{\phi^2(|x|)}{|x|^2}\right)\frac{|x'|^2}{|x|^2}} \\
&\leqslant \left(\frac{\phi(|x|)}{|x|}\right)^{n-2} \phi'(|x|) \leqslant \phi'(|x|) \leqslant 1 + \sigma_h + 2^n(n-1)\sigma_h.
\end{aligned}
$$

In order to estimate $I_{2,h}$, we use the area formula for maps between rectifiable sets Theorem A.2.4, thus getting

$$
\begin{aligned}
I_{2,h} = P(E_h; \overline{B}_r) - P(\widetilde{E}_h; \overline{B}_r) &= \int_{\partial^* E_h \cap \overline{B}_r} d\mathcal{H}^{n-1} - \int_{\partial^* E_h \cap \overline{B}_r} J^{E_h}\Phi(x)\, d\mathcal{H}^{n-1} \\
&= \int_{\partial^* E_h \cap \overline{B}_r \backslash B_{r/2}} \big(1 - J^{E_h}\Phi(x)\big)\, d\mathcal{H}^{n-1} + \int_{\partial^* E_h \cap B_{r/2}} \big(1 - J^{E_h}\Phi(x)\big)\, d\mathcal{H}^{n-1}.
\end{aligned}
$$

Notice that the last integral in the above formula is non-negative since Φ is a contraction in $B_{r/2}$, hence $J^{E_h}\Phi(x) < 1$ in $B_{r/2}$, while from (**??**) we have

$$
\int_{\partial^* E_h \cap \overline{B}_r \backslash B_{r/2}} \big(1 - J^{E_h}\Phi(x)\big)\, d\mathcal{H}^{n-1} \geqslant -2^n n P(E_h; \overline{B}_r)\sigma_h \geqslant -2^n n \Theta \sigma_h\,,
$$

thus concluding that

$$
I_{2,h} \geqslant -2^n n \Theta \sigma_h\,. \tag{2.25}
$$

Step 4. *Estimate of* $I_{3,h}$. To estimate $I_{3,h}$ we recall (2.14), (2.15) and (2.20), thus getting

$$
\begin{aligned}
I_{3,h} &= \lambda_h \int_{E_h \cap B_r \backslash B_{r/2}} (J\Phi(x) - 1)\, dx + \lambda_h \int_{E_h \cap B_{r/2}} (J\Phi(x) - 1)\, dx \\
&\geqslant \lambda_h \left(\frac{\omega_n}{2^{n+2}} - \varepsilon\right)\sigma_h r^n - \lambda_h \big[1 - \big(1 - (2^n - 1)\sigma_h\big)^n\big]\varepsilon r^n \\
&\geqslant \lambda_h \sigma_h r^n \left[\frac{\omega_n}{2^{n+2}} - \varepsilon - (2^n - 1)n\varepsilon\right].
\end{aligned}
$$

Therefore, if we choose $0 < \varepsilon < \varepsilon_0(n)$, we have that

$$
I_{3,h} \geqslant \lambda_h C_3 \sigma_h r^n,
$$

for some positive $C_3 = C_3(n)$. From this inequality, recalling (2.16), (2.24) and (2.25), we obtain

$$
\mathcal{F}_{\lambda_h}(E_h, u_h) - \mathcal{F}_{\lambda_h}(\widetilde{E}_h, \tilde{u}_h) \geqslant \sigma_h\big(\lambda_h C_3 r^n - \Theta(C_2(n, N, L_D) + 2^n n)\big) > 0,
$$

if λ_h is sufficiently large. This contradicts the minimality of (E_h, u_h), thus concluding the proof.

□

The previous theorem motivates the following definition.

Definition 2.2.2 (Λ-minimizers). *The energy pair (E, u) is a Λ-**minimizer** in Ω of the functional $\mathcal{F}$, defined in* (2.1)*, if and only if for every $B_r(x_0) \subset \Omega$ it holds:*

$$\mathcal{F}(E, u; B_r(x_0)) \leqslant \mathcal{F}(F, v; B_r(x_0)) + \Lambda |F \Delta E|,$$

whenever (F, v) is an admissible test pair, namely, F is a set of finite perimeter with $F \Delta E \subset\subset B_r(x_0)$ and $v - u \in H^1_0(B_r(x_0))$.

2.3 Higher integrability results

In this section we quote higher integrability results both for local minimizers of functional (2.1) and for comparison functions that we will use later in the paper. It is worth mentioning that the following lemmata can be applied in general to minimizers of integral functionals of the type

$$\mathcal{H}(u; \Omega) := \int_\Omega F(x, u, \nabla u)\, dx, \tag{2.26}$$

assuming that the energy density only satisfies the structure condition (2.2) and the growth conditions (2.5) and (2.6), without assuming any continuity on the coefficients. Therefore, functionals of the type (2.1) belong to this class and in addition the involved estimates only depend on the constants appearing in (2.5) and (2.6) but do not depend on E accordingly.

Lemma 2.3.1. *Let $u \in H^1(\Omega)$ be a local minimizer of the functional $\mathcal{H}$ defined in* (2.26)*, where F satisfies the structure condition* (2.2) *and the growth conditions* (2.5) *and* (2.6)*. Then, for every $B_{2R}(x_0) \subset\subset \Omega$, it holds*

$$\fint_{B_R(x_0)} |\nabla u|^2\, dx \leqslant C_1 \left[\left(\fint_{B_{2R}(x_0)} |\nabla u|^{2m}\, dx \right)^{\frac{1}{m}} + 1 \right], \tag{2.27}$$

where $m = \frac{n}{n+2}$ and $C_1 = C_1(n, \nu, N, L)$ is a positive constant.

Proof. Whithout loss of generality we may assume that $x_0 = 0$. Let $R < t < s < 2R$ and choose $\eta \in C^\infty_0(B_s)$ such that $\eta \equiv 1$ in B_t and $|\nabla \eta| \leqslant 2/(s-t)$. We choose a test function $v = u - \phi$, where $\phi = \eta(u - u_s)$ and u_s denotes the average of u in B_s, $u_s = \fint_{B_s} u\, dx$. Testing the minimality of u with v and using growth condition (2.10) we deduce that

$$\begin{aligned}
&\frac{\nu}{2} \int_{B_s} \left[|\nabla u|^2 - \frac{2L^2}{\nu^2} \right] dx \leqslant \mathcal{H}(u; B_s) \leqslant \mathcal{H}(v; B_s) \\
&\leqslant 2 \int_{B_s} \left[(N+1) |\nabla u (1 - \eta) - \nabla \eta (u - u_s)|^2 + L(L+1) \right] dx \\
&\leqslant 4(N+1) \int_{B_s \setminus B_t} |\nabla u|^2\, dx + 4(N+1) \int_{B_s} |u - u_s|^2 |\nabla \eta|^2\, dx
\end{aligned}$$

$$+ 2 \int_{B_s} L(L+1)\, dx.$$

Adding to both sides $4(N+1) \int_{B_t} |\nabla u|^2 \, dx$ we deduce

$$\left[4(N+1) + \frac{\nu}{2}\right] \int_{B_t} |\nabla u|^2 \, dx$$
$$\leqslant 4(N+1) \int_{B_s} |\nabla u|^2 \, dx + 4(N+1) \int_{B_s} |u - u_s|^2 |\nabla \eta|^2 \, dx + \int_{B_s} C(L, \nu)\, dx.$$

Thus we get

$$\int_{B_t} |\nabla u|^2 \, dx \leqslant \theta \int_{B_s} |\nabla u|^2 \, dx + \frac{C(n,\nu,N,L)}{(s-t)^2} \int_{B_s} |u - u_s|^2 \, dx + C(n,\nu,N,L),$$

where $\frac{\theta = 4(N+1)}{4(N+1)+\frac{\nu}{2}} < 1$. Using Sobolev-Poincaré's inequality

$$\int_{B_s} |u - u_s|^2 \, dx \leqslant C(n) \left(\int_{B_s} |\nabla u|^{2m} \, dx \right)^{\frac{1}{m}} \leqslant C(n) \left(\int_{B_{2R}} |\nabla u|^{2m} \, dx \right)^{\frac{1}{m}},$$

with $m = \frac{n}{n+2}$, we eventually obtain

$$\int_{B_t} |\nabla u|^2 \, dx \leqslant \theta \int_{B_s} |\nabla u|^2 \, dx + \frac{C(n,\nu,N,L)}{(s-t)^2} \left(\int_{B_{2R}} |\nabla u|^{2m} \, dx \right)^{\frac{1}{m}} + C(n,\nu,N,L),$$

Iterating the previous estimate using Lemma 2.1.3, we deduce that there exists a constant $C = C(\theta) = C(\nu, N) > 0$ such that

$$\int_{B_R} |\nabla u|^2 \, dx \leqslant C(\nu, N) \left[\frac{C(n,\nu,N,L)}{R^2} \left(\int_{B_{2R}} |\nabla u|^{2m} \, dx \right)^{\frac{1}{m}} + C(n,\nu,N,L) \right].$$

We get the thesis if we divide by R^n.

□

Remark 2.3.2. *We observe that the reverse Hölder inequality stated in the previous lemma can be also proved exactly in the same way replacing the balls with the cubes. The reverse Hölder inequality written on cubes is the suitable version in order to apply Calderòn-Zygmund decomposition and Gehring's lemma (see [37, Proposition 6.1]), thus obtaining the higher integrability estimate on cubes. Finally, the higher integrability estimate on balls stated below, which is suitable in our setting, can be deduced by a covering argument.*

Lemma 2.3.3. *Let $u \in H^1(\Omega)$ be a local minimizer of the functional $\mathcal{H}$ defined in* (2.26)*, where F satisfies the structure condition* (2.2) *and the growth conditions* (2.5) *and* (2.6)*. There exists $s = s(n,\nu,N,L) > 1$ such that, for every $B_{2R}(x_0) \subset\subset \Omega$, it holds*

$$\fint_{B_R(x_0)} |\nabla u|^{2s} \, dx \leqslant C_2 \left(\fint_{B_{2R}(x_0)} \left(1 + |\nabla u|^2\right) dx \right)^{s},$$

where $C_2 = C_2(n,\nu,N,L)$ is a positive constant.

In the next section we will prove some energy density estimates by using a standard comparison argument. For this purpose we will need a reverse Hölder inequality for the comparison function defined below.

Definition 2.3.4 (Comparison function). *Let $u \in H^1(\Omega)$ be a local minimizer of the functional $\mathcal{H}$ defined in* (2.26) *and $B_{2R} \subset\subset \Omega$. We shall denote by v the solution of the following problem*

$$v := \underset{w \in u + H_0^1(B_R)}{\operatorname{argmin}} \int_{B_R} \tilde{F}(x, \nabla w)\, dx, \tag{2.28}$$

where $\tilde{F}(x,z) := F(x,u(x),z)$ satisfies the structure condition (2.2) *and the growth conditions* (2.5) *and* (2.6).

For the comparison function v defined in (2.28) we can state the following reverse Hölder inequality up to the boundary of B_R.

Lemma 2.3.5. *Let $u \in H^1(\Omega)$ be a local minimizer of the functional $\mathcal{H}$ defined in* (2.26), *where F satisfies the structure condition* (2.2) *and the growth conditions* (2.5) *and* (2.6). *Let v be the comparison function defined above and $B_{2R} \subset\subset \Omega$. Let us consider the following extension of v:*

$$V(x) := \begin{cases} v(x) & \text{for } x \in B_R, \\ u(x) & \text{for } x \in \Omega \backslash B_R. \end{cases}$$

Let $B_\rho(x_0) \subset B_{2R}$ with $x_0 \in B_R$ and $\rho < \frac{R}{2}$. Then

$$\fint_{B_\rho(x_0)} |\nabla V|^2\, dx \leqslant C_3 \left[\left(\fint_{B_{2\rho}(x_0)} |\nabla V|^{2m}\, dx \right)^{\frac{1}{m}} + \left(\fint_{B_{2\rho}(x_0)} |\nabla u|^{2m} \right)^{\frac{1}{m}} + 1 \right], \tag{2.29}$$

where $m = \frac{n}{n+2}$ and $C_3 = C_3(n, \nu, N, L)$ is a positive constant.

Proof. Let $x_0 \in B_R$ and $\rho \leqslant s < t \leqslant r < 2\rho < R$, where $r = \frac{3}{2}\rho$; then, the following alternatives may occur:

$$\begin{cases} i)\ B_r(x_0) \subset\subset B_R \\ ii)\ \overline{B}_r(x_0) \cap (\Omega \backslash B_R) \neq \varnothing. \end{cases}$$

In the case i) we can proceed exactly as in Lemma 2.3.1 to get the desired estimate. Let us then consider the case ii) which is slightly different.

Choose $\eta \in C_0^\infty(B_t(x_0))$, such that $0 \leqslant \eta \leqslant 1$, $\eta \equiv 1$ in $B_s(x_0)$ and $|\nabla \eta| \leqslant 2/(t-s)$. Now we can use the function $\varphi := \eta(V-u)$ to test the minimality of v with the aim of estimating the difference $\int_{B_s(x_0)} |\nabla(V-u)|^2$. In the following it would be useful to keep in mind that $\nabla\varphi = \nabla(V-u)$ in $B_s(x_0)$, which is the quantity we are interested to estimate.

In order to simplify the notation, let us denote

$$\tilde{a}_{ij}(x) := a_{ij}(x, u(x)), \quad \tilde{a}_i(x) := a_i(x, u(x)), \quad \tilde{a}(x) := a(x, u(x)),$$

$$\tilde{\mathcal{F}}(w;A) := \int_A \tilde{F}(x,\nabla w)\,dx.$$

We start comparing the energy of φ and v inside a generic set $A \subset B_R$. We have that

$$\begin{aligned}
&\tilde{\mathcal{F}}(\nabla\varphi;A) - \tilde{\mathcal{F}}(\nabla v;A) \\
&= \int_A \tilde{a}_{ij}(x)\nabla_i\varphi\nabla_j\varphi\,dx + \int_A \tilde{a}_i(x)\nabla_i\varphi\,dx + \int_A \tilde{a}(x)\,dx \\
&- \int_A \tilde{a}_{ij}(x)\nabla_i v\nabla_j v\,dx - \int_A \tilde{a}_i(x)\nabla_i v\,dx - \int_A \tilde{a}(x)\,dx \\
&= \int_A \tilde{a}_{ij}(x)\nabla_i(\varphi - v)\nabla_j(\varphi - v)\,dx + 2\int_A \tilde{a}_{ij}(x)\nabla_i v\nabla_j(\varphi - v)\,dx \\
&+ \int_A \tilde{a}_i(x)\nabla_i(\varphi - v)\,dx.
\end{aligned}$$

Using the growth conditions (2.5) and (2.6) and Young's inequality we deduce that

$$\tilde{\mathcal{F}}(\nabla\varphi;A) - \tilde{\mathcal{F}}(\nabla v;A) \leqslant \Big(N + N^2 + \frac{1}{2}\Big)\int_A |\nabla(v-\varphi)|^2 dx + \int_A |\nabla v|^2 dx + \int_A \frac{L^2}{2}\,dx. \tag{2.30}$$

Recalling the growth condition (2.10) we estimate

$$\int_A |\nabla v|^2 dx \leqslant \frac{2}{\nu}\int_A \tilde{F}(x,\nabla v)\,dx + \int_A \frac{2L^2}{\nu^2}\,dx.$$

We can conclude from (2.30) that

$$\begin{aligned}
\tilde{\mathcal{F}}(\nabla\varphi;A) \leqslant \Big(1 + \frac{2}{\nu}\Big)\tilde{\mathcal{F}}(\nabla v;A) + \Big(N + N^2 + \frac{1}{2}\Big)\int_A |\nabla(v-\varphi)|^2 dx \\
+ \int_A \Big(\frac{L^2}{2} + \frac{2L^2}{\nu^2}\Big)\,dx.
\end{aligned} \tag{2.31}$$

We compute now the previous integrals on the set $B_t(x_0) \cap B_R$. We use again the growth condition (2.10) and the minimality of v with respect to $v - \varphi$ in order to estimate further on the right hand side of the previous inequality:

$$\begin{aligned}
&\tilde{\mathcal{F}}(\nabla v; B_t(x_0) \cap B_R) \\
&\leqslant \tilde{\mathcal{F}}(\nabla(v-\varphi); B_t(x_0) \cap B_R) \\
&\leqslant \big(N+1\big)\int_{B_t(x_0)\cap B_R} |\nabla(v-\varphi)|^2\,dx + \int_{B_t(x_0)\cap B_R} L(L+1)\,dx.
\end{aligned}$$

Finally we can resume (2.31) to conclude this first part concerning the energy estimate of φ, using again (2.10),

$$\frac{\nu}{2}\int_{B_t(x_0)}\Big(|\nabla\varphi|^2 - \frac{2L^2}{\nu^2}\Big)\,dx \leqslant \int_{B_t(x_0)\cap B_R} \tilde{F}(x,\nabla\varphi)\,dx$$

$$\leqslant C(\nu, N, L) \int_{B_t(x_0)} \big(|\nabla(V-\varphi)|^2\, dx + 1\big)\, dx.$$

We summarize the previous estimate as follows:

$$\frac{\nu}{2} \int_{B_t(x_0)} |\nabla\varphi|^2\, dx \leqslant C(\nu, N, L) \int_{B_t(x_0)} \big(|\nabla(V-\varphi)|^2 + 1\big)\, dx. \tag{2.32}$$

Now we observe that $|\nabla(V-\varphi)| \leqslant |\nabla u| + (1-\eta)|\nabla(V-u)| + \frac{2}{t-s}|V-u|$; then by (2.32) we deduce

$$\int_{B_s(x_0)} |\nabla(V-u)|^2\, dx \leqslant C(\nu, N, L) \int_{B_t(x_0)\setminus B_s(x_0)} |\nabla(V-u)|^2\, dx$$
$$+ \frac{C(\nu, N, L)}{(t-s)^2} \int_{B_t(x_0)} |V-u|^2\, dx + C(\nu, N, L) \int_{B_t(x_0)} \big(|\nabla u|^2 + 1\big)\, dx. \tag{2.33}$$

Now we use the "hole-filling" technique adding $C(\nu, N, L) \int_{B_s(x_0)} |\nabla(V-u)|^2\, dx$ to both sides of (2.33) getting

$$\int_{B_s(x_0)} |\nabla(V-u)|^2\, dx \leqslant \theta \int_{B_t(x_0)} |\nabla(V-u)|^2\, dx$$
$$+ C(\nu, N, L)\left[\frac{1}{(t-s)^2} \int_{B_t(x_0)} |V-u|^2\, dx + \int_{B_t(x_0)} \big(|\nabla u|^2 + 1\big)\, dx\right],$$

where $\theta = \frac{C(\nu,N,L)}{C(\nu,N,L)+1}$. Using Lemma 2.1.3 we obtain

$$\int_{B_\rho(x_0)} |\nabla(V-u)|^2\, dx \leqslant \frac{C(\nu, N, L)}{(r-\rho)^2} \int_{B_r(x_0)} |V-u|^2\, dx$$
$$+ C(\nu, N, L) \int_{B_r(x_0)} \big(1 + |\nabla u|^2\big)\, dx.$$

Therefore, being $r = \frac{3}{2}\rho$ and by condition *ii)*, we have

$$|B_{2\rho}(x_0)\setminus B_R| \geqslant C|B_\rho(x_0)|,$$

for some universal constant $C = C(n) > 0$. We can now use Sobolev-Poincaré's inequality for functions vanishing on a set of positive measure (see Theorem A.3.3) to deduce

$$\fint_{B_\rho(x_0)} |\nabla(V-u)|^2\, dx$$
$$\leqslant C(n, \nu, N, L)\left[\left(\fint_{B_{2\rho}(x_0)} (|\nabla(V-u)|^{2m}\, dx\right)^{\frac{1}{m}} + \fint_{B_{2\rho}(x_0)} \big(1 + |\nabla u|^2\big)\, dx\right]$$

Finally we can apply reverse Hölder inequality (2.27) for u in the last estimate to get (2.29). □

Reasoning in a similar way as above, the higher integrability for v can be obtained by means of Gehring's lemma (see [37, Proposition 6.1]).

Lemma 2.3.6. *Let $u \in H^1(\Omega)$ be a local minimizer of the functional $\mathcal{H}$ defined in* (2.26)*, where F satisfies the structure condition* (2.2) *and the growth conditions* (2.5) *and* (2.6)*. Let $v \in H^1(B_R(x_0))$ be the comparison function defined in* (2.28)*. Denoting by $s = s(n, \nu, N, L) > 1$ the same exponent given in Lemma 2.3.3, it holds*

$$\fint_{B_R(x_0)} |\nabla v|^{2s}\, dx \leqslant C_4 \left(\fint_{B_{2R}(x_0)} \big(1 + |\nabla u|^2\big)\, dx \right)^s,$$

where $C_4 = C_4(n, \nu, N, L)$ is a positive constant.

2.4 A decay estimate for elastic minima

In this section we prove a decay estimate for elastic minima that will be crucial for the proof strategy. Indeed, we show that if (E, u) is a Λ-minimizer of the functional $\mathcal{F}$ defined in (2.1) and x_0 is a point in Ω, where either the density of E is close to 0 or 1, or the set E is asymptotically close to a hyperplane, then for ρ sufficiently small we have

$$\int_{B_\rho(x_0)} |\nabla u_E|^2\, dx \leqslant C\rho^{n-\delta},$$

for any $\delta > 0$. A preliminary result, which will be used later, provides an upper bound for $\mathcal{F}$. It is rather standard and is related to the threshold Hölder exponent $\frac{1}{2}$ of the function u, when (E, u) is either a solution of the constrained problem (P_c) or a solution of the penalized problem (P) defined in Section 1. Its proof is contained in [45, Lemma 2.3], [35], and we recall it here for the sake of completeness.

Theorem 2.4.1. *Let (E, u) be a Λ-minimizer of $\mathcal{F}$ in Ω. Then for every open set $U \subset\subset \Omega$ there exists a constant $C_3 = C_3\big(n, \Lambda, U, \|\nabla u\|_{L^2(\Omega)}\big) > 0$ such that for every $B_r(x_0) \subset U$ it holds*

$$\mathcal{F}(E, u; B_r(x_0)) \leqslant C_3 r^{n-1}.$$

Proof. Fixing $B_r(x_0) \subset U \subset\subset \Omega$, we compare (E, u) with $(E \backslash B_r(x_0), u)$ thus obtaining

$$\begin{aligned} \mathcal{F}(E, u; \Omega) &\leqslant \mathcal{F}(E \backslash B_r(x_0), u; \Omega) + \Lambda |E \Delta (E \backslash B_r(x_0)) \cap \Omega| \\ &\leqslant \mathcal{F}(E \backslash B_r(x_0), u; \Omega) + \Lambda |B_r(x_0)|. \end{aligned}$$

Making $\mathcal{F}$ explicit and getting rid of the common terms, we obtain:

$$\int_{B_r(x_0) \cap E} G(x, u, \nabla u)\, dx + P(E; B_r(x_0)) \leqslant P(E \cap \partial B_r(x_0); \Omega) + c(n, \Lambda) r^n$$

$$\leqslant \mathcal{H}^{n-1}(\partial B_r(x_0)) + c(n,\Lambda)r^{n-1}$$
$$\leqslant c(n,\Lambda)r^{n-1}. \tag{2.34}$$

Now we want to prove that there exist $\tau \in \left(0, \frac{1}{2}\right)$ and $\delta \in (0,1)$ such that for every $M > 0$ there exists $h_0 \in \mathbb{N}$ such that, for any $B_r(x_0) \subset U$, we have

$$\int_{B_r(x_0)} |\nabla u|^2 \leqslant h_0 r^{n-1} \quad \text{or} \quad \int_{B_{\tau r}(x_0)} |\nabla u|^2\,dx \leqslant M\tau^{n-\delta}\int_{B_r(x_0)} |\nabla u|^2\,dx.$$

Step 1: Arguing by contradiction, for $\tau \in \left(0, \frac{1}{2}\right)$ and $\delta \in (0,1)$, we choose $M > \tau^{\delta-n}$ and we assume that, for every $h \in \mathbb{N}$, there exists a ball $B_{r_h}(x_h) \subset U$ such that

$$\int_{B_{r_h}(x_h)} |\nabla u|^2\,dx > h r_h^{n-1} \tag{2.35}$$

and

$$\int_{B_{\tau r_h}(x_h)} |\nabla u|^2\,dx > M\tau^{n-\delta}\int_{B_{r_h}(x_h)} |\nabla u|^2\,dx. \tag{2.36}$$

Note that estimates (2.34) and (2.35) yield

$$\int_{B_{r_h}(x_h)\cap E} |\nabla u|^2\,dx + P(E; B_{r_h}(x_h)) \leqslant c_0 r_h^{n-1} < \frac{c_0}{h}\int_{B_{r_h}(x_h)} |\nabla u|^2\,dx, \tag{2.37}$$

and so

$$\int_{B_{r_h}(x_h)\cap E} |\nabla u|^2\,dx < \frac{c_0}{h}\int_{B_{r_h}(x_h)} |\nabla u|^2\,dx, \tag{2.38}$$

for some positive constant c_0.

Step 2: We will prove our aim by means of a blow-up argument. We set

$$\varsigma_h^2 := \fint_{B_{r_h}(x_h)} |\nabla u|^2\,dx$$

and, for $y \in B_1$, we introduce the sequence of rescaled functions defined as

$$v_h(y) := \frac{u(x_h + r_h y) - a_h}{\varsigma_h r_h}, \quad \text{with} \quad a_h := \fint_{B_{r_h}(x_h)} u\,dx.$$

We have $\nabla u(x_h + r_h y) = \varsigma_h \nabla v_h(y)$ and a change of variable yields

$$\int_{B_1} |\nabla v_h(y)|^2\,dy = \frac{1}{\varsigma_h^2}\fint_{B_{r_h}(x_h)} |\nabla u(x)|^2\,dx = 1.$$

Therefore, there exist a (not relabeled) subsequence of v_h and $v \in H^1(B_1)$ such that $v_h \rightharpoonup v$ in $H^1(B_1)$ and $v_h \to v$ in $L^2(B_1)$. Moreover, the semicontinuity of the norm implies

$$\fint_{B_1} |\nabla v(y)|^2\,dy \leqslant \liminf_{h\to+\infty}\fint_{B_1} |\nabla v_h(y)|^2\,dy = 1. \tag{2.39}$$

We rewrite the inequalities (2.35), (2.36) and (2.38). They become, respectively,

$$\varsigma_h^2 > \frac{h}{r_h}, \tag{2.40}$$

$$\fint_{B_\tau} |\nabla v_h(y)|^2\, dy > M\tau^{-\delta}, \tag{2.41}$$

$$\int_{B_1 \cap E_h^*} |\nabla v_h(y)|^2\, dy < \frac{c_0}{h} \int_{B_1} |\nabla v_h(y)|^2\, dy = \frac{c_0 \omega_n}{h}. \tag{2.42}$$

Of course, (2.40) implies that $\varsigma_h \to +\infty$, as $h \to +\infty$.
Step 3: We claim that the L^2-norm of v_h converges to the L^2-norm of v. Consider the sets

$$E_h^* := \frac{E - x_h}{r_h} \cap B_1, \quad \forall h \in \mathbb{N}.$$

Since $r_h^{n-1} P(E_h^*; B_1) = P(E; B_{r_h}(x_h))$, by (2.37), we have that the sequence $\{P(E_h^*; B_1)\}_{h\in\mathbb{N}}$ is bounded. Therefore up to a not relabeled subsequence, $\mathbb{1}_{E_h} \to \mathbb{1}_{E^*}$ in $L^1(B_1)$, for some set $E^* \subset B_1$ of locally finite perimeter. By (2.42) and Fatou's Lemma,

$$\int_{B_1 \cap E^*} |\nabla v(y)|^2\, dy = 0.$$

By the Λ-minimality of (E, u) with respect to $(E, u + \phi)$ we get, for $\phi \in H_0^1(B_{r_h}(x_h))$,

$$\begin{aligned} &\int_{B_{r_h}(x_h)} \big[F(x, u, \nabla u) + \mathbb{1}_E G(x, u, \nabla u)\big]\, dx \\ &\leqslant \int_{B_{r_h}(x_h)} \big[F(x, u + \phi, \nabla u + \nabla\phi) + \mathbb{1}_E G(x, u + \phi, \nabla u + \nabla\phi)\big]\, dx. \end{aligned}$$

Using the change of variable $x = x_h + r_h y$, we deduce for every $\psi \in H_0^1(B_1)$,

$$\begin{aligned} &\int_{B_1} \big[F(x_h + r_h y, u(x_h + r_h y), \varsigma_h \nabla v_h) \\ &+ \mathbb{1}_{E_h^*} G(x_h + r_h y, u(x_h + r_h y), \varsigma_h \nabla v_h)\big]\, dy \\ &\leqslant \int_{B_1} F(x_h + r_h y, u(x_h + r_h y) + r_h\psi, \varsigma_h \nabla v_h + \nabla\psi)\, dy \\ &+ \int_{B_1} \mathbb{1}_{E_h^*} G(x_h + r_h y, u(x_h + r_h y) + r_h\psi, \varsigma_h \nabla v_h + \nabla\psi)\, dy. \end{aligned}$$

Let $\eta \in C_c^\infty(B_1)$ such that $0 \leqslant \eta \leqslant 1$. We choose as a test function $\psi_h = \varsigma_h \eta(v - v_h)$ and exploit $\nabla v_h + \nabla\psi_h$ for reader's convenience,

$$\nabla v_h + \nabla\psi_h = \varsigma_h \eta \nabla v + \varsigma_h(1 - \eta)\nabla v_h + \varsigma_h(v - v_h)\nabla\eta.$$

For simplicity of notation we denote $w_h := u(x_h + r_h y) + r_h \varsigma_h \eta(v - v_h)$ so that the previous inequality can be read as

$$\int_{B_1} \big[F(x_h + r_h y, u(x_h + r_h y), \varsigma_h \nabla v_h) + \mathbb{1}_{E_h^*} G(x_h + r_h y, u(x_h + r_h y), \varsigma_h \nabla v_h)\big] dy$$

$$\leqslant \int_{B_1} F(x_h + r_h y, w_h, \varsigma_h \eta \nabla v + \varsigma_h (1-\eta)\nabla v_h + \varsigma_h (v - v_h)\nabla \eta)\, dy$$
$$+ \int_{B_1} \mathbb{1}_{E_h^*} G(x_h + r_h y, w_h, \varsigma_h \eta \nabla v + \varsigma_h (1-\eta)\nabla v_h + \varsigma_h (v - v_h)\nabla \eta)\, dy.$$

Using the quadratic structure of F and G, we can pull out the term $\varsigma_h(v - v_h)\nabla\eta$ in the last argument of F and G, in order to use the convexity in the next step.

$$\int_{B_1} \big[F(x_h + r_h y, u(x_h + r_h y), \varsigma_h \nabla v_h)$$
$$+ \mathbb{1}_{E_h^*} G(x_h + r_h y, u(x_h + r_h y), \varsigma_h \nabla v_h)\big]\, dy$$
$$\leqslant \int_{B_1} F(x_h + r_h y, w_h, \varsigma_h \eta \nabla v + \varsigma_h (1-\eta)\nabla v_h)\, dy$$
$$+ \int_{B_1} \mathbb{1}_{E_h^*} G(x_h + r_h y, w_h, \varsigma_h \eta \nabla v + \varsigma_h (1-\eta)\nabla v_h)\, dy$$
$$+ c(N, L) \int_{B_1} \big(|\varsigma_h \nabla v| + |\varsigma_h \nabla v_h| + |\varsigma_h (v - v_h))|\big)\varsigma_h |v - v_h|\, dy.$$

Using the convexity of F and G and rearranging the terms we obtain

$$\int_{B_1} \eta F(x_h + r_h y, w_h, \varsigma_h \nabla v_h) \leqslant \int_{B_1} \eta F(x_h + r_h y, w_h, \varsigma_h \nabla v)\, dy$$
$$+ \int_{B_1} \big[F(x_h + r_h y, w_h, \varsigma_h \nabla v_h) - F(x_h + r_h y, u(x_h + r_h y), \varsigma_h \nabla v_h)\big]\, dy$$
$$+ \int_{B_1} \mathbb{1}_{E_h^*} \big[G(x_h + r_h y, w_h, \varsigma_h \nabla v_h) - G(x_h + r_h y, u(x_h + r_h y), \varsigma_h \nabla v_h)\big]\, dy$$
$$+ \int_{B_1} \mathbb{1}_{E_h^*} \eta \big[G(x_h + r_h y, w_h, \varsigma_h \nabla v) - G(x_h + r_h y, w_h, \varsigma_h \nabla v_h)\big]\, dy$$
$$+ c(N, L) \int_{B_1} \big(|\varsigma_h \nabla v| + |\varsigma_h \nabla v_h| + |\varsigma_h (v - v_h))|\big)\varsigma_h |v - v_h|\, dy.$$

The last term and the second to last term can be treated in a standard way using (2.39), Hölder's inequality, the strong convergence of v_h to v and the weak convergence of ∇v_h to ∇v. The remaining two terms, which differ only in the second argument, can be treated as follows.

We remark that, by definition of v_h and Hölder continuity of u_h, it immediately follows $r_h \varsigma_h v_h \to 0$. Therefore, being $r_h \varsigma_h \to 0$ where $v \neq 0$, we deduce also $w_h - u(x_h + r_h y) = r_h \varsigma_h \eta (v - v_h) \to 0$ for a.e. $y \in B_1$. Finally, using the equi-integrability of $|\nabla v_h|^2$, resulting from the weak convergence of ∇v_h, and the uniform boundedness of the coefficients a_{ij}, a_i, a, we conclude that

$$\int_{B_1} \big[F(x_h + r_h y, w_h, \varsigma_h \nabla v_h) - F(x_h + r_h y, u(x_h + r_h y), \varsigma_h \nabla v_h)\big]\, dy$$
$$\leqslant \varsigma_h^2 \int_{B_1} |a_{ij}(x_h + r_h y, w_h) - a_{ij}(x_h + r_h y, u(x_h + r_h y)| \nabla_i v_h| |\nabla_j v_h|\, dy$$

$$+ \varsigma_h \int_{B_1} |a_i(x_h + r_h y, w_h) - a_i(x_h + r_h y, u(x_h + r_h y)|\nabla_i v_h|\, dy + c(n, L)$$
$$= \varsigma_h^2 \varepsilon_h.$$

Combining the previous inequalities, we get

$$\int_{B_1} \eta F(x_h + r_h y, w_h, \varsigma_h \nabla v_h)\, dy \leqslant \int_{B_1} \eta F(x_h + r_h y, w_h, \varsigma_h \nabla v)\, dy + \varsigma_h^2 \varepsilon_h.$$

Dividing by ς_h^2, the linear terms in F tend to 0, thus getting

$$\int_{B_1} \eta a_{ij}(x_h + r_h y, w_h) \nabla_i v_h \nabla_j v_h\, dy \leqslant \int_{B_1} \eta a_{ij}(x_h + r_h y, w_h) \nabla_i v \nabla_j v\, dy + \varepsilon_h.$$

Since $B_{r_h}(x_h) \subset U \subset\subset \Omega$ for all $h \in \mathbb{N}$, we may assume that $x_h \to \overline{x}$, as $h \to +\infty$. Letting $\eta \downarrow 1$ in the previous inequality, passing to the lower limit, as $h \to +\infty$, by lower semicontinuity, we finally get

$$\lim_{h \to +\infty} \int_{B_1} a_{ij}(\overline{x}, u(\overline{x})) \nabla_i v_h \nabla_j v_h\, dy = \int_{B_1} a_{ij}(\overline{x}, u(\overline{x})) \nabla_i v \nabla_j v\, dy.$$

Since the matrix $a_{ij}(\overline{x}, u(\overline{x}))$ is elliptic and bounded, it induces a norm which is equivalent to the euclidean norm. Thus we get

$$\lim_{h \to +\infty} \fint_{B_\tau} |\nabla v_h|^2\, dy = \fint_{B_\tau} |\nabla v|^2\, dy \leqslant \frac{1}{\tau^n} \fint_{B_1} |\nabla v|^2\, dy \leqslant \frac{1}{\tau^n},$$

which contradicts (2.41), provided we choose $M > \tau^{\delta - n}$.

Step 4: We conclude that there exists $\tau \in \left(0, \frac{1}{2}\right)$ and $\delta \in (0, 1)$ such that, setting $M = 1$, there exists $h_0 \in \mathbb{N}$ such that, for any $B_r(x_0) \subset \Omega$, we have

$$\int_{B_r(x_0)} |\nabla u|^2 \leqslant h_0 r^{n-1} \quad \text{or} \quad \int_{B_{\tau r}(x_0)} |\nabla u|^2\, dx \leqslant \tau^{n-\delta} \int_{B_r(x_0)} |\nabla u|^2\, dx.$$

Hence,

$$\int_{B_{\tau r}(x_0)} |\nabla u|^2\, dx \leqslant \tau^{n-\delta} \int_{B_r(x_0)} |\nabla u|^2\, dx + h_0 r^{n-1},$$

and, using Lemma 2.1.4, we obtain that

$$\int_{B_\rho(x_0)} |\nabla u|^2\, dx \leqslant c \left\{ \left(\frac{\rho}{r} \right)^{n-1} \int_{B_r(x_0)} |\nabla u|^2\, dx + h_0 \rho^{n-1} \right\}, \quad \forall\, 0 < \rho < r \leqslant R,$$

and so we conclude

$$\int_{B_\rho(x_0)} |\nabla u|^2\, dx \leqslant c \rho^{n-1}.$$

□

As a consequence of the previous theorem, thanks to Lemma 2.1.1, we can infer that $u \in C^{0,\frac{1}{2}}$. We deduce the following remark.

Remark 2.4.2. *Let (E, u) be a Λ-minimizer of the functional $\mathcal{F}$ defined in (2.1). For every open set $U \subset\subset \Omega$ there exists a constant $C = C\big(n, \Lambda, U, \|\nabla u\|_{L^2(\Omega)}\big) > 0$ such that*

$$\sup_{x,y \in U} \frac{|u(x) - u(y)|}{|x - y|^{\frac{1}{2}}} \leqslant C \|\nabla u\|_{L^2(\Omega)} . \tag{2.43}$$

Notation 2.4.3. *In the sequel $E \subset \Omega$ will denote any given subset of Ω with finite perimeter. We denote by u_E, or simply by u if no confusion arises, any local minimizers of the functional $\mathcal{F}_E(v; \Omega)$.*

- *If $x \in \mathbb{R}^n$ we write $x = (x', x_n)$, where $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$. Accordingly, we denote $\nabla' = (\partial_{x_1}, \ldots, \partial_{x_{n-1}})$ the gradient with respect to the first $n-1$ components.*
- *We will denote $H = \{x \in \mathbb{R} : x_n > 0\}$.*

In what follows, we will use the following lemma, whose proof can be found in [7, Theorem 7.51].

Lemma 2.4.4. *Let $u \in L^p(B_{2R}(x_0))$ for some $p \in [1, +\infty)$ and let us assume that, for some $\alpha \in (0, 1]$ and $\gamma > 0$,*

$$\fint_{B_\rho(x)} |u(y) - u_{x,\rho}|^p \, dy \leqslant \gamma^p \left(\frac{\rho}{R}\right)^{p\alpha},$$

for any $B_\rho(x)$ with $\rho \leqslant R$ and $x \in B_R(x_0)$. Then (a representative of) u is Hölder continuous in $B_R(x_0)$ with Hölder exponent α and

$$|u(x) - u(y)| \leqslant c\gamma \left(\frac{|x - y|}{R}\right)^{\alpha}, \quad \forall x, y \in B_R(x_0),$$

$$\max_{B_R(x_0)} |u| \leqslant c\gamma + |u_{x_0,R}|,$$

for some positive constant $c = c(n, \alpha)$.

In order to prove the main lemma of this section we introduce the following preliminary result. For reader's convenience we give here a sketch of the proof, which can be found in [49]. Actually we state here a weaker version that is suitable for our aim.

Lemma 2.4.5. *Let $v \in H^1(B_1)$ be a solution of*

$$-\operatorname{div}(A\nabla u) = \operatorname{div} G, \qquad \text{in } \mathcal{D}'(B_1),$$

where

$$G^+ := \mathbb{1}_H G \in C^{0,\alpha}(H \cap B_1), \qquad G^- := \mathbb{1}_{H^c} G \in C^{0,\alpha}(H^c \cap B_1),$$

for some $\alpha > 0$ *and* A *is a uniformly elliptic matrix satisfying*

$$\nu|z|^2 \leqslant A_{ij}(x)z_i z_j \leqslant N|z|^2$$

and

$$A^+ := \mathbb{1}_H A \in C^{0,\alpha}(\overline{H} \cap B_1), \qquad A^- := \mathbb{1}_{H^c} A \in C^{0,\alpha}(\overline{H}^c \cap B_1),$$

for some $\nu, N > 0$*. Let us denote*

$$C_A = \max\{\|A^+\|_{C^{0,\alpha}}, \|A^-\|_{C^{0,\alpha}}\}, \qquad C_G = \max\{\|G^+\|_{C^{0,\alpha}}, \|G^-\|_{C^{0,\alpha}}\}.$$

Then $\nabla v \in L^{2,n}_{loc}(B_1)$ *(see* (2.9)*). Moreover, there exist two positive constants* $C = C(n, \nu, N, C_A, C_G)$ *and* $r_0 = r_0(n, \nu, N, \|G\|_{L^\infty}, C_A, C_G)$ *such that, for any* $r < r_0$ *with* $B_r(x_0) \subset B_1$*,*

$$\int_{B_\rho(x_0)} |\nabla v|^2\,dx \leqslant C\Big(\frac{\rho}{r}\Big)^n \int_{B_r(x_0)} |\nabla v|^2\,dx + C\rho^n, \quad \forall\, \rho < \frac{r}{4}.$$

Proof. Fix $x_0 \in B_1$ and let r be such that $B_r(x_0) \subset B_1$. Let us denote by a^+ and a^- the averages of A in $H \cap B_r(x_0)$ and $H^c \cap B_r(x_0)$ respectively. In an analogous way we define g^+ and g^- the averages of G in $H \cap B_r(x_0)$ and $H^c \cap B_r(x_0)$. For $x \in B_r(x_0)$ we define

$$\overline{A} := a^+\mathbb{1}_H + a^-\mathbb{1}_{H^c}, \qquad \overline{G} := g^+\mathbb{1}_H + g^-\mathbb{1}_{H^c}.$$

Notice that by assumption

$$|A(x) - \overline{A}(x)| \leqslant C_A r^\alpha \quad \text{and} \quad |G(x) - \overline{G}(x)| \leqslant C_G r^\alpha. \tag{2.44}$$

Let w be the solution of

$$\begin{cases} -\operatorname{div}(\overline{A}\nabla w) = \operatorname{div}\overline{G} & \text{in } B_r(x_0),\\ w = v & \text{on } \partial B_r(x_0). \end{cases}$$

The last equation can be rewritten as

$$\begin{cases} -\operatorname{div}(a^+\nabla w^+) = 0 & \text{in } B_r(x_0) \cap H,\\ -\operatorname{div}(a^-\nabla w^-) = 0 & \text{in } B_r(x_0) \cap H^c,\\ w^+ = w^- & \text{on } B_r(x_0) \cap \partial H,\\ a^+\nabla w^+ \cdot e_n - a^-\nabla w^- \cdot e_n = g^+ \cdot e_n - g^- \cdot e_n, & \text{on } B_r(x_0) \cap \partial H, \end{cases} \tag{2.45}$$

where $w^+ := w\mathbb{1}_{B_r(x_0)\cap H}$, $w^- := w\mathbb{1}_{B_r(x_0)\cap H^c}$. Set

$$\overline{D}_c w := \sum_{i=1}^n \overline{A}_{in}\nabla_i w + \overline{G}\cdot e_n,$$

where $\overline{A}_{in}$ is the (i,n)-th entry of the matrix $\overline{A}$. We notice that $\overline{D}_c w$ has no jumps on the boundary thanks to the transmission condition in (2.45). This allows us to prove that the distributional gradient of $\overline{D}_c w$ coincides with the point-wise one.

Step 1: *Tangential derivatives of* w. Let us denote with τ the general direction tangent to the hyperplane ∂H. Since $\overline{A}$ and $\overline{G}$ are both constant along the tangential directions, Theorem 1.2.10 gives that $\nabla_\tau w \in H^1_{loc}(B_r(x_0))$ and

$$\operatorname{div}(\overline{A}\nabla(\nabla_\tau w)) = 0 \quad \text{in } B_r(x_0).$$

Hence, Caccioppoli's inequality holds:

$$\int_{B_\rho(x)} |\nabla(\nabla_\tau w)|^2 \, dy \leqslant \frac{c(n,\nu,N)}{\rho^2} \int_{B_{2\rho}(x)} |\nabla_\tau w - (\nabla_\tau w)_{x,2\rho}|^2 \, dy, \tag{2.46}$$

for all balls $B_{2\rho}(x) \subset B_r(x_0)$ (see Theorem 1.2.2) and, by De Giorgi's regularity theorem (see Theorem 1.2.6), $\nabla_\tau w$ is Hölder continuous and there exists $\gamma = \gamma(n,\nu,N) > 0$ such that if $B_s(x) \subset B_r(x_0)$

$$\int_{B_\rho(x)} |\nabla_\tau w - (\nabla_\tau w)_{x,\rho}|^2 \, dy \leqslant c(n,\nu,N) \left(\frac{\rho}{s}\right)^{n+2\gamma} \int_{B_s(x)} |\nabla_\tau w - (\nabla_\tau w)_{x,s}|^2 \, dy, \tag{2.47}$$

for any $\rho \in \left(0, \frac{s}{2}\right)$ and

$$\max_{B_{\frac{\rho}{2}}(x)} |\nabla_\tau w|^2 \leqslant \frac{c(n,\nu,N)}{\rho^n} \int_{B_\rho(x)} |\nabla_\tau w|^2 \, dy. \tag{2.48}$$

Step 2: *Regularity of* $\overline{D}_c w$. First of all observe that $\nabla_\tau(\overline{D}_c w) = \overline{D}_c(\nabla_\tau w) - \overline{G} \cdot e_n$. This implies by step 1 that the tangential derivatives of $\overline{D}_c w$ belong to $L^2_{loc}(B_r(x_0))$. Furthermore we can estimate directly by definition of $\overline{D}_c w$:

$$|\nabla_n(\overline{D}_c w)| \leqslant c(n,N)|\nabla\nabla_\tau w|,$$

which implies again by step 1

$$|\nabla \overline{D}_c w| \leqslant c(n,N)|\nabla\nabla_\tau w|.$$

We can conclude that $\overline{D}_c w \in H^1_{loc}(B_r(x_0))$. Using Poincaré's inequality and (2.46), we have

$$\begin{aligned}
\int_{B_\rho(x)} |\overline{D}_c w - (\overline{D}_c w)_{x,\rho}|^2 \, dy &\leqslant c(n)\rho^2 \int_{B_\rho(x)} |\nabla(\overline{D}_c w)|^2 \, dy \\
&\leqslant c(n,N)\rho^2 \int_{B_\rho(x)} |\nabla(\nabla_\tau w)|^2 \, dy \\
&\leqslant c(n,\nu,N) \int_{B_{2\rho}(x)} |\nabla_\tau w - (\nabla_\tau w)_{x,2\rho}|^2 \, dy,
\end{aligned}$$

for any $B_{2\rho}(x) \subset B_r(x_0)$. By (2.47) we infer

$$\int_{B_\rho(x)} |\overline{D}_c w - (\overline{D}_c w)_{x,\rho}|^2 \, dy$$

$$
\begin{aligned}
&\leqslant c(n,\nu,N)\left(\frac{\rho}{r}\right)^{n+2\gamma}\int_{B_{\frac{r}{2}}(x)}|\nabla_\tau w-(\nabla_\tau w)_{x,\frac{r}{2}}|^2\,dy\\
&\leqslant c(n,\nu,N)\left(\frac{\rho}{r}\right)^{n+2\gamma}\int_{B_r(x_0)}|\nabla_\tau w|^2\,dy,
\end{aligned}
$$

for any $x\in B_{\frac{r}{4}}(x_0)$ and $\rho\leqslant\frac{r}{4}$. Hence by Lemma 2.4.4, $\overline{D}_c w$ is Hölder continuous and, by (2.48), we get

$$
\begin{aligned}
\max_{B_{\frac{r}{4}}(x_0)}|\overline{D}_c w|^2 &\leqslant c(n,\nu,N)\int_{B_r(x_0)}|\nabla_\tau w|^2\,dy+\left|\fint_{B_{\frac{r}{4}}(x_0)}\overline{D}_c w(y)\,dy\right|^2\\
&\leqslant \frac{c(n,\nu,N)}{r^n}\int_{B_r(x_0)}|\nabla w|^2\,dy+2\,\|G\|^2_{L^\infty}\,. \qquad (2.49)
\end{aligned}
$$

Step 3: *Comparison between v and w.* Subtracting the equation for w from the equation for v we get

$$
\begin{aligned}
&\int_{B_r(x_0)}\overline{A}_{ij}(x)\big(\nabla_i v-\nabla_i w\big)\nabla_j\varphi\,dx\\
&=\int_{B_r(x_0)}\big(\overline{A}_{ij}(x)-A_{ij}(x)\big)\nabla_i v\nabla_j\varphi\,dx+\int_{B_r(x_0)}\big(\overline{G}_i-G_i\big)\nabla_i\varphi\,dx
\end{aligned}
$$

for any $\varphi\in W^{1,2}_0(B_r(x_0))$. Choosing $\varphi=v-w$ in the previous equation and using assumption (2.44) we have

$$
\nu\int_{B_r(x_0)}|\nabla v-\nabla w|^2\,dx\leqslant C_A r^\alpha\int_{B_r(x_0)}|\nabla v|^2\,dy+C_G r^{n+\alpha}.
$$

Finally we can estimate

$$
\begin{aligned}
&\int_{B_\rho(x_0)}|\nabla v|^2\,dy\leqslant 2\int_{B_\rho(x_0)}|\nabla w|^2\,dy+2\int_{B_\rho(x_0)}|\nabla v-\nabla w|^2\,dy\\
&\leqslant 2\omega_n\rho^n\sup_{B_{\frac{r}{4}}}|\nabla w|^2+2\int_{B_\rho(x_0)}|\nabla v-\nabla w|^2\,dy,
\end{aligned}
$$

for any $\rho\leqslant\frac{r}{4}$, and observing that

$$
\begin{aligned}
\sup_{B_{\frac{r}{4}}(x_0)}|\nabla w|^2&=\sup_{B_{\frac{r}{4}}(x_0)}|\nabla_\tau w|^2+\sup_{B_{\frac{r}{4}}(x_0)}|\nabla_n w|^2\\
&\leqslant c(n,\nu,N)\sup_{B_{\frac{r}{4}}(x_0)}|\nabla_\tau w|^2+c(\nu)\sup_{B_{\frac{r}{4}}(x_0)}|\overline{D}_c w|^2+c(\nu,\|G\|_\infty),
\end{aligned}
$$

by (2.48), (2.49), the minimality of w and Young's inequality we gain

$$
\begin{aligned}
&\int_{B_\rho(x_0)}|\nabla v|^2\,dy\\
&\leqslant c(n,\nu,N)\left(\frac{\rho}{r}\right)^n\int_{B_r(x_0)}|\nabla w|^2\,dy
\end{aligned}
$$

$$
\begin{aligned}
&+ c(n,\nu,\|G\|_\infty, C_A, C_G)\left[r^\alpha \int_{B_r(x_0)} |\nabla v|^2\, dy + r^n\right]\\
&\leqslant C(n,\nu,N,\|G\|_\infty, C_A, C_G)\left\{\left[\left(\frac{\rho}{r}\right)^n + r^\alpha\right]\int_{B_r(x_0)} |\nabla v|^2\, dy + r^n\right\},
\end{aligned}
$$

which leads to our aim if we apply Lemma 2.1.4. □

The next lemma is inspired by [35, Proposition 2.4] and is the main result of this section.

Lemma 2.4.6. *Let (E,u) be a Λ-minimizer of the functional $\mathcal{F}$ defined in (2.1). There exists $\tau_0 \in (0,1)$ such that the following statement is true: for all $\tau \in (0,\tau_0)$ there exists $\varepsilon_0 = \varepsilon_0(\tau) > 0$ such that if $B_r(x_0) \subset\subset \Omega$ with $r^{\frac{1}{2n}} < \tau$ and one of the following conditions holds:*

(i) $|E \cap B_r(x_0)| < \varepsilon_0 |B_r(x_0)|$;

(ii) $|B_r(x_0)\backslash E| < \varepsilon_0 |B_r(x_0)|$;

(iii) *there exists a halfspace H such that* $\frac{|(E\Delta H)\cap B_r(x_0)|}{|B_r(x_0)|} < \varepsilon_0$,

then

$$
\int_{B_{\tau r}(x_0)} |\nabla u|^2\, dx \leqslant C_4\left[\tau^n \int_{B_r(x_0)} |\nabla u|^2\, dx + r^n\right],
$$

for some positive constant $C_4 = C_4\big(n,\nu,N,L,L_D,\|\nabla u\|_{L^2(\Omega)}\big)$.

Proof. Let us fix $B_r(x_0) \subset\subset \Omega$ and $0 < \tau < 1$. Without loss of generality, we may assume that $\tau < 1/4$ and $x_0 = 0$. We start proving the assertion in the case (i), being the proof in the case (ii) similar. Let us define

$$
A^0_{ij} := a_{ij}(x_0, u_{r/2}(x_0)), \quad B^0_i := a_i(x_0, u_{r/2}(x_0)), \quad f^0 := a(x_0, u_{r/2}(x_0)),
$$

and

$$
F_0(\xi) := A^0\xi\cdot\xi + B^0\cdot\xi + f^0, \quad \forall \xi \in \mathbb{R}^n.
$$

Let us denote by v the solution of the following problem:

$$
\min_{w \in u + H^1_0(B_{r/2})} \mathcal{F}_0(w; B_{r/2}),
$$

where

$$
\mathcal{F}_0(w; B_{r/2}) := \int_{B_{r/2}} F_0(\nabla w)\, dx.
$$

Now we use the following identity

$$
A^0\xi\cdot\xi - A^0\eta\cdot\eta = [A^0(\xi-\eta)]\cdot(\xi-\eta) + 2A^0\eta\cdot(\xi-\eta), \quad \forall \xi,\eta \in \mathbb{R}^n,
$$

in order to deduce that

$$
\mathcal{F}_0(u) - \mathcal{F}_0(v)
$$

$$
\begin{aligned}
&= \int_{B_{r/2}} \big[A^0\nabla u \cdot \nabla u - A^0\nabla v \cdot \nabla v\big]\,dx + \int_{B_{r/2}} B^0 \cdot \nabla u - \nabla v\,dx \\
&= \int_{B_{r/2}} [A^0(\nabla u - \nabla v)] \cdot (\nabla u - \nabla v)\,dx \\
&+ 2\int_{B_{r/2}} A^0\nabla v \cdot (\nabla u - \nabla v)\,dx + \int_{B_{r/2}} B^0 \cdot (\nabla u - \nabla v)\,dx. \qquad (2.50)
\end{aligned}
$$

By the Euler-Lagrange equation for v we deduce that the sum of the last two integrals in the previous identity is zero, being also $u = v$ on $\partial B_{r/2}$. Therefore, using the ellipticity assumption of A^0, we finally achieve that

$$
\nu \int_{B_{r/2}} |\nabla u - \nabla v|^2\,dx \leqslant \mathcal{F}_0(u) - \mathcal{F}_0(v). \qquad (2.51)
$$

Now we prove that u is an ω-minimizer of $\mathcal{F}_0$. We start writing

$$
\begin{aligned}
\mathcal{F}_0(u) &= \mathcal{F}(E,u) + [\mathcal{F}_0(u) - \mathcal{F}(E,u)] \\
&\leqslant \mathcal{F}(E,v) + [\mathcal{F}_0(u) - \mathcal{F}(E,u)] \\
&= \mathcal{F}_0(v) + [\mathcal{F}_0(u) - \mathcal{F}(E,u)] + [\mathcal{F}(E,v) - \mathcal{F}_0(v)]. \qquad (2.52)
\end{aligned}
$$

Estimate of $\mathcal{F}_0(u) - \mathcal{F}(E,u)$. We use (2.6) and (2.43) to infer

$$
\begin{aligned}
\mathcal{F}_0(u) - \mathcal{F}(E,u) &= \int_{B_{r/2}} \big(a_{ij}(x_0, u_{r/2}(x_0)) - a_{ij}(x, u(x))\big)\nabla_i u \nabla_j u\,dx \\
&+ \int_{B_{r/2}} \big(a_i(x_0, u_{r/2}(x_0)) - a_i(x, u(x))\big)\nabla_i u\,dx \\
&+ \int_{B_{r/2}} \big(a(x_0, u_{r/2}(x_0)) - a(x, u(x))\big)\,dx - \int_{B_{r/2}\cap E} G(x, u, \nabla u)\,dx \\
&\leqslant c\big(n, L_D\, \|\nabla u\|_{L^2(\Omega)}\big)\bigg(r^{\frac{1}{2}} \int_{B_{r/2}} |\nabla u|^2\,dx + r^{n+\frac{1}{2}}\bigg) \qquad (2.53) \\
&+ C(N, L)\bigg(\int_{B_{r/2}\cap E} |\nabla u|^2\,dx + r^n\bigg),
\end{aligned}
$$

where we denoted with L_D the greatest Lipschitz constant of the data $a_{ij}, b_{ij}, a_i, b_i, a, b$, defined in (2.4). Now we use Hölder's inequality and Lemma 2.3.3 to estimate

$$
\begin{aligned}
\int_{B_{r/2}\cap E} |\nabla u|^2\,dx &\leqslant |E \cap B_r|^{1-1/s}|B_r|^{1/s}\bigg(\fint_{B_{r/2}} |\nabla u|^{2s}\bigg)^{1/s} \\
&\leqslant C_1^{1/s}\bigg(\frac{|E \cap B_r|}{|B_r|}\bigg)^{1-1/s} \int_{B_r} \big(1 + |\nabla u|^2\big)\,dx. \qquad (2.54)
\end{aligned}
$$

Merging the last estimate in (2.53) we deduce

$$
\mathcal{F}_0(u) - \mathcal{F}(E,u) \leqslant \Big(c\big(n, L_D, \|\nabla u\|_{L^2(\Omega)}\big) + C(N, L)C_1^{1/s}\Big)
$$

$$\times \left(r^{\frac{1}{2}} + \varepsilon_0^{1-1/s}\right) \int_{B_r} |\nabla u|^2\,dx$$
$$+ \left(C_1^{1/s} + 2L + c\big(n, L_D, \|\nabla u\|_{L^2(\Omega)}\big)\right) r^n. \tag{2.55}$$

Estimate of $\mathcal{F}(E,v) - \mathcal{F}_0(v)$. We have

$$\begin{aligned}\mathcal{F}(E,v) - \mathcal{F}_0(v) = &\int_{B_{r/2}} \big(a_{ij}(x,v(x)) - a_{ij}(x_0, u_{r/2}(x_0))\big)\nabla_i v \nabla_j v\,dx \\ &+ \int_{B_{r/2}} \big(a_i(x,v(x)) - a_i(x_0,u_{r/2}(x_0))\big)\nabla_i v\,dx \\ &+ \int_{B_{r/2}} \big(a(x,v(x)) - a(x_0,u_{r/2}(x_0))\big)\,dx \\ &+ \int_{B_{r/2}\cap E} G(x,v,\nabla v)\,dx.\end{aligned} \tag{2.56}$$

If we choose now $z \in \partial B_{r/2}$, recalling that $u(z) = v(z)$ we deduce

$$\begin{aligned}&\big|a_{ij}(x,v(x)) - a_{ij}(x_0,u_{r/2}(x_0))\big| \\ &= \big|a_{ij}(x,v(x)) - a_{ij}(x,v(z)) + a_{ij}(x,u(z)) - a_{ij}(x_0,u_{r/2}(x_0))\big| \\ &\leqslant L_D\big(|v(x)-v(z)| + Cr^{\frac{1}{2}}\|\nabla u\|_{L^2(\Omega)} + r\big) \\ &\leqslant L_D\big(\mathrm{osc}(u;\partial B_{r/2}) + C(n,\nu,N,L)r + Cr^{\frac{1}{2}}\|\nabla u\|_{L^2(\Omega)} + r\big) \\ &\leqslant C\big(n,\nu,N,L,L_D,\|\nabla u\|_{L^2(\Omega)}\big)r^{\frac{1}{2}},\end{aligned}$$

where we used the fact that $\mathrm{osc}(v;B_{r/2}) \leqslant \mathrm{osc}(u;\partial B_{r/2}) + C(n,\nu,N,L)r$ (see Proposition 1.2.7). Analogously we can estimate the other differences in (2.56), deducing

$$\begin{aligned}\mathcal{F}(E,v) - \mathcal{F}_0(v) \leqslant C\big(n,\nu,N,L,L_D,\|\nabla u\|_{L^2(\Omega)}\big)r^{\frac{1}{2}}\left(\int_{B_{r/2}}|\nabla v|^2\,dx + r^n\right) \\ + C(N,L)\left(\int_{B_{r/2}\cap E}|\nabla v|^2\,dx + r^n\right),\end{aligned}$$

Reasoning in a similar way as in (2.54), we can apply the higher integrability for v given by Lemma 2.3.6 and infer

$$\int_{B_{r/2}\cap E}|\nabla v|^2\,dx \leqslant C(n,\nu,N,L)\varepsilon_0^{1-1/s}\left(\int_{B_r}|\nabla u|^2\,dx + r^n\right).$$

Therefore we obtain

$$\begin{aligned}&\mathcal{F}(E,v) - \mathcal{F}_0(v) \\ &\leqslant C\big(n,\nu,N,L,L_D,\|\nabla u\|_{L^2(\Omega)}\big)\left[\left(r^{\frac{1}{2}} + \varepsilon_0^{1-1/s}\right)\int_{B_r}|\nabla u|^2\,dx + r^n\right].\end{aligned} \tag{2.57}$$

Finally, collecting (2.51), (2.52), (2.55) and (2.57), if we choose ε_0 such that $\varepsilon_0^{1-\frac{1}{s}} = \tau^n$, recalling that $r^{\frac{1}{2n}} < \tau$, we conclude that

$$\int_{B_{r/2}} |\nabla u - \nabla v|^2 \, dx \leqslant C\left[\tau^n \int_{B_r} |\nabla u|^2 \, dx + r^n\right], \tag{2.58}$$

for some constant $C = C\big(n, \nu, N, L, L_D, \|\nabla u\|_{L^2(\Omega)}\big)$. On the other hand v is the solution of a uniformly elliptic equation with constant coefficients, so we have

$$\begin{aligned}\int_{B_{\tau r}} |\nabla v|^2 \, dx &\leqslant C(n,\nu,N)\tau^n \int_{B_{r/2}} |\nabla v|^2 \, dx \\ &\leqslant C(n,\nu,N,L)\left[\tau^n \int_{B_{r/2}} |\nabla u|^2 \, dx + r^n\right],\end{aligned} \tag{2.59}$$

(see Theorem 1.2.11). Hence we may estimate, using (2.58) and (2.59),

$$\begin{aligned}\int_{B_{\tau r}} |\nabla u|^2 \, dx &\leqslant 2\int_{B_{\tau r}} |\nabla v - \nabla u|^2 \, dx + 2\int_{B_{\tau r}} |\nabla v|^2 \, dx \\ &\leqslant C\left[\tau^n \int_{B_r} |\nabla u|^2 \, dx + r^n\right],\end{aligned}$$

for some constant $C = C\big(n, \nu, N, L, L_D, \|\nabla u\|_{L^2(\Omega)}\big)$.

We are left with the case (iii). Let H be the half-space from our assumption and let us denote accordingly

$$\begin{aligned}&A^0_{ij}(x) := a_{ij}(x,u(x)) + \mathbb{1}_H b_{ij}(x,u(x)),\\ &B^0_{ij}(x) := a_i(x,u(x)) + \mathbb{1}_H b_i(x,u(x)),\\ &f^0(x) := a(x,u(x)) + \mathbb{1}_H b(x,u(x)),\\ &F_0(x,z) := A^0(x)z\cdot z + B^0(x)\cdot z + f^0(x).\end{aligned}$$

Let us denote by v_H the solution of the following problem

$$\min_{w\in u+H^1_0(B_{r/2})} \mathcal{F}_0(w; B_{r/2}),$$

where

$$\mathcal{F}_0(w; B_{r/2}) := \int_{B_{r/2}} F_0(x, \nabla w)\, dx.$$

Let us point out that v_H solves the Euler-Lagrange equation

$$-2\mathrm{div}(A^0\nabla v_H) = \mathrm{div}\, B^0 \quad \text{in } \mathcal{D}'(B_{r/2}). \tag{2.60}$$

Therefore we are in a position to apply Lemma 2.4.5 to the function v_H. Indeed, from the Hölder continuity of u (see Remark 2.4.2) we deduce that the restrictions of A^0 and B^0 onto $H \cap B_r$ and $B_r \backslash H$ respectively are Hölder continuous. We can conclude using also (2.43) that there exist two constants

$C = C\big(n, \nu, N, L, L_D, \|\nabla u\|_{L^2(\Omega)}\big)$ and $\tau_0 = \tau_0\big(n, \nu, N, L, L_D, \|\nabla u\|_{L^2(\Omega)}\big)$ such that for $\tau < \tau_0$

$$\int_{B_{\tau r}} |\nabla v_H|^2\, dx \leqslant C\left[\tau^n \int_{B_{r/2}} |\nabla v_H|^2\, dx + r^n\right]. \tag{2.61}$$

In addition, using the uniform ellipticity condition of A^0 we can argue as in (2.50) to deduce, using also the fact that v_H satisfies (2.60),

$$\nu \int_{B_{r/2}} |\nabla u - \nabla v_H|^2\, dx \leqslant \mathcal{F}_0(u) - \mathcal{F}_0(v_H). \tag{2.62}$$

One more time we can prove that u is an ω-minimizer of $\mathcal{F}_0$. We start as above writing

$$\begin{aligned}
\mathcal{F}_0(u) &= \mathcal{F}(E, u) + [\mathcal{F}_0(u) - \mathcal{F}(E, u)] \\
&\leqslant \mathcal{F}(E, v_H) + [\mathcal{F}_0(u) - \mathcal{F}(E, u)] \\
&= \mathcal{F}_0(v_H) + [\mathcal{F}_0(u) - \mathcal{F}(E, u)] + [\mathcal{F}(E, v_H) - \mathcal{F}_0(v_H)].
\end{aligned}$$

We can estimate the differences $\mathcal{F}_0(u)-\mathcal{F}(E,u)$ and $\mathcal{F}(E,v_H)-\mathcal{F}_0(v_H)$ exactly as before using this time the higher integrability given in Lemma 2.3.6. We conclude that

$$\int_{B_{r/2}} |\nabla u - \nabla v_H|^2\, dx \leqslant C\left[\tau^n \int_{B_r} |\nabla u|^2\, dx + r^n\right],$$

for some constant $C = C\big(n, \nu, N, L, L_D, \|\nabla u\|_{L^2(\Omega)}\big)$. From the last estimate we can conclude the proof as before using (2.61) and (2.62). □

2.5 Energy density estimates

This section is devoted to proving a lower bound esimate for the functional $\mathcal{F}(E, u; B_r(x_0))$. Cases (i) and (ii) of Lemma 2.4.6 are the main tool to achieve such a result.

Lemma 2.5.1 (Scaling of Λ-minimizers)**.** *Let $B_r(x_0) \subset \Omega$ and let (E, u) be a Λ-minimizer of $\mathcal{F}$ in $B_r(x_0)$. Then $(E_{x_0,r}, u_{x_0,r})$ is a Λr-minimizer of $\mathcal{F}_r$ in B_1, where*

$$E_{x_0,r} := \frac{E - x_0}{r}, \qquad u_{x_0,r}(y) := r^{-\frac{1}{2}} u(x_0 + ry), \quad \textit{for } y \in B_1,$$

$$\begin{aligned}
\mathcal{F}_r(E_{x_0,r}, u_{x_0,r}; B_1) &:= r \int_{B_1} \big[F(x_0 + ry, r^{\frac{1}{2}} u_{x_0,r}(y), r^{-\frac{1}{2}} \nabla u_{x_0,r}(y)) \\
&+ \mathbb{1}_{E_{x_0,r}}(y) G(x_0 + ry, r^{\frac{1}{2}} u_{x_0,r}(y), r^{-\frac{1}{2}} \nabla u_{x_0,r}(y))\big]\, dy + P(E_{x_0,r}; B_1).
\end{aligned}$$

Proof. Since $\nabla u_{x_0,r}(y) = r^{\frac{1}{2}}\nabla u(x_0+ry)$, for any $y \in B_1$, we rescale

$$\begin{aligned}&\mathcal{F}(E,u;B_r(x_0)) = r^n \int_{B_1} \big[F(x_0+ry, u(x_0+ry), \nabla u(x_0+ry))\\ &+ \mathbb{1}_E(x_0+ry)G(x_0+ry, u(x_0+ry), \nabla u(x_0+ry))\big]\,dy + r^{n-1}P(E_{x_0,r};B_1)\\ &= r^{n-1}\mathcal{F}_r(E_{x_0,r}, u_{x_0,r}; B_1).\end{aligned}$$

Thus, if $\tilde{F} \subset \mathbb{R}^n$ is a set of finite perimeter with $\tilde{F}\Delta E_{x_0,r} \subset\subset B_1$ and $\tilde{v} \in H^1(B_1)$ is such that $\tilde{v} - u_{x_0,r} \in H_0^1(B_1)$, then

$$\begin{aligned}\mathcal{F}_r(E_{x_0,r}, u_{x_0,r}; B_1) &= \frac{\mathcal{F}(E,u;B_r(x_0))}{r^{n-1}} \leqslant \frac{\mathcal{F}(F,v;B_r(x_0)) + \Lambda|F\Delta E|}{r^{n-1}}\\ &= \mathcal{F}_r(\tilde{F},\tilde{v};B_1) + \Lambda r|\tilde{F}\Delta E_{x_0,r}|,\end{aligned}$$

where $F := x_0 + r\tilde{F}$ and $v(x) = r^{\frac{1}{2}}\tilde{v}\big(\frac{x-x_0}{r}\big)$, for $x \in B_r(x_0)$. □

We shall prove that the energy $\mathcal{F}$ decays “fast” if the perimeter of E is “small”.

Lemma 2.5.2. *Let (E,u) be a Λ-minimizer of $\mathcal{F}$ in Ω. For every $\tau \in (0,1)$ there exists $\varepsilon_1 = \varepsilon_1(n,\tau) > 0$ such that, if $B_r(x_0) \subset \Omega$ and $P(E;B_r(x_0)) < \varepsilon_1 r^{n-1}$, then*

$$\mathcal{F}(E,u;B_{\tau r}(x_0)) \leqslant C_5\tau^n\big(\mathcal{F}(E,u;B_r(x_0)) + r^n\big),$$

for some constant $C_5 = C_5\big(n,\nu,N,L,L_D,\Lambda,\|\nabla u\|_{L^2(\Omega)}\big)$ independent of τ and r.

Proof. Let $\tau \in (0,1)$ and $B_r(x_0) \subset \Omega$. Without loss of generality we may assume that $\tau < \frac{1}{2}$. We may also assume that $x_0 = 0$, $r = 1$ by scaling $E_{x_0,r} = \frac{E-x_0}{r}$, $u_{x_0,r}(y) = r^{-\frac{1}{2}}u(x_0+ry)$, for $y \in B_1$, and replacing Λ with Λr. Thus, we have that $(E_{x_0,r}, u_{x_0,r})$ is a Λr-minimizer of $\mathcal{F}_r$ in $\frac{\Omega - x_0}{r}$. For simplicity of notation we may still denote $E_{x_0,r}$ by E, $u_{x_0,r}$ by u and then we have to prove that there exists $\varepsilon_1 = \varepsilon_1(\tau)$ such that, if $P(E;B_1) < \varepsilon_1$, then

$$\mathcal{F}_r(E,u;B_\tau) \leqslant C_5\tau^n\Big(\mathcal{F}_r(E,u;B_1) + r\Big).$$

Note that, since $P(E;B_1) < \varepsilon_1$, by the relative isoperimetric inequality,

$$\min\{|B_1 \cap E|, |B_1\backslash, \tilde{E}|\} \leqslant c(n)P(E;B_1)^{\frac{n}{n-1}}.$$

Thus Lemma 2.4.6 holds. Choosing the set of density one points of E as a representative of E, we get by Fubini's theorem that

$$|B_1\backslash E| \geqslant \int_\tau^{2\tau} \mathcal{H}^{n-1}(\partial B_\rho \backslash E)\,d\rho.$$

Combining the previous inequalities, we can choose $\rho \in (\tau, 2\tau)$ such that

$$\mathcal{H}^{n-1}(\partial B_\rho \backslash E) \leqslant \frac{|B_1 \backslash E|}{\tau} \leqslant \frac{c(n)}{\tau} P(E; B_1)^{\frac{n}{n-1}} \leqslant \frac{c(n)\varepsilon_1^{\frac{1}{n-1}}}{\tau} P(E; B_1). \tag{2.63}$$

Now we set $F = E \cup B_\rho$. Using (2.63), we observe that

$$\begin{aligned} P(F; B_1) &\leqslant P(E; B_1 \backslash \overline{B}_\rho) + \mathcal{H}^{n-1}(\partial B_\rho \backslash E) \\ &\leqslant P(E; B_1 \backslash \overline{B}_\rho) + \frac{c(n)\varepsilon_1^{\frac{1}{n-1}}}{\tau} P(E; B_1). \end{aligned}$$

If we choose (F, u) to test the Λr-minimality of (E, u) we get

$$\begin{aligned} & r \int_{B_1} \left[F(x_0 + ry, r^{\frac{1}{2}} u(y), r^{-\frac{1}{2}} \nabla u(y)) + \mathbb{1}_E G(x_0 + ry, r^{\frac{1}{2}} u(y), r^{-\frac{1}{2}} \nabla u(y)) \right] dy \\ & + P(E; B_1) = \mathcal{F}_r(E, u; B_1) \leqslant \mathcal{F}_r(F, u; B_1) + \Lambda r |E \Delta F| \\ & \leqslant r \int_{B_1} \left[F(x_0 + ry, r^{\frac{1}{2}} u(y), r^{-\frac{1}{2}} \nabla u(y)) + \mathbb{1}_F G(x_0 + ry, r^{\frac{1}{2}} u(y), r^{-\frac{1}{2}} \nabla u(y)) \right] dy \\ & + P(F; B_1) + \Lambda r |F \backslash E| \\ & \leqslant r \int_{B_1} \left[F(x_0 + ry, r^{\frac{1}{2}} u(y), r^{-\frac{1}{2}} \nabla u(y)) + \mathbb{1}_F G(x_0 + ry, r^{\frac{1}{2}} u(y), r^{-\frac{1}{2}} \nabla u(y)) \right] dy \\ & + P(E; B_1 \backslash \overline{B}_\rho) + \frac{c(n)\varepsilon_1^{\frac{1}{n-1}}}{\tau} P(E; B_1) + \Lambda r |B_\rho|. \end{aligned}$$

Then, getting rid of the common terms we obtain

$$\begin{aligned} P(E; B_\rho) \leqslant r & \int_{B_1 \cap (B_\rho \backslash E)} G(x_0 + ry, r^{\frac{1}{2}} u(y), r^{-\frac{1}{2}} \nabla u(y)) \, dy \\ & + \frac{c(n)\varepsilon_1^{\frac{1}{n-1}}}{\tau} P(E; B_1) + \Lambda r |B_\rho|. \end{aligned}$$

Now if we choose $\varepsilon_1 = \varepsilon_1(n, \tau) > 0$ such that $c(n)\varepsilon_1^{\frac{1}{n-1}} \leqslant \tau^{n+1}$ we infer

$$\begin{aligned} P(E; B_\rho) \leqslant r & \int_{B_\rho \cap (B_1 \backslash E)} G(x_0 + ry, r^{\frac{1}{2}} u(y), r^{-\frac{1}{2}} \nabla u(y)) \, dy \\ & + \tau^n P(E; B_1) + \Lambda r |B_\rho|. \end{aligned}$$

Then, we choose $\varepsilon_1 = \varepsilon_1(n, \tau) > 0$ satisfying $c(n)\varepsilon_1^{\frac{n}{n-1}} \leqslant \varepsilon_0(2\tau)|B_1|$ to obtain, using Lemma 2.4.6, growth conditions (2.5) and (2.6),

$$\begin{aligned} & \int_{B_1 \cap (B_\rho \backslash E)} G(x_0 + ry, r^{\frac{1}{2}} u(y), r^{-\frac{1}{2}} \nabla u(y)) \, dy \\ & \qquad \leqslant C(N, L) \int_{B_\rho} (|\nabla u|^2 + r) \, dy \\ & \qquad \leqslant C\big(n, \nu, N, L, L_D, \|\nabla u\|_{L^2(\Omega)}\big) \tau^n \int_{B_1} (|\nabla u|^2 + r) \, dy. \end{aligned}$$

Finally, we recall that $\rho \in (\tau, 2\tau)$ to get

$$\begin{aligned} P(E;B_\tau) &\leqslant C\big(n,\nu,N,L,L_D,\|\nabla u\|_{L^2(\Omega)}\big)\tau^n \int_{B_1} (|\nabla u|^2 + r)\,dy \\ &\quad + \tau^n P(E;B_1) + \Lambda r|B_\rho| \\ &\leqslant C\big(n,\nu,N,L,L_D,\|\nabla u\|_{L^2(\Omega)}\big)\tau^n \big[\mathcal{F}_r(E,u;B_1) + r + \Lambda r\big]. \end{aligned}$$

From this estimate, applying again Lemma 2.4.6, we deduce that

$$\begin{aligned} &r\int_{B_1} \big[F(x_0+ry, r^{\frac12}u(y), r^{-\frac12}\nabla u(y)) + \mathbb{1}_E(y)G(x_0+ry, r^{\frac12}u(y), r^{-\frac12}\nabla u(y))\big]\,dy \\ &\leqslant C\big(n,\nu,N,L,L_D,\|\nabla u\|_{L^2(\Omega)}\big)\tau^n \bigg[r\int_{B_1} \big[F(x_0+ry, r^{\frac12}u(y), r^{-\frac12}\nabla u(y)) \\ &+ \mathbb{1}_E(y)G(x_0+ry, r^{\frac12}u(y), r^{-\frac12}\nabla u(y))\big]\,dy\bigg], \end{aligned}$$

and thus

$$\mathcal{F}_r(E,u;B_\tau) \leqslant C\big(n,\nu,N,L,L_D,\|\nabla u\|_{L^2(\Omega)}\big)\tau^n \big(\mathcal{F}_r(E,u;B_1) + r\big).$$

□

Theorem 2.5.3 (Density lower bound)**.** *Let (E,u) be a Λ-minimizer of $\mathcal{F}$ in Ω and $U \subset\subset \Omega$ be an open set. Then there exists a constant $C_6 = C_6\big(n,\nu,N,L,L_D,\Lambda,\|\nabla u\|_{L^2(\Omega)},U\big) > 0$, such that, for every $x_0 \in \partial E$ and $B_r(x_0) \subset U$, it holds*

$$P(E;B_r(x_0)) \geqslant C_6 r^{n-1}.$$

Moreover, $\mathcal{H}^{n-1}((\partial E \backslash \partial^* E) \cap \Omega) = 0$.

Proof. We start assuming that $x_0 \in \partial^* E$. Without loss of generality we may also assume that $x_0 = 0$. Let

$$\tau \in \left(0, \frac14\right) \text{ such that } 2C_5\tau^{\frac12} < 1,$$

$$\sigma \in (0,1) \text{ such that } 2C_5C_3\sigma < \varepsilon_1(\tau),\ 2\omega_n\frac{L^2}{\nu}\sigma < \varepsilon_1(\tau),$$

$$0 < r_0 < \min\big\{1, C_3, \varepsilon_1(\tau)\big\},$$

where C_5 and ε_1 come from Lemma 2.5.2, C_3 comes from Theorem 2.4.1. We point out that $\tau, \sigma, r_0, \varepsilon_1(\sigma)$ depend on $n,\nu,N,L,L_\alpha,L_D,\|\nabla u\|_{L^2(\Omega)}$ through the constants C_3 and C_5 only. Let us suppose by contradiction that there exists $B_r \subset U$, with $r < r_0$, such that $P(E;B_r) < \varepsilon_1(\sigma)r^{n-1}$. We shall prove that

$$\mathcal{F}(E,u;B_{\sigma\tau^h r}) \leqslant \varepsilon_1(\tau)\tau^{\frac{h}{2}}(\sigma\tau^h r)^{n-1}, \tag{2.64}$$

for any $h \in \mathbb{N}_0$, reaching a contradiction afterward.
For $h = 0$, using Lemma 2.5.2 with $\varepsilon_1 = \varepsilon_1(\sigma)$, Theorem 2.4.1, $r < r_0 < C_3$ and $2C_5C_3\sigma < \varepsilon_1(\tau)$, we get:

$$\begin{aligned}\mathcal{F}(E,u;B_{\sigma r}) &\leqslant C_5\big(\sigma^n\mathcal{F}(E,u;B_r) + (\sigma r)^n\big)\\ &\leqslant C_5C_3\sigma^n r^{n-1} + C_5\sigma^n r^{n-1} r\\ &\leqslant 2C_5C_3\sigma^n r^{n-1} \leqslant \varepsilon_1(\tau)(\sigma r)^{n-1}.\end{aligned}$$

In order to prove the induction step we have to ensure to be in position to apply Lemma 2.5.2, that is by proving smallness of the perimeter. In such regard, let us observe that, by the definition of $\mathcal{F}(E,u;B_\rho)$ and the growth condition given in (2.10),

$$P(E;B_\rho) \leqslant \mathcal{F}(E,u;B_\rho) + 2\omega_n\frac{L^2}{\nu}\rho^n,$$

for any $B_\rho \subset \Omega$. Assuming that the induction hypothesis (2.64) holds true for some $h \in \mathbb{N}$ and, being $2\omega_n\frac{L^2}{\nu}\sigma < \varepsilon_1(\tau)$, $\tau < \frac{1}{4}$ and $r < 1$, we infer

$$\begin{aligned}P(E;B_{\sigma\tau^h r}) &\leqslant \mathcal{F}(E,u;B_{\sigma\tau^h r}) + 2\omega_n\frac{L^2}{\nu}(\sigma\tau^h r)^n\\ &\leqslant (\sigma\tau^h r)^{n-1}\left(\varepsilon_1(\tau)\tau^{\frac{h}{2}} + 2\omega_n\frac{L^2}{\nu}\sigma\tau^h r\right) \leqslant (\sigma\tau^h r)^{n-1}\varepsilon_1(\tau)(\tau^{\frac{h}{2}} + \tau^h)\\ &\leqslant (\sigma\tau^h r)^{n-1}\varepsilon_1(\tau)2\tau^{\frac{1}{2}} \leqslant (\sigma\tau^h r)^{n-1}\varepsilon_1(\tau).\end{aligned}$$

We are now in position to apply Lemma 2.5.2 with $\varepsilon_1 = \varepsilon_1(\tau)$. Using also the induction hypothesis and, since $r < r_0 \leqslant \varepsilon_1(\tau)$ and $2C_5\tau^{\frac{1}{2}} < 1$, we estimate:

$$\begin{aligned}\mathcal{F}(E,u;B_{\sigma\tau^{h+1}r}) &\leqslant C_5\big[\tau^n\mathcal{F}(E,u;B_{\sigma\tau^h r}) + \tau^n(\sigma\tau^h r)^n\big]\\ &\leqslant C_5\big[\tau^n\varepsilon_1(\tau)\tau^{\frac{h}{2}}(\sigma\tau^h r)^{n-1} + \tau^n(\sigma\tau^h r)^n\big]\\ &= \tau^{\frac{h}{2}}(\sigma\tau^h r)^{n-1}C_5\big[\tau^n\varepsilon_1(\tau) + \tau^n\sigma\tau^{\frac{h}{2}}r\big]\\ &\leqslant \tau^{\frac{h}{2}}(\sigma\tau^h r)^{n-1}\tau^n\big[C_5\varepsilon_1(\tau) + C_5\tau^{\frac{1}{2}}r\big]\\ &\leqslant \tau^{\frac{h}{2}}(\sigma\tau^h r)^{n-1}\tau^n 2C_5\varepsilon_1(\tau) \leqslant \tau^{\frac{h}{2}}(\sigma\tau^h r)^{n-1}\tau^n\varepsilon_1(\tau)\tau^{-\frac{1}{2}}\\ &= \tau^{\frac{h+1}{2}}(\sigma\tau^{h+1}r)^{n-1}\varepsilon_1(\tau).\end{aligned}$$

We conclude that (2.64) holds for any $h \in \mathbb{N}_0$. Thus, we gain

$$\begin{aligned}P(E;B_{\sigma\tau^h r}) &\leqslant \varepsilon_1(\tau)\tau^{\frac{h}{2}}(\sigma\tau^h r)^{n-1} + 2\omega_n\frac{L^2}{\nu}(\sigma\tau^h r)^n\\ &\leqslant (\sigma\tau^h r)^{n-1}\tau^{\frac{h}{2}}\left(\varepsilon_1(\tau) + 2\omega_n\frac{L^2}{\nu}\sigma\tau^{\frac{h}{2}}\right)\\ &\leqslant (\sigma\tau^h r)^{n-1}\tau^{\frac{h}{2}}\varepsilon_1(\tau)\big(1 + \tau^{\frac{h}{2}}\big)\\ &\leqslant 2(\sigma\tau^h r)^{n-1}\tau^{\frac{h}{2}}\varepsilon_1(\tau).\end{aligned}$$

We finally get

$$\lim_{\rho\to 0^+}\frac{P(E;B_\rho)}{\rho^{n-1}}=\lim_{h\to+\infty}\frac{P(E;B_{\sigma\tau^h r})}{(\sigma\tau^h r)^{n-1}}\leqslant\lim_{h\to+\infty}2\varepsilon_1(\tau)\tau^{\frac{h}{2}}=0,$$

which implies that $x_0\notin\partial^*E$, that is a contradiction. We recall that we chose the representative of ∂E such that $\partial E=\overline{\partial^*E}$. Thus, if $x_0\in\partial E$, there exists $\{x_h\}_{h\in\mathbb{N}}\subset\partial^*E$ such that $x_h\to x_0$ as $h\to+\infty$,

$$P(E;B_r(x_h))\geqslant c\big(n,\nu,N,L,L_D,\Lambda,\|\nabla u\|_{L^2(\Omega)}\big)r^{n-1}$$

and $B_r(x_h)\subset U$, for h large enough. Passing to the limit as $h\to+\infty$, we get the thesis. □

2.6 Compactness for sequences of minimizers

In this section we basically follow the path given in [46, Part III]. We start proving a standard compactness result.

Lemma 2.6.1 (Compactness). *Let (E_h,u_h) be a sequence of Λ_h-minimizers of the functional $\mathcal{F}$ in Ω such that $\sup_h\mathcal{F}(E,u_h;\Omega)<+\infty$ and $\Lambda_h\to\Lambda\in\mathbb{R}^+$. There exist a (not relabelled) subsequence and a Λ-minimizer of $\mathcal{F}$ in Ω, (E,u), such that for every open set $U\subset\subset\Omega$, it holds*

$$u_h\to u\text{ in }H^1(U),\quad E_h\to E\text{ in }L^1(U),\quad P(E_h;U)\to P(E;U).$$

In addition, $\mu_{E_h\cap U}\overset{\star}{\rightharpoonup}\mu_{E\cap U}$, $|\mu_{E_h}|\overset{\star}{\rightharpoonup}|\mu_E|$ in U and the following assertions hold:

$$\text{if } x_h\in\partial E_h\cap U\text{ and }x_h\to x\in U,\text{ then }x\in\partial E\cap U,\tag{2.65}$$

$$\text{if } x\in\partial E\cap U,\text{ there exists }x_h\in\partial E_h\cap U\text{ such that }x_h\to x.\tag{2.66}$$

Finally, if we assume also that $\nabla u_h\rightharpoonup 0$ weakly in $L^2_{loc}(\Omega;\mathbb{R}^n)$ and $\Lambda_h\to 0$, as $h\to+\infty$, then E is a local minimizer of the perimeter, that is

$$P(E;B_r(x_0))\leqslant P(F;B_r(x_0)),$$

for every set F such that $F\Delta E\subset\subset B_r(x_0)\subset\Omega$.

Proof. We start observing that, by the uniform boundedness condition on $\mathcal{F}(E_h,u_h;\Omega)$, we may assume that u_h weakly converges to u in $H^1(U)$ and strongly in $L^2(U)$, and $\mathbb{1}_{E_h}$ converges to $\mathbb{1}_E$ in $L^1(U)$, as $h\to+\infty$. By lower semicontinuity we are going to prove the Λ-minimality of (E,u).

Let us fix $B_r(x_0)\subset\subset\Omega$ and assume for simplicity of notation that $x_0=0$. Let (F,v) be a test pair such that $F\Delta E\subset\subset B_r$ and $\mathrm{spt}(u-v)\subset\subset B_r$. We can handle the perimeter term as in [46], that is, eventually passing to a subsequence and using Fubini's theorem, we may choose $\rho<r$ such that, once again, $F\Delta E\subset\subset B_\rho$ and $\mathrm{spt}(u-v)\subset\subset B_\rho$, and, in addition,

$$\mathcal{H}^{n-1}(\partial^*F\cap\partial B_\rho)=\mathcal{H}^{n-1}(\partial^*E_h\cap\partial B_\rho)=0,$$

and
$$\lim_{h\to 0}\mathcal{H}^{n-1}(\partial B_\rho \cap (E\Delta E_h)) = 0. \tag{2.67}$$
Now we choose a cut-off function $\psi \in C^1_c(B_r)$ such that $\psi \equiv 1$ in B_ρ and define $v_h = \psi v + (1-\psi)u_h$, $F_h := (F \cap B_\rho) \cup (E_h \backslash B_\rho)$ to test the minimality of (E_h, u_h). Thanks to the Λ_h-minimality of (E_h, u_h) we have

$$\begin{aligned}
&\int_{B_r}\big(F(x,u_h,\nabla u_h) + \mathbb{1}_{E_h}G(x,u_h,\nabla u_h)\big)\,dx + P(E_h;B_r) \leqslant \\
&\leqslant \int_{B_r}\big(F(x,v_h,\nabla v_h) + \mathbb{1}_{F_h}G(x,v_h,\nabla v_h)\big)\,dx + P(F_h;B_r) + \Lambda_h|F_h\Delta E_h| \\
&\leqslant \int_{B_r}\big(F(x,v_h,\nabla v_h) + \mathbb{1}_{F_h}G(x,v_h,\nabla v_h)\big)\,dx + P(F;B_\rho) + \Lambda_h|F_h\Delta E_h| \\
&+ P(E_h;B_r\backslash \overline{B}_\rho) + \varepsilon_h.
\end{aligned} \tag{2.68}$$

The mismatch term $\varepsilon_h = \mathcal{H}^{n-1}(\partial B_\rho \cap (F^{(1)}\Delta E_h^{(1)}))$ appears because F is not in general a compact variation of E_h. Nevertheless, we have that $\varepsilon_h \to 0$ because of the assumption (2.67).

Now we use the convexity of F and G with respect to the last variable to deduce

$$\begin{aligned}
&\int_{B_r}\Big(F(x,v_h,\nabla v_h) + \mathbb{1}_{F_h}G(x,v_h,\nabla v_h)\Big)\,dx \\
&\leqslant \int_{B_r}\big(F(x,v_h,\psi\nabla v + (1-\psi)\nabla u_h) + \mathbb{1}_{F_h}G(x,v_h,\psi\nabla v + (1-\psi)\nabla u_h)\big)\,dx \\
&+ \int_{B_r}\langle \nabla_z F(x,v_h,\nabla v_h), \nabla\psi(v-u_h)\rangle\,dx \\
&+ \int_{B_r}\mathbb{1}_{F_h}\langle \nabla_z G(x,v_h,\nabla v_h), \nabla\psi(v-u_h)\rangle\,dx,
\end{aligned}$$

where the last two terms in the previous estimate tend to zero as $h \to +\infty$. Indeed, the term $\nabla\psi(v-u_h)$ strongly converges to zero in L^2, being $u = v$ in $B_r\backslash B_\rho$ and the first part in the scalar product weakly converges in L^2. Then using again the convexity of F and G with respect to the z variable we obtain, for some infinitesimal σ_h,

$$\begin{aligned}
&\int_{B_r}\big(F(x,v_h,\nabla v_h) + \mathbb{1}_{F_h}G(x,v_h,\nabla v_h)\big)\,dx \\
&\leqslant \int_{B_r}\psi\big(F(x,v_h,\nabla v) + \mathbb{1}_{F_h}G(x,v_h,\nabla v)\big)\,dx \\
&+ \int_{B_r}(1-\psi)\big(F(x,v_h,\nabla u_h) + \mathbb{1}_{F_h}G(x,v_h,\nabla u_h)\big)\,dx + \sigma_h.
\end{aligned} \tag{2.69}$$

Finally, we combine (2.68) and (2.69) and pass to the limit as $h \to +\infty$, using the lower semicontinuity on the left-hand side. For the right-hand side we observe that $\mathbb{1}_{E_h} \to \mathbb{1}_E$ and $\mathbb{1}_{F_h} \to \mathbb{1}_F$ in $L^1(B_r)$, as $h \to +\infty$, and we use also the equi-integrability of $\{\nabla u_h\}_{h\in\mathbb{N}}$ to conclude,

$$\begin{aligned}
&\int_{B_r} \psi\big(F(x,u,\nabla u) + \mathbb{1}_E G(x,u,\nabla u)\big)\,dx + P(E;B_\rho) \\
&\quad\leqslant \int_{B_r} \psi\big(F(x,v,\nabla v) + \mathbb{1}_F G(x,v,\nabla v)\big)\,dx + P(F;B_\rho) + \Lambda|F\Delta E|.
\end{aligned}$$

Letting $\psi \downarrow \mathbb{1}_{B_\rho}$ we finally get

$$\begin{aligned}
&\int_{B_\rho} \big(F(x,u,\nabla u) + \mathbb{1}_E G(x,u,\nabla u)\big)\,dx + P(E;B_\rho) \\
&\quad\leqslant \int_{B_\rho} \big(F(x,v,\nabla v) + \mathbb{1}_F G(x,v,\nabla v)\big)\,dx + P(F;B_\rho) + \Lambda|F\Delta E|,
\end{aligned}$$

and this proves the Λ-minimality of (E,u).

To prove the strong convergence of ∇u_h to ∇u in $L^2(B_r)$ we start observing that by (2.68) and (2.69) applied using (E_h,u) to test the Λ-minimality of (E_h,u_h) we get

$$\begin{aligned}
&\int_{B_r} \psi\big(F(x,u_h,\nabla u_h) + \mathbb{1}_{E_h} G(x,u_h,\nabla u_h)\big)\,dx \\
&\quad\leqslant \int_{B_r} \psi\big(F(x,u,\nabla u) + \mathbb{1}_{E_h} G(x,u,\nabla u)\big)\,dx + \sigma_h.
\end{aligned}$$

Then from the equi-integrability of $\{\nabla u_h\}_{h\in\mathbb{N}}$ in $L^2(U)$ and recalling that $\mathbb{1}_{E_h} \to \mathbb{1}_E$ in $L^1(U)$, we obtain

$$\begin{aligned}
&\limsup_{h\to+\infty} \int_{B_r} \psi\big(F(x,u_h,\nabla u_h) + \mathbb{1}_{E_h} G(x,u_h,\nabla u_h)\big)\,dx \\
&\quad\leqslant \int_{B_r} \psi\big(F(x,u,\nabla u) + \mathbb{1}_E G(x,u,\nabla u)\big)\,dx.
\end{aligned}$$

The opposite inequality can be obtained by semicontinuity. Thus we get

$$\begin{aligned}
&\lim_{h\to+\infty} \int_{B_r} \psi\big(F(x,u_h,\nabla u_h) + \mathbb{1}_{E_h} G(x,u_h,\nabla u_h)\big)\,dx \\
&\quad= \int_{B_r} \psi\big(F(x,u,\nabla u) + \mathbb{1}_E G(x,u,\nabla u)\big)\,dx.
\end{aligned}$$

From the uniform ellipticity condition in (2.5) we infer, for some $\sigma_h \to 0$,

$$\begin{aligned}
\nu \int_{B_r} \psi|\nabla u_h - \nabla u|^2\,dx &\leqslant \int_{B_r} \psi\big(F(x,u_h,\nabla u_h) - F(x,u,\nabla u)\big)\,dx \\
&\quad+ \int_{B_r} \psi\mathbb{1}_E\big(G(x,u_h,\nabla u_h) - G(x,u,\nabla u)\big)\,dx + \sigma_h.
\end{aligned}$$

Passing to the limit we obtain

$$\lim_{h\to+\infty} \int_{B_r} \psi|\nabla u_h - \nabla u|^2\,dx = 0.$$

Finally, testing the minimality of (E_h, u_h) with respect to the pair (E, u) we also get

$$\lim_{h\to+\infty} P(E_h; B_\rho) = P(E; B_\rho).$$

With a usual argument we can deduce $u_h \to u$ in $H^1(U)$ and $P(E_h; U) \to P(E; U)$, for every open set $U \subset\subset \Omega$.

Let us prove that $\mu_{E_h\cap U} \overset{\star}{\rightharpoonup} \mu_{E\cap U}$. Let us fix $\mathbf{C}_r(x,\nu) \subset\subset U$ such that $\mathcal{H}^{n-1}(\partial^* E \cap \partial\mathbf{C}_r(x,\nu)) = 0$. Then, we have

$$|\mu_E|(\mathbf{C}_r(x,\nu)) = \lim_{h\to+\infty} |\mu_{E_h}|(\mathbf{C}_r(x,\nu)).$$

On the other hand, since $E_h \cap U \to E \cap U$, we easily get, by applying Proposition 1.1.2, that

$$\begin{aligned}\lim_{h\to+\infty}\int_{\mathbb{R}^n} T\cdot d\mu_{E_h\cap U} &= \lim_{h\to+\infty}\int_{E_h\cap U} \operatorname{div} T\,dx = \int_{E\cap U}\operatorname{div} T\,dx\\ &= \int_{\mathbb{R}^n} T\cdot d\mu_{E\cap U}, \quad \forall T\in C^1_c(\mathbb{R}^n;\mathbb{R}^n). \qquad (2.70)\end{aligned}$$

If $T \in C_c(\mathbb{R}^n;\mathbb{R}^n)$, then we can find a sequence $\{T_h\}_{h\in\mathbb{N}} \subset C^1_c(\mathbb{R}^n;\mathbb{R}^n)$ such that $T_h \to T$ uniformly and $\bigcup_{h\in\mathbb{N}} \operatorname{spt} T_h \cup \operatorname{spt} T \subset K$, for some compact set $K \subset \mathbb{R}^n$. Fixing $\varepsilon > 0$, for h and k sufficiently large, we get

$$\begin{aligned}&\left|\int_{\mathbb{R}^n} T\cdot d\mu_{E_h\cap U} - \int_{\mathbb{R}^n} T\cdot d\mu_{E\cap U}\right|\\ &\leqslant \left|\int_{\mathbb{R}^n}(T-T_k)\cdot d\mu_{E_h\cap U}\right| + \left|\int_{\mathbb{R}^n} T_k\cdot d\mu_{E_h\cap U} - \int_{\mathbb{R}^n} T_k\cdot d\mu_{E\cap U}\right|\\ &+\left|\int_{\mathbb{R}^n} T_k\cdot d\mu_{E\cap U} - \int_{\mathbb{R}^n} T\cdot d\mu_{E\cap U}\right| < \|T-T_k\|_\infty\,|\mu_{E_h\cap U}|(\mathbb{R}^n) + \frac{2}{3}\varepsilon < \varepsilon,\end{aligned}$$

since $|\mu_{E_h\cap U}|(\mathbb{R}^n) = P(E_h; U) + P(U; E_h) \leqslant P(E_h; U) + P(U) < +\infty$. Therefore, we conclude that (2.70) holds for $T \in C_c(\mathbb{R}^n;\mathbb{R}^n)$, i.e. $\mu_{E_h\cap U} \overset{\star}{\rightharpoonup} \mu_{E\cap U}$.

In order to prove the other weak-star convergence, thanks to Proposition A.1.14, we need to show that $|\mu_{E_h}|$ is lower semicontinuous on open sets and upper semicontinuous on compact sets. We preliminarily observe that, if $G \subset U$ is a $|\mu_{E_h}|$-measurable set, by Theorem 1.1.18, we have

$$|\mu_{E_h\cap U}|(G) = |\mu_{E_h}|(G) + |\mu_U|(G\cap E_h) = |\mu_{E_h}|(G),$$

i.e. $|\mu_{E_h\cap U}| = |\mu_{E_h}|$ on U. The same argument can be used to prove that $|\mu_{E\cap U}| = |\mu_E|$ on U. Using the lower semicontinuity property of Radon measures with respect to open sets (see Proposition A.1.14), we infer

$$|\mu_E|(A) = |\mu_{E\cap U}|(A) \leqslant \liminf_{h\to+\infty} |\mu_{E_h\cap U}|(A) = \liminf_{h\to+\infty} |\mu_{E_h}|(A), \qquad (2.71)$$

for every open set $A \subset U$.

We are left to prove the upper semicontinuity on compact sets. Let $K \subset U$

be a compact set. Since $P(E_h;U) \to P(E;U)$ as $h \to +\infty$, then, by the lower semicontinuity of $|\mu_{E_h}|$ on open sets (see Proposition A.1.14),

$$\begin{aligned}|\mu_E|(K) &= |\mu_E|(U) - |\mu_E|(U\backslash K) \geqslant \lim_{h\to+\infty} |\mu_{E_h}|(U) - \liminf_{h\to+\infty} |\mu_{E_h}|(U\backslash K) \\ &= \limsup_{h\to+\infty}(|\mu_{E_h}|(U) - |\mu_{E_h}|(U\backslash K)) = \limsup_{h\to+\infty} |\mu_{E_h}|(K). \qquad (2.72)\end{aligned}$$

Putting (2.71) and (2.72) together we get $|\mu_{E_h}| \overset{\star}{\rightharpoonup} |\mu_E|$ in U.

Now we prove (2.65). If $s > 0$ is such that $B_{2s}(x) \subset\subset U$, then, for h sufficiently large, we have $B_s(x_h) \subset B_{2s}(x)$. Thus, the upper semicontinuity property of Radon measures with respect to closed sets (see Proposition A.1.14) and the density lower bound estimate (see Theorem 2.5.3) give

$$P(E;\overline{B}_{2s}(x)) \geqslant \limsup_{h\to+\infty} P(E_h;\overline{B}_{2s}(x)) \geqslant P(E_h;\overline{B}_s(x_h)) \geqslant c(n)s^{n-1} > 0.$$

In particular, $x \in \operatorname{spt}\mu_E = \partial E$.

Finally, we prove (2.66). We recall that we chose a representative of ∂E such that $\partial E = \operatorname{spt}\mu_E$. Let us fix $\varepsilon > 0$ such that $B_\varepsilon(x) \subset A_0$. Assume by contradiction that there exists a divergent subsequence $\{h(k)\}_{k\in\mathbb{N}} \subset \mathbb{N}$ such that $\operatorname{spt}\mu_{E_{h(k)}} \cap B_\varepsilon(x) = \varnothing$, for any $k \in \mathbb{N}$. Since $\mu_{E_{h(k)}} \overset{\star}{\rightharpoonup} \mu_E$, by the lower semicontinuity property of Radon measures with respect to open sets (see Proposition A.1.14), we finally get

$$\mu_E(B_\varepsilon(x)) \leqslant \liminf_{k\to+\infty} \mu_{E_{h(k)}}(B_\varepsilon(x)) = 0,$$

which implies that $x \notin \operatorname{spt}\mu_E$, that is a contradiction. □

Proposition 2.6.2 (Lower semicontinuity of the excess)**.** *If A, $A_0 \subset \mathbb{R}^n$ are open sets with $A_0 \subset\subset A$, $P(A_0) < +\infty$, and if $\{E_h\}_{h\in\mathbb{N}}$ is a sequence of Λ-minimizers of $\mathcal{F}$ in A such that $A_0 \cap E_h \to E$, then, for every $\mathbf{C}_r(x,\nu) \subset\subset A_0$, we have*

$$\mathbf{e}^C(E,x,r,\nu) \leqslant \liminf_{h\to+\infty} \mathbf{e}^C(E_h,x,r,\nu).$$

In fact, if $\mathbf{C}_r(x,\nu)$ is such that $\mathcal{H}^{n-1}(\partial^* E \cap \partial \mathbf{C}_r(x,\nu)) = 0$, then we have exactly

$$\mathbf{e}^C(E,x,r,\nu) = \lim_{h\to+\infty} \mathbf{e}^C(E_h,x,r,\nu).$$

Proof. **Step 1:** Let us fix $\mathbf{C}_r(x,\nu) \subset\subset A_0$ such that $\mathcal{H}^{n-1}(\partial^* E \cap \partial\mathbf{C}_r(x,\nu)) = 0$. Then, by Lemma 2.6.1, we have

$$|\mu_E|(\mathbf{C}_r(x,\nu)) = \lim_{h\to+\infty} |\mu_{E_h}|(\mathbf{C}_r(x,\nu)).$$

On the other hand, since $A_0 \cap E_h \to E$, we easily get, by applying Proposition 1.1.2, that

$$\lim_{h\to+\infty} \int_{\mathbb{R}^n} T \cdot d\mu_{A_0\cap E_h} = \int_{\mathbb{R}^n} T \cdot d\mu_E, \quad \forall T \in C_c^1(\mathbb{R}^n;\mathbb{R}^n). \qquad (2.73)$$

If $T \in C_c(\mathbb{R}^n;\mathbb{R}^n)$, then we can find a sequence $\{T_h\}_{h\in\mathbb{N}} \subset C^1_c(\mathbb{R}^n;\mathbb{R}^n)$ such that $T_h \to T$ uniformly and $\bigcup_{h\in\mathbb{N}} \operatorname{spt} T_h \cup \operatorname{spt} T \subset K$, for some compact set $K \subset \mathbb{R}^n$. Fixing $\varepsilon > 0$, for h and k sufficiently large, we get

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} T \cdot d\mu_{A_0\cap E_h} - \int_{\mathbb{R}^n} T \cdot d\mu_E\right| \\
&\leqslant \left|\int_{\mathbb{R}^n} (T - T_k) \cdot d\mu_{A_0\cap E_h}\right| + \left|\int_{\mathbb{R}^n} T_k \cdot d\mu_{A_0\cap E_h} - \int_{\mathbb{R}^n} T_k \cdot d\mu_E\right| \\
&+ \left|\int_{\mathbb{R}^n} T_k \cdot d\mu_E - \int_{\mathbb{R}^n} T \cdot d\mu_E\right| < \|T - T_k\|_\infty \, |\mu_{A_0\cap E_h}|(\mathbb{R}^n) + \frac{2}{3}\varepsilon < \varepsilon,
\end{aligned}$$

since $|\mu_{A_0\cap E_h}|(\mathbb{R}^n) < +\infty$. Therefore we conclude that (2.73) holds for $T \in C_c(\mathbb{R}^n;\mathbb{R}^n)$, i.e. $\mu_{A_0\cap E_h} \overset{\star}{\rightharpoonup} \mu_E$. By Proposition A.1.14 we infer

$$\mu_E(\mathbf{C}_r(x,\nu)) = \lim_{h\to+\infty} \mu_{A_0\cap E_h}(\mathbf{C}_r(x,\nu)).$$

Since $\mathbf{C}_r(x,\nu) \subset\subset A_0$, by Theorem 1.1.18 we have

$$\mu_{A_0\cap E_h}(\mathbf{C}_r(x,\nu)) = \mu_{E_h}(\mathbf{C}_r(x,\nu))$$

and thus

$$\begin{aligned}
\lim_{h\to+\infty} \mathbf{e}^C(E_h,x,r,\nu) &= \lim_{h\to+\infty} \frac{|\mu_{E_h}|(\mathbf{C}_r(x,\nu)) - \nu\cdot\mu_{E_h}(\mathbf{C}_r(x,\nu))}{r^{n-1}} \\
&= \frac{|\mu_E|(\mathbf{C}_r(x,\nu)) - \nu\cdot\mu_E(\mathbf{C}_r(x,\nu))}{r^{n-1}} = \mathbf{e}^C(E_h,x,r,\nu).
\end{aligned}$$

Step 2: We observe that the function $r \mapsto \mathbf{e}^C(E,x,r,\nu)$ is left-continuous on $(0,+\infty)$. By the foliation's property by Borel sets, we can choose a sequence $\{r_k\}_{k\in\mathbb{N}}$ with $r_k \to r^-$ as $k \to +\infty$ such that $\mathcal{H}^{n-1}(\partial^* E \cap \partial\mathbf{C}(x,r_k,\nu)) = 0$ and $\mathbf{C}(x,r_k,\nu) \subset\subset A_0$, for all $k \in \mathbb{N}$. By step 1 and Proposition 1.1.22, we find

$$\mathbf{e}^C(E,x,r_k,\nu) = \lim_{h\to+\infty} \mathbf{e}^C(E_h,x,r_k,\nu) \leqslant \left(\frac{r}{r_k}\right)^{n-1} \liminf_{h\to+\infty} \mathbf{e}^C(E_h,x,r,\nu).$$

Finally, we let $k \to +\infty$ and obtain the thesis. □

2.7 Height bound lemma

Now we introduce a usual quantity involved in regularity theory. We define the **rescaled Dirichlet integral** of u as

$$\mathcal{D}(x,r) := \frac{1}{r^{n-1}} \int_{B_r(x)} |\nabla u|^2 dy.$$

The proof of the height bound is rather standard and it can be found in [46, Chapter 22]. We first need the following two lemmata.

Lemma 2.7.1 (Small excess position). *For any $t_0 \in (0,1)$ there exists a constant $\omega = \omega(n,t_0) > 0$ with the following property. If (E,u) is a Λ-minimizer of $\mathcal{F}$ in $\mathbf{C}_2$, $0 \in \partial E$ and*

$$\mathbf{e}_n(2) \leqslant \omega(n,t_0),$$

then

$$|qx| < t_0, \quad \forall x \in \mathbf{C}_1 \cap \partial E,$$
$$|\{x \in \mathbf{C}_1 \cap E \,:\, qx > t_0\}| = 0,$$
$$|\{x \in \mathbf{C}_1 \backslash E \,:\, qx < -t_0\}| = 0.$$

Proof. Let $t_0 \in (0,1)$. Assume by contradiction that there exists a sequence $\{(E_h,u_h)\}_{h\in\mathbb{N}}$ of Λ-minimizers of $\mathcal{F}$ in $\mathbf{C}_2$ with equibounded energies such that $0 \in \partial E_h$,

$$\lim_{h\to+\infty} \mathbf{e}(E_h,0,2,e_n) = 0,$$

and for inifinitely many $h \in \mathbb{N}$ at least one of the following conditions holds true:

$$\{x \in \mathbf{C}_1 \cap \partial E_h \,:\, t_0 \leqslant |qx| \leqslant 1\} \neq \varnothing, \tag{2.74}$$
$$|\{x \in \mathbf{C}_1 \cap E_h \,:\, qx > t_0\}| > 0, \tag{2.75}$$
$$|\{x \in \mathbf{C}_1 \backslash E_h \,:\, qx < -t_0\}| > 0. \tag{2.76}$$

Up to subsequences, there exists a Λ-minimizer of $\mathcal{F}$ in $\mathbf{C}_2$ such that $E_h \to E$ in $L^1_{loc}(\mathbf{C}_2)$, $u_h \to u$ in $H^1_{loc}(\mathbf{C}_2)$, $P(E_h;U) \to P(E;U)$, for any open set $U \subset\subset \Omega$, as $h \to +\infty$. Furthermore, since $0 \in \partial E_h \cap \mathbf{C}_{\frac{5}{3}}$ for any $h \in \mathbb{N}$, by (2.65) we deduce that $0 \in \partial E$. In particular, E is a Λ-minimizer of $\mathcal{F}$ in $\mathbf{C}_{\frac{5}{3}}$ and we may assume that $E_h \cap \mathbf{C}_{\frac{5}{3}} \to E$.

By the semicontinuity of the excess (see Proposition 2.6.2) and the comparison between excess at different scales (see Proposition 1.1.22), we deduce

$$\mathbf{e}\left(E,0,\frac{4}{3},e_n\right) \leqslant \liminf_{h\to+\infty} \mathbf{e}\left(E_h,0,\frac{4}{3},e_n\right) \leqslant \left(\frac{3}{2}\right)^{n-1} \lim_{h\to+\infty} \mathbf{e}(E_h,0,2,e_n) = 0.$$

Thus, using Proposition 1.1.23 we infer that

$$E \cap \mathbf{C}_{\frac{4}{3}} \text{ is equivalent to } \mathbf{C}_{\frac{4}{3}} \cap \{qx < 0\}. \tag{2.77}$$

If (2.74) were valid for infinitely many values of $h \in \mathbb{N}$, then, up to extract a further subsequence, we may construct $\{x_h\}_{h\in\mathbb{N}}$ with $x_h \in \mathbf{C}_1 \cap \partial E_h$, $t_0 \leqslant |qx_h| \leqslant 1$ and, by (2.66), $x_h \to x_0 \in \overline{\mathbf{C}}_1 \cap \partial E$, as $h \to +\infty$. Then, it would be that

$$\overline{\mathbf{C}}_1 \cap \partial E \cap \{|qx| \geqslant t_0\} \subset \mathbf{C}_{\frac{4}{3}} \cap \partial E \cap \{|qx| \geqslant t_0\} \neq \varnothing,$$

in contradiction with (2.77). Thus there exists $h_0 \in \mathbb{N}$ such that

$$\{x \in \mathbf{C}_1 \cap \partial E_h \,:\, t_0 \leqslant |qx| \leqslant 1\} = \varnothing, \quad \forall h \geqslant h_0.$$

Since Theorem 1.1.18 guaranties that

$$\begin{aligned}|\mu_{\mathbf{C}_1\cap E_h}| &= |\mu_{E_h}|\llcorner \mathbf{C}_1 + |\mu_{\mathbf{C}_1}|\llcorner E_h^{(1)} + \mathcal{H}^{n-1}\llcorner\{\nu_{\mathbf{C}_1} = \nu_{E_h}\}\\ &= |\mu_{\mathbf{C}_1}|\llcorner E_h^{(1)} + |\mu_{E_h}|\llcorner(\mathbf{C}_1 \cup \{\nu_{\mathbf{C}_1} = \nu_{E_h}\}),\end{aligned}$$

we find that, for every $h \geqslant h_0$,

$$\begin{aligned}&|\mu_{\mathbf{C}_1\cap E_h}|(\{x \in \mathbf{C}_1 \,:\, t_0 < |qx| < 1\})\\ &= \mathcal{H}^{n-1}(E_h^{(1)} \cap \partial\mathbf{C}_1 \cap \{x \in \mathbf{C}_1 \,:\, t_0 < |qx| < 1\})\\ &+ |\mu_{E_h}|(\{x \in \mathbf{C}_1 \,:\, t_0 < |qx| < 1\} \cap \{\nu_{\mathbf{C}_1} = \nu_{E_h}\})\\ &\leqslant |\mu_{E_h}|(\{x \in \mathbf{C}_1 \,:\, t_0 < |qx| < 1\}) = 0.\end{aligned}$$

Decomposing the set $\{x \in \mathbf{C}_1 : t_0 < |qx| < 1\}$ in the union of the two connected open sets $\{x \in \mathbf{C}_1 : t_0 < qx < 1\}$ and $\{x \in \mathbf{C}_1 : -1 < qx < -t_0\}$, we have that

$$\begin{aligned}0 &= |\mu_{\mathbf{C}_1\cap E_h}|(\{x \in \mathbf{C}_1 \,:\, t_0 < qx < 1\})\\ &= \sup_{\substack{\phi\in C_c^\infty(\{x\in\mathbf{C}_1\,:\,t_0<qx<1\})\\ \|\phi\|_\infty\leqslant 1}} \int_{\mathbb{R}^n} \mathbb{1}_{\mathbf{C}_1\cap E_h} \mathrm{div}\phi \, dx.\end{aligned}$$

Thus, we infer that $\mathbb{1}_{C\cap E_h}$ is equivalent to a constant in $\{x \in \mathbf{C}_1 : t_0 < qx < 1\}$. With the same argument, $\mathbb{1}_{C\cap E_h}$ is equivalent to a (possibly different) constant in $\{x \in \mathbf{C}_1 : -1 < qx < -t_0\}$. Since $\mathbf{C}_1 \cap E_h \to E$ as $h \to +\infty$ and (2.77) holds true, we deduce

$$\mathbb{1}_{C\cap E_h} = 0 \quad \text{a.e. in } \{x \in \mathbf{C}_1 \,:\, t_0 < qx < 1\},$$

$$\mathbb{1}_{C\cap E_h} = 1 \quad \text{a.e. in } \{x \in \mathbf{C}_1 \,:\, -1 < qx < -t_0\},$$

that is a contradiction to (2.75) and (2.76). □

Lemma 2.7.2 (Excess measure). *If $E \subset \mathbb{R}^n$ is a set of locally finite perimeter, with $0 \in \partial E$, and such that, for some $t_0 \in (0, 1)$,*

$$|qx| < t_0, \quad \forall x \in \mathbf{C}_1 \cap \partial E,$$

$$|\{x \in \mathbf{C}_1 \cap E \,:\, qx > t_0\}| = 0, \tag{2.78}$$

$$|\{x \in \mathbf{C}_1 \backslash E \,:\, qx < -t_0\}| = 0, \tag{2.79}$$

then, setting $M := \mathbf{C}_1 \cap \partial^* E$, we have

$$\mathcal{H}^{n-1}(G) \leqslant \mathcal{H}^{n-1}(M \cap p^{-1}(G)), \tag{2.80}$$

$$\mathcal{H}^{n-1}(G) = \int_{M\cap p^{-1}(G)} (\nu_E \cdot e_n) \, d\mathcal{H}^{n-1}, \tag{2.81}$$

$$\int_{\mathbf{D}_1} \phi \, dx = \int_M (\phi \circ p)(\nu_E \cdot e_n) d\mathcal{H}^{n-1}, \tag{2.82}$$

$$\int_{E_t \cap \mathbf{D}_1} \phi \, dx = \int_{M \cap \{qx > t\}} (\phi \circ p)(\nu_E \cdot e_n) d\mathcal{H}^{n-1}, \tag{2.83}$$

for every Borel set $G \subset \mathbf{D}_1$, $\phi \in C_c(\mathbf{D}_1)$ and $t \in (-1, 1)$. The set function

$$\zeta(G) := P(E; \mathbf{C} \cap p^{-1}(G)) - \mathcal{H}^{n-1}(G) = \mathcal{H}^{n-1}(M \cap p^{-1}(G)) - \mathcal{H}^{n-1}(G),$$

*for $G \subset \mathbb{R}^{n-1}$, defines a Radon measure on $\mathbb{R}^{n-1}$, concentrated on $\mathbf{D}_1$. The Radon measure ζ is called the **excess measure** of E over $\mathbf{D}_1$ since $\zeta(\mathbf{D}_1) = \mathbf{e}^C(E, 0, 1, e_n)$.*

Proof. We first remark that (2.82) implies (2.81), which in turn implies (2.80) by observing that $|\nu_E \cdot e_n| \leqslant 1$. Indeed, if (2.82) holds true, then approximating $\mathbb{1}_G$, where $G \subset \mathbf{D}_1$ is a Borel set, with a sequence of functions $\{\phi_h\}_{h \in \mathbb{N}} \subset C_c(\mathbf{D}_1)$, we get (2.81). We are led to prove (2.82) and (2.83). By a density argument we may assume that $\phi \in C^1_c(\mathbf{D}_1)$. Using the foliation's property by Borel sets, we have

$$\mathcal{H}^{n-1}(\partial^* E \cap (\partial \mathbf{D}_r \times \mathbb{R})) = 0, \tag{2.84}$$

for a.e. $r \in (0, 1)$. By hypothesis (2.78), thanks to Fubini's theorem, we get

$$0 = \int_{\mathbf{C}_1 \cap E \cap \{qx > t_0\}} dx = \int_{t_0}^1 \mathcal{H}^{n-1}(E \cap (\mathbf{D}_1 \times \{s\})) ds.$$

This implies that

$$\mathcal{H}^{n-1}(E \cap (\mathbf{D}_1 \times \{s\})) = 0, \quad \text{for a.e. } s \in (t_0, 1). \tag{2.85}$$

By assumption (2.79), with the same argument we infer

$$\mathcal{H}^{n-1}(E \cap (\mathbf{D}_1 \times \{t\})) = \mathcal{H}^{n-1}(\mathbf{D}_1), \quad \text{for a.e. } t \in (-1, -t_0). \tag{2.86}$$

We let $r \in (0, 1)$ and $s \in (t_0, 1)$ satisfy respectively (2.84) and (2.85). Given $t \in (-1, s)$, we define a set of finite perimeter F as $F := E \cap (\mathbf{D}_r \times (t, s))$. By (2.84) we have $\{\nu_E = \nu_{\mathbf{D}_r \times (t,s)}\} = \varnothing$, thus obtaining by Theorem 1.1.18

$$\mu_F = \mu_E \llcorner (\mathbf{D}_r \times (t, s)) + \mu_{\mathbf{D_r} \times (t,s)} \llcorner E.$$

Now we make the Gaus-Green measure $\mu_{\mathbf{D}_r \times (t,s)}$ explicit. If we set $\nu(x) := \frac{px}{|px|}$ for every $x \in \mathbb{R}^n$ such that $px \neq 0$ (so that $\nu(x)$ is the outer normal to the cylinder $\mathbf{D}_r \times \mathbb{R}$ at $x \in \partial \mathbf{D}_r \times \mathbb{R}$), then

$$\mu_{\mathbf{D}_r \times (t,s)} = e_n \mathcal{H}^{n-1} \llcorner (\mathbf{D}_r \times \{s\}) + \nu \mathcal{H}^{n-1} \llcorner (\partial \mathbf{D}_r \times (t, s)) - e_n \mathcal{H}^{n-1}(\mathbf{D}_r \times \{t\}), \tag{2.87}$$

that is the sum of the Gauss-Green measures of the bottom, the lateral surface and the top of the cylinder. Since $\nu \cdot e_n = 0$ and, by (2.85), $\mathcal{H}^{n-1} \llcorner (E \cap (\mathbf{D}_r \times \{s\})) = 0$, multiplying by e_n the equality (2.87), we get

$$e_n \cdot \mu_F = (e_n \cdot \nu_E) \mathcal{H}^{n-1} \llcorner (\partial^* E \cap (\mathbf{D}_r \times (t, s))) - \mathcal{H}^{n-1} \llcorner (E \cap (\mathbf{D}_r \times \{t\})).$$

Hence, given $\phi \in C^1_c(\mathbf{D}_1)$ we may define a vector field $T \in C^1(\mathbb{R}^n;\mathbb{R}^n)$ by setting $T(x) = \phi(px)e_n$, for $x \in \mathbb{R}^n$. We apply the distributional Gauss-Green theorem to obtain

$$\begin{aligned} 0 &= \int_F \operatorname{div} T\, dx = \int_{\partial^* F} T \cdot d\mu_F \\ &= \int_{\partial^* E \cap (\mathbf{D}_r \times (t,s))} (\phi \circ p)(e_n \cdot \nu_E)\, d\mathcal{H}^{n-1} - \int_{E \cap (\mathbf{D}_r \times \{t\})} (\phi \circ p)\, d\mathcal{H}^{n-1}. \end{aligned}$$

We first let $r \to 1^-$ and then $s \to 1^-$ to prove (2.83), that is

$$\begin{aligned} \int_{E_t \cap \mathbf{D}_1} \phi\, dx &= \int_{E \cap (\mathbf{D}_1 \times \{t\})} (\phi \circ p)\, d\mathcal{H}^{n-1} = \int_{\partial^* E \cap (\mathbf{D} \times (t,1))} (\phi \circ p)(e_n \cdot \nu_E)\, d\mathcal{H}^{n-1} \\ &= \int_{M \cap \{qx > t\}} (\phi \circ p)(\nu_E \cdot e_n) d\mathcal{H}^{n-1}. \end{aligned}$$

Finally, by letting $t \to 1^-$ and by (2.86), we prove (2.82). □

Remark 2.7.3. *We observe that* (2.80) *ensures that* $\mathbf{C}_1 \cap \partial^* E$ *"leaves no holes" over* $\mathbf{D}_1$*. Furthermore the measure* ζ *defined in the previous lemma measures the flatness of subsets of* $\mathbf{C}_1 \cap \partial^* E$*. The name "excess measure" is justified by the fact that, by* (2.81),

$$\zeta(D) = \mathcal{H}^{n-1}(\mathbf{C}_1 \cap \partial^* E) - \int_{\mathbf{C}_1 \cap \partial^* E} (\nu_E \cdot e_n) d\,\mathcal{H}^{n-1} = \mathbf{e}^C(E, 0, 1, e_n).$$

Thus, if $\mathbf{e}^C(E,0,1,e_n)$ *is small, by the monotonicity property of measures, then every subset of* G *is almost flat, that can be explicited as*

$$\mathcal{H}^{n-1}(G) \leqslant \mathcal{H}^{n-1}(\mathbf{C}_1 \cap \partial^* E \cap p^{-1}(G)) \leqslant \mathcal{H}^{n-1}(G) + \mathbf{e}^C(E,0,1,e_n),$$

for every Borel set $G \subset \mathbf{D}_1$.

The following height bound lemma is a standard step in the proof of regularity.

Lemma 2.7.4 (Height bound)**.** *Let* (E,u) *be a* Λ*-minimizer of* $\mathcal{F}$ *in* $\mathbf{C}_{4r}(x_0,\nu)$ *and* $\nu \in \mathbb{S}^{n-1}$*. There exist two positive constants* $C_7 = C_7\big(n,\nu,N,L,L_D,\Lambda,\|\nabla u\|_{L^2(\Omega)}\big)$ *and* $\varepsilon_2 = \varepsilon_2(n)$ *such that if* $x_0 \in \partial E$ *and*

$$\mathbf{e}(x,4r,\nu) < \varepsilon_2,$$

then

$$\sup_{y \in \partial E \cap \mathbf{C}_r(x_0)} \frac{|\nu \cdot (y - x_0)|}{r} \leqslant C_7 \mathbf{e}(x,4r,\nu)^{\frac{1}{2(n-1)}}.$$

Proof. **Step 1:** Up to replacing (E,u) with $\Big(\frac{E-x_0}{2r}, (2r)^{-\frac{1}{2}}u(x_0 + r\cdot)\Big)$ and rotating in a way such that $\nu = e_n$, by Lemma 2.5.1, we have that (E,u) is a Λr-minimizer of $\mathcal{F}_r$ in $\mathbf{C}_2$, $0 \in \partial E$ and, by Proposition 1.1.21,

$$\mathbf{e}(E,0,2,e_n) < \varepsilon_2.$$

Assuming that $\varepsilon_2(n) < \omega\left(n, \frac{1}{4}\right)$, with $\omega\left(n, \frac{1}{4}\right)$ as in Lemma 2.7.1, and defining $M := \mathbf{C}_1 \cap \partial E$, by Lemma 2.7.1, Remark 2.7.3 and Proposition 1.1.22 we deduce that

$$|qx| < \frac{1}{4}, \quad \forall x \in M,$$

$$0 \leqslant \mathcal{H}^{n-1}(M) - \mathcal{H}^{n-1}(D_1) \leqslant \mathbf{e}_n(1) \leqslant 2^{n-1}\mathbf{e}_n(2), \tag{2.88}$$

$$0 \leqslant \mathcal{H}^{n-1}(M \cap \{qx > t\}) - \mathcal{H}^{n-1}(E_t \cap D_1) \leqslant \mathbf{e}_n(1) \leqslant 2^{n-1}\mathbf{e}_n(2). \tag{2.89}$$

Step 2: Let $f\colon (-1,1) \to [0, \mathcal{H}^{n-1}(M)]$ be the left-continuous nonincreasing function defined as

$$f(t) := \mathcal{H}^{n-1}(M \cap \{qx > t\}).$$

By Lemma 2.7.1 we get that

$$f(t) = \mathcal{H}^{n-1}(M), \quad \forall t \in \left(-1, -\frac{1}{4}\right),$$

$$f(t) = 0, \quad \forall t \in \left(\frac{1}{4}, 1\right).$$

Since f is non-increasing, there exists $t_0 \in (-1,1)$ such that

$$f(t) \leqslant \frac{\mathcal{H}^{n-1}(M)}{2}, \quad \text{if } t \geqslant t_0, \tag{2.90}$$

$$f(t) \geqslant \frac{\mathcal{H}^{n-1}(M)}{2}, \quad \text{if } t \leqslant t_0,$$

for $t \in (-1,1)$. As $0 \in \mathbf{C}_{\frac{1}{2}} \cap \partial E$ and, so,

$$|qx| \leqslant |qx - t_0| + |t_0 - q0|,$$

it is enough to prove that

$$qx - t_0 \leqslant C(n)\mathbf{e}_n(2)^{\frac{1}{2(n-1)}}, \quad \forall x \in \mathbf{C}_{\frac{1}{2}} \cap \partial E.$$

Indeed, applying the same argument with $\mathbb{R}^n \backslash E$ in place of E, we shall then deduce that $t_0 - qx \leqslant C(n)\mathbf{e}_n(2)^{\frac{1}{2(n-1)}}$, for any $x \in \mathbf{C}_{\frac{1}{2}} \cap \partial E$.

Step 3: Let $t_1 \in \left(t_0, \frac{1}{4}\right)$ be such that $f(t) \leqslant \sqrt{\mathbf{e}_n(2)}$ for any $t \geqslant t_1$. Then it holds true that

$$qy - t_1 \leqslant C(n)\mathbf{e}_n(2)^{\frac{1}{2(n-1)}}, \quad \forall y \in \mathbf{C}_{\frac{1}{2}} \cap \partial E.$$

Indeed if $qy < t_1$, the assertion is trivial. If $qy > t_1$, then $B(y, qy - t_1) \subset\subset \mathbf{C}_2$, with $qy - t_1 < \frac{1}{2}$. Since E is a Λr-minimizer of $\mathcal{F}_r$ in $\mathbf{C}_2$, then, using Theorem 2.5.3 and observing that $B(t, qy - t_1) \subset \mathbf{C}_1 \cap \{qx > t_1\}$, we get

$$\mathbf{C}_3(qy - t_1)^{n-1} \leqslant P(E; B(y, qy - t_1)) = \mathcal{H}^{n-1}(M \cap \{qx > t_1\}) = f(t) \leqslant \sqrt{\mathbf{e}_n(2)},$$

obtaining the assertion.
Step 4: By step 3, we are left to prove that

$$t_1 - t_0 \leqslant C(n)\mathbf{e}_n(2)^{\frac{1}{2(n-1)}}.$$

We shall prove the following chain of inequalities:

$$c(n)\sqrt{\mathbf{e}_n(2)} \geqslant \int_{t_0}^{t_1} \mathcal{H}^{n-1}(E_t \cap D_1)^{\frac{n-2}{n-1}}\, dt \geqslant c(n)(t_1 - t_0)\sqrt{\mathbf{e}_n(2)}^{\frac{n-2}{n-1}}, \qquad (2.91)$$

where E_t is the horizontal section of E at level t. Let us prove the first inequality in (2.91). Since, by (2.90), (2.88) and choosing $\varepsilon_2(n) \leqslant \frac{\mathcal{H}^{n-1}(D_1)}{2^n}$,

$$\begin{aligned}\mathcal{H}^{n-1}(E_t \cap D_1) \leqslant \mathcal{H}^{n-1}(M \cap \{qx > t\}) = f(t) &\leqslant \frac{\mathcal{H}^{n-1}(M)}{2} \\ \leqslant \frac{\mathcal{H}^{n-1}(D_1) + 2^{n-1}\mathbf{e}_n(2)}{2} &\leqslant \frac{3}{4}\mathcal{H}^{n-1}(D_1),\end{aligned}$$

we can apply the relative isoperimetric inequality in dimension $n-1$ (see Proposition 1.1.9) to obtain that

$$\mathcal{H}^{n-2}(\partial^* E_t \cap D_1) = P(E_t; D_1) \geqslant c(n)\mathcal{H}^{n-1}(E_t \cap D_1)^{\frac{n-2}{n-1}}. \qquad (2.92)$$

Since $\varepsilon_2(n) \leqslant \omega\big(n, \frac{1}{4}\big)$, we estimate

$$\mathcal{H}^{n-1}(M) \leqslant \mathcal{H}^{n-1}(D_1) + 2^{n-1}\mathbf{e}_n(2) \leqslant c(n). \qquad (2.93)$$

Furthermore, since $\mathcal{H}^{n-2}(\partial^* E_t \Delta (\partial^* E)_t) = 0$ for almost every $t \in \mathbb{R}$, we infer

$$\mathcal{H}^{n-2}(D_1 \cap \partial^* E_t) = \mathcal{H}^{n-2}(D_1 \cap (\partial^* E)_t) = \mathcal{H}^{n-2}((\mathbf{C}_1 \cap \partial^* E)_t) = \mathcal{H}^{n-2}(M_t). \qquad (2.94)$$

Therefore, applying (2.92), (2.94) by Fubini's Theorem, Coarea formula (see Theorem A.2.6) and (2.93), we get:

$$\begin{aligned}\int_{t_0}^{t_1} \mathcal{H}^{n-1}(E_t \cap D_1)^{\frac{n-2}{n-1}}\, dt &\leqslant c(n)\int_{-1}^{1} \mathcal{H}^{n-2}(D_1 \cap \partial^* E_t)\, dt \\ &= c(n)\int_{\mathbb{R}} \mathcal{H}^{n-2}(M_t)\, dt = c(n)\int_{\mathbb{R}} dt \int_{\partial^* E_t} \mathbb{1}_C \, d\mathcal{H}^{n-2} \\ &= c(n)\int_{\partial^* E} \mathbb{1}_C \sqrt{1 - (\nu_E \cdot e_n)^2}\, d\mathcal{H}^{n-1} \\ &= c(n)\int_M \sqrt{1 - (\nu_E \cdot e_n)^2}\, d\mathcal{H}^{n-1} \\ &\leqslant c(n)\int_M \sqrt{1 - \nu_E \cdot e_n}\, d\mathcal{H}^{n-1} \\ &\leqslant c(n)\sqrt{\mathcal{H}^{n-1}(M)}\left(\int_M (1 - \nu_E \cdot e_n)\, d\mathcal{H}^{n-1}\right)^{\frac{1}{2}} \\ &\leqslant c(n)\sqrt{\mathbf{e}_n(2)}\end{aligned}$$

We now prove the second inequality in (2.91). By (2.89), the choice of t_1, using that f is decreasing, we deduce that

$$\begin{aligned}\int_{t_0}^{t_1} \mathcal{H}^{n-1}(E_t \cap D_1)^{\frac{n-2}{n-1}}\, dt &\geqslant \int_{t_0}^{t_1} [\mathcal{H}^{n-1}(M \cap \{qx > t\}) - 2^{n-1}\mathbf{e}_n(2)]^{\frac{n-2}{n-1}}\, dt \\ &= \int_{t_0}^{t_1} [f(t) - 2^{n-1}\mathbf{e}_n(2)]^{\frac{n-2}{n-1}}\, dt \\ &\geqslant \int_{t_0}^{t_1} [\sqrt{\mathbf{e}_n(2)} - 2^{n-1}\mathbf{e}_n(2)]^{\frac{n-2}{n-1}}\, dt \\ &\geqslant c(n)\sqrt{\mathbf{e}_n(2)}^{\frac{n-2}{n-1}}(t_1 - t_0),\end{aligned}$$

for $\varepsilon_2(n)$ sufficiently small. □

2.8 Lipschitz approximation theorem

Proceeding as in [46], we give the proof of the following Lipschitz approximation lemma, which is a consequence of the height bound lemma.

Theorem 2.8.1 (Lipschitz approximation). *There exist three positive constants $C_8 = C_8\big(n, \nu, N, L, L_D, \Lambda, \|\nabla u\|_{L^2(\Omega)}\big)$, $\varepsilon_3 = \varepsilon_3(n)$ and $\delta_0(n)$, with the following property. If (E, u) is a Λ-minimizer of $\mathcal{F}$ in $\mathbf{C}_{9r}(x_0)$, with $x_0 \in \partial E$ and*

$$\mathbf{e}_n(x_0, 9r) < \varepsilon_3,$$

and if we set

$$M := \mathbf{C}_r(x_0) \cap \partial E, \qquad M_0 := \left\{ y \in M \;:\; \sup_{0<s<8r} \mathbf{e}_n(y, s) \leqslant \delta_0(n) \right\},$$

then there exists a Lipschitz function $f : \mathbb{R}^{n-1} \to \mathbb{R}$ with

$$\sup_{x' \in \mathbb{R}^{n-1}} \frac{|f(x')|}{r} \leqslant C_8 \mathbf{e}_n(x_0, 9r)^{\frac{1}{2(n-1)}}, \qquad \|\nabla' f\|_{L^\infty} \leqslant 1$$

such that a suitable translation Γ of the graph of f over D_r contains M_0,

$$M_0 \subset M \cap \Gamma, \qquad \text{with } \Gamma = x_0 + \{(z, f(z)) \;:\; z \in D_r\}$$

and covers a large portion of M in terms of $\mathbf{e}_n(x_0, 9r)$, that is

$$\frac{1}{r^{n-1}} \mathcal{H}^{n-1}(M \Delta \Gamma) \leqslant C_8 \mathbf{e}_n(x_0, 9r).$$

Moreover,

$$\frac{1}{r^{n-1}} \int_{D_r} |\nabla' f|^2 \, dx' \leqslant C_8 \mathbf{e}(x_0, 9r).$$

Proof. **Step 1:** Up to replacing (E,u) with $\left(\frac{E-x_0}{r}, u(x_0+r\cdot)\right)$ and f with $\frac{f(r\cdot)}{r}$, we shall prove that if (E,u) is a Λr-minimizer for $\mathcal{F}$ in $\mathbf{C}_9$, with $0 \in \partial E$ and $\mathbf{e}_n(9) \leqslant \varepsilon_3$, and

$$M = \mathbf{C}_1 \cap \partial E, \qquad M_0 = \left\{ y \in M \,:\, \sup_{0<s<8} \mathbf{e}_n(y,s) \leqslant \delta_0(n) \right\},$$

then there exists a Lipschitz function $f\colon \mathbb{R}^{n-1} \to \mathbb{R}$, with $\|\nabla f\|_\infty \leqslant 1$ such that

$$\sup_{\mathbb{R}^{n-1}} |f| \leqslant C_8 \mathbf{e}_n(9)^{\frac{1}{2(n-1)}}, \tag{2.95}$$

$$M_0 \subset M \cap \Gamma, \qquad \text{with } \Gamma = \{(z, f(z)) \,:\, z \in D_1\}$$

$$\mathcal{H}^{n-1}(M \Delta \Gamma) \leqslant C_8 \mathbf{e}_n(9) \tag{2.96}$$

$$\int_{D_1} |\nabla' f|^2 \, dx' \leqslant C_8 \mathbf{e}_n(9). \tag{2.97}$$

Let ε_2 and C_7 the constants from Lemma 2.7.4. Assuming that

$$\varepsilon_3 \leqslant \min\left\{ \varepsilon_2, \omega\left(n, \frac{1}{4}\right) \right\},$$

by Lemma 2.7.4, Lemma 2.7.1 and Proposition 1.1.22 it holds

$$\sup_{x \in \mathbf{C}_2 \cap \partial E} |qx| \leqslant \sup_{x \in \mathbf{C}_{\frac{9}{4}} \cap \partial E} |qx| \leqslant C_7 \mathbf{e}_n(9)^{\frac{1}{2(n-1)}}$$

$$0 \leqslant \mathcal{H}^{n-1}(M \cap p^{-1}(G)) - \mathcal{H}^{n-1}(G) \leqslant \mathbf{e}_n(1) \leqslant 9^{n-1} \mathbf{e}_n(9),$$

$$\left\{ x \in \mathbf{C}_2 \,:\, qx < -\frac{1}{4} \right\} \subset \mathbf{C}_2 \cap E \subset \left\{ x \in \mathbf{C}_2 \,:\, qx < \frac{1}{4} \right\},$$

for any Borel set $G \subset D_1$.

Step 2: We now prove that f is invertible on M_0, by showing that

$$|qy - qx| \leqslant L|py - px|, \quad \forall y \in M_0, \forall x \in M, \tag{2.98}$$

for some positive constant $L < 1$. Let $y \in M_0$ and $x \in M$. Scaling

$$F := \frac{E - y}{\|y - x\|}, \qquad v(\cdot) := \|y - x\|^{-\frac{1}{2}} u(x_0 + r\cdot),$$

where $\|z\| := \max\{|pz|, |qz|\}$ for $z \in \mathbb{R}^n$ (and consequently $\mathbf{C}_s(y) = \{z \in \mathbb{R}^n : \|z - y\| < s\}$), we have that F is a $(\Lambda r \|y - x\|)$-minimizer of $\mathcal{F}_{r\|y-x\|}$ in $\mathbf{C}_{\frac{9}{\|y-x\|}}$. Since $\frac{9}{\|y-x\|} > 4$ and $0 \in \partial F$, by Proposition 1.1.21, we infer

$$\mathbf{e}_n(F, 0, 4) = \mathbf{e}_n(E, y, 4\|y - x\|) \leqslant \sup_{s \in (0,8)} \mathbf{e}_n(y, s) \leqslant \delta_0.$$

Assuming $\delta_0(n) \leqslant \varepsilon_2(n)$, by Lemma 2.7.4 we get

$$\sup_{w \in \mathbf{C}_1 \cap \partial F} |qw| \leqslant C_7 \delta_0^{\frac{1}{2(n-1)}}$$

and, choosing $w = \frac{x-y}{\|x-y\|} \in \mathbf{C}_1 \cap \partial F$, we obtain (2.98) with $L = C_7\delta_0^{\frac{1}{2(n-1)}}$, depending on $n, \nu, N, L, L_D, \|\nabla u\|_{L^2(\Omega)}$. Furthermore, if δ_0 is sufficiently small, we have that $L < 1$. We define $f\colon p(M_0) \to \mathbb{R}$ as

$$f(px) = qx,$$

for $x \in M_0$. We can rewrite (2.98) as

$$|f(py) - f(px)| \leqslant L|py - px|, \quad \forall x, y \in M_0.$$

Thus $\|\nabla' f\|_\infty \leqslant 1$ and, since $M_0 \subset M$, we get

$$\sup_{x \in M_0} |f(px)| \leqslant C_7 \mathbf{e}_n(9)^{\frac{1}{2(n-1)}}.$$

By McShane's lemma (see Lemma A.3.2), we can extend f to the whole $\mathbb{R}^{n-1}$ without changing its Lipschitz constant. Up to truncating f, we may also assume that (2.95) holds. Furthermore, we have that

$$M_0 = \{(px, f(x)) \,:\, x \in M_0\} \subset M \cap \Gamma.$$

Let us prove (2.96). Since $M\backslash\Gamma \subset M\backslash M_0$, it suffices to show that

$$\mathcal{H}^{n-1}(M\backslash M_0) \leqslant C_8 \mathbf{e}_n(9). \tag{2.99}$$

By definition of M_0, for any $y \in M\backslash M_0$ there exists $s \in (0,4)$ such that

$$\delta_0 s^{n-1} < \int_{\mathbf{C}_s(y) \cap \partial E} \frac{|\nu_E - \mathbf{e}_n|^2}{2} \, d\mathcal{H}^{n-1}.$$

By [46, Corollary 5.2] there exists a family of disjoint ball contained in $\mathbf{C}_9$, $\{B(y_h, \sqrt{2}s_h)\}_{h\in\mathbb{N}}$, with centers $y_h \in M\backslash M_0$ satisfying the above estimate, such that

$$\begin{aligned}
\mathcal{H}^{n-1}(M\backslash M_0) &\leqslant c(n) \sum_{h\in\mathbb{N}} \mathcal{H}^{n-1}((M\backslash M_0) \cap B(y_h, \sqrt{2}s_h)) \\
&\leqslant c(n) \sum_{h\in\mathbb{N}} \mathcal{H}^{n-1}(M \cap B(y_h, \sqrt{2}s_h)) \\
&\leqslant c(n) \sum_{h\in\mathbb{N}} P(E; B(y_h, \sqrt{2}s_h)) \\
&\leqslant c(n) \sum_{h\in\mathbb{N}} s_h^{n-1} \\
&\leqslant c(n) \sum_{h\in\mathbb{N}} \int_{\mathbf{C}_{s_h}(y_h) \cap \partial E} \frac{|\nu_E - \mathbf{e}_n|^2}{2} \, d\mathcal{H}^{n-1} \\
&\leqslant c(n) \int_{\mathbf{C}_9 \cap \partial E} \frac{|\nu_E - \mathbf{e}_n|^2}{2} \, d\mathcal{H}^{n-1} \\
&\leqslant c(n) \mathbf{e}_n(9).
\end{aligned}$$

Therefore, we finally obtain (2.99). Now we prove that

$$\mathcal{H}^{n-1}(\Gamma\backslash M) \leqslant C_8 \mathbf{e}_n(9).$$

By the area formula (see Theorem A.2.5) and (2.99), we deduce that

$$\begin{aligned}\mathcal{H}^{n-1}(\Gamma\backslash M) &= \int_{p(\Gamma\backslash M)} \sqrt{1+|\nabla' f|^2}\, dx' \leqslant \sqrt{1+\|\nabla' f\|_\infty^2}\,\mathcal{H}^{n-1}(p(\Gamma\backslash M)) \\ &\leqslant \sqrt{2}\mathcal{H}^{n-1}(M \cap p^{-1}(p(\Gamma\backslash M))) \leqslant \sqrt{2}\mathcal{H}^{n-1}(M\backslash\Gamma) \\ &\leqslant c(n)\mathbf{e}_n(9).\end{aligned}$$

Step 3: We finaly prove (2.97). We split the integral in two addends:

$$\int_{D_1} |\nabla' f|^2\, dx' = \int_{p(M\Delta\Gamma)} |\nabla' f|^2\, dx' + \int_{p(M\cap\Gamma)} |\nabla' f|^2\, dx'$$

The first integral is easily estimated by (2.96); indeed,

$$\int_{p(M\Delta\Gamma)} |\nabla' f|^2\, dx' \leqslant \mathcal{H}^{n-1}(p(M\Delta\Gamma)) \leqslant \mathcal{H}^{n-1}(M\Delta\Gamma) \leqslant c(n)\mathbf{e}_n(9).$$

We are left to estimate the second integral. We observe that, since f is Lipschitz continuous, the normal to its graph is $\nu(z) = (-\nabla' f(z), 1)$ and $T_{(z,f(z))}\Gamma = \nu(z)^\perp$, for almost every $z \in \mathbb{R}^{n-1}$. On the other side, by Theorem 1.1.13, $T_x(\partial^* E) = \nu_E(x)^\perp$, for any $x \in M$. Thus, since $\partial^* E$ and Γ are locally $\mathcal{H}^{n-1}$-rectifiable, we infer that

$$T_x(\partial^* E) = T_x\Gamma,$$

for $\mathcal{H}^{n-1}$-a.e. $x \in M \cap \Gamma$, and, consequently, there exists $\lambda(x) \in \{-1, 1\}$ such that

$$\nu_E(x) = \lambda(x)\nu(px) = \lambda(x)\frac{(-\nabla' f(px), 1)}{\sqrt{1+|\nabla' f(px)|^2}},$$

for $\mathcal{H}^{n-1}$-a.e. $x \in M \cap \Gamma$. By Proposition 1.1.22 and Theorem A.2.5, we get

$$\begin{aligned}9^{n-1}\mathbf{e}_n(9) \geqslant \mathbf{e}_n(1) &= \int_M \frac{|\nu_E - e_n|^2}{2}\, d\mathcal{H}^{n-1} \geqslant \frac{1}{2}\int_{M\cap\Gamma} |p\nu_E|^2\, d\mathcal{H}^{n-1} \\ &= \frac{1}{2}\int_{M\cap\Gamma} \frac{|\nabla' f(px)|^2}{1+|\nabla' f(px)|^2}\, d\mathcal{H}^{n-1} = \frac{1}{2}\int_{p(M\cap\Gamma)} \frac{|\nabla' f(z)|^2}{\sqrt{1+|\nabla' f(z)|^2}}\, dz \\ &\geqslant \frac{1}{2\sqrt{2}}\int_{p(M\cap\Gamma)} |\nabla' f(z)|^2\, dz,\end{aligned}$$

where we used the fact that

$$\frac{|\nu_E - e_n|}{2} = 1 - \nu_E \cdot e_n \geqslant \frac{1-(\nu_E\cdot e_n)^2}{2} = \frac{|p\nu_E|^2 + |q\nu_E|^2 - (\nu_E\cdot e_n)^2}{2} = \frac{|p\nu_E|^2}{2}.$$

□

2.9 Reverse Poincaré inequality

In this section we shall prove a reverse Poincaré inequality following the path traced in [46, Chapter 24]. We will use the following theorem, which is useful to construct "good" comparison sets. The proof is purely geometric and can be found in [46, Lemma 24.8].

Lemma 2.9.1 (Cone-like comparison sets)**.** *If $s > 0$ and E is an open set with smooth boundary such that*

$$|qx| < \frac{1}{4}, \quad \forall x \in \mathbf{K}_s \cap \partial E,$$

$$\left\{x \in \mathbf{K}_s \,:\, qx < -\frac{1}{4}\right\} \subset \mathbf{K}_s \cap E \subset \left\{x \in \mathbf{K}_s \,:\, qx < \frac{1}{4}\right\},$$

then, for every $\lambda \in \left(0, \frac{1}{4}\right)$ and $|c| < \frac{1}{4}$, there exist $r \in \left(\frac{2}{3}, \frac{3}{4}\right)$ and an open set F of locally finite perimeter satisfying the "boundary conditions"

$$F \cap \partial\mathbf{K}_{rs} = E \cap \partial\mathbf{K}_{rs}, \tag{2.100}$$

$$\mathbf{K}_{\frac{s}{2}} \cap \partial F = \mathbf{D}_{\frac{s}{2}} \times \{c\}, \tag{2.101}$$

and the "excess-flatness estimate"

$$\begin{aligned} &P(F;\mathbf{K}_{rs}) - \mathcal{H}^{n-1}(\mathbf{D}_{rs}) \\ &\leqslant C\left\{\lambda(P(E;\mathbf{K}_s) - \mathcal{H}^{n-1}(\mathbf{D}_s)) + \frac{1}{\lambda}\int_{\mathbf{K}_s\cap\partial E}\frac{|qx-c|^2}{s^2}\,d\mathcal{H}^{n-1}(x)\right\}, \end{aligned} \tag{2.102}$$

with $C = C(n) > 0$. In fact, given s, E, λ and c as above, there exists $I \subset \left(\frac{2}{3}, \frac{3}{4}\right)$ with $|I| \geqslant \frac{1}{24}$ such that for every $r \in I$ there exists an open set F of locally finite perimeter satisfying (2.100)*,* (2.101) *and* (2.102)*.*

We will need a weak form of the reverse Poincaré inequality.

Lemma 2.9.2 (Weak reverse Poincaré inequality)**.** *If (E, u) is a Λ-minimizer of $\mathcal{F}$ in $\mathbf{C}_4$ such that*

$$|qx| < \frac{1}{8}, \quad \forall x \in \mathbf{C}_2 \cap \partial E,$$

$$\left|\left\{x \in \mathbf{C}_2\backslash E \,:\, qx < -\frac{1}{8}\right\}\right| = \left|\left\{x \in \mathbf{C}_2 \cap E \,:\, qx > \frac{1}{8}\right\}\right| = 0,$$

and if $z \in \mathbb{R}^{n-1}$ and $s > 0$ are such that

$$\mathbf{K}_s(z) \subset \mathbf{C}_2, \qquad \mathcal{H}^{n-1}(\partial E \cap \partial\mathbf{K}_s(z)) = 0, \tag{2.103}$$

then, for every $|c| < \frac{1}{4}$,

$$\begin{aligned} P(E;\mathbf{K}_{\frac{s}{2}}(z)) - \mathcal{H}^{n-1}(D_{\frac{s}{2}}(z)) \leqslant c(n,N,L)\Bigg\{\Bigg[\,\big(P(E;\mathbf{K}_s(z)) - \mathcal{H}^{n-1}(D_s(z))\big) \\ \times \int_{\mathbf{K}_s(z)\cap\partial^* E}\frac{(qx-c)^2}{s^2}d\,\mathcal{H}^{n-1}\Bigg]^{\frac{1}{2}} + \Lambda s + \int_{\mathbf{K}_s}|\nabla u|^2\,dx\Bigg\}. \end{aligned}$$

Proof. The proof of this lemma is fairly standard and it is inspired by [46, Lemma 24.9]. We start assuming that $z = 0$.
Step 1: The set function

$$\zeta(G) = P(E; \mathbf{C}_2 \cap p^{-1}(G)) - \mathcal{H}^{n-1}(G), \quad \text{for } G \subset D_2,$$

defines a Radon measure on $\mathbb{R}^{n-1}$, concentrated on D_2.
Step 2: Since E is a set of locally finite perimeter, by Theorem 1.1.10 there exist a sequence $\{E_h\}_{h\in\mathbb{N}}$ of open subsets of $\mathbb{R}^n$ with smooth boundary and a vanishing sequence $\{\varepsilon_h\}_{h\in\mathbb{N}} \subset \mathbb{R}^+$ such that

$$E_h \overset{loc}{\to} E, \quad \mathcal{H}^{n-1} \llcorner \partial E_h \to \mathcal{H}^{n-1} \llcorner \partial E, \quad \partial E_h \subset I_{\varepsilon_h}(\partial E),$$

where $I_{\varepsilon_h}(\partial E)$ is a tubolar neighborhood of ∂E with half-lenght ε_h. By Coarea formula we get

$$\mathcal{H}^{n-1}(\partial \mathbf{K}_{rs} \cap (E^{(1)} \Delta E_h)) \to 0, \quad \text{for a.e. } r \in \left(\frac{2}{3}, \frac{3}{4}\right).$$

Moreover, provided h is large enough, by $\partial E_h \subset I_{\varepsilon_h}(\partial E)$,

$$|qx| < \frac{1}{4}, \quad \forall x \in \mathbf{C}_2 \cap \partial E_h,$$

$$\left\{x \in \mathbf{C}_2 \,:\, qx < -\frac{1}{4}\right\} \subset \mathbf{C}_2 \cap E_h \subset \left\{x \in \mathbf{C}_2 \,:\, qx < \frac{1}{4}\right\}.$$

Therefore, given $\lambda \in \left(0, \frac{1}{4}\right)$ and $|c| < \frac{1}{4}$, we are in position to apply Lemma 2.9.1 to every E_h to deduce that there exists $I_h \subset \left(\frac{2}{3}, \frac{3}{4}\right)$, with $|I_h| \geqslant \frac{1}{24}$, and, for any $r \in I_h$, there exists an open subset F_h of $\mathbb{R}^n$ of locally finite perimeter such that

$$F_h \cap \partial \mathbf{K}_{rs} = E_h \cap \partial \mathbf{K}_{rs}, \tag{2.104}$$
$$\mathbf{K}_{\frac{r}{2}} \cap \partial F_h = D_{\frac{s}{2}} \times \{c\},$$

$$P(F_h; \mathbf{K}_{rs}) - \mathcal{H}^{n-1}(D_{rs}) \leqslant c(n) \Bigg\{ \lambda \left(P(E_h; \mathbf{K}_s) - \mathcal{H}^{n-1}(D_s) \right) + \frac{1}{\lambda} \int_{\mathbf{K}_s \cap \partial E_h} \frac{|qx - c|^2}{s^2} \, d\mathcal{H}^{n-1} \Bigg\}. \tag{2.105}$$

Clearly $\displaystyle\bigcap_{h\in\mathbb{N}} \bigcup_{k\geqslant h} |I_k| \geqslant \frac{1}{24} > 0$ and thus there exist a divergent subsequence $\{h_k\}_{k\in\mathbb{N}}$ and $r \in \left(\frac{2}{3}, \frac{3}{4}\right)$ such that

$$r \in \bigcap_{k\in\mathbb{N}} I_{h_k} \quad \text{and} \quad \lim_{k\to+\infty} \mathcal{H}^{n-1}(\partial \mathbf{K}_{rs} \cap (E^{(1)} \Delta E_{h_k})) = 0.$$

We write F_k in lieu of F_{h_k}. Now we test the Λ-minimality of (E,u) in $\mathbf{C}_4$ with (G_k,u), where $G_k = (F_k \cap \mathbf{K}_{rs}) \cup (E\backslash\mathbf{K}_{rs})$, as $E\Delta G_k \subset\subset \mathbf{K}_s \subset\subset B_4$. By Theorem 1.1.19 we infer:

$$\begin{aligned} P(E;\mathbf{K}_{rs}) &\leqslant P(G_k;\mathbf{K}_{rs}) + \Lambda|(E\Delta F_k)\cap\mathbf{K}_{rs}| + \int_{\mathbf{K}_{rs}} G(x,u,\nabla u)[\mathbb{1}_{G_k} - \mathbb{1}_E]\,dx \\ &\leqslant P(F_k;\mathbf{K}_{rs}) + \sigma_k + \Lambda|(E\Delta F_k)\cap\mathbf{K}_{rs}| \\ &\quad + c(n,N,L)\int_{\mathbf{K}_{rs}} (|\nabla u|^2+1)\,dx, \end{aligned}$$

with $\sigma_k = \mathcal{H}^{n-1}(\partial\mathbf{K}_{rs}\cap(E^{(1)}\Delta F_k)) = \mathcal{H}^{n-1}(\partial\mathbf{K}_{rs}\cap(E^{(1)}\Delta E_{h_k})) \to 0$, thanks to (2.104), as $k\to+\infty$. Thus, since ζ is increasing and $r \geqslant \frac{2}{3}$, by (2.105), we deduce that

$$\begin{aligned} &P(E;\mathbf{K}_{\frac{s}{2}}) - \mathcal{H}^{n-1}(D_{\frac{s}{2}}) = \zeta(D_{\frac{s}{2}}) \leqslant \zeta(D_{rs}) = P(E;\mathbf{K}_{rs}) - \mathcal{H}^{n-1}(D_{rs}) \\ &\leqslant P(F_k;\mathbf{K}_{rs}) - \mathcal{H}^{n-1}(D_{rs}) \\ &+ \sigma_k + \Lambda|(E\Delta F_k)\cap\mathbf{K}_{rs}| + c(n,N,L)\int_{\mathbf{K}_{rs}} (|\nabla u|^2+1)\,dx \\ &\leqslant c(n)\left\{\lambda\left(P(E_{h_k};\mathbf{K}_s) - \mathcal{H}^{n-1}(D_s)\right) + \frac{1}{\lambda}\int_{\mathbf{K}_s\cap\partial E_{h_k}} \frac{|qx-c|^2}{s^2}\,d\mathcal{H}^{n-1}\right\} \\ &+ c(n,N,L)\left(\Lambda s^{n-1} + \int_{\mathbf{K}_s}|\nabla u|^2\,dx\right). \end{aligned}$$

Letting $k\to+\infty$, (2.103) implies that $P(E_{h(k)};\mathbf{K}_s) \to P(E;\mathbf{K}_s)$ and therefore

$$\begin{aligned} &P(E;\mathbf{K}_{\frac{s}{2}}) - \mathcal{H}^{n-1}(D_{\frac{s}{2}}) \\ &\leqslant c(n)\left\{\lambda\left(P(E;\mathbf{K}_s) - \mathcal{H}^{n-1}(D_s)\right) + \frac{1}{\lambda}\int_{\mathbf{K}_s\cap\partial E} \frac{|qx-c|^2}{s^2}\,d\mathcal{H}^{n-1}\right\} \\ &+ c(n,N,L)\left(\Lambda s^{n-1} + \int_{\mathbf{K}_s}|\nabla u|^2\,dx\right), \end{aligned} \tag{2.106}$$

for any $\lambda \in \left(0,\frac{1}{4}\right)$. If $\lambda > \frac{1}{4}$,

$$\begin{aligned} &P(E;\mathbf{K}_{\frac{s}{2}}) - \mathcal{H}^{n-1}(D_{\frac{s}{2}}) = \zeta(D_{\frac{s}{2}}) \leqslant \zeta(D_{rs}) \\ &\leqslant 4\lambda P(E;\mathbf{K}_{rs}) - \mathcal{H}^{n-1}(D_{rs}) \leqslant c(n)\lambda\left(P(E;\mathbf{K}_s) - \mathcal{H}^{n-1}(D_s)\right) \end{aligned}$$

and thus (2.106) holds true for $\lambda > 0$, provided we choose $c(n) \geqslant 4$. Minimizing over λ, we get the thesis. □

Theorem 2.9.3 (Reverse Poincaré Inequality)**.** *There exists a positive constant $C_9 = C_9(n,N,L)$ such that if (E,u) be a Λ-minimizer of $\mathcal{F}$ in $\mathbf{C}_{4r}(x_0,\nu)$ with $x_0 \in \partial E$ and*

$$\mathbf{e}(x_0,4r,\nu) < \omega\left(n,\frac{1}{8}\right),$$

then

$$\mathbf{e}(x_0,r,\nu) \leqslant C_9\left(\frac{1}{r^{n+1}}\int_{\partial E\cap\mathbf{C}_{2r}(x_0,\nu)} |\langle\nu,x-x_0\rangle - c|^2 d\mathcal{H}^{n-1}\right.$$

$$+ \Lambda r + \frac{1}{r^{n-1}} \int_{\mathbf{K}_{2r}} |\nabla u|^2 \, dx \Big),$$

for every $c \in \mathbb{R}$.

Proof. Up to replacing (E, u) with $\left(\frac{E-x_0}{r}, r^{-\frac{1}{2}} u(x_0 + ry) \right)$ (see Lemma 2.5.1) we may assume that (E, u) is a Λr-minimizer of $\mathcal{F}_r$ in $\mathbf{C}_4$, $0 \in \partial E$ and, by Proposition 1.1.21,

$$\mathbf{e}_n(4) \leqslant \omega \left(n, \frac{1}{8} \right).$$

Applying Lemma 2.7.1 and Lemma 2.7.2, we get that

$$|qx| < \frac{1}{4}, \quad \forall x \in \mathbf{C}_2 \cap \partial E,$$

$$\left| \left\{ x \in \mathbf{C}_2 \backslash E \,:\, qx < -\frac{1}{8} \right\} \right| = \left| \left\{ x \in \mathbf{C}_2 \cap E \,:\, qx > \frac{1}{8} \right\} \right|,$$

$$\mathcal{H}^{n-1}(G) = \int_{\mathbf{C}_2 \cap \partial^* E \cap p^{-1}(G)} \nu_E \cdot e_n \, d\mathcal{H}^{n-1}, \quad \forall G \subset D_2.$$

Since

$$\begin{aligned} \mathbf{e}_n(1) &= \int_{\mathbf{C}_1 \cap \partial^* E} (1 - \nu_E \cdot e_n) \, d\mathcal{H}^{n-1} = P(E; \mathbf{C}_1) - \int_{\mathbf{C}_1 \cap \partial^* E} (\nu_E \cdot e_n) \, d\mathcal{H}^{n-1} \\ &= P(E; \mathbf{C}_1) - \mathcal{H}^{n-1}(D_1), \end{aligned}$$

then our aim is to show

$$P(E; \mathbf{C}_1) - \mathcal{H}^{n-1}(D_1) \leqslant C_9 \left(\int_{\mathbf{C}_2 \cap \partial E} |qx - c|^2 \, d\mathcal{H}^{n-1} + \Lambda r + \int_{\mathbf{K}_2} |\nabla u|^2 \, dx \right),$$

for any $c \in \mathbb{R}$. Actually it suffices to prove it only for $|c| < \frac{1}{4}$; indeed, for $|c| \geqslant \frac{1}{4}$, we have:

$$\int_{\mathbf{C}_2 \cap \partial E} |qx - c|^2 \, d\mathcal{H}^{n-1} \geqslant \int_{\mathbf{C}_2 \cap \partial E} (|c| - |qx|)^2 \, d\mathcal{H}^{n-1} \geqslant \frac{P(E; \mathbf{C}_2)}{64} \geqslant \frac{P(E; \mathbf{C}_1)}{64}.$$

Step 2: the set function $\zeta(G) = P(E; \mathbf{C}_2 \cap p^{-1}(G)) - \mathcal{H}^{n-1}(G)$, for $G \subset D_2$, defines a Radon measure on $\mathbb{R}^{n-1}$, concentrated on D_2. We apply Lemma 2.9.2 to E in every cylinder $\mathbf{K}_s(z)$ with $z \in \mathbb{R}^{n-1}$ and $s > 0$ such that

$$D_{2s}(z) \subset D_2, \qquad \mathcal{H}^{n-1}(\partial E \cap \partial \mathbf{K}_{2s}(z)) = 0, \tag{2.107}$$

to get that

$$\zeta(D_s(z)) \leqslant C(n, N, L) \left\{ (\zeta(D_{2s}(z)) h)^{\frac{1}{2}} + \Lambda r s^{n-1} + \int_{\mathbf{K}_{2s}(z)} |\nabla u|^2 \, dx \right\},$$

where

$$h := \inf_{|c| < \frac{1}{4}} \int_{\mathbf{C}_2 \cap \partial E} |qx - c|^2 \, d\mathcal{H}^{n-1}.$$

Multiplying by s^2 and using an approximation argument to remove the second assumption in (2.107), we obtain:

$$s^2\zeta(D_s(z)) \leqslant c(n,N,L)\left(\sqrt{s^2\zeta(D_{2s}(z))h} + \Lambda r + \int_{\mathbf{K}_{2s}(z)} |\nabla u|^2\,dx\right), \quad (2.108)$$

for $D_{2s}(z) \subset D_2$, where we used that $s < 1$. In order to prove the thesis, we use a covering argument by setting

$$Q = \sup_{D_{2s}(z)\subset D_2} s^2\zeta(D_s(z)) < +\infty.$$

We cover $D_s(z)$ by finitely many balls $\{D\left(z_k, \frac{s}{4}\right)\}_{k\in\{1,\dots,\tilde{N}\}}$ with centers $z_k \in D_s(z)$. Of course, this can be done with $\tilde{N} \leqslant \tilde{N}(n)$, for some $\tilde{N}(n) \in \mathbb{N}$. Hence, by the sub-additivity of ζ and (2.108) for $\frac{s}{4}$, since $D_s(z_k) \subset D_2$, we have:

$$\begin{aligned} s^2\zeta(D_s(z)) &\leqslant s^2 \sum_{k=1}^{\tilde{N}} \zeta\left(D_{\frac{s}{4}}(z_k)\right) = 16 \sum_{k=1}^{\tilde{N}} \left(\frac{s}{4}\right)^2 \zeta\left(D_{\frac{s}{4}}(z_k)\right) \\ &\leqslant c(n,N,L) \sum_{k=1}^{\tilde{N}} \left(\sqrt{\left(\frac{s}{2}\right)^2 \zeta\left(D_{\frac{s}{2}}(z_k)\right) h} + \Lambda r + \int_{\mathbf{K}_{2s}(z)} |\nabla u|^2\,dx\right) \\ &\leqslant c(n,N,L)\left(\sqrt{Qh} + \Lambda r + \int_{\mathbf{K}_{2s}(z)} |\nabla u|^2\,dx\right). \end{aligned}$$

Passing to the supremum for $D_{2s}(z) \subset D_2$ we infer that

$$Q \leqslant c(n,N,L)\left(\sqrt{Qh} + \Lambda r + \int_{\mathbf{K}_2} |\nabla u|^2\,dx\right).$$

If $\sqrt{Qh} \leqslant \Lambda r + \int_{\mathbf{K}_2} |\nabla u|^2\,dx$, then $Q \leqslant c(n,N,L)\left(\Lambda r + \int_{\mathbf{K}_2} |\nabla u|^2\,dx\right)$. If $\sqrt{Qh} > \Lambda r + \int_{\mathbf{K}_2} |\nabla u|^2\,dx$, then $Q \leqslant c(n,N,L)\sqrt{Qh}$ and thus $Q \leqslant c(n,N,L)h$. In both cases we obtain:

$$Q \leqslant c(n,N,L)\left(h + \Lambda r + \int_{\mathbf{K}_2} |\nabla u|^2\,dx\right),$$

which leads to the thesis. □

2.10 Weak Euler-Lagrange equation

The last ingredient to prove the excess improvement is the following Euler-Lagrange equation that we state for Λr-minimizers of the rescaled functional $\mathcal{F}_r$. For the sake of simplicity we will denote with A_1 the matrix whose entries are a_{hk}, A_2 the vector of components a_h, $A_3 = a$ and similarly for B_i, $i = 1,2,3$. Accordingly, we can write

$$\mathcal{F}_r(w;D) = \int_{B_1} \left[F_r(x,w,\nabla w) + \mathbb{1}_D G_r(x,w,\nabla w)\right] dx$$

$$= \int_{B_1} \big[(A_{1r} + \mathbb{1}_D B_{1r})\nabla w \cdot \nabla w + \sqrt{r}(A_{2r} + \mathbb{1}_D B_{2r}) \cdot \nabla w \\ + r(A_{3r} + \mathbb{1}_D B_{3r})\big]\, dx,$$

where $r > 0$, $x_0 \in \Omega$, $A_{ir} := A_i(x_0 + ry, \sqrt{r}w)$, $B_{ir} := B_i(x_0 + ry, \sqrt{r}w)$, for $i = 1, 2, 3$. The argument used to prove the next result is similar to the one in [7, Theorem 7.35]. We recall a useful result that can be found in [39, Theorem 3.2].

Theorem 2.10.1. *Let $A \subset \mathbb{R}^n$ be an open set, $E \subset \mathbb{R}^n$ be a set of locally finite perimeter and $\Phi_t(x) := x + tX(x)$, for some fixed $X \in C^1_c(A; \mathbb{R}^n)$, be a local variation in A, i.e. $\{x \neq \Phi_t(x)\} \subset K \subset A$, for some compact set $K \subset A$ and for $|t| < \varepsilon_0$. Then*

$$\lim_{t\to 0^+} \frac{|\Phi_t(E)\Delta E|}{t} \leqslant \int_{\partial E} |X \cdot \nu_E|\, d\mathcal{H}^{n-1}.$$

Theorem 2.10.2 (Weak Euler-Lagrange equation)**.** *Let (E, u) be a Λr-minimizer of $\mathcal{F}_r$ in B_1. For every vector field $X \in C^1_c(B_1; \mathbb{R}^n)$ and for some constant $C_{10} = C_{10}(N, L_D, \sup |X|, \sup |\nabla X|) > 0$ it holds*

$$\int_{\partial E} \mathrm{div}_\tau X\, d\mathcal{H}^{n-1} \leqslant C_{10} \int_{B_1} \big(|\nabla u|^2 + r\big)\, dx + \Lambda r \int_{\partial E} |X|\, d\mathcal{H}^{n-1}, \tag{2.109}$$

where div_τ denotes the tangential divergence on ∂E, i.e.

$$\mathrm{div}_\tau X = \mathrm{div} X - \nu_E \cdot \nabla X \nu_E.$$

Proof. Let us fix $X \in C^1_c(B_1, \mathbb{R}^n)$. We set $\Phi_t(x) := x + tX(x)$, $E_t := \Phi_t(E)$ and $u_t := u \circ \Phi_t^{-1}$, for any $t > 0$. From the Λr-minimality it follows that

$$[P(E_t; B_1) - P(E; B_1)] + \Lambda r |E_t \Delta E| \\ + \int_{B_1} [F_r(y, u_t, \nabla u_t) + \mathbb{1}_{E_t}(y) G_r(y, u_t, \nabla u_t)]\, dy \\ - \int_{B_1} [F_r(x, u, \nabla u) + \mathbb{1}_E(x) G_r(x, u, \nabla u)]\, dx \geqslant 0. \tag{2.110}$$

In order to obtain (2.109) we will divide by t and pass to the upper limit as $t \to 0^+$. Let us study these terms separately. The first variation of the area gives

$$\lim_{t\to 0^+} \frac{1}{t}[P(E_t; B_1) - P(E; B_1)] = \int_{\partial E} \mathrm{div}_\tau X\, d\mathcal{H}^{n-1}. \tag{2.111}$$

In regard to the second term, we apply Theorem 2.10.1, obtaining

$$\lim_{t\to 0^+} \frac{|E_t \Delta E|}{t} \leqslant \int_{\partial E} |X \cdot \nu_E|\, d\mathcal{H}^{n-1}. \tag{2.112}$$

In the first bulk term we make the change of variables $y = \Phi_t(x)$ with $x \in B_1$ and $t > 0$, taking into account that

$$\nabla \Phi_t^{-1}(\Phi_t(x)) = I - t\nabla X(x) + o(t), \quad \mathrm{J}\Phi_t(x) = 1 + t\mathrm{div} X(x) + o(t).$$

Thus we gain

$$\begin{aligned}
&\int_{B_1} \big[F_r(y, u_t, \nabla u_t) + \mathbb{1}_{E_t}(y) G_r(y, u_t, \nabla u_t)\big]\, dy \\
&= \int_{B_1} \big[F_r(\Phi_t(x), u, \nabla u) + \mathbb{1}_E(x) G_r(\Phi_t(x), u, \nabla u)\big](1 + t\mathrm{div} X)\, dx \\
&- t\int_{B_1} \big[2(C_1 \nabla u \nabla X)\cdot \nabla u + \sqrt{r} C_2 \cdot (\nabla u \nabla X)\big]\, dx + o(t),
\end{aligned}$$

where we set

$$C_i := \tilde{A}_{ir} + \mathbb{1}_E \tilde{B}_{ir} = A_{ir}(\Phi_t(x), u) + \mathbb{1}_E(x) B_{ir}(\Phi_t(x), u),$$

for $i = 1, 2, 3$. By simple calculations we obtain

$$\begin{aligned}
&\int_{B_1} \big[F_r(y, u_t, \nabla u_t) + \mathbb{1}_{E_t}(y) G_r(y, u_t, \nabla u_t)\big]\, dy \\
&- \int_{B_1} \big[F_r(x, u, \nabla u) + \mathbb{1}_E(x) G_r(x, u, \nabla u)\big]\, dx \\
&= \int_{B_1} \big\{F_r(\Phi_t(x), u, \nabla u) + \mathbb{1}_E(x) G_r(\Phi_t(x), u, \nabla u) \\
&- [F_r(x, u, \nabla u) + \mathbb{1}_E(x) G_r(x, u, \nabla u)]\big\}\, dx \\
&+ t\bigg[\int_{B_1} \big[F_r(\Phi_t(x), u, \nabla u) + \mathbb{1}_E(x) G_r(\Phi_t(x), u, \nabla u)\big]\mathrm{div} X\, dx \\
&- \int_{B_1} \big[2(C_1 \nabla u \nabla X)\cdot \nabla u + \sqrt{r} C_2 \cdot (\nabla u \nabla X)\big]\, dx\bigg] + o(t).
\end{aligned}$$

Let us estimate the first of the three terms. By Lipschitz continuity and Young's inequality we get

$$\begin{aligned}
&\int_{B_1} \Big\{F_r(\Phi_t(x), u, \nabla u) + \mathbb{1}_E(x) G_r(\Phi_t(x), u, \nabla u) \\
&- [F_r(x, u, \nabla u) + \mathbb{1}_E(x) G_r(x, u, \nabla u)]\Big\}\, dx \\
&\leqslant c(L_D) t \int_{B_1} |X|[|\nabla u|^2 + \sqrt{r}|\nabla u| + r]\, dx \leqslant c(L_D) t \int_{B_1} |X|[|\nabla u|^2 + r]\, dx.
\end{aligned}$$

Finally, dividing by t and passing to the upper limit as $t \to 0^+$ we infer

$$\begin{aligned}
&\limsup_{t\to 0^+} \frac{1}{t}\bigg[\int_{B_1} [F_r(y, u_t, \nabla u_t) + \mathbb{1}_{E_t}(y) G_r(y, u_t, \nabla u_t)]\, dy \\
&- \int_{B_1} [F_r(x, u, \nabla u) + \mathbb{1}_E G_r(x, u, \nabla u)]\, dx\bigg] \\
&\leqslant c(L_D) \int_{B_1} |X|[|\nabla u|^2 + r]\, dx + \int_{B_1} [F_r(x, u, \nabla u) + \mathbb{1}_E G_r(x, u, \nabla u)]\mathrm{div} X\, dx \\
&- \int_{B_1} \big[2\big((A_{1r} + \mathbb{1}_E B_{1r})\nabla u \nabla X\big)\cdot \nabla u + \sqrt{r}(A_{2r} + \mathbb{1}_E B_{2r})\cdot (\nabla u \nabla X)\big]\, dx.
\end{aligned} \tag{2.113}$$

Passing to the upper limit as $t \to 0^+$ in (2.110) and putting (2.111), (2.112), (2.113) together we get

$$\begin{aligned}
&\int_{\partial E} \mathrm{div}_\tau X \, d\mathcal{H}^{n-1} \\
&\leqslant c(L_D) \int_{B_1} |X|[|\nabla u|^2 + r] \, dx + \left| \int_{B_1} [F_r(x,u,\nabla u) + \mathbb{1}_E G_r(x,u,\nabla u)] \mathrm{div} X \, dx \right| \\
&+ \int_{B_1} \left| 2\big((A_{1r} + \mathbb{1}_E B_{1r})\nabla u \nabla X\big) \cdot \nabla u + \sqrt{r}(A_{2r} + \mathbb{1}_E B_{2r}) \cdot (\nabla u \nabla X) \right| dx \\
&+ \Lambda r \int_{\partial E} |X| \, d\mathcal{H}^{n-1} \leqslant C \int_{B_1} \big(|\nabla u|^2 + r\big) \, dx + \Lambda r \int_{\partial E} |X| \, d\mathcal{H}^{n-1},
\end{aligned}$$

where $C = C(N, L_D, \sup|X|, \sup|\nabla X|)$. □

2.11 Excess improvement

The last ingredient we need to prove the main theorem of the chapter is the excess improvement theorem. Its proof is inspired by [35, Proposition 4.10].

Theorem 2.11.1 (Excess improvement). *For every $\tau \in \big(0, \frac{1}{2}\big)$ and $M > 0$ there exists a constant $\varepsilon_4 = \varepsilon_4(\tau, M) \in (0,1)$ such that if (E,u) is a Λ-minimizer of $\mathcal{F}$ in $B_r(x_0)$ with $x_0 \in \partial E$ and*

$$\mathbf{e}(x_0, r) \leqslant \varepsilon_4, \quad \mathcal{D}(x_0, r) + r \leqslant M\mathbf{e}(x_0, r),$$

then there exists a positive constant $C_{11} = C_{11}\big(n, \nu, N, L, L_D, \Lambda, \|\nabla u\|_{L^2(\Omega)}\big)$ such that

$$\mathbf{e}(x_0, \tau r) \leqslant C_{11}(\tau^2 \mathbf{e}(x_0, r) + \mathcal{D}(x_0, 2\tau r) + \tau r).$$

Proof. Without loss of generality we may assume that $\tau < \frac{1}{8}$. Let us rescale and assume by contradiction that there exist an infinitesimal sequence $\{\varepsilon_h\}_{h\in\mathbb{N}} \subset \mathbb{R}^+$, a sequence $\{r_h\}_{h\in\mathbb{N}} \subset \mathbb{R}^+$ and a sequence $\{(E_h, u_h)\}_{h\in\mathbb{N}}$ of Λr_h-minimizers of $\mathcal{F}_{r_h}$ in B_1, with equibounded energies, such that, denoting by $\mathbf{e}_h$ the excess of E_h and by $\mathcal{D}_h$ the rescaled Dirichlet integral of u_h, we have

$$\mathbf{e}_h(0,1) = \varepsilon_h, \quad \mathcal{D}_h(0,1) + r_h \leqslant M\varepsilon_h \tag{2.114}$$

and

$$\mathbf{e}_h(0,\tau) > C_{11}(\tau^2 \mathbf{e}(0,1) + \mathcal{D}(0,2\tau) + \tau r_h),$$

with some positive constant C_{11} to be chosen. Up to rotating each E_h we may also assume that, for all $h \in \mathbb{N}$,

$$\mathbf{e}_h(0,1) = \frac{1}{2} \int_{\partial E_h \cap B_1} |\nu_{E_h} - e_n|^2 \, d\mathcal{H}^{n-1}.$$

Step 1. Thanks to the Lipschitz approximation theorem, for h sufficiently large, there exists a 1-Lipschitz function $f_h\colon \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$\sup_{\mathbb{R}^{n-1}}|f_h| \leqslant C_8\varepsilon_h^{\frac{1}{2(n-1)}}, \quad \mathcal{H}^{n-1}((\partial E_h \Delta \Gamma_{f_h}) \cap B_{\frac{1}{2}}) \leqslant C_8\varepsilon_h, \quad \int_{D_{\frac{1}{2}}}|\nabla' f_h|^2\,dx' \leqslant C_8\varepsilon_h. \tag{2.115}$$

We define

$$g_h(x') := \frac{f_h(x') - a_h}{\sqrt{\varepsilon_h}}, \quad \text{where} \quad a_h = \fint_{D_{\frac{1}{2}}} f_h\,dx',$$

and we assume, up to a subsequence, that $\{g_h\}_{h\in\mathbb{N}}$ converges weakly in $H^1(D_{\frac{1}{2}})$ and strongly in $L^2(D_{\frac{1}{2}})$ to a function g.

We prove that g is harmonic in $D_{\frac{1}{2}}$. It is enough to show that

$$\lim_{h\to+\infty}\frac{1}{\sqrt{\varepsilon_h}}\int_{D_{\frac{1}{2}}}\frac{\nabla' f_h\cdot\nabla'\phi}{\sqrt{1+|\nabla' f_h|^2}}\,dx' = 0, \tag{2.116}$$

for all $\phi \in C^1_c(D_{\frac{1}{2}})$; indeed, if $\phi \in C^1_c(D_{\frac{1}{2}})$, by weak convergence we have

$$\int_{D_{\frac{1}{2}}}\nabla' g\cdot\nabla'\phi\,dx' = \lim_{h\to+\infty}\frac{1}{\sqrt{\varepsilon_h}}\int_{D_{\frac{1}{2}}}\nabla' f_h\cdot\nabla'\phi dx'$$
$$= \lim_{h\to+\infty}\frac{1}{\sqrt{\varepsilon_h}}\left\{\int_{D_{\frac{1}{2}}}\frac{\nabla' f_h\cdot\nabla'\phi}{\sqrt{1+|\nabla' f_h|^2}}\,dx' + \int_{D_{\frac{1}{2}}}\left[\nabla' f_h\cdot\nabla'\phi - \frac{\nabla' f_h\cdot\nabla'\phi}{\sqrt{1+|\nabla' f_h|^2}}\right]dx'\right\}.$$

Using the Lipschitz continuity of f_h and the third inequality in (2.115), we infer that the second term in the previous equality is infinitesimal:

$$\limsup_{h\to+\infty}\frac{1}{\sqrt{\varepsilon_h}}\left|\int_{D_{\frac{1}{2}}}\left[\nabla' f_h\cdot\nabla'\phi - \frac{\nabla' f_h\cdot\nabla'\phi}{\sqrt{1+|\nabla' f_h|^2}}\right]dx'\right|$$
$$\leqslant \limsup_{h\to+\infty}\frac{1}{\sqrt{\varepsilon_h}}\int_{D_{\frac{1}{2}}}|\nabla' f_h||\nabla'\phi|\frac{\sqrt{1+|\nabla' f_h|^2}-1}{\sqrt{1+|\nabla' f_h|^2}}\,dx'$$
$$\leqslant \limsup_{h\to+\infty}\frac{1}{\sqrt{\varepsilon_h}}\int_{D_{\frac{1}{2}}}|\nabla'\phi||\nabla' f_h|^2\,dx' \leqslant \lim_{h\to+\infty} C_8\,\|\nabla'\phi\|_\infty\sqrt{\varepsilon_h} = 0.$$

Therefore, we should prove (2.116). We fix $\delta > 0$ so that $\operatorname{spt}\phi\times[-2\delta, 2\delta] \subset B_{\frac{1}{2}}$, choose a cut-off function $\psi\colon \mathbb{R} \to [0,1]$, with $\operatorname{spt}\psi \subset (-2\delta, 2\delta)$, $\psi = 1$ in $(-\delta,\delta)$ and apply to E_h the weak Euler-Lagrange equation with $X = \phi\psi e_n$. By the height bound, for h sufficiently large it holds that $\partial E_h \cap B_{\frac{1}{2}} \subset D_{\frac{1}{2}} \times (-\delta,\delta)$.

Plugging X in the weak Euler-Lagrange equation and using the assumption in (2.114), we have

$$-\int_{\partial E_h\cap B_{\frac{1}{2}}}(\nu_{E_h}\cdot e_n)(\nabla'\phi\cdot\nu'_{E_h})\,d\mathcal{H}^{n-1}$$
$$\leqslant c(N, L_D, \phi, \psi)\int_{B_{\frac{1}{2}}}\big(|\nabla u_h|^2 + r_h\big)\,dx + \Lambda r_h\int_{\partial E_h\cap B_{\frac{1}{2}}}|\phi\psi|\,d\mathcal{H}^{n-1}$$

$$\leqslant c(n, N, \Lambda, L_D, M, \phi, \psi)\varepsilon_h.$$

Therefore, if we replace ϕ by $-\phi$, we infer

$$\lim_{h\to+\infty} \frac{1}{\sqrt{\varepsilon_h}} \int_{\partial E_h \cap B_{\frac{1}{2}}} (\nu_{E_h} \cdot e_n)(\nabla'\phi \cdot \nu'_{E_h})\, d\mathcal{H}^{n-1} = 0. \tag{2.117}$$

Decomposing $\partial E_h \cap B_{\frac{1}{2}} = \big([\Gamma_{f_h} \cup (\partial E_h \backslash \Gamma_{f_h})] \backslash (\Gamma_{f_h} \backslash \partial E_h)\big) \cap B_{\frac{1}{2}}$, we deduce

$$\begin{aligned}
&- \frac{1}{\sqrt{\varepsilon_h}} \int_{\partial E_h \cap B_{\frac{1}{2}}} (\nu_{E_h} \cdot e_n)(\nabla'\phi \cdot \nu'_{E_h})\, d\mathcal{H}^{n-1} \\
&= \frac{1}{\sqrt{\varepsilon_h}} \Big[- \int_{\Gamma_{f_h} \cap B_{\frac{1}{2}}} (\nu_{E_h} \cdot e_n)(\nabla'\phi \cdot \nu'_{E_h})\, d\mathcal{H}^{n-1} \\
&- \int_{(\partial E_h \backslash \Gamma_{f_h}) \cap B_{\frac{1}{2}}} (\nu_{E_h} \cdot e_n)(\nabla'\phi \cdot \nu'_{E_h})\, d\mathcal{H}^{n-1} \\
&+ \int_{(\Gamma_{f_h} \backslash \partial E_h) \cap B_{\frac{1}{2}}} (\nu_{E_h} \cdot e_n)(\nabla'\phi \cdot \nu'_{E_h})\, d\mathcal{H}^{n-1} \Big].
\end{aligned}$$

Since by the second inequality in (2.115) we have

$$\left| \frac{1}{\sqrt{\varepsilon_h}} \int_{(\partial E_h \backslash \Gamma_{f_h}) \cap B_{\frac{1}{2}}} (\nu_{E_h} \cdot e_n)(\nabla'\phi \cdot \nu'_{E_h})\, d\mathcal{H}^{n-1} \right| \leqslant C_8 \sqrt{\varepsilon_h} \sup_{\mathbb{R}^{n-1}} |\nabla'\phi|,$$

$$\left| \frac{1}{\sqrt{\varepsilon_h}} \int_{(\Gamma_{f_h} \backslash \partial E_h) \cap B_{\frac{1}{2}}} (\nu_{E_h} \cdot e_n)(\nabla'\phi \cdot \nu'_{E_h})\, d\mathcal{H}^{n-1} \right| \leqslant C_8 \sqrt{\varepsilon_h} \sup_{\mathbb{R}^{n-1}} |\nabla'\phi|,$$

then by (2.117) and the area formula, we infer

$$\begin{aligned}
0 &= \lim_{h\to+\infty} \frac{-1}{\sqrt{\varepsilon_h}} \int_{\Gamma_{f_h} \cap B_{\frac{1}{2}}} (\nu_{E_h} \cdot e_n)(\nabla'\phi \cdot \nu'_{E_h})\, d\mathcal{H}^{n-1} \\
&= \lim_{h\to+\infty} \frac{1}{\sqrt{\varepsilon_h}} \int_{D_{\frac{1}{2}}} \frac{\nabla' f_h \cdot \nabla'\phi}{\sqrt{1+|\nabla' f_h|^2}}\, dx'.
\end{aligned}$$

This proves that g is harmonic.

Step 2. The proof of this step now follows exactly as in [35] using the height bound lemma and the reverse Poincaré inequality. We give it here to be thorough. By the mean value property of harmonic functions (see for example [46, Lemma 25.1], Jensen's inequality, semicontinuity and the third inequality in (2.115) we deduce that

$$\begin{aligned}
&\lim_{h\to+\infty} \frac{1}{\varepsilon_h} \int_{D_{2\tau}} |f_h(x') - (f_h)_{2\tau} - (\nabla' f_h)_{2\tau} \cdot x'|^2\, dx' \\
&= \int_{D_{2\tau}} |g(x') - (g)_{2\tau} - (\nabla' g)_{2\tau} \cdot x'|^2\, dx'
\end{aligned}$$

$$
\begin{aligned}
&= \int_{D_{2\tau}} |g(x') - g(0) - \nabla' g(0) \cdot x'|^2 \, dx' \\
&\leqslant c(n)\tau^{n-1} \sup_{x' \in D_{2\tau}} |g(x') - g(0) - \nabla' g(0) \cdot x'|^2 \\
&\leqslant c(n)\tau^{n+3} \int_{D_{\frac{1}{2}}} |\nabla' g|^2 \, dx' \leqslant c(n)\tau^{n+3} \liminf_{h \to +\infty} \int_{D_{\frac{1}{2}}} |\nabla' g_h|^2 \, dx' \\
&\leqslant \tilde{C}(n, C_8)\tau^{n+3}.
\end{aligned}
$$

On one hand, using the area formula, the mean value property, the previous inequality and setting

$$
c_h := \frac{(f_h)_{2\tau}}{\sqrt{1 + |(\nabla' f_h)_{2\tau}|^2}}, \quad \nu_h := \frac{(-(\nabla' f_h)_{2\tau}, 1)}{\sqrt{1 + |(\nabla' f_h)_{2\tau}|^2}},
$$

we have

$$
\begin{aligned}
&\limsup_{h \to +\infty} \frac{1}{\varepsilon_h} \int_{\partial E_h \cap \Gamma_{f_h} \cap B_{2\tau}} |\nu_h \cdot x - c_h|^2 \, d\mathcal{H}^{n-1} \\
&= \limsup_{h \to +\infty} \frac{1}{\varepsilon_h} \int_{\partial E_h \cap \Gamma_{f_h} \cap B_{2\tau}} \frac{|-(\nabla' f_h)_{2\tau} \cdot x' + f_h(x') - (f_h)_{2\tau}|^2}{1 + |(\nabla' f_h)_{2\tau}|^2} \sqrt{1 + |\nabla' f_h(x')|^2} \, dx' \\
&\leqslant \lim_{h \to +\infty} \frac{1}{\varepsilon_h} \int_{D_{2\tau}} |f_h(x') - (f_h)_{2\tau} - (\nabla' f_h)_{2\tau} \cdot x'|^2 \, dx' \leqslant \tilde{C}(n, C_8)\tau^{n+3}.
\end{aligned}
$$

On the other hand, arguing as in step 1, we immediately get from the height bound lemma and the first two inequalities in (2.115) that

$$
\lim_{h \to +\infty} \frac{1}{\varepsilon_h} \int_{(\partial E_h \setminus \Gamma_{f_h}) \cap B_{2\tau}} |\nu_h \cdot x - c_h|^2 \, d\mathcal{H}^{n-1} = 0.
$$

Hence, we conclude that

$$
\limsup_{h \to +\infty} \frac{1}{\varepsilon_h} \int_{\partial E_h \cap B_{2\tau}} |\nu_h \cdot x - c_h|^2 \, d\mathcal{H}^{n-1} \leqslant \tilde{C}(n, C_8)\tau^{n+3}. \tag{2.118}
$$

We claim that the sequence $\{\mathbf{e}_h(0, 2\tau, \nu_h)\}_{h \in \mathbb{N}}$ is infinitesimal; indeed, by the definition of excess, Jensen's inequality and the third inequality in (2.115) we have

$$
\begin{aligned}
&\limsup_{h \to +\infty} \int_{\partial E_h \cap B_{2\tau}} |\nu_{E_h} - \nu_h|^2 \, d\mathcal{H}^{n-1} \\
&\leqslant \limsup_{h \to +\infty} \left[2 \int_{\partial E_h \cap B_{2\tau}} |\nu_{E_h} - e_n|^2 \, d\mathcal{H}^{n-1} + 2|e_n - \nu_h|^2 \mathcal{H}^{n-1}(\partial E_h \cap B_{2\tau}) \right] \\
&\leqslant \limsup_{h \to +\infty} \left[4\varepsilon_h + 2\mathcal{H}^{n-1}(B_{2\tau}) \frac{|((\nabla' f_h)_{2\tau}, \sqrt{1 + |(\nabla' f_h)_{2\tau}|^2} - 1)|^2}{1 + |(\nabla' f_h)_{2\tau}|^2} \right] \\
&\leqslant \limsup_{h \to +\infty} \left[4\varepsilon_h + 4\mathcal{H}^{n-1}(B_{2\tau}) |(\nabla' f_h)_{2\tau}|^2 \right] \\
&\leqslant \limsup_{h \to +\infty} \left[4\varepsilon_h + 4 \int_{D_{\frac{1}{2}}} |\nabla' f_h|^2 \, dx' \right] \leqslant \lim_{h \to +\infty} [4\varepsilon_h + 4C_8\varepsilon_h] = 0.
\end{aligned}
$$

Therefore, applying the reverse Poincaré inequality and (2.118), we have, for h large, that

$$\mathbf{e}_h(0,\tau) \leqslant \mathbf{e}_h(0,\tau,\nu_h) \leqslant C_9(\tilde{C}\tau^2\mathbf{e}_h(0,1) + \mathcal{D}(0,2\tau) + 2\tau r_h),$$

which is a contradiction if we choose $C_{11} > C_9 \max\{\tilde{C}, 2\}$. □

2.12 Proof of the optimal theorem

Before proving Theorem 2.0.3, for reader's convenience we recall a well-known result, which can be found in [46, Theorem 26.5 and Theorem 28.1]

Theorem 2.12.1. *If $A \subset \mathbb{R}^n$ is an open set and E is a perimeter minimizer in A, then $A \cap \partial^* E$ is a $C^{1,\gamma}$-hypersurface for every $\gamma \in \left(0, \frac{1}{2}\right)$ that is relatively open in $A \cap \partial E$. Moreover, defining*

$$\Sigma(E;A) = A \cap (\partial E \backslash \partial^* E),$$

the following statements are true:

i) if $2 \leqslant n \leqslant 7$, then $\Sigma(E;A) = \varnothing$;

ii) if $n = 8$, then $\Sigma(E;A)$ has no accumulation points in A;

iii) if $n \geqslant 9$, then $\mathcal{H}^s(\Sigma(E;A)) = 0$, for every $s > n-8$.

There exists a perimeter minimizer $E \subset \mathbb{R}^8$ with $\mathcal{H}^0(\Sigma(E;\mathbb{R}^8)) = 1$. If $n \geqslant 9$, then there exists a perimeter minimizer $E \subset \mathbb{R}^n$ with $\mathcal{H}^{n-8}(\Sigma(E;\mathbb{R}^8)) = +\infty$.

Proof of Theorem 2.0.3. The proof works exactly as in [35]. We give here some details to emphasize the dependence of the constant ε appearing in the statement of Theorem 2.0.3 from the structural data of the functional. The proof is divided into four steps.

Step 1. We show that for every $\tau \in (0,1)$ there exists $\varepsilon_5 = \varepsilon_5(\tau) > 0$ such that if $\mathbf{e}(x,r) \leqslant \varepsilon_5$, then

$$\mathcal{D}(x,\tau r) \leqslant C_4 \tau \mathcal{D}(x,r),$$

where C_4 is from Lemma 2.4.6. Assume by contradiction that for some $\tau \in (0,1)$ there exist two positive sequences $\{\varepsilon_h\}_{h\in\mathbb{N}}$ and $\{r_h\}_{h\in\mathbb{N}}$ and a sequence (E_h, u_h) of Λr_h-minimizers of $\mathcal{F}_{r_h}$ in B_1 with equibounded energies such that, denoting by $\mathbf{e}_h$ the excess of E_h and by $\mathcal{D}_h$ the rescaled Dirichlet integral of u_h, we have that $0 \in \partial E_h$,

$$\mathbf{e}_h(0,1) = \varepsilon_h \to 0 \quad \text{and} \quad \mathcal{D}_h(0,\tau) > C_4 \tau \mathcal{D}_h(0,1). \tag{2.119}$$

Thanks to the energy upper bound (Theorem 2.4.1) and the compactness lemma (Lemma 2.6.1), we may assume that $E_h \to E$ in $L^1(B_1)$ and $0 \in \partial E$.

Since, by lower semicontinuity, the excess of E at 0 is null, it follows that E is a half-space in B_1, say H (see Proposition 1.1.23). In particular, for h large, it holds

$$|(E_h \Delta H) \cap B_1| < \varepsilon_0(\tau)|B_1|,$$

where ε_0 is from Lemma 2.4.6, which gives a contradiction with the inequality (2.119).

Step 2. Let $U \subset\subset \Omega$ be an open set. Prove that for every $\tau \in (0,1)$ there exist two positive constants $\varepsilon_6 = \varepsilon_6(\tau, U)$ and C_{12} such that if $x_0 \in \partial E$, $B_r(x_0) \subset U$ and $\mathbf{e}(x_0,r) + \mathcal{D}(x_0,r) + r < \varepsilon_6$, then

$$\mathbf{e}(x_0,\tau r) + \mathcal{D}(x_0,\tau r) + \tau r \leqslant C_{12}\tau(\mathbf{e}(x_0,r) + \mathcal{D}(x_0,r) + r). \tag{2.120}$$

Fix $\tau \in (0,1)$ and assume without loss of generality that $\tau < \frac{1}{2}$. We can distinguish two cases.

Case 1: $\mathcal{D}(x_0,r) + r \leqslant \tau^{-n}\mathbf{e}(x_0,r)$. If $\mathbf{e}(x_0,r) < \min\{\varepsilon_4(\tau,\tau^{-n}), \varepsilon_5(2\tau)\}$ it follows from Theorem 2.11.1 and step 1 that

$$\begin{aligned}\mathbf{e}(x_0,\tau r) &\leqslant C_{11}(\tau^2\mathbf{e}(x_0,r) + \mathcal{D}(x_0,2\tau r) + \tau r)\\ &\leqslant C_{11}\tau(\mathbf{e}(x_0,r) + 2C_4\mathcal{D}(x_0,r) + r)\end{aligned}$$

Case 2: $\mathbf{e}(x_0,r) \leqslant \tau^n(\mathcal{D}(x_0,r) + r)$. By the property of the excess at different scales, we infer

$$\mathbf{e}(x_0,\tau r) \leqslant \tau^{1-n}\mathbf{e}(x_0,r) \leqslant \tau(\mathcal{D}(x_0,r) + r).$$

We conclude that choosing $\varepsilon_6 = \min\{\varepsilon_4(\tau,\tau^{-n}), \varepsilon_5(2\tau), \varepsilon_5(\tau)\}$, inequality (2.120) is verified.

Step 3. Fix $\sigma \in (0,\frac{1}{2})$ and choose $\tau_0 \in (0,1)$ such that $C_{12}\tau_0 \leqslant \tau_0^{2\sigma}$. Let $U \subset\subset \Omega$ be an open set. We define

$$\begin{aligned}\Gamma \cap U := \{x \in \partial E \cap U \,:\, &\mathbf{e}(x,r) + \mathcal{D}(x,r) + r < \varepsilon_6(\tau_0,U),\\ &\text{for some } r > 0 \text{ such that } B_r(x_0) \subset U\}.\end{aligned}$$

Note that $\Gamma \cap U$ is relatively open in ∂E. We show that $\Gamma \cap U$ is a $C^{1,\sigma}$-hypersurface. Indeed, inequality (2.120) implies via standard iteration argument that if $x_0 \in \Gamma \cap U$ there exist $r_0 > 0$ and a neighborhood V of x_0 such that for every $x \in \partial E \cap V$ it holds:

$$\mathbf{e}(x,\tau_0^k r_0) + \mathcal{D}(x,\tau_0^k r_0) + \tau_0^k r_0 \leqslant \tau_0^{2\sigma k}, \quad \text{for } k \in \mathbb{N}_0.$$

In particular $\mathbf{e}(x,\tau_0^k r_0) \leqslant \tau_0^{2\sigma k}$ and, arguing as in [35], we obtain that for every $x \in \partial E \cap V$ and $0 < s < t < r_0$ it holds

$$|(\nu_E)_s(x) - (\nu_E)_t(x)| \leqslant ct^\sigma,$$

for some constant $c = c(n,\tau_0,r_0)$, where

$$(\nu_E)_t(x) = \fint_{\partial E \cap B_t(x)} \nu_E \, d\mathcal{H}^{n-1}.$$

The previous estimate first implies that $\Gamma \cap U$ is C^1. By a standard argument we then deduce again from the same estimate that $\Gamma \cap U$ is a $C^{1,\sigma}$-hypersurface. Finally we define $\Gamma := \bigcup_i (\Gamma \cap U_i)$, where $\{U_i\}_{i\in\mathbb{N}}$ is an increasing sequence of open sets such that $U_i \subset\subset \Omega$ and $\Omega = \bigcup_i U_i$.

Step 4. The proof of this final step basically follows as in [35] (see also [6], [20] and [22]). Finally we prove that there exists $\varepsilon > 0$ such that

$$\mathcal{H}^{n-1-\varepsilon}(\partial E \backslash \Gamma) = 0.$$

Setting

$$\Sigma = \{x \in \partial E \backslash \Gamma \,:\, \lim_{r\to 0^+} \mathcal{D}(x,r) = 0\},$$

by Lemma 2.3.3, $\nabla u \in L^{2s}_{loc}(\Omega)$ for some $s > 1$, depending only on ν, N, L, n. Since Hölder's inequality implies that

$$\mathcal{D}(x,r) = r^{1-n} \int_{B_r(x_0)} |\nabla u|^2 \, dx \leqslant c(n) \left(r^{s-n} \int_{B_r(x_0)} |\nabla u|^{2s} \, dx \right)^{\frac{1}{s}},$$

the following inclusion is true:

$$\begin{aligned} E^{n-1} &:= \left\{ x \in \Omega \,:\, \limsup_{r\to 0^+} \mathcal{D}(x,r) > 0 \right\} \\ &\subset \left\{ x \in \Omega \,:\, \limsup_{r\to 0^+} r^{s-n} \int_{B_r(x_0)} |\nabla u|^{2s} \, dx > 0 \right\}. \end{aligned}$$

Applying Proposition A.1.10 with $\mu(B_r(x_0)) := \displaystyle\int_{B_r(x_0)} |\nabla u|^{2s} \, dx$, we have that

$$\dim_{\mathcal{H}}\big(E^{n-1}\big) \leqslant n - s.$$

As $(\partial E \backslash \Gamma) \backslash E^{n-1} = \Sigma$, it is clear that

$$\dim_{\mathcal{H}}(\partial E \backslash \Gamma) \leqslant \max\{\dim_{\mathcal{H}}(\Sigma), \dim_{\mathcal{H}}(E^{n-1})\} \leqslant \max\{\dim_{\mathcal{H}}(\Sigma), n - s\}$$

If we show that $\Sigma = \varnothing$ when $n \leqslant 7$ and $\dim_{\mathcal{H}}(\Sigma) \leqslant n - 8$ for $n > 7$, we will have that

$$\dim_{\mathcal{H}}(\partial E \backslash \Gamma) \leqslant \begin{cases} \max\{0, n-s\} & \text{if } n \leqslant 7, \\ \max\{n-8, n-s\} & \text{if } n > 7 \end{cases} = n - s,$$

which is the thesis of the theorem.

Case 1: $n \leqslant 7$. Suppose by contradiction that $\Sigma \neq \varnothing$ and, up to translations, that $0 \in \Sigma$. If $\{r_h\}_{h\in\mathbb{N}}$ is infinitesimal, denoting by $E_h := \frac{E}{r_h}$ and by $u_h(x) := r_h^{-\frac{1}{2}} u(r_h x)$, by scaling we get that (E_h, u_h) is a Λr_h-minimizer of $\mathcal{F}$ in $\frac{\Omega}{r_h}$. Furthermore $\nabla u_h \to 0$ in $L^2(B_1)$ as $h \to +\infty$; indeed, since $0 \in \Sigma$,

$$\lim_{h\to+\infty} \int_{B_1} |\nabla u_h|^2 \, dy = \lim_{h\to+\infty} \frac{1}{r_h^{n-1}} \int_{B_{r_h}} |\nabla u|^2 \, dy = \lim_{h\to+\infty} \mathcal{D}(0, r_h) = 0.$$

By Lemma 2.6.1 there exists a local minimizer of the perimeter $E_\infty \subset \mathbb{R}^n$ such that $0 \in \partial E_\infty$,

$$E_h \to E_\infty \quad \text{and} \quad P(E_h; U) \to P(E_\infty; U),$$

for every open set $U \subset B_1$. In dimension $n \leqslant 7$, ∂E_∞ is a smooth manifold, since, by Theorem 2.12.1, $\partial^* E$ is smooth and $\partial E_\infty = \partial^* E_\infty$. By Proposition 1.1.24, for any $\varepsilon > 0$ there exists $r > 0$ such that $\mathbf{e}(E_\infty, 0, r) < \varepsilon$. Applying Proposition 2.6.2, there exists $h_0 \in \mathbb{N}$ such that

$$\mathbf{e}(E_h, 0, r_h) < \varepsilon, \quad \forall h > h_0,$$

which implies the contradiction $0 \notin \Sigma$.
Case 2: $n > 7$. Assume by contradiction that $\mathcal{H}^s(\Sigma) > 0$ for some $s > n - 8$. Then, $\mathcal{H}^s_\infty(\Sigma) > 0$ and furthermore, by Proposition A.1.12, we have that

$$\limsup_{\rho \to 0^+} \frac{\mathcal{H}^s_\infty(\Sigma \cap B_\rho(x))}{\omega_s \rho^s} \geqslant 2^{-s}, \quad \text{for } \mathcal{H}^s\text{-a.e. } x \in \Sigma.$$

We choose $x \in \Sigma$ such that $\limsup_{\rho \to 0^+} \frac{\mathcal{H}^s_\infty(\Sigma \cap B_\rho(x))}{\omega_s \rho^s} \geqslant 2^{-s}$. Let $\Sigma_h := \frac{\Sigma}{\rho_h}$. We find an infinitestimal sequence $\{\rho_h\}_{h \in \mathbb{N}} \subset \mathbb{R}^+$ such that

$$\limsup_{\rho \to 0^+} \mathcal{H}^s_\infty(\Sigma_h \cap \overline{B}_1) = \limsup_{\rho \to 0^+} \frac{\mathcal{H}^s_\infty(\Sigma \cap \overline{B}_{\rho_h}(x))}{\rho_h^s} \geqslant 2^{-s-1}\omega_s. \tag{2.121}$$

Reasoning as in the previous case, there exists a local minimizer of the perimeter $E_\infty \subset \mathbb{R}^n$ such that $0 \in \partial E_\infty$,

$$E_h \to E_\infty \quad \text{and} \quad P(E_h; U) \to P(E_\infty; U),$$

for every open set $U \subset B_1$. If we show that

$$\mathcal{H}^s((\partial E_\infty \backslash \partial^* E_\infty) \cap \overline{B}_1) \geqslant \mathcal{H}^s_\infty(\Sigma_h \cap \overline{B}_1), \tag{2.122}$$

for any h sufficiently large, using (2.121), we get:

$$\begin{aligned} \mathcal{H}^s((\partial E_\infty \backslash \partial^* E_\infty) \cap \overline{B}_1) &\geqslant \mathcal{H}^s_\infty((\partial E_\infty \backslash \partial^* E_\infty) \cap \overline{B}_1) \\ &\geqslant \limsup_{h \to +\infty} \mathcal{H}^s_\infty(\Sigma_h \cap \overline{B}_1) \geqslant 2^{-s-1}\omega_s > 0, \end{aligned}$$

which implies that $\dim_{\mathcal{H}}((\partial E_\infty \backslash \partial^* E_\infty) \cap \overline{B_1}) = s > n - 8$, which is a contradiction (see Theorem 2.12.1). In order to prove (2.122), it suffices to show that if $A \subset \mathbb{R}^n$ is an open set such that $(\partial E_\infty \backslash \partial^* E_\infty) \cap \overline{B}_1 \subset A$, then there exists $h_0 \in \mathbb{N}$ such that

$$\Sigma_h \cap \overline{B}_1 \subset A, \quad \forall h \geqslant h_0.$$

Since $\overline{B}_1$ is compact and A is open, we may assume by contradiction that there exists $\{x_{h_j}\}_{j \in \mathbb{N}} \subset (\Sigma_{h_j} \cap \overline{B}_1) \backslash A$ such that $x_{h_j} \to x_0 \in \overline{B}_1 \backslash A$. Furthermore

$x_0 \in \partial E_\infty$ thanks to Lemma 2.6.1. By our assumption on A, we deduce that $x_0 \in \partial^* E_\infty$. Using Proposition 1.1.24, there exists $\rho > 0$ such that

$$\mathbf{e}(E_\infty, x_0, \rho) \leqslant \varepsilon_6.$$

Applying Proposition 2.6.2, there exists $j_0 \in \mathbb{N}$ such that

$$\mathbf{e}(E_{h_j}, x_0, r_{h_j}) < \varepsilon_6, \quad \forall j > j_0.$$

Since $\Sigma \subset \partial E_{h_j} \backslash \Gamma_{h_j}$, where Γ_{h_j} is the singular set of ∂E_{h_j}, we have the contradiction $x_{h_j} \notin \Sigma_{h_j}$. □

Chapter 3

The p-polynomial growth case

In this chapter we deal with the following energy functional:

$$\mathcal{F}(v;E) := \int_\Omega [F(\nabla v) + \mathbb{1}_E G(\nabla v) + f_E(x,v)]\,dx + P(E;\Omega),$$

with $(v,E) \in \big(u_0 + W_0^{1,p}(\Omega)\big) \times \mathcal{A}(\Omega)$, for $p > 1$, where $u_0 \in W^{1,p}(\Omega)$ and $\mathcal{A}(\Omega)$ is the set of all subsets of Ω with finite perimeter. Here we consider $F, G \in C^1(\mathbb{R}^n)$ and $f_E = g + \mathbb{1}_E h$, for $E \subset \mathbb{R}^n$, where $g, h\colon \Omega \times \mathbb{R} \to \mathbb{R}$ are two Borel measurable and lower semicontinuous functions with respect to the real variable.

With regard to the hypotheses on the integrands, we assume that there exist some positive constants l, L, α, β and $\mu \geqslant 0$ such that

- F and G have p-growth:

$$0 \leqslant F(\xi) \leqslant L(\mu^2 + |\xi|^2)^{\frac{p}{2}}, \tag{F1}$$

$$0 \leqslant G(\xi) \leqslant \beta L(\mu^2 + |\xi|^2)^{\frac{p}{2}},$$

  for all $\xi \in \mathbb{R}^n$.

- F and G are strongly quasi-convex:

$$\int_\Omega F(\xi + \nabla\varphi)\,dx \geqslant \int_\Omega [F(\xi) + l(\mu^2 + |\xi|^2 + |\nabla\varphi|^2)^{\frac{p-2}{2}}|\nabla\varphi|^2]\,dx,$$

$$\int_\Omega G(\xi + \nabla\varphi)\,dx \geqslant \int_\Omega [G(\xi) + \alpha l(\mu^2 + |\xi|^2 + |\nabla\varphi|^2)^{\frac{p-2}{2}}|\nabla\varphi|^2]\,dx,$$

  for all $\xi \in \mathbb{R}^n$ and $\varphi \in C_c^1(\Omega)$.

- there exist two positive constants t_0, a and $0 < m < p$ such that for every $t > t_0$ and $\xi \in \mathbb{R}^n$ with $|\xi| = 1$, it holds

$$\left| F_p(\xi) - \frac{F(t\xi)}{t^p} \right| \leqslant \frac{a}{t^m}, \tag{F3}$$

$$\left| G_p(\xi) - \frac{G(t\xi)}{t^p} \right| \leqslant \frac{a}{t^m}, \tag{G3}$$

where F_p and G_p are the p-recession functions of F and G (see Definition 3.1.1).

We remark that the proximity conditions (F3) and (G3) are trivially satisfied if F and G are positively p-homogeneous.

The first of the following assumptions on g and h is essential to prove the existence of a minimal configuration. The same condition turns out to be crucial in the proof of the regularity result as well. We assume that there exist a function $\gamma \in L^1(\Omega)$ and two constants $C_0 > 0$ and $k \in \mathbb{R}$, with $k < \frac{l}{2^{p-1}\lambda}$, being $\lambda = \lambda(\Omega)$ the first eigenvalue of the p-Laplacian on Ω with boundary datum u_0, such that

- g and h satisfy the following assumptions:

$$g(x,s) \geqslant \gamma(x) - k|s|^p, \quad h(x,s) \geqslant \gamma(x) - k|s|^p, \tag{3.1}$$

for almost all $(x,s) \in \Omega \times \mathbb{R}$.

- g and h satisfy the following growth conditions:

$$|g(x,s)| \leqslant C_0(1+|s|^q), \quad |h(x,s)| \leqslant C_0(1+|s|^q), \tag{3.2}$$

for all $(x,s) \in \Omega \times \mathbb{R}$, with the exponent

$$q \in \begin{cases} [p, +\infty) & \text{if } n = 2, \\ [p, p^*) & \text{if } n > 2 \end{cases}$$

fixed.

We study the following problem:

$$\min_{(v,E) \in \left(u_0 + W_0^{1,p}(\Omega)\right) \times \mathcal{A}(\Omega)} \mathcal{F}(v,E). \tag{P}$$

The main result of this chapter is the following theorem about the regularity of solutions of problem (P).

Theorem 3.0.1. *Let* (A,u) *be a solution of* (P). *Then*

1. *u is locally Hölder continuous;*
2. *A is equivalent to an open set $\tilde{A}$, that is*

$$|A \Delta \tilde{A}| = 0 \quad \text{and} \quad P(A;\Omega) = P(\tilde{A};\Omega) = \mathcal{H}^{n-1}(\partial \tilde{A} \cap \Omega).$$

The idea of its proof is similar to that of [5, Theorem 2.2], which in turns relies on the ideas introduced in [19]. The regularity of u is proved in Theorem 3.3.1 and the regularity of A follows from Proposition 3.4.1. The proof will be discussed in the final section.

The same arguments can be used to treat also the volume-costrained problem

$$\min_{\substack{(v,E)\in\left(u_0+W_0^{1,p}(\Omega)\right)\times\mathcal{A}(\Omega)\\|E|=d}} \mathcal{F}(v,E), \tag{Q}$$

for some $0 < d < |\Omega|$. The following theorem holds true.

Theorem 3.0.2. *There exists $\lambda_0 > 0$ such that if (A,u) is a minimizer of the functional*

$$\mathcal{F}_\lambda(v,E) = \int_\Omega [F(\nabla v) + \mathbb{1}_E G(\nabla v) + f_E(x,v)]\,dx + P(E;\Omega) + \lambda||E| - d|,$$

for some $\lambda \geqslant \lambda_0$ and among all configurations (v,E) such that $v \in u_0 + W_0^{1,p}(\Omega)$ and $E \in \mathcal{A}(\Omega)$, then $|A| = d$ and (A,u) is a minimizer of problem (Q) *. Conversely, if (A,u) is a minimizer of the problem* (Q)*, then it is a minimizer of $\mathcal{F}_\lambda$, for all $\lambda \geqslant \lambda_0$.*

The proof of the previous theorem is a a straightforward adaptation of the proof of [13, Theorem 1.4], that is a generalization of the proof of Theorem 2.2.1. The term concerning the function f_E can be treated as a constant, thanks to the boundedness stated in Theorem 3.3.1. We finally remark that the term $\lambda||E| - d|$ in the functional $\mathcal{F}_\lambda$ can be inglobed in f_E, since it is bounded. For this reason, Theorem 3.0.1 is still valid also for minimal configurations of $\mathcal{F}_\lambda$ and, consequently, for solutions of problem (Q).

We give here an outline of this chapter. In Section 3.1, we recall some well-known lemmata concerning general functionals with p-polynomial growth.

Section 3.2 is entirely devoted to the proof of the existence of solutions of problem (P) by means of a standard argument.

In the subsequent two sections we address to regularity properties of minimizing couples (A,u) of problem (P). In particular, in Section 3.3, we quote the classical regularity result proved by De Giorgi and, in addition, the usual higher integrability property of ∇u. Section 3.4 is devoted to proving regularity properties of E. The main ingredient is Proposition 3.4.1, which states that if in a ball of radius ρ the energy associated with a minimal configuration is controlled by ρ^{n-1}, then it decays faster than ρ^{n-1}.

Finally, in Section 3.5 we give the proof of the main theorem of the chapter.

3.1 Some auxiliary results

Throughout this section we denote with H a function belonging to $C^1(\mathbb{R}^n)$ and satisfying for some positive constants $\tilde{l}$ and $\tilde{L}$ the same kind of assumptions imposed on F and G:

$$0 \leqslant H(\xi) \leqslant \tilde{L}(\mu^2 + |\xi|^2)^{\frac{p}{2}},$$

$$\int_\Omega H(\xi + \nabla\varphi)\, dx \geqslant \int_\Omega [H(\xi) + \tilde{l}(\mu^2 + |\xi|^2 + |\nabla\varphi|^2)^{\frac{p-2}{2}}|\nabla\varphi|^2]\, dx,$$

for all $\xi \in \mathbb{R}^n$ and $\varphi \in C_c^1(\Omega)$. We collect some definitions and well-known results that will be used later. We start giving the definition of p-recession function of H.

Definition 3.1.1. *The p-recession function of H is defined by*

$$H_p(\xi) := \limsup_{t \to +\infty} \frac{H(t\xi)}{t^p},$$

for all $\xi \in \mathbb{R}^n$.

Remark 3.1.2. *It is clear that* H_p *is positively p-homogeneous, which means that*

$$H_p(s\xi) = s^p H_p(\xi),$$

for all $\xi \in \mathbb{R}^n$ *and* $s > 0$. *It is also true that the growth condition of H implies the following growth condition of* H_p*:*

$$0 \leqslant H_p(\xi) \leqslant \tilde{L}|\xi|^p,$$

for any $\xi \in \mathbb{R}^n$.

The next lemma esthabilishes strong quasi-convexity of H_p, provided H verifies an appropriate growth condition. Although its proof can be found in [31, Lemma 2.8], we illustrate it here for the sake of completeness.

Lemma 3.1.3. *Let H be as above. If there exist two positive constants* $\tilde{t}_0$, $\tilde{d}$ *and* $0 < \tilde{m} < p$ *such that for every* $t > \tilde{t}_0$ *and* $\xi \in \mathbb{R}^n$ *with* $|\xi| = 1$, *it holds*

$$\left| H_p(\xi) - \frac{H(t\xi)}{t^p} \right| \leqslant \frac{\tilde{d}}{t^{\tilde{m}}},$$

then

$$\int_\Omega H_p(\xi + \nabla\varphi)\, dx \geqslant \int_\Omega [H_p(\xi) + \tilde{l}(|\xi|^2 + |\nabla\varphi|^2)^{\frac{p-2}{2}}|\nabla\varphi|^2]\, dx,$$

for all $\xi \in \mathbb{R}^n$ *and* $\varphi \in C_c^1(\Omega)$.

Proof. Fix $\lambda > 1$. It holds true that, for $t > t_0\lambda$ and $z \in \mathbb{R}^n$ such that $\lambda^{-1} < |z| < \lambda$, we have

$$\left| F_p(z) - \frac{F(tz)}{t^p} \right| \leqslant \frac{c_0 \lambda^{p-m}}{t^m}. \tag{3.3}$$

Indeed,

$$\left|F_p(z) - \frac{F(tz)}{t^p}\right| = |z|^p \left|F_p\left(\frac{z}{|z|}\right) - \frac{1}{(t|z|)^p} F\left(t|z|\frac{z}{|z|}\right)\right|$$
$$\leqslant |z|^p \frac{c_0}{(t|z|)^m} = \frac{c_0|z|^{p-m}}{t^m} \leqslant \frac{c_0 \lambda^{p-m}}{t^m}.$$

Fix $z \in \mathbb{R}^n$, $\phi \in C_c^1(\Omega)$ and take an increasing divergent sequence $\{t_h\}_{h\in\mathbb{N}}$ such that

$$F_p(z) = \lim_{h\to+\infty} \frac{F(t_h z)}{t_h^p}.$$

Fix $\lambda > \max\{1, |z| + \|\nabla\phi\|_\infty\}$. We recall that, by Remark 3.1.2, F_p is nonnegative. Thus, if $t_h > t_0\lambda$, from (3.3) and by virtue of the strong quasiconvexity of F we have

$$\begin{aligned}
&\int_\Omega F_p(z+\nabla\phi)\,dx \\
&\geqslant \int_{\Omega\cap\{\lambda^{-1}<|z+\nabla\phi|\}} F_p(z+\nabla\phi)\,dx \\
&\geqslant \frac{1}{t_h^p}\int_{\Omega\cap\{\lambda^{-1}<|z+\nabla\phi|\}} F(t_h z + t_h\nabla\phi)\,dx - \frac{c_0\lambda^{p-m}}{t_h^m}|\Omega| \\
&= \frac{1}{t_h^p}\int_\Omega F(t_h z + t_h\nabla\phi)\,dx - \frac{1}{t_h^p}\int_{\Omega\cap\{\lambda^{-1}\geqslant|z+\nabla\phi|\}} F(t_h z + t_h\nabla\phi)\,dx - \frac{c_0\lambda^{p-m}}{t_h^m}|\Omega| \\
&\geqslant \frac{1}{t_h^p}\int_\Omega F(t_h z + t_h\nabla\phi)\,dx - \frac{\tilde{L}}{t_h^p}\int_{\Omega\cap\{\lambda^{-1}\geqslant|z+\nabla\phi|\}} (\mu^2 + |t_h z + t_h\nabla\phi|^p)\,dx \\
&- \frac{c_0\lambda^{p-m}}{t_h^m}|\Omega| \\
&\geqslant \int_\Omega \left[\frac{F(t_h z)}{t_h^p} + \tilde{l}\left(\frac{\mu^2}{t_h^2} + |z|^2 + |\nabla\phi|^2\right)^{\frac{p-2}{2}} |\nabla\phi|^2\right] dx - \frac{\tilde{L}}{t_h^p}\left(\mu^2 + \frac{t_h^p}{\lambda^p}\right)|\Omega| \\
&- \frac{c_0\lambda^{p-m}}{t_h^m}|\Omega|.
\end{aligned}$$

The result follows by letting $h \to +\infty$ and then $\lambda \to +\infty$. □

Let us recall some other useful lemmata. The proof of Lemma 3.1.4 can be found in [37, Lemma 5.2], while Lemma 3.1.5 is proved in [13, Lemma 2.3].

Lemma 3.1.4. *Let H be as above. It holds that*

$$|\nabla H(\xi)| \leqslant 2^p \tilde{L}(\mu^2 + |\xi|^2)^{\frac{p-1}{2}},$$

for all $\xi \in \mathbb{R}^n$.

Lemma 3.1.5. *Let H be as above. There exists a positive constant $\tilde{c} = \tilde{c}(p, \tilde{l}, \tilde{L}, \mu)$ such that*

$$H(\xi) \geqslant \frac{\tilde{l}}{2}(\mu^2 + |\xi|^2)^{\frac{p}{2}} - \tilde{c},$$

for all $\xi \in \mathbb{R}^n$.

Proof. The strong quasi-convexity of H is equivalent to the convexity of the function

$$K(\xi) := H(\xi) - \tilde{l}\big(\mu^2 + |\xi|^2\big)^{\frac{p}{2}}, \quad \forall \xi \in \mathbb{R}^n,$$

which, in turn, implies

$$K(\xi) \geqslant K(0) + \nabla_z K(0) \cdot \xi, \quad \forall \xi \in \mathbb{R}^n.$$

Let us fix $\xi \in \mathbb{R}^n$. The previous inequality can be written in terms of H as

$$H(\xi) \geqslant \tilde{l}\big(\mu^2 + |\xi|^2\big)^{\frac{p}{2}} + H(0) - \tilde{l}\mu^p + \nabla_z H(0) \cdot \xi.$$

Since $H(0) \geqslant 0$ and, by Schwartz's and Young's inequality we infer

$$\begin{aligned} |\nabla_z H(0) \cdot \xi| &= \Big|\big(\tilde{l}^{-\frac{1}{p}} \nabla_z H(0)\big) \cdot \big(\tilde{l}^{\frac{1}{p}} \xi\big)\Big| \leqslant c(p, \tilde{l}) |\nabla_z H(0)|^{\frac{p}{p-1}} + \frac{\tilde{l}}{2} |\xi|^p \\ &\leqslant c(p, \tilde{l}, \tilde{L}) \mu^p + \frac{\tilde{l}}{2}\big(\mu^2 + |\xi|^2\big)^{\frac{p}{2}}, \end{aligned}$$

then we conclude

$$\begin{aligned} H(\xi) &\geqslant \tilde{l}\big(\mu^2 + |\xi|^2\big)^{\frac{p}{2}} - c(p, \tilde{l}, \tilde{L}) \mu^p - \frac{\tilde{l}}{2}\big(\mu^2 + |\xi|^2\big)^{\frac{p}{2}} - \tilde{l}\mu^p \\ &= \frac{\tilde{l}}{2}\big(\mu^2 + |\xi|^2\big)^{\frac{p}{2}} - c(p, \tilde{l}, \tilde{L}, \mu). \end{aligned}$$

□

We define the auxiliary function

$$V(\xi) = (\mu^2 + |\xi|^2)^{\frac{p-2}{4}} \xi,$$

for all $\xi \in \mathbb{R}^n$.

In order to prove Lemma 3.1.7, we need the following auxiliary result proved in [36, Lemma 2.1] in the case $\delta > 0$ and in [1, Lemma 2.1] in the case $\delta \in \big(-\frac{1}{2}, 0\big)$.

Lemma 3.1.6. *For any $\delta > -\frac{1}{2}$, the following estimate holds:*

$$4^{-(1+\delta)} \leqslant \frac{\displaystyle\int_0^1 \big(\mu^2 + |t\xi + (1-t)\eta|^2\big)^{\frac{\delta}{2}} \, dt}{\big(\mu^2 + |\xi|^2 + |\eta|^2\big)^{\frac{\delta}{2}}} \leqslant \max\left\{2^{\frac{\delta}{2}}, \frac{8}{2\delta + 1}\right\},$$

for any $\xi, \eta \in \mathbb{R}^n$.

Proof. We need to distinguish the cases $\delta \geqslant 0$ and $\delta \in \big(\frac{1}{2}, 0\big)$.
Case 1: $\delta \geqslant 0$. The estimate from above is straightforward; indeed, since, for $t \in (0,1)$,

$$\mu^2 + |t\xi + (1-t)\eta|^2 \leqslant \mu^2 + 2\big(t^2|\xi|^2 + (1-t)^2|\eta|^2\big) \leqslant 2\big(\mu^2 + |\xi|^2 + |\eta|^2\big),$$

then

$$\frac{\int_0^1 \left(\mu^2 + |t\xi + (1-t)\eta|^2\right)^{\frac{\delta}{2}}\, dt}{\left(\mu^2 + |\xi|^2 + |\eta|^2\right)^{\frac{\delta}{2}}} \leqslant \frac{\int_0^1 \left[2\left(\mu^2 + |\xi|^2 + |\eta|^2\right)\right]^{\frac{\delta}{2}}\, dt}{\left(\mu^2 + |\xi|^2 + |\eta|^2\right)^{\frac{\delta}{2}}} = 2^{\frac{\delta}{2}}.$$

For the estimate from below, it is not restrictive to assume $|\xi| \geqslant |\eta|$. For $t \in \left(\frac{3}{4}, 1\right)$, we estimate

$$\begin{aligned}|t\xi + (1-t)\eta| \geqslant t|\xi| - (1-t)|\eta| = t(|\xi| + |\eta|) - |\eta| = t(|\xi| + |\eta|) - \frac{|\eta|}{2} - \frac{|\eta|}{2} \\ = \frac{1}{4}(|\xi| + |\eta|),\end{aligned}$$

in order to obtain

$$\begin{aligned}\frac{\int_0^1 \left(\mu^2 + |t\xi + (1-t)\eta|^2\right)^{\frac{\delta}{2}}\, dt}{\left(\mu^2 + |\xi|^2 + |\eta|^2\right)^{\frac{\delta}{2}}} &\geqslant \frac{\int_{\frac{3}{4}}^1 \left[\mu^2 + \frac{1}{16}\left(|\xi| + |\eta|\right)^2\right]^{\frac{\delta}{2}}\, dt}{\left(\mu^2 + |\xi|^2 + |\eta|^2\right)^{\frac{\delta}{2}}} \\ &\geqslant \frac{\int_{\frac{3}{4}}^1 \left(\mu^2 + |\xi|^2 + |\eta|^2\right)^{\frac{\delta}{2}}\, dt}{16^{\frac{\delta}{2}}\left(\mu^2 + |\xi|^2 + |\eta|^2\right)^{\frac{\delta}{2}}} = 4^{-(\delta+1)}.\end{aligned}$$

Case 2: $\delta \in \left(-\frac{1}{2}, 0\right)$. By the convexity of the map $t \mapsto |\eta + t(\xi - \eta)|^2$, we infer

$$\frac{\int_0^1 \left(\mu^2 + |t\xi + (1-t)\eta|^2\right)^{\frac{\delta}{2}}\, dt}{\left(\mu^2 + |\xi|^2 + |\eta|^2\right)^{\frac{\delta}{2}}} \geqslant 1 \geqslant 4^{-(1+\delta)}.$$

Regarding the upper bound, it is not restrictive to assume $|\xi| \leqslant |\eta|$ and $\xi \neq \eta$. Let $\xi_0 \in \mathbb{R}^n$ be the point on the line connecting ξ and η with the lowest norm, i.e.

$$|\xi_0| = \min_{t \in [0,1]} |\eta + t(\xi - \eta)|.$$

We define

$$t_0 := \frac{|\xi_0 - \eta|}{|\xi - \eta|} \geqslant \frac{1}{2},$$

$$\phi_\lambda(t) := (\mu^2 + |\eta + t(\lambda - \eta)|^2)^\delta, \quad \forall t \in (0,1).$$

If $t_0 \geqslant 1$, by the minimality of ξ_0, $\phi_\xi \leqslant \phi_{\xi_0}$ and thus

$$\int_0^1 \phi_\xi(t)\, dt \leqslant \int_0^1 \phi_{\xi_0}(t)\, dt.$$

If $t_0 \in \left[\frac{1}{2}, 1\right)$, we obtain a similar estimate:

$$\int_0^1 \phi_\xi(t)\, dt \leqslant 2\int_0^{t_0} \phi_\xi(t)\, dt = 2t_0 \int_0^1 \left(\mu^2 + |\eta + t(\xi_0 - \eta)|^2\right)^\delta dt = 2\int_0^1 \phi_{\xi_0}(t)\, dt.$$

Thus, we can write, for a general $t_0 \geqslant \frac{1}{2}$,

$$\begin{aligned}\int_0^1 \phi_\xi(t)\,dt &\leqslant 2\int_0^1 \phi_{\xi_0}(t)\,dt \leqslant 2\int_0^1 \phi_0(t)\,dt \leqslant 2\int_0^1 \left(\mu^2 + \frac{t^2}{2}\big(|\xi|^2+|\eta|^2\big)\right)^\delta dt \\ &\leqslant 2^{1-\delta}\int_0^1 \big[\mu^2 + t^2\big(|\xi|^2+|\eta|^2\big)\big]\,dt \\ &\leqslant 2^{1-2\delta}\int_0^1 \big[\mu + t\big(|\xi|^2+|\eta|^2\big)^{\frac{1}{2}}\big]^{2\delta}\,dt \\ &\leqslant 4\int_0^1 \big[\mu + t\big(|\xi|^2+|\eta|^2\big)^{\frac{1}{2}}\big]^{2\delta}\,dt. \end{aligned} \tag{3.4}$$

where we used $\phi_{\xi_0} \leqslant \phi_0$. We remark that, for $0 \leqslant b \leqslant a$, we have

$$\int_0^1 (a+tb)^{2\delta}\,dt \leqslant a^{2\delta} \leqslant 2^\delta(a^2+b^2).$$

A similar estimate can be obtained for $0 \leqslant a < b$, that is

$$\int_0^1 (a+tb)^{2\delta}\,dt \leqslant \frac{(a+b)^{2\delta+1}}{(2\delta+1)b} \leqslant \frac{2}{2\delta+1}(a+b)^{2\delta} \leqslant \frac{2}{2\delta+1}(a^2+b^2)^\delta.$$

Applying to (3.4) the previous inequalities for $a=\mu$ and $b=\big(|\xi|^2+|\eta|^2\big)^{\frac{1}{2}}$, we finally get

$$\int_0^1 \phi_\xi(t)\,dt \leqslant \frac{8}{2\delta+1}(\mu^2+|\xi|^2+|\eta^2)^\delta,$$

which concludes the proof. □

The next lemma has been proved in [36, Lemma 2.2] for $p \geqslant 2$ and in [1, Lemma 2.2] for $1 < p < 2$.

Lemma 3.1.7. *There exists a constant $c = c(n,p)$ such that*

$$\frac{1}{c}(\mu^2+|\xi|^2+|\eta|^2)^{\frac{p-2}{2}} \leqslant \frac{|V(\xi)-V(\eta)|^2}{|\xi-\eta|^2} \leqslant c(\mu^2+|\xi|^2+|\eta|^2)^{\frac{p-2}{2}},$$

for all $\xi,\eta \in \mathbb{R}^n$.

Proof. Let us distinguish the cases $p \geqslant 2$ and $1 < p < 2$.
Case 1: $p \geqslant 2$. We start proving the esitmate from above. For reader's convenience, we compute

$$\frac{d}{dz}V(z) = \frac{p-2}{2}\big(\mu^2+|z|^2\big)^{\frac{p-6}{4}}|z|^2 + \big(\mu^2+|z|^2\big)^{\frac{p-2}{4}} \leqslant \frac{p}{2}\big(\mu^2+|z|^2\big)^{\frac{p-2}{4}}.$$

Applying Lemma 3.1.6 for $\delta = \frac{p-2}{4}$, we easily get

$$|V(\xi)-V(\eta)| \leqslant \int_0^1 \left|\frac{d}{dt}V(t\xi+(1-t)\eta)\right|\,dt$$

$$\leqslant \frac{p}{2}\int_0^1 \left(\mu^2+|t\xi+(1-t)\eta|^2\right)^{\frac{p-2}{4}} dt|\xi-\eta|$$
$$\leqslant \frac{p}{2}2^{\frac{p-2}{4}}\left(\mu^2+|\xi|^2+|\eta|^2\right)^{\frac{p-2}{4}}|\xi-\eta|.$$

It is not restrictive to assume $|\xi|\geqslant|\eta|$. If $|\xi|\geqslant 2|\eta|$, then

$$|\xi-\eta|\leqslant|\xi|+|\eta|\leqslant\frac{3}{2}|\xi|,$$

$$\left(\mu^2+|\xi|^2+|\eta|^2\right)^{\frac{p-2}{4}}\leqslant\left(\mu^2+2|\xi|^2\right)^{\frac{p-2}{4}}\leqslant 2^{\frac{p-2}{4}}\left(\mu^2+|\xi|^2\right)^{\frac{p-2}{4}}.$$

Since $|V(z)|=\left(\mu^2+|z|^2\right)^{\frac{p-2}{4}}|z|$ is increasing in $|z|$, then

$$|V(\xi)-V(\eta)|\geqslant|V(\xi)|-\left|V\left(\frac{\xi}{2}\right)\right|\geqslant\frac{|V(\xi)|}{2}\geqslant\frac{\left(\mu^2+|\xi|^2+|\eta|^2\right)^{\frac{p-2}{4}}|\xi-\eta|}{2^{\frac{p-2}{4}}\cdot 3}$$
$$\geqslant 5^{-\frac{p+2}{4}}\left(\mu^2+|\xi|^2+|\eta|^2\right)^{\frac{p-2}{4}}|\xi-\eta|.$$

If $|\eta|\leqslant|\xi|\leqslant 2|\eta|$, then, for $\tau\geqslant 1$, $|\tau\xi-\eta|\geqslant|\xi-\eta|$ and so

$$|V(\xi)-V(\eta)|=\left(\mu^2+|\eta|^2\right)^{\frac{p-2}{4}}\left|\left(\frac{\mu^2+|\xi|^2}{\mu^2+|\eta|^2}\right)^{\frac{p-2}{4}}\xi-\eta\right|$$
$$\geqslant 5^{-\frac{p-2}{4}}\left(\mu^2+|\xi|^2+|\eta|^2\right)^{\frac{p-2}{4}}|\xi-\eta|.$$

Case 2: $1<p<2$. For $z\in\mathbb{R}^n$, we define

$$F(z):=\frac{2}{p+2}\left(\mu^2+|z|^2\right)^{\frac{p+2}{4}},$$

so that

$$\nabla F(z)=V(z),\quad \nabla^2F(z)=\left(\mu^2+|z|^2\right)^{\frac{p-2}{4}}\left(I+\frac{p-2}{2\left(\mu^2+|z|^2\right)}z\otimes z\right),$$

for any $z\in\mathbb{R}^n$. It holds:

$$(\nabla^2F(z)\lambda)\cdot\lambda\geqslant p\left(\mu^2+|z|^2\right)^{\frac{p-2}{4}}|\lambda|^2,\quad |\nabla^2F(z)|\leqslant\sqrt{n+1}\left(\mu^2+|z|^2\right)^{\frac{p-2}{4}},$$

for any $z,\lambda\in\mathbb{R}^n$. Applying Lemma 3.1.6, we infer

$$|V(\xi)-V(\eta)||\xi-\eta|\geqslant(\nabla F(\xi)-\nabla F(\eta))\cdot(\xi-\eta)$$
$$=\left(\int_0^1\nabla^2F(\eta+t(\xi-\eta))\,dt(\xi-\eta)\right)\cdot(\xi-\eta)$$
$$\geqslant\int_0^1 p\left(\mu^2+|\eta+t(\xi-\eta)|^2\right)^{\frac{p-2}{4}}|\xi-\eta|^2\,dt$$
$$\geqslant p\left(\mu^2+|\xi|^2+|\eta|^2\right)^{\frac{p-2}{4}}|\xi-\eta|^2.$$

and

$$|V(\xi) - V(\eta)| = |\nabla F(\xi) - \nabla F(\eta)| \leqslant \int_0^1 |\nabla^2 F(\eta + t(\xi - \eta))| \, ds |\xi - \eta|$$
$$\leqslant \frac{8\sqrt{n+1}}{p} \left(\mu^2 + |\xi|^2 + |\eta|^2\right)^{\frac{p-2}{4}} |\xi - \eta|.$$

□

We need also the following result.

Lemma 3.1.8. *Let* $\{u_h\}_{h\in\mathbb{N}} \subset W^{1,p}(B_1)$ *and* $u \in W^{1,p}(B_1)$ *such that* $u_h \rightharpoonup u$ *in* $W^{1,p}(B_1)$. *Assume that* $\{\nabla u_h\}_{h\in\mathbb{N}}$ *is bounded in* $L^p(B_1)$. *If*

$$\lim_{h\to+\infty} \int_{B_1} \psi |V(\nabla u_h) - V(\nabla u)|^2 \, dy = 0, \qquad \forall \psi \in C_c^\infty(B_1) \quad s.t. \quad 0 \leqslant \psi \leqslant 1,$$

then $u_h \to u$ *in* $W^{1,p}_{loc}(B_1)$.

Proof. We proceed as in [13]. By Lemma 3.1.7, the convergence in our assumption yields to

$$\lim_{h\to+\infty} \int_{B_1} \eta\left(\mu^2 + |\nabla u_h|^2 + |\nabla u|^2\right)^{\frac{p-2}{2}} |\nabla u_h - \nabla u|^2 \, dy = 0$$

We distinguish two cases. If $p \geqslant 2$, then

$$\int_{B_1} \eta |\nabla u_h - \nabla u|^p \, dy \leqslant \int_{B_1} (\mu + |\nabla u_h| + |\nabla u|)^{p-2} |\nabla u_h - \nabla u|^2 \, dy$$
$$\leqslant c(p) \int_{B_1} \left(\mu^2 + |\nabla u_h|^2 + |\nabla u|^2\right)^{\frac{p-2}{2}} |\nabla u_h - \nabla u|^2 \, dy \to 0,$$

as $h \to +\infty$. If $1 < p < 2$, by Hölder's inequality, we infer

$$\int_{B_1} \eta |\nabla u_h - \nabla u|^p \, dy$$
$$= \int_{B_1} \eta\left(\mu^2 + |\nabla u_h|^2 + |\nabla u|^2\right)^{\frac{p(p-2)}{4}} |\nabla u_h - \nabla u|^2 \left(\mu^2 + |\nabla u_h|^2 + |\nabla u|^2\right)^{\frac{p(2-p)}{4}} dy$$
$$\leqslant \left(\int_{B_1} \eta\left(\mu^2 + |\nabla u_h|^2 + |\nabla u|^2\right)^{\frac{p-2}{2}} |\nabla u_h - \nabla u|^2 \, dy \right)^{\frac{p}{2}}$$
$$\times \left(\int_{B_1} \eta\left(\mu^2 + |\nabla u_h|^2 + |\nabla u|^2\right)^{\frac{p}{2}} dy \right)^{\frac{2-p}{2}}$$
$$\leqslant c(n,p,\mu) \left(\int_{B_1} \eta\left(\mu^2 + |\nabla u_h|^2 + |\nabla u|^2\right)^{\frac{p-2}{2}} |\nabla u_h - \nabla u|^2 \, dy \right)^{\frac{p}{2}} \to 0,$$

as $h \to +\infty$. □

Starting from Theorem 1.2.10, by means of an approximation argument, the following theorem has been proved in [31, Theorem 2.2]. The subsequent corollary is immediate.

Theorem 3.1.9. *Let H be as above and let $v \in W^{1,p}(\Omega)$ be a local miniminizer of the functional*

$$\mathcal{H}(w;\Omega) = \int_\Omega H(\nabla w)\,dx,$$

where $w \in v + W_0^{1,p}(\Omega)$. Then v is locally Lipschitz-continuous in Ω and there exists a constant $c = c(n,p,\tilde{l},\tilde{L}) > 0$ such that

$$\operatorname*{ess\,sup}_{B_{\frac{R}{2}}(x_0)} (\mu^2 + |\nabla v|^2)^{\frac{p}{2}} \leqslant c \fint_{B_R(x_0)} (\mu^2 + |\nabla v|^2)^{\frac{p}{2}}\,dy,$$

for all $B_R(x_0) \subset \Omega$.

Corollary 3.1.10. *Let $\mathcal{H}$ and $v \in W^{1,p}(\Omega)$ be as in Theorem 3.1.9. Then there exists a constant $c_{\mathcal{H}} = c_{\mathcal{H}}(n,p,\tilde{l},\tilde{L}) > 0$ such that*

$$\int_{B_r(x_0)} (\mu^2 + |\nabla v|^2)^{\frac{p}{2}}\,dy \leqslant c_{\mathcal{H}} \left(\frac{r}{R}\right)^n \int_{B_R(x_0)} (\mu^2 + |\nabla v|^2)^{\frac{p}{2}}\,dy,$$

for all $B_R(x_0) \subset \Omega$ and $0 < r < R$.

3.2 Existence of minimizing couples

In order to prove the existence of a solution of problem (P), we recall here a semicontinuity result by Ioffe, which can be found in [7, Theorem 5.8] and [40].

Theorem 3.2.1 (Ioffe's semicontinuity result)**.** *Let $f\colon \Omega \times \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}$ be $\mathscr{L}^n \times \mathcal{B}(\mathbb{R}^{m+k})$-measurable and lower semicontinuous in $\mathbb{R}^{m+k}$ a.e. in Ω. Assume that $f(x,s,\cdot)$ convex in $\mathbb{R}^k$ for any $(x,s) \in \Omega \times \mathbb{R}^m$. Then*

$$\liminf_{h\to+\infty} \int_\Omega f(x,u_h,z_h)\,dx \geqslant \int_\Omega f(x,u,z)\,dx,$$

if $\{u_h\}_{h\in\mathbb{N}} \subset \big(L^1(\Omega)\big)^m$ and $\{z_h\}_{h\in\mathbb{N}} \subset \big(L^1(\Omega)\big)^k$ are such that $u_k \to u$ in $\big(L^1(\Omega)\big)^m$ and $v_h \to v$ in $\big(L^1(\Omega)\big)^k$, as $k \to +\infty$.

Theorem 3.2.2. *The minimum problem* (P) *admits at least a solution.*

Proof. We initially remark that problem (P) can be written as follows:

$$\min_{E\in\mathcal{A}(\Omega)} \{\mathcal{E}(E) + P(E;\Omega)\}, \tag{3.5}$$

where

$$\mathcal{E}(E) = \min_{v\in u_0+W_0^{1,p}(\Omega)} \int_\Omega [F(\nabla v) + \mathbb{1}_E G(\nabla v) + f_E(x,v)]\,dx \tag{3.6}$$

Since F, G are strongly quasi-convex and g, h are lower semicontinuous in the real variable s, the functional $\mathcal{F}$ is lower semicontinuous with respect to the weak convergence of ∇v_h in L^p and the strong convergence of v_h in L^p. Moreover, the coerciveness of

$$\int_\Omega [F(\nabla v) + \mathbb{1}_E G(\nabla v)]\,dx$$

is granted by Lemma 3.1.5. Therefore the minimum problem (3.6) admits a solution. Let $\{A_h\}_{h\in\mathbb{N}} \subset \mathcal{A}(\Omega)$ be a minimizing sequence for problem (3.5). It follows that the sequence $\{P(A_h;\Omega)\}_{h\in\mathbb{N}}$ is bounded and so, by compactness, there exists $A \in \mathcal{A}(\Omega)$ such that $\mathbb{1}_{A_h} \to \mathbb{1}_A$ in $L^1_{loc}(\Omega)$. Let $u_h \in u_0 + W^{1,p}_0(\Omega)$ a solution of problem (3.6) associated with A_h, for all $h \in \mathbb{N}$. The sequence $\{u_h\}_{h\in\mathbb{N}}$ is bounded in $W^{1,p}(\Omega)$; indeed, by (3.1) and Poincaré's inequality we obtain

$$\begin{aligned}
&\min_{v\in u_0+W^{1,p}_0(\Omega)} \mathcal{F}(v,\Omega)\\
&\geqslant \mathcal{F}(A_h,u_h) \geqslant l\int_\Omega |\nabla u_h|^p\,dx + \int_\Omega \gamma\,dx - k\int_\Omega |u_h|^p\,dx\\
&\geqslant l\int_\Omega |\nabla u_h|^p\,dx + \int_\Omega \gamma\,dx - 2^{p-1}k\int_\Omega |u_h-u_0|^p\,dx - 2^{p-1}k\int_\Omega |u_0|^p\,dx\\
&\geqslant (l-2^{p-1}k\lambda)\int_\Omega |\nabla u_h|^p\,dx + \int_\Omega \gamma\,dx - 2^{p-1}k\int_\Omega |u_0|^p\,dx.
\end{aligned}$$

Hence, we can extract a subsequence (not relabelled) such that $u_h \rightharpoonup u$ in $W^{1,p}(\Omega)$. By definition of minimum we infer

$$\mathcal{E}(A) \leqslant \int_\Omega [F(\nabla u) + \mathbb{1}_A G(\nabla u) + f_A(x,u)]\,dx.$$

Applying Ioffe lower semicontinuity result, Theorem 3.2.1, to the integrand

$$\Phi(x,s_1,s_2,\xi) := F(\xi) + s_1 G(\xi) + g(x,s_2) + s_1 h(x,s_2),$$

where $x \in \Omega$, $s_1 \in [0,1]$, $s_2 \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$, we obtain

$$\begin{aligned}
\mathcal{E}(A) &\leqslant \int_\Omega [F(\nabla u) + \mathbb{1}_A G(\nabla u) + f_A(x,u)]\,dx = \int_\Omega \Phi(x,\mathbb{1}_A,u,\nabla u)\,dx\\
&\leqslant \liminf_{h\to+\infty}\int_\Omega \Phi(x,\mathbb{1}_{A_h},u_h,\nabla u_h)\,dx = \liminf_{h\to+\infty}\mathcal{E}(A_h).
\end{aligned}$$

Therefore, by the lower semicontinuity of perimeter we finally gain

$$\mathcal{E}(A) + P(A;\Omega) \leqslant \liminf_{h\to+\infty}[\mathcal{E}(A_h) + P(A_h;\Omega)],$$

which proves that A is a minimizer of problem (3.5) and so (A,u) is a minimizing couple of problem (P). □

3.3 Higher integrability and Hölder continuity of minimizers

The following theorem shows that local minimizers of the functional $\mathcal{F}(\cdot, E)$, with $E \in \mathcal{A}(\Omega)$ fixed, are Hölder continuous and a higher integrability property for the gradient holds true. The proof of this result is standard and can be carried out adopting the obvious adaptation in the proof of Lemma 2.27 and applying Gehring's Lemma (see Lemma 2.3.3). The local boundedness and the Hölder continuity of solutions of problem (P) is easily obtained if one follows the same argument of De Giorgi's regularity theorem (Theorem 1.2.6).

Theorem 3.3.1. *Let (A, u) be a solution of* (P). *Then the following facts hold:*

- *u is locally bounded in Ω by a constant depending only on $n, p, q, \alpha, \beta, l, L, \mu, C_0, \|u\|_{L^p(\Omega)}$ and is locally Hölder continuous in Ω;*
- *Let $\Omega_0 \Subset \Omega$, $\tau = \operatorname{dist}(\Omega_0, \partial\Omega)$ and $K = \{x \in \Omega : \operatorname{dist}(x, \Omega_0) \leqslant \frac{\tau}{2}\}$. Then there exist two constants $\gamma > 0$ and $r > p$ depending only on $n, p, q, \beta, l, L, \mu, C_0, \|u\|_{L^\infty(K)}$ such that*

$$\int_{Q_{\frac{R}{2}}(y)} |\nabla u|^r \, dx \leqslant \gamma \left[R^{n\left(1-\frac{r}{p}\right)} \left(\int_{Q_R(y)} |\nabla u|^p \, dx \right)^{\frac{r}{p}} + R^n \right],$$

for all $y \in \Omega_0$ and $Q_R(y) \subset K$.

3.4 Regularity of the set

The following proposition is the main result of this section and also the main ingredient to prove Theorem 3.0.1.

Proposition 3.4.1. *Let (A, u) be a solution of* (P). *Then for every compact set $K \subset \Omega$ there exists a constant $\xi \in (0, \operatorname{dist}(K, \partial\Omega))$ such that if $y \in K$ and for some $\rho < \xi$ it holds*

$$\int_{B_\rho(y)} [F_p(\nabla u) + \mathbb{1}_A G_p(\nabla u)] \, dx + P(A; B_\rho(y)) < \xi \rho^{n-1},$$

then

$$\lim_{\eta \to 0} \eta^{1-n} \left[\int_{B_\eta(y)} [F_p(\nabla u) + \mathbb{1}_A G_p(\nabla u)] \, dx + P(A; B_\eta(y)) \right] = 0.$$

The proof of the previous proposition relies on Proposition 3.4.5, which is an iteration of the decay estimate in Theorem 3.4.4. The following definition is crucial in the rescaling argument used in the proof of Theorem 3.4.4 (see (3.17)).

Definition 3.4.2 (Asymptotically minimizing sequence). *Let $\{(A_h, u_h)\}_{h\in\mathbb{N}} \subset W^{1,p}(B_1) \times \mathcal{A}(B_1)$ and $\{\lambda_h\}_{h\in\mathbb{N}} \subset \mathbb{R}^+$. We say that the sequence $\{(A_h, u_h)\}_{h\in\mathbb{N}}$ is λ_h-**asymptotically minimizing** if and only if for any compact set $K \subset B_1$ and any couple $\{(u'_h, A'_h)\} \subset W^{1,p}(B_1) \times \mathcal{A}(B_1)$ formed by a bounded sequence $\{u'_h\}_{n\in\mathbb{N}}$ in $W^{1,p}(B_1)$ with $\mathrm{spt}(u_h - u'_h) \subset K$ and a sequence of sets $\{A'_h\}_{n\in\mathbb{N}}$ with $A_h \Delta A'_h \subset K$, we have*

$$\begin{aligned}&\int_{B_1} [F_p(\nabla u_h) + \mathbb{1}_{A_h} G_p(\nabla u_h)]\, dy + \lambda_h P(A_h; B)\\&\leqslant \int_{B_1} [F_p(\nabla u'_h) + \mathbb{1}_{A'_h} G_p(\nabla u'_h)]\, dy + \lambda_h P(A'_h; B) + \eta_h,\end{aligned} \tag{3.7}$$

where $\{\eta_h\}_{h\in\mathbb{N}} \subset \mathbb{R}$ is an infinitesimal sequence.

In the proof of Theorem 3.4.4 we will show that the sequence of appropriately rescaled miniminal configurations of problem (P) is asymptotically minimizing. The following theorem is concerned with the behaviour of asimptotically minimizing sequences.

Theorem 3.4.3. *Let $\{\lambda_h\}_{h\in\mathbb{N}} \subset \mathbb{R}^+$ and $\{(A_h, u_h)\}_{h\in\mathbb{N}} \subset W^{1,p}(B_1) \times \mathcal{A}(B_1)$. Assume that (A_h, u_h) is λ_h-asymptotically miminimizing and that*

i) $\left\{\int_{B_1} [F_p(\nabla u_h) + \mathbb{1}_{A_h} G_p(\nabla u_h)]\, dy + \lambda_h P(A_h; B_1)\right\}_{h\in\mathbb{N}}$ *is bounded;*

ii) $u_h \rightharpoonup u$ *in* $W^{1,p}(B_1)$;

iii) $\mathbb{1}_{A_h} \to \mathbb{1}_A$ *in* $L^1(B_1)$ *and* $\lambda_h \to +\infty$;

iv) $G_p(\nabla u_h)$ *is locally equi-integrable in* B_1.

Then

a) $u_h \to u$ *in* $W^{1,p}_{loc}(B_1)$;

b) $\lambda_h P(A_h; B_\rho) \to 0$, *for all* $\rho \in (0, 1)$;

c) $A = \varnothing$ *or* $A = B_1$ *and* u *minimizes the functional*

$$\int_{B_1} [F_p(\nabla v) + \mathbb{1}_A G_p(\nabla v)]\, dy,$$

among all $v \in u + W^{1,p}_0(B_1)$.

Proof. Let us prove *a)*. The hypothesis *iv)* implies that

$$\lim_{h\to+\infty} \int_{B_1} \psi[\mathbb{1}_A G_p(\nabla u_h) - \mathbb{1}_{A_h} G_p(\nabla u_h)]\, dy = 0, \quad \forall \psi \in C^\infty_c(B_1). \tag{3.8}$$

Let $\tilde{u}_h := (1-\psi)u_h + \psi u$, $\psi \in C_c^\infty(B_1)$, with $0 \leqslant \psi \leqslant 1$. Then $\nabla\tilde{u}_h = (u-u_h)\nabla\psi + (1-\psi)\nabla u_h + \psi\nabla u$. Testing $(A_h, \tilde{u}_h)$, we have

$$\int_{B_1} [F_p(\nabla u_h) + \mathbb{1}_{A_h} G_p(\nabla u_h)]\, dy \leqslant \int_{B_1} [F_p(\nabla\tilde{u}_h) + \mathbb{1}_{A_h} G_p(\nabla\tilde{u}_h)]\, dy + \eta_h, \tag{3.9}$$

where $\{\eta_h\}_{h\in\mathbb{N}} \subset \mathbb{R}$ is the infinitesimal sequence in (3.7). By the convexity of F_p and G_p and Lemma 3.1.4, it follows that

$$\begin{aligned}
&\int_{B_1} [F_p(\nabla\tilde{u}_h) + \mathbb{1}_{A_h} G_p(\nabla\tilde{u}_h)]\, dy \\
&\leqslant \int_{B_1} [F_p((1-\psi)\nabla u_h + \psi\nabla u) + \mathbb{1}_{A_h} G_p((1-\psi)\nabla u_h + \psi\nabla u)]\, dy \\
&+ \int_{B_1} [\nabla F_p((u-u_h)\nabla\psi + (1-\psi)\nabla u_h + \psi\nabla u)] \cdot [(u-u_h)\nabla\psi]\, dy \\
&+ \int_{B_1} [\nabla G_p((u-u_h)\nabla\psi + (1-\psi)\nabla u_h + \psi\nabla u)] \cdot [(u-u_h)\nabla\psi]\, dy \\
&\leqslant \int_{B_1} [(1-\psi)F_p(\nabla u_h) + \psi F_p(\nabla u) + \mathbb{1}_{A_h}[(1-\psi)G_p(\nabla u_h) + \psi G_p(\nabla u)]\, dy \\
&+ c(p,L,\beta) \int_B (\mu^2 + |(u-u_h)\nabla\psi + (1-\psi)\nabla u_h + \psi\nabla u|^2)^{\frac{p-1}{2}} |(u-u_h)\nabla\psi|\, dy.
\end{aligned}$$

Using the previous one in (3.9), we obtain

$$\begin{aligned}
&\int_{B_1} \psi[F_p(\nabla u_h) + \mathbb{1}_{A_h} G_p(\nabla u_h)]\, dy \\
&\leqslant \int_{B_1} \psi[F_p(\nabla u) + \mathbb{1}_{A_h} G_p(\nabla u)]\, dy + \eta_h \\
&+ c(p,L,\beta) \int_B (\mu^2 + |(u-u_h)\nabla\psi + (1-\psi)\nabla u_h + \psi\nabla u|^2)^{\frac{p-1}{2}} |(u-u_h)\nabla\psi|\, dy.
\end{aligned} \tag{3.10}$$

The second term in the right hand side is infinitesimal; indeed, using Hölder's inequality, we have

$$\begin{aligned}
&\int_B (\mu^2 + |(u-u_h)\nabla\psi + (1-\psi)\nabla u_h + \psi\nabla u|^2)^{\frac{p-1}{2}} |(u-u_h)\nabla\psi|\, dy \\
&\leqslant \|u-u_h\|_{L^p(B_1)} \left(\int_{B_1} (\mu^p + |(u-u_h)\nabla\psi|^p + |(1-\psi)\nabla u_h|^p + |\psi\nabla u|^p)\, dy \right)^{\frac{p-1}{p}},
\end{aligned}$$

which tends to 0 as h approaches $+\infty$. So we can inglobe the second term in the right hand side of (3.10) in η_h. Add $\int_{B_1} \psi \mathbb{1}_A G_p(\nabla u_h)\, dy$ to both sides in (3.10) in order to obtain

$$\int_{B_1} \psi[F_p(\nabla u_h) + \mathbb{1}_A G_p(\nabla u_h)]\, dy \leqslant \int_{B_1} \psi[F_p(\nabla u) + \mathbb{1}_{A_h} G_p(\nabla u)]\, dy$$

$$+\int_{B_1}\psi[\mathbb{1}_A G_p(\nabla u_h)-\mathbb{1}_{A_h}G_p(\nabla u_h)]\,dy+\tilde{\eta}_h,$$

where $\{\tilde{\eta}_h\}_{h\in\mathbb{N}}\subset\mathbb{R}$ is infinitesimal. Thanks to (3.8), we can pass to the upper limit and obtain

$$\limsup_{h\to+\infty}\int_{B_1}\psi[F_p(\nabla u_h)+\mathbb{1}_A G_p(\nabla u_h)]\,dy\leqslant\int_{B_1}\psi[F_p(\nabla u)+\mathbb{1}_A G_p(\nabla u)]\,dy.$$

Finally, by lower semicontinuity, we gain

$$\lim_{h\to+\infty}\int_{B_1}\psi[F_p(\nabla u_h)+\mathbb{1}_A G_p(\nabla u_h)]\,dy=\int_{B_1}\psi[F_p(\nabla u)+\mathbb{1}_A G_p(\nabla u)]\,dy. \tag{3.11}$$

By the strong quasi-convexity of F_p and G_p and Lemma 3.1.7, we have

$$\begin{aligned}&\int_{B_1}\psi|V(\nabla u_h)-V(\nabla u)|^2\,dy \qquad (3.12)\\ &\leqslant c(n,p)\int_{B_1}(\mu^2+|\nabla u_h|^2+|\nabla u|^2)^{\frac{p-2}{2}}|\nabla u_h-\nabla u|^2\,dy\\ &\leqslant c(n,p,l)\Bigg[\int_{B_1}[\psi(F_p(\nabla u_h)-F_p(\nabla u))-\nabla F_p(\nabla u)\cdot[\psi(\nabla u_h-\nabla u)]]\,dy\\ &+\int_{B_1}[\psi 1_A(G_p(\nabla u_h)-G_p(\nabla u))-\mathbb{1}_A\nabla G_p(\nabla u)\cdot[\psi(\nabla u_h-\nabla u)]]\,dy\Bigg].\end{aligned}$$

Let $h\to+\infty$ in (3.12). By the *ii)* and (3.11), we infer

$$\lim_{h\to+\infty}\int_{B_1}\psi|V(\nabla u_h)-V(\nabla u)|^2\,dy=0.$$

Thanks to Lemma 3.1.8 and the arbitrariety of ψ, we conclude that $u_h\to u$ in $W^{1,p}_{loc}(B_1)$.

Let us prove *b)*. Since $\lambda_h\to+\infty$ and the energies are bounded by an appropriate constant c, it holds that

$$P(A_h;B_1)\leqslant\frac{c}{\lambda_h}.$$

Let $h\to+\infty$ in the previous inequality. By semicontinuity we infer that $P(A;B_1)=0$. Thanks to isoperimetric inequality it follows that $A=\varnothing$ or $A=B$. We'll discuss the case $A=\varnothing$, being the other one similar. For h large enough, by the isoperimetric inequality we have

$$|A_h|=\min\{|A_h|,|B_1\backslash A_h|\}\leqslant c(n)\left(\frac{c}{\lambda_h}\right)^{\frac{n}{n-1}}.$$

Denoting $\mathbb{1}_h(\rho)=\mathbb{1}_{A_h\cap\partial B_\rho}$, for all $h\in\mathbb{N}$ and $\rho\in(0,1)$, the Coarea formula provides that

$$|A_h|=\int_0^1 d\rho\int_{\partial B_\rho}\mathbb{1}_h(\rho)\,d\mathcal{H}^{n-1}\leqslant c(n)\left(\frac{c}{\lambda_h}\right)^{\frac{n}{n-1}},$$

which means that the sequence of functions $\left\{\lambda_h \int_{\partial B_\rho} \mathbb{1}_h(\rho)\, d\mathcal{H}^{n-1}\right\}_{h\in\mathbb{N}}$ converges to 0 in $L^1(0,1)$. Thus, it converges to 0 for almost every $\rho \in (0,1)$. Then, for every $\rho \in (0,1)$ fixed, we can find a sequence $\{\rho_h\}_{h\in\mathbb{N}} \subset \left(\rho, \frac{1+\rho}{2}\right)$ such that

$$\lambda_h \int_{\partial B_{\rho_h}} \mathbb{1}_h(\rho_h)\, d\mathcal{H}^{n-1} \to 0, \tag{3.13}$$

as h approaches $+\infty$. Comparing $\{(A_h, u_h)\}_{h\in\mathbb{N}}$ and $\{(A_h, u_h \backslash \overline{B}_{\rho_h})\}_{h\in\mathbb{N}}$, using (3.13) and the equality

$$P(A_h \backslash B_{\rho_h}; B_1) = P(A_h; B_1 \backslash \overline{B}_{\rho_h}) + \int_{\partial B_{\rho_h}} \mathbb{1}_h(\rho_h)\, d\mathcal{H}^{n-1},$$

there exists an infinitesimal sequence $\{\eta_h\}_{h\in\mathbb{N}} \subset \mathbb{R}$ such that

$$\begin{aligned}
\lambda_h P(A_h; B_{\rho_h}) \leqslant \lambda_h P(A_h; B_1) &\leqslant \lambda_h P(A_h \backslash \overline{B}_{\rho_h}; B_1) + \eta_h \\
&= \lambda_h P(A_h; B_1 \backslash \overline{B}_{\rho_h}) + \lambda_h \int_{\partial B_{\rho_h}} \mathbb{1}_h(\rho_h)\, d\mathcal{H}^{n-1} + \eta_h \\
&= \lambda_h \int_{\partial B_{\rho_h}} \mathbb{1}_h(\rho_h)\, d\mathcal{H}^{n-1} + \eta_h,
\end{aligned}$$

provided h is so large that $A_h \subset \overline{B}_{\frac{\rho+1}{2}}$. Thus, thanks to (3.13) the sequence $\{\lambda_h P(A_h; B_{\rho_h})\}_{h\in\mathbb{N}}$ is infinitesimal and we can conclude that

$$\lambda_h P(A_h; B_\rho) \to 0,$$

as h approaches $+\infty$, since $\rho_h > \rho$.

Let us prove *c)*. Comparing (A_h, u_h) with $(A_h, \tilde{u}_h) = (A_h, u_h + \varphi)$, where $\varphi \in C^1(B_1)$ and spt$(\varphi) \subset B_\rho$, we have

$$\int_{B_\rho} [F_p(\nabla u_h) + \mathbb{1}_{A_h} G_p(\nabla u_h)]\, dy \leqslant \int_{B_\rho} [F_p(\nabla \tilde{u}_h) + \mathbb{1}_{A_h} G_p(\nabla \tilde{u}_h)]\, dy + \eta_h,$$

with $\{\eta_h\}_{h\in\mathbb{N}} \subset \mathbb{R}$ infinitesimal and $\rho \in (0,1)$ arbitrary. Thanks to *a)*, we can use the dominated convergence theorem in order to pass to the limit as h approaches $+\infty$, obtaining

$$\int_{B_\rho} [F_p(\nabla u) + \mathbb{1}_A G_p(\nabla u)]\, dy \leqslant \int_{B_\rho} [F_p(\nabla(u+\varphi)) + \mathbb{1}_A G_p(\nabla(u+\varphi))]\, dy.$$

By the arbitrariety of ρ and φ we can conclude the proof. □

The following theorem is the main tool for proving Proposition 3.4.1.

Theorem 3.4.4 (Energy decay estimate)**.** *Let $K \subset \Omega$ be a compact set, $\delta = \operatorname{dist}(K, \partial\Omega) > 0$ and $\varepsilon \in (0,1)$. Let $\tilde{c} = \tilde{c}(p, l, L, \alpha, \beta, \mu)$ and $c_{\mathcal{H}} = c_{\mathcal{H}}(n, p, l, L, \alpha, \beta)$ the constants of Lemma 3.1.5 and Corollary 3.1.10 for*

$$\mathcal{H}(w) = \int_{B_1} [F_p(\nabla w) + G_p(\nabla w)]\, dx.$$

Moreover, let $\tau \in (0,1)$ be such that $\tau^\varepsilon < \frac{1}{2(1+\omega_n \tilde{c})}$. Then there exist two positive constants γ and θ such that for any solution (A, u) of the problem (P) *and for any ball $B_\rho(y)$ with $y \in K$ and $\rho \in (0, \frac{\delta}{2})$ the two estimates*

$$\int_{B_\rho} [F_p(\nabla u) + \mathbb{1}_A G_p(\nabla u)]\, dx + P(A; B_\rho) \leqslant \gamma \rho^{n-1},$$

$$\rho^n \leqslant \theta \Big[\int_{B_\rho} [F_p(\nabla u) + \mathbb{1}_A G_p(\nabla u)]\, dx + P(A; B_\rho) \Big],$$

imply that

$$\begin{aligned} &\int_{B_{\tau\rho}(y)} [F_p(\nabla u) + \mathbb{1}_A G_p(\nabla u)]\, dx + P(A; B_{\tau\rho}(y)) \\ &\leqslant \frac{c_{\mathcal{H}}(1+\beta)L}{l} \tau^{n-\varepsilon} \Big[\int_{B_\rho} [F_p(\nabla u) + \mathbb{1}_A G_p(\nabla u)]\, dx + P(A; B_\rho) \Big]. \end{aligned}$$

Proof. Let us suppose by contradiction that there exist two sequences $\{\gamma_h\}_{h\in\mathbb{N}}$ and $\{\theta_h\}_{h\in\mathbb{N}}$ which tend to 0, a sequence of minimizing couples $\{(D_h, w_h)\}_{h\in\mathbb{N}}$ of (P) and a sequence of balls $\{B_{\rho_h}(x_h)\}_{h\in\mathbb{N}}$, with $x_h \in K$ and $\rho_h \in (0, \frac{\delta}{2})$, for all $h \in \mathbb{N}$, such that these estimates hold:

$$\int_{B_{\rho_h}(x_h)} [F_p(\nabla w_h) + \mathbb{1}_{D_h} G_p(\nabla w_h)]\, dx + P(D_h; B_{\rho_h}(x_h)) = \gamma_h \rho_h^{n-1}, \quad (3.14)$$

$$\rho_h^n \leqslant \theta_h \Big[\int_{B_{\rho_h}(x_h)} [F_p(\nabla w_h) + \mathbb{1}_{D_h} G_p(\nabla w_h)]\, dx + P(D_h; B_{\rho_h}(x_h)) \Big], \quad (3.15)$$

$$\begin{aligned} &\int_{B_{\tau\rho_h}(x_h)} [F_p(\nabla w_h) + \mathbb{1}_{D_h} G_p(\nabla w_h)]\, dx + P(D_h; B_{\tau\rho_h}(x_h)) \quad (3.16) \\ &> \frac{c_{\mathcal{H}}(1+\beta)L}{l} \tau^{n-\varepsilon} \Big[\int_{B_{\rho_h}(x_h)} [F_p(\nabla w_h) + \mathbb{1}_{D_h} G_p(\nabla w_h)]\, dx + P(D_h; B_{\rho_h}(x_h)) \Big]. \end{aligned}$$

In what follows it will be important that the sequence $\{w_h\}_{h\in\mathbb{N}}$ is locally equibounded in Ω. It descends from Theorem 3.3.1 once we have proved that $\{w_h\}_{h\in\mathbb{N}}$ is bounded in $W^{1,p}(\Omega)$, which holds true; indeed, by the minimality of (D_h, w_h), (F1), (3.1) and Poincaré's inequality it follows that

$$\begin{aligned} &\min_{v \in u_0 + W^{1,p}(\Omega)} \mathcal{F}(v, \Omega) \\ &\geqslant \mathcal{F}(w_h, D_h) \geqslant l \int_\Omega |\nabla w_h|^p\, dx + \int_\Omega \gamma\, dx - k \int_\Omega |w_h|^p\, dx \\ &\geqslant l \int_\Omega |\nabla w_h|^p\, dx + \int_\Omega \gamma\, dx - 2^{p-1} k \int_\Omega |w_h - u_0|^p\, dx - 2^{p-1} k \int_\Omega |u_0|^p\, dx \\ &\geqslant (l - 2^{p-1} k \lambda) \int_\Omega |\nabla w_h|^p\, dx + \int_\Omega \gamma\, dx - 2^{p-1} k \int_\Omega |u_0|^p\, dx, \end{aligned}$$

since $k < \frac{l}{2^{p-1}\lambda}$. Rescale the functions w_h; define

$$u_h(y) := \frac{w_h(x_h + \rho_h y) - \overline{w}_h}{\rho_h^{\frac{p-1}{p}} \gamma_h^{\frac{1}{p}}} \in W^{1,p}(B_1), \quad A_h := \frac{D_h - x_h}{\rho_h}, \quad \lambda_h := \frac{1}{\gamma_h}, \tag{3.17}$$

where $\overline{w}_h = \fint_{B_1} w_h(x_h + \rho_h y)\, dy$, for all $h \in \mathbb{N}$. By the usual change of variables $x := x_h + \rho_h y$, we have:

$$\begin{aligned}
&\int_{B_{\rho_h}(x_h)} [F_p(\nabla w_h) + \mathbb{1}_{D_h} G_p(\nabla w_h)]\, dx + P(D_h; B_{\rho_h}(x_h)) \\
&= \gamma_h \rho_h^{n-1} \left[\int_{B_1} [F_p(\nabla u_h) + \mathbb{1}_{A_h} G_p(\nabla u_h)]\, dy + \lambda_h P(A_h; B_1) \right].
\end{aligned}$$

Rescale the estimates (3.14), (3.15) and (3.16), obtaining

$$\int_{B_1} [F_p(\nabla u_h) + \mathbb{1}_{A_h} G_p(\nabla u_h)]\, dy + \lambda_h P(A_h; B_1) = 1, \tag{3.18}$$

$$\rho_h \leqslant \theta_h \gamma_h, \tag{3.19}$$

$$\int_{B_\tau} [F_p(\nabla u_h) + \mathbb{1}_{A_h} G_p(\nabla u_h)]\, dy + \lambda_h P(A_h; B_\tau) > \frac{c_{\mathcal{H}}(1+\beta)L}{l} \tau^{n-\varepsilon}. \tag{3.20}$$

We want to apply Theorem 3.4.3 to the sequence $\{(A_h, u_h)\}_{n \in \mathbb{N}}$.
Firstly, let us prove that $\{(A_h, u_h)\}_{n \in \mathbb{N}}$ is λ_h-asymptotically minimizing. Let $K' \subset B_1$ be a compact set and $\{(A'_h, u'_h)\}_{h \in \mathbb{N}}$ such that $\{u'_h\}_{h \in \mathbb{N}}$ is a bounded sequence in $W^{1,p}(B_1)$ with spt$(u'_h - u_h) \subset K'$ and $A'_h \subset B_1$ with $A'_h \Delta A_h \subset K'$.
Rescale the functions u'_h:

$$w'_h(x) := \rho_h^{\frac{p-1}{p}} \gamma_h^{\frac{1}{p}} u'_h\left(\frac{x - x_h}{\rho_h}\right) + \overline{w}_h \in W^{1,p}(B_{\rho_h}(x_h)), \quad D'_h = x_h + \rho_h A'_h.$$

Compare the two sequences $\{(D_h, w_h)\}_{h \in \mathbb{N}}$ and $\{(D'_h, w'_h)\}_{h \in \mathbb{N}}$: by the minimality of $\{(D_h, w_h)\}_{h \in \mathbb{N}}$ and by (3.2) we have

$$\begin{aligned}
&\int_{B_1} [F_p(\nabla u'_h) + \mathbb{1}_{A'_h} G_p(\nabla u'_h)]\, dy + \lambda_h P(A'_h; B) \\
&= \frac{1}{\gamma_h \rho_h^{n-1}} \left[\int_{B_{\rho_h}(x_h)} [F_p(\nabla w'_h) + \mathbb{1}_{D'_h} G_p(\nabla w'_h)]\, dx + P(D'_h; B_{\rho_h}(x_h)) \right] \\
&\geqslant \frac{1}{\gamma_h \rho_h^{n-1}} \left[\int_{B_{\rho_h}(x_h)} [F(\nabla w_h) + \mathbb{1}_{D_h} G(\nabla w_h)]\, dx + P(D_h; B_{\rho_h}(x_h)) \right. \\
&+ \int_{B_{\rho_h}(x_h)} [f_{D_h}(x, w_h) - f_{D'_h}(x, w'_h)]\, dx \\
&\left. + \int_{B_{\rho_h}(x_h)} \{[F_p(\nabla w'_h) - F(\nabla w'_h)] + \mathbb{1}_{D'_h}[G_p(\nabla w'_h) - G(\nabla w'_h)]\}\, dx \right] \\
&\geqslant \frac{1}{\gamma_h \rho_h^{n-1}} \left[\int_{B_{\rho_h}(x_h)} [F(\nabla w_h) + \mathbb{1}_{D_h} G(\nabla w_h)]\, dx + P(D_h; B_{\rho_h}(x_h)) \right.
\end{aligned}$$

$$- C_0 \int_{B_{\rho_h}(x_h)} [2 + |w_h|^q + |w'_h|^q]\, dx$$
$$+ \int_{B_{\rho_h}(x_h)\cap\{|\nabla w'_h|\geqslant t_0\}} \left\{[F_p(\nabla w'_h) - F(\nabla w'_h)] + \mathbb{1}_{D'_h}[G_p(\nabla w'_h) - G(\nabla w'_h)]\right\} dx$$
$$+ \int_{B_{\rho_h}(x_h)\cap\{|\nabla w'_h|< t_0\}} \left\{[F_p(\nabla w'_h) - F(\nabla w'_h)] + \mathbb{1}_{D'_h}[G_p(\nabla w'_h) - G(\nabla w'_h)]\right\} dx\Bigg]$$

In the sixth line of the previous inequality we need F_p and G_p in place of F and G, so that by (F3) and (G3) we infer

$$\begin{aligned}
&\int_{B_{\rho_h}(x_h)} [F(\nabla w_h) + \mathbb{1}_{D_h} G(\nabla w_h)]\, dx \\
&\geqslant \int_{B_{\rho_h}(x_h)\cap\{|\nabla w_h|\geqslant t_0\}} [F(\nabla w_h) + \mathbb{1}_{D_h} G(\nabla w_h)]\, dx \\
&\geqslant \int_{B_{\rho_h}(x_h)\cap\{|\nabla w_h|\geqslant t_0\}} [F_p(\nabla w_h) + \mathbb{1}_{D_h} G_p(\nabla w_h)]\, dx - 2a \int_{B_{\rho_h}(x_h)} |\nabla w_h|^{p-m}\, dx \\
&\geqslant \int_{B_{\rho_h}(x_h)} [F_p(\nabla w_h) + \mathbb{1}_{D_h} G_p(\nabla w_h)]\, dx - c(n,p,L,\beta,t_0)\rho_h^n \\
&- 2a \int_{B_{\rho_h}(x_h)} |\nabla w_h|^{p-m}\, dx.
\end{aligned}$$

Thus by homogeneity, (F3) and (G3), we get

$$\begin{aligned}
&\int_{B_1} [F_p(\nabla u'_h) + \mathbb{1}_{A'_h} G_p(\nabla u'_h)]\, dy + \lambda_h P(A'_h; B) \\
&\geqslant \int_{B_1} [F_p(\nabla u_h) + \mathbb{1}_{D_h} G_p(\nabla u_h)]\, dx + \lambda_h P(A_h; B) \\
&- \frac{C_0}{\gamma_h \rho_h^{n-1}} \int_{B_{\rho_h}(x_h)} (|w_h|^q + |w'_h|^q)\, dx - c(n,p,L,\beta,t_0)\frac{\rho_h}{\gamma_h} \\
&- \frac{2a}{\gamma_h \rho_h^{n-1}} \int_{B_{\rho_h}(x_h)} [|\nabla w'_h|^{p-m} + |\nabla w_h|^{p-m}]\, dx.
\end{aligned}$$

In order to prove that $\{(A_h, u_h)\}_{h\in\mathbb{N}}$ is λ_h-asymptotically minimizing, we need to show that

$$\lim_{h\to+\infty} \Bigg[\frac{C_0}{\gamma_h \rho_h^{n-1}} \int_{B_{\rho_h}(x_h)} (|w_h|^q + |w'_h|^q)\, dx + c(n,p,L,\beta,t_0)\frac{\rho_h}{\gamma_h}$$
$$+ \frac{2a}{\gamma_h \rho_h^{n-1}} \int_{B_{\rho_h}(x_h)} [|\nabla w'_h|^{p-m} + |\nabla w_h|^{p-m}]\, dx\Bigg] = 0.$$

By (3.19) it is clear that $\lim_{h\to+\infty} \frac{\rho_h}{\gamma_h} = 0$. Since $\{w_h\}_{n\in\mathbb{N}}$ is locally equibounded in Ω, also

$$\lim_{h\to+\infty} \frac{1}{\gamma_h \rho_h^{n-1}} \int_{B_{\rho_h}(x_h)} |w_h|^q\, dx = 0.$$

It remains to prove that

$$\lim_{h\to+\infty}\frac{1}{\gamma_h\rho_h^{n-1}}\int_{B_{\rho_h}(x_h)}|w_h'|^q\,dx=0, \tag{3.21}$$

$$\lim_{h\to+\infty}\frac{1}{\gamma_h\rho_h^{n-1}}\int_{B_{\rho_h}(x_h)}[|\nabla w_h'|^{p-m}+|\nabla w_h|^{p-m}]\,dx=0. \tag{3.22}$$

Let us prove (3.21). Since $\{w_h\}_{h\in\mathbb{N}}$ is locally equibounded by a constant $M>0$, substituting the expression of $\overline{w}_h$ from (3.17) it follows that

$$\begin{aligned}
&\frac{1}{\gamma_h\rho_h^{n-1}}\int_{B_{\rho_h}(x_h)}|w_h'|^q\,dx=\frac{\rho_h}{\gamma_h}\int_{B_1}|\rho_h^{\frac{(p-1)}{p}}\gamma_h^{\frac{1}{p}}u_h'+\overline{w}_h|^q\,dy\\
&\leqslant c(q)\left[\frac{\rho_h}{\gamma_h}\rho_h^{\frac{(p-1)q}{p}}\gamma_h^{\frac{q}{p}}\int_{B_1}|u_h'-u_h|^q\,dy+\frac{\rho_h}{\gamma_h}\int_{B_1}|w_h(x_h+\rho_h y)|^q\,dy\right]\\
&\leqslant c(n,p,q)\frac{\rho_h}{\gamma_h}\rho_h^{\frac{(p-1)q}{p}+1}\gamma_h^{\frac{q}{p}-1}\Big(\|u_h'\|^q_{W^{1,p}(B_1)}+\|u_h\|^q_{W^{1,p}(B_1)}\Big)+c(n,q,M)\frac{\rho_h}{\gamma_h},
\end{aligned}$$

where we used the Sobolev embedding theorem. Since $q\geqslant p$, $\{u_h'\}_{h\in\mathbb{N}}$ and $\{u_h\}_{h\in\mathbb{N}}$ are bounded in $W^{1,p}(B_1)$ and $\lim_{h\to+\infty}\frac{\rho_h}{\gamma_h}=0$, we conclude that (3.21) holds true. We are left to prove (3.22). By Hölder's inequality we get

$$\begin{aligned}
&\frac{1}{\gamma_h\rho_h^{n-1}}\int_{B_{\rho_h}(x_h)}[|\nabla w_h'|^{p-m}+|\nabla w_h|^{p-m}]\,dx\\
&\leqslant\frac{c(n,p,m)}{\gamma_h\rho_h^{n-1}}\left[\left(\int_{B_{\rho_h}(x_h)}|\nabla w_h'|^p\,dx\right)^{1-\frac{m}{p}}+\left(\int_{B_{\rho_h}(x_h)}|\nabla w_h|^p\,dx\right)^{1-\frac{m}{p}}\right]\rho_h^{\frac{nm}{p}}\\
&=\frac{c(n,p,m)}{\gamma_h\rho_h^{n-1}}\big(\gamma_h\rho_h^{n-1}\big)^{1-\frac{m}{p}}\left[\left(\int_{B_1}|\nabla u_h'|^p\,dy\right)^{1-\frac{m}{p}}+\left(\int_{B_1}|\nabla u_h|^p\,dy\right)^{1-\frac{m}{p}}\right]\rho_h^{\frac{nm}{p}}\\
&\leqslant c(n)\left(\frac{\rho_h}{\gamma_h}\right)^{\frac{m}{p}}\big(\|u_h'\|^{p-m}_{W^{1,p}(B_1)}+\|u_h\|^{p-m}_{W^{1,p}(B_1)}\big).
\end{aligned}$$

Since $\lim_{h\to+\infty}\frac{\rho_h}{\gamma_h}=0$ and $\{u_h'\}_{h\in\mathbb{N}}$, $\{u_h\}_{h\in\mathbb{N}}$ are bounded in $W^{1,p}(B_1)$, we obtain (3.22).

Thanks to (3.18) there exist a function $u\in W^{1,p}(B_1)$ and a set of finite perimeter $A\subset B_1$ such that

$$u_h\rightharpoonup u \text{ in } W^{1,p}(B_1)\quad\text{and}\quad \mathbb{1}_{A_h}\to\mathbb{1}_A \text{ in } L^1(B_1).$$

We are finally in position to apply Theorem 3.4.3 to $\{(A_h,u_h)\}_{h\in\mathbb{N}}$. It remains only to prove that $G_p(\nabla u_h)$ is locally equi-integrable, which we will prove later. As a consequence of Theorem 3.4.3 we have that $A=\varnothing$ or $A=B_1$. We'll discuss the case $A=\varnothing$, being the other one similar. Thanks to Corollary 3.1.10 and Lemma 3.1.5, by lower semicontinuity we infer

$$\int_{B_\tau}|\nabla u|^p\,dy\leqslant\int_{B_\tau}(\mu^2+|\nabla u|^2)^{\frac{p}{2}}\,dy\leqslant c_{\mathcal{H}}\tau^n\int_{B_1}(\mu^2+|\nabla u|^2)^{\frac{p}{2}}\,dy \tag{3.23}$$

$$\leqslant \frac{2c_{\mathcal{H}}}{l}\tau^n\left(\int_{B_1} F_p(\nabla u)\,dy + \omega_n\tilde{c}\right)$$
$$\leqslant \frac{2c_{\mathcal{H}}}{l}\tau^n\left(\liminf_{h\to+\infty}\int_{B_1} F_p(\nabla u_h)\,dy + \omega_n\tilde{c}\right).$$

Using inequality (3.18), (3.20) and the *b)* of Theorem 3.4.3, we gain

$$\begin{aligned}
&\frac{2c_{\mathcal{H}}}{l}\tau^n\left(\liminf_{h\to+\infty}\int_{B_1} F_p(\nabla u_h)\,dy + \omega_n\tilde{c}\right)\\
&= \frac{2c_{\mathcal{H}}}{l}\tau^n\left(1 - \limsup_{h\to+\infty}\lambda_h P(A_h; B_1) + \omega_n\tilde{c}\right)\\
&\leqslant \frac{2c_{\mathcal{H}}}{l}\tau^n(1+\omega_n\tilde{c}) < \frac{c_{\mathcal{H}}}{l}\tau^{n-\varepsilon} < \frac{1}{(1+\beta)L}\int_{B_\tau} F_p(\nabla u)\,dy \leqslant \int_{B_\tau}|\nabla u|^p\,dy.
\end{aligned}$$

Comparing the previous estimate with (3.23) we reach a contradiction.
We are only left to prove the equi-integrability of $G_p(\nabla u_h)$ in B_1. It is enough to prove that for all $t \in (0,1)$ there exists $r > p$ such that

$$\sup_{h\in\mathbb{N}}\int_{B_t}|\nabla u_h|^r\,dy < +\infty. \tag{3.24}$$

Indeed, fix $\varepsilon > 0$, a compact set $K' \subset B_1$ and $A \subset K'$. Then by the growth condition of G_p and Hölder's inequality, it follows that

$$\sup_{n\in\mathbb{N}}\int_A G_p(\nabla u_h)\,dy \leqslant \beta L\int_A|\nabla u_h|^p\,dy \leqslant \beta L|A|^{1-\frac{p}{r}}\left(\sup_{h\in\mathbb{N}}\int_{B_t}|\nabla u_h|^r\,dy\right)^{\frac{p}{r}}.$$

In order to prove (3.24), we can apply Theorem 3.3.1: there exist two constants $\gamma > 0$ and $r > p$ depending only on $n, p, q, \beta, l, L, \mu, C_0, \|w_h\|_{L^\infty(K)}$ such that for all $h \in \mathbb{N}$ and $y \in K$, with $\mathrm{dist}(Q_{2\rho_h}(y), K) \leqslant \frac{\delta}{2}$ we have the following local higher summability:

$$\int_{Q_{\rho_h}(y)}|\nabla w_h|^r\,dx \leqslant \gamma\left[\rho_h^{n\left(1-\frac{r}{p}\right)}\left(\int_{Q_{2\rho_h}(y)}|\nabla w_h|^p\,dx\right)^{\frac{r}{p}} + \rho_h^n\right].$$

It can be also shown that the dependence of γ and r on $\|w_h\|_{L^\infty(K)}$ is uniform with respect to h, since $\{w_h\}_{h\in\mathbb{N}}$ is locally equibounded in Ω.
Fix $t \in (0,1)$. By a covering argument it follows that

$$\int_{B_{t\rho_h}(x_h)}|\nabla w_h|^r\,dx \leqslant c(n,t)\gamma\left[\rho_h^{n\left(1-\frac{r}{p}\right)}\left(\int_{B_{\rho_h}(x_h)}|\nabla w_h|^p\,dx\right)^{\frac{r}{p}} + \rho_h^n\right].$$

Rescale and write the estimate in terms of u_h:

$$\int_{B_t}|\nabla u_h|^r\,dy \leqslant c(n,t)\gamma\left[\left(\int_{B_1}|\nabla u_h|^p\,dy\right)^{\frac{r}{p}} + \left(\frac{\rho_h}{\gamma_h}\right)^{\frac{r}{p}}\right]$$

$$\leqslant c(n,t,r,M')\gamma\left[1+\left(\frac{\rho_h}{\gamma_h}\right)^{\frac{r}{p}}\right],$$

where $M'>0$ is an upper bound for $\{\|u_h\|_{W^{1,p}(\Omega)}\}_{h\in\mathbb{N}}$. Using (3.19) we prove our assertion. □

The last proposition that we need to prove Proposition 3.4.1 follows from the previous result and is based on an iteration argument.

Proposition 3.4.5. *Let K,γ,θ,δ be given by Theorem 3.4.4 and let (A,u) be a solution of* (P)*. Let $y\in K$ and denote*

$$\Psi(\rho)=\int_{B_\rho(y)}[F_p(\nabla u)+\mathbb{1}_A G_p(\nabla u)]\,dx+P(A;B_\rho(y)),\quad \forall\rho\in\left(0,\frac{\delta}{2}\right).$$

Moreover, let $\varepsilon\in(0,1)$ and $\sigma\in(n-1,n-\varepsilon)$ such that there exists $\tau\in(0,1)$ satisfying $\frac{c_{\mathcal{H}}(1+\beta)L}{l}\tau^{n-\varepsilon}<\tau^\sigma$ and $\tau^\varepsilon<\frac{1}{2(1+\omega_n\tilde{c})}$. Set

$$\xi=\min\{\mathrm{dist}(y,\partial\Omega),\gamma,\tau^\sigma\gamma\theta\}.$$

If $\Psi(\rho)<\xi\rho^{n-1}$ for some $\rho\in(0,\xi)$, then

$$\Psi(\eta)<\tau^{-\sigma}\gamma\rho^{n-1}\left(\frac{\eta}{\rho}\right)^\sigma,\quad \forall\eta\in(0,\rho].$$

In particular,

$$\lim_{\eta\to0}\eta^{1-n}\Psi(\eta)=0.$$

Proof. Let us assume that $\Psi(\rho)<\xi\rho^{n-1}$ for some $\rho\in(0,\xi)$. Since Ψ is non-decreasing, it suffices to show by induction on $j\in\mathbb{N}_0$ that

$$\Psi(\eta_j)<\gamma\rho^{n-1}\left(\frac{\eta_j}{\rho}\right)^\sigma,$$

where $\eta_j=\tau^j\rho$. Since we chose $\xi<\gamma$, the inequality holds true if $j=0$. Let us assume that it holds true for $j>0$. By induction we state

$$\frac{\Psi(\eta_j)}{\eta_j^{n-1}}<\gamma\left(\frac{\eta_j}{\rho}\right)^{\sigma-n+1}<\gamma,$$

that is $\Psi(\eta_j)<\gamma\eta_j^{n-1}$. If $\theta\Psi(\eta_j)>\eta_j^n$, thanks to the choice $\xi<\mathrm{dist}(y,\partial\Omega)$, we can apply Theorem 3.4.4 and the inductive hypothesis in order to obtain

$$\Psi(\eta_{j+1})\leqslant\tau^\sigma\Psi(\eta_j)<\tau^\sigma\gamma\rho^{n-1}\left(\frac{\eta_j}{\rho}\right)^\sigma=\gamma\rho^{n-1}\left(\frac{\eta_{j+1}}{\rho}\right)^\sigma.$$

If $\theta\Psi(\eta_j)\leqslant\eta_j^n$, then we can state

$$\frac{\eta_j^n}{\theta}<\gamma\rho^{n-1}\left(\frac{\eta_{j+1}}{\rho}\right)^\sigma.$$

Indeed

$$\frac{\eta_j^n \rho^\sigma}{\gamma\theta\rho^{n-1}\eta_{j+1}^\sigma} = \frac{\tau^{-n}\rho^{\sigma-n+1}\eta_{j+1}^{n-\sigma}}{\gamma\theta} = \frac{\tau^{nj-\sigma j-\sigma}\rho}{\gamma\theta} < \tau^{(n-\sigma)j} < 1,$$

since $\xi < \tau^\sigma\gamma\theta$. Finally, using that Ψ is non-decreasing, we have

$$\Psi(\eta_{j+1}) \leqslant \Psi(\eta_j) \leqslant \frac{\eta_j^n}{\theta} < \gamma\rho^{n-1}\left(\frac{\eta_{j+1}}{\rho}\right)^\sigma,$$

which concludes the proof. □

Finally, we can prove Proposition 3.4.1 choosing

$$\xi = \min\{\mathrm{dist}(K,\partial\Omega), \gamma, \tau^\sigma\gamma\theta\},$$

where $\gamma, \tau, \sigma, \theta$ are given by Proposition 3.4.5.

3.5 Proof of the main theorem

In this section we give the proof of Theorem 3.0.1, which makes use of the results we obtained in the previous sections.

Proof of Theorem 3.0.1. The assertion *1.* follows from Theorem 3.3.1. Let us prove the statement *2.*
Define

$$\Omega_0 = \left\{ y \in \Omega \,:\, \lim_{\rho\to 0}\rho^{1-n}\left[\int_{B_\rho(y)} [F_p(\nabla u) + \mathbb{1}_A G_p(\nabla u)]\,dx + P(A; B_\rho(y))\right] = 0\right\}.$$

Thanks to Proposition 3.4.1 we infer that Ω_0 is an open set. Setting

$$\hat{\partial} A := \left\{ x \in \Omega \,:\, \limsup_{\rho\to 0^+} \frac{P(A; B_\rho(x))}{\rho^{n-1}} > 0\right\},$$

by De Giorgi's structure theorem (Theorem 1.1.13) it holds that $P(A;\cdot) = \mathcal{H}^{n-1} \llcorner \hat{\partial} A$. It is clear that $\Omega_0 \subset \Omega\backslash\hat{\partial} A$.
Let $x \in \Omega_0$. Since Ω_0 is an open set, choose $\rho > 0$ such that $B_\rho(x) \subset \Omega_0$. By the isoperimetric inequality, we infer

$$\min\{|A \cap B_\rho(x)|, |B_\rho(x)\backslash A|\} \leqslant c(n)P(A; B_\rho(x))^{\frac{n}{n-1}} = 0,$$

which implies that $\mathbb{1}_A = 1$ a.e. in $B_\rho(x)$ or $\mathbb{1}_A = 0$ a.e. in $B_\rho(x)$. Define the open set

$$\tilde{A} = \{x \in \Omega_0 \,:\, \mathbb{1}_A = 1 \text{ a.e. in a neighborhood of } x\}.$$

Let us prove that $\mathcal{H}^{n-1}((\Omega\backslash\Omega_0)\Delta\hat{\partial} A) = 0$. Since $\hat{\partial} A \subset \Omega\backslash\Omega_0$, it is clear that $\mathcal{H}^{n-1}(\hat{\partial} A\backslash(\Omega\backslash\Omega_0)) = 0$. It remains to prove that $\mathcal{H}^{n-1}((\Omega\backslash\Omega_0)\backslash\hat{\partial} A) = 0$. Define

$$S_\varepsilon := \left\{ y \in \Omega \,:\, \limsup_{\rho\to 0^+}\rho^{1-n}\int_{B_\rho(y)} [F_p(\nabla u) + \mathbb{1}_A G_p(\nabla u)]\,dx > \varepsilon\right\},$$

for $\varepsilon > 0$. It is clear that

$$(\Omega\backslash\Omega_0)\backslash\widehat{\partial}A \subset \bigcup_{\varepsilon>0} S_\varepsilon. \tag{3.25}$$

Using a density argument, thanks to Lemma A.1.9 we can estimate

$$\varepsilon\mathcal{H}^{n-1}(S_\varepsilon) \leqslant c(n)\int_{S_\varepsilon}\left[F_p(\nabla u) + \mathbb{1}_A G_p(\nabla u)\right] dx, \quad \forall\varepsilon > 0.$$

We deduce that $\mathcal{H}^{n-1}(S_\varepsilon) < +\infty$. It implies that $|S_\varepsilon| = 0$ and so, from the previous inequality, we finally infer that $\mathcal{H}^{n-1}(S_\varepsilon) = 0$, for all $\varepsilon > 0$. Thanks to (3.25) we prove our claim.

Let us prove that A and $\tilde{A}$ are equivalent. One one hand, by the definition of $\tilde{A}$ we have

$$|\tilde{A}| = \int_{\tilde{A}} \mathbb{1}_A \, dx = |\tilde{A} \cap A|,$$

which implies that $|\tilde{A}\backslash A| = 0$; on the other hand, since $\mathcal{H}^{n-1}(\Omega\backslash\Omega_0) = \mathcal{H}^{n-1}(\widehat{\partial}A) < +\infty$, we deduce that $|\Omega\backslash\Omega_0| = 0$ and hence

$$|A\backslash\tilde{A}| = |(A\backslash\tilde{A}) \cap \Omega_0| = \int_{\Omega_0\backslash\tilde{A}} \mathbb{1}_A \, dx = 0.$$

Since $|A\Delta\tilde{A}| = 0$, we infer that $P(A;\Omega) = P(\tilde{A};\Omega)$. Moreover, since $\Omega\cap\partial\tilde{A} \subset \Omega\backslash\Omega_0$ and $\mathcal{H}^{n-1}((\Omega\backslash\Omega_0)\Delta\widehat{\partial}A) = 0$, we have

$$\mathcal{H}^{n-1}(\Omega\cap\partial\tilde{A}) \leqslant \mathcal{H}^{n-1}(\Omega\backslash\Omega_0) = \mathcal{H}^{n-1}(\widehat{\partial}A) = P(A;\Omega) = P(\tilde{A};\Omega).$$

The converse inequality can be obtained from the following one that holds true for any Borel set $C \subset \mathbb{R}^n$ and can be obtained by De Giorgi's structure theorem:

$$P(C;\Omega) \leqslant \mathcal{H}^{n-1}(\Omega\cap\partial C).$$

Choosing $C = \tilde{A}$, we conclude the proof. □

Part II

A frustrated lattice system

Chapter 4

The one-dimensional case

In this chapter we study a one-dimensional and two-dimensional frustrated lattice system, whose spins take values in the unit sphere in $\mathbb{R}^3$. In the one-dimensional case, we set the problem in the lattice

$$\mathcal{I}^n(I) := \mathbb{Z}^n(I) \backslash \left\{ \left\lfloor \frac{1}{\lambda_n} \right\rfloor - 1, \left\lfloor \frac{1}{\lambda_n} \right\rfloor \right\},$$

where $\mathbb{Z}_n = \{i \in \mathbb{Z} \,:\, \lambda_n i \in \overline{I}\}$ and $I := (0,1)$ and $\{\lambda_n\}_{n\in\mathbb{N}}$ is a vanishing sequence of lattice spacings. The set $\mathcal{I}^n(I)$ is the domain of the spins.

Fixing $v_1, v_2 \in S^2$ and $R \in (0,1)$, we define the two circles centred in $v_i\sqrt{1-R^2}$:

$$S_i := \left\{ w \in \mathbb{S}^2 \,:\, |\pi_{v_i^\perp}(w)| = R,\, w \cdot v_i > 0 \right\}, \quad \text{for } i \in \{1,2\}.$$

We assume that for $0 < R < R_{Max} := \sqrt{\frac{1-v_1\cdot v_2}{2}}$ so that the sets S_1 and S_2 are disjoint. The set $S_1 \cup S_2$ is the codomain of the spins.

We introduce the class of functions valued in $S_1 \cup S_2$ which are piecewise constant on the edges of the lattice $\mathbb{Z}_n(I)$ and satisfy a joint boundary condition:

$$\mathcal{PC}_{\lambda_n} := \Big\{ v\colon I \to S_1 \cup S_2 \,:\, v(t) = v(\lambda_n i) \text{ for } t \in \lambda_n[i + [0,1)],$$
$$v^0 \cdot v^1 = v^{\lfloor \frac{1}{\lambda_n} \rfloor - 1} \cdot v^{\lfloor \frac{1}{\lambda_n} \rfloor} \Big\}.$$

We identify a piecewise function $v\colon I \to S_1 \cup S_2$ with the function defined on the points of the lattice given by $i \in \mathbb{Z}_n(I) \mapsto v^i := v(\lambda_n i)$. Conversely, given values $v^i \in S_1 \cup S_2$ for any $i \in \mathbb{Z}_n(I)$, we define $v\colon I \to S_1 \cup S_2$ by $v(t) := v^i$ for $t \in \lambda_n[i + [0,1)]$.

We deal with an energy $\mathcal{E}_n\colon L^\infty(I;\mathbb{R}^3) \to (-\infty, +\infty]$ defined as

$$\mathcal{E}_n = E_n + P_n,$$

with

$$E_n(u) = \begin{cases} \displaystyle\sum_{i\in\mathcal{I}^n(I)} \lambda_n \left(-\alpha u^i \cdot u^{i+1} + u^i \cdot u^{i+2}\right) & \text{for } u \in \mathcal{PC}_{\lambda_n}, \\ +\infty & \text{for } u \in L^\infty(I;\mathbb{R}^3) \backslash \mathcal{PC}_{\lambda_n}, \end{cases}$$

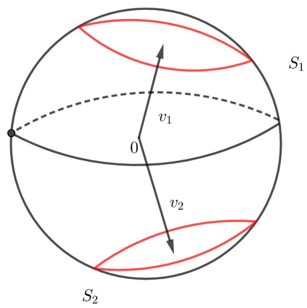

Figure 4.1: S_1 and S_2 circles of anisotropic transitions.

and

$$P_n(u) = \begin{cases} \lambda_n k_n |D\mathcal{A}(u)|(I) & \text{for } u \in \mathcal{PC}_{\lambda_n}, \\ +\infty & \text{for } u \in L^\infty(I;\mathbb{R}^3)\backslash\mathcal{PC}_{\lambda_n}. \end{cases}$$

where $\alpha \in (0, +\infty)$, $\{k_n\} \subset \mathbb{R}^+$ is such that $k_n \to +\infty$, as $n \to +\infty$. The map $\mathcal{A}\colon \mathcal{PC}_{\lambda_n} \to \{v_1, v_2\}$ projects the spin on the circles and is defined as follows:

$$\mathcal{A}(u(t)) = \begin{cases} v_1 & \text{if } u(t) \in S_1, \\ v_2 & \text{otherwise.} \end{cases} \tag{4.1}$$

We explain here the structure of the chapter and how we carry out our analysis. In Section 4.1 we characterize the minimizers of the energy $\mathcal{E}_n$. We point out that they are confined in only one circle of magnetic anisotropy S_i and two different situations may occur:

- for $\alpha \geqslant 4$, minimizers are constant;
- for $\alpha \in (0, 4)$, minimizers are made up of rotating vectors with a constant angle $\psi = \pm \arccos(\alpha/4)$.

In the latter case, a symmetric and rigid structure for minimizers arise: they rotate with a signed constant angle, which determines their chirality. If minimizers rotate clockwise they are said to have a **positive chirality**; if they rotate counterclockwise they are said to have a **negative chirality**.

In the next sections, we are mainly interested in computing the amount of energy the system pays to allow general spins to break the simmetry of minimizers. Hence, we concentrate our analysis in two directions. One one hand, we compute how much energy is spent to allow spins to switch their chiralities (**chirality transitions**). On the other hand, we calculate the quantity of energy needed to let spins "jump" from one magnetic anisotropy circle to the other one (**magnetic anisotropy transitions**). We achieve our purpose in the next sections by means of a techinque based on the notion of Γ-convergence.

Section 4.2 is devoted to the computation of the Γ-limit of the energy $\mathcal{E}_n$. The convergence theorem can be obtained by means of an abstract Γ-convergence result proved in [3] (see Theorem 4.2.1). As a result of Theorem

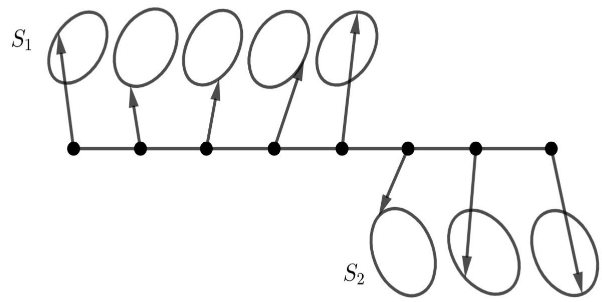

Figure 4.2: Magnetic anisotropic transitions.

4.2.8, the Γ-limit of $\mathcal{E}_n$ does not provide a detailed description of the two fenomena. This suggests that, in order to get more information, we need to consider higher order Γ-limits (see [10] and [11]). This is done in Section 4.3, where we address to the same system, when it is close the helimagnet/ferromagnet transition point as the number of particles diverges. This means that the parameter α depends on n and it converges to the threshold 4 from below, as $n \to +\infty$. In this case, we call the energy E_n as E_n^{hf}. We expand

$$E_n^{hf} = \min E_n^{hf} + \sum_j (R_n)_j + \sqrt{2}\lambda_n \delta_n^{\frac{3}{2}} \mathcal{H}_n,$$

where the functionals $(R_n)_j$ and $\mathcal{H}_n$ are defined in (4.21) and (4.22) respectively. The two fenomena can be detected at different orders and scales. In Subsection 4.3.1, we study the first order Γ-limit, that is the asymptotic behavior of a (rescaling of the) new functional H_n^{hf} defined as

$$H_n^{hf} = E_n^{hf} - \min E_n^{hf}.$$

Rescaling H_n^{hf} by λ_n, magnetic anisotropy transitions can be made by the spin on a scale of order $\lambda_n k_n$, for n large enough (see Theorem 4.3.2).

In Subsection 4.3.2, in order to continue the analysis at the next order, we restrict every spin u on some intervals I_j that partition I such that on $\mathbb{Z}_n(I_j)$ it takes values only in one S_j. We need to modify such restrictions $u_{|I_j}$ in a way that they are well-connected on the boundary of the interval I_j, denotating them as $\widetilde{u}_{I_j}$.

The functional H_n^{hf} can be split in two terms:

$$H_n^{hf}(u) = \sum_j MM_n(\widetilde{u}_{I_j}) + \sum_j (R_n)_j(u).$$

As long as we consider a remainder $(R_n)_j$ for each modification of the spin, the analysis of the global process can be localized in each S_j with the associated energy $MM_n(\widetilde{u}_{I_j})$. The two sums need to be rescaled in different ways, being the first sum a higher order term. Thus, at the second order we deal with the rescaled energy

$$\mathcal{H}_n(u) = \frac{1}{\sqrt{2}\lambda_n\delta_n^{\frac{3}{2}}}\left[H_n(u) - \sum_j (R_n)_j(u)\right] = \frac{1}{\sqrt{2}\lambda_n\delta_n^{\frac{3}{2}}}\sum_j MM_n(\tilde{u}_{I_j}).$$

We transpose the problem valued in the 3d-sphere into a finite number of problems valued in 2d-circles with functionals MM_n of Modica-Mortola type, thus generalizing the result contained in [17]. In each S_j several regimes are possible (see Theorem 4.3.5). For n large enough, the spin system makes a chirality transition on a scale of order $\lambda_n/\sqrt{\delta_n}$. As a result, depending on the value of $\lim_n \lambda_n/\sqrt{\delta_n} := l \in [0, +\infty]$ different scenarios may occur. If $l = +\infty$, chirality transitions are forbidden. If $l > 0$, the spin system may have diffuse and regular macroscopic (on an order one scale) chirality transitions in each S_j whose limit energy is finite on $H^1(I_j)$ (provided some boundary conditions are taken into account). When $l = 0$, transitions on a mesoscopic scale are allowed. In this case, the continuum limit energy is finite on $BV(I_j)$ and counts the number of jumps of the chirality of the system.

4.1 Minimizers of the energy

In order to characterize the minimizers of our energy $\mathcal{E}_n$, we define the auxiliary functional $H_n \colon \mathcal{PC}_{\lambda_n} \to [0, +\infty)$ as

$$H_n(u) := \begin{cases} \dfrac{1}{2}\lambda_n \displaystyle\sum_{i \in \mathcal{I}^n(I)} \left| u^{i+2} - \frac{\alpha}{2}u^{i+1} + u^i \right|^2 & \text{for } u \in \mathcal{PC}_{\lambda_n}, \\ +\infty & \text{for } u \in L^\infty(I;\mathbb{R}^3)\backslash\mathcal{PC}_{\lambda_n}. \end{cases}$$

If $u \in \mathcal{PC}_{\lambda_n}$, since $|u^i| = 1$ for all $i \in \mathbb{Z}_n(I)$, thanks to the boundary condition contained in the definition of $\mathcal{PC}_{\lambda_n}$, we may rewrite the energy $\mathcal{E}_n$ in terms of H_n as

$$\mathcal{E}_n = H_n + P_n - \lambda_n\left(1 + \frac{\alpha^2}{8}\right)\#\mathcal{I}^n(I). \tag{4.2}$$

Thanks to this decomposition, we characterize the ground states of E_n.

Proposition 4.1.1. *Let $0 < \alpha \leqslant 4$. Then there exists $k_0 = k_0(n, \alpha) > 0$ such that, for $k_n \geqslant k_0$, we have that*

$$\min_{u \in L^\infty(I;\mathbb{R}^3)} \mathcal{E}_n(u) = -\lambda_n\#\mathcal{I}^n(I)\left[R^2\left(1 + \frac{\alpha^2}{8}\right) + (\alpha - 1)(1 - R^2)\right].$$

Furthermore, a minimizer u_n of E_n over $L^\infty(I;\mathbb{R}^3)$ takes values only in one circle S_d, with $d \in \{1, 2\}$, and satisfies

$$\pi_{v_d^\perp}u_n^i \cdot \pi_{v_d^\perp}u_n^{i+1} = \frac{\alpha}{4} \quad \text{and} \quad \pi_{v_d^\perp}u_n^i \cdot \pi_{v_d^\perp}u_n^{i+2} = \frac{\alpha^2}{8} - 1, \quad \forall i \in \mathcal{I}^n(I).$$

Proof. We start observing that there exists $k_0 = k_0(n, \alpha) > 0$ such that

$$E_n(u) \leqslant \lambda_n k_0 |v_1 - v_2|, \quad \forall u \in \mathcal{PC}_{\lambda_n}.$$

Assuming that $k_n \geqslant k_0$, we get that $E_n \leqslant P_n$ on $\mathcal{PC}_{\lambda_n}$. Thus

$$\min_{u \in L^\infty(I;\mathbb{R}^3)} \mathcal{E}_n(u) = \min_{\substack{u \in \mathcal{PC}_{\lambda_n} \\ u(I) \subset S_1 \text{ or } u(I) \subset S_2}} E_n(u).$$

We prove that

$$\min_{\substack{u \in \mathcal{PC}_{\lambda_n} \\ u(I) \subset S_1 \text{ or } u(I) \subset S_2}} E_n(u) = -\lambda_n \# \mathcal{I}^n(I) \left[R^2 \left(1 + \frac{\alpha^2}{8} \right) + (\alpha - 1)(1 - R^2) \right].$$

Fix $d \in \{1, 2\}$ and consider $u_n \in \mathcal{PC}_{\lambda_n}$ such that $u_n(I) \subset S_d$. By geometrical and trigonometric identities we deduce that

$$u_n^i \cdot u_n^{i+1} = 1 - R^2 + R^2 \pi u_n^i \cdot \pi u_n^{i+1},$$

where $\pi u_n^i := \pi_{v_d^\perp} u_n^i$. Thus

$$\begin{aligned} E_n(u_n) &= \sum_{i \in \mathcal{I}^n(I)} \lambda_n [-\alpha \pi u_n^i \cdot \pi u_n^{i+1} + \pi u_n^i \cdot \pi u_n^{i+2}] - (\alpha - 1)(1 - R^2) \lambda_n \# \mathcal{I}^n(I) \\ &=: \widetilde{E}_n(u_n) - (\alpha - 1)(1 - R^2) \lambda_n \# \mathcal{I}^n(I). \end{aligned} \tag{4.3}$$

Now we minimize $\widetilde{E}_n$, following the same argument in [17]. We remark that

$$\begin{aligned} \widetilde{E}_n(u_n) &= \frac{1}{2} \lambda_n \sum_{i \in \mathcal{I}^n(I)} \left| \pi u_n^{i+2} - \frac{\alpha}{2} \pi u_n^{i+1} + \pi u_n^i \right|^2 - R^2 \left(1 - \frac{\alpha^2}{8} \right) \lambda_n \# \mathcal{I}^n(I) \\ &= \widetilde{H}_n(u_n) - R^2 \left(1 - \frac{\alpha^2}{8} \right) \lambda_n \# \mathcal{I}^n(I). \end{aligned} \tag{4.4}$$

Fix $\phi \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$ so that $\cos \phi = \frac{\alpha}{4}$. We may assume for simplicity of notation that $v_d = e_n$. Let

$$u_n^i := (\cos(\phi i), \sin(\phi i), \sqrt{1 - R^2}), \quad \forall i \in \mathbb{Z}_n(I),$$

so that $\pi u_n^i = (\cos(\phi i), \sin(\phi i), 0)$. By trigonometric identities, we have that

$$\pi u_n^i + \pi u_n^{i+2} = 2 \cos \phi = \frac{\alpha}{2} \pi u_n^{i+1}, \quad \forall i \in \mathcal{I}^n(I).$$

Remarking that $\widetilde{H}_n(u_n) = 0$, we combine the previous identity with (4.4) to get that

$$\min_{\substack{u \in \mathcal{PC}_{\lambda_n} \\ u(I) \subset S_1 \text{ or } u(I) \subset S_2}} \widetilde{E}_n(u) = -R^2 \left(1 - \frac{\alpha^2}{8} \right) \lambda_n \# \mathcal{I}^n(I).$$

The computation of the minimum follows from (4.3).

Consider now a minimizer $u_n \in L^\infty(I;\mathbb{R}^3)$ of $\mathcal{E}_n$. For $k_n \geqslant k_0$, it must hold that $u_n \in \mathcal{PC}_{\lambda_n}$, $u_n(I) \subset S_d$, for some $d \in \{1,2\}$, and

$$\widetilde{E}_n(u_n) = R^2\left(1 - \frac{\alpha^2}{8}\right)\lambda_n \#\mathcal{I}^n(I),$$

thus implying that $\widetilde{H}_n(u_n) = 0$. It follows that

$$\pi u_n^{i+1} = \frac{2}{\alpha}\big(\pi u_n^i + \pi u_n^{i+2}\big), \quad \forall i \in \mathcal{I}^n(I).$$

Squaring both sides of the previous equality, we infer

$$\pi u_n^i \cdot \pi u_n^{i+2} = \frac{\alpha^2}{8} - 1.$$

Hence

$$\pi u_n^i \cdot \pi u_n^{i+1} = \frac{2}{\alpha}\pi u_n^i \cdot \pi u_n^i + \pi u_n^{i+2} = \frac{2}{\alpha}\big(1 + \pi u_n^i \cdot \pi u_n^{i+2}\big) = \frac{\alpha}{4},$$

which concludes the proof. □

Remark 4.1.2. *The case $\alpha > 4$ is trivial. Indeed, the ground states are all ferromagnetic, i.e. $u_n^i = \bar{u} \in S_1 \cup S_2$ for all $i \in \mathbb{Z}_n(I)$. Denoting with $\mathcal{E}_n^{(\alpha=4)}$ the energy of formula* (4.2) *for $\alpha = 4$, we have that, for all $u \in \mathcal{PC}_{\lambda_n}$,*

$$\mathcal{E}_n(u) = \mathcal{E}_n^{(\alpha=4)}(u) - \lambda_n(\alpha - 4)\sum_{i \in \mathcal{I}^n(I)} u^i \cdot u^{i+1}.$$

By the above proposition, the energy $\mathcal{E}_n^{(\alpha=4)}$ is minimized on ferromagnetic states, which trivially also holds true for the second term in the above sum. The minimal value of $\mathcal{E}$ is

$$\min_{u \in L^\infty(I;\mathbb{R}^3)} \mathcal{E}_n(u) = -\lambda_n(\alpha - 1)\#\mathcal{I}^n(I).$$

4.2 Zero order Γ-convergence of the energy $\mathcal{E}_n$

4.2.1 An auxiliary abstract theorem

Here we cite an abstract Γ-convergence result proved in [3] that will be applied later on in this subsection. For this purpose, we introduce the following notation. Let $K \subset \mathbb{R}^N$ be a compact set and for all $\xi \in \mathbb{Z}$ let $f^\xi \colon \mathbb{R}^{2N} \to \mathbb{R}$ be a function such that

(H1) $f^\xi(x,y) = f^{-\xi}(y,x)$, for any $(x,y) \in \mathbb{R}^{2N}$;

(H2) for all $\xi \in \mathbb{Z}$, $f^\xi(x,y) = +\infty$ if $(x,y) \notin K^2$;

(**H3**) for all $\xi \in \mathbb{Z}$ there exists $C^\xi \geqslant 0$ such that

$$\sup_{(x,y)\in K^2} |f^\xi(x,y)| \leqslant C^\xi \quad \text{and} \quad \sum_{\xi\in\mathbb{Z}} C^\xi < +\infty.$$

For any $n \in \mathbb{N}$ we define the functional space

$$D_n(I;\mathbb{R}^N) := \{u\colon \mathbb{R} \to \mathbb{R}^N : u \text{ is constant in } \lambda_n(i+[0,1)) \text{ for all } i \in \mathbb{Z}_n(I)\}$$

and the sequence of functions $F_n\colon L^\infty(I,\mathbb{R}^N) \to (-\infty,+\infty]$ as follows:

$$F_n(u) := \begin{cases} \displaystyle\sum_{\xi\in\mathbb{Z}} \sum_{i\in R_n^\xi(I)} \lambda_n f^\xi(u^i, u^{i+\xi}) & \text{for } u \in D_n(I;\mathbb{R}^N), \\ +\infty & \text{for } u \in L^\infty(I;\mathbb{R}^N)\backslash D_n(I;\mathbb{R}^N), \end{cases}$$

where $R_n^\xi(I) := \{i \in \mathbb{Z}_n(I) : i+\xi \in \mathbb{Z}_n(I)\}$. For any open and bounded set $A \subset \mathbb{R}$ and for every $v\colon \mathbb{Z} \to \mathbb{R}^N$, we define the discrete average of v in A as

$$(v)_{1,A} := \frac{1}{\#(\mathbb{Z}\cap A)} \sum_{i\in\mathbb{Z}\cap A} v^i.$$

Theorem 4.2.1. *Let $\{f^\xi\}_{\xi\in\mathbb{Z}}$ a family of functions that satisfies **H1**, **H2**, **H3**. Then the sequence F_n converges, as $n \to +\infty$ with respect to the weak-star topology of $L^\infty(I;\mathbb{R}^N)$, in the sense of the Γ-convergence to*

$$F(u) := \begin{cases} \displaystyle\int_I f_{hom}(u(t))\,dt & \text{for } u \in L^\infty(I;co(K)), \\ +\infty & \text{for } u \in L^\infty(I;\mathbb{R}^N)\backslash L^\infty(I;co(K)), \end{cases}$$

where $f_{hom}\colon \mathbb{R}^N \to \mathbb{R}$ is given by the following homogenization formula:

$$f_{hom}(z) = \lim_{\rho\to 0}\lim_{k\to+\infty} \frac{1}{k} \inf\left\{ \sum_{\xi\in\mathbb{Z}} \sum_{\beta\in R_1^\xi((0,k))} f^\xi(v(\beta), v(\beta+\xi)) : (v)_{1,(0,k)} \in \overline{B(z,\rho)} \right\}.$$

4.2.2 The zero-order Γ-limit

Before stating the main result of this subsection, we need to introduce some notation. We call a collection $\mathcal{C}$ of subsets of an open set S an **open partition** of S if and only if $\mathcal{C}$ does not contain empty sets and

$$\overline{S} = \bigcup_{C\in\mathcal{C}} \overline{C}, \quad C_1 \cap C_2 = \varnothing, \quad \forall C_1, C_2 \in \mathcal{C}.$$

We observe that if $u \in \mathcal{PC}_{\lambda_n}$, the interval I can be partitioned in regions where the function u takes values only in one of the two circles. In other words, there exist $M(u) \in \mathbb{N}$ and a collection of open intervals, $\{I_j^d\}_{j\in\{1,\dots,M(u)\}}$, such that

$$\{I_j^d\}_{j\in\{1,\dots,M(u)\}} \text{ is an open partition of } I, \tag{4.5}$$

$$u(t) \in S_d, \quad \forall t \in I_j^d, \forall j \in \{1, \ldots, M(u)\}. \tag{4.6}$$

These two properties imply that this partition is unique. We observe that

$$M(u) = \frac{|D\mathcal{A}(u)|(I)}{|v_1 - v_2|} + 1.$$

The following definition will be useful throughout the section.

Definition 4.2.2. *Let $u \in \mathcal{PC}_{\lambda_n}$. We say that*

$$\mathcal{C}_n(u) = \{I_j^d \mid j \in \{1, \ldots, M(u)\}\}$$

*is **the open partition associated with** u if $M(u) = \frac{|D\mathcal{A}(u)|(I)}{|v_1 - v_2|} + 1$ and the collection of open intervals $\{I_j^d\}_{j\in\{1,\ldots,M(u)\}}$ satisfies* (4.5) *and* (4.6).

In view of the study of the Γ-convergence of the energy $\mathcal{E}_n$ defined in (4.2) at the zero order, we consider the space

$$\mathfrak{D} := \Big\{ u \in L^\infty(I; co(S_1) \cup co(S_2)) \; : \; \exists \mathcal{C}_n(u) \text{ finite open partition associated with } u \Big\}. \tag{4.7}$$

We observe that $\mathcal{A}(\mathfrak{D}) = BV(I; \{v_1, v_2\})$, where $\mathcal{A}$ is the function defined in (4.1). The following convergence law will be used.

Definition 4.2.3 (Convergence law)**.** *Let $\{f_n\}_{n\in\mathbb{N}} \subset \mathfrak{D}$ and $f \in \mathfrak{D}$. We say that f_n L-converges to f (we write $f_n \xrightarrow{L} f \in \mathfrak{D}$) as $n \to +\infty$ if and only if $f_n \overset{*}{\rightharpoonup} f$ in the weak-star topology of $L^\infty(I; \mathbb{R}^3)$ and $\mathcal{A}(f_n) \to \mathcal{A}(f)$ in the weak-star topology of $BV(I; \{v_1, v_2\})$, as $n \to +\infty$.*

Remark 4.2.4. *We observe that the convergence law introduced in the definition above is induced by the topology on $\mathfrak{D}$ defined as the smaller topology containing the set*

$$\{A : A \text{ is open set of weak-star topology of } L^\infty \text{ or of the } BV(I; \{v_1, v_2\}) \text{ topology}\}.$$

Firstly, we study the Γ-convergence of the energy E_n. The following theorem relies on a straightforward application of Theorem 4.2.1.

Theorem 4.2.5. *The sequence E_n converges in the sense of the Γ-convergence to the functional*

$$E(u) := \begin{cases} \displaystyle\int_I f_{hom}(u(t))\, dt & \text{if } u \in L^\infty(I; co(S_1 \cup S_2)), \\ +\infty & \text{otherwise,} \end{cases}$$

with respect to the weak-star topology of $L^\infty(I;\mathbb{R}^3)$, where $f_{hom}\colon co(S_1 \cup S_2) \to \mathbb{R}$ is defined as

$$f_{hom}(z) = \lim_{\rho\to 0}\lim_{k\to+\infty}\frac{1}{k}\inf\left\{\sum_{i=1}^{k-2}\left[-\alpha u^i\cdot u^{i+1} + u^i\cdot u^{i+2}\right] \,:\, (u)_{1,(0,k)} \in \overline{B(z,\rho)}\right\}. \tag{4.8}$$

Proof. The result immediately follows applying Theorem 4.2.1 to

$$f^\xi(u,v) = \begin{cases} -\frac{\alpha}{2}u\cdot v & \text{if } |\xi| = 1,\\ \frac{1}{2}u\cdot v & \text{if } |\xi| = 2,\\ 0 & \text{otherwise,}\end{cases}$$

extended to $+\infty$ outside K, where $u, v \in K := S_1 \cup S_2$. □

Remark 4.2.6. *The function f_{hom} defined in* (4.8) *does not depend on the parameter λ_n. Therefore, in the theorem above the Γ-limit does not depend on the choice of λ_n.*

Remark 4.2.7. *An analogous statement of Theorem 4.2.5 can be obtained if the functional E_n is defined only in $L^\infty(I;S_i)$ for some $i \in \{1,2\}$ (see [17, Theorem 3.4]). The Γ-limit has the same form and it is finite on $L^\infty(I;co(S_i))$.*

The following theorem is the main result of this section.

Theorem 4.2.8. *Assume that there exists $\lim_{n\to+\infty}\lambda_n k_n =: \eta \in (0,+\infty]$. Then the following Γ-convergence and compactness results hold true:*

(i) if $\eta \in (0,+\infty)$, then $\mathcal{E}_n$ converges in the sense of Γ-convergence to the functional

$$\mathcal{E}(u) = \begin{cases} \displaystyle\int_I f_{hom}(u(t))\,dt + \eta|D\mathcal{A}(u)|(I) & \text{if } u \in \mathfrak{D},\\ +\infty & \text{otherwise,}\end{cases}$$

with respect to the L-convergence of Definition 4.2.3, where f_{hom} is defined in (4.8) *and $\mathfrak{D}$ is the set defined in* (4.7)*. Moreover, if $\{u_n\}_{n\in\mathbb{N}} \subset L^\infty(I;\mathbb{R}^3)$ satisfies*

$$\sup_{n\in\mathbb{N}}\mathcal{E}_n(u_n) < +\infty,$$

then, up to a subsequence, $u_n \xrightarrow{L} u \in \mathfrak{D}$;

(ii) if $\eta = +\infty$, then E_n converges in the sense of Γ-convergence to the functional

$$\mathcal{E}(u) := \begin{cases} \displaystyle\int_I f_{hom}(u(t))\,dt & \text{if } u \in L^\infty(I;co(S_1)) \text{ or } u \in L^\infty(I;co(S_2)),\\ +\infty & \text{otherwise,}\end{cases}$$

with respect to the weak-star topology of $L^\infty(I;\mathbb{R}^3)$, where f_{hom} is defined in (4.8)*. Moreover, for all $\{u_n\}_{n\in\mathbb{N}} \subset L^\infty(I;\mathbb{R}^3)$ such that*

$$\sup_{n\in\mathbb{N}} \mathcal{E}_n(u_n) < +\infty,$$

then, up to a subsequence, $u_n \overset{*}{\rightharpoonup} u$ for some $u \in L^\infty(I;co(S_1))$ or $u \in L^\infty(I;co(S_2))$.

Proof. (i) We start to prove the compactness result. Let $\{u_n\}_{n\in\mathbb{N}} \subset L^\infty(I;\mathbb{R}^3)$ such that

$$\sup_{n\in\mathbb{N}} \mathcal{E}_n(u_n) < H, \tag{4.9}$$

for some $H > 0$. Thus we have that $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{PC}_{\lambda_n}$. Moreover, by the definition of the space $\mathcal{PC}_{\lambda_n}$, we have that, for all $u_n \in \mathcal{PC}_{\lambda_n}$, there exists a finite open partition $\mathcal{C}_n(u_n) = \{(I_j^d)_n \,|\, j \in \{1,\dots,M(u_n)\}\}$ associated with u_n, where $M(u_n) = \frac{|D\mathcal{A}(u_n)|(I)}{|v_1-v_2|} + 1 \in \mathbb{N}$. By (4.2) and by the definition of the function $\mathcal{A}$, we compute

$$\begin{aligned}\mathcal{E}_n(u_n) &= H_n(u_n) + P_n(u_n) - \lambda_n\left(1+\frac{\alpha^2}{8}\right)\#\mathcal{I}^n(I) \\ &\geqslant P_n(u_n) - \lambda_n\left(1+\frac{\alpha^2}{8}\right)\#\mathcal{I}^n(I) \\ &= k_n\lambda_n\,|D\mathcal{A}(u_n)|\,(I) - \lambda_n\left(1+\frac{\alpha^2}{8}\right)\#\mathcal{I}^n(I) \\ &= k_n\lambda_n(M(u_n)-1)|v_1-v_2| - \lambda_n\left(1+\frac{\alpha^2}{8}\right)\#\mathcal{I}^n(I) \\ &\geqslant -C(\alpha) + k_n\lambda_n(M(u_n)-1)|v_1-v_2|,\end{aligned} \tag{4.10}$$

for some constant $C = C(\alpha) > 0$, where the last inequality is obtained by observing that $\lambda_n\#\mathcal{I}^n(I) = \lambda_n\left\lfloor\frac{1}{\lambda_n}\right\rfloor - \lambda_n \to 1$ as $n \to +\infty$ and thus it is bounded. Therefore, by (4.9) and (4.10), we obtain that

$$\sup_{n\in\mathbb{N}} M(u_n) < C(\eta, H, \alpha, |v_1-v_2|).$$

Hence the sequence $\{u_n\}_{n\in\mathbb{N}}$ satisfies the hypotheses of the Proposition C.0.3, and so we deduce the existence of $u \in \mathfrak{D}$ such that, up to a subsequence, $u_n \xrightarrow{L} u$.

Now we prove the liminf inequality. Let $\{u_n\}_{n\in\mathbb{N}}$ such that $u_n \xrightarrow{L} u \in \mathfrak{D}$ as $n \to +\infty$, i.e. $u_n \overset{*}{\rightharpoonup} u$ in L^∞ and $\mathcal{A}(u_n) \to \mathcal{A}(u)$ in $BV(I;\{v_1,v_2\})$. By the liminf inequality of Theorem 4.2.5 we have

$$\liminf_{n\to+\infty} E_n(u_n) \geqslant \int_I f_{hom}(u(t))\,dt. \tag{4.11}$$

On the other hand, by the lower semicontinuity of the total variation with respect to the convergence in $BV(I;\{v_1,\, v_2\})$ (see Proposition 1.1.26) we have

$$\liminf_{n\to+\infty} P_n(u_n) = \liminf_{n\to+\infty} k_n\lambda_n|D\mathcal{A}(u_n)|(I) \geqslant \eta|D\mathcal{A}(u)|(I). \tag{4.12}$$

Hence, by (4.11) and (4.12), we obtain

$$\liminf_{n\to+\infty} \mathcal{E}_n(u_n) \geqslant \liminf_{n\to+\infty} E_n(u_n) + \liminf_{n\to+\infty} P_n(u_n) \geqslant \int_I f_{hom}(u(t))\,dt + \eta|D\mathcal{A}(u)|.$$

We finally prove the limsup inequality. Let $u \in L^\infty(I; co(S_1) \cup co(S_2))$. We may assume that $u \in \mathfrak{D}$ and furthermore, by a standard density argument and the locality of the construction, we may assume that

$$u(t) = \begin{cases} a_1 & \text{if } t \in \left[0, \frac{1}{2}\right], \\ a_2 & \text{if } t \in \left(\frac{1}{2}, 1\right], \end{cases}$$

where $a_1 \in co(S_1)$ and $a_2 \in co(S_2)$. For $j \in \{1,2\}$, let $\{v_n^j\}_{n\in\mathbb{N}} \subset L^\infty(I; S_j)$ be the recovery sequence for the constant function a_j obtained by the Γ-convergence result in Remark 4.2.7 with $2\lambda_n$ as the spacing of the lattice (see Remark 4.2.6), i.e.

$$f_{hom}(a_j) = \lim_{n\to+\infty} E_n(v_n^j) = \lim_{n\to+\infty} 2\lambda_n \sum_{i=1}^{\left\lfloor \frac{1}{2\lambda_n}\right\rfloor - 2} \left[-\alpha(v_n^j)^i \cdot (v_n^j)^{i+1} + (v_n^j)^i \cdot (v_n^j)^{i+2}\right]. \tag{4.13}$$

We define

$$u_n(t) = \begin{cases} v_n^1(2t) & \text{if } t \in \left[0, \frac{1}{2}\right], \\ v_n^2(2t-1) & \text{if } t \in \left(\frac{1}{2}, 1\right], \end{cases}$$

and compute

$$\begin{aligned} E_n(u_n) = {} & \frac{1}{2} \sum_{i=1}^{\left\lfloor \frac{1}{2\lambda_n}\right\rfloor - 2} 2\lambda_n \left[-\alpha(v_n^1)^i \cdot (v_n^1)^{i+1} + (v_n^1)^i \cdot (v_n^1)^{i+2}\right] \\ & + \frac{1}{2} \sum_{i=1}^{\left\lfloor \frac{1}{2\lambda_n}\right\rfloor - 2} 2\lambda_n \left[-\alpha(v_n^2)^i \cdot (v_n^2)^{i+1} + (v_n^2)^i \cdot (v_n^2)^{i+2}\right] \\ & + \sum_{i=\left\lfloor \frac{1}{2\lambda_n}\right\rfloor - 1}^{\left\lfloor \frac{1}{2\lambda_n}\right\rfloor} \lambda_n \left[-\alpha u_n^i \cdot u_n^{i+1} + u_n^i \cdot u_n^{i+2}\right]. \end{aligned} \tag{4.14}$$

We observe that

$$\left| \sum_{i=\left\lfloor \frac{1}{2\lambda_n}\right\rfloor - 1}^{\left\lfloor \frac{1}{2\lambda_n}\right\rfloor} \lambda_n \left[-\alpha u_n^i \cdot u_n^{i+1} + u_n^i \cdot u_n^{i+2}\right] \right| \leqslant C(\alpha)\lambda_n \to 0, \quad \text{as } n \to +\infty. \tag{4.15}$$

By (4.13), (4.14), (4.15), we obtain

$$E_n(u_n) \to \frac{f_{hom}(a_1) + f_{hom}(a_2)}{2} = \int_I f_{hom}(u(t))\,dt, \quad \text{as } n \to +\infty. \tag{4.16}$$

Observing that for all $n \in \mathbb{N}$

$$\mathcal{A}(u_n)(t) = \mathcal{A}(u)(t) = \begin{cases} v_1 & \text{if } t \in [0, \frac{1}{2}], \\ v_2 & \text{if } t \in (\frac{1}{2}, 1], \end{cases}$$

then $|D\mathcal{A}(u_n)|(I) = |D\mathcal{A}(u)|(I) = |v_1 - v_2|$ and

$$\lim_{n\to+\infty} P_n(u_n) = \lim_{n\to+\infty} \lambda_n k_n |v_1 - v_2| = \eta|v_1 - v_2|. \tag{4.17}$$

Combining (4.16) and (4.17), we deduce the limsup inequality.
(ii) Let us prove first the compactness result. Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{PC}_{\lambda_n}$ such that

$$\sup_{n\in\mathbb{N}} \mathcal{E}_n(u_n) < H,$$

for some constant $H > 0$. With the same compactness argument used in the previous case, we deduce the existence of $u \in \mathfrak{D}$ such that $u_n \xrightarrow{L} u$ as $n \to +\infty$. By the lower semicontinuity of the map

$$u \to |D\mathcal{A}(u)|(I)$$

with respect to the L-convergence (see Definition 4.2.3), we get

$$\begin{aligned} 0 = \liminf_{n\to+\infty} \frac{H}{\lambda_n k_n} &> \liminf_{n\to+\infty} \frac{1}{\lambda_n k_n} \left[E_n(u_n) + \lambda_n k_n |D\mathcal{A}(u_n)|(I)\right] \\ &\geqslant \liminf_{n\to+\infty} \left(\frac{C(\alpha)}{\lambda_n k_n} + |D\mathcal{A}(u_n)|(I) \right) \geqslant |D\mathcal{A}(u)|(I), \end{aligned}$$

hence $u \in L^\infty(I; co(S_1)) \cup L^\infty(I; co(S_2))$.

Let us prove the liminf inequality. Let $u_n \xrightarrow{*} u$ and suppose that

$$\liminf_{n\to+\infty} \mathcal{E}_n(u_n) < +\infty.$$

Up to the extraction of a subsequence, we may assume that the previous lower limit is actually limit. By compactness, we infer that $u_n \xrightarrow{L} u \in L^\infty(I; co(S_1)) \cup L^\infty(I; co(S_2))$. Hence, by Theorem 4.2.5, we obtain

$$\liminf_{n\to+\infty} \mathcal{E}_n(u_n) \geqslant \liminf_{n\to+\infty} E_n(u_n) \geqslant \int_I f_{hom}(u(t))\, dt.$$

We finally prove the limsup inequality. Let $u \in L^\infty(I; co(S_1))$, being the case $u \in L^\infty(I; co(S_2))$ fully analogous. The recovery sequence obtained from Remark 4.2.7, $\{u_n\}_{n\in\mathbb{N}} \subset L^\infty(I; S_1)$, satisfies the limsup inequality. □

4.3 First and second order Γ-convergence of the energy $\mathcal{E}_n$

In this section we study the system when it is close to the helimagnet/ferromagnet transition point as the number of particles diverges. In what follows we denote $\alpha_n := 4(1 - \delta_n)$, where $\delta_n > 0$ and $\delta_n \to 0$ as $n \to +\infty$.

Regarding the notation, in this section we denote with d a number that can be 1 or 2. This number stands for the index of the two circles S_1 and S_2.

4.3.1 First order Γ-convergence of the energy $\mathcal{E}_n$

We define the renormalized energy and introduce a new functional whose asymptotic behavior will better describe the spin's magnetic anisotropy transitions. More precisely we define

$$E_n^{hf}\colon L^\infty(I;\mathbb{R}^3) \to (-\infty,+\infty] \quad \text{and} \quad H_n^{hf}\colon L^\infty(I;\mathbb{R}^3) \to [0,+\infty]$$

as

$$E_n^{hf}(u) := \begin{cases} \lambda_n \displaystyle\sum_{i\in\mathcal{I}^n(I)} [-\alpha_n u^i\cdot u^{i+1} + u^i \cdot u^{i+2}] & \text{if } u \in \mathcal{PC}_{\lambda_n}, \\ +\infty & \text{otherwise,} \end{cases}$$

$$H_n^{hf}(u) := \begin{cases} \dfrac{1}{2}\lambda_n \displaystyle\sum_{i\in\mathcal{I}^n(I)} \left|u^{i+2} - \frac{\alpha_n}{2}u^{i+1} + u^i\right|^2 & \text{if } u \in \mathcal{PC}_{\lambda_n}, \\ +\infty & \text{otherwise.} \end{cases}$$

By (4.2) we observe that

$$H_n^{hf}(u) = E_n^{hf}(u) + \lambda_n\left(1+\frac{\alpha_n^2}{8}\right)\#\mathcal{I}^n(I).$$

The functional we are interested in is

$$G_n^{hf}(u) = E_n^{hf}(u) - \min E_n^{hf}.$$

At this point we need to introduce a modified spin chain in order to understand better the asymptotic behaviour of the renormalized energy G_n^{hf}. Let $u \in \mathcal{PC}_{\lambda_n}$ and $\mathcal{C}_n(u) = \{I_j^d \,|\, j \in \{1,\ldots,M(u)\}\}$ the open partition associated with u, with $\overline{I}_j^d = [(t_j^d)_1,(t_j^d)_2]$. For simplicity of notation, henceforth we omit to write the superscript d in the next formulas. We define the auxiliary function $\widetilde{u}_{I_j}\colon \overline{I}_j = [(t_j)_1,(t_j)_2] \to S_d$ as

$$\widetilde{u}_{I_s}(t) = \begin{cases} u_{|[(t_j)_1,(t_j)_2)}(t) & \text{if } t \in [(t_j)_1,(t_j)_2), \\ w_j & \text{if } t = (t_s)_2, \end{cases} \tag{4.18}$$

where $w_j \in S_d$ is a vector such that the following joint boundary condition is satisfied:

$$u^{\frac{(t_j)_2}{\lambda_n}-1}\cdot w_j = u^{\frac{(t_j)_1}{\lambda_n}}\cdot u^{\frac{(t_j)_1}{\lambda_n}+1}.$$

We split the energy G_n^{hf} as follows:

$$G_n^{hf}(u) = E_n^{hf}(u) - \min E_n^{hf}(u) = \sum_{j=1}^{M(u)} MM_n(\widetilde{u}_{I_j}) + \sum_{j=1}^{M(u)-1}(R_n)_j(u), \quad \forall u \in \mathcal{PC}_{\lambda_n}, \tag{4.19}$$

where

$$MM_n(\widetilde{u}_{I_j}) = \lambda_n \sum_{i\in\mathcal{I}^n(I_j)}\left[-\alpha_n \widetilde{u}^i_{I_j}\cdot\widetilde{u}^{i+1}_{I_j} + \widetilde{u}^i_{I_j}\cdot\widetilde{u}^{i+2}_{I_j}\right] + \lambda_n\left(1+\frac{\alpha_n^2}{8}\right)\#\mathcal{I}^n(I_j)$$

is the local energy associated with the modified spin chain $\widetilde{u}_{I_j}$, and

$$(R_n)_j(u) := -\lambda_n\Big[-\alpha_n u^{\frac{(t_j)_2}{\lambda_n}-1}\cdot u^{\frac{(t_j)_2}{\lambda_n}} + u^{\frac{(t_j)_2}{\lambda_n}-2}\cdot u^{\frac{(t_j)_2}{\lambda_n}} + u^{\frac{(t_j)_2}{\lambda_n}-1}\cdot u^{\frac{(t_j)_2}{\lambda_n}+1}$$
$$+ u^{\frac{(t_j)_2}{\lambda_n}-2}\cdot w_j\Big] + \frac{\lambda_n}{M(u)-1}\Big[u^{\frac{\left(t_{M(u)}\right)_2}{\lambda_n}-2}\cdot u^{\frac{\left(t_{M(u)}\right)_2}{\lambda_n}} - u^{\frac{\left(t_{M(u)}\right)_2}{\lambda_n}-2}\cdot w_{M(u)}\Big]$$
$$+ \lambda_n \#\mathcal{I}^n(I)(1-R^2)\left(\alpha_n - 2 - \frac{\alpha_n^2}{8}\right),$$

is the remainder for each modification. Note that $(R_n)_j$ consists of three addends: the first sum is related to the interactions between spins with values in two neighboring intervals I_j and I_{j+1}, for $j \in \{1,\dots,M(u)-1\}$, the second sum refers to the last interval $I_{M(u)}$ and third one is a corrective constant.

Remark 4.3.1. *For all $u \in \mathcal{PC}_{\lambda_n}$ we have that $MM_n(\widetilde{u}_{I_j}) \geqslant 0$ for all $j \in \{1,\dots, M(u)\}$, since it is simple to verify that*

$$MM_n(\widetilde{u}_{I_j}) = \frac{1}{2}\lambda_n \sum_{i\in\mathcal{I}^n(I)} \left|\widetilde{u}_{I_j}^{i+2} - \frac{\alpha_n}{2}\widetilde{u}_{I_j}^{i+1} + \widetilde{u}_{I_j}^{i}\right|^2.$$

Theorem 4.3.2. *Assume that there exists $\lim_{n\to+\infty} \lambda_n k_n =: \eta \in (0,+\infty)$ and let*

$$R := \inf\left\{\liminf_{n\to+\infty} \frac{(R_n)_j(u_n)}{\lambda_n} \ :\ \mathcal{A}(u_n) \xrightarrow{BV} v_1 \mathbb{1}_{(0,\frac{1}{2})} + v_2 \mathbb{1}_{[\frac{1}{2},1)}\right\}.$$

Then the following compactness and Γ-convergence results hold true:

(i) (Compactness) if for $\{u_n\}_{n\in\mathbb{N}} \subset L^\infty(I;\mathbb{R}^3)$ there exists $C > 0$ independent of n such that

$$G_n^{hf}(u_n) \leqslant \lambda_n P_n(u_n) \leqslant \lambda_n C \tag{4.20}$$

then, up to subsequence, $\mathcal{A}(u_n) \to v \in BV(I;\{v_1, v_2\})$ as $n \to +\infty$ in the weak-star topology of $BV(I;\{v_1, v_2\})$;

*(ii) (*lim inf *inequality) For all $v \in BV(I;\{v_1, v_2\})$ and for all $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{PC}_{\lambda_n}$ such that*

$$\mathcal{A}(u_n) \to v, \quad \text{as } n \to +\infty \text{ in the weak-star topology of } BV(I;\{v_1, v_2\}),$$

and

$$G_n^{hf}(u_n) \leqslant \lambda_n P_n(u_n) \leqslant \lambda_n C, \quad \text{for some } C > 0,$$

then

$$\liminf_{n\to+\infty} \frac{G_n^{hf}(u_n)}{\lambda_n} \geqslant R\frac{|Dv|(I)}{|v_1 - v_2|};$$

*(iii) (*lim sup *inequality) For all $v \in BV(I;\{v_1,v_2\})$ there exists $\{u_n\}_{n\in N} \subset \mathcal{PC}_{\lambda_n}$ such that*

$$\mathcal{A}(u_n) \to v, \quad \text{as } n \to +\infty \text{ in the weak-star topology of } BV(I;\{v_1,v_2\}),$$

and

$$G_n^{hf}(u_n) \leqslant \lambda_n P_n(u_n) \leqslant \lambda_n C, \quad \text{for some } C > 0,$$

satisfying

$$\lim_{n\to+\infty} \frac{G_n^{hf}(u_n)}{\lambda_n} = R\frac{|Dv|(I)}{|v_1 - v_2|}.$$

Proof. We start proving (i). Since $\eta \in (0,+\infty)$, by the second inequality of (4.20), we deduce that the sequence $\{|D\mathcal{A}(u_n)|(I)\}_{n\in\mathbb{N}}$ is bounded. Accordingly the sequence $\{\mathcal{A}(u_n)\}_{n\in\mathbb{N}}$ is bounded in the space $BV(I;\{v_1,v_2\})$ (see Proposition C.0.3). Thus, up to subsequence, it converges to a function $v \in BV(I;\{v_1,v_2\})$ in the weak-star topology of $BV(I;\{v_1,v_2\})$.

Let us prove (ii). By assumption, $\{D\mathcal{A}(u_n)(I)\}_{n\in\mathbb{N}}$ is bounded. Let $\mathcal{C}_n(u_n) = \{(I_j^d)_n \,|\, j \in \{1,\dots,M(u_n)\}\}$ be the open partition associated with u_n. By Remark 4.3.1 and by the definition of R we have

$$\begin{aligned}
\liminf_{n\to+\infty} \frac{G_n^{hf}(u_n)}{\lambda_n} &\geqslant \liminf_{n\to+\infty} \sum_{j=1}^{M(u_n)} \frac{MM_n(\widetilde{u}_{n(I_j^d)_n})}{\lambda_n} + \liminf_{n\to+\infty} \sum_{j=1}^{M(u_n)-1} \frac{R_n(u_n)}{\lambda_n} \\
&\geqslant \liminf_{n\to+\infty} \sum_{j=1}^{M(u_n)-1} \frac{R_n(u_n)}{\lambda_n} \\
&\geqslant \liminf_{n\to+\infty} R\frac{|D\mathcal{A}(u_n)|(I)}{|v_1 - v_2|} \geqslant R\frac{|D\mathcal{A}(u)|(I)}{|v_1 - v_2|},
\end{aligned}$$

where in the last step we have used that $M(u_n) - 1 = \frac{|D\mathcal{A}(u_n)|(I)}{|v_1 - v_2|}$ and the lower semicontinuity of total variation (see Proposition 1.1.26).

We finally prove (iii). By a standard density argument we can choose u such that $\mathcal{A}(u) = v_1\mathbb{1}_{(0,\frac{1}{2})} + v_2\mathbb{1}_{[\frac{1}{2},1)}$. By the definition of R and by [17, Theorem 4.2] we gain the existence of $\{u_n\}_{n\in\mathbb{N}}$ such that

$$\frac{(R_n)_j(u_n)}{\lambda_n} \to R \text{ as } n \to +\infty, \qquad \mathcal{A}(u_n) \xrightarrow{BV} \mathcal{A}(u) \text{ as } n \to +\infty,$$

$$u_n\mathbb{1}_{(0,\frac{1}{2})} \in S_1, \quad u_n\mathbb{1}_{[\frac{1}{2},1)} \in S_2, \qquad \frac{MM_n(u_n\mathbb{1}_{(0,\frac{1}{2})})}{\lambda_n\delta_n^{\frac{3}{2}}}, \frac{MM_n(u_n\mathbb{1}_{[\frac{1}{2},1)})}{\lambda_n\delta_n^{\frac{3}{2}}} < C.$$

Therefore

$$\lim_{n\to+\infty} \frac{G_n^{hf}(u_n)}{\lambda_n} = R = R\frac{|D\mathcal{A}(u)|(I)}{|v_1 - v_2|}.$$

□

4.3.2 Second order Γ-convergence of the energy $\mathcal{E}_n$ as $n \to +\infty$

At the second order we split the global functional on the 2-dimensional sphere in a finite number of functionals localized in circles, where we repeat the analysis lead in [17].

We need the following theorem proved in [17, Theorem 2.2], which states that the discrete functional F_n has the same Γ-limit of the Modica-Mortola functional.

Theorem 4.3.3. *Let $F_n \colon L^1(I) \to [0, +\infty)$ defined as*

$$F_n(u) = \begin{cases} \varepsilon_n \sum_i \lambda_n \left(\dfrac{u^{i+1} - u^i}{\lambda_n} \right)^2 + \dfrac{1}{\eta_n} \sum_i \lambda_n \big[1 - (u^i)^2\big]^2 & \text{if } u \in C_n(I; \mathbb{S}^1), \\ +\infty & \text{otherwise,} \end{cases}$$

where ε_n and $\frac{\lambda_n}{\varepsilon_n}$ are infinitesimal, $\lim_{n \to +\infty} \dfrac{\varepsilon_n}{\eta_n} = 1$ *and*

$$C_n(I; \mathbb{S}^1) := \{u \colon \mathbb{Z}_n(I) \to \mathbb{S}^1 \,:\, u \text{ satisfies } (0.16), \\ u \text{ is constant on } \lambda_n(i + [0,1)),\ \forall i \in \mathbb{Z}_n(I)\}.$$

Then, with respect to the $L^1(I)$-convergence,

$$\Gamma\text{-}\lim_n F_n(u) = \begin{cases} \frac{4}{3}|Du|(I), & \text{if } u \in BV(I; \{-1, 1\}), \\ +\infty & \text{otherwise.} \end{cases}$$

Let $w = (w^1, w^2, 0)$, $\overline{w} = (\overline{w}^1, \overline{w}^2, 0)$ two vectors of $\mathbb{R}^3$, we define the function

$$\chi[w, \overline{w}] := \operatorname{sign}(w^1 \overline{w}^2 - w^2 \overline{w}^1).$$

For each S_i we define a convenient order parameter.

Let $u \in \mathcal{PC}_{\lambda_n}$. We associate each pair $\widetilde{u}^i_{I^d_j}, \widetilde{u}^{i+1}_{I^d_j}$ (see (4.18)) with the corresponding oriented angle $\theta^{ij} \in [-\pi, \pi)$ with vertex the center of the circle S_j given by

$$\theta^i_{I_j} := \chi\left[\widetilde{u}^i_{I^d_j} - \pi_{v_d}(\widetilde{u}^i_{I^d_j}), \widetilde{u}^{i+1}_{I^d_j} - \pi_{v_d}(\widetilde{u}^{i+1}_{I^d_j})\right] \\ \times \arccos\left(\frac{1}{R} (\widetilde{u}^i_{I^d_j} - \pi_{v_d}(\widetilde{u}^i_{I^d_j})) \cdot (\widetilde{u}^{i+1}_{I^d_j} - \pi_{v_d}(\widetilde{u}^{i+1}_{I^d_j})) \right).$$

Furthermore, we set

$$w^i_{I_j} := \sqrt{\frac{2}{\delta_n}} \sin \frac{\theta^i_{I_j}}{2}$$

and

$$w(t) := w^i_{I_j}$$

if $t \in \lambda_n\{i + [0,1)\}$, $i \in \left\{\frac{(t_j^d)_1}{\lambda_n}, \dots, \frac{(t_j^d)_2}{\lambda_n} - 1\right\}$ and $j \in \{1, \dots, M(u) - 1\}$. Note that we may define a map

$$T_n \colon u \in \mathcal{PC}_{\lambda_n} \mapsto (w, \mathcal{A}(u))$$

and denote $\widetilde{\mathcal{PC}}_{\lambda_n} := T_n(\mathcal{PC}_{\lambda_n})$. We observe that if $h = T_n(u) = T_n(v)$ then u and v take values in the same S_i and differ by a constant rotation so that $H_n^{hf}(u) = H_n^{hf}(v)$ and the same holds for the functionals $(R_n)_j^d, MM_n, P_n$.

Therefore, with a slight abuse of notation, we now regard $H_n^{hf}, (R_n)_j, MM_n, P_n$ as functionals defined on $h \in L^1(I; \mathbb{R} \times \{v_1, v_2\})$:

$$\begin{aligned} H_n^{hf}(h) &:= \begin{cases} H_n^{hf}(u) & \text{if } h \in \widetilde{\mathcal{PC}}_{\lambda_n}, \\ +\infty & \text{otherwise,} \end{cases} \\ (R_n)_j(h) &:= \begin{cases} R_{n,j}(u) & \text{if } h \in \widetilde{\mathcal{PC}}_{\lambda_n}, \\ +\infty & \text{otherwise,} \end{cases} \\ MM_n(h_{|I_j^d}) &:= \begin{cases} MM_n(\tilde{u}_{I_j^d}) & \text{if } h \in \widetilde{\mathcal{PC}}_{\lambda_n}, \\ +\infty & \text{otherwise,} \end{cases} \\ P_n(h) &:= \begin{cases} P_n(u) & \text{if } h \in \widetilde{\mathcal{PC}}_{\lambda_n}, \\ +\infty & \text{otherwise,} \end{cases} \end{aligned} \tag{4.21}$$

for $j \in \{1, \dots, M(h)\}$, where $h = T(u)$ and $M(h) := M(u)$.

We want to study the convergence of the functional

$$\mathcal{H}_n(h) = \frac{1}{\sqrt{2}\lambda_n \delta_n^{\frac{3}{2}}} \left[H_n(h) - \sum_{j=1}^{M(h)-1} (R_n)_j(h) \right], \tag{4.22}$$

for $h \in L^1(I; \mathbb{R} \times \{v_1, v_2\})$. In order to establish the related result, we need a notion of convergence.

Definition 4.3.4. *Let* $\{h_n\}_{n \in \mathbb{N}} \subset \widetilde{\mathcal{PC}}_{\lambda_n}$ *and* $h \in L^1(I; \mathbb{R} \times \{v_1, v_2\})$. *We say that* h_n L_θ*-converges to* h *(we write* $h_n \xrightarrow{L_\theta} h$*) if and only if the following conditions are satisfied:*

- *there exist* $\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{PC}_{\lambda_n}$ *and a positive constant* C *such that if* $\mathcal{C}_n(u_n) = \{(I_j^d)_n \,|\, j \in \{1, \dots, M(u_n)\}\}$ *is the open partition associated with* u_n*, the following two conditions are satisfied:*
  - $h_n = T_n(u_n)$ *and* $P_n(h_n) < C$*;*
  - $M(u_n) \to M \in \mathbb{N}$ *as* $n \to +\infty$*;*
  - $(I_j^d)_n \to I_j^d$ *in the Hausdorff sense, as* $n \to +\infty$*, for any* $j \in \{1, \dots, M\}$ *(see Definition A.1.15);*
- $h_n \mathbb{1}_{(I_j^d)_n} \to h \mathbb{1}_{I_j^d}$ *in* $L^1(I; \mathbb{R} \times \{v_d\})$*, a* $n \to +\infty$*, for all* $j \in \{1, \dots, M\}$*.*

Theorem 4.3.5. *Assume that there exist* $\lim_{n\to+\infty} \lambda_n k_n =: \eta \in (0,+\infty)$ *and* $l := \lim_{n\to+\infty} \frac{\lambda_n}{(2\delta_n)^{\frac{1}{2}}} \in [0,+\infty]$. *Then the following statements are true:*

(i) (Compactness) if for $\{h_n\}_{n\in\mathbb{N}} \subset L^1(I;\mathbb{R}\times\{v_1,v_2\})$ *there exists a constant* $C>0$ *such that*

$$\mathcal{H}_n(h_n) \leqslant P_n(h_n) \leqslant C, \tag{4.23}$$

then, up to a subsequence, $h_n \xrightarrow{L_\theta} h$ *as* $n\to+\infty$, *where*

- *if* $l=0$, $h \in BV(I;\{-1,1\}\times\{v_1,v_2\})$;
- *if* $l\in(0,+\infty)$, $h_{|I_j^d} \in \mathrm{H}^1_{|per|}(I_j^d;\mathbb{R}\times\{v_1,v_2\})$ *for all* $j\in\{1,\ldots,M(h)\}$;
- *if* $l=+\infty$, h *is piecewise-constant with values in* $\mathbb{R}\times\{v_1,v_2\}$.

The space $\mathrm{H}^1_{|per|}((a,b);\mathbb{R}\times\{v_1,v_2\})$ *is equal to*

$$\left\{h\in H^1((a,b);\mathbb{R}\times\{v_1,v_2\}) : |w(a)| = |w(b)| \text{ where } h=(w,\mathcal{A}(u))\right\};$$

(ii) ($\liminf$ *inequality)*

- *If* $l=0$, *for all* $h=(w,\mathcal{A}(u)) \in BV(I;\{-1,1\}\times\{v_1,v_2\})$ *and for all* $\{h_n\}_{n\in\mathbb{N}} \subset \widetilde{PC}_{\lambda_n}$ *such that*

  $$h_n \xrightarrow{L_\theta} h \in BV(I;\{-1,1\}\times\{v_1,v_2\}), \quad \text{as } n\to+\infty,$$

  and

  $$\mathcal{H}_n(h_n) \leqslant P_n(h_n) \leqslant C, \tag{4.24}$$

  for some $C>0$, *then*

  $$\liminf_{n\to+\infty} \mathcal{H}_n(h_n) \geqslant \frac{4}{3}R^2 \sum_{j=1}^{M(h)} |Dw|\,(I_j^d).$$

- *If* $l\in(0,+\infty)$, *for all* $h=(w,\mathcal{A}(u))\in L^1(I;\mathbb{R}\times\{v_1,v_2\})$ *such that* $h_{|I_j^d} \in \mathrm{H}^1_{|per|}(I_j^d,\mathbb{R}\times\{v_1,\, v_2\})$ *for all* $j\in\{1,\ldots,M(h)\}$, *and for all* $\{h_n\}_{n\in\mathbb{N}} \subset \widetilde{PC}_{\lambda_n}$ *such that*

  $$h_n \xrightarrow{L_\theta} h, \quad \text{as } n\to+\infty,$$

  and satisfies formula (4.24), then

  $$\liminf_{n\to+\infty} \mathcal{H}_n(h_n) \geqslant R^2 \sum_{j=1}^{M(h)} \left[\frac{1}{l}\int_{I_j} (w^2(x)-1)^2\,dx + l\int_{I_j} (w'(x))^2\,dx\right].$$

– *If $l = +\infty$, for all h piecewise-constant function with values in $\mathbb{R} \times \{v_1, v_2\}$ and for all $\{h_n\}_{n\in\mathbb{N}} \subset \widetilde{PC}_{\lambda_n}$ such that*

$$h_n \xrightarrow{L_\theta} h, \quad \text{as } n \to +\infty,$$

and satisfies formula (4.24), then

$$\liminf_{n\to+\infty} \mathcal{H}_n(h_n) \geqslant 0;$$

(iii) ($\limsup$ *inequality)*

– *If $l = 0$, for all $h = (w, \mathcal{A}(u)) \in BV(I; \{-1, 1\} \times \{v_1, v_2\})$ there exists $\{h_n\}_{n\in N} \subset \widetilde{\mathcal{PC}}_{\lambda_n}$ such that*

$$h_n \xrightarrow{L_\theta} h, \quad \text{as } n \to +\infty,$$

satisfies formula (4.24) and

$$\lim_{n\to+\infty} \mathcal{H}_n(h_n) = \frac{4}{3} R^2 \sum_{j=1}^{M} |Dw| (I_j).$$

– *If $l \in (0, +\infty)$, for all $h = (w, \mathcal{A}(u)) \in L^1(I; \mathbb{R} \times \{v_1, v_2\})$ such that $h_{|I_j} \in \mathrm{H}^1_{|per|}(I_j; \mathbb{R} \times \{v_1, v_2\})$ for all $j \in \{1, \ldots, M(h)\}$, there exists $\{h_n\}_{n\in N} \subset \widetilde{\mathcal{PC}}_{\lambda_n}$ such that*

$$h_n \xrightarrow{L_\theta} h, \quad \text{as } n \to +\infty,$$

satisfies formula (4.24) and

$$\lim_{n\to+\infty} \mathcal{H}_n(h_n) = R^2 \sum_{j=1}^{M} \left[\frac{1}{l} \int_{I_j} (w^2(x) - 1)^2 \, dx + l \int_{I_j} (w'(x))^2 \, dx \right].$$

– *If $l = +\infty$, for all h piecewise-constant function with values in $\mathbb{R} \times \{v_1, v_2\}$ there exists $\{h_n\}_{n\in\mathbb{N}} \subset \widetilde{\mathcal{PC}}_{\lambda_n}$ such that*

$$h_n \xrightarrow{L_\theta} h, \quad \text{as } n \to +\infty,$$

satisfies formula (4.24) and

$$\liminf_{n\to+\infty} \mathcal{H}_n(h_n) = 0.$$

Proof. We prove the statement only in case $l = 0$, being the other cases fully analogous. We start proving (i). By formulas (4.23), (4.19) we infer

$$MM_n(h_{n|I_j^n}) \leqslant \lambda_n \delta_n^{\frac{3}{2}} C, \quad \text{for all } j \in \{1, \ldots, M(h_n)\} \text{ and } n \in \mathbb{N}.$$

It is easy to see that, up to subsequences, $M(h_n)$ is independent of $n \in \mathbb{N}$ and that the intervals $(I_j^d)_n \to I_j = (t_{j-1}, t_j)$ in the Hausdorff sense (it may happen that some limit intervals are empty). In the following computations we drop for simplicity the dependence on n writing I_j^d in place of $(I_j^d)_n$. If $M = 1$, the proof can be led exactly as in [17, Theorem 4.2]. Let us assume that $M \geqslant 2$. By the definition of $\widetilde{u}^i_{nI_j^d}$ (see formula (4.18)), observing that

$$1 - \widetilde{u}^i_{nI_j^d} \cdot \widetilde{u}^{i+1}_{nI_j^d} = 2R^2 \sin^2\left(\frac{\theta^i_{I_j}}{2}\right),$$

$$1 - \widetilde{u}^I_{nI_j^d} \cdot \widetilde{u}^{i+2}_{nI_j^d} = R^2[1 - \cos(\theta^i_{I_j} + \theta^{i+1}_{I_j})],$$

$$\sum_{j=1}^{M} \#\mathcal{I}^n(I_j^d) = \#\mathcal{I}^n(I) - M - 1,$$

we gain

$$\begin{aligned}
\sqrt{2}\lambda_n\delta_n^{\frac{3}{2}}\mathcal{H}_n(h_n) &= \sum_{j=1}^{M}\lambda_n \sum_{i\in\mathcal{I}^n(I_j^d)} \left\{\alpha_n\left[1 - \widetilde{u}^i_{nI_j^d}\cdot\widetilde{u}^{i+1}_{nI_j^d}\right] - \left[1 - \widetilde{u}^i_{nI_j^d}\cdot\widetilde{u}^{i+2}_{nI_j^d}\right]\right\} \\
&+ \lambda_n\left(1 + \frac{\alpha_n^2}{8}\right)\#\mathcal{I}^n(I) + \lambda_n(1-\alpha_n)\sum_{j=1}^{M}\#\mathcal{I}^n(I_j^d) \\
&= R^2\sum_{j=1}^{M}\lambda_n\sum_{i\in\mathcal{I}^n(I_j^d)}\left\{2\alpha_n\sin^2\left(\frac{\theta^i_{I_j}}{2}\right) - \left[1 - \cos(\theta^i_{I_j} + \theta^{i+1}_{I_j})\right]\right\} \\
&+ \lambda_n\left(1 + \frac{\alpha_n^2}{8}\right)\#\mathcal{I}^n(I) + \lambda_n(1-\alpha_n)(\#\mathcal{I}^n(I) - M - 1) \\
&\geqslant R^2\sum_{j=1}^{M}\lambda_n\sum_{i\in\mathcal{I}^n(I_j^d)}\left\{2\alpha_n\sin^2\left(\frac{\theta^i_{I_j}}{2}\right) - \left[1 - \cos(\theta^i_{I_j} + \theta^{i+1}_{I_j})\right]\right\} \\
&+ \lambda_n\left(2 - \alpha_n + \frac{\alpha_n^2}{8}\right)\#\mathcal{I}^n(I),
\end{aligned}$$

where we used that $M > 1$. Repeating the same computations shown in [17, Theorem 4.2], we eventually obtain

$$\begin{aligned}
&\sum_{j=1}^{M} MM_n(\widetilde{u}_{nI_j}) \\
&\geqslant R^2\sum_{j=1}^{M}\sum_{i\in\mathcal{I}^n(I_j^d)}\lambda_n\left\{2\delta_n^2\left[(w^i_{nI_j^d})^2 - 1\right]^2 + (1-\gamma_n)\delta_n(w^{i+1}_{nI_j^d} - w^i_{nI_j^d})^2\right\}.
\end{aligned} \tag{4.25}$$

If $\varepsilon > 0$ is sufficiently small we have that $I_j^\varepsilon := (t_{j-1} + \varepsilon, t_j - \varepsilon) \subset (I_j^d)_n$, for all $n \in \mathbb{N}$, then we obtain that

$$MM_n(w_{n|I_j^\varepsilon}) \leqslant \lambda_n\delta_n^{\frac{3}{2}}C$$

and the formula (4.25) holds with I_j^ε in place of I_j^d. Therefore, by Theorem 4.3.3, $\{w_n \mathbb{1}_{I_j^\varepsilon}\}_{n\in\mathbb{N}}$, up to subsequence, converge in L^1 to $w \in BV(I_j)$. Thus we deduce the existence of $h \in BV(I;\{-1,1\} \times \{v_1, v_2\})$ such that $h_n := (w_n, \mathcal{A}(u_n)) \xrightarrow{L_\theta} h$ as $n \to +\infty$.

Now we prove (ii). Let $h = (w, \mathcal{A}(u)) \in BV(I;\{-1,1\} \times \{v_1, v_2\})$ By Definition 4.3.4 we have that, up to a subsequence, $M(h_n)$ is independent of n and for $\varepsilon > 0$ sufficiently small $I_j^\varepsilon := (t_j + \varepsilon, t_{j+1} - \varepsilon) \subset (I_j^d)_n$ for all $j \in \{1, \dots, M(h_n)\}$ and $n \in \mathbb{N}$, where $(t_j, t_{j+1}) = I_j$. By (4.19), we have

$$\liminf_{n\to+\infty} \mathcal{H}_n(h_n) = \liminf_{n\to+\infty} \frac{\sum_{j=1}^{M} MM_n(h_{n_{|I_j}})}{\sqrt{2}\lambda_n \delta_n^{\frac{3}{2}}} \geqslant \frac{4}{3} R^2 \sum_{j=1}^{M} |Dw|\,(I_j^\varepsilon),$$

where in the last step we have used the Γ-liminf inequality of Theorem 0.0.11. Letting $\varepsilon \to 0^+$, we obtain the liminf inequality.

We finally prove (iii). Let $h \in BV(I;\{-1,1\} \times \{v_1, v_2\})$. We can find $M > 0$ and an open partition of I made by the intervals $\mathcal{C} = \{I_j\}_{j\in\{1,\dots,M\}}$ such that $h_{|I_j} = (z_j, \overline{v}_j)$ for some $\overline{v}_j \in \{v_1, v_2\}$ and $z_j \in BV(I_j;\{-1,1\})$. For all $j \in \{1, \dots, M\}$ there exists a sequence $\{(z_j)_n\}_{n\in\mathbb{N}} \subset L^1(I_j;\mathbb{R})$ (see Theorem 0.0.11), such that

$$\lim_{n\to+\infty} (z_j)_n = z_j \ \text{ in } L^1(I_j;\mathbb{R}) \quad \text{and} \quad \lim_{n\to+\infty} \frac{MM_n(h_{n|I_j})}{\sqrt{2}\lambda_n \delta_n^{\frac{3}{2}}} = \frac{4}{3} R^2 |Dz|(I_j), \tag{4.26}$$

where $h_{n|I_j} = ((z_j)_n, \overline{v}_j)$. By (4.19) and (4.26) we gain

$$\lim_{n\to+\infty} \mathcal{H}_n(h_n) = \lim_{n\to+\infty} \frac{\sum_{j=1}^{M} MM_n(h_{n_{|I_j}})}{\sqrt{2}\lambda_n \delta_n^{\frac{3}{2}}} = \frac{4}{3} R^2 \sum_{j=1}^{M} |Dz|\,(I_j),$$

that is the thesis. □

Chapter 5

The two-dimensional case

In this chapter we analyze the two-dimensional version of the frustrated lattice system investigated in Chapter 4. Therefore we need to introduce proper notation and new definitions.

Let $\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ be an infinitesimal sequence of lattice spacings. The energy of a given spin of the system $u\colon i \in \Omega \cap \lambda_n\mathbb{Z}^2 \to S_1 \cup S_2$ is

$$H_n(u;\Omega) := \frac{1}{2}\lambda_n^2 \sum_{(i,j)\in\mathcal{I}^n(\Omega)} \left[\left| u^{i+2,j} - \frac{\alpha_n}{2}u^{i+1,j} + u^{i,j} \right|^2 + \left| u^{i,j+2} - \frac{\alpha_n}{2}u^{i,j+1} + u^{i,j} \right|^2 \right],$$

under the assumption that the functional

$$P_n(u;\Omega) := \lambda_n k_n |D\mathcal{A}(u)|(\Omega)$$

is bounded. In the previous formulas, the frustation parameter α_n of the system is close to the helimagnet/ferromagnet transition point as the number of particles diverges, i.e. $\alpha_n < 4$ for any $n \in \mathbb{N}$ and $\alpha_n \to 4$, as $n \to +\infty$. Furthermore, $\{k_n\} \subset \mathbb{R}^+$ is a divergent sequence and $\mathcal{A}\colon \mathcal{PC}_{\lambda_n}(S_1 \cup S_2) \to \{v_1, v_2\}$ is defined as

$$\mathcal{A}(u(t)) = \begin{cases} v_1 & \text{if } u(t) \in S_1, \\ v_2 & \text{otherwise,} \end{cases}$$

where $\mathcal{PC}_{\lambda_n}(S_1 \cup S_2)$ is the space of piecewise constants functions defined in (5.1).

Also in the two-dimensional setting we can repeat the previous construction and restrict every spin u to some open connected sets C_s that partition Ω in such a way that u takes values only in one magnetic anisotropy circle S_i. In order to avoid more complicated notation, we do not impose boundary conditions on $\partial\Omega$ and we will state the result by means of a local convergence.

While in the one-dimensional setting the partition associated with a spin was made by intervals, which guaranty the compactness results stated, in this case the sets C_s could be very wild, as the spacing of the lattice shrinks. Therefore, we require an additional regularity conditions for the components C_s, that is the BVG regularity (see Definition 5.2.1). Such conditions are satisfied by minimizing sequences, almost minimizing sequences and minimizing sequences under volume preserving hypotheses for the domain with the same anisotropy magnetization.

If the number of magnetic anisotropy transitions is finite, we may apply the Γ-convergence result proved in [15] in each component C_s for the rescaled functional

$$\mathcal{H}_n(u;\Omega) := \frac{1}{\sqrt{2}\lambda_n\delta_n^{\frac{3}{2}}}\Big[H_n(u,\Omega) - \sum_s R_{n,C_s}(u)\Big] = \frac{1}{\sqrt{2}\lambda_n\delta_n^{\frac{3}{2}}}\Big[\sum_s H_n(u;C_s)\Big],$$

as it is shown in Theorem 5.3.2.

This chapter is divided in three sections. In Section 5.1 we introduce discrete functions on the equi-spaced lattice on $\mathbb{R}^n$ and the notion of discrete derivatives. In Section 5.2 we introduce new notation and formulate the assumptions of our problem. Finally, Section 5.3 is devoted to the proof of the main result of this chapter (Theorem 5.3.2), where we compute the energy that the system spends for any chirality transition that the spin chain of the system makes.

5.1 Discrete functions

In this section we denote with d a number that can be 1 or 2. This number is the index of the two circles S_1 and S_2.

Given $i, j \in \mathbb{Z}$, we denote with $Q_{\lambda_n}(i,j) := (\lambda_n i, \lambda_n j)+[0,\lambda_n)^2$ the half-open square with left-bottom corner in $(\lambda_n i, \lambda_n j)$. For a given set S, we introduce the class of functions with values in S which are piecewise constant on the squares of the lattice $\lambda_n\mathbb{Z}^2$:

$$\mathcal{PC}_{\lambda_n}(S) := \{v \colon \mathbb{R}^2 \to S \,:\, v(x) = v(\lambda_n i, \lambda_n j) \text{ for } x \in Q_{\lambda_n}(i,j)\}. \tag{5.1}$$

We identify a function $v \in \mathcal{PC}_{\lambda_n}(S)$ with the function defined on the points of the lattice $\lambda_n\mathbb{Z}^2$ given by $(i,j) \mapsto v^{i,j} := v(\lambda_n i, \lambda_n j)$ for $i, j \in \mathbb{Z}$. Conversely, given values $v^{i,j} \in S$ for $i,j \in \mathbb{Z}$, we define $v \in \mathcal{PC}_{\lambda_n}(S)$ by $v(x) := v^{i,j}$ for $x \in Q_{\lambda_n}(i,j)$.

Given a function $v \in \mathcal{PC}_{\lambda_n}(\mathbb{R})$, we define the discrete partial derivatives $\partial_i^d v, \partial_2^d v \in \mathcal{PC}_{\lambda_n}(\mathbb{R})$ by

$$\partial_1^d v^{i,j} := \frac{v^{i+1,j} - v^{i,j}}{\lambda_n} \quad \text{and} \quad \partial_2^d v^{i,j} := \frac{v^{i,j+1} - v^{i,j}}{\lambda_n}, \quad \forall i,j \in \mathbb{Z},$$

and we denote the discrete gradient of v by $\nabla^d v$, defined as usual. Note that the two derivatives commute. Thus we may define the second order discrete partial derivatives $\partial_{11}^d v$, $\partial_{12}^d v = \partial_{21}^d v$, $\partial_{22}^d v$ by iterative application of these operators in arbitrary order. Similarly, we define higher order discrete partial derivatives.

5.2 Assumptions on the model

Our model involves an energy on discrete spin fields defined on square lattices inside a given sufficiently regular domain $\Omega \subset \mathbb{R}^2$. The following definition is given in [15, Section 3] and [52].

Definition 5.2.1 (*BVG* domains)**.** *A set $\Omega \subset \mathbb{R}^2$ is called a BVG domain if and only if for any $x \in \partial\Omega$ there exist a neighborhood $U_x \subset \mathbb{R}^2$, a function $\psi_x \in W^{1,\infty}(\Omega)$, with $\nabla\psi_x \in BV(\Omega;\mathbb{R}^2)$, and a rigid motion $R_x\colon \mathbb{R}^2 \to \mathbb{R}^2$ satisfying*

$$R_x(\Omega \cap U_x) = \{y = (y_1, y_2) \in \mathbb{R}^2 \,:\, y_1 > \psi_x(y_2)\} \cap R_x(U_x).$$

We assume that the domain $\Omega \subset \mathbb{R}^2$ belongs to the following class:

$$\mathfrak{A}_0 := \{\Omega \subset \mathbb{R}^2 \,:\, \Omega \text{ is an open, bounded, simply connected, } BVG \text{ domain}\}. \tag{5.2}$$

To define the energies in our model, we introduce the set of indices

$$\mathcal{I}^n(\Omega) := \{(i,j) \in \mathbb{Z}^2 \,:\, \overline{Q}_{\lambda_n}(i,j), \overline{Q}_{\lambda_n}(i+1,j), \overline{Q}_{\lambda_n}(i,j+1) \subset \Omega\}, \tag{5.3}$$

for $\Omega \in \mathfrak{A}_0$. Let $\alpha_n := 4(1-\delta_n)$, where $\{\delta_n\} \subset \mathbb{R}^+$ is an infinitesimal sequence, and $\{k_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ a divergent sequence. In the following we shall assume that $\varepsilon_n := \frac{\lambda_n}{\sqrt{\delta_n}} \to 0$ and $\lambda_n k_n \to \eta \in (0,+\infty)$ as $n \to +\infty$.
We consider the functionals $H_n, P_n\colon L^\infty(\mathbb{R}^2; S_1 \cup S_2) \times \mathfrak{A}_0 \to [0,+\infty]$ defined by

$$\begin{aligned} H_n(u;\Omega) :&= \frac{1}{2}\lambda_n^2 \sum_{(i,j)\in\mathcal{I}^n(\Omega)} \Bigg[\left| u^{i+2,j} - \frac{\alpha_n}{2} u^{i+1,j} + u^{i,j} \right|^2 \\ &\quad + \left| u^{i,j+2} - \frac{\alpha_n}{2} u^{i,j+1} + u^{i,j} \right|^2 \Bigg], \\ P_n(u;\Omega) :&= \lambda_n k_n |D\mathcal{A}(u)|(\Omega), \end{aligned}$$

for $u \in \mathcal{PC}_{\lambda_n}(S_1 \cup S_2)$, and extended to $+\infty$ elsewhere, where $\mathcal{A}(u)$ is the function defined in (4.1).
Similarly to the one-dimensional case, we observe that if $u \in \mathcal{PC}_{\lambda_n}(S_1 \cup S_2)$, the set Ω can be uniquely partitioned in regions where the function u takes values only in one of the two magnetic anisotropy circles. We now make the notation clear. There exist $M(u) \in \mathbb{N}$ and a collection of open connected sets, $\{C_s^d\}_{s\in\{1,\dots,M(u)\}}$, such that

$$\{C_s^d\}_{s\in\{1,\dots,M(u)\}} \text{ is an open partition of } \Omega, \tag{5.4}$$

$$u(x) \in S_d, \quad \forall x \in C_s^d,\ \forall s \in \{1,\dots,M(u)\}. \tag{5.5}$$

For the notation of open partition, we address the reader to Subsection 4.2.2.
The following definition will be useful throughout the following section.

Definition 5.2.2. *Let $u \in \mathcal{PC}_{\lambda_n}(S_1 \cup S_2)$. We say that $\mathcal{C}_n(u) = \{C_s^d \,|s \in \{1, \dots, M(u)\}\}$ is the* ***open partition associated with*** *u if $M(u) \in \mathbb{N}$ and the collection of open connected sets, $\{C_s^d\}_{s\in\{1,\dots,M(u)\}}$ satisfies* (5.4), (5.5). *We call $\mathcal{C}_n(u)$ the* ***open BVG partition associated with*** *u if C_s^d is also BVG, for all $s \in \{1, \dots, M(u)\}$.*

5.3 The Γ-convergence result

In this section we prove the main result of the chapter. Following the same idea adopted in the one-dimensional case, we split the functional H_n in two addends:

$$H_n(u;\Omega) = \sum_{s=1}^{M(u)} \Big[H_n(u; C_s^d) + R_{nC_s^d}(u) \Big],$$

where

$$R_{n,C_s^d}(u) := \frac{1}{2}\lambda_n^2 \sum_{(i,j)\in(C_s^d\cap\mathcal{I}^n(\Omega))\setminus\mathcal{I}^n(C_s^d)} \Bigg[\Big| u^{i+2,j} - \frac{\alpha_n}{2} u^{i+1,j} + u^{i,j} \Big|^2 + \Big| u^{i,j+2} - \frac{\alpha_n}{2} u^{i,j+1} + u^{i,j} \Big|^2 \Bigg]$$

is the remainder associated with the partition $\mathcal{C}_n(u)$ of u, which consists of the interactions between the vectors of the spin field in different elements of $\mathcal{C}_n(u)$.

We now introduce the chirality order parameter associated with a spin field. Let $u \in \mathcal{PC}_{\lambda_n}(S_1 \cup S_2)$ and $\mathcal{C}_n(u) = \{C_s^d \,|s \in \{1, \dots, M(u)\}\}$ be the open partition associated with u. For $(i,j) \in \mathcal{I}^n(C_s^d)$, we define the horizontal and vertical oriented angles between two adjacent spins by

$$\widetilde{\theta}^{i,j}_{C_s^d} := \operatorname{sign}(u^{i,j} \times u^{i+1,j}) \arccos\left(\frac{1}{R}\big(u^{i,j} - \pi_{v_d}(\widetilde{u}^{i,j}_{I_j^d})\big) \cdot \big(u^{i+1,j} - \pi_{v_d}(\widetilde{u}^{i+1,j}_{I_j^d})\big) \right),$$

$$\breve{\theta}^{i,j}_{C_s^d} := \operatorname{sign}(u^{i,j} \times u^{i,j+1}) \arccos\left(\frac{1}{R}\big(u^{i,j} - \pi_{v_d}(\widetilde{u}^{i,j}_{I_j^d})\big) \cdot \big(u^{i,j+1} - \pi_{v_d}(\widetilde{u}^{i,j+1}_{I_j^d})\big) \right),$$

both belonging to $[-\pi, \pi)$. Denoting with $\mathcal{D}(\Omega; \{v_1, v_2\})$ the space of functions defined in Ω with values in $\{v_1, v_2\}$, we define the order parameter $(w, z, \mathcal{A}(u)) \in \mathcal{PC}_{\lambda_n}(\mathbb{R}^2) \times \mathcal{D}(\Omega; \{v_1, v_2\})$ as

$$w^{i,j} := \begin{cases} \sqrt{\frac{2}{\delta_n}} \sin \frac{\widetilde{\theta}^{i,j}_{C_s^d}}{2} & \text{if } (i,j) \in \mathcal{I}^n(C_s^d) \text{ for some } s \in \{1, \dots, M(u)\}, \\ 0 & \text{otherwise,} \end{cases}$$

$$z^{i,j} := \begin{cases} \sqrt{\frac{2}{\delta_n}} \sin \frac{\breve{\theta}^{i,j}_{C_s^d}}{2} & \text{if } (i,j) \in \mathcal{I}^n(C_s^d) \text{ for some } s \in \{1, \dots, M(u)\}, \\ 0 & \text{otherwise.} \end{cases}$$

It is convenient to introduce the transformation

$$T_n \colon \mathcal{PC}_{\lambda_n}(S_1 \cup S_2) \to \mathcal{PC}_{\lambda_n}(\mathbb{R}^2) \times \mathcal{D}(\Omega; \{v_1, v_2\})$$

given by

$$T_n(u) := \big(w, z, \mathcal{A}(u)\big).$$

With a slight abuse of notation we define the functional $H_n\colon L^1_{loc}(\mathbb{R}^2;\mathbb{R}^2 \times \{v_1, v_2\}) \times \mathfrak{A}_0 \to [0, +\infty)$ by

$$H_n(h;\Omega) = \begin{cases} H_n(u;\Omega) & \text{if } T_n(u) = h \text{ for some } u \in \mathcal{PC}_{\lambda_n}(S_1 \cup S_2), \\ +\infty & \text{otherwise.} \end{cases}$$

Notice that H_n does not depend on the particular choice of u, since it is rotation-invariant. The same notation can be adopted for P_n and R_{n,C^d_s}.

We study the convergence of the rescaled functional

$$\mathcal{H}_n(h;\Omega) := \frac{1}{\sqrt{2}\lambda_n\delta_n^{\frac{3}{2}}}\left[H_n(h,\Omega) - \sum_{s=1}^{M(h)} R_{n,C^d_s}(h)\right] = \frac{1}{\sqrt{2}\lambda_n\delta_n^{\frac{3}{2}}}\sum_{s=1}^{M(h)} H_n(h;C^d_s),$$

where $h = T_n(u)$ and $M(h) := M(u)$, for some $u \in \mathcal{PC}_{\lambda_n}(S_1 \cup S_2)$. Hence, we introduce the functional $\mathcal{H}\colon L^1_{loc}(\mathbb{R}^2;\mathbb{R}^2 \times \{v_1, v_2\}) \times \mathfrak{A}_0 \to [0, +\infty)$ by setting

$$\mathcal{H}(h;\Omega) := \begin{cases} \dfrac{4}{3}\displaystyle\sum_{s=1}^{M(h)} (|D_1 w|(C^d_s) + |D_2 z|(C^d_s)) & \text{if } h = (w, z, \alpha) \in \mathrm{Dom}(\mathcal{H};\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$\mathrm{Dom}(\mathcal{H};\Omega) := \Big\{(w, z, \alpha) \in L^1_{loc}(\mathbb{R}^2;\mathbb{R}^2 \times \{v_1, v_2\}) :$$
$$\exists \{C^d_s\}_{s\in\{1,\ldots,M\}} \text{ open partition of } \Omega \text{ s.t.}$$
$$(w_{|C^d_s}, z_{|C^d_s}, \alpha_{|C^d_s}) \in BV(C^d_s;\{-1, 1\}^2\times\{v_d\}), \mathrm{curl}(w_{|C^d_s}, z_{|C^d_s}) = 0 \text{ in } \mathcal{D}'(C^d_s;\mathbb{R}^2)\Big\},$$

We have denoted by $\mathcal{D}'(C^d_s;\mathbb{R}^2)$ the space of distributions on C^d_s and for all $T \in \mathcal{D}'(C^d_s;\mathbb{R}^2)$ we have

$$\langle \mathrm{curl}(T)_{h,k}, \xi\rangle := -\langle T^k, \partial_h\xi\rangle + \langle T^h, \partial_k\xi\rangle \quad \text{for every } \xi \in C^\infty_c(C^d_s).$$

Definition 5.3.1 (Convergence law). *Let $\{h_n\}_{n\in\mathbb{N}} \subset L^1_{loc}(\mathbb{R}^2;\mathbb{R}^2 \times \{v_1, v_2\})$. We say that h_n L_θ-converges to $h \in L^1_{loc}(\mathbb{R}^2;\mathbb{R}^2 \times \{v_1, v_2\})$ (we write $h_n \xrightarrow{L_\theta} h$) if the following conditions are satisfied:*

- *there exist $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{PC}_{\lambda_n}(S_1 \cup S_2)$, a positive constant C and, for any $n \in \mathbb{N}$, an open BVG partition associated with u_n, $\mathcal{C}_n(u_n) = \{(C^d_s)_n | s \in \{1, \ldots, M(h_n)\}\}$, such that*
  - *$h_n = T_n(u_n)$ and $P_n(u_n;\Omega) < C$;*
  - *$M(u_n) \to M \in \mathbb{N}$ as $n \to +\infty$;*

– $(C_s^d)_n \to C_s^d$ *in the Hausdorff sense, as* $n \to +\infty$, *for any* $s \in \{1, \ldots, M\}$ *(see Definition A.1.15);*

- $h_n \mathbb{1}_{(C_s^d)_n} \to h \mathbb{1}_{C_s^d}$ *in* $L^1_{loc}(\mathbb{R}^2; \mathbb{R}^2 \times \{v_d\})$, *as* $n \to +\infty$, $\forall s \in \{1, \ldots, M\}$.

Now we state the main theorem of this section. Therein the hypotheses assumed are satisfied by minimizing sequences, almost minimizing sequences and minimizing sequences under volume preserving hypotheses. Thus by a standard argument of the Γ-convergence, we obtain that a minimizing sequences converge to a minimizer of the Γ-lim functional and that the minimal values of the functionals $\mathcal{H}_n$ converge to the minimal values of the functional $\mathcal{H}$.

Theorem 5.3.2. *Assume that* $\Omega \in \mathfrak{A}_0$. *Then the following statements hold true:*

i) (Compactness) Let $\{h_n = T_n(u_n)\}_{n\in\mathbb{N}} \subset T_n(\mathcal{PC}_{\lambda_n} \times \mathcal{D}(v_1, v_2))$ *be a sequence such that*

$$\mathcal{H}_n(h_n; \Omega) \leqslant P_n(h_n; \Omega) < C, \tag{5.6}$$

for some constant $C > 0$. *Assume that the BVG partition associated with* u_n, $\mathcal{C}_n(u_n) = \{(C_s^d)_n |\, s \in \{1, \ldots, M(u_n)\}\}$, *is such that*

$$M(u_n) \to M \in \mathbb{N} \quad \text{as } n \to +\infty,$$

$$(C_s^d)_n \to C_s^d \quad \text{in the Hausdorff sense, as } n \to +\infty, \forall s \in \{1, \ldots, M\}.$$

Then there exists $h \in \mathrm{Dom}(\mathcal{H}, \Omega)$ *such that, up to a subsequence,* $h_n \xrightarrow{L_\theta} h$, *as* $n \to +\infty$;

ii) (Liminf inequality) Let $\{h_n\}_{n\in\mathbb{N}} \subset L^1_{loc}(\mathbb{R}^2; \mathbb{R}^2 \times \{v_1, v_2\})$ *and* $h \in L^1_{loc}(\mathbb{R}^2; \mathbb{R}^2 \times \{v_1, v_2\})$. *Assume that* $P_n(h_n; \Omega) < C$ *for some constant* $C > 0$ *and* $h_n \xrightarrow{L_\theta} h$, *as* $n \to +\infty$. *Then*

$$\mathcal{H}(h; \Omega) \leqslant \liminf_{n\to+\infty} \mathcal{H}_n(h_n; \Omega);$$

iii) (Limsup inequality) Let $h \in L^1_{loc}(\mathbb{R}^2; \mathbb{R}^2 \times \{v_1, v_2\})$. *Then there exists a sequence* $\{h_n\}_{n\in\mathbb{N}} \subset L^1_{loc}(\mathbb{R}^2; \mathbb{R}^2 \times \{v_1, v_2\})$ *such that* $h_n \xrightarrow{L_\theta} h$ *and*

$$\limsup_{n\to+\infty} \mathcal{H}_n(h_n; \Omega) \leqslant \mathcal{H}(h; \Omega).$$

Proof. We start proving i). Let $\{h_n = (w_n, z_n, \mathcal{A}(u_n))\}_{n\in\mathbb{N}} \subset L^1_{loc}(\mathbb{R}^2; \mathbb{R}^2 \times \{v_1, v_2\})$ be a sequence satisfiying (5.6). Fixing $\varepsilon > 0$ sufficiently small, we have that for all $n \in \mathbb{N}$, up to a subsequence, $(C_s^d)_\varepsilon := \{x \in C_s^d : \mathrm{dist}(x, \partial C_s^d) > \varepsilon\} \subset (C_s^d)^n$ and $u_{n_{|(C_s^d)_\varepsilon}}$ takes values only in one circle. We infer that

$$\frac{1}{\sqrt{2}\lambda_n \delta_n^{\frac{3}{2}}} \sum_{s=1}^{M} H_n(h_n; (C_s^d)_\varepsilon) \leqslant \mathcal{H}_n(h_n; \Omega) < C.$$

which of course implies that $H_n(h_n;(C_s^d)_\varepsilon) < C\sqrt{2}\lambda_n\delta_n^{\frac{3}{2}}$, for all $s \in \{1,\dots,M\}$. We are in position to apply [15, Theorem 2.1 i)] to deduce the existence of $(w_{(C_s^d)_\varepsilon}, z_{(C_s^d)_\varepsilon}) \in BV((C_s^d)_\varepsilon;\{-1,1\}^2)$ such that, up to subsequences, $(w_n,z_n) \to (w_{(C_s^d)_\varepsilon}, z_{(C_s^d)_\varepsilon})$ in $L^1_{loc}((C_s^d)_\varepsilon;\mathbb{R}^2)$ and $\mathrm{curl}(w_{(C_s^d)_\varepsilon}, z_{(C_s^d)_\varepsilon}) = 0$ in $\mathcal{D}'((C_s^d)_\varepsilon;\mathbb{R}^2)$. The couples $(w_{(C_s^d)_\varepsilon}, z_{(C_s^d)_\varepsilon})$ can be extended to 0 in $C_s^d \backslash (C_s^d)_\varepsilon$. We preliminary observe that

$$(w_{(C_s^d)_{\varepsilon_2}}, z_{(C_s^d)_{\varepsilon_2}}) = (w_{(C_s^d)_{\varepsilon_1}}, z_{(C_s^d)_{\varepsilon_1}}) \quad \text{a.e. on } (C_s^d)_{\varepsilon_2}, \tag{5.7}$$

for any $0 < \varepsilon_1 < \varepsilon_2$. Indeed, since $(C_s^d)_{\varepsilon_2} \subset (C_s^d)_{\varepsilon_1}$, we have that

$$(w_n,z_n) \to (w_{(C_s^d)_{\varepsilon_1}}, z_{(C_s^d)_{\varepsilon_1}}) \quad \text{in } L^1_{loc}((C_s^d)_{\varepsilon_2};\mathbb{R}^2).$$

The uniqueness of the limit in the L^1_{loc}-topology implies (5.7). We now define the couples $(w_{C_s^d}, z_{C_s^d})\colon C_s^d \to \mathbb{R}^2$ as

$$(w_{C_s^d}, z_{C_s^d}) := \lim_{\varepsilon\to 0^+} (w_{(C_s^d)_\varepsilon}, z_{(C_s^d)_\varepsilon}).$$

The definition is well-posed; indeed, since by (5.7),

$$\lim_{\varepsilon'\to 0^+} (w_{(C_s^d)_{\varepsilon'}}, z_{(C_s^d)_{\varepsilon'}}) = (w_{(C_s^d)_{\frac{1}{n}}}, z_{(C_s^d)_{\frac{1}{n}}}) \quad \text{a.e. in } (C_s^d)_{\frac{1}{n}},$$

for all $n \in \mathbb{N}$, then

$$\begin{aligned}&\left|\left\{x \in C_s^d \ : \ \nexists \lim_{\varepsilon'\to 0^+} (w_{(C_s^d)_{\varepsilon'}}(x), z_{(C_s^d)_{\varepsilon'}}(x))\right\}\right| \\ &= \left|\bigcup_{n=1}^{+\infty}\left\{x \in (C_s^d)_{\frac{1}{n}} \ : \ \nexists \lim_{\varepsilon'\to 0^+} (w_{(C_s^d)_{\varepsilon'}}(x), z_{(C_s^d)_{\varepsilon'}}(x))\right\}\right| = 0.\end{aligned}$$

Furthermore we set $(w,z)\colon \Omega \to \mathbb{R}^2$

$$(w,z)(x) = (w_{C_s^d}, z_{C_s^d})(x),$$

for a.e. $x \in \Omega$ with $x \in C_s^d$, for some $s \in \{1,\dots,M\}$. Of course $(w_{|C_s^d}, z_{|C_s^d}) = (w_{C_s^d}, z_{C_s^d}) \in BV(C_s^d;\{-1,1\}^2)$, as it is the limit of BV functions. In order to show the L^1_{loc}-convergence, we fix $A \subset\subset C_s^d$. Since $\mathrm{dist}(A,\partial C_s^d) > 0$, there exists $\varepsilon > 0$ such that $A \subset\subset (C_s^d)_\varepsilon$. We obtain:

$$\left\|(w_n,z_n) - (w_{C_s^d}, z_{C_s^d})\right\|_{L^1(A;\mathbb{R}^2)} = \left\|(w_n,z_n) - (w_{(C_s^d)_\varepsilon}, z_{(C_s^d)_\varepsilon})\right\|_{L^1(A;\mathbb{R}^2)},$$

which vanishes as $n \to +\infty$, up to subsequences. This leads to the convergence

$$(w_n,z_n) \to (w_{C_s^d}, z_{C_s^d}) \quad \text{in } L^1_{loc}(C_s^d;\mathbb{R}^2).$$

Finally, we prove that $\mathrm{curl}(w_{C_s^d}, z_{C_s^d}) = 0$ in $\mathcal{D}'(C_s^d)$. If $\xi \in C_c^\infty(C_s^d)$, then $\mathrm{spt}\xi \subset (C_s^d)_\varepsilon$ for some $\varepsilon > 0$ and so

$$\left\langle \mathrm{curl}(w_{C_s^d}, z_{C_s^d}), \xi\right\rangle = -\int_{(C_s^d)_\varepsilon} w_{(C_s^d)_\varepsilon}\partial_2\xi\,dx + \int_{(C_s^d)_\varepsilon} z_{(C_s^d)_\varepsilon}\partial_1\xi\,dx$$

$$= \left\langle \mathrm{curl}(w_{(C_s^d)_\varepsilon}, z_{(C_s^d)_\varepsilon}), \xi \right\rangle = 0.$$

Now we prove ii). Let $\{h_n\}_{n\in\mathbb{N}} \subset L^1_{loc}(\mathbb{R}^2;\mathbb{R}^2 \times \{v_1, v_2\})$ and $h \in L^1_{loc}(\mathbb{R}^2;\mathbb{R}^2 \times \{v_1, v_2\})$ such that $P_n(h_n;\Omega) < C$ and $h_n \overset{L_\theta}{\to} h$, as $n \to +\infty$. Up to subsequences we may assume that the lower limit in the right hand side of the liminf inequality is actually a limit. Furthermore we may assume that it is finite, the inequality being otherwise trivial. In particular, we have

$$\mathcal{H}_n(h_n;\Omega) < C,$$

with a possibly larger C. By the definition of L_θ-convergence, $h_n = (w_n, z_n, \mathcal{A}(u_n)) = T_n(u_n)$ for some $u_n \in \mathcal{PC}_{\lambda_n}(S_1 \cup S_2)$. We may assume, up to subsequences, that $M(h_n)$ is independent of n and, by the Hausdorff convergence, for $\varepsilon > 0$ sufficiently small, $(C_s^d)_\varepsilon \subset (C_s^d)^n$ and $u_{n_{|(C_s^d)_\varepsilon}}$ takes values only on one circle, for all $n \in \mathbb{N}$. By the positivity of $\mathcal{H}_n$, we infer that

$$\mathcal{H}_n(h_n;\Omega) \geqslant \frac{1}{\sqrt{2}\lambda_n\delta_n^{\frac{3}{2}}} \sum_{s=1}^{M} H_n(h_n; (C_s^d)_\varepsilon).$$

Since $h_n \to h$ in $L^1((C_s^d)_\varepsilon;\mathbb{R}^2 \times \{v_d\})$, as $n \to +\infty$, we are in position to apply [15, Theorem 2.1 ii)] so that, passing to the lower limit, we get

$$\begin{aligned}\liminf_{n\to+\infty} \mathcal{H}_n(h_n;\Omega) &\geqslant \sum_{s=1}^{M} \liminf_{n\to+\infty} H_n(h_n; (C_s^d)_\varepsilon) \\ &\geqslant \sum_{s=1}^{M} \frac{4}{3}[|D_1 w|((C_s^d)_\varepsilon) + |D_2 z|((C_s^d)_\varepsilon)],\end{aligned}$$

where $h = (w, z, \alpha)$. Letting $\varepsilon \to 0^+$ we get the thesis.
Let us prove iii). Let $h \in L^1_{loc}(\mathbb{R}^2;\mathbb{R}^2 \times \{v_1, v_2\})$. We may assume that $h \in \mathrm{Dom}(H;\Omega)$. This means that $h = (w, z, \alpha) \in L^1_{loc}(\mathbb{R}^2;\mathbb{R}^2 \times \{v_1, v_2\})$ and the existence of an open partition of Ω, $\mathcal{C} = \{C_s^d | \, s \in \{1, \dots, M\}\}$, such that

$$(w_{|C_s^d}, z_{|C_s^d}, \alpha_{|C_s^d}) \in BV(C_s^d; \{-1, 1\}^2 \times \{v_d\}),\ \mathrm{curl}(w_{|C_s^d}, z_{|C_s^d}) = 0 \text{ in } \mathcal{D}'(C_s^d;\mathbb{R}^2).$$

Applying [15, Theorem 2.1 iii)] to any $(w_{|C_s^d}, z_{|C_s^d})$, we get the existence of a sequence $(w_{nC_s^d}, z_{nC_s^d}) \in L^1_{loc}(\mathbb{R}^2;\mathbb{R}^2)$ such that $(w_{nC_s^d}, z_{nC_s^d}) \to (w_{|C_s^d}, z_{|C_s^d})$ in $L^1(C_s^d;\mathbb{R}^2)$ and

$$\limsup_{n\to+\infty} H_n(w_{nC_s^d}, z_{nC_s^d}, v_d) \leqslant \frac{4}{3}(|D_1 w|(C_s^d) + |D_2 z|(C_s^d)).$$

Defining $(w_n, z_n, \alpha_n)\colon \mathbb{R}^2 \to \mathbb{R}^2 \times \{v_1, v_2\}$ by

$$(w_n, z_n, \alpha_n)(x) = (w_{nC_s^d}(x), z_{nC_s^d}(x), v_d),$$

if $x \in \Omega$ such that $x \in C_s^d$ for some $s \in \{1, \dots, M\}$, and arbitrarily extended outside Ω, and summing on $s \in \{1, \dots, M\}$ the previous inequality we get the thesis. □

Appendix A

Some tools from measure theory

In this appendix we recall some classical results that we need in our manuscript, which can be found in the books of [7], [32], [46] and [54].

A.1 Radon measures

In this section we recall the definition of Radon measures and some of their properties.

Definition A.1.1 (Radon measure). *An outer measure μ on $\mathbb{R}^n$ is a **Radon measure** if it is*

- *locally finite, i.e., for every compact set $K \subset \mathbb{R}^n$, $\mu(K) < +\infty$;*
- *Borel regular, i.e. μ is a Borel measure and regular, that is, for every set $F \subset \mathbb{R}^n$, there exists a Borel set $E \subset \mathbb{R}^n$ such that $F \subset E$ and $\mu(E) = \mu(F)$.*

Radon measures are linked with bounded linear functionals via total variation measures. Indeed, as stated in Riesz's theorem, every bounded linear functional L defined on $C_c(\mathbb{R}^n; \mathbb{R}^m)$ has an integral representation with respect to the total variation measure associated with L. We start giving the following definition.

Definition A.1.2 (Total variation of a linear functional). *Let L be a linear functional on $C_c(\mathbb{R}^n; \mathbb{R}^m)$. We define the **total variation** $|L|$ of L as the set function $|L|\colon \mathcal{P}(\mathbb{R}^n) \to [0, +\infty]$ such that, for $A \subset \mathbb{R}^n$ open,*

$$|L|(A) = \sup\{\langle L, \phi\rangle \,:\, \phi \in C_c(A; \mathbb{R}^m),\ |\phi| \leqslant 1\},$$

and, for $E \subset \mathbb{R}^n$ arbitrary,

$$|L|(E) = \inf\{|L|(A) \,:\, E \subset A \text{ and } A \text{ is open}\}.$$

Next, we state Riesz's theorem.

Theorem A.1.3 (Riesz's theorem)**.** *If $L\colon C_c(\mathbb{R}^n;\mathbb{R}^m)\to\mathbb{R}$ is a bounded linear functional, then its total variation $|L|$ is a Radon measure on $\mathbb{R}^n$ and there exists a $|L|$-measurable function $g\colon \mathbb{R}^n\to\mathbb{R}^m$, with $|g|=1$ $|L|$-a.e. on $\mathbb{R}^n$, such that*

$$\langle L,\phi\rangle=\int_{\mathbb{R}^n}(\phi\cdot g)\,d|L|,\quad \forall\phi\in C_c(\mathbb{R}^n;\mathbb{R}^m),$$

that is $L=g|L|$. Moreover, for every open set $A\subset\mathbb{R}^n$,

$$|L|(A)=\sup\left\{\int_{\mathbb{R}^n}(\phi\cdot g)\,d|L|\;:\;\phi\in C_c(A;\mathbb{R}^m),\ |\phi|\leqslant 1\right\}.$$

Remark A.1.4 (Total variation of a Radon measure)**.** *Radon measures can be unambiguously identified with monotone linear functionals on $C_c(\mathbb{R}^n)$. Hence, the total variation $|\mu|$ of a Radon measure μ in $\mathbb{R}^n$ can be defined as*

$$|\mu|(A):=\sup\left\{\int_{\mathbb{R}^n}\phi\cdot d\mu\;:\;\phi\in C_c(A;\mathbb{R}^m),\ |\phi|\leqslant 1\right\}.$$

Here we introduce the operation of push-forward of a measure through a function and we recall a useful property.

Definition A.1.5 (Push-forward of a measure)**.** *Given a function $f\colon\mathbb{R}^n\to\mathbb{R}^m$ and a measure μ on $\mathbb{R}^n$, the* ***push-forward*** *of μ through f is the outer measure $f_\sharp\mu$ on $\mathbb{R}^m$ defined by the formula*

$$f_\sharp\mu(E):=\mu\big(f^{-1}(E)\big),\quad \forall E\subset\mathbb{R}^n.$$

Proposition A.1.6 (Push-forward of a Radon measure)**.** *If μ is a Radon measure on $\mathbb{R}^n$, and $f\colon\mathbb{R}^n\to\mathbb{R}^m$ is continuous and proper, then $f_\sharp\mu$ is a Radon measure on $\mathbb{R}^m$, $\mathrm{spt}(f_\sharp\mu)=f(\mathrm{spt}\mu)$, and for every Borel measurable function $u\colon\mathbb{R}^m\to[0,+\infty]$ we have*

$$\int_{\mathbb{R}^m}u\,d(f_\sharp\mu)=\int_{\mathbb{R}^n}(u\circ f)\,d\mu.$$

We recall the Radon-Nikodym theorem, a well-known result related to the differentiation of Radon measures. It states that every Radon measure ν have an integral representation with respect to any Radon measure μ such that ν is absolutely continuous with respect to μ.

Definition A.1.7. *Let μ and ν be two Borel measures on $\mathbb{R}^n$. We say that ν is* ***absolutely continuous*** *with respect to μ, written $\nu<<\mu$, if and only if*

$$\mu(A)=0\quad\textit{implies}\quad\nu(A)=0,$$

for any $A\subset\mathbb{R}^n$.

Theorem A.1.8 (Radon-Nikodym theorem). *Let ν and μ be two Radon measures on $\mathbb{R}^n$, with $\nu << \mu$. Then*

$$\nu(A) = \int_A D_\mu \nu \, d\mu,$$

for all μ-measurable sets $A \subset \mathbb{R}^n$, where, for any $x \in \mathbb{R}^n$,

$$D_\mu \nu(x) := \begin{cases} \lim\limits_{r \to 0^+} \dfrac{\nu(B_r(x))}{\mu(B_r(x))} & \textit{if } \mu(B_r(x)) > 0, \, \forall r > 0, \\ +\infty & \textit{if } \mu(B_r(x)) = 0, \textit{ for some } r > 0, \end{cases}$$

*is the **derivative** (or the **density**) of ν with respect to μ.*

The derivative function enjoys of the following properties.

Lemma A.1.9. *Let μ and ν be two Radon measures on $\mathbb{R}^n$, $A \subset \mathbb{R}^n$ and fix $\alpha \in (0, +\infty)$. Then*

i) $A \subset \left\{ x \in \mathbb{R}^n \, : \, \liminf\limits_{r \to 0^+} \dfrac{\nu(B_r(x))}{\mu(B_r(x))} \leqslant \alpha \right\}$ *implies* $\nu(A) \leqslant \alpha \mu(A)$*;*

ii) $A \subset \left\{ x \in \mathbb{R}^n \, : \, \limsup\limits_{r \to 0^+} \dfrac{\nu(B_r(x))}{\mu(B_r(x))} \geqslant \alpha \right\}$ *implies* $\nu(A) \geqslant \alpha \mu(A)$

Proposition A.1.10. *Let $A \subset \mathbb{R}^n$ be an open set, and let μ be a positive Radon measure in A, with $\mu(A) < +\infty$. For $0 < \alpha < n$ let*

$$E^\alpha := \left\{ x \in A \, : \, \limsup_{\rho \to 0^+} \frac{\mu(B_\rho(x))}{\rho^\alpha} > 0 \right\}.$$

Then $\dim_{\mathcal{H}}(E^\alpha) \leqslant \alpha$.

We quote below some density results for the Hausdorff measures.

Proposition A.1.11. *If $s \in (0, n)$ and $M \subset \mathbb{R}^n$ is a Borel set with $\mathcal{H}^s(M \cap K) < +\infty$ for every compact set $K \subset \mathbb{R}^n$, then for $\mathcal{H}^s$-a.e. $x \in \mathbb{R}^n \backslash M$,*

$$\lim_{r \to 0^+} \frac{\mathcal{H}^s(M \cap B_r(x))}{\omega_s r^s} = 0.$$

Proposition A.1.12. *Let $E \subset \mathbb{R}^n$. If $\mathcal{H}^s_\infty(E) < +\infty$, then, for $\mathcal{H}^s$-a.e. $x \in E$,*

$$\limsup_{r \to 0^+} \frac{\mathcal{H}^s_\infty(E \cap B_r(x))}{\omega_s r^s} \geqslant 2^{-s}.$$

We conclude this section by giving the definition of weak-star convergence of Radon measures and its characterization.

Definition A.1.13 (Weak-star convergence of Radon measures)**.** *Let $\{\mu_h\}_{h\in\mathbb{N}}$ and μ be $\mathbb{R}^m$-valued Radon measures on $\mathbb{R}^n$. We say that μ_h **weak-star converges** to μ, and we write $\mu_h \overset{\star}{\rightharpoonup} \mu$, if and only if*

$$\int_{\mathbb{R}^n} \phi \cdot d\mu = \lim_{h\to+\infty} \int_{\mathbb{R}^n} \phi \cdot d\mu_h, \quad \forall \phi \in C_c(\mathbb{R}^n; \mathbb{R}^m).$$

Proposition A.1.14. *If $\{\mu_h\}_{h\in\mathbb{N}}$ and μ are Radon measures on $\mathbb{R}^n$, then the following three statements are equivalent.*

(i) $\mu_h \overset{\star}{\rightharpoonup} \mu$.

(ii) If $K \subset \mathbb{R}^n$ is compact and $A \subset \mathbb{R}^n$ is open, then

$$\mu(K) \geqslant \limsup_{h\to+\infty} \mu_h(K), \quad \textit{and} \quad \mu(A) \leqslant \liminf_{h\to+\infty} \mu_h(A).$$

(iii) If $K \subset \mathbb{R}^n$ is a bounded Borel set with $\mu(\partial E) = 0$, then

$$\mu(E) = \lim_{h\to+\infty} \mu_h(E).$$

Definition A.1.15 (Hausdorff convergence)**.** *Let $C \subset \mathbb{R}^n$ and a the sequence $\{C_n\}_{n\in\mathbb{N}}$ of sets in $\mathbb{R}^n$. We say that **Hausdorff converges** to C if and only if there exists $\varepsilon_0 > 0$ such that $C \subset (C_n)_\varepsilon$ and $C_n \subset C_\varepsilon$, for every $\varepsilon < \varepsilon_0$. We have denoted*

$$D_\varepsilon := \{x \in D \,:\, \mathrm{dist}(x, \partial D) > \varepsilon\},$$

for some set $D \subset \mathbb{R}^n$.

A.2 Area and Coarea formulas

In this section we recall the Area formula for surfaces, for which we need the notion of tangential differentiability, and its application to graph of Lipschitz functions of codimension one. We also give the Coarea formula on locally $\mathcal{H}^{n-1}$-rectifiable sets.

We start giving the definition of $\mathcal{H}^k$-rectifiable sets.

Definition A.2.1. *Given a $\mathcal{H}^k$-measurable set $M \subset \mathbb{R}^n$, we say that M is **countably $\mathcal{H}^k$-rectifiable** if there exist countably many Lipschitz maps $f_h\colon \mathbb{R}^k \to \mathbb{R}^n$ such that*

$$\mathcal{H}^k\Big(M \setminus \bigcup_{h\in\mathbb{N}} f_h(\mathbb{R}^k)\Big) = 0;$$

*we say that M is **locally $\mathcal{H}^k$-rectifiable** provided $\mathcal{H}^k(M \cap K) < +\infty$, for every compact set $K \subset \mathbb{R}^n$; finally, if $\mathcal{H}^k(M) < +\infty$, then M is simply called **$\mathcal{H}^k$-rectifiable**.*

For locally $\mathcal{H}^k$-rectifiable sets an approximate tangent space can be defined.

Theorem A.2.2 (Approximate tangent space)**.** *If $M \subset \mathbb{R}^n$ is a locally $\mathcal{H}^k$-rectifiable set, then, for $\mathcal{H}^k$-a.e. $x \in M$, there exists a unique k-dimensional space T_xM such that, as $r \to 0^+$,*

$$\frac{(\Phi_{x,r})_\sharp(\mathcal{H}^k \llcorner M)}{r^k} = \mathcal{H}^k \llcorner \left(\frac{M-x}{r}\right) \xrightharpoonup{*} \mathcal{H}^k \llcorner T_xM,$$

*where $\Phi_{x,r}(y) := \frac{y-x}{r}$, for $y \in \mathbb{R}^n$. The space T_xM is called the **approximate tangent space** to M at x.*

Definition A.2.3. *Let $M \subset \mathbb{R}^n$ be a countably $\mathcal{H}^k$-rectifiable set and $f\colon \mathbb{R}^n \to \mathbb{R}^m$ a Lipschitz function. We say that f is **tangentially differentiable** at $x \in M$ if the restriction of f to the affine space $x + T_xM$ is differentiable at x. In this case, there exists a linear function $\nabla^M f(x)\colon T_xM \to \mathbb{R}^m$ such that*

$$\lim_{h\to 0} \frac{f(x+hv)-f(x)}{h} = \nabla^M f(x)v.$$

*The **tangential Jacobian** of f with respect to M at x is defined by*

$$J^M f(x) = \sqrt{\det(\nabla^M f(x)^* \nabla^M f(x))},$$

where ${}^*\nabla^M f(x)$ is the adjoint of the map $\nabla^M f(x)$.

We are now in position to recall the Area and Coarea formulas.

Theorem A.2.4 (Area formula for injective maps)**.** *If $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is an injective Lipschitz function and $M \subset \mathbb{R}^n$ is a countably $\mathcal{H}^k$-rectifiable set ($m \leqslant k$), then*

$$\mathcal{H}^k(M) = \int_E J^M f(x)\, dx.$$

Theorem A.2.5 (Area formula of a graph of codimension one)**.** *If $u\colon \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function, then for every Lebesgue measurable set $G \subset \mathbb{R}^{n-1}$,*

$$\mathcal{H}^{n-1}(\Gamma_u(G)) = \int_G \sqrt{1+|\nabla' u(z)|^2}\, dz.$$

In fact, $\mathcal{H}^{n-1} \llcorner \Gamma_u$ is a Radon measure on $\mathbb{R}^n$ and, for every $\phi \in C_c(\mathbb{R}^n)$,

$$\int_{\Gamma_u} \phi\, d\mathcal{H}^{n-1} = \int_{\mathbb{R}^{n-1}} \phi(z, u(z))\sqrt{1+|\nabla' u(z)|^2}\, dz.$$

Theorem A.2.6 (Coarea formula on locally $\mathcal{H}^{n-1}$-rectifiable sets)**.** *If $M \subset \mathbb{R}^n$ is a locally $\mathcal{H}^{n-1}$-rectifiable set and $f\colon \mathbb{R}^n \to \mathbb{R}$ is a Lipschitz function, then*

$$\int_{\mathbb{R}} \mathcal{H}^{n-2}(M \cap \{f = t\})\, dt = \int_M |\nabla^M f|\, d\mathcal{H}^{n-1}.$$

In particular, if $g\colon M \to [-\infty, +\infty]$ is a Borel function and either $g \geqslant 0$ or $g \in L^1(\mathbb{R}^n, \mathcal{H}^{n-1} \llcorner M)$, then

$$\int_{\mathbb{R}} dt \int_{M\cap\{f=t\}} g\, d\mathcal{H}^{n-2} = \int_M g|\nabla^M f|\, d\mathcal{H}^{n-1}.$$

A.3 Other useful results

We present here three well-known useful results. The first one is an obvious consequence of Fubini's theorem and the fundamental theorem of Calculus. The second and the third results are well-known.

Lemma A.3.1. *If $\phi \in C_c^1(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} \nabla\phi \, dx = 0.$$

Lemma A.3.2 (McShane's Lemma)**.** *If $E \subset \mathbb{R}^n$ and $f\colon E \to \mathbb{R}$ is a Lipschitz function, then there exists an extension $g\colon \mathbb{R}^n \to \mathbb{R}$ of f in $\mathbb{R}^n$ with the same Lipschitz constant.*

Theorem A.3.3 (Sobolev-Poincaré's inequality for functions vanishing on a set of positive measure)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded connected set with Lipschitz continuous boundary. For every $u \in W^{1,p}(\Omega)$, $p < n$, taking the value zero in a set A of positive measure, we have*

$$\|u\|_{L^{p^\star}(\Omega)} \leqslant c\left(\frac{|\Omega|}{|A|}\right)^{\frac{1}{p^\star}} \|\nabla u\|_{L^p(\Omega)} ,$$

for some constant positive $c = c(n, p, \Omega)$.

Appendix B

The notion of Γ-convergence

The Γ-convergence is a convegence law introduced by De Giorgi in the 1970s. It is well-suited for the description of the asymptotic behaviour of variational problems, which depend on some parameters. The latter ones could have a geometric nature or derive from an approximation procedure or a discretization argument.

No a priori assumptions on the form of minimizers are needed, so that Γ-convergence can be applied to diversified contexts: from the study of problems with discontinuities in computer vision as well as to the description of the overall properties of nonlinear composites, to the formalization of the passage from discrete systems to continuum theories, homogenization theory, phase transitions and boundary value problems in wildly perturbed domains.

We now recall the mathematical definition of Γ-convergence.

Definition B.0.1. *Let X be a metric space. We say that a sequence $f_j\colon X \to \overline{R}$* ***Γ-converges*** *in X to $f_\infty\colon X \to \overline{R}$ if, for all $x \in X$, we have*

i) ($\liminf$ inequality) for every sequence $\{x_j\}_{j\in\mathbb{N}} \subset X$ converging to x

$$f_\infty(x) \leqslant \liminf_{j\to+\infty} f_j(x_j);$$

ii) ($\limsup$ inequality) there exists a sequence $\{x_j\}_{j\in\mathbb{N}} \subset X$ (called ***"recovery" sequence****) converging to x such that*

$$f_\infty(x) \geqslant \limsup_{j\to+\infty} f_j(x_j).$$

The function f_∞ is called the Γ-limit of $\{f_j\}_{j\in\mathbb{N}}$, and we write $f_\infty = \Gamma\text{-}\lim_j f_j$.

As it is evident in the previous definition and in Part II, the choice of the metric on X is clearly a crucial step in problems involving Γ-limits.

Γ-convergence is designed so that it implies the convergence of "compact" minimum problems, as stated in the following theorem.

Theorem B.0.2. *Let X be a metric space and let $\{f_j\}_{j\in\mathbb{N}}$ be a sequence of equi-mildly coercive functions on X, i.e. there exists a non-empty compact set $K \subset X$ such that $\inf_X f_j = \inf_K f_j$, for all $j \in \mathbb{N}$. Let $f_\infty := \Gamma\text{-}\lim_j f_j$; then f_∞ admits minimum and*

$$\min_X f_\infty = \lim_{j\to+\infty} \inf_X f_j.$$

Moreover, if $\{x_j\}_{j\in\mathbb{N}} \subset X$ is a precompact sequence such that $\lim_{j\to+\infty} f_j(x_j) = \lim_{j\to+\infty} \inf_X f_j$, then every limit of a subsequence of $\{x_j\}_{j\in\mathbb{N}}$ is a minimum point of f_∞.

In Part II we employ the notion of Γ-convergence (with respect different topologies) in the study of the asymptotic behaviour of a functional deriving from a discrete system.

Appendix C

Some properties of L^∞ functions with values in a compact set

In this appendix we recall some classical properties of the Lebesgue space $L^\infty(I;K)$, where $K \subset \mathbb{R}^N$ is a compact set.

Proposition C.0.1. *Let $K \subset \mathbb{R}^N$ be a compact set and let $\{f_n\}_{n\in\mathbb{N}} \subset L^\infty(I;K)$. Then, up to subsequences, $f_n \overset{*}{\rightharpoonup} f \in L^\infty(I;co(K))$ as $n \to +\infty$ in the weak-star topology of $L^\infty(I;\mathbb{R}^N)$. Moreover, for all $u \in L^\infty(I;co(K))$, there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset L^\infty(I;K)$ of piecewise functions such that $u_n \overset{*}{\rightharpoonup} u$ as $n \to +\infty$.*

Proof. Since the set K is bounded then, up to a subsequence, there exists $f \in L^\infty(I;\mathbb{R}^N)$ such that $f_n \overset{*}{\rightharpoonup} f$ as $n \to +\infty$. We now prove that $f(t) \in co(K)$ for almost every $t \in (0,1)$. For every $\xi \notin co(K)$ there exist an affine function $h_\xi : \mathbb{R}^N \to \mathbb{R}$ and $\alpha < 0$ such that

$$h_\xi(\xi) > 0 > \alpha > h_\xi(x), \quad \forall x \in co(K).$$

By the weak-star convergence of $\{f_n\}_{n\in\mathbb{N}}$ we have that for any measurable set $A \subset (0,1)$

$$\int_A h_\xi(f(t))dt = \lim_{n\to+\infty} \int_A h_\xi(f_n(t))dt \leqslant |A|\alpha < 0.$$

Hence by the arbitrariness of A we obtain

$$h_\xi(f(t)) < 0, \quad \text{for a.e. } t \in (0,1). \tag{C.1}$$

Recalling that

$$co(K) = \bigcap_{j\in\mathbb{N}} \left\{y \in \mathbb{R}^N : \ h_{\xi_j}(y) < 0, \ \xi_j \in \mathbb{Q}^N \backslash co(K)\right\},$$

by formula (C.1) we obtain

$$f(t) \in co(K), \quad \text{for a.e. } t \in (0,1).$$

Now we can prove the second statement of the proposition. By a standard density argument, it is enough to prove the claim for $u = a\mathbb{1}_J$, where J is an open interval and $a \in co(K)$. We define the following function:

$$h(t) := \begin{cases} a_1 & \text{if } t \in (0, \lambda), \\ a_2 & \text{if } t \in [\lambda, 1), \end{cases}$$

where $a = \lambda a_1 + (1 - \lambda)a_2$ with a_1, $a_2 \in K$ and for some $\lambda \in [0, 1]$. Then the sequence $u_n(t) := h(nt)$ converges to u in the weak-star topology of L^∞ by Riemann-Lebesgue's lemma. □

Corollary C.0.2. *Let $K \subset \mathbb{R}^N$ be a compact set. The closure of the set $L^\infty(I; K)$ with respect to the weak-star topology of $L^\infty(I; \mathbb{R}^N)$ is the set $L^\infty(I; co(K))$.*

Proof. Since the space $L^1(I; \mathbb{R}^N)$ is separable, every bounded subset of $L^\infty(I; \mathbb{R}^N)$ is metrizable with respect to the weak-star topology of $L^\infty(I; \mathbb{R}^N)$. Hence the set $L^\infty(I; K)$ is metrizable. Therefore, by the above proposition, we have that the set $L^\infty(I; co(K))$ is the weak-star closure of the set $L^\infty(I; K)$. □

Proposition C.0.3. *Let K_1, $K_2 \subset \mathbb{R}^N$ be two compact sets and let $\{f_n\}_{n\in\mathbb{N}} \subset L^\infty(I; K_1 \cup K_2)$ be such that for all $n \in \mathbb{N}$ exist $M(n) \in \mathbb{N}$ and $0 = t_1^{(n)} < \cdots < t_{M(n)}^{(n)} = 1$ for which*

$$f_n(t) \in K_j, \text{ for some } j \in \{1, 2\} \text{ and a.e. } t \in (t_i^{(n)}, t_{i+1}^{(n)}),$$

for all $i \in \{1, \ldots, M(n) - 1\}$. Finally we suppose that

$$\sup_{n\in\mathbb{N}} M(n) < +\infty. \tag{C.2}$$

Then, up to subsequences, $f_n \overset{*}{\rightharpoonup} f$ in the weak-star topology of $L^\infty(I; \mathbb{R}^N)$ and $f \in L^\infty(I; co(K_1) \cup co(K_2))$. Moreover if $co(K_1) \cap co(K_2) = \varnothing$, there exist $M \in \mathbb{N}$ and $0 = t_1 < \cdots < t_M = 1$ such that

$$f(t) \in co(K_j), \text{ for some } j \in \{1, 2\} \text{ and a.e. } t \in (t_i, t_{i+1}),$$

for all $i \in \{1, \ldots, M\}$,

Proof. By Proposition C.0.1, we have, up to a subsequence, that $f_n \overset{*}{\rightharpoonup} f \in L^\infty(I; co(K_1 \cup K_2))$. Accordingly, by assumption (C.2), we can find $M \in \mathbb{N}$, independent of $n \in \mathbb{N}$, such that for every $n \in \mathbb{N}$ there exist $0 = t_1^{(n)} < \cdots < t_M^{(n)} = 1$ for which $f_n(t) \in K_j$ for some $j \in \{1, 2\}$ and a.e. $t \in (t_i^{(n)}, t_{i+1}^{(n)})$, for all $i \in \{1, \ldots, M - 1\}$. Up to subsequence, we can calculate

$$\lim_{n\to+\infty} t_i^{(n)} = t_i, \quad \forall i \in \{1, \cdots, M\},$$

so that $0 = t_1 < \cdots < t_M = 1$. Let us fix $i \in \{1, \cdots, M-1\}$. For all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that we have

$$(t_i + \varepsilon, t_{i+1} - \varepsilon) \subset (t_i^{(n)}, t_{i+1}^{(n)}) \quad \forall n \geqslant n_0.$$

We define the following two sets:

$$A_1 = \{n \geqslant n_0 \,:\, f_n(t) \in K_1 \text{ for a.e. } t \in (t_i + \varepsilon, t_{i+1} - \varepsilon)\},$$
$$A_2 = \{n \geqslant n_0 \,:\, f_n(t) \in K_2 \text{ for a.e. } t \in (t_i + \varepsilon, t_{i+1} - \varepsilon)\}.$$

One of the following three alternatives may occur:

1. $\#A_1 = \infty$, $\#A_2 < \infty$;
2. $\#A_1 < \infty$, $\#A_2 = \infty$;
3. $\#A_1 = \infty$, $\#A_2 = \infty$.

In the first case we have that $f_n \in L^\infty((t_i + \varepsilon, t_{i+1} - \varepsilon); K_1)$ for all $n \geqslant n_0$, up to a finite number of indices of the sequence. Thus, by Proposition C.0.1 $f_n \overset{*}{\rightharpoonup} f \in L^\infty((t_i + \varepsilon, t_{i+1} - \varepsilon); co(K_1))$ and hence, by the arbitrariness of $\varepsilon > 0$, we obtain $f \in L^\infty((t_i, t_{i+1}); co(K_1))$.

The second case is fully analogous to the first case. In the third case we can find two subsequences $\{n_k^{(1)}\}_{k\in\mathbb{N}}$ and $\{n_k^{(2)}\}_{k\in\mathbb{N}}$ such that $f_{n_k^{(1)}} \in L^\infty((t_i+\varepsilon, t_{i+1} - \varepsilon); K_1)$ and $f_{n_k^{(2)}} \in L^\infty((t_i+\varepsilon, t_{i+1}-\varepsilon); K_2)$ for all $k \in \mathbb{N}$. By Proposition C.0.1, there exist $f_1 \in L^\infty((t_i+\varepsilon, t_{i+1}-\varepsilon); co(K_1))$ and $f_2 \in L^\infty((t_i+\varepsilon, t_{i+1}-\varepsilon); co(K_2))$ such that $f_{n_k^{(1)}} \overset{*}{\rightharpoonup} f_1$ and $f_{n_k^{(2)}} \overset{*}{\rightharpoonup} f_2$. One the other hand, we recall that $f_n \overset{*}{\rightharpoonup} f \in L^\infty(I; co(K_1 \cup K_2))$. Thus, by the uniqueness of the limit in the weak-star topology, we have $f_1(t) = f_2(t) = f(t)$ for almost every $t \in (t_i + \varepsilon, t_{i+1} - \varepsilon)$ and so $f \in L^\infty((t_i + \varepsilon, t_{i+1} - \varepsilon); co(K_1) \cap co(K_2))$. Hence, by the arbitrariness of $\varepsilon > 0$, we have $f \in L^\infty((t_i, t_{i+1}); co(K_1) \cap co(K_2))$.

If we repeat the above argument for all $i \in \{1, \cdots, M\}$ we obtain the thesis. If $co(K_1) \cap co(K_2) = \varnothing$, the case $\#A_1 = \#A_2 = \infty$ cannot occur and therefore we obtain the last claim of the statement. □

List of Symbols

$\mathbb{N}$	set of all positive natural numbers
$\mathbb{Z}$	set of all integer numbers
$\mathbb{R}$	set of all real numbers
$\mathbb{R}^n$	euclidean n-dimensional space
$\mathcal{P}(E)$	power set of the set E
$co(E)$	convex hull of the set $E \subset \mathbb{R}^n$
$\mathbb{S}^{n-1}$	unit sphere of $\mathbb{R}^n$
$\lfloor x \rfloor$	integer part of $x \in \mathbb{R}$
$x \bigotimes y$	the tensor product between $x, y \in \mathbb{R}^n$
$x \cdot y$	inner euclidean product between the vectors $x, y \in \mathbb{R}^n$
$\lvert x \rvert$	euclidean norm of $x \in \mathbb{R}^n$;
$E_{x,r}$	blow-up $\frac{E-x}{r}$ of the set $E \subset \mathbb{R}^n$ of centre $x \in \mathbb{R}^n$ and radius $r > 0$
$I_\varepsilon(E)$	tubular neighborhood of $E \subset \mathbb{R}^n$ of half-lenght $\varepsilon > 0$
$A \subset\subset B$	the closure of the set $A \subset \mathbb{R}^n$ is contained in the set $B \subset \mathbb{R}^n$
$B_r(x)$	open ball of $\mathbb{R}^n$ with centre $x \in \mathbb{R}^n$ and radius $r > 0$
B_r	open ball of $\mathbb{R}^n$ with centre 0 and radius $r > 0$
$\mathbf{C}_r(x, \nu)$	open cylinder of $\mathbb{R}^n$ with centre $x \in \mathbb{R}^n$, radius and half-height $r > 0$ and oriented in the direction $\nu \in \mathbb{S}^{n-1}$
$\mathbf{C}_r(x)$	open cylinder of $\mathbb{R}^n$ with centre $x \in \mathbb{R}^n$, radius and half-height $r > 0$, oriented in the direction e_n
$\mathbf{C}_r$	open cylinder of $\mathbb{R}^n$ with centre 0, radius and half-height $r > 0$, oriented in the direction e_n
$\mathbf{K}_r(x)$	open cylinder of $\mathbb{R}^n$ with centre $x \in \mathbb{R}^n$, radius $r > 0$, half-height 2 and oriented in the direction e_n
$\mathbf{K}_r$	open cylinder of $\mathbb{R}^n$ with radius $r > 0$, half-height 2 and oriented in the direction e_n
$\lvert E \rvert$	Lebesgue measure of the set $E \subset \mathbb{R}^n$
ω_n	Lebesgue measure of B_1
$E^{(t)}$	set of points of density $t \in [0, 1]$ of $E \subset \mathbb{R}^n$
$\mathcal{H}^s(E)$	s-dimensional Hausdorff measure of the set $E \subset \mathbb{R}^n$
$\dim_{\mathcal{H}}(E)$	Hausdorff dimension of the set $E \subset \mathbb{R}^n$

$\#E$	cardinality of the set E
$\|\mu\|$	total variation measure associated with μ
$\text{spt}\mu$	support of the measure μ
$\mu \llcorner E$	restriction of the measure μ to the set $E \subset \mathbb{R}^n$
$\mu << \nu$	the measure μ is absolutely continuous with respect the measure ν
$\mu_h \overset{*}{\rightharpoonup} \mu$	the sequence of measures $\{\mu_h\}_{h\in\mathbb{N}}$ weak-star converges to the measure μ
$p_v w$	the projection of the vector $w \in \mathbb{R}^n$ on the vector $v \in \mathbb{R}^n$
qw	the projection of the vector $w \in \mathbb{R}^n$ on the vector e_n
$p_{v^\perp} w$	the rojection of the vector $w \in \mathbb{R}^n$ the orthogonal complement of the vector $v \in \mathbb{R}^n$
pw	the projection of the vector $w \in \mathbb{R}^n$ on the orthogonal complement of the vector e_n
u_+	the positive part of the function u
$\text{spt}f$	the support of the function f
$\mathbb{1}_E$	the characteristic function of the set $E \subset \mathbb{R}^n$
$u_{x,r}$	the mean value of u in the ball $B_r(x)$, i.e. $\fint_{B_r(x)} u(z)\,dz$
u_r	the mean value of u in the ball B_r
$L^p(\Omega)$	the space of Lebesgue measurable functions f in the open set $\Omega \subset \mathbb{R}^n$, with $p \in [1, +\infty]$
$W^{1,p}(\Omega)$	the space of Sobolev functions in the open set $\Omega \subset \mathbb{R}^n$
$W_0^{1,p}(\Omega)$	the space of Sobolev functions $W^{1,p}(\Omega)$ with zero trace on Ω
$H^1(\Omega)$	the space of Sobolev functions $W^{1,2}(\Omega)$
$H_0^1(\Omega)$	the space of Sobolev functions $W_0^{1,2}(\Omega)$
$C(\Omega)$	the space of continuous functions in the open set $\Omega \subset \mathbb{R}^n$
$C^k(\Omega)$	the space of continuously k-differentiable functions in the open set $\Omega \subset \mathbb{R}^n$
$C^\alpha(\Omega)$	the space of Hölder continuous functions with Hölder exponent $\alpha \in [0, 1)$ in the open set $\Omega \subset \mathbb{R}^n$
$\text{osc}(f; \Omega)$	oscillation of the function f in the open set Ω
$C^{0,1}(\Omega)$	the space of Lipschitz functions in the open set $\Omega \subset \mathbb{R}^n$, with $k \in \mathbb{N}$
$BV(\Omega)$	the space of BV functions in Ω
$\mathcal{D}'(\Omega)$	the space of distributions on the open set $\Omega \subset \mathbb{R}^n$
$T_x M$	the approximate tangent space to M at the point $x \in \mathbb{R}^n$
$\\|f\\|_{L^p(\Omega)}$	the L^p norm of the function $f \in L^p(\Omega)$
$[f]_\alpha$	the Hölder continuous seminorm of $f \in C^\alpha(\Omega)$
$\langle L, f\rangle$	dual scalar product between the functional L and the element f

$\vert L\vert$	total variation measure of the linear functional $L\colon C_c(\mathbb{R}^n;\mathbb{R}^n)$
$\to$	strong convergence of functions or points
$E_h \to E$	the sequence of functions $\{\mathbb{1}_{E_h}\}_{h\in\mathbb{N}}$ of $\mathbb{R}^n$ conveges to $\mathbb{1}_E$
$f_h \rightharpoonup f$	the sequence of functions $\{f_h\}_{h\in\mathbb{N}}$ weakly converges to the function f
$f_h \overset{*}{\rightharpoonup} f$	the sequence of functions $\{f_h\}_{h\in\mathbb{N}}$ weakly-star converges to the function f
$\Gamma_f(G)$	the graph of the function $f\colon \Omega \to \mathbb{R}$ lying on $G \subset \Omega$
Γ_f	the set $\Gamma_f(\Omega)$

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