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A characterisation of *mathematics education creativity* inspired by active and popular pedagogy

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Ai miei genitori e a mio fratello,
la mia luce, il mio scudo, le mie ali

Alla nonna Elvira,
mio angelo custode da sempre
e a zia Nunzia,
esempio di coraggio e fantasia

A tutti i sogni realizzati
e a tutti quelli ancora in volo

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INTRODUCTION

The present PhD thesis has been developed through an interweaving of reflections and experiences in the field of mathematics education that have, gradually, steered the research interest in a specific direction. Initially, the focus on studies and research about *informal mathematics education* (Nemirovsky, Kelton & Civil, 2017) and the field of *critical mathematics education* (Skovsmose & Penteado, 2012) allowed for an in-depth study of these two strands, with a view to a connection between mathematics education and ideals of democracy and active citizenship. Contextually, particular educational experiences, carried out during the PhD course, have stimulated a reflection in this same sense. In particular, my involvement in two editions of the "Proud of You"[1] project fostered the development of some issues concerning the creation of educational activities designed in *informal mathematics education* contexts and the potential, through them, to mediate mathematical content in students from socio-culturally disadvantaged backgrounds, acting effectively and sustainably with respect to preventing and combating early school drop-out. The curiosity about these two issues generated an opening of the horizon of interests that led to explore the field of active and popular pedagogy. The literature references taken into consideration turned out to be valuable and inspiring, and opened the way to a reflection on the possible tools for implementing a motivating didactic design and the possibility of intertwining it with the desire for social redemption and liberation from any kind of oppression. With these assumptions, the research then moved in a more specific and clear direction. In particular, the focus was on the design and implementation of teaching activities that were meaningful, from a mathematical point of view, and inclusive and attentive to the ideals of democracy, from a social point of view. The focus therefore shifted to the process of designing and implementing mathematical teaching activities of the kind made explicit, paving the way for the research and the characterisation of the *mathematics education creativity*. With this in mind, studies and research into mathematical creativity have been investigated in detail, enabling the object of investigation to be framed. With the aim of investigating the kind of mathematical creativity involved, effective methodologies were researched for data collection and subsequent

[1] The "Proud of You" (PoY) project aims to prevent the risk of school drop-out and encourage learning and formative growth, through the implementation of Italian and mathematics educational actions directed to primary and middle school students. It is a project implemented by the "Next-Level" association, which works in the field of social and cultural promotion of young people and which has involved, over the course of its three editions, primary and middle schools located in different neighbourhoods of Naples, characterised by socio-economic and cultural difficulties.

analysis. Inspired by the research (Hadamard, 1945; Liljedahl, 2004), interviews were conducted, involving teachers who showed great creativity in teaching design, also with reference to socially and culturally disadvantaged contexts.

In the following, the organisation into chapters of this work is presented, which will allow, through the salient points that characterised this stimulating and dynamic interweaving, to come to points of arrival, for the research conducted, that open up new directions to be explored, in a broader research perspective.

In **Chapter 1**, the main literary references concerning active and popular pedagogy will be presented, allowing a broad outline of the theoretical framework. In particular, the most significant theories and experiences mentioned will lead to the emergence of a new idea of school: an *active school*, which places pupils, their interests and their relationship with the world at the centre. The idea of the school as part of society, the search for an educational proposal that reflects the needs and interests of pupils and values material and sensitive experiences, the role of the teacher as a guide in the process of discovery and learning of students will be presented as key points of active and popular pedagogy, also representing a meeting point with the educational proposal of Paulo Freire, one of the most authoritative pedagogues of the 20th century. In order to have a deeper understanding of his thought, ample space will be dedicated to Freire's pedagogical studies and research, which, with his most popular work *Pedagogy of the Oppressed* (Freire, 1970; 2018), will represent a point of reference for the research activity conducted. In particular, attention will be focused on the *problem-posing* model of education, proposed by Freire, and its opposition to the banking model of education. Later on, considering the conception of human as a *being-in-situation* (Freire, 1970) who cannot prescind from the interactions and the relationship with the world around him, from the concrete, existential and historical situations belonging to his reality, attention will be focused on the programmatic content of an educational proposal and on the search for the " game of the people" (Freire, 1989).

Starting with the studies presented on active and popular pedagogy and deepened through the Freireian perspective, possible connections to the specific area of mathematics education will be explored in **Chapter 2**. In particular, an initial focus will be on the field of *critical mathematics education* (see e.g. Skovsmose, 1994) - for which Freire's vision is of great inspiration - which recognises an intertwined development of a critical awareness of citizenship and mathematical competence. Although the line of research referred to is very

young and the material in the literature is fairly recent, it will nevertheless make it possible to steer reflection in the direction of interest, enriched by the prolific ideas in the field of mathematics education introduced by Emma Castelnuovo (1963; 1966; 1967; 2003) and Eric Gutstein (2006; 2007; 2008; 2009). The research material that will be taken into consideration will make it possible to deepen the discourse on *active schools*, from the perspective of mathematics education inspired by ideals of democracy and active citizenship. The considerations that emerge will provide the basis for introducing the experiences within the "Proud of You" project in the succeeding chapter.

Chapter 3 will be dedicated to the narration of the "Proud of You" (PoY) project, framed within the field of *critical mathematics education*. In particular, the general idea of the project will be proposed and two different editions, in which I took part with different roles and responsibilities, will be recounted in more detail. The PoY project will be presented through a particular lens that takes into account Freire's pedagogical vision and consequently, the educational visions of Emma Castelnuovo and Eric Gutstein. Through this lens, attention will be drawn to methodological choices and aspects of the construction of teaching activities in which mathematical objectives are intertwined with objectives of education for active citizenship and participation in a democratic discourse. The chapter will close with references to a more specific direction of research, aimed at investigating the creativity at the basis of teaching design and implementation processes.

With the aim of defining the kind of mathematical creativity to be investigated, several studies on creativity, in general (see, for example, Levenson, 2022; Pehkonen, 1997; Sriraman, 2004), and on mathematical creativity, in particular (Hadamard, 1945; Liljedahl, 2004; Joklitschke, Rott & Schindler, 2021), will be cited and explored in **Chapter 4**, focusing mainly on the creative process of the mathematician engaged in solving mathematical problems of various nature and complexity. As will be seen, consulting the different researches will provide access to a variety of definitions of creativity, allowing, firstly, to frame the object of research in a broader context and, subsequently, to clarify the type of creativity to be investigated. The creativity we are interested in will be defined as *mathematics education creativity*. In this regard, the reasons for this definition will be clarified and the research question will be made explicit.

In **Chapter 5**, the focus will be on the choice of an effective methodology for the purposes of the research conducted. In particular, in order to investigate the particular kind of

mathematical creativity involved, I chose to interview mathematics teachers from different school grades. To this end, the first section will introduce the participating teachers and argue the reasons that motivated their choice. Then, the method used for data collection, the Explicitation Interview (Vermersch, 1994) will be presented, adopted to structure the interview and elaborate its questions. The interview was used with the aim of identifying and describing any visible connections between the teaching design and implementation process and aspects of teachers' mathematical creativity. Finally, in order to analyse the content of the narratives, also taking into account the meaning attributed by the teachers to the shared experiences, the methodology used to analyse the data collected - the Content Analysis method (Berelson, 1952) - will be discussed.

Chapter 6 will be entirely focused on the analysis of the interviews collected. The data to be analysed and, step by step, the various stages of the content analysis will be presented. Attention will be particularly focused on the two crucial phases of which the analysis is composed: a first quantitative phase, intended in a more exploratory sense, and a second phase, of a more qualitative and interpretative nature, oriented towards the identification of *recurring themes*.

Finally, **Chapter 7** will aim to draw conclusions, taking into account all the aspects analysed in the course of the previous chapters. In particular, focal points in the research path will be taken up and critically reviewed in order to bring out broader reflections and shed light on significant findings. Particular emphasis will be placed on the structural definition of three dimensions that will enable the characterisation of *mathematics education creativity* and answer to the research question of the present study.

CHAPTER 1

ACTIVE AND POPULAR PEDAGOGY: A NEW IDEA OF SCHOOL

"The education we propose [...] cannot be based on a conception of humans as 'empty' beings that the world 'fills' with content [...]. It cannot be a depositary of content, but problematising for humans in their relations with the world". (Freire, 2018, p. 87)

Active pedagogy was born at the end of the 19th century and it is based on a set of studies and experiences conducted by pedagogues and educators from all over the world, in which the student is considered the active protagonist of the educational process. The collection and the study of research material relating to pedagogical activism have made it possible to identify in various pedagogues and educators a point of reference for this research work and to trace the main characteristics and cornerstones of the educational methodology in question. In order to outline a large part of the theoretical frame of reference, this chapter focuses on the most significant theories and experiences in the field of active and popular pedagogy, with a particular emphasis, in the following paragraphs, on relevant arguments in the research literature.

The emblem of pedagogical activism is certainly represented by the theory of John Dewey (Burlington, October 20, 1859 - New York, June 1, 1952), an American philosopher, psychologist, and educational reformer. According to Dewey, education consists of a constant reconstruction and reorganisation of personal and social experience. From this perspective, man's thinking comes from experience, in the sense of social experience, because it is determined by the human-environment relationship. Starting from the idea of the school as part of society, Dewey believes that the educational process must try to involve both the pupil and the social environment in which he or she is embedded (Dewey, 1973). With this in mind, the interests of the pupils and the activities that are most familiar and close to their everyday experience should inspire teaching practice. It is interesting to dwell on this important aspect of Dewey's pedagogical analysis, from which he derives an idea of a school that has the task of mediating between society and pupils. The introduction into the school of work and of material and sensible experiences that put pupils in contact with real things seems to respond to this aim: the school becomes an active environment, capable of giving the pupil the opportunity to live and not only to prepare for life (Dewey,

1954). The experience conducted in the Laboratory school, founded by Dewey in 1896 and attached to the University of Chicago, enabled him to put into practice what he had theorised, through the activation of a didactic path and the involvement of pupils aged between four and eighteen. The curriculum proposed by Dewey, at the basis of his experimental schooling, is divided into three areas: active occupations (expressive or productive activities), knowledge of the social sphere (history and geography) and knowledge of cultural symbols (reading, writing, calculating,...), which are not clearly separated but in continuous communication with each other, favouring increasingly interdisciplinary learning. Starting from interesting problem situations, pupils are invited to reason and develop hypotheses for solutions and to verify them through practical activity, another central element of pedagogical activism. Through didactic laboratories and concrete experiences, pupils stimulate and develop their operational intelligence, freely constructing their own knowledge. The study topics are thus linked to the learners' everyday experience and the knowledge content, appropriately selected and organised by the teacher, represents a goal to be reached and not a starting point.

The focus on learning that takes place through personal and concrete experience with the surrounding reality can also be found in the pedagogical vision of Célestin Freinet (Gars, October 15, 1896 - Saint-Paul-de-Vence, October 8, 1966), a French educator and exponent of popular pedagogy, aimed at the social rescue of pupils from culturally and economically disadvantaged backgrounds. Freinet pays particular attention to the connection between the school and the pupils' backgrounds, aiming through its teaching action to integrate into the school poorer young people from disadvantaged backgrounds. The teaching techniques promoted by Freinet (Legrand, 1993) opened the way to learning based on experience with the surrounding reality, on cooperation between pupils and teachers, but also on collaboration between peers, promoting a collective sharing and review of what was done. In fact, the lessons proposed are strongly linked to real-life situations and often consist of laboratory activities, walks in the countryside or visits to local artisan shops. Such experiences, profoundly far removed from the traditional occidental school practices of the time, with the teacher's guidance and orientation, represent a starting point and allow for the introduction of subjects in geography, history, mathematics, etc. (Freinet, 1969). The school is thus viewed as a community of cooperation between teachers and pupils, which places the interests, needs and aspirations of the pupil at the center of the educational process and in which learning is strongly motivated by sensitive experience with the surrounding reality.

The idea of the school as part of society, the search for an educational proposal that reflects the needs and interests of pupils and considers material and sensible experiences as central elements of the work between teachers and pupils, the role of the teacher who, instead of transmitting knowledge, becomes a guide in the process of discovery and learning, are key points of pedagogical activism and represent a meeting point with the educational proposal of Paulo Freire (Recife, September 19, 1921 - São Paulo, May 2, 1997). Freire is considered one of the most influential exponents of popular pedagogy of the 20th century; with his pedagogical vision, he does not aim to propose an alternative methodology to the existing one, but to initiate an emancipatory pedagogical process aimed especially at that part of society that lives in a situation of oppression (Freire, 2018). In accordance with Dewey's and Freinet's vision, it is also important for Freire that the topics of study resonate with the everyday life and lived experiences of the learners. The link between a person and the world in which he or she lives, the social and cultural context, aspirations, doubts, hopes and fears, are fundamental, even in the field of education (Freire, 1970). Freire proposes a *problem-posing* method of education, based on the dialogue between learners and educator, in which the search for the "game of the people", i.e. meaningful playful practices or themes that are habitual for the learners or able to interest them, assumes a fundamental role in teaching and learning (Freire, 1989). Through the search for the "game of the people", the teacher develops the programmatic content of his or her educational proposal and, taking into account the needs and interests of his or her pupils, selects original and meaningful themes and returns them as concrete, challenging and motivating problem situations. According to Freire, there can be no education except through the liberation of people from oppression and it is only through this type of education, which takes into account man and the world in which he or she lives, that it is possible to implement an emancipatory pedagogical process that invites students to learn how to deal with their world by putting knowledge into action, developing creativity and constant reflective critical capacities (Freire, 2018). Concerning the current research, I have chosen to focus on Freire. In this regard, in order to gain a deeper understanding of his thought, the following paragraphs take a closer look at his studies and research. The motivation for this choice can be found in the strong connection between education and society, which is the basis of Freire's pedagogical vision, but also one of the key points of active and popular pedagogy. In this sense, the educational project proposed by Freire was very interesting for the study conducted because, on the one hand, it offers possible tools to implement a motivating teaching design for students, and on the other hand, it sets the basis for a project aimed at social redemption and liberation from any kind of oppression.

The interest in the human-environment relationship, the subject of critical reflection by numerous popular pedagogues, is reflected in the conception of education as active participation in social life, opening the way to a new idea of school. Dewey himself, following his experience in Chicago, realises the important role that schools play in the education and intellectual and moral development of pupils (Incollingo, 2019), arriving at a conception of education as participation of the individual in social activity and democratic life. From this perspective, the school is understood as an embryonic democratic society, populated by the citizens of tomorrow (Pezzano, 2013), in which each pupil can collect experiences that are useful for his or her formative and personal growth, design his or her own activity and participate democratically in group life (Dewey, 1916). What emerges, therefore, is a vision of a school that bases its activities on the interests of its pupils, taking into account the surrounding environment and social reality, and that provides the indispensable condition for the emergence of a society in which pupils, future citizens, can actively experience democracy. The relationship between democracy and education, at the basis of the relationship between school and society, is also discussed by Maria Montessori (Chiaravalle, August 31, 1870 - Noordwijk, May 6, 1952), an Italian educator, pedagogist, philosopher, child neuropsychiatrist and scientist, best known for her educational method, still adopted today in many schools around the world. Montessori criticises the strong detachment between school mechanisms and the social issues of those years: just as the school appeared alien to social life, so the latter seemed to be excluded from educational contexts (Montessori, 2016). In order to overcome this obstacle, Montessori proposes an education that takes life as the center of its function (Montessori, 2016), valuing the school's role in the education of pupils and in the development of their autonomy and their unique potential. In the light of this conviction, the education of the newborn child also gains importance and the participation of the family in school life appears essential. With active pedagogy, the infant period acquires value for the intellectual development of the pupil; already from infancy, the child is considered to be the builder of the adult he or she will become, he or she absorbs from the environment around him or her and the stimuli he or she receives and shapes the future man from himself or herself (Montessori, 2016). With these assumptions, in 1907, Montessori founded the first Children's House in Rome, destined for the children of the people living in the San Lorenzo district, today a symbol of the scientific and social commitment that characterises the Montessori initiative. Recognising the school as a learning environment, its interior and exterior spaces also acquire an important educational value. In the Montessori Children's House, in fact, the spaces, furnishings and scientific materials used for the sensory and cognitive growth of the pupils

are all designed on a child's scale (Montessori, 2016). Classrooms are designed to encourage pupils' learning, which occurs through experiences in the environment (Montessori, 2016). Once again, the relationship between subject and environment cannot be overlooked: since the educational experience cannot be separated from the everyday life in which the subject lives, educational and school contexts are organised in such a way that the environment can foster creativity, plurality of opinions and the freedom of pupils to experiment independently.

It seems useful to reflect on how Dewey's experience in the Chicago school, Freinet's need to create a continuity between school and life, Freire's search for the "game of the people", the child-friendly environment proposed by Montessori, once again place the pupil at the center of the educational process that does not exclude, indeed validates, the importance of influences coming from the external environment and the society in which the pupil lives. The educational model that emerges moves away from the sterile formalisms adopted by traditional school practices and lays the foundations for a new concept of school: an active school, which starts from the needs and interests of the pupils, and considers their relationship with the world as central. In this view, the school organises and prepares educational contexts to foster learning, takes into consideration not only classrooms, but all environments inside and outside the school for teaching activities. The pupil takes an active role in the learning process, right from childhood, and is supported and accompanied by the teacher in the discovery and construction of the knowledge. The teacher does not transmit knowledge but facilitates learning, helping pupils to give meaning to their work. The teacher acts as a guide: he or she knows his or her students, their personalities, their interests, and from constant dialogue with them, research and propose original topics that may appear challenging and motivating in their eyes, fostering creativity and curiosity. It seems fair to assume that the active school no longer appears to be separate from life, but prepares for it: teaching laboratories and practical activities stimulate the operational intelligence of pupils, who are free to experiment on their own; attention to social aspects and the pupil's surroundings promotes a process of education and maturation that develops reflective critical capacities and prepares each pupil to be a good citizen of tomorrow, conscious of democratic ideals.

1. 1 Paulo Freire

The following paragraphs present the pedagogical vision of Paulo Freire, considered one of the most authoritative exponents of popular pedagogy in the 20th century. We have

consciously chosen to start by evoking salient stages of Freire's life, on the one hand to frame the historical, social and political context in which he lived, and on the other to trace the roots and understand the reasons that strongly motivated his political choices and pedagogical commitment. The conception of education as the practice of freedom and "the faith in man and women and in the creation of a world in which it will be easier to love" [2] characterise his most popular work, *Pedagogy of the Oppressed* (1970; 2018), and animate his denunciation and his struggle against all kinds of oppression. As the quotation above shows, Freire strongly criticises the banking model of education, which fills students with content, leaving no way for reflection and critical thinking. Instead, he proposes a problem-posing model of education, based on dialogue and oriented towards building a democratic society and a critical consciousness. The method of alphabetisation, which began with the first experiments in 1962 that enabled three hundred Brazilian farm workers to learn reading and writing in just forty-five days, consists of a method of conscientisation and liberation, considered by Freire to be an indispensable vehicle for social justice, which allows education to elevate itself above the status of a mere didactic means (Vlieghe, 2014). It is in this scenario that Freire's pedagogical project, we intend to present in the following paragraphs, is set. It probably represents the most important tool that the Brazilian pedagogue made available to create a process of historical and human redemption that would free the oppressed from injustice and the oppressors from the myth of power and control over the masses (Tolomelli, 2012).

1.1.1 Biographical notes

Paulo Freire represents one of the most influential pedagogues of the 20th century. He was born on September 19, 1921, in Brazil, where he developed extensive literacy programs addressed to farmworkers in northeast of Brazil. The great economic collapse of the 1930s forced Freire and his family to move to Morro de Saúde, a poor area populated mostly by labourers and poor rural families, where Freire spent his childhood and adolescence. His life experiences profoundly influenced his research interests and work (Bentley, 1999), fuelling his courage and determination to denounce a society founded on oppression and class division, to the point of radically rejecting it. So, he proposed to start an emancipatory pedagogical process aimed above all at that part of society living in a situation of oppression (Freire, 2018). Nevertheless, his educational proposal and revolutionary ideas in

[2] (Freire, 2018, p. 204)

the pedagogical field were considered subversive for those years and, in June 1964, after a military coup d'etat, he was put in jail for seventy days and was later forced into a fifteen-year exile (Gadotti & Torres, 1997). After a brief stay in Bolivia, he lived in Chile for five years, where he continued his research, working for an international organization in the context of the *Christian Democratic Agrarian Reform Movement* (Bentley, 1999).

Freire spent the last years of his exile in the United States of America, working at Harvard University: this stage was an opportunity for Freire to come into contact with the reality of the United States, and to come to the conclusion that repression and the exclusion of poverty from economic and political life were not exclusive to culturally and economically disadvantaged countries (Freire, 2018). Even in apparently free contexts, by virtue of race, class, gender, religious belief, political affiliation, physical and intellectual handicap, people tend to be victims or perpetrators of oppression (Gadotti & Torres, 1997). During this period, he wrote *Pedagogy of the Oppressed*, his most popular work, published in Spanish and English in 1970, but in Brazil only in 1974. In this work, Freire juxtaposes his pedagogical proposal with an account of a strong commitment and intense political activity. The publication of Pedagogy of the Oppressed calls for dialogue, and ultimately, for conscientization as a way to overcome oppression among and between human beings (Gadotti & Torres, 1997). Starting in 1970, Freire worked in Geneva for ten years as a special education advisor to the World Congress of Churches. During this time, he had the opportunity to travel around the world, implementing popular education and literacy reforms in many countries. One of his most influential works was conducted in Guinea-Bissau, a West African country, where he advised on national literacy. This experience contributed to the publication of *Pedagogy in Process: The Letters to Guinea-Bissau*. In 1980, after sixteen years, Freire returned to Brazil, where he joined the Workers' Party in São Paulo and supervised the adult literacy project. In 1989 he was appointed Secretary of Public Education in the State of São Paulo, where, a few years later, in 1991, the *Paulo Freire Institute* was established, which still welcomes scholars and pedagogues with the aim of fostering new educational theories and concrete actions in reality and through which, Freire's commitment and innovations continue to have an effect even to the present day.

1.1.2 Freire's Pedagogy: banking and problem-posing models of education

Freire's pedagogical proposal and the principles in which it is rooted have been a strong inspiration for this research work, stimulating a profound reflection on the role of the socio-

cultural dimension in the teaching and learning of mathematics. The current society, characterised by numerous situations of economic, social and cultural hardship, accentuated by the Covid-19 pandemic emergency experienced, mirrors a reality full of contradictions and conflicts, in which differences often turn into inequity and oppression. Against the backdrop of these considerations, we questioned the role of schools in terms of equity, inclusion and fair opportunities, issues that were already considered crucial by researchers in the field of mathematics education and that became more urgent during the Covid-19 pandemic (Bakker, Cai & Zenger, 2021). Freire's denunciation of oppression and his desire to create an emancipatory and liberating pedagogical process has made it possible to include, in the discourse already begun in the field of mathematics education, attention to the socio-cultural dimension in the design of meaningful and inclusive educational paths. With these assumptions, the theoretical and methodological innovations underlying the popular pedagogy proposed by Freire are presented below. The development of his ideas in the educational field appears to be strongly motivated by the conception of education as a practice of freedom and as an act of knowledge through which it is possible to approach reality more and more critically (Freire, 2018). Freire's pedagogy is not limited to just reading the word, but also to reading the world, with the aim of awakening knowledge, creativity and critical capacity in the oppressed. Specifically, Freire talks about *conscientização*: the development in people of a critical conscience that allows them to read their world and question the nature of their own historical and social situation, taking active action in the creation of a democratic society (Bentley, 1999). The *conscientização* consists in a process of conscientization that allows people to acquire the critical tools necessary to recognise a situation of oppression and understand how equality, fair opportunities, access and justice are being taken away from them (Freire, 2018). The liberation to which the educational practice proposed by Freire aspires, resolves the contradictions between oppressor and oppressed, allowing man to be neither more oppressor nor more oppressed, but to liberate himself. In order to understand education as a process of liberation, it is essential for Freire to rethink the role of the educator and the educator-learner relationship. According to Freire, the educator's action is characterised by two essential moments: the preparation of the lessons and the meeting with the students; it is by analysing these two practices that it is possible to distinguish a *banking model of education* from a *problem-posing model of education*. In the first case, the educator, during the preparation of his lessons, performs an act of knowledge in relation to the knowable object (educator as a *knowing subject*). Then, during the meeting with the students, he narrates and discusses the object on which he has performed his act of

knowledge, becoming a *narrating subject*. In this practice, the emphasis of education falls heavily on the action of narrating, which involves, on the one hand, a narrating subject, the educator, and on the other hand the patient objects, the learners, who listen, learn and memorise the narrated content (Freire, 1970). In this perspective, the narration, of which the educator is the subject, appears static and disciplined, characterised by a word emptied of the concrete dimension it should have and deprived of its transformative power (Freire, 2018). Education is transformed into the act of depositing, into a *banking education*, in which the learners appear as vessels to be filled and the educator as the one who deposits his or her knowledge. In this perspective, the learners play the role of *patients* or *docile recipients of content*, since their role is only to know or learn by heart what is narrated and delivered to them by the educator. According to Freire, the *banking model* of education doesn't allow the students to perform a true act of knowledge, since they are only required to archive the knowledge that is delivered to them; knowledge that comes from a narrated or transmitted experience and that represents a possession of the educator and not a mediator of the critical reflection of both. Thus, *banking education* seems to move away from the vision proposed by active pedagogy, but also away from the *conscientização* process towards which Freire is oriented and suggests a conception of man as a passive being and spectator of the world and not as an active and conscious part of it. In this regard, Freire argues that: «*The education […] cannot be based on a conception of men as "empty" beings that the world "fills" with content [...] It cannot be a depository of content, but problematizing for men in their relations with the world*» (Freire, 2018, p.87). To a banking model of education, Freire contrasts a liberating *problem-posing model of education*, which rethinks the role of the educator and bases the educator-learner relationship on dialogue between the two parties. As far as the educator's action is concerned, for the *problem-posing model of education* there isn't a clear distinction between the moment of preparing the lessons and the moment of meeting the learners. The educator, both when preparing the lesson and when meeting with the learners, is always a *knowing subject*, in constant dialogue with the learners. Dialogue, an essential element of problem-posing practice, is the key to the construction of knowledge and turns out to be fundamental in order to conceive of education as a practice of freedom. In the banking model of education there is an absence of dialogue: the depository-educator does not ask himself what he will dialogue with his students about, but only about the program about which he will narrate to them. On the other hand, in problem-posing education, the dialogue between educator and students already begins when the educator asks himself what he will dialogue with his students about and starts to develop

the programmatic content of his educational proposal. At this stage, according to Freire, the problematizing-educator develops the programmatic content of his educational proposal, taking into account the needs and interests of the students. The educator mingles with the community, asking questions and collecting a list of interesting and relevant themes which he returns as concrete, challenging and motivating problem situations. The encounter with the learners is also characterised by dialogue, which is fundamental for the educator in order to review – through the re-visioning of the students – what was objectified in the preparation of the lesson. During this dialogic exchange between educators and students, both learn, question, reflect, and both participate in the creation of meaning.

> The educator is not only the one who educates, but the one who, while educating, is educated in dialogue with the learner, who in turn, while being educated, also educates.
> (Freire, 2018, p. 89).

Both subjects, therefore, become protagonists in the teaching-learning process, in which they grow together. It is useful to emphasise that, with problem-posing practice, the learner is no longer a *patient* or a *docile recipient of content*, but becomes a *critical researcher*, in a permanent dialogue with the educator, who is also a *critical researcher*. According to Freire (1970), it is only through a problem-posing education that it is possible to achieve an emancipatory pedagogical process that encourages and challenges the students for learning to face their world (Freire, 2018).

The problem-posing model of education proposed by Freire stands in stark contrast to the banking model of education, which is closer to the traditional educational practices currently in use. Banking education seems to aim more at the reproduction of facts than the construction of knowledge, neglecting dialogue and limiting students' creativity. Educational practice is thus reduced to an act of depositing, which overemphasises the role of the educator, reducing the student to a passive container that merely stores the content deposited in his or her head (Vlieghe, 2014); educator and learners assume the roles of oppressor and oppressed respectively. While the educator represents the one who knows, the learner represents the one who does not know, yet the educator appears almost completely alienated and alienated, estranged from the socio-cultural reality of the learner (Freire, 2018). With this practice, there is a kind of anaesthesia (Freire, 2018) in which the only possibility of action for the learners consists in listening and filing away what the educator has deposited. Freire's critique, however, does not seem to concern the validity, in terms of motivation, of traditional forms of classroom teaching or the ability or otherwise to create

more stimulating contexts, it rather concerns a type of pedagogy that treats students as passive containers and silently contributes to perpetuating an oppressive social order (Vlieghe, 2014). Through dialogue, problem-posing education promotes an active and creative engagement of learners and overcomes the role distinction between educator and learner: both contribute and actively participate in the creation of meaning. From this perspective, learners are not considered empty beings to be "filled", but "conscious bodies", in relationship and communication with the world (Freire, 2018). This educational model encourages learners to construct their knowledge actively, autonomously and on the basis of what they themselves consider relevant (Vlieghe, 2014), welcoming and exploring through teaching practice the socio-cultural dimension, from which neither the educator nor the learners are excluded.

1.1.3 The conception of human as a *being-in-situation* and the research of the "game of the people"

With the background of what has been presented in the previous paragraph, it seems legitimate to assume that Freire's ideas are inspired by the fundamental principle that there can be no education except through the liberation of people from oppression. In his book Pedagogy of the Oppressed (1970, 2018), a point of reference for this research work, he deepens his position, coherently connecting pedagogical theory and practice, educational experiences and political and civil commitment, giving life to a liberation process moved by the principles of equality, justice, inclusion and solidarity and oriented towards restoring dignity to man, starting from his awareness. The proposal of a problem-posing education that invites continuous reflection, promoting an ever more critical understanding of reality, is placed in this scenario. In the problem-posing education model, the educator-learner dialogue begins when the educator develops the programmatic content of his educational proposal. This content does not appear as a set of knowledge to be deposited or handed over to the learners, but is the result of research and reflection on the part of the educator. In the planning phase, the educator problematises questions and, through dialogue with the learners, collects a number of interesting and relevant topics that can lead to class discussion. In this process of enquiry, collection and reflection, reality is understood as a mediator, through which and in which the programmatic content of an educational proposal is to be sought: the educator proposes to his or her students significant dimensions of their reality, on which to critically discuss and reflect together (Freire, 2018). In contrast to banking practice that considers reality static and fixed, emphasising

permanence, problem-posing education is based on *change*. Reality no longer appears crystallised and education represents a process through which humans reflect on their reality and critically perceive how they are in the world. The human being, and implicitly his relationship to the world, represents the starting point for problem-posing education, which identifies in human beings a historical character and historicity and recognises them as *beings in the process of becoming*, in other words, historical beings who know that they are incomplete and therefore, eager and capable of change. Recognising history as a time of possibilities and not something already determined, Freire (2018) argues that man's historical awareness is a necessary condition for human advancement and the construction of a better future. From this perspective, man is no longer a motionless subject, but plans, looks and walks forward, necessarily finding himself immersed in a specific reality and embedded in a specific situation. With these assumptions, problem-posing practice aims to propose man his situation as a challenge that does not limit him: man, through dialogue, becomes capable of objectifying the situation in which he is immersed and transforming it. In pursuit of this aim, the task of the problem-posing educator consists in researching the issues to be tackled, which therefore cannot disregard the conception of men as beings in situation, i.e. men who are immersed in spatial and temporal situations from which they receive and to which they give an imprint. The educator, considering the students as beings in situation, reflects on what to propose to his or her students and, from the constant dialogue with them, imagines original themes to be returned as concrete problematic situations, stimulating and motivating, which require an intellectual response and action and which allow critical reflection on significant dimensions of reality. According to Freire (1970), in order to actively involve students, the search for themes to be addressed must start precisely from present, existential and concrete situations that take into account the personal, social and cultural dimensions in which students are immersed. The desire, already expressed in the previous paragraph, to strive for an education that takes into account the socio-cultural dimension and that considers the issues of equity, inclusion and equal opportunities as central, seems to be reflected in the implementation of the problematising practice. What has been presented, however, necessarily leads to new questions: How can the educator design meaningful and inclusive didactic paths that do not limit the students and do not place them in the position of "oppressed"? How can an educational design that aims to be motivating for students and not alien to the socio-cultural context be didactically effective and implementable in difficult social contexts? To what extent can an educational design include students experiencing severe socio-cultural disadvantage? Taking into account the principles behind Freire's popular pedagogy and the narration of his experiences in both

the educational and political fields, various aspects related to the questions raised are clarified and explored in depth and, in some cases, a possible answer is also proposed. In the following chapters, however, an attempt will be made to grapple with these questions by analysing the considerations of various teachers, with reference to their teaching practice, with the aim of outlining common characteristics underlying effective didactic designs to be implemented in difficult social contexts and adapted to the needs, requirements and interests of students. Taking into account Freire's thought (1970), the failure of many educational and political projects is attributed to the proposal of an overly personal vision of reality that does not take into account human as *being-in-situation*. Emblematic are a number of examples presented by the Brazilian pedagogue (Freire, 2018) of popular projects of an educational and political nature that, although studied with commitment, then failed because the worldview proposed to the rural or urban masses was not in accordance with the true needs of the people or did not reflect what they most wanted to know. In the case analysed by Freire (2018), the revolutionary vanguard groups, in their efforts to gain the people's adherence, approached the rural or urban masses with projects often motivated by purely personal reasons that, although good and honest, were not always able to reflect the needs and desires of the people. Rather than dialoguing with the people, they spoke to the people, proposing to them a vision of the world that, albeit with the best of intentions, they attempted to impose. The gap between what one proposes and what one wants to know can be overcome, in the design phase, by mixing with the community, asking questions, getting to know, through dialogue, not only the objectivity in which the community finds itself, but also the consciousness it has of this objectivity (Freire, 2018).

> You cannot approach workers, urban or rural (and these generally immersed in a colonial context, almost viscerally linked to the world of nature of which they feel more a part than a transformer), to deliver them knowledge as you would deposit a sum in a bank.
> (Freire, 2018, p. 104).

Therefore, not only the themes, the subject of the discourse to be addressed, but also the language is important. For communication to be effective, it is necessary that the language used is in syntony with that of one's interlocutor. The politician who talks to the people or the educator who talks to his or her students must be aware of the concrete situation and the conditions in which the thoughts and language of the people they are addressing are constituted (Freire, 2018). From this perspective, for a political action programme to be

successful or an educational design to be effective and feasible, even in disadvantaged socio-cultural contexts, it is necessary to consider men as *beings-in-situation*, taking into account their personal, social, cultural context, their roots and aspirations, their doubts, hopes and fears (Freire, 1970). In particular, in the educational sphere, through the problem-posing practice, the educator is called upon to respect the worldviews that the student has, or is developing, and design an educational programme that reflects them in its content. In this regard, Freire, in presenting the principles of his pedagogy, speaks of the "game of the people" (Freire, 1989), understood as a set of playful practices or significant themes that are habitual for a given audience or that are able to involve them, excite them and make them authentically curious. According to Freire, there is no universal "game of the people": it may not always be the same and may vary depending on the case, the group of people or the type of society being analysed.

> In a modern society like this it is obvious that the game is a different one, it cannot be the game of the peasant in the North-East of Brazil [...]. You should rather start with the technologies, the absence of communication, the emphasis on broadcasting, the lack of curiosity, the fear of freedom. There are so many issues that lie within this type of social relations.
> (Freire, 1989, January 23).

In this sense, searching and knowing the "game of the people" comes to be an educational task and a cultural action that necessarily takes into account the conception of man as a *being-in-situation*. For educators, and even more so for popular educators, it appears necessary to ask themselves what the "game of the people" might be for their students, in order to create and design effective and meaningful didactic paths, and to select methodologies and didactic strategies in line with the educational proposal to be offered. The search for the "game of the people" and thus, of the themes to be addressed, can be understood as a useful tool for designing effective teaching activities and cannot be done without the people, but together with them. This research is carried out through observation, reflection and constant dialogue with the students and allows access to a *thematic universe* and the set of its *generating themes*, which are themes that contain within themselves the possibility of giving rise to new interrelated themes (Freire, 1970). It is possible to imagine the set of themes as being located in concentric circles (Figure 1) that go from the most general to the most particular: the widest circle contains the generating theme, of a universal kind, from which other more and more specific themes, contained in the innermost circles, develop.

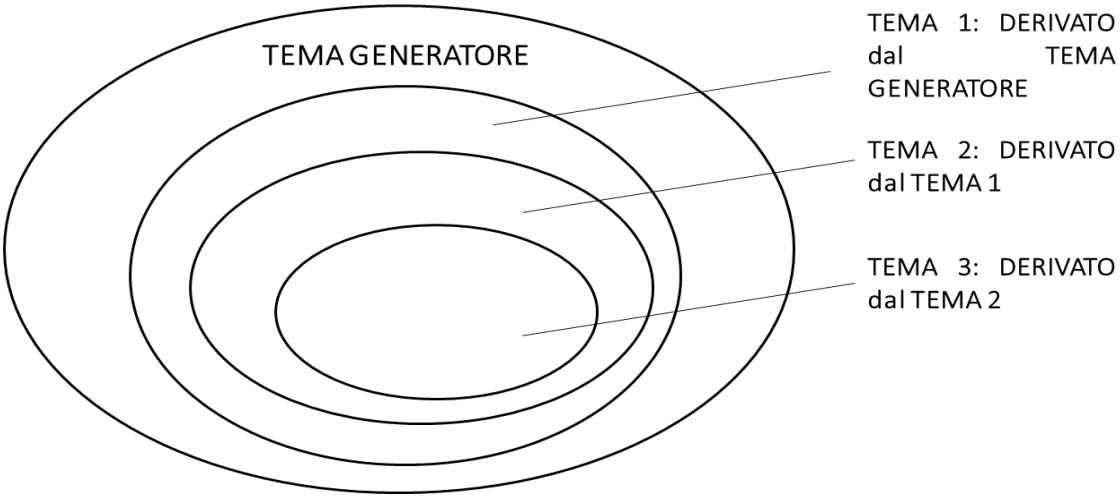


Figure 1.

Schematic representation of themes in concentric circles.

Referring to the model of concentric circles shown in Figure 1, Freire (2018) takes as an example the theme of underdevelopment, which can be imagined as a universal theme, located in the widest circle. Starting from this thematic, one can consider others that originate from it, located in the smaller circles, and which differ depending on the society and context being considered. The search for the *generating theme* must, therefore, take place through a critical reflection on the relationship between man and the world. Man, understood as *being-in-situation*, cannot prescind from his relationship with the world around him and from coming into contact with concrete and historical situations that present themselves to his eyes as *limit situations* (Freire, 2018). In this regard, Freire proposes two possible scenarios: in the first, the *generative themes* are enveloped by the *limit situations*, which appear to men as determined, insuperable and with no possibility of resolution; in the second, the *generative themes* envelop the *limit situations*, of which man takes critical note, committing himself to overcome them through actions called *limit-acts*[3], which lay the foundations for human advancement and the construction of a better future. The second scenario is only possible through critical reflection from which a climate of hope and trust develops, motivating human beings to commit themselves to overcoming *limit situations* (Freire, 2018). Starting from these considerations, the search for *generative themes* seems to represent the very starting point of educational action that does not place the student in the position of the "oppressed" and lays the foundations for an educational

[3] Á. Vieira Pinto analyses with care the concept of the *limit situation*, which he defines as "the frontier between being and being more" (Pinto, 1960, p.284) and calls *limit-acts* those actions that refuse the passive acceptance of a *limit situation* and aim at overcoming it.

process understood as the practice of freedom. This vision of education is rooted in the conviction that the programmatic content of an educational proposal must be searched in the dialogue between educator and students, and must not be understood as a set of objectives to be imposed on students, but as a reflection of their needs and interests. The questions raised and stated above are answered in the search for the "game of the people" that motivates exploration in this sense. In the following chapters, the research questions will be made explicit and the methodology of investigation will be clarified, which will allow, through a semi-structured interview with teachers of different school grades, to focus on the search for the "game of the people", in relation to the creation and design of meaningful and inclusive educational pathways that affect everyday school practice nowadays.

CHAPTER 2

ACTIVE AND POPULAR PEDAGOGY IN THE FIELD OF MATHEMATICS EDUCATION

"The school thus becomes the place to experience the actions of the individual, his relating to other individuals and, above all, it becomes the cultural space needed to understand how to use education in a perspective "of the common good" and, therefore, of democracy"
(Pezzano, 2013, p.78)

The educational model that has recently emerged moves away from traditional school practices and goes in the direction of a new idea of school, recognised as an *active school*. In this chapter, starting with active and popular pedagogy and, in particular, with Freire's studies, I have chosen to explore possible connections, traces and influences in the field of mathematics education. The problem-posing model of education and the teaching methodology proposed by Freire represent true theoretical and methodological innovations, which are also of great inspiration in the field of *critical mathematics education* (see for example Skovsmose, 1994). In the following, this very young line of research, which recognises an intertwined development of a critical awareness of citizenship and mathematical competence, is explored. In particular, the pedagogical visions presented above are enriched in the light of the studies and considerations that emerge from the following chapter, allowing a deeper discourse on *active schools* and giving them a more profound relevance in the context of mathematics education. In this regard, I have considered the research works and educational visions of Emma Castelnuovo and Eric Gutstein, which lay the foundations for the "Proud of You" (PoY) project story, which is explored in more detail in the next chapter.

2.1 Freire's ideas in the field of *critical mathematics education*

The problem-posing education and the didactic methodology proposed by Freire represent theoretical and methodological innovations implemented not only in the studies of adults' education, but also in the field of mathematics teaching (Gadotti & Torres, 1997).

In particular, Freire's ideas have been a great inspiration for the field of *critical mathematical education* (Skovsmose, 1994). From the studies referred to above, we have seen the role attributed by Freire to literacy, considered as a process of conscientisation and liberation, strongly motivated by principles of social justice. In this sense, the literacy programmes conducted by Freire connect the educational dimension with a political dimension. While literacy, by enabling people to learn to read and write, is a necessary condition for being part of a society - "for informing people about their obligations and for people to be used in essential work-processes" - it can also be used for the purpose of personal empowerment, as a not only educational but also useful means to "organise and reorganise interpretations of social institutions, traditions and proposals for political reforms" (Skovsmose, 1994). Consequently, literacy is conceived as a competence that could be developed from the perspective of critical education. With reference to this last point, Skovsmose (1994) wonders whether the role of literacy can also be played by mathematics, understood as a capacity for calculation and the use of formal techniques. In this reflection, Skovsmose certainly recognises the role of mathematical education as preparation for the world of work and, more generally, for economic growth. But he also wonders if and in what terms mathematics could be used for the purpose of personal empowerment, involving, as in the case of literacy, two different dimensions.

Therefore, with the aim of recognising mathematics education as part of a democratic effort, in which mathematical skills are useful for critically understanding and interpreting reality, Skovsmose (1994) distinguishes three types of knowledge towards which a mathematics education can be oriented:

- *Mathematical knowing*, which includes mathematical skills and the ability to master algorithms and procedures and to reproduce theorems and proofs;
- *Technological knowing*, which includes mathematical application abilities and model-building competences:
- *Reflective knowing*, which includes the competence to reflect and evaluate on a certain use of mathematics.

In distinguishing the three *knowledges*, Skovsmose traces reflective knowing, referring, by the term "reflection", not to the processes through which students formulate mathematical ideas and develop their comprehension, but to the processes that, through the comprehension of mathematics, lead students to choose to use it. Thus, by considering mathematics as consisting of three different competences - *mathematical, technological and reflective* - it appears to be taken into account in the educational context in a similar way to Freire's literacy. Skovsmose (1998) traces, therefore, a connection between

mathematics education and democracy, in which the term "mathematics" involves several dimensions. In this sense, mathematics is used to reflect a "kind of competence in which mathematics-oriented, model-oriented and context-oriented reflections are brought together as an epistemic unit [...] in parallel with literacy, in such a way that mathemacy becomes a kind of competence for acting in the world structured by mathematics" (Skovsmose, 1998, p. 200). In addition to the didactic dimension, there is also a critical dimension, which emerges in correspondence with the development of reflective knowing. According to the author, it is only by considering mathematical competence in this perspective, as a typified competence, that mathematics education can be understood as critical and consequently occupy a relevant position also in the promotion of individual empowerment. In this regard, one of the issues of *critical mathematics education* is the promotion of individual empowerment through citizenship development (Carotenuto, 2021). In this sense, the author sees the need for a discussion on the relationship between mathematics education and democracy, in terms of citizenship and critical consciousness. The pupil is considered to be a member of society and, from this perspective, reflection on the social functions of mathematics appears necessary, both in a broader sense and in the micro-society constituted by the classroom (Skovsmose, 1998). In this scenario, critical education also includes the reclamation of a fair distribution of educational opportunities within a society, in line with the principles of democracy.

I have chosen to focus on two aspects of critical mathematics education, the dialogue and the learning space, that will be found below as an important connection to this line of research. In the following chapter, on the presentation of the "Proud of You" project, these two aspects will be found as a democratic trait promoted by critical mathematics education. With Freire's problem-posing education, the potential of dialogue between educator and pupils had already emerged. Dialogue, a necessary element in conceiving education as a practice of freedom, was oriented towards building a democratic society. With reference to the field of *critical mathematics education*, inspired by Freire's work, Alrø and Skovsmose (2004) give relevance to educational dialogue. The authors describe dialogue as a "process of enquiry that includes an exploration of the participants' perspectives as well as a willingness to suspend one's own pre-understanding - at least for a moment" (Alrø & Skovsmose, 2004, p. 15). In the sense made explicit, participating in a dialogue means listening to the arguments of others and proposing one's own in a collective exchange that necessarily involves suspending one's own perspectives in order to accommodate those of others. Such a conceptualisation of dialogue requires respectful participation by all the components involved: the teacher is called upon to share power

with all the students in the classroom - in this case, the unexpected could be perceived as risky - the students, involved in a collective enquiry, take control of the activities, developing responsibilities and democratic citizenship skills (Carotenuto, 2021).
With regard to the focus on the learning space, Skovsmose (2019) speaks of *inclusive landscapes of investigation* to delineate learning spaces that facilitate all kinds of encounters, moving beyond differences. These spaces engage students in inquiry processes, are designed to be accessible to everybody and facilitate collaborations. This is because *inclusive landscapes of investigation* are thought to pursue social justice and pupil empowerment.

In view of what has been presented, in the following paragraphs, the discussion in the field of mathematics education is deepened, taking up the considerations about active schools and the possibility of an intertwined development of a critical awareness of citizenship and mathematical competence.

2.2 Emma Castelnuovo and a democratic vision of mathematics education

Emma Castelnuovo (Rome, 12 December 1913 - Rome, 13 April 2014), an Italian teacher and mathematician, fits perfectly into the framework of active and popular pedagogy for the proposed educational model that aims at the direct involvement of pupils in explorations (Arzarello, Bartolini Bussi & Bazzini, 2013). In her book "Didattica della matematica" (Castelnuovo, 1963), she deepens the discourse on active schools, identifying two fundamental principles for them: cycle teaching method and intuitive-constructive method. The idea of a cycle teaching method is attributed to Jan Amos Komenski, popularly Comenius, and motivated by his desire for a school that was for everyone (Comenius, 1960). In this view, in fact, while distinguishing many stages, according to age, and attributing to each one a specific educational programme, the topics to be covered are the same, but the ways in which they are addressed are different. For example, the same subject is imagined as being extended in concentric circles and proposed to the students from increasingly broader viewpoints, so that what is learnt today can then be reinforced by what was learnt yesterday and lays the foundations for what will be learnt tomorrow (Comenius, 1960). In this sense, the cyclical method is intended as a priceless opportunity for the student who prepares to be a citizen of tomorrow and to actively participate in society. Each cycle of study proposes and prepares for a "complete culture",

which can then be further enriched and deepened in the next cycle (Castelnuovo, 1963). The second fundamental principle, the idea of an intuitive-constructive method, has been attributed to Enrico Pestalozzi, according to whom "an education is only true and educational when it comes from the children's own activity" (Pestalozzi, 1963). He mentions activity, factual energy and intuition, intended in a dynamic sense, as arising from work, from an operation, which leads one to consider intuition as a construction. "Every scientific study whose definitions [...] have been blown into [the students'] ears as by a theatrical prompter, is no more valuable than the study destined to produce wretched comedians" (Pestalozzi, 1963). This sentence by Pestalozzi seems reminiscent of Freire's vision and his opposition between banking and problem-posing models of education. Like Freire, also in Pestalozzi's vision - developed long before the Brazilian pedagogist - emerges the need for an education that is not purely verbal teaching, which would result in passive learning. Instead, education must actively involve the learner who, through exploration, experimentation, direct experience of things and about things, builds increasingly clear and solid concepts. The principles of the active school are then found in the activities designed by Emma Castelnuovo, but also in the didactic paths inspired by her work, in which the idea of "doing mathematics with dirty hands" is present (Castelnuovo, 1963). With this expression Castelnuovo intends to emphasise, through teaching practice, concrete experience, the manipulation of artefacts, the involvement of pupils with the whole body and interaction with the material world and movement (Carotenuto, Mellone, Sabena & Lattaro, 2020). In relation to the principles of active school, in the article from 1946, Emma Castelnuovo highlights the importance of engaging with pupils' knowledge and experiences in primary school, with a view to a method that is not descriptive but allows for active and continuous knowledge construction. Emma Castelnuovo's educational vision gives relevance to the relationship between mathematics and everyday life, also finding many points of encounter with active and popular pedagogy in this respect. Castelnuovo's didactic proposal starts from the surrounding reality, from a non-artificial concrete, in which students can find motivation and, through mathematical exploration, bring theory to life from the concrete and then, finally, return to reality as an application of the theory itself (Castelnuovo, 1966). In this perspective, Castelnuovo believes that an education in "knowing how to see" is possible in mathematics, which educates students to look, starting from the observation of a concrete and then arriving at the abstract. "Knowing how to see" is not intended as an equal step for all - for Castelnuovo, students from a working-class context look better and with more confidence than their peers from a bourgeois context - but in order to make it a possible step, it is necessary that the concrete to be observed is dynamic and constructive

(Castelnuovo, 1967). Just as a child is fascinated by a mill rotating under the thrust of water or a crane rising and falling, but does not appear interested when these same elements, mill and crane, are no longer in motion, so for Castelnuovo (1967) a child is fascinated by mathematical questions "in motion", which allow the evidence of elements that vary and elements that are invariant. Asking to transform an articulable square into a rhombus, or to construct through a tied string, held between the forefinger and thumb of both hands, a rectangle-variable may not yet guarantee " to see ". However, by exploring the "borderline" cases and showing that the square-rhombus can be squashed, and the rectangle-variable can be reduced to two overlapping strings, the movable objects acquire dynamism, become a mathematical problem and allow the students to "see", in each case, that the area of the figure changes. The example given underlines how the material used by the students does not represent the object of their interest, but it is its mobility and therefore, the transformation operation of the material that becomes the object of observation and interest, allowing the transition from the concrete to the abstract. In this sense, "knowing how to see" a figure can be transformed into "knowing how to see" a formula, being able to interpret it in the concrete. Montessori's idea of using materials for the sensory and cognitive growth of the pupils appears to be developed in a deeper direction. If with Montessori, children learn through their experiences with the environment and with what they directly manipulate, with Castelnuovo, learning materials take on a new role, an operative function, in which it is the transformations from configuration to configuration that must be of interest to the pupil, and not the configurations themselves (Arzarello et al., 2013). The material and tools used are thus intended as a cultural and historical product that incorporates precise meanings and purposes. The choice of object and action at the centre of the learning experience takes into account the needs of the class each time, finding a connection with Freire's ideas. In fact, it is the teacher who, by observing and trying to grasp the needs of the students, designs and proposes learning activities that can be meaningful. In this way, the teacher shakes and captures the instinctive curiosity of his students, accompanying them in their discovery. The students on their part, participating in such teaching activities personally, are inside mathematics, constructing it through observation, imagination and the ability to "see" that precedes any systematisation process. The students' many interests are therefore a starting point and a motive for the active development of the programme. Education in "knowing how to see", the idea of "doing mathematics with dirty hands" and Castelnuovo's conception of the pupil as a curious investigator and "miniature scientist" seem to come together in the theoretical construct of mathematics laboratory.

> Mathematics laboratory is not a physical place other than the classroom, it is rather a structured set of activities aimed at constructing meanings of mathematical objects. The laboratory, therefore, involves people (students and teachers), structures (classrooms, tools, organisation of space and time), ideas (projects, plans of teaching activities, experiments). The environment of the mathematics laboratory is in some ways comparable to that of the Renaissance workshop, in which apprentices learn by doing and seeing, communicating with each other and with experts. The construction of meanings, in the mathematics laboratory, is closely linked, on the one hand, to the use of the tools used in the various activities, and on the other, to the interactions between people that develop during the exercise of these activities. It is necessary to remember that a tool is always the result of cultural evolution, that it is produced for specific purposes and that, consequently, it incorporates ideas. In educational terms, this has some important implications: firstly, meaning cannot reside solely in the instrument nor can it emerge from the interaction between student and instrument alone. Meaning resides in the purposes for which the tool is used, in the plans that are drawn up for using the tool; the appropriation of meaning also requires individual reflection on the objects of study and the proposed activities. (Materiali UMI-CIIM, Matematica 2003).

The mathematical laboratory is not understood as a physical place but can be constituted by the classroom itself (Castelnuovo, 1963) which, in the spirit of collaboration and cooperation, works towards a common goal. The work begins with the observation of concrete forms and continues through the exploration of specially designed artefacts and, more generally, tools intended as a product of cultural evolution. In this perspective, material and sensitive experiences are central elements of the work of students engaged in mathematical activities, but also central elements of the work of teachers who design and prepare the conditions for meaningful learning. Two components of Castelnuovo's constructive method strongly characterise these new learning environments: the dynamism and the development and support of pupils' imagination (Arzarello et al., 2013). As previously emphasised, the dynamic and manipulative aspect appears fundamental in the didactic methodology proposed by Emma Castelnuovo, which gives relevance to concrete experiences connected to the students' everyday life. At the same time, students, through direct observation, activate their fantasy and imagination to "do mathematics with their dirty hands". Thus, an idea of freedom emerges - in the pupil and the teacher - in exploring, interpreting and imagining from a mathematical point of view (Castelnuovo, 1963).

An important spin-off of Castelnuovo's methodologies in teaching practice is certainly the "mathematics exhibitions", which have proved to be a very effective learning tool for students.
As Emma makes clear:

> The verb "exhibit" has a double meaning:
> 1° - to show something, objects
> 2° - to explain verbally.
> An exhibition of mathematics, by pupils, must have both meanings. Now, to make this possible, the pupil must have made the concept his own; he must have created the concrete and the abstract, in other words, he must have created the material to be expounded and the argument to be verbally expounded.
> (Castelnuovo, 2003, p. 133)

Thus, the "mathematics exhibition" involves, on the one hand, the showing of something and, on the other, an accompanying verbal explanation. Both actions, however, imply a work of exploration, reflection and production on the part of the students who, at mathematical exhibitions, introduce to a not necessarily expert audience explanatory posters, teaching materials, objects and dynamic models they have worked on (Figure 2).

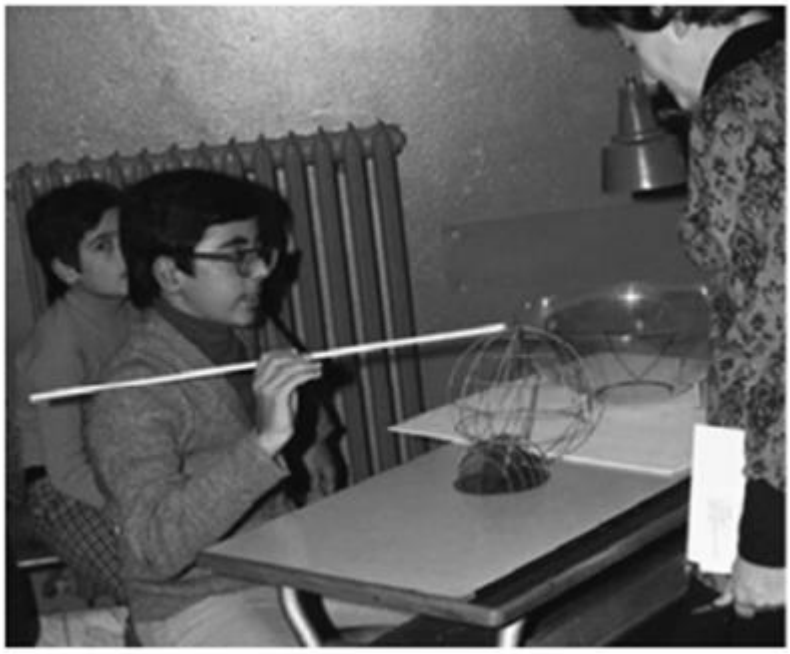


Figure 2.
1974 "mathematics exhibition" - A boy illustrates air routes (Menghini, 2013, p. 61).

The preparation of a "mathematics exhibition" starts with the choice of a topic, carefully selected from the most suggestive and interesting ones discussed in class, followed by a considerable amount of additional activity in the afternoons. For example, during this activity Emma Castelnuovo was often supported by young graduates and undergraduates attending her classes (Menghini, 2013). The moment of preparation is experienced in groups and through individual and cooperative activities that allow the class to be transformed into a laboratory, in the sense explained above. Based on the considerations made, the importance of "mathematics exhibition" can be recognised not only on a didactic level, but also on a psychological and social level. Through preparation in two directions - creating the concrete and the abstract - students become aware of what they know and what they are learning, overcoming, in many cases, an attitude of subjection and distance towards mathematics. The collective and confrontational experience engages students in a formative and social discourse in which mathematics appears connected to reality. The creation of a community of practice in which teachers and students work towards a common goal stimulates joy of learning and creative and imaginative processes of encounter with mathematics and activates critical and reflective capacities. In general, the didactic methodology proposed by Emma Castelnuovo involves pupils in experiences of physical and mental observation and exploration - individual and collective - through the use of mathematical artefacts and taking into account the surrounding space. The objective recognised by Emma Castelnuovo (1966) and attributed to school mathematics teaching is precisely that of forming students who are able to reflect critically and understand the environment in which they live, being able to identify the mathematical aspects also underlying biological, economic and other phenomena. In Emma Castelnuovo's method it is possible to find many links with the democratic values of the research current of *critical mathematical education*. In particular, points of contact are identified with the idea of an *inclusive landscape of investigation* (Skovsmose, 2019). Already in Comenius and Pestalozzi, attention to a social and cultural component is recognised. In fact, in both cases, the considerations arise from the needs of society, which, from their point of view, can only be reorganised through the education and upbringing of people. In fact, their reflections appear strongly impregnated with a vision of education above all politics and capable of directing the destiny of human communities (Castelnuovo, 1963). The conception of mathematical education proposed by Castelnuovo and inspired by the principles of Comenius and Pestalozzi is based on the desire to give all students, in the same way, the opportunity to reach discoveries that can encourage and enthuse them. Mathematics is proposed as "something alive in the realm

of thought" (Castelnuovo, 1967) that holds great inclusive and democratic potential. Wanting to clearly oppose the conception of mathematics teaching as a "selective weapon", capable of accentuating the differences between students from a more prosperous and developed social context and students from more fragile backgrounds, Castelnuovo proposes a teaching of mathematics that, starting from the concrete, assumes a crucial role in the sense of social equality (Castelnuovo, 1966). All students are in fact equal when faced with a mathematical situation, which only requires them the faculty of observation, imagination and intuitive and "natural" thinking.

2.3 Eric Gutstein and a Freirean approach to teaching and learning mathematics for social justice

In the discourse that we intend to conduct and that allows us to connect mathematics education with the development of principles of democracy and social justice, it makes sense to involve and introduce the research work of Eric Gutstein, researcher and Professor of Mathematics Education. Gutstein currently works on Freirean approaches to mathematics teaching and learning with a particular interest in teaching for social justice and critical literacies in an urban and multicultural context. He has taught mathematics for social justice at a Chicago public middle school and high school, collaborating with mathematics teachers and students on projects in which mathematics was aimed at social justice. He worked with a group of students who were co-researchers, public advocates and spokespersons for the teaching and learning of mathematics for social justice, and together they studied the process of creating a critical mathematics curriculum for schools, focused on developing students' sociopolitical awareness, their sense of social agency and their strong cultural and social identity. Gutstein's educational ideas are deeply intertwined with the ideals of Freire's pedagogy. Indeed, his didactic action appears to be strongly inspired by the desire to liberate people from oppression, from a genuinely inclusive perspective and in line with the most recent research in *critical mathematics education*. In its educational proposal, mathematics education becomes a tool to investigate and critique social injustices - such as racism and language discrimination - allowing students to actively challenge oppressive structures and acts through words and actions.

> [...] mathematics should be a vehicle for students to deepen their grasp of the sociopolitical contexts of their lives, and through the process of studying their realities - using mathematics -

> they should strengthen their conceptual understanding and procedural proficiencies in mathematics.
> (Gutstein, 2007, p. 109)

Mathematics, understood as a "key analytical tool" (Gutstein, 2006), enables students to investigate, make sense of, and act on issues of social justice related to their own lives or the community in which they live. With this in mind, through mathematical education and reflection, students prepare themselves to face society and their world, critically understanding its socio-political and cultural-historical conditions. In this action, teachers acquire a delicate position, aimed to conceive their role as part of a broader social movement. Indeed, teachers are called upon to seek out issues of interest to students - mirroring the cultural, political, social needs of their community - and explicitly create the conditions for students to become active participants in society's change (Gutstein, 2006). In relation to this aspect, it is possible to note the influence of Freire's pedagogical vision that is rooted in the conviction that the programmatic content of an educational proposal should start from students' needs, operationally translated into the search for *generative themes* (Section 1.1.3). In this regard, Gutstein raises a question, asking how teachers can actually help to activate this social movement and enable, through their classroom practice, a dialectical development and interweaving of critical thinking and mathematical competence. In this regard, Gutstein outlines and describes in his research contribution (Gutstein, 2007) three types of knowledge: *community*, *critical*, and *classical knowledge*.

In order to characterise *community knowledge*, Gutstein recalls an example from Freire's "Pedagogy of Hope" (1994), which I thought it appropriate to present in order to better understand what this type of knowledge includes and what it doesn't. In his book (Freire, 1994, pp. 44-49) Freire narrates about a game with a group of Chilean farmers, used as a challenge during a rousing discussion. The challenge is to stump each other with questions that the other thinks the other cannot answer. Freire starts by asking the farmers: "What is Socratic maieutic?". The farmers laugh and unable to answer, ask Freire: "What is a contour curve?". The game continues in this way until the score is 10-10. This example aims to emphasise the validity of both sets of knowledge: that of Freire and that of the Chilean farmers. Each side knows what the other does not know and respects their own and the other's knowledge (Gutstein, 2007). The knowledge of the farmers, derived from their personal and collective experience, represents the *community knowledge* defined by

Gutstein. This knowledge therefore includes all the things that people already know and have learnt outside school, through their personal experience within the community in which they live. Also included in this knowledge are the ways in which people make sense of their experiences, their language and the way people "understand their lives, their communities, power relationships, and their society" (Gutstein, 2007, p. 110).

Critical knowledge, on the other hand, includes the knowledge of the historical, economic, cultural and socio-political conditions and motivations of the various phenomena that characterise reality. As the author argues, it is possible that sometimes, the boundaries between *community knowledge* and *critical knowledge* are not defined. It is possible to find a reason for this blurring of boundaries in Freire's pedagogical vision. In particular, as discussed above (Section 1.1.3), the search for *generating themes* and the proposal of meaningful situations represent for Freire the beginning of an educational process, understood as practice of freedom. From this perspective, reflecting on significant themes that come from personal experience within a community means reflecting on what Gutstein characterises as *community knowledge*. When this reflection helps to recognise one's own situation, laying the foundations for human advancement and the construction of a better future, then knowledge becomes critical and *community knowledge* becomes *critical knowledge*.

Finally, *classical knowledge* includes all knowledge that is more formal and learned in a school context. With reference to the mathematical field, *classical mathematical knowledge* involves the knowledge selected and contained in the school curriculum in terms of mathematical competences and skills, probably finding an affinity with the *mathematical knowing* proposed by Skovsmose (1994).

On the basis of what has been presented, if the task of teachers is to propose mathematically meaningful situations to their students, through which they can engage the principles of democracy and social justice and stimulate critical reflection, then an effort appears necessary in trying to connect the three types of knowlegde proposed by Gutstein. The author himself notes a difficulty in this connection, which presupposes knowledge of many factors. For example, in order to design a teaching action along these lines, teachers would have to be aware not only of the three types of student knowledge, but also of historical, cultural, local and global social and political movements that influence students' lives (Gustein, 2008). And if such types of knowledge are known, it should be ensured that the mathematics that is proposed is meaningful. Design in this sense appears to be very

complex, not least because it is difficult to generalise due to the specific situations that may characterise one community but not another. This complexity is certainly also noted by Gutstein:

> […] for many reasons, it is quite complex to create curriculum that starts from students' and their communities' lived experiences/knowledge and then simultaneously and with rich interconnections supports both mathematical power/ classical mathematical knowledge and a critical awareness of one's social context.
> (Gutstein, 2007, p.113)

Currently, there are several examples of educational designs in which mathematical learning objectives are combined with social justice objectives (see, for example, Buendía, Molfino & Ochoviet, 2017). In the following, I have chosen to focus on one of the designs documented by Gutstein. In his research work (Gutstein, 2009), the author describes the experience conducted in a public high school in Chicago, 30% of which was attended by African American students and 70% by Latinos. Through social justice teaching, the author and his mathematics team tried to connect *community*, *critical*, and *classical knowledge*. The project started as a consequence of visible tensions between the Mexican immigrant community and the African American community that lived in the school neighbourhood and attended it. Tensions, already typical between the two communities, escalated following a press conference by a local Latino politician in which he proposed a public referendum to exclude African American students from school. The aim of the project was to find a fair solution for both communities through mathematics. The students were involved in various mathematical problems, all closely related to the social issue they experienced at that time. For example, they were asked to determine the number of students from the different communities in the school, knowing the total number of students and also using different ratios than 30:70. They then focused on determining the probability of acceptance into a lottery, considering each community and using various ratios. Or again, the students explored local maps, examined data from neighbouring schools and generally tried to mathematicise the problem. While recognising limitations in the design presented, the author also identifies strengths (Gutstein, 2007). The activity conducted in the Chicago school made it possible to start from the students' community knowledge and to develop classical and critical mathematical knowledge from it. In particular, starting from the direct and concrete experience lived by the students - *community knowledge* - through mathematics - *classical knowledge* - the two communities worked together, through

dialogue and confrontation, bringing out stories and stereotypes that contributed to the development of critical consciousness - *critical knowledge* (Gutstein, 2007). The choice to connect to a real and contemporary issue for the students allowed them to engage in a reflection on their own experiences that brought them to deepen their socio-political awareness and simultaneously learn mathematics.

Driven by the idea that mathematics can help to understand the injustices and inequalities inside a society, the idea of education, and more generally of school, proposed by Gutstein appears in line with *critical mathematics education* and even more so with Freire's pedagogical vision. As we have seen, problem-posing education and the methodology for thematic research are two of the theoretical and methodological innovations resulting from Freire's work (Gadotti & Torres, 1997), which in this case, are not implemented in adult education studies, but in the field of mathematics education. In fact, the idea of conscientização (Section 1.1.2) seems to be found in Gutstein's desire to guide, through mathematics education, the understanding of socio-political, historical and cultural conditions. This challenge not only concerns students, who are invited to engage in critical reflection, but also teachers, who are called upon to interweave mathematical concepts with aspects of social justice that characterise the community in which they work. In this sense, attention to the three knowledges in the teaching design process appears fundamental for teachers, in order for them to be able to propose activities that are mathematically stimulating but also oriented towards social justice. This attention, however, also appears fundamental for the students who, through the development of a critical consciousness, acquire, in a Freirean vision, the necessary tools to recognise situations of oppression and make a change.

CHAPTER 3

AN EXAMPLE OF TEACHING DESIGN: THE "PROUD OF YOU" PROJECT

"*We too, when we talk about education, are preaching a revolution, because through it everything we know today will be transformed*"
(Montessori, 2016, p. 36)

The studies considered in the field of active and popular pedagogy and the research presented in the previous chapter allow placing the "Proud of You" project (PoY) in the framework of *critical mathematics education*. In the following paragraphs, the general idea of the project is presented and two different editions in which I took part with different roles and responsibilities are described in more detail. The pedagogical vision of Paulo Freire and consequently, the educational visions of Emma Castelnuovo and Eric Gutstein make it possible to emphasise neuralgic aspects of the design and implementation of PoY's educational activities. In this sense, the project can be understood as an expression of *active schools*, but it does not stand as an example in this direction, nor does it show what *critical mathematics education* is. Rather, through the description of PoY, the intention is to focus on methodological choices and the construction of activities in which mathematical objectives are intertwined with objectives of active citizenship education and participation in a democratic discourse.

3.1 The "Proud of You" project

The "Proud of You" project (PoY) was set up with the aim of preventing the risk of early school leaving and encouraging learning and educational growth through the implementation of educational actions of italian and mathematics. It was promoted by the "Next-Level" association, which works in the field of social and cultural promotion of young people. Over the course of its three editions, PoY has involved primary and secondary schools located in various districts of Naples, the capital of a region that records school drop-out rates of up to 19%, reflecting a serious social alarm when compared with the national average of 14% (Il Mattino, 2019). The territories addressed by PoY are often

characterised by a low level of education and by situations of economic and socio-cultural deprivation that, in many cases, lead to crime, judicial problems and drug addiction. This context inevitably also has repercussions on the everyday life and liveability of minors in these areas. For these reasons, school drop-outs and drop-outs in these contexts represent a real danger for young people who, by moving away from education and the school environment, run the risk of becoming involved in local organised crime. So, the PoY project can be seen as an educational experiment aimed at preventing and counteracting student attrition in this sense (Carotenuto, 2021).

The project can be situated in the framework of *informal mathematics education* (Nemirovsky, Kelton & Civil, 2017), an emerging field of research that explores appropriately designed learning spaces - science museum exhibitions, summer camps, after-school programmes. In this perspective, school is also experienced outside the classroom, for instance in gardens, museums and public spaces of the school's surroundings.

As noted by (Nemirovsky et al., 2017), informal education spaces differ from traditional school settings with respect to the following characteristics:

- the voluntariness of the student's participation in the proposed activities and their freedom in choosing to follow their own interests, even influencing the course of the same activities;
- the fluidity of the disciplinary boundaries: the boundaries between mathematics and other disciplines are more blurred, the connections of mathematics with art, literature, other sciences and technology are usual;
- the absence of traditional forms of assessment: although learning is documented, students are not assessed individually.

For these reasons, informal contexts offer students the possibility of alternative visions of mathematics, engaging learners creatively (Nemirovsky et al., 2017) and aiming to create in them the conditions to develop and maintain a positive attitude (Di Martino & Zan, 2011) towards the discipline and, more generally, towards school.

The principles that inspired the PoY design choices certainly take into account the valuable contribution in the field of research in mathematics education by Emma Castelnuovo and her "doing mathematics with dirty hands" (1963). In line with this vision, the PoY path activities were implemented in a laboratory mode, in large groups or through small group work. Students were involved in activities, both individual and collective, which allowed them to

start from direct observation - from a non-artificial concrete in which motivation can be found - which then developed through mathematical exploration, eventually supported by the use of artefacts. All PoY laboratories were characterised by manipulative activities and the involvement of pupils with the whole body, promoting and emphasising interaction with the material world, movement, concrete experience and the manipulation of artefacts (Carotenuto et al., 2020). All PoY activities were found to be genuinely inclusive, even in the more specific case of students with difficulties, disabilities or special educational needs - including also students with a very critical cultural disadvantage which, in the PoY areas, often corresponds to a poor knowledge of the Italian mother tongue, due to the exclusive use of the local dialect in non-school contexts (Carotenuto, 2021). In this view, sensitive experience has been considered an excellent ally in terms of an inclusive and democratic perspective. As an effective expressive tool, it helped especially those students with severe language difficulties and very little fluency in everyday language, making the learning of mathematics accessible to all. More generally and in a genuinely inclusive and democratic perspective (Skovsmose & Penteado, 2012), the problems proposed in the PoY activities were mathematical problems that the students themselves could perceive as meaningful and in relation to which they could feel sufficiently competent. Both of these goals are very ambitious and challenging, especially in the areas in which PoY operates, but not impossible to pursue.

PoY activities also emphasised the use of the body, with reference to studies on the *embodied mind* (Lakoff & Núñez, 2000) and the enactivist view of knowledge (Varela, Thompson & Rosch, 1991). In particular, the activities, with different modalities and objectives, required students to move and participate with the whole body, valuing the dynamic aspect of the learning experience. This involved a necessary reorganisation of spaces and in particular, of the more traditional classroom setting, which was modified to promote movement, teamwork and dynamism. When possible, activities were conducted in the large spaces available in the school - gymnasium, theatre, courtyard - and in other cases, significant locations in the city were involved. During the PoY project, many historical and natural sites in the city were explored, providing a backdrop for the proposed mathematical activities. In this sense, PoY represented a precious opportunity for the students involved. Many of them, for the first time, had the chance to get out of their suburbs and discover the cultural and natural heritage of their city. In this aspect we can observe the project's declared desire to interweave mathematics education objectives with social objectives, finding points of contact with *critical mathematics education* and Freire's pedagogical vision, declined in the mathematical context by Gutstein. In this case, the

discovery of the city of Naples, in addition to motivating the proposal of mathematical problems, represented a way to pursue transversal objectives oriented towards the emancipation of students from geographical and social boundaries, in order to create a sense of belonging to the entire city community (Carotenuto et al., 2020). In general, all PoY activities are animated by the desire to respond, through didactic action, to the territory's need for legality, which translates into the choice of involving in the mathematical discourse themes close to the students, the community and the territory in general, which can lead to increasingly critical reflection.

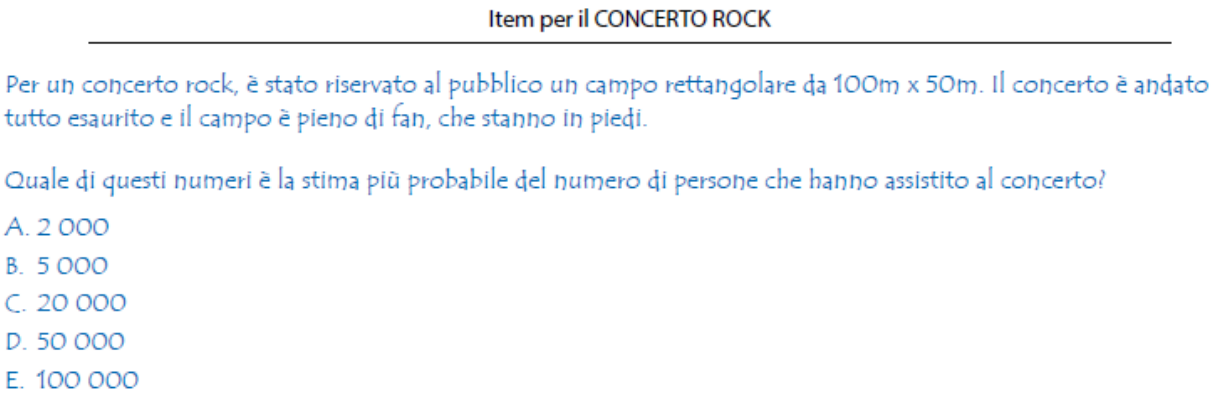
To achieve this, the implementation of the teaching activities was preceded by a training and co-design phase with the teachers of the classes involved, which took place at different times and in different ways during the two editions in which I participated.

3.1.1 Second Edition 2019 – 2020

In the year 2019-2020, I took part in the second edition of the PoY project as the tutor in charge of the group of tutors in mathematics education. This experience, which took place during my PhD first year, gave me the opportunity to be involved at first hand in the captivating mathematical activities conducted, working alongside the students in a course that engaged them mathematically and gave them the opportunity to reflect, experiment and be amazed, in entirely new terms. For the second edition, the PoY project was financed by the Benefit Fund of the Italian bank “Intesa San Paolo” and the company “Gestione dei Servizi Aeroportuali Campani" (GESAC). The design and support for the implementation of the mathematical paths was entrusted to the Department of Mathematics and Applications of the University of Naples "Federico II" and the Department of Philosophy and Educational Sciences of the University of Turin. The two institutions were supported by the Department of Educational Sciences of the University "Suor Orsola Benincasa" for the recruitment of the educational tutors and by the Department of Psychology of the University of Campania "Luigi Vanvitelli" for the impact assessment of the project. PoY involved teachers and students from the fourth and fifth grade of the primary school and the first grade of the middle school (sixth grade) of the Istituto Comprensivo "Radice Sanzio Ammaturo" in Naples. The school comprises four plexuses located in three different neighbourhoods of Naples, between Poggioreale and Capodichino, situated along the dorsal that separates the third, fourth and seventh municipalities of the City of Naples. During my first experience in the project - the second PoY edition - the training interventions were carried out over a long period of time, through regular meetings throughout the

duration of the project. In fact, I participated, as a tutor, in the training meetings aimed at teachers and tutors and dedicated to the co-design of activities. The meetings made it possible to identify the didactic objectives of the PoY action, taking into account both the needs and requirements of the pupils - as perceived by the teachers of the schools involved - and the thematic areas for primary school mathematics specified in the National Indications for the first cycle of education (MIUR, 2012): Numbers, Space and Figures, Relations, Data and Predictions. In a spirit of community and dialogue, starting from the design drafts gradually proposed by the project team (Carotenuto et al., 2020), it was possible to reflect on the methodologies and didactic objectives involved in the realisation of the project. In the operational phase, PoY involved students and teachers two days a week, one day dedicated to the italian education and another day dedicated to mathematics education. During this edition, the two disciplines were developed in different and disconnected directions. Initially, the project was supposed to end in April, with a total of 32 meetings planned, but it ended prematurely in March, due to the covid-19 pandemic. During all the meetings, each class involved was supervised by at least one teacher - a class teacher in primary school and a maths and science teacher in middle school and if necessary a support teacher - and a tutor. With the aim of responding to the strong demand for legality in the local area, the PoY activities were designed from the perspective of *critical mathematics education*, deliberately trying to intertwine mathematics education with citizenship education. This aim was pursued through the narrative expedient of an epistolary correspondence between the students and various - imaginary - figures from the institutional and cultural sphere of the city of Naples. The pupils, in the role of little architects, were invited through letters to solve complex mathematical problems concerning their city, which involved group work, discussion and confrontation, the collection of information, the formulation of hypotheses and strategies, the construction of appropriate mathematical models and the visit to the places involved.

One of the activities proposed with the PoY project was inspired by the OECD-PISA question (Figure 3), "Rock concert", administered in the field test that preceded the 2003 cycle.



Item per il CONCERTO ROCK

Per un concerto rock, è stato riservato al pubblico un campo rettangolare da 100m x 50m. Il concerto è andato tutto esaurito e il campo è pieno di fan, che stanno in piedi.

Quale di questi numeri è la stima più probabile del numero di persone che hanno assistito al concerto?

A. 2 000
B. 5 000
C. 20 000
D. 50 000
E. 100 000

Figure 3.

Taken from (OECD, 2013). The problem: "For a rock concert, a rectangular field measuring 100m x 50m was reserved for the audience. The concert was sold out and the field is full of fans, who stand around. Which of these numbers is the most likely estimate of the number of people who attended the concert?".

The question requires selecting the correct answer (C.) from the options provided and it is considered to be "moderately difficult" (OECD, 2013), since only 28% of the students can answer correctly. The main difficulty associated with this question relates to the "translation" of the information provided (size and shape of the field, "sold-out" concert, standing-room-only fans) into mathematical form. The item asks students to estimate the total number of attendees at a concert, based on the size of the rectangular field in which the concert takes place. In particular, students are asked to identify missing information - which can be derived on the basis of deductions from practical experience - to use mathematical concepts, facts, procedures and reasoning to determine a connection between the area of the field and the area occupied by a spectator or group of spectators, and to interpret, apply and evaluate the mathematical results obtained, in relation to the proposed context (OECD, 2013).

In connection with the PoY project, the mathematical activity, inspired by the question "Rock concert", was conducted with the motivational background in mind. Through a letter, the municipal geometer of Naples requested the students' help in organising the New Year's Eve concert at the "Terrazza del Belvedere" of the Royal Palace of Capodimonte. Specifically, the geometer, declaring himself little expert in mathematics, asked the students to determine the maximum number of possible guests for the concert. The activity was implemented by alternating phases in which the students worked in large groups and phases in which they were divided into teams of architects, each consisting of four or five students and supervised by an adult (teacher or tutor). After a careful reading of the missive, the problem posed was tackled in spacious but internal school

environments such as the theatre, the gymnasium or the schoolyard, which although large spaces were still considerably smaller than the terrace (Figures 4 - 5).

Figure 4.

In the schoolyard, the pupils, divided into small groups, explore the proposed problem with their whole bodies, experiment by implementing measurement and estimation processes.

Figure 5.

Students construct a smaller concert place for a simulation in the schoolyard.

Each student explored the spaces with their whole body, formulated hypotheses and took measurements using chalk, a tape measure and scotch paper, and then noted what they discovered in their team notebook. At the end of this exploratory phase, the teachers brought the students together in a large group and conducted a discussion by formulating questions and inviting contributions from everyone, in a democratic perspective (Figure 6). On this occasion, the students discussed their strategies for trying to answer questions such as: How many people are there in a certain unit area? How do we measure this space against the surface unit we have chosen?

The aim of this discussion was to share their own strategies with their peers and understand those of others, in order to arrive ready for the visit to the Bosco di Capodimonte planned for the next meeting.

Figure 6.

The students, gathered in the schoolyard, through the stimuli of the teacher, confront each other in large groups and share the strategies worked out in the small groups of architects.

After discussing the mathematical problem at school, the students, together with their teachers and tutors, visited the "Terrazza del Belvedere" of the Royal Palace of Capodimonte, the concert location. Each group of young architects was given a map of

the Royal Palace of Capodimonte in order to help them make a conscious orientation in the host space (macro-space). In general, with the aim to promote emancipation from geographical boundaries, the use of maps also accompanied other activities and explorations in the PoY project. For example, by exploring the map of the city of Naples, through mathematical games, students were able to understand its organisation into municipalities and districts and geographically locate their school and the various places visited.

At the Terrazza del Belvedere, in large groups, the pupils, with the support of the teachers and tutors, reviewed the task given to them and the strategies developed at school, in order to proceed, divided into small groups, to explore and measure the place with their whole bodies (Figure 7).

Figure 7.

At the top, a picture of the "Terrazza del Belvedere", available on the official website of the Museo e Real Bosco di Capodimonte. In the bottom pictures, students at the "Terrazza del Belvedere", intent on measuring and collecting data useful for solving the problem.

During the activity, the students wrote down the data collected and the mathematical steps adopted to solve the complex mathematical problem. The activity, conducted in a significant location for the city of Naples, fascinated and involved everyone, adults and children alike, who through fruitful teamwork arrived at a solution to the proposed problem. The work ended with a large group discussion phase. Each team shared its resolution and results with the others. Together, the students produced an answer to the geometer's letter, arguing the chosen strategy, both through words and graphical representations.
In the next meetings, again motivated by the organisation of the concert at the "Terrazza del Belvedere" in Capodimonte, the students tried to answer a new letter, concerning the cost of the concert ticket, considering a number of expenses to be incurred. With a view to proposing mathematically meaningful problem situations to the students, but also attentive to the principles of democracy and social development, the transversal objective of the activity concerned the fight against mafias. For possible performances at the New Year's Eve concert, the students were introduced to the Modena City Ramblers band and their song "I Cento Passi". The song, inspired by the figure of Peppino Impastato, a Sicilian journalist and activist who was a victim of the mafia, was used to introduce the delicate topic to the class and to stimulate a dialogue that turned out to be particularly meaningful, especially in the places where PoY acted.

In general, the idea of the designers (Carotenuto et al., 2020) to involve figures from the institutional and cultural worlds in the mathematical discourse, however fictitiously, arose from the desire to create a pleasant encounter with these worlds, which are often perceived, in the territories PoY addressed, as enemies, due to the economic sanctions or periods of detention they impose. In this sense, mathematics enabled the students to observe with a different lens and understand the social, cultural and political contexts of their lives, finding in this a strong connection with Gutstein's educational proposal.

3.1.2 Third Edition 2020 – 2021

In the year 2020-2021, I was involved in the third edition of the PoY project with a different role. I took part in it, in the preparation phase, as a member of the design team for the mathematics educational activities and, in the operational phase, as a local coordination referent. For the third edition, the PoY project was financed by the Charity Fund and social and cultural works of the Italian bank “Intesa San Paolo”. The design and support for the

implementation of the mathematical paths was entrusted to professors and researchers from the University "Suor Orsola Benincasa" in Naples and to designers collaborating with the Holden School in Turin for the Italian section. Also in this edition, the two institutions were supported by the Department of Psychology of the University of Campania 'Luigi Vanvitelli' for the project's impact assessment. The third edition of PoY reconfirmed the involvement of the Istituto Comprensivo "Radice Sanzio Ammaturo" in Naples, with the participation of teachers and students from the fourth and fifth classes of the primary school. The Istituto Comprensivo 83° "Porchiano Bordiga" in Ponticelli, with the participation of teachers and students from the fourth and fifth classes of the primary school, was also involved. In June, before PoY was activated in the schools, the project was delivered in Summer Camp mode in Polistena (RC), involving teachers and students of the Istituto Comprensivo "Capoluogo Brogna", hosted in the gardens of the "Luigi Monti" Community in Polistena. Once again, the areas covered by PoY action are characterised by socio-cultural disadvantages. Therefore, also in this edition, the educational interventions proposed were aimed at promoting a positive attitude towards mathematics (Di Martino & Zan, 2011) through the provision of extra-curricular learning in line with the characteristics of informal mathematics education contexts and the *critical mathematics education* approach. The project involved 90 Calabrian students who participated in the Summer Camp formula and a total of approximately 320 Neapolitan students from the two schools in question. Compared to the previous edition, there were two important differences. In this new edition, the mathematics education was connected to the Italian education, through intertwining in the teaching design that led to the creation of a path characterised by strong communication between the two parts. Thanks to the advice of an external expert, more attention was paid to planning from an inclusive perspective, with the production of selected teaching materials. Moreover, due to the Covid-19 pandemic events that forced the closure of schools in Campania for a long period of time, it was impossible to meet the teachers involved in the project and implement an exchange of ideas and group reflection with them, as was the case for the last edition. The teaching objectives were identified through an on-line meeting with the teacher-referent of the Istituto Comprensivo "Radice Sanzio Ammaturo", representing the teachers involved, which made it possible to orient and focus the teaching design on specific points. There were then other on-line meetings to share with the teacher-referents (also from the schools that were added later) the planning drafts and to tackle difficulties and potentials together. On the other hand, the meetings between the mathematics project team and the Italian project team, where it was possible to benefit from the different disciplinary competences and professional experiences, turned out to be

valuable. The continuous dialogue, cultivated through numerous meetings, allowed the construction of a joint educational path, enriched by the contribution of the various professionals involved.

In conjunction with the project team, I also dealt with the training of teachers and tutors involved in the project. For the Istituto Comprensivo "Capoluogo Brogna" in Polistena and the Istituto Comprensivo "Radice Sanzio Ammaturo" in Naples, the teacher training was condensed into just two meetings (one day after the other), prior to the start of the project with the students, in which the planned activities were shared. During the two training meetings, teachers and tutors were personally involved and experienced the activities through an immersive experience (Figure 8).

Figure 8.

A group of teachers and tutors participate in one of the planned activities, moving around the large space made available by the Juvenile Penitentiary Institute of Nisida (NA), which hosted the training for the Istituto Comprensivo "Radice Sanzio Ammaturo". Photo taken from (Next Level, s.d.).

For the Istituto Comprensivo 83° "Porchiano Bordiga", the teacher training took place in three different moments spread out over time: an introductory phase, the presentation of a first block of activities and the presentation of a second and final block of activities - to which were added several moments of confrontation and on-line meeting with the teacher-referent. The training meetings were also attended by teachers who were not directly involved in the project, such as secondary school maths and science teachers. In the first meeting, the project was introduced in a general framework and the teachers, divided into small groups, were able to experience one of the planned activities first-hand. In preparation for the subsequent meetings, the activity design sheets were previously shared with the teachers, who were able to consult them and on which an initial reflection began. The time-diluted meetings allowed for a sharing of the design in blocks, making it possible to focus on a few activities at a time that the teachers could explore and experiment with. These meetings provided a valuable opportunity to discuss and refine the planned activities, taking into account the on-the-ground experience of the teachers involved. For example, after the first modules were proposed in class, one difficulty noted by the teachers, both in Italian and in mathematics, concerned the limited time for revision at the end of each activity (metacognition and introjection of learning). In the view of what emerged, the modules were revised together with the teachers, making appropriate changes where necessary in order to allow more space for a final revision moment. In general, sharing specific needs and difficulties encountered with the designers made it possible to intervene on the activities and improve them, before they were proposed to the pupils.

Again, the design of the activities was motivated by a strong desire to respond to the needs of the local area, from the perspective of *critical mathematics education*. In particular, with the mathematics project team, we pondered for a long time which social issues to involve and, above all, how to do so, without losing sight of the mathematical content to be proposed and ensuring that the activities made "sense" in the eyes of the students. Thus, for the mathematics design, the theoretical framework outlined (Chapter 1-2), the needs of the local area and the reality in which pupils are placed were taken into account. With the aim of proposing students to explore meaningful problem situations, activities were designed with a view to finding a "game of the people" that, in a fantasy scenario, offered the opportunity to reflect and argue within the mathematical discourse about important issues related to environmental education and gender prejudices. In concert with the Italian educational design team, we constructed a didactic path modulated according to the

game-book format and organised in ten modules, through which joint Italian and mathematical activities were proposed. Through the telling of a "two-way" story, the students were able to actively participate in a journey through time, choosing at each encounter which way to take (Figure 9).

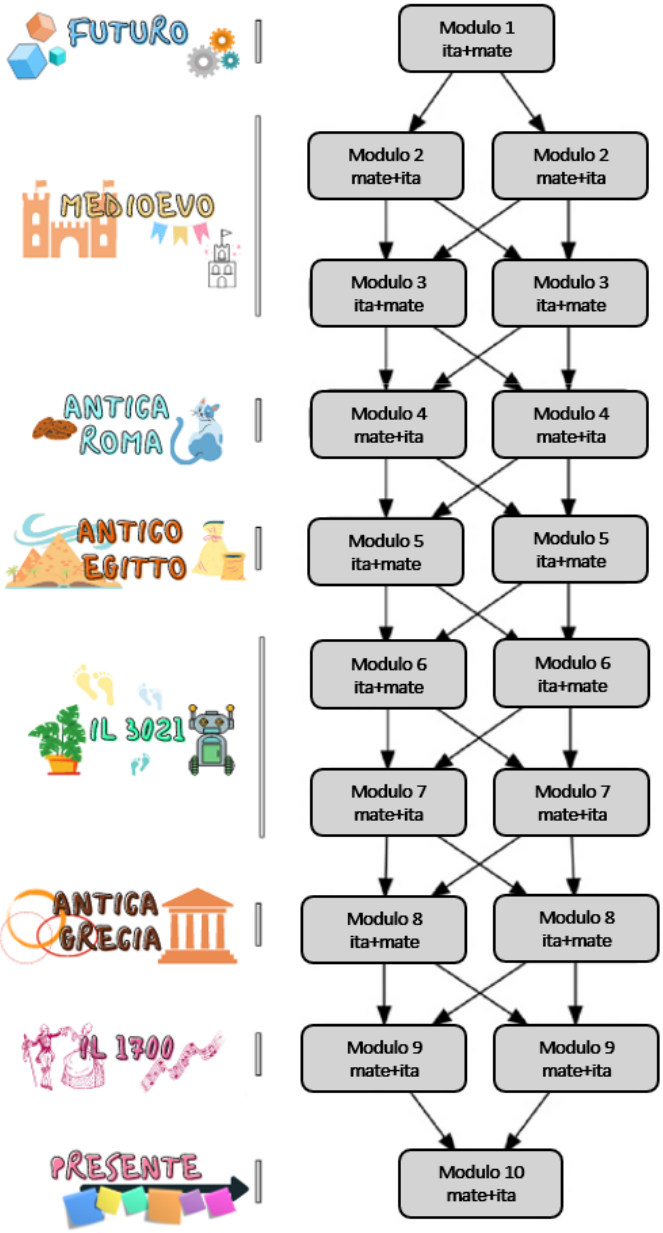


Figure 9.

The diagram shows in order all the stages of the PoY journey through time, each linked to a specific historical period: The Future; The Middle Ages; Ancient Rome; Ancient Egypt; The 3021; Ancient Greece; The 1700s; The Present.

In contrast to the last edition, the first activities proposed were implemented during curricular teaching so that all students could benefit from them and then freely choose whether or not to participate in the afternoon meetings. The response from the pupils was very satisfactory: in almost all cases, participation in the extra-curricular meetings involved the entire class, and furthermore, in the classes involved, there were fewer absences on the days dedicated to the project. In many cases, in fact, it was the pupils themselves who told the teachers that they enjoyed going to school, knowing that the afternoon would be devoted to PoY activities. For the mathematics education, the methodological choices were in line with those of the second edition of the project and thus, strongly inspired by Emma Castelnuovo's method and aimed to emphasise sensitive experience, active exploration and the collaboration of the class group. Bearing in mind Gutstein's research and the studies on which it is based, mathematics education was also understood as a tool for investigating and critically understanding problems and issues that characterise the everyday life of pupils and their surroundings. The decision to use an integrating background, whose idea originated with Paolo Zanelli (1990) in the context of school integration of pupils with disabilities, was significant. This is a didactic design that uses mediators, for example narrative contexts, through which disciplinary knowledges are proposed by teachers as strongly intertwined and thus discovered, constructed and explored by the students in a framework of meaning and great sharing of meanings (Zanelli, 1990). In the case of the PoY project, the integrating background is provided by the narrative, which makes it possible to propose engaging mathematical problems and to stimulate the pupils' metaphorical thinking, understood as a powerful cognitive tool that helps to construct concepts and solve problems. In particular, through the expedient of a journey through time, it was possible to interweave mathematics not only with the teaching of Italian, but also with other disciplines and with objectives of active citizenship and participation in democratic dialogue. The path designed for the didactics of mathematics involved various didactic objectives, with the declared intention of taking into account all the subject areas for primary schooling set out in the National Indications for the first cycle of education (MIUR, 2012).

Some of the PoY project activities considered most significant from the perspective of the research conducted are presented below.

Landed in the Middle Ages, the students were asked to decorate a castle where a wedding was about to take place. In each corner of each terrace was a stone battlement. The request was to hang a festoon to connect each battlements with all the others, except the

two that were immediately next to it (that is, if two battlements were not consecutive, then a festoon should be stretched between them). Together, the students pondered how many festoons were needed to decorate the roof terraces of three buildings with pentagonal, hexagonal and heptagonal plans respectively. Arranged in a circle, first in groups of 5, then 6 and then 7 (pentagonal, hexagonal and heptagonal terrace), the students dramatised the situation, exploring the proposed problem with their whole bodies. The arms of the "round dance" represented the contours of the terraces of the buildings (sides of the polygon), the intersections using long ribbons represented the festoons to be hung (diagonals of the polygon) (Figure 10). Material and sensory resources such as gestures, posture, motor actions, artefacts and signs were conceptualised as central elements of students' and teachers' mathematical thinking (Radford, Arzarello, Edwards & Sabena, 2017).

Figure 10.
In the image, students explore the proposed problem situation with the use of their whole body. Image taken from (Next Level, s.d.).

In order to recognise the geometry in the story, with the support of a suitably designed educational board, the students were asked to note the correspondences between the elements of the story, their spatial configurations in the dramatisation and the geometric elements of the polygons represented in the board. Following a comparison, the students then completed special tables and looked for relationships between the numbers that appeared in them. The proposed activity preceded the educational outing to Castel dell'Ovo and Castel Sant'Elmo - one of the two castles was chosen for each class - two evocative and historically rich destinations belonging to the artistic and cultural heritage of the city of Naples (Figures 11 - 12).

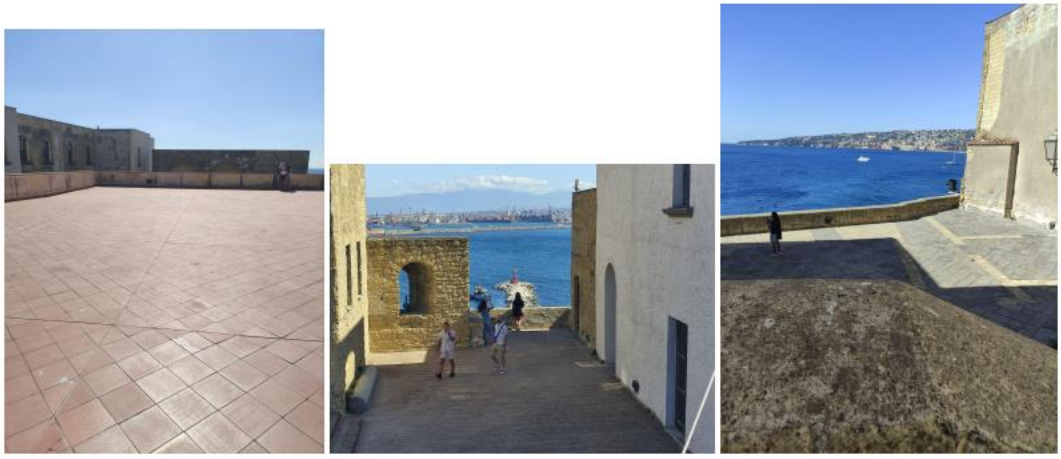


Figure 11.
Photos from the terraces of Castel dell'Ovo, taken during an inspection for the design of the teaching activity.

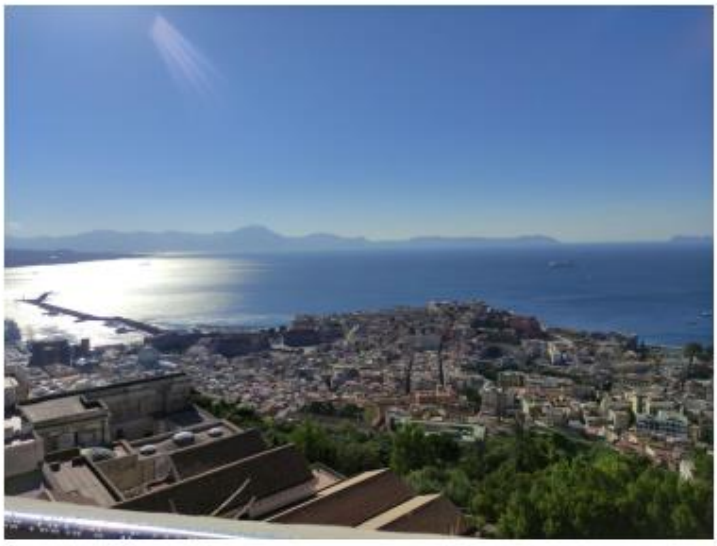


Figure 12.
Panorama from Castel Sant'Elmo. Photo taken during an inspection for the design of the teaching activity.

The in-class activity was a preparation for the outgoing activity. On this occasion, the students were able to dramatise the problem addressed in class in a larger space, allowing themselves to be captivated by the beauty of the places they visited (Figure 13). Together they tried to generalise what they had observed to obtain the formula for calculating the number of diagonals in a polygon of n sides. The narrative context that introduced the maths laboratory created the perfect setting for the visit to the castles. Pupils, teachers and tutors were fascinated by the enchanting views of the Neapolitan city of Castel dell'Ovo and Castel Sant'Elmo, leaving time for stories and legends related to the places visited.

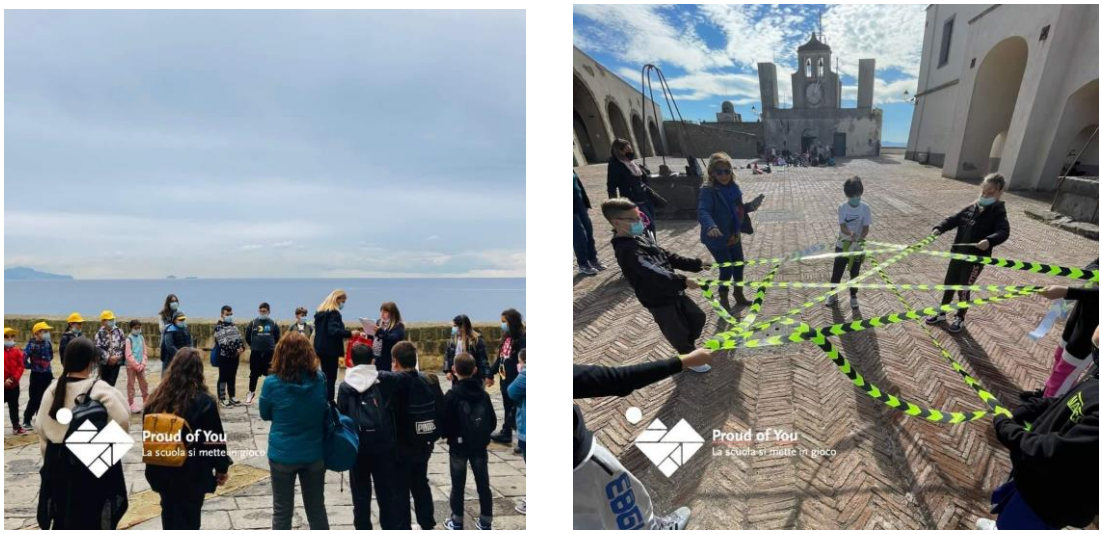


Figure 13.
In the pictures, taken from (Next Level, s.d.), from left to right, at Castel dell'Ovo and Castel Sant'Elmo, students explore the mathematical problem.

During the PoY project, there were also other moments in which the didactic activity and the narrative context framed an educational outing. Through the involvement of a cultural association that operates in the area and organises guided tours, the educational outings were planned in such a way that they could fit in coherently with the planned path. The collaboration with the association made it possible to combine each educational outing with a guided visit to the museum or host location, enabling the students to get to know, also from a historical and cultural point of view, places in the city, many of which they had never visited before. Through the PoY project, the students had the opportunity to visit the National Archaeological Museum of Naples (MANN), the Museum and Real Bosco of Capodimonte and, for the classes of the Istituto Comprensivo 83° 'Pochiano Bordiga', there

was the possibility to add a final stage at the Palazzo Reale in Naples, which also allowed them to visit adjacent historical places and streets.

The MANN provided the perfect setting for the Ancient Egypt stage. With the support of specially prepared educational boards and in the rooms provided by the museum, the students explored the seven symbols used in hieroglyphic writing to represent numbers and their relations, trying out coding and decoding (Figure 14).

Figure 14. Students visiting the MANN and working with Egyptian symbols. Photo taken from (Next Level, s.d.).

One of the aims of the proposed activity was to introduce students to notation systems that are or have been in use in places, times and cultures other than our own. In a collaborative work of exploration and discovery, our numeration system was compared with the Egyptian one, noting points of encounter and differences.

The educational outing to the Museum and Real Bosco di Capodimonte was the coclusive moment of an activity already started, set in 1700s France. In class, the pupils invented the choreography of a minuet, a French dance characterised by small steps, letting themselves be helped by the harmony of geometry. The pupils, in groups of four and disposed in the vertices of an imaginary square, on the notes of a minuet, performed position exchanges corresponding to particular geometric transformations of the square: symmetries and rotations. Choosing between the different transformations, each group freely invented their own choreography (Figure15).

Figure 15.
In the classroom, in turn, each group of four pupils interprets the invented choreography through whole-body movements. Images taken from (Next Level, s.d.).

Once at the Capodimonte Museum, following the guided tour, each small group of pupils was given a version of the "Choreographer's Notebook": a choreography in the form of a diagram, showing a succession of particular geometric transformations of the square. Taking into account the transformations of the square in the notebook, each small group tried to interpret the dance, then performed it in one of the large rooms inside the Museum (Figure 16).

Figure 16.
One of the small group of students performs at the Capodimonte Museum, interpetating on the notes of a minuet the choreography from the "Choreographer's Notebook". Photo taken from (Next Level, s.d.).

The dimension of the body and movement significantly characterised this activity, representing a resource and a privileged channel for learning mathematics. On the one hand, the dynamism aroused pleasant and positive emotions in relation to the proposed mathematical activity, on the other, it made the activity accessible also to those pupils with serious language deficiencies.

For the pupils of the Istituto Comprensivo 83° "Porchiano Bordiga", the visit to the Palazzo Reale in Naples and the surrounding historical streets represented the final moment of the PoY project. Through the guided tour, modulated in the formula of a treasure hunt through the rooms of the Palazzo Reale, they retraced the stages of the journey through time, recalling experiences and discoveries made (Figure 17).

Figure 17.
The photo shows one of the moments of the visit to the Palazzo Reale in Naples. Photo taken from (Next Level, s.d.).

During the design phase, an attempt was also made to involve social issues that students could reflect on and at the same time be meaningful in their eyes. It was decided to focus on issues related to environmental education and gender prejudice. In this regard, at the 3021 stage, the students were involved in a debate: on one side were the robots, fascinated by speed and technological progress, and on the other side were the plants, interested in

environmental protection and the preservation of their species. The students were asked to analyse, understand and reprocess the data provided by the two factions and then choose which side they supported. Firstly, the students, all together, analysed the bar graphs provided by the bionic community in favour of the use of teleportation, containing, through a one-year forecast, information on the number of tourists entering the country, the average daily time spent by children on the home-school journey and the average salary of a worker. So that the robot's reasons could not remain hidden, the students, divided into small groups, prepared an argument in support of their idea, which was later shared with the whole class. Then, starting from the argument prepared by the plants, the students were asked to make a graph that would give a clear and more concise view of the data.

During the planning phase, much thought was given to which topic could particularly engage the students and raise their environmental awareness. In this regard, the reality in which the students, and in general the territory, are immersed was taken into account and, with a view to the search for a "game of the people", an attempt was made to identify a theme that, through dialogue in the classroom, could give rise to new reflections. From the perspective of the search for a *generating theme* (Freire, 1970), the focus was on pollution, a complicated and worrying wound in Campania, where, due to illegal waste disposal and the consequent pollution caused by illicit fires, a vast area is sadly known as the "land of fires" (Peluso, 2015). As shown in Figure 18, imagining the set of themes as located in concentric circles (Chapter 1), we gradually reflected on consequential themes (from the most general to the most particular), with reference to the specificity of the context to which the activities were addressed. Given the presence, both in Naples and in Polistena, of a coastal area to which the inhabitants, especially the children, are particularly connected, the choice was made to refer to the problem of coastal erosion - containing in itself connections with other possible themes (for example: melting glaciers, disappearance of animal species, floods).

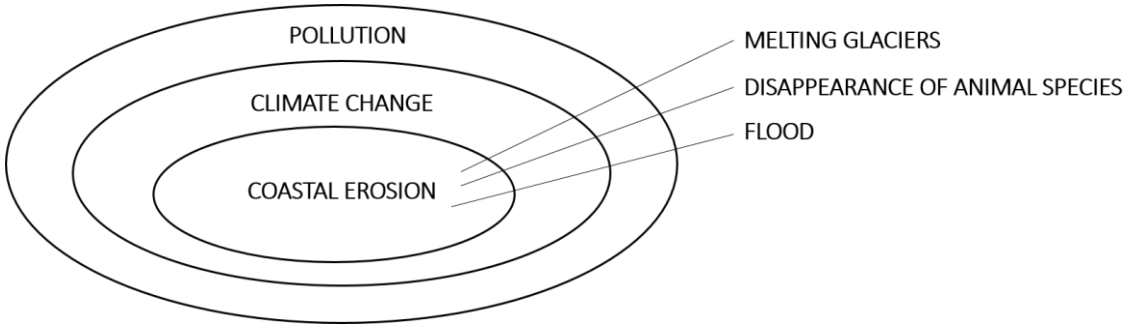


Figure 18.

The diagram reproduces that proposed in Chapter 1 for the representation of themes in concentric circles.

The defence speech of the plants concerned pollution and the consequent possibility of coastal erosion. The students were asked to represent a bar graph that would allow a clearer view of the data on the length of the country's sandy shoreline contained in the plants' argument and referring to the year 3021 and forecasts for the years 3041 and 3100 (Figures 19).

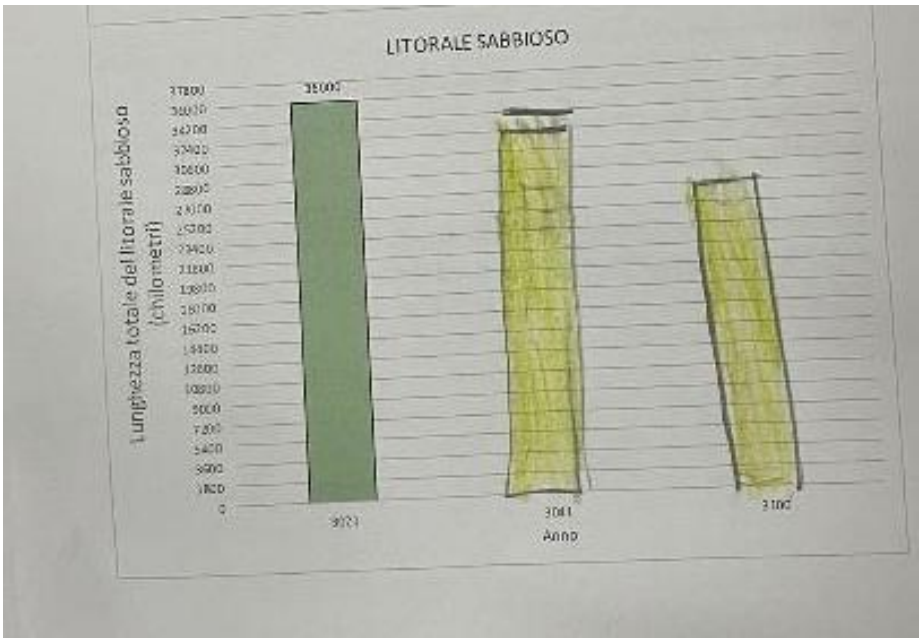


Figure 19.
Graphical representation using a bar graph of the data provided by the plants.

The two moments, the first dedicated to robots and the second to plants, provided the students with an opportunity to critically read, analyse and understand data, initiating reflection on issues directly and meaningfully connected to their lives. This moment gave the students the chance to choose which faction to support - all classes chose to defend plants - by making an informed decision and participating in a democratic discourse.

With the stage in Ancient Greece, the choice was made instead to address the discourse on gender prejudices, in order to direct reflection towards the phenomenon of the gender gap. In particular, the 2017 OECD report "The Pursuit of Gender Equality: An Uphill Battle" highlighted a gender gap in the social and economic spheres at a global level (OECD, 2017). From the data, it was found that young women in OECD countries, despite studying longer than men, are less likely to be employed and become entrepreneurs, and are also underrepresented in private and public leadership. The gender gap phenomenon has also been highlighted with reference to STEM (acronym for Science, Technology, Engineering and Mathematics) studies and careers, with respect to which a preponderance of males

over females has emerged in participation in scientific-technological disciplines (Boccomino & Halo, 2021). For these reasons, the design of the teaching activity, through the involvement of an illustrious female example, tried to approach the issue of gender inequality. The mathematics laboratory provided an opportunity to introduce the significant figure of Ipazia, Greek mathematician, philosopher and astronomer. Through the reading of selected short stories, Ipazia's courage and audacity to have been a woman and a scientist at a time in history when knowledge was exclusively entrusted to men was highlighted. With a reference to one of Ipazia's most important works - an eight-book treatise on the "Conics" of Apollonius - the mathematics laboratory was introduced. The students, divided into small groups of three or four, tried to draw a circumference, having various artefacts at their disposal, including chalk, markers, string, plastic cones and scotch paper (Figure 20).

Figure 20.
The picture shows a student tracing a circumference with string, scotch paper and a marker. The circumference is obtained as a trace of a movement. The marker is tied to the end of a string, kept taut at all times, the other end of which remains fixed at a point chosen beforehand. Image taken from (Next Level, s.d.).

From a mathematical point of view, the laboratory provided an opportunity to experience, through body movement, the figure of the circle and to grasp the definition of a circumference as a place of points equidistant from a fixed point that constitutes its centre. In general, the decision to use a "two-way" story arose from the desire to promote a democratic dialogue in the classroom, open to all, at each meeting. Even in the activities that were not specifically constructed with a social objective in mind, the identification of a way forward always took place as a result of a confrontation, in which each pupil participated, arguing his or her position and listening to that of the others, with a view to suspending one's own perspectives in order to welcome those of others (Alrø & Skovsmose, 2004).

The design of the activities, through which an attempt was made to favour the workshop aspect and reinforce the ability to cooperate, collaborate and share, was also oriented with a view to educating to citizenship. The educational outings, strongly intertwined with the mathematical activities proposed, represented an opportunity for pupils to look towards an "external" world, because it is often perceived as external and distant from their everyday life, despite belonging to a local area. The choice, at the design stage, to combine the educational outings with a guided tour was motivated by the desire to create an opportunity for the pupils to get to know and discover more about their territory, understanding its traditions and historical and cultural aspects, in order to be able to place themselves in a broader context. The desire to link the mathematical discourse with moments in which students could benefit from historical and cultural information about their own city was strongly motivated by Gutstein's (2006) idea that it is precisely through knowledge of their own history that students can develop positive cultural and social identities. In fact, these occasions represented a real resource for students that allowed them to get out of the suburbs, often perceived as ghettos.

3.2 References on research direction

Participation in the two PoY editions certainly allowed for a deeper reflection on the didactic-methodological aspects involved in the activities and on the impact on the students, both in terms of mathematics education and in terms of developing a critical consciousness. In particular, my first experience in the project raised a number of questions that guided my project choices and, more generally, my research direction during the third edition. Firstly, I wondered how the creation of teaching activities designed in informal

mathematics education contexts could mediate mathematical content in students from disadvantaged socio-cultural backgrounds. And also, how it was possible to act so that an educational design could be effective and lasting in preventing and contrasting early school drop-out. From an external evaluation, a design like the PoY one seemed to be actually effective in the context on which it acted. For both editions in fact, the PoY project included a final impact evaluation, conducted by the Department of Psychology of the University of Campania "Luigi Vanvitelli". Through direct observations, questionnaires and focus groups with teachers and tutors, a positive impact of the project on the students was revealed. In particular, concerning the mathematics education, teachers and tutors appreciated the pragmatic nature of the content and the strong connection with aspects of city and everyday life. The activities were also engaging and inclusive with regard to pupils with disabilities and special educational needs. Teachers who took part in the project reported that their colleagues who did not participate were intrigued by the pupils' conversations in PoY. On the other hand, as far as the students were concerned, there was an increase in school attendance on project days and the climate of cooperation and free expression fostered by the proposed activities was highly appreciated. Hence, a possible solution for the first of the issues that emerged could be a design in line with the principles of active schools and inspired by critical mathematics education. On the other hand, the meeting and dialogue between researchers-trainers, tutors and teachers, each holding different educational and professional experiences, has certainly enriched me, giving me the opportunity to reflect even more on how important and necessary it is to involve teachers in the design of educational interventions that can prevent and combat the phenomenon of school drop-out. Therefore, with reference to the second issue that emerged, the efficacy over time of meaningful instructional design in disadvantaged contexts, but also in more fortunate ones, necessarily involved teachers, their design and classroom practice. In particular, this experience has opened the way to a more precise research direction, oriented towards focusing attention on teachers and their process of teaching design and implementation in everyday classroom practice. Therefore, interest shifted to investigating the creativity at the root of these processes that lead teachers to design mathematically meaningful teaching activities that are responsive to the socio-cultural context - more or less disadvantaged - in which they operate.

CHAPTER 4

MATHEMATICS TEACHERS' CREATIVITY

"The act of creation of new entities has two aspects: the generation of new possibilities, for which we might attempt a stochastic description, and the selection of what is valuable from among them. However, the importation of biological metaphors to explain cultural evolution is dubious…both creation and selection are acts of design within a social context".
(Muir, 1988, p.33).

In the previous chapters, we reflected on the role that school plays in today's society, in terms of equity, inclusion and fair opportunities, providing, through the vision of active and popular pedagogy, a new idea of school, which starts from the needs and interests of pupils and considers their relationship with the world as central. On the basis of these considerations, in the following paragraphs, we reflect on the design and implementation of mathematics teaching paths that promote inclusion and foster a positive attitude towards school (Di Martino & Zan, 2010), even in those areas with worrying drop-out rates. In particular, considering the role of the teacher and his or her teaching practice as central, teachers' mathematical creativity, referred to as *mathematics education creativity*, is explored in the research action, in order to define a new construct. To clarify the meaning of this terminology and what is meant by mathematical creativity, it seemed appropriate to begin this chapter by recalling studies and findings already known in the research literature concerning the idea of creativity and mathematical creativity in general. The first paragraph of this chapter is devoted to systematic overview of notions of creativity identified in the current field of research, also taking into consideration several studies of a psychological nature on the mathematical creativity of professional mathematicians. It is thanks to these assumptions that, in the second paragraph, the research direction can be made explicit and the type of mathematical creativity that is to be explored and on which, at present, very few studies can be found about it (see for example Levenson, 2021).

4.1 The state of art in mathematical creativity: a starting point for a different direction

In the research literature, interest in creativity in the field of mathematics education is growing steadily, however, at present, there does not seem to be a commonly accepted definition of creativity and the ideas that emerge are often related to different backgrounds or theoretical assumptions (Joklitschke, Rott & Schindler, 2021). Sriraman's work (2004) sheds light on the conception of creativity in the field of psychology research and allows to reconstruct, through a roundup of studies and reflections, the progression of interest in the thought processes involved in the mathematician's creative act. The research studies (Levenson, 2021; Pehkonen, 1997) provide access to a variety of definitions of creativity, which are useful for then delineating the direction of research and making explicit the type of creativity to be explored. The work of Joklitschke et al. (2021) also provides a systematic overview of the notions of creativity discussed in recent empirical research on mathematics education, with the intention of indicating and deducing the different ways in which creativity is understood and defined. From the perspective of psychology, the notion of creativity can be understood in different ways, depending on the approach used to define it. The Hungarian psychologist Mihály Csikszentmihalyi (1998; 2000) considers creativity as arising from a psychological process that takes into account social and cultural dimensions and not only the individual subject. Through the Systems Approach theorisation, Csikszentmihalyi argues that creativity can only manifest itself when *individual*, *domain* and *field* interact and, more precisely, whenever an *individual* proposes a change within a given *domain* (Csikszentmihalyi, 2000). The *domain* represents the cultural dimension that preserves and transmits creative products to *individuals* in the *field*. The latter represents the social aspect and consists of the *individuals* who influence the *domain*. An example (Sriraman, 2004) that allows a full understanding of these three components is to consider the publishers of a research journal and a subject that submits a contribution to them. In this case, the *field* can be represented by the publishers of research journals concerning mathematics and the *domain* can be represented by mathematics itself, on which the *field* has influence. An *individual's* personal background and position within the *domain* may influence the likelihood of making or not a contribution. For example, if the *individual*, in this case the person submitting a contribution, is a mathematician who is involved in research, he or she is more probable to produce research articles, since he or she has more opportunities for reflection, discussion and exchange of ideas (Sriraman, 2004). The condition that the creative process only occurs when the *individual* proposes a change

within a given *domain* attributes a fundamental role for creativity: it would be impossible to introduce a change without referring to an existing model (Csikzentmihalyi, 2000).

The *Evolving systems approach* (*ESA*), proposed by psychologists Howard Ernest Gruber and Doris B. Wallace (2000) considers each individual as a unique system of evolving creativity and ideas. The model aims to investigate the evolution of creative work, leaving aside the identification of the origins of creativity or the characteristics of the creative individual. By studying people's behaviour when a creative process is in progress and how they use the available resources to realise something unique, creative work is defined as the result of intentional behaviour that may take several years (months, years, decades) to achieve something new and meaningful.

According to the *Investment theory approach*, theorised by psychologists Robert Sternberg and Todd I. Lubart (1996), creativity is defined as the final, new and original product resulting from the interaction between person, task and environment. This product is unique and is regarded as the convergence of six elements (Sriraman, 2004):

- *Intelligence*, in the sense of analytical and practical skills and the ability to generate new ideas suited to the task;
- *Knowledge*, understood as knowing enough about a particular topic to the extent that it can be further developed and deepened;
- *Thinking style*, understood as the ability to think in an original way and to be able to distinguish important from less important questions;
- *Personality*, which promotes creative functioning, representing the willingness to take risks, overcome obstacles and tolerate ambiguities;
- *Motivation* and environment, fundamental elements supporting creativity.

According to this model, a creative individual is a good investor who buys at a low price and resells at a high price: he evokes ideas that are unpopular or underappreciated and invests time in convincing other people of the intrinsic value of these ideas. Influenced by the work of Sternberg and Lubart (1996; 2000) who define creativity as the ability to produce original, unexpected, useful and adaptable work, Sriraman (2004) proposes his own definition of creativity with reference to the field of mathematics education. Specifically, he defines mathematical creativity as the ability to produce new and original work through a

process that originates unusual and acute solutions, regardless of the level of complexity of the problem being solved.

In general, studies concerning mathematical creativity focus mainly on the creative process of the mathematician engaged in solving mathematical problems of varying nature and complexity. The first attempt to study the mathematician's creativity dates back to 1902, when an extensive questionnaire was published in the French periodical *L'Enseigement Mathematique*, which, together with a lecture on creativity by the famous mathematician Henri Poincaré, inspired the mathematician Jaques Hadamard to investigate the psychology behind mathematical creativity (Sriraman, 2004). To define mathematical creativity, Poincaré resorts to the metaphor of choice, according to which creating consists in not making useless associations, in other words, creating consists in the mathematician's ability to accurately choose and select questions or problems that can help make appropriate connections and that can lead to something new. Later, in his investigation into mathematical creativity, Hadamard (1945) involved prominent mathematicians and scientists of the time, including George Birkhoff, George Polya and Albert Einstein. By collecting the perceptions of the scientists involved about the mechanisms through which they produce new mathematics, he outlines the existence of unconscious mental processes, attempting to grasp the phenomenon of mathematical illumination (Lo Sapio, Carotenuto, Coppola & Mellone, 2022). Hadamard, influenced by research in psychology, uses the four stages of the *Gestalt Model* (Wallas, 1926) to theorise creative processes in mathematicians:

- *Preparatory stage*: in this first stage, the mathematician reflects and works on the problem;
- *Incubatory stage*: the problem is set aside and the mathematician concentrates on other issues;
- *Illuminatory stage*: at this stage, the mathematician, while involved in other activities, not necessarily related to the initial problem, is caught by the phenomenon of mathematical illumination and the solution to the problem suddenly appears;
- *Final stage*: consistency is given to the result obtained, which is verified and expressed through language or writing.

The American educational scientist, J. M. Rhodes (1961) is also inspired by the ideas of Wallas (1926) which lead him to develop the *4Ps Model* (Figure 21) to explain the factors that, according to him, constitute the creative system. He identifies four strands or perspectives on creativity that he calls the 4Ps of creativity: *Person*, *Process*, *Product* and *Press*.

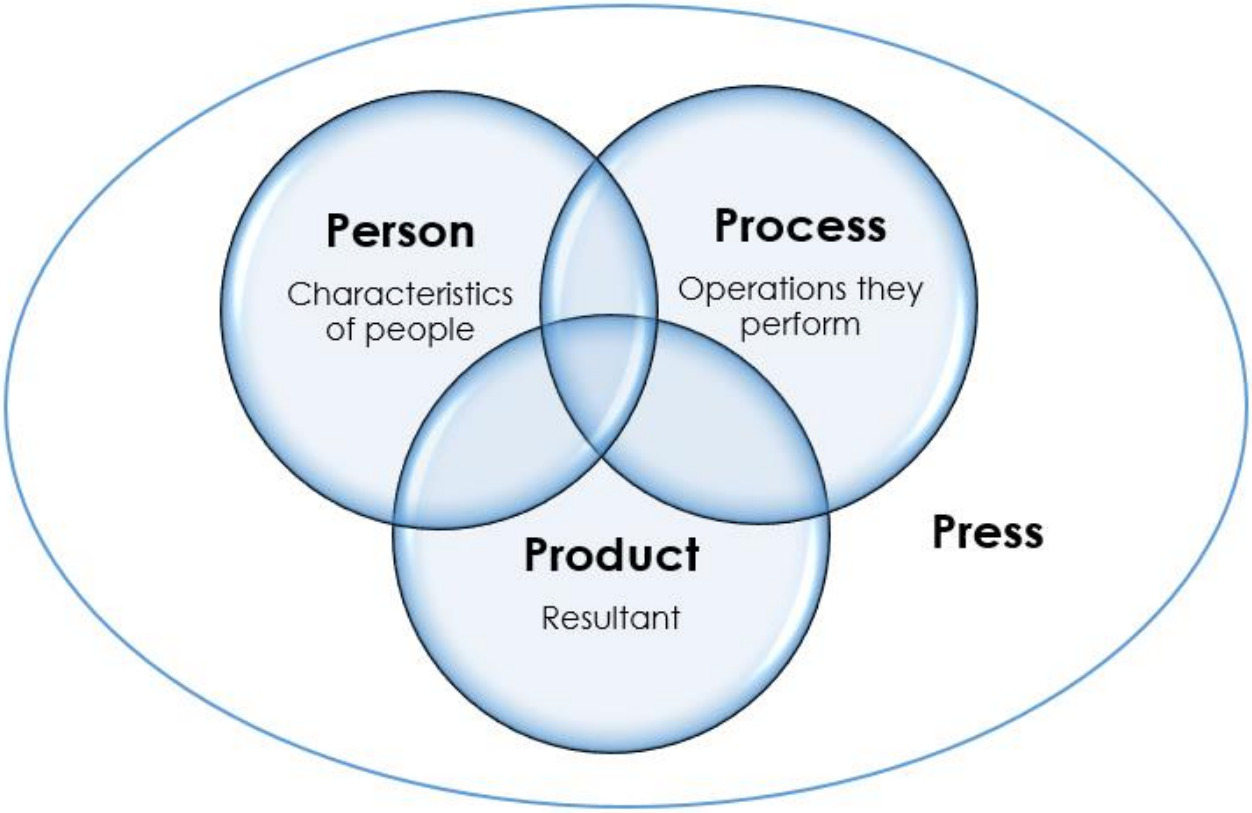


Figure 21. Schematic representation of the four strands underlying the 4Ps Model developed by J.M. Rhodes (1961).

The Person area includes the characteristics of creative people: personality, traits, habits, attitudes, value systems, defence mechanisms and behaviour. The *Process* area includes the processes used by creative people and the way they think, learn and communicate. The *Product* area, on the other hand, refers to the results obtained, what is achieved when an idea materialises. Finally, considering the origin of the term *Press*, which derives from the Latin word *pressus*, meaning box or container, the *Press* area represents the environment in which the 3Ps coexist and the element that helps or hinders creative expression, also evoking the idea of press as an element that imparts pressure. These four strands are not separate from each other, but coexist and operate in a functional manner (Rhodes, 1961). In this view, creativity is seen as a characteristic of a *Person*, a *Process*, a *Product* and a *Press* (Joklitschke et al., 2021). According to Rhodes (1961), in fact, a creative individual

possesses mental flexibility, is sensitive to problems, is able to think in a divergent way, and is also able to redefine existing objects and concepts; but this creative capacity is also influenced by the surrounding environment, which influences the way ideas are generated.

It is interesting to note how in some of the proposed models (Csikszentmihalyi, 1988 and 2000; Rhodes, 1961; Sternberg & Lubart, 1996), the cultural and social dimension of human activity is recognised as a powerful tool in the creative process. These considerations inevitably lead one to reflect on the role that these components play with regard to creative thinking and the ability to produce new and original ideas. In the research literature, however, creativity is defined in different ways and currently, there does not seem to be a commonly accepted definition (Haylock, 1987). In particular, the studies on mathematical creativity known in the literature refer mainly to the mathematician's performance when solving a new task. Kiesswetter (1983) states that, in his experience, flexible thinking is a fundamental component of creativity, and one of the skills a successful problem solver should have. The Finnish neurophysiologist Matti Bergström (1984), defines creativity as "performance where the individual is producing something new and unpredictable" (Bergström, 1984, p.159), introducing the concepts of "everyday creativity" and "Sunday creativity". The former concerns the search for new associations, which can be expected, between already known elements. “Sunday creativity”, on the other hand, cannot be predicted, nor can it be achieved by mechanical methods, since it is considered a type of spontaneous and unintentional creativity. From this perspective, mathematical creativity is defined (Pehkonen, 1997) as a combination of logical thinking and divergent thinking, the latter being more intuitive but with a conscious purpose. In the concrete situation of problem-solving, Pehkonen (1997) observes an initial exploration phase in which the mathematician proceeds with random experimentation that gradually moves in a specific direction. On the basis of the experiments carried out, the mathematician formulates a hypothesis which he or she tries to prove. In this case, divergent thinking helps the mathematician to produce many ideas, some of which may be useful in finding a solution. The creative process is considered an essential part of doing mathematics (Pehkonen, 1997). Ervynck (1991), trying to describe the process through which ideas and questions evolve in the mathematician, proposes a model in terms of three stages:

- *Stage 0*: preliminary technical stage, in which mathematical rules and procedures are applied without any awareness of the theoretical foundation;

- *Stage 1*: algorithmic activity consisting of the execution of mathematical techniques as an explicit and repeated application of an algorithm (Sriraman, 2004);
- *Stage 2*: creative phase consisting of a non-algorithmic decision process.

Ervynck's description of mathematical creativity appears to be very similar to Poincaré's: Ervynck's non-algorithmic decision process echoes Poincaré's metaphor of choice.

Subsequently, Liljedahl (2004), updating the work of Hadamard (1945), studies the same type of beliefs, involving outstanding mathematicians of present time and using the modern tools of mathematics education. Analysing the logbooks of pre-service teachers, anecdotes and reflections of university mathematics students and prominent mathematicians, he focuses on mathematical problem-solving, confirming the phenomenon of mathematical illumination (Hadamard, 1945), perceived as caused by the sudden arising of an idea. In more recent studies (Levenson, 2021), mathematical creativity is defined as a disposition that promotes the ability to generate different solutions and paths, but also to change direction of attention and produce new and original solutions. In this view, mathematical creativity is associated with three components: fluency, flexibility, and originality. Specifically, the generation of ideas and potential approaches to find solutions for a given problem is considered a sign of fluent thinking (Mann, Chamberlin & Graefe, 2017). The ability to readapt and change direction or objective, the use of different representations, and the ability to connect different branches of mathematics are seen instead, as signs of flexibility (Leikin, 2009). Finally, the level of intuition, conventionality and novelty of a proposed solution provides a measure of its originality.

The present research work aims to explore a different kind of mathematical creativity from the one intended in the reported studies, but which needs to be framed within a broader context in order to be defined. As discussed in detail in the following section, the mathematical creativity under investigation is that activated by teachers in the process of design and implementation of mathematics teaching activities, in line with the principles of active and popular pedagogy and Freire's pedagogical ideas.

4.2 The kind of mathematical creativity involved

4.2.1 Starting points

In recent years, the Italian school has been undergoing continuous and numerous changes, mainly due to institutional factors and their not always positive practical implications. In the previous chapters, we reflected on the role that schools play in today's society, in terms of equity, inclusion and fair opportunities. The Covid-19 pandemic has revealed even more clearly the inability of school education to properly include pupils from disadvantaged contexts (Lo Sapio, et al., 2022), underlining how, especially in places of severe socio-cultural deprivation, schools do not always have sufficient resources to accommodate and include children and adolescents in difficulty (Carotenuto, Mellone, Sabena, & Lattaro, 2020). The theories and experiences in the field of active and popular pedagogy, discussed in the previous chapters, lay the foundations for a new idea of school, which starts from the needs and interests of pupils and considers their relationship with the world as central. The "new" school, attentive to democratic ideals and responding to the specific needs of the territory, assumes a crucial role from both a social and educational point of view, contrasting phenomena of marginalisation, inequity and social exclusion and contributing to the personal, intellectual and moral development of the pupil. In actual factual terms, however, the school does not always appear to be close to this vision: in many cases, the school results of the younger generations are still largely determined by the socio-economic background of their parents rather than by their potential and, in the suburbs of many cities, the school, by failing to adequately include pupils who are in conditions of socio-cultural hardship, ends up losing its institutional value in the eyes of pupils and their families (Lo Sapio, et al., 2022). In these cases, the risk of school drop-outs and the danger for young people of being approached by organised crime gangs becomes increasingly alarming. For this reason, it is crucial to improve educational contexts to prevent and contrast this phenomenon and to try to (re)build through teaching practice a positive attitude towards school and, in particular, towards mathematics (Di Martino & Zan, 2010). In socially disadvantaged contexts, this means proposing pupils tasks that are meaningful in their eyes and improving their perceived competence in mathematics, putting in place the conditions for them to create their own identity within the school. In this regard, the questions already explained, concerning the ways in which to design inclusive didactic paths that are effective and feasible even in difficult social contexts, find a possible answer in Freire's educational proposal, which finds in the problem-posing practice and in the search for the

"game of the people" the key to the realisation of a teaching design that is motivating for the students and, at the same time, responsive to the socio-cultural context in which they are involved. The search for the "game of the people" is fundamental for the design of teaching activities, even more so in those disadvantaged contexts in which it appears necessary to create and design effective and meaningful teaching paths, selecting methodologies and teaching strategies in line with the educational proposal to be offered.

4.2.2 Mathematics education creativity

In the research literature, there are numerous testimonies of actions and didactic paths in the field of mathematics education aimed at improving attitudes towards mathematics (Di Martino & Zan, 2010) and developed to also involve students experiencing severe socio-cultural disadvantage in a genuinely inclusive perspective. Starting from this assumption, it was decided to focus on mathematics education projects considered to be effective and which, from the point of view of active and popular pedagogy, appear to be highly inclusive and capable of critically connecting students to the territory in which they live. These projects oriented the research interest towards the teaching practice of teachers, focusing on their processes of design and implementation of mathematics teaching paths in line with the principles of active and popular pedagogy, also in terms of researching the "game of the people". With these assumptions in mind, the research action chose to explore teachers' mathematical creativity, which is different from that analysed so far. The studies referred to in the first paragraph, concerning creativity in general, allow framing the research object in a broader context that helps to clarify what is meant by teachers' mathematical creativity. First of all, it is worth emphasising that I have chosen to refer to creativity because what we want to investigate is precisely the process of creation and the capacity to produce work that is new and original, but also useful and adaptable, in the sense proposed by Sternberg and Lubart (2000) and extended by Sriraman (2004). Whereas the ESA model was limited to investigating the evolution of the creative process, the present research work aims to investigate the sources of creativity, attempting to outline its characteristics. The reference to the mathematical field and the focus on teachers came from the fact that, at first, the interest was aimed to explore the creativity enacted by teachers in the design of mathematics teaching activities that were in line with the principles of active and popular pedagogy and with the ideas underlying Freire's pedagogical proposal. However, recent studies on mathematical creativity (Levenson, 2021) have added to the ability to generate new and original solutions, also the ability to identify and

change direction, signs of fluent thinking (Mann et al., 2017) and flexibility (Leikin, 2009) respectively. In the light of these considerations, the interest in teaching design is combined with the interest in the implementation of the teaching activities designed, which contains within itself the ability to readapt and to consider new directions, taking up the concepts of flexibility and fluidity. Thus, while, on the one hand, the teacher invents, creates and designs inclusive mathematics teaching activities, on the other, the teacher also implements them, taking into account the possibility of changing direction, and initiating an on-going dialogue with the students and the areas in which they live.

With this in mind, interest has developed in searching for a possible intertwining of the highly engaging design approach of these educational projects with curricular teaching.

Therefore, the mathematical creativity attempted to be captured is what has been deemed more appropriate to call *mathematics education creativity*, meaning, in the sense just explained, the kind of creativity required for teaching in challenging contexts, and elsewhere. By taking a closer look at the process of teaching design and implementation, this research work tries to answer the following question:

> *What are the aspects that make it possible to characterise mathematics education creativity?*

Therefore, with the aim of outlining and describing possible aspects that characterise *mathematics education creativity*, it was decided to interview teachers-researchers, who are considered to be active on both the educational and social level and involved in the design of teaching activities, in the sense specified previously. The interview questions, analysed in detail in the following chapter, are designed with the aim of investigating the motivation that each teacher attributes to the origin of his or her creative process. In particular, taking up the concept of mathematical illumination, introduced by Hadamard (1945) and further explored by Liljedahl (2004), the unconscious mental processes are explored in an attempt to grasp the causes of the sudden arising of an idea, no longer with reference to the solving of mathematical problems, but with reference to the design and implementation of mathematics teaching activities. In addition, since the recent literature in psychology recognises the social component of human activity as a powerful tool in the creative process, through the interview questions, an attempt is made to guide teachers' reflection on the role that personal, social and cultural components occupy in relation to their design and teaching action choices.

CHAPTER 5

METHODOLOGY

"Always remaining loyal to a vision of history as possibility, and always firm in the hope that a less discriminatory, less dehumanising, more just and humane world is something possible [...]"
(Freire, 2018, p.9)

In the present chapter, the methodology considered most effective for attempting to answer the research question and characterise *mathematics education creativity* is described. In particular, with the choice of conducting interviews, the following paragraphs explain the motivations that led to the selection of the participants, who are also presented in relation to evidence of mathematically significant actions and didactic paths and in line with the educational perspective outlined in the previous chapters (Chapter 1 - 2). Next, the methodology used to structure the interview and elaborate the questions is clarified and then, the methodology used to analyse the data collected is presented.

5.1 Participants

In order to investigate the particular kind of creativity of the mathematicians involved, I chose to interview mathematics teachers from different school grades who were considered to be active both on the creative level, with reference to the strictly intertwined processes of designing and implementing mathematical education activities, and on the social level, that is, responding to a specific context, often difficult from a socio-economic and cultural point of view. The choice of each participant arose from the consideration of testimonies, designs and teaching experiences that I considered innovative and significant from a mathematical point of view and strongly inclusive in terms of active and popular pedagogy. In fact, the teachers interviewed are all teacher-researchers with some familiarity in the field of education and considerable experience in the field of mathematics education. In particular, they are teacher-researchers who are or have been involved, at different times in their training and teaching career, in teaching experiences lived in disadvantaged territories and socio-economic-cultural contexts that required them to

design and implement highly inclusive mathematics teaching activities in line with the vision of active and popular pedagogy. In accordance with the research goal, the choice of participants is aimed, through the proposed interview, to have access to different experiential content regarding the processes of design and implementation of mathematics teaching activities, in relation to which an attempt is made to identify meeting points and common characteristics.

The first didactic project taken into consideration arose in the context of the cultural association "Matematici per la Città" (MPC), founded in 2013 by a group of mathematics graduates from the University of Naples "Federico II", with the aim of realising *urban mathematical walks* (Figure 22) among the charming and fascinating streets of Naples, bringing mathematics out of the classroom, in a less formal, more comfortable and enveloping scenario.

Figure 22.
The image - taken from (Matematici Per la Città, 2019) - shows some moments of a mathematical walk documented on the official MPC page. The pupils, solving mathematical questions while walking around the city of Naples, go in search of the egg of the siren Parthenope, finding it at the Virgiliano Park.

Currently, MPC collaborates with several primary and secondary schools in the Naples area, often located in disadvantaged districts of the city. The association promotes curricular and extra-curricular educational courses with the aim of stimulating and encouraging children and young people to observe reality with curiosity and a critical sense. One of the main characteristics of the projects proposed by MPC is the intertwining of mathematics and other disciplines, such as art, music, history, science, and above all, education for active citizenship. With reference to this project, I chose to involve the persons responsible for the

didactic design of the mathematics activities: Annunziata (Nunzia) Di Maria, Paola Lattaro and Teresa Gianni, secondary school teachers in Naples and members of the MPC association. The teachers interviewed are teacher-researchers who promote, through MPC's activities and their classroom practice, a highly inclusive didactics that puts pupils and their concrete experience at the centre. During the period of the pandemic, on the occasion of the celebration of the 700th anniversary of Dante Alighieri's death, thanks to the collaboration with cooperatives and associations for the fight against child educational poverty, MPC, together with adolescents from various schools in Naples, created a long and laborious process that led to the realisation of the "Tombola Infernale", a special declination of the famous Neapolitan game in which the tradition of the classic Neapolitan tombola was able to intertwine with mathematics and the Inferno narrated by Dante.

The second project taken into consideration, "Next-Land", is an Italian informal education project that offers very innovative mathematical problematising education activities to secondary students from socio-culturally disadvantaged areas in the city of Turin. "Next-Land", was created as an innovative didactic project promoted by the Next-Level cultural association and supported by the Intesa San Paolo Benefit Fund, Enel Cuore Onlus, the Vodafone Italia Foundation, the Fondazione Compagnia di San Paolo and the Fondazione CRT. The project, which started in October 2020 and lasted two years, involved over a thousand secondary school students per year from the most disadvantaged Turin's neighbourhoods, offering them educational courses on STEM disciplines (science, technology, engineering and mathematics). The project, whose transversal objectives include combating the gender gap in study and work contexts, is characterised above all by an educational laboratory action strongly intertwined with art and artistic disciplines. For the mathematics teaching activities, the Department of Philosophy and Educational Sciences of the University of Turin and the University of Naples "Suor Orsola Benincasa" collaborated with the education departments of the museums of the city of Turin. The educational activities, hosted in the city's museums, open spaces, research centres and digital spaces, made a significant cultural contribution, helping to potential positive perceptions of STEM subjects, especially in pupils from disadvantaged backgrounds (Next Level, s.d.). With reference to this project, I chose to interview Chiara Pizzarelli and Raffaele Casi, two of the researchers involved in the design of the mathematics teaching activities for the "Next-Land" project. Chiara and Raffaele, secondary school mathematics teachers in Turin, designed, with the mathematics didactics design team, supervised by Cristina Sabena - University of Turin and Maria Mellone - University of Naples, "Federico II", four mathematics laboratory activities, each conducted in a museum in the city of Turin. One of

the four planned activities was organised at the PAV, Parco Arte Vivente, an experimental centre for contemporary art which, in addition to offering an interactive museum space, also has an open-air exhibition site (PAV, s.d.). The activity, inspired by the short story "Flatland", was conceived as a pathway between art, nature and ecology, in which the students were invited to explore the *Trèfle* cloverleaf and Piero Gilardi's *Labyrinth Anthropocene*, some of the installations in the PAV open space (Figure 23).

Figure 23.
The image - taken from (Next Level, s.d.) - shows students during one of the explorations at PAV. In this immersive exploration phase, the students move and walk freely in order to sense the shape of the installations and create a map and a three-dimensional model of the plant labyrinths.

The two plant labyrinths are perceived as a three-dimensional world whose perimeter shape is difficult to grasp immediately. Through a free and immersive exploration, which allowed them to experience the space and at the same time test the students' orientation, visualisation and spatial representation skills, the students were asked to make maps and models, moving from observing reality from different points of view to two- and three-dimensional representations of it (Figure 24).

Figure 24.
In the image - taken from (Next Level, s.d.) - students assemble pre-built components to make a three-dimensional model of the labyrinth. This last phase allows a reflection on the importance of such a tool to more easily find a way out of the labyrinth.

The second of the four planned activities took place in the National Museum of the Italian Risorgimento in Turin, located inside Palazzo Carignano and dedicated to the Italian Risorgimento, offering a variety of paintings, documents, prints, manuscripts and sculptures (Museo Nazionale del Risorgimento Italiano, s.d.). During the activity, the students, in the role of members of a Carbonara society conquering Risorgimento freedoms, were able to learn different cryptographic methods and solve cryptographic enigmas (Figure 25).

Figure 25.
The image - taken from (Next Level, s.d.) - shows a small group of students intent on discovering, by means of a perforated grid, the hidden message within the 1821 Declaration document.

Through a chronological tour of the museum's rooms, ending with a treasure hunt (Figure 26), codes and messages were deciphered, the decoding of which allowed for the conquest of different types of freedom: of association, of the press, of thought and expression, of movement, of voting, giving the students the opportunity to dialogue and discuss the theme of freedom, rights and duties (Casi, Leo, Pizzarelli & Sabena, 2022).

Figure 26.
The image - taken from (Next Level, s.d.) - shows the scìtala, a stick around which the ribbon, originally a strip of leather, is wrapped in order to reveal the hidden message.

The third planned activity took place at Palazzo Madama, a historical building in the city of Turin, built in 1861 with the aim of collecting and preserving the region's artistic heritage. The building houses the collections of the Museo Civico d'Arte Antica and offers visitors an itinerary dedicated to the history of the rooms and another focused on the significance of the works of art exhibited there (Palazzo Madama, s.d.). The laboratory-visit to the museum was articulated in the search for spirals, present among the architectural structures and decorative elements of Palazzo Madama (Casi, Leo, Pizzarelli & Sabena, 2022) (Figure 27).

Figure 27.
In the image - taken from (Next Level, s.d.) - students are positioned along a circular shape at Palazzo Madama. Moving in groups, the students explore translations and rotations with their bodies, until together they form a spiral.

Walking freely through the rooms of the Palazzo, which houses collections from the Middle Ages to artistic testimonies from the Gothic period and the Renaissance, from Baroque rooms to decorative arts from various historical eras, the students searched for spirals and spiral staircases, discovering the geometric transformations of translation and rotation through perceptual-motor experiences (Casi, Leo, Pizzarelli & Sabena, 2022), and then challenging themselves in their reproduction, both in the plane and in space, and their construction using mathematical machines (Figure 28).

Figure 28.
In the image - taken from (Next Level, s.d.) - a student reproduces the circular spiral staircase observed in the Museum, through the rotation and translation of wooden dowels: rotation around the centre of the base plane and translation along the central axis.

The fourth of the planned activities was conducted in the Egyptian Museum in Turin, one of the largest and most comprehensive museums in the world dedicated to Egyptian civilisation (Museo Egizio Torino, s.d.). The artisans' village of Deir-el-Medina provided the backdrop for the laboratory tour in the museum rooms. The laboratory-visit allowed a first approach to the Egyptian numerical system (Figure 29), through which similarities and differences with the Indo-Arabic numerical system used by the students were identified.

Figure 29.
In the picture - taken from (Next Level, s.d.) - a student consults a bookmark on which hieroglyphs with their respective translation into Indo-Arabic numerals are displayed.

The discovery of the Egyptian algorithms of addition, subtraction and multiplication allowed the students, divided into groups and in the roles of scribes, labourers, painters and foremen, to calculate their weekly wages and then choose whether or not to strike (Figure 30).

Figure 30.
The image - taken from (Next Level, s.d.) - shows a student solving operations by applying Egyptian algorithms.

The design for the laboratories presented, which led to the choice of interviewing Chiara Pizzarelli and Raffaele Casi, was considered highly interesting due to the obvious connection between mathematical argumentation and other disciplines. In a perspective of informal mathematical education, starting from architectural elements and installations in museum rooms, the mathematical laboratories involved students in activities of exploration, discovery and manipulation, in which mathematical questions were interwoven with aspects of citizenship education (Casi, Leo, Pizzarelli & Sabena, 2022). In the four laboratory activities mentioned, it is evident how mathematical objectives were mixed with objectives concerning education in art, history, culture and active citizenship, in the spirit of creating an attitude of curiosity towards the artistic world and developing a profound dialogue between the various disciplines involved.

The decision to involve Franco Lorenzoni and Marina Spadea in the interviews came from the consideration of mathematics education experiences in line with the "Movement of Educational Cooperation" (MCE). This Movement was born in Italy in 1951 and continues to involve teachers whose design choices and teaching practice reflect the pedagogical and social thinking of the French teacher Célestin Freinet. Marina Spadea is an active member

of the Movement of Educational Cooperation. She was a kindergarten teacher in Naples and still a tutor for future primary teachers (undergraduate students of the University course for becoming primary teachers). Over the years she has been involved in teaching and research in mathematics education, also accumulating experience in the field of teacher training. The decision to involve her in the interviews was motivated by the great originality shown in the design of her teaching actions, which are strongly influenced by the ideals of MCE and the particular way of doing school towards which they are directed. The testimonies of Marina's didactic designs and implementations show a particular attention to the creation of opportunities for listening and confrontation between students, aimed at enhancing the creativity and expressive freedom of each pupil, in an inclusive, democratic perspective that is sensitive to the socio-cultural context.

Franco Lorenzoni is also an active member of the Movement of Educational Cooperation. A pupil of Emma Catelnuovo, Franco was a primary school teacher in Giove, Umbria, where he founded and currently coordinates the Casa-Laboratorio in Cenci. It is a centre for educational experimentation that, through activities for adults, young people and children, interweaves scientific discourse with ecological, intercultural and inclusion issues. The educational opportunities offered by Casa-Laboratorio in Cenci take the form of residential stages for educators and teachers of all levels and school camps for primary and secondary school girls and boys. In the training stages for educators and teachers, the laboratories offered have different objectives and involve various themes. In the school camps for young people, through the valorisation of movement and sensitive experiences, the proposed ecological paths aim to trace possible correspondences with the elements of the planet and the cosmos. The activities are organised in small groups and the construction of knowledge takes place together - as a practice of democracy - through collaboration, comparison and sharing of experiences (Figure 31).

Figure 31.
The picture - taken from www.cencicasalab.it - shows children engaged in one of the activities promoted within Casa-Laboratorio in Cenci.

The activities are designed not only from a democratic perspective, but also from an integration perspective. Indeed, the designs take into account the singularities of each participating pupil from a genuinely inclusive perspective. These aspects also represented cornerstones in Franco's school practice, strongly inspired by his experience with Emma Castelnuovo, representing a motivation for the decision to involve him in the interview. The classroom, and therefore the school, is understood by Franco as a place of encounter where it is possible to foster inclusion and break down prejudices. A process that requires effort and changes, both physical and in intentions: arrangement of desks, use of time and materials, valorisation of pupils' experiences, ideas and proposals.

The choice of interviewing Giancarlo Artiano, Marco Marino and Maria Pezzia was motivated by their significant experience in educational design in socially disadvantaged contexts. Giancarlo Artiano, a mathematics and science teacher in a middle school in Naples and currently a PhD student in mathematical engineering and physics at the University of Campania "Luigi Vanvitelli", has taken part in numerous projects to combat school drop-out and educational poverty. Since 2018, he has in fact been involved in several educational pathways with this purpose and which use science subjects as a tool to reach students, school communities and families, with the aim of trying to give a better and different future to those who, for economic, social and cultural reasons, tend to drift away from school. His commitment through these projects has focused on the historical centre of Naples, Sanità. The Sanità district represents one of the popular and folkloristic areas of Naples, a theatre of Neapolitan history, tradition and culture, often victim of unfair prejudice. While the district boasts the presence of numerous historical and artistic heritage sites, it also has areas of urban decay, often associated with situations of economic and social poverty. Through the involvement of a number of schools in the Sanità district and a number of third sector associations - from theatre to sustainable mobility - Giancarlo's educational and research work has involved projects with both educational and social aims. In these contexts, he has taken part in territorial education laboratories for children identified as socio-economically fragile, acting with scientific educational activities both at school - formal context - and in territorial educational centres - informal context. In parallel to the students, Giancarlo's educational experience also involved the students' families and teachers' training. On these occasions, practices, strategies and teaching activities that worked positively with students were shared with the adult component of the educational world, in a spirit of collective comparison and reflection.

Marco Marino, currently a maths and science teacher in a middle school in Naples, has worked for several years in Scampia, one of the most densely populated suburbs of Naples, located in the far north of the city. Scampia, which has one of the highest unemployment rates in Italy, is sadly known for the presence and activity of organised crime that has been operating in the area for years, negatively affecting the social and cultural context of the territory. Within the neighbourhood there are several Roma camps, often illegal and degraded from a sanitary point of view, and in many cases also occupied by Italian families (Del Giudice, 2009). In the period preceding his career as a teacher, together with a group of citizens, free from any third sector association, Marco built the "Scola jungla" (that means jungle school) in that place, a space - now no longer in existence - dedicated to designing educational interventions and welcoming Roma and Neapolitan children. These children, who were often excluded at school and isolated at the last desks, at the "Scola jungla" were instead involved in laboratory activities, inspired by the principles of active schooling and the pedagogy of Freire and Freinet. The "Scola jungla" not only represented an almost survival necessity in that context, but also contributed to the educational development and identity-building of the children, through activities that were highly responsive to the context. In his first years as a teacher, Marco continued to work in Scampia, and then moved to a middle school in another area of Naples, where he currently works. The experience with the "Scola jungla" was recognised by Marco as highly formative and inspirational for his subsequent choice to become a teacher, characterised by a desire to put into practice, in a more formal context, project ideas, reflections and activities that had worked there. Currently, in his school practice with his students, Marco uses Emma Castelnuovo's textbooks, favouring an active, laboratory-based didactics that uses mathematics to involve other topics - for example, social injustice, architecture, art.

Maria Pezzia is a primary school teacher involved in research in mathematics education, with significant teaching experience in challenging contexts, in her case, at primary school level. Before she became a teacher, Maria worked for several years at the department of primary education, collaborating on the chairs of mathematics education and also dealing with teacher training. She was trained as a Montessori and taught for a few years in a public school inspired by Montessori's pedagogical vision, where she was able to put into practice the principles of the Montessori method, strongly motivated by the context. For the past two years, however, she has been a teacher in a primary school in the Barriera di Milano district, an old district of Turin, located on the northern suburbs of the city centre. The neighbourhood owes its name to its having been a border area due to the presence of a

passage, guarded by barriers - for this reason "Barriera" -, facing the city of Milan and which allowed entry into the city only after customs control. Today, Barriera di Milano is characterised by hardship and poverty and is predominantly populated by foreigners. Even the school where Maria teaches is attended by very few Italian children and mostly by students of North African or Nigerian origin. The decision to involve Maria was determined by her many years of experience in mathematics education. Her design choices and classroom practice take into account the particular context of the students they address, trying to break down geographical, ethnic and, more generally, social stereotypes through mathematics. With the period of the covid-19 pandemic, Maria chose to have a new experience at school and work on support, creating small groups of students in which the use of the body and movement for learning mathematics is emphasised.

As mentioned at the beginning of this paragraph, all the teachers were chosen for the originality they demonstrated in their teaching design, both in terms of their design and methodological choices, in the perspective of active and popular pedagogy, and in their intention to engage, through mathematical argumentation, with social and cultural issues close to their students' contexts, in line with the intertwined development of mathematical competences and critical consciousness promoted by *critical mathematics education*.

5.2 The Explicitation Interview method

In recent years, the use of written and oral narratives has become increasingly relevant in the field of social science research, representing a valuable investigative and observational tool (Lieblich, Tuval-Mashiach & Tamar, 1998). Narrative, understood as an organised discourse designed to refer to a series of interrelated happenings (Dictionary, 1966) is also often used in the field of mathematics education research, both as an object of study and as a vehicle for exploring other issues. With reference to the present research work, it was felt that narrative, in the sense explained, could provide access to elements and information useful for identifying and describing any visible connections between the process of teaching design and implementation and aspects of mathematical creativity in teachers. Therefore, in order to analyse narrative material, I chose to resort to the semi-structured interview and the recording and transcription of the resulting narratives. In particular, the methodological tool I considered most appropriate to use is the Explicitation Interview method (Vermersch, 1994), conceived as a form of guided introspection capable of

creating, in the interviewee, the conditions for the possibility of an awakening provoked by the recollection and evocation of past experiences (Maurel, 2009). The examples of explicitation interviews reported in Maurel's work (2009), applied to different contexts, show how this methodological approach can indeed represent an effective investigative instrument. If, on the one hand, such a tool allows the interviewer-researcher to collect a new field of data through which to analyse and explore the subjective experience of the interviewee, on the other hand, it allows the interviewee to reflect a posteriori on the shared and evoked experience, accessing implicit or already explicit information and beliefs. Specifically, this methodology is based on techniques of formulating relays aimed at facilitating and assisting the a posteriori verbalisation - in the sense of putting into words - of a particular experience (Vermersch, 1994). Through the intervention and the questions asked by the interviewer-researcher, in fact, the interviewee can access a pre-reflective knowledge, in other words a knowledge of which he or she is not yet aware but which he or she tries to explore and make explicit. The interviewee, through first-person narratives, recounts, in his or her own words and from his or her own perspective, personal experiences that he or she considers relevant, becoming aware, through reflection, of the connections of meaning between discontinuous moments in his or her career (Faingold, 2001). First-person narration, in accordance with Vermersch's view, becomes an embodied speaking that allows one to enter into a regime of expression that relates concrete reality with what one is experiencing at the moment one is formulating it (Depraz, 2011). The choice of this research methodology comes precisely from the desire to bring to light the moments of professional teaching practice that the teachers interviewed consider most significant, allowing aspects and issues that matter most to them to emerge. By means of appropriate questions the intention is to open up a space that allows the teachers interviewed to reflect on the processes connected with the planning and implementation of mathematics teaching activities with the aim of grasping and outlining traits and characteristics of *mathematics education creativity*, in the sense explained in the previous chapter. While on the one hand, an attempt is made to bring out and analyse particular experiences of the interviewed teachers, an attempt is also made to grasp and understand the way in which each teacher has processed these experiences and the repercussions these have had on his or her training and teaching practice over time. Between capturing aspects and details of the lived experience and understanding its repercussions over time, narrative enquiry is composed of two narrative regimes (Breton, 2020): a biographical regime (Baudouin, 2010) which, through a work of "time compression" and "smoothing effects" that eliminate details in the description, allows for the capture of the lived experience over time, and a micro-

phenomenological regime (Depraz, 2012) which consists of a granular and detailed description of events and an exploration of short sequences of experiences. As will be discussed in more detail in the following section, the proposed teacher interview tries to involve both regimes. Questions referring to more general situations, possibly placed at different temporal moments or referring to a broader time scale aim to bring out the biographical component in the narrative. The request to retrieve and bring to light anecdotes or to reflect upon particular and specific moments aims to obtain a more meticulous and detailed narrative and thus, to bring out the micro-phenomenological component in the narrative.

Although the questions for teachers are elaborated after deep and careful reflection, this kind of oral and semi-structured interview also gives space to more open and flexible moments that lead to different directions and developments, depending on the interviewee.

5.3 The Interview

The preparation of the initial structure of the interview and the drafting of the questions for the teachers required a reflective work, oriented to bring out from the narratives of the interviewed teachers possible connections between creativity and the process of teaching design and implementation. In this regard, I chose to use open-ended questions, for which the different specific techniques of the explicitation interview were taken into account, trying to foster the conditions for the reflective act and the emergence of a narrative. By proposing to the teachers interviewed to reconnect with past experiences, possibly referring to specific situations or anecdotes, the aim is to involve both the biographical and micro-phenomenological regimes, in the sense described in the previous paragraph. With these assumptions in mind, considering the research work of Paolo Freire (Freire, 1998) as central, and wishing to be responsive to the didactic vision proposed by active and popular pedagogy, an interview was devised for the teachers, concentrated in four questions.

> Q1. *In designing a teaching activity, do you feel more like ideas emerge all of a sudden or that they originate from a process of investigation?*

The first question, Q1, allows an initial exploratory study of teachers' mathematical creativity. In detail, Q1 is inspired by a particular question used by Hadamard (1945) and later by Liljedahl (2004). This question refers to the dichotomy, already emerged in the community

of mathematicians at the beginning of the 20th century, between a vision of mathematical activity that emphasizes intuition and another one that tends to emphasize the rigour and formalism of deductive reasoning (Lo Sapio et al., 2022). In this context, in which we are reflecting on the design of teaching activities in mathematics, the investigation is whether teachers attribute their creative acts to a spontaneous arising of ideas or to a work of reflection that could be considered more methodical and goal-oriented. The question is oriented to proceed through a biographical narrative and is intended to make each teacher reflect on his or her own experiences. With this question the aim is to collect teachers' perceptions of how the creative acts underlying their teaching work come about.

> Q2. *Referring to your design choices, do you think that, in general, they also consider your personal background and the social and cultural dimension? If so, to what extent do you think that the personal, social and cultural dimension can influence the creation of educational activities? Could you refer to a specific anecdote or a particular moment to support your thinking?*

Question Q2 aims to investigate teachers' perceptions about the causes of the processes involved in the creation of teaching activities and, in particular, whether and how these creative acts can be linked to the social and cultural dimensions of their lives. The initial stimulus leads to a biographical narrative aimed at making each teacher reflect on how their own past and long-term experiences have influenced or still influence their choices in the teaching context. The request to refer to a specific anecdote or moment shifts the narrative from the biographical register to the micro-phenomenological register. The question is structured in this way in order to encourage in the teacher-interviewee a deeper reflection and re-elaboration of past personal, social and cultural experiences in relation to current and future design choices. The answers to Q2 could therefore provide information on the personal and professional experiences of the teachers-interviewee, related to their educational training and teaching design work. With this question it is expected to have access to more details that would allow, during the analysis phase, to capture the sensitive components of the experience lived and recounted by the teachers (Breton, 2020) and, at the same time, to outline certain aspects that are at the origin of the creativity of the teacher-interviewed and that determine its uniqueness.

> Q3. *Reflecting on your design activity, can you distinguish steps that you usually follow? If you consider it appropriate, you can refer to the creation process of one of the teaching activities you have recently designed.*

Question Q3 invites teachers to reflect in more detail on their design work, possibly also referring to the process of creating one of the teaching activities designed. An attempt is made to involve the micro-phenomenological register in the narrative, with the aim of focusing attention on the work of teaching design, trying to distinguish and possibly characterise different phases of work.

> Q4. *Paulo Freire, one of the most influential pedagogists of the 20th century, refers to the "game of the people". By this expression, he refers to a set of playful practices and themes that are familiar to a certain group of people and that are able to engage, fascinate and intrigue them. What could be the "games of the people" for your students? For the creation and design of mathematics teaching activities, do you usually ask yourself about the "game of the people" for your students?*

Considering Paulo Freire's research work as central and with the purpose of exploring the role of mathematical creativity in the design of didactic paths that can also include students experiencing severe socio-cultural disadvantage, question Q4 aims to investigate the attention paid to the "game of the people" and its possible pursuit in the creation and design of meaningful and inclusive didactic paths. With this question, teachers are explicitly invited to reflect on the importance of a teaching design that is, on the one hand, meaningful and motivating for the students, from the perspective of active and popular pedagogy and, on the other hand, also responsive to the context in which it is proposed. The answers to this question could provide information on the type of creativity behind effective teaching designs that can be implemented in challenging social contexts and that are adapted to the needs, requirements and interests of the students.

5.4 The Content Analysis method

As mentioned above, the Explicitation Interview Method was chosen in order to collect narratives, whose subsequent analysis could allow access to useful elements and information to characterise mathematics education creativity. For the analysis of the collected data, I searched for a methodological tool that could allow, on the one hand, the analysis of the content of the interviewed teachers' narratives and, on the other hand, also take into account the meaning assigned to the shared experiences. It was considered that Content Analysis (Berelson, 1952) could be the most appropriate methodological tool for analysing the collected narratives. Content Analysis is described (Devi Prasad, 2008) as

the scientific study of content of communication that allows for the analysis of various messages - such as letters, diaries, newspapers, stories, documents, texts - respecting three basic principles of scientific method:

- *Objectivity*: the analysis is pursued on the basis of explicit rules and, starting from the same assumptions and analysing the same document, different researchers obtain the same result (Berelson, 1952);

- *Systematic*: the rules are consistently applied to include or exclude content and to eliminate the possibility of including only materials which support the researcher's ideas;

- *Generalizability*: The results obtained can be applied to other similar situations.

It is a method of analysis most widely used in social science and mass communication research (Berelson, 1952) however, this method has also developed and spread in relation to different disciplinary fields. Franco Rositi (1988), one of the researchers responsible for the introduction of Content Analysis in the Italian research scenario, defines this methodology as a set of methods that use procedures of analytical decomposition and classification of texts or other symbolic sets, in order to check certain hypotheses on communication facts. Although this methodology is intended as a quantitative method, the results it will allow to be obtained in the present research work will be used in order to capture more qualitative aspects and contents.

5.4.1 *Units of analysis* and frequency analysis

Content Analysis can be understood as a set of procedures that can be applied on many different research "objects" (Losito, 1996). Quantitative content analysis procedures use *units of analysis*, which can be identified in *words*, *key-symbols*, *themes*, *propositions*. Using the characterisation proposed by Losito (1996), the units of analysis are set out below in increasing order of extension and semantic complexity:

- the *word* represents the simplest *unit of analysis*. The researcher, taking into account the research objectives and hypotheses, selects and traces words within the analysed texts;

- the *key-symbol* (Lasswell, 1949) can coincide with a word, but also with a sequence of multiple words, or with an expression. It requires a more relevant intervention by the researcher;

- the *proposition* represents a statement or phrase extrapolated directly from the text, in the form in which it appears;

- the *theme*, while representing a statement or sentence about a given element in the text, remains external to it and derives from a re-elaboration by the researcher, carried out during the analysis phase.

In the following chapter, a quantitative analysis is presented. This analysis takes into account the frequency of occurrence of *words* and *key-symbols* inside the texts of the collected transcripts. Frequency analysis focuses on the number of times a *word* or *key-symbol* appears inside a text, based on the assumption that the frequency of a given *unit* can indicate its relative interest. In order to clarify this analysis procedure, one could imagine wanting to measure the interest of the press in political leaders. For this purpose, it is possible to choose to analyse the frequencies with which the names of leaders appear in the principal newspapers, referring to a specific time span or, possibly, to distant periods in order to detect a trend of references over time (Losito, 1996). Various examples in the research literature show the application of Content Analysis with reference to studies of a different kind and nature. For instance, Berelson and Salter (1957), during the period 1937-1943, analysed the content of 198 stories published in eight popular magazines in order to highlight the media under-representation of minority groups and to study discrimination against minority groups of Americans. Later, Taviss (1969) uses Content Analysis to study social alienation and self-alienation in the American middle class. Through the analysis of popular fiction in the 1900s and the 1950s, an overall rise in the appearance of alienation themes is noted. In particular, there is a slight decrease in social alienation and a large increase in self-alienation. In a study of a more political nature (Devi Prasad & Sampath Kumar, 1991), the editorials and letters to the editor published in four dailies in India before the 1991 elections are analysed with the intention of identifying election-related *themes*, also with reference to election coverage. It must be emphasised that the Content Analysis cannot be limited to counting individual *units*, but must also take into account their meaning. With reference to the first proposed example, in addition to their names, political leaders could be mentioned in a text through the position they hold, pronouns or other expressions that, in conjunction with the *units of analysis*, necessarily require an attribution

of meaning on the part of the researcher. For *key-symbols* this requirement appears even more evident. In addition, it may also be necessary to consider the context in which the *units* appear. For example, if the *unit* to be searched for is the *word* "democracy", the number of times the *word* appears in the analysed text can be counted. But the meaning and attitude with which this *word* is involved in the text cannot be inferred solely on the basis of its frequency of occurrence (Devi Prasad, 2008). In fact, we speak of *recording units* and *context units*. The *recording unit* is the *unit* that is searched for in the text and whose occurrence is counted. The *context unit* is the larger body of content that may be searched to characterise the *recording unit* (Berelson, 1952). For example, if the *recording unit* is the *word*, then the *context unit* may be the sentence or the part of speech in which the *word* appears. The *context units* may overlap and contain many *recording units*. In general, the frequency analysis of *words* and *key-symbols* is used above all in an initial exploratory phase of analysis, providing a starting point for the development of a more complex and sophisticated discourse. With thematic analysis, for instance, the frequencies of *themes* are recorded, allowing the synthesis of more articulated phrases or references actually present in the analysed text, and then moving on to the next level.

5.4.2 Development of content *categories*

In general, the analysis procedure almost always involves a consecutive classification of the *units of analysis* into *content categories*. These *categories* are constructed from the research questions, in order to also create a link to the relevant studies and literature. The construction of the *categories* is guided by the idea of aiming to find a classification that can most efficiently generate the data needed to answer the research questions raised (Devi Prasad, 2008). Even before starting the analysis, therefore, the researcher should be clear about the research questions, so that the analysis can focus on the aspects related to the objective to be achieved. It seems necessary, in order to identify the *categories*, a specific statement of the investigation objectives and the study topic that can be solved through Content Analysis. Therefore, a researcher intending to use this methodology to conduct a content study must select the *units of analysis* he/she intends to involve and develop appropriate categories, useful for sampling the data. From this perspective, *categories* can be defined as compartments into which the *units of analysis* are collected. These compartments are disjointed and the boundaries are explicitly stated. The choice of appropriate *categories* represents the heart of the analysis (Berelson, 1952). Each *category*, to be useful, must fulfil three basic requirements (Marradi, 1984):

- exhaustiveness, every *unit* must be attributed to a *category*;
- mutual exclusivity, *categories* must exclude each other so that a *unit* can be in one and only one *category*;
- the uniqueness of the *fundamentum divisionis*, the attribution of *units* to *categories* must take place according to a unique rule.

After constructing the *categories*, defining them in terms of research objectives, one proceeds with the classification of *units* under *categories*.

Content Analysis is thus understood as a method through which to analyse the content of a message - in the specific case of the present research work we are dealing with the transcripts of the collected interviews - by evaluating references, themes and facts that permeate the message, in order to be able to draw specific conclusions about the content. Through valid, replicable and objective inferences about the message being analysed, the researcher interprets its content on the basis of explicit rules. As already stated, this type of analysis can allow for different developments and also lead to more complex analyses in which relationships are explored and data are compared.

CHAPTER 6

DATA COLLECTION AND ANALYSIS

> *"Maybe this is what teaching is all about: [...] making sure that with each lesson comes the hour of awakening"*. (Pennac, 2010, p. 137)

In the light of the methodological choices discussed in the previous chapter, the analysis of the teachers' interviews is now described. Through the study of the collected narrative materials, an attempt was made to take a closer look at the process of teaching design and implementation, trying to answer the following question:

> *What are the aspects that make it possible to characterise mathematics education creativity?*

With the aim of characterising this creativity, the first phase of analysis involved the identification of *units of analysis* and the construction of *content categories*. As presented in the following paragraphs, *words* and *key-symbols* were initially chosen to be searched in the narrative material available and *content categories* were developed to help find answers to the research question. Through frequency analysis, a count of the occurrence of the single *units* inside each transcription was performed. This first phase, intended in a more exploratory sense, laid the foundations for a subsequent level. Attention was also paid to the context in which the selected *units* appeared, thus enabling an understanding of their meaning, not limited to just counting. Taking into consideration the selected *units of analysis* and their meanings, all transcripts were analysed in parallel, extrapolating *recurring themes*. These *themes* were used in order to synthesise and re-elaborate particular statements shared by the interviewed teachers surrounding the *word* or *key-symbol* research. With the aim of identifying a characterisation that could meet the explicit research objectives, these *themes* were then coded into *content categories*.

Recognising the difficulty in conducting an analysis that allows, on the one hand, to find and analyse words, phrases and expressions useful for the research purpose and, on the other hand, to look at their meaning, considering the context in which they appear, the following paragraphs discuss the nodal points on which the analysis was focused, recalling fundamental and important aspects.

6.1 Data collection: the interview transcripts

An initial data collection was performed through the audio recording of the teachers' interviews. Data analysis was conducted using the narrative material from the interview transcripts. For more immediate reference, the transcript of each teacher interviewed has been associated with an abbreviation - initials of first name and surname - which, in some cases, may be followed by a number - 1, 2, 3 or 4 - to indicate the question to which the answer refers.

Below are the abbreviations in alphabetical order, with the association to the corresponding interview:

CP: Reference to the transcript of the interview involving Chiara Pizzarelli.

FL: Reference to the transcript of the interview involving Franco Lorenzoni.

GA: Reference to the transcript of the interview involving Giancarlo Artiano.

MM: Reference to the transcript of the interview involving Marco Marino.

MP: Reference to the transcript of the interview involving Maria Pezzia.

MS: Reference to the transcript of the interview involving Marina Spadea.

NDM: Reference to the transcript of the interview involving Annunziata (Nunzia) Di Maria.

PL: Reference to the transcript of the interview involving Paola Lattaro.

RC: Reference to the transcript of the interview involving Raffaele Casi.

TG: Reference to the transcript of the interview involving Teresa Gianni.

Although the semi-structured nature of the interview with open-ended questions, which were pre-determined and the same for all teachers, ensured a certain conformity, there was still variability in the time and discourse covered. The minimum duration of an interview was 12 minutes, while the maximum duration was one hour. Having chosen to use the methodology of the Explicitation Interview, based on techniques of formulating relays, aimed at facilitating, stimulating and assisting verbalisation, a possible variability in times was already foreseen, as an indicator of the singularity of each teacher's experiences, of

their freedom in exploring, recalling and explicating significant moments and of the mutability of developments and possible connections of meaning. The collected interviews were transcribed and subsequently analysed through Content Analysis (Berelson, 1952).

6.2 The choice of *units of analysis*

The Content Analysis began with the choice of *words* and *key-symbols* to search for in each transcript, in the sense of *recording units*. This choice was made from the different development and exploration intentions that led to the elaboration of the four interview questions (Chapter 5). For an easier reading, the questions used are reproduced below and, for each of them, the *words* and *key-symbols* identified are made explicit. It should be pointed out that in order to include all valid references in the research, it was sometimes useful to associate the *word*, meant as a *unit of analysis*, with several words. As will be shown shortly, in some cases it was the same word in its singular and plural variants, in other cases several words with the same root and in still other cases different words with the same meaning. The units of analysis will be clearly distinguishable because they are separated by a semicolon.

6.2.1 *Units of analysis* form question Q1

> Q1. *In designing a teaching activity, do you feel more like ideas emerge all of a sudden or that they originate from a process of investigation?*

In order to find references in the narratives to the more intuitive and spontaneous aspect of education design, it was assumed that the teachers interviewed could use terms such as: spontaneity, intuition, improvisation and their synonyms or derivatives in their narratives. Therefore, the following *words* were chosen as *units of analysis*: *sudden/ -ly*, *improvis-e/ -ation*; *instinctive*; *intuiti-on/ -ve*; *spontane-ity/ -ous/ - ously*.

With the aim of also revealing possible references to project ideas arising from a process of investigation, it was imagined that the teachers could use expressions in their discussion that referred to the idea of research or reflection, hinting at more structured processes of investigation with defined and explicit objectives. For this reason, the following *words* were selected as *units of analysis*: *reflect/ -ion*; *investiga-te/ -tion*, *research*.

6.2.2 *Units of analysis* form question Q2

> Q2. *Referring to your design choices, do you think that, in general, they also consider your personal background and the social and cultural dimension? If so, to what extent do you think that the personal, social and cultural dimension can influence the creation of educational activities? Could you refer to a specific anecdote or a particular moment to support your thinking?*

Question Q2 was developed with the intention of bringing out from the teachers' accounts the possible motivations - implicit or already explicit - that they put at the basis of their creative processes, through a reflection on possible external influences. In order to capture the teachers' references to influences attributable to their personal, social and cultural backgrounds, it was imagined that in their accounts they might bring to light personal or professional life experiences that have inspired or still inspire their classroom practice and the design of mathematics educational activities. For this reason, the following *words* were selected as units of analysis: *experience/ -s*; *cultur-e/ -al*; *social*; *education/ -al*; *condition/ -ed/ -ing*, *influence/ -d*, *inspir-ed/ -ation*.

6.2.3 *Units of analysis* form question Q3

> Q3. *Reflecting on your design activity, can you distinguish steps that you usually follow? If you consider it appropriate, you can refer to the creation process of one of the teaching activities you have recently designed.*

Question Q3 drew attention to instructional design, asking the teachers interviewed to try to distinguish possible phases of work, also by evoking examples of designs they had implemented or worked on. It was imagined that the teachers, through a posteriori reflection, could refer to a possible organisation in phases, sharing examples of their own projects and recalling fundamental steps in their process of creating teaching activities. For these reasons, the following *words* were chosen as *units of analysis*: *phase/ -s*; *project/ -s*, *activit-y/ -ies* and as a *key-symbol*: *when I design*.

6.2.4 *Units of analysis* form question Q4

> Q4. *Paulo Freire, one of the most influential pedagogists of the 20th century, refers to the "game of the people". By this expression, he refers to a set of playful practices and themes that are familiar to a certain group of people and that are able to engage, fascinate and intrigue them. What could be the "games of the people" for your students? For the creation and design of mathematics teaching activities, do you usually ask yourself about the "game of the people" for your students?*

The purpose of question Q4 was to focus on the search for the "game of the people" in the instructional design phase. For this reason, the *word game/ -s* was chosen as the first *unit of analysis*. It was preferred to choose the *word game*, rather than the *key-symbol game of the people*, in order to capture and analyse all the teachers' references to the use of the game - intended as a playful element - in their teaching practice. The search in the text for the *symbol-key game of the people* could have excluded the teachers' references to simple game, not necessarily identified with Freire's expression. On the other hand, the *unit of analysis game* makes it possible to include references to the simple game in the search, without excluding references to the "game of the people" from the frequency of occurrence analysis.

In question Q4, but also in the course of the interview conducted, Freire's thoughts on the "game of the people" and his intention in referring with this expression also to themes and situations that are significant in the eyes of the students and that, in the educational sphere, manage to involve and fascinate them, were made explicit. Consequently, it was also chosen to include the *words*: *fascinat-es/ -ing*, *engag-e/ -ing/ -ement*; *context/ -s*.

6.3 The construction of *content categories*

In order to identify a classification that could help answer the research questions raised, the following *content categories* were defined:

- **Conditioned Spontaneity**
- **Attention to the context**
- **Mathematical education objectives intertwined with citizenship education objectives**.

These *categories* were constructed taking into account the research question and attempting to include connections with the theoretical frame of reference. In addition, the three basic requirements (Marradi, 1984) of exhaustiveness, mutual exclusivity and uniqueness of the fundamentum divisionis were respected for their construction, in order to obtain disjointed categories with well-defined rules and boundaries. The reflections that led to their definition were also read in terms of rules to ensure what is or is not included in a given *category*. Thus, the explanation provided below for each *content category* was intended as an operational definition of them, in order to avoid any ambiguity and allow for a legitimate and coherent coding of the *context units of analysis*.

Conditioned Spontaneity

The choice to indicate the first *category* with the term "conditioned spontaneity" emerged as a result of a series of considerations developed from the studies presented in Chapter 4 and oriented towards exploring and characterising teachers' mathematical creativity. In fact, it was seen how these studies made it possible to frame the object of research - to which we chose to refer with the term *mathematics education creativity* - in a broader context. For the definition of the first *category*, research in the field of psychology concerning creativity was taken into account. In general, the existence of possible unconscious mental processes was outlined, associated, in studies concerning mathematical creativity in particular, with the phenomenon of mathematical illumination (Hadamard, 1945). With the aim then to identify possible origins to creativity, the models presented (Csikszentmihalyi, 1998, 2000; Rhodes, 1961; Sternerg & Lubart, 1996, 2000) recognised the presence of various factors that, interacting with each other, seemed to condition the thought processes involved in the creative act. In this regard, it was considered possible to find potential connections with the investigation conducted and thus, to trace in the collected interview transcripts references to spontaneity and its influences in the educational design and implementation processes of the interviewed teachers. These assumptions prompted the desire to develop a *content category* that could include all the *units of analysis* referring, on the one hand, to more spontaneous, sudden and instinctive ideas and, on the other, to conditioning. Hence the decision to indicate a content category with the expression “conditioned spontaneity”.

Attention to the context

The second *content category* was referred to as "attention to context". Here again, the definition of the category came about by taking into account the theoretical framework outlined, in particular the studies concerning active and popular pedagogy presented in Chapter 1 - 2. In reference to this area, the centrality of the pupil-environment link had emerged: the educational experience does not prescind from the everyday life in which the subject lives and his relationship with the environment that surrounds him and with which he interacts. Through Freire's pedagogical vision, and consequently Gutstein's, it was possible to highlight the importance of developing the programmatic content of an educational proposal, taking into account the needs and interests of the students. Starting from the conception of man as a being in situation (Freire, 1970), who cannot prescind from the interactions and the relationship with the world around him, from the concrete, existential and historical situations belonging to his reality, it was then introduced the concept of the game of the people (Freire, 1989), related to the design of effective and meaningful didactic paths. In general, the theories and experiences in the field of active and popular pedagogy had made it possible to outline the idea of a new school, developed from the interests, needs and aspirations of the pupils and in which the sensitive experience with the surrounding reality is valued. Due to the considerations outlined above, attention to the context has been understood in a broader sense, which has allowed, again, to provide an operational definition of the category. In particular, by "attention to the context" it has been chosen to indicate attention to the individual student, to his or her ideas, interests and needs, but also attention to the class group and to a series of situations and methodological choices oriented towards fostering creativity, plurality of opinions and the student's freedom to experiment on his or her own. By "attention to the context" we have therefore also chosen to indicate an attention to the educational and scholastic context that is intended to be built and offered, in terms of the learning experience. The reference to context, however, was also meant in terms of the social, cultural and economic context in which students live and with which they relate, as a reflection of their everyday life, their aspirations, doubts, hopes and fears (Freire, 1970).

Mathematical education objectives intertwined with citizenship education objectives

Finally, the third *content category* was developed with the intention of capturing possible references to didactic-educational objectives involved in the interviewed teachers' project planning. In particular, the reference to a new idea of school was seen as focusing on the

role played in the process of students' education and maturation. In this sense, by proposing learning that was increasingly interdisciplinary and strongly motivated by the students' experience, in the world and with others, the school was perceived as a community of cooperation, with a central role in the education and intellectual and moral development of the students (Incollingo, 2019). Referring exclusively to the field of mathematics education, the research work of Castelnuovo (1963) and Gutstein (2006; 2007) has shown a desire to conceive mathematics as a useful tool for investigating, making sense of and understanding issues related to one's own life or the community in which one lives. Mathematics teaching activities are part of broader didactic pathways, also aimed at active citizenship education goals, in the sense of *critical mathematics education* (Skovmose, 1994). There emerged, therefore, a desire to detect, in the interviewed teachers' stories of their educational design, a possible attention to the relationship between mathematics education and democracy. In this sense, therefore, it was chosen to include in this *content category*, all references to teaching activities in which mathematics education is recognised as intertwined with the development of a critical awareness of citizenship. In particular, it was chosen to include in this *content category* the units of analysis referring to genuinely inclusive teaching situations aimed at encounters, confrontation and dialogue between students.

6.4 The frequency analysis

Once the *units of analysis* had been selected, through an automated search process performed using a computer, the frequency analysis phase was implemented. The technique of searching for words within a written text made it possible to identify the *units of analysis* inside the body of the single transcripts. For each *unit of analysis*, the number of times it appears in each transcript was recorded.

Although each question made it possible to extrapolate certain *words* and precise *key-symbols*, it is appropriate to specify that, considering the single transcript, the frequency of occurrence of each *unit* was measured over the entire narrative material and not with reference to the respective question alone. This choice arose from the fact that, having used the methodology of the Explicitation Interview and having given the interviewed teachers freedom in answering and following different directions, some topics were taken up or explored in depth at different points and times of the interview.

In order to have a clearer view of the analysis conducted, it was decided to separate the results obtained by considering the question that led to the selection of the *units of analysis*.

6.4.1 Frequency analysis of Q1 *units*

Units of analysis	Occurrence frequency of the *unit* with reference to the indicated transcript									
	CP	FL	GA	MM	MP	MS	NDM	PL	RC	TG
Sudden/ -ly, improvis-e/ -ation	1		3	2	4			1	4	1
instinctive			1					3		
intuiti-on/ -ve		5						3		
spontane-ity/ -ous/ –ously			2	3	1	1	8			
reflect/ -ion			3	8	6		8	1	5	1
Investiga-te/-tion research	5	3	2	2	8	3	4			4

Table 1. The left column shows the *units of analysis* extrapolated from question Q1 and used for the frequency analysis. For each *unit*, the frequency of occurrence within the reference transcript is given in the corresponding boxes on the right.

The *words sudden/ -ly*, *improvis-e/ -ation*; *instinctive*; *intuiti-on/ -ve*; *spontane-ity/ -ous/ -ously* all refer to the concept of spontaneity and naturalness. Considering, for example, the NDM transcription column in Table 1, it emerges that in the analysed text, the first three *units of analysis* do not appear, while the *word spontane-ity/ -ous/ -ously* appears 8 times. Similarly, with reference to the *units of analysis reflect/ -ion*; *investiga-te/ -tion* and *research*, chosen because they are related to the idea of reflection and research, it emerges that the word reflect-ion has a frequency of occurrence equal to 8, investiga-te/ -tion, research equal to 4. According to the results obtained from the frequency analysis, in Nunzia's transcription,

the reference to spontaneity occurs 8 times, while the reference to the aspect of investigation and reflection - considering the selected *units* - occurs a total of 12 times. Wanting to summarise for all transcripts the results obtained as references to spontaneity and references to the aspect of investigation and reflection, we have that:

in CP, the reference to spontaneity occurs a total of 1 time, while the reference to the aspect of investigation and reflection occurs a total of 5 times;

in FL, the reference to spontaneity occurs a total of 5 times, while the reference to the aspect of investigation and reflection occurs a total of 3 times;

in GA, the reference to spontaneity occurs a total of 6 times, while the reference to the aspect of investigation and reflection occurs a total of 5 times;

in MM, the reference to spontaneity occurs a total of 5 times, while the reference to the aspect of investigation and reflection occurs a total of 10 times;

in MP, the reference to spontaneity occurs a total of 5 times, while the reference to the aspect of investigation and reflection occurs a total of 14 times;

in MS, the reference to spontaneity occurs a total of 1 time, while the reference to the aspect of investigation and reflection occurs a total of 3 times;

in PL, the reference to spontaneity occurs a total of 7 times, while the reference to the aspect of enquiry and reflection occurs a total of 1 times;

in RC, the reference to spontaneity occurs a total of 4 times, while the reference to the aspect of investigation and reflection occurs a total of 5 times;

in TG, the reference to spontaneity occurs a total of 1 time, while the reference to the aspect of investigation and reflection occurs a total of 5 times.

Through the reported summary, it can be seen that, in transcripts CP, MM, MP, MS, NDM, RC and TG, more references were counted to the aspect of search and reflection, and fewer occurrences of *words* related to the concept of spontaneity. Conversely, for the FL, GA and PL transcripts, more references to the idea of spontaneity and fewer occurrences of *words* related to the aspect of research and reflection were counted. In some cases (MM, MP and PL) the difference between the recorded occurrences is quite significant. Considering the frequency of occurrences recorded as indicators of interest and involvement with a

particular topic (Losito, 1996), from an initial analysis, which only includes the data collected, it would seem that in the CP, MM, MP, MS, NDM, RC and TG transcripts there is a tendency to involve in the speech references to reflection and research work. Similarly, for the FL, GA and PL transcripts, the result obtained from the analysis of the frequency of occurrence can be read as an indicator of an interest in involving references to spontaneity in the discourse. It should be pointed out, however, that the *units* sought are single words that are not necessarily used in the discourse in the intended sense or can be traced back to the perceived aspect. Therefore, the result returned to us by the analysis of frequencies certainly provided interesting data on the emergence of particular topics in the analysed discourse, making it possible to circumscribe attention to certain points, but it was not considered a definitive point of arrival for the research conducted as it is not able to give indications as to the direction of the interest revealed. It seemed necessary to analyse the recorded occurrences by understanding how the *units of analysis* are used and inserted into the discourse.

6.4.2 Frequency analysis of Q2 *units*

Units of analysis	Occurrence frequency of the *unit* with reference to the indicated transcript									
	CP	FL	GA	MM	MP	MS	NDM	PL	RC	TG
experience/ -s	3	14	3	6	2	4	11		6	3
cultur–e/ -al	5	5	4	3	2		4	1	4	3
social	4		3	10	2	2	3		9	3
education/ -al		3	1	7	1	2	8	1	1	
condition/ -ed/ -ing *influence/ -d* *inspir-ed/ -ation*	3	3	2		4		18	1	1	2

Table 2. The left column shows the *units of analysis* extrapolated from question Q2 and used for the frequency analysis. For each *unit*, the frequency of occurrence within the reference transcript is given in the corresponding boxes on the right.

The analysis of the frequency of occurrence made it possible, once again, to gain an indication of the use, in the narrative of each teacher interviewed, of the *words* identified. Taking into account the occurrences reported in Table 2, it can be read that almost all the teachers refer to experiences in their discourse. Among the transcripts in which the *word experience/ -s* appears, there was a minimum frequency of occurrence equal to 2 and a maximum frequency of occurrence equal to 14. Thus, the teachers, in different ways and numbers, in the guided reflection through the Explication Interview, recalled their experience and evoked past events. From Table 2, it also emerges that in almost all transcripts the *words social* and *cultur-e/ -al* appear. Only in half of the transcripts analysed, on the other hand, is it possible to identify a direct reference to the *word education/ -al*, with a minimum frequency of occurrence of 1 and a maximum frequency of occurrence of 8. It can also be read that the frequencies of occurrence relating to references to conditioning, influences and inspirational elements, range from a minimum of 1 to a maximum of 18 - excluding zero occurrences. Comparing these frequencies of occurrence, it can be seen that, excluding null occurrences, they all remain below 5, except in one case (NDM) in which 18 occurrences of the *unit of analysis* sought were found. If one focuses only on the NDM column, one can see that the *units of analysis* with a higher frequency of occurrence are *experience/ -s* (11), *education/ -al* (8) and *condition/ -ed/ -ing*, *influence/ -d* and *inspir-ed/ -ation* (18). From the perspective of Content Anlysis, the *units of analysis* with a higher frequency of occurrence delimit the boundaries of the central topic of the text analysed, therefore, with regard to the question explored, Nunzia's narrative seems to focus around the theme of experience, education and conditioning that inspires and influences design choices and classroom practice. Reading from this perspective, it was also possible to analyse the other transcripts in which there is a clear difference between the frequencies of occurrence of different *units of analysis*. With respect to the specific purpose of the investigation, which made it possible to extrapolate the *units of analysis* in Table 2, in Franco's transcript (FL column) the theme of experience appears to be central, Marco's transcript (MM column), on the other hand, is concentrated around the theme of experience and on the social and educational training aspect, and finally, in Raffaele's transcript (RC) a propensity towards the experiential and social component is detected.

Considering all the transcripts, it is noted that among the *words cultur-e/ -al*, *social* and *education/ -al*, the latter has a higher frequency of occurrence among all of them, amounting to 36. This result can be read in terms of interest in this aspect, inevitably

prompting reflection on the role that the social and cultural, as well as the personal, components occupy in the teachers' creative process. In this case too, then, the analysis of the frequencies returned an interesting result that cannot, however, be understood as complete or definitive. In fact, this first phase of analysis made it possible to detect the occurrence of certain *words* within the discourse of each teacher interviewed, thus highlighting a focus on particular themes involving the personal, cultural, social and educational spheres. The results obtained, however, do not make explicit to an external observer the value with which the *words* sought were understood during the interview. Therefore, it was decided to investigate this aspect by trying to investigate the reference context in which the *words* identified appear.

6.4.3 Frequency analysis of Q3 *units*

Units of analysis	Occurrence frequency of the *unit* with reference to the indicated transcript									
	CP	FL	GA	MM	MP	MS	NDM	PL	RC	TG
phase/ -s	6	1	7	8	2			10	7	2
project/ -s *activit-y/ -ies*	35	5	27	16	23	9	46	23	23	13
when I design		2	2	2		2	7	2	1	2

Table 3. The left column shows the *units of analysis* extrapolated from question Q3 and used for the frequency analysis. For each *unit*, the frequency of occurrence within the reference transcript is given in the corresponding boxes on the right.

Table 3 shows the results obtained from the frequency analysis of the *units* extrapolated from question Q3. Looking in detail at the individual columns, in each one it appears that the highest frequency of occurrence was recorded at the *unit project/ -s*, *activit-y/ -ies*. With respect to the *units of analysis* researched, the narrative of each teacher seems to focus around examples of projects and teaching activities, which give meaning to the discourse, in the sense of Content Analysis. Only in a few cases and with less frequency than that associated with the *unit of analysis project/ -s*, *activit-y/ -ies*, the teachers use the expression

"*when I design*". Except in two cases, the rest of the teachers interviewed involved the *unit of analysis phase/ -s* in the discourse at least once, which always appears with a lower frequency than that associated with the second *unit of analysis* in Tabel 3.

The result obtained provided an indication of the central topic of the discourse, allowing for references to examples of project design and teaching activities mentioned by the teachers. Again, it was not possible to consider this result as exhaustive because, while it made it possible to verify the quantification of the *units of analysis* identified, it also overlooked their context, precluding a full understanding of the meaning conveyed. The occurrences recorded in correspondence with the *word phase/ -s* did not provide deeper indications as to the meaning of the discourse. Similarly, the frequency recorded in reference to the activities recalled and to the moment of teaching design did not provide access to the teachers' subjective experience. Therefore, these observations prompted a more detailed exploration of the interviewed teachers' narratives in order to find the *units* in the analysed text and gain access to contextual information, nuances and deeper meanings. Having placed the focus on teaching design and on the chance to try to distinguish and characterise possible phases of work, also through the use of practical examples, it was decided to continue the analysis with the aim of understanding in what terms the projects and activities mentioned are included in the narrative and what motivation led them to be the central topic of the discourse, in the sense of Content Analysis.

6.4.4 Frequency analysis of Q4 *units*

Units of analysis	Occurrence frequency of the *unit* with reference to the indicated transcript									
	CP	FL	GA	MM	MP	MS	NDM	PL	RC	TG
game/ -s	4	7	5	9	15		40	24	15	6
fascinat-es/ -ing engag-e/ -ing/ -ement	10	5		3	3	2	2	5	2	1
context/ -s	10		4	2	5	2	11		7	

Table 4. The left column shows the *units of analysis* extrapolated from question Q4 and used for the frequency analysis. For each *unit*, the frequency of occurrence within the reference transcript is given in the corresponding boxes on the right.

Question Q4 arose from the desire to stimulate reflection on the "game of the people" in order to obtain information regarding the creativity involved in the design of mathematically meaningful, but also inclusive, motivating and context-sensitive teaching activities. The frequencies of occurrence present in Table 4, in correspondence with the *word game/ -s*, give information about the teachers' references to the use of the game, not necessarily understood in the sense of the Brazilian pedagogist. This reference is totally absent in the MS transcript, with respect to which, even the occurrences in the other units of analysis do not seem to point in any particular direction. Considering instead the CP column, it can be read that the frequency of occurrence of the *word game/ -s* is equal to 4, remaining, however, lower than the other two recorded frequencies that, from the perspective of Content Analysis and with reference to the question explored, delimit the boundaries of the dominant subject of the text analysed. In the other cases, instead, it can be read that the highest frequency of occurrence is associated with the *word game/ -s*, which appears predominant in the speech and in reference to the particular objective explicated. Again, the focus on single isolated *words* cannot be taken as an exhaustive result. The counting of references to the *game*, understood both as "games of the people" and as an essentially ludic practice, reveals each teacher's interest in this point, but it does not shed light on the interpretative aspect and the teachers' intentions behind this interest.

Therefore, it was decided to continue the analysis by also taking into account the context in which the units of analysis appear.

In general, in all the transcripts analysed, the outcomes provided as a result of the frequency analysis gave an indication of the involvement of certain words or expressions in the speech of the interviewed teachers. The detected occurrence frequencies were in fact understood, in the sense of Content Analysis, as an indication of an interest of the interviewed teachers in certain aspects and topics. If, on the one hand, the detected occurrences made it possible to direct attention towards interesting results, on the other hand, this first analysis stimulated new questions, bringing out the need to also pay attention to the context, in order to trace aspects that can characterise *mathematics education creativity*. This first step of analysis represented an exploratory phase that made it possible to detect and analyse the frequencies of occurrence of the identified *units*, without, however, making explicit the underlying meanings and values, which are also a motive of interest. For these reasons, in order to deepen the analysis, it was decided to pursue the investigation in a more qualitative and interpretative perspective. Starting from the results obtained, which provide indications of the most significant points in each teacher's discourse, it seemed useful to explore the context in which the units of analysis operate, in order to grasp through the teachers' words their vision and the essence of their experience. In the following section, this second phase of analysis is presented, conducted with the aim of extrapolating ideas and themes present in all transcripts and common to all teachers.

6.5 Recurring themes

The process of analysis presented in the previous section originated a second phase of work that allowed the discourse initiated to be deepened through an analysis of meanings and the consequent determination of context units. The choice of an exploration that necessarily also involved an interpretative paradigm set in motion a long and complex process that required numerous re-readings of the available material. Starting from the results of the frequency analysis, which provided indications of the involvement of certain topics in the narratives examined, all the interview transcripts were analysed in parallel. In particular, the contexts in which the *units of analysis* act were first explored and subsequently, the meanings attributed to the *units of analysis* in the different transcripts were compared. The motivation that guided the analysis was to extrapolate from the collected

narratives the most recurring issues and themes, leading to the delineation of *recurring themes*, used as new *units of analysis* to be codified in the predefined *content categories*.

Below are excerpts from transcripts considered particularly significant from the perspective of the analysis conducted. In them, the *units of analysis*, made explicit at the beginning of the chapter, and searched for through frequency analysis, are underlined in order to make their identification in the text easier.

6.5.1 Context analysis of Q1 *units*

Considering the *words* extrapolated from question Q1, the frequency analysis showed in some cases (CP, MM, MP, MS, NDM, RC and TG) a tendency to include in the discourse references to a work of investigation and reflection, in other cases (FL, GA and PL) a greater occurrence of references to the idea of spontaneity and improvisation.

In the NDM transcript, although a higher occurrence of references to the idea of research and reflection was detected, the *word spontane-ity/ -ous/ -ously* appears 8 times, registering, compared to all the transcripts analysed, the highest frequency of occurrence of references to the idea of spontaneity. Considering the body of the text of the NDM transcription, the references to the *word spontane-ity/ -ous/ -ously* are all concentrated in response to question Q1 therefore, effectively linked to a reflection on the creative process and its attribution to a spontaneous emergence of ideas or to a more methodical process of investigation.

> Nunzia (NDM1): […] this approach of building activities in this way, in my opinion, leads me to think spontaneously about things, but it is a spontaneous in some way conditioned […] To me it seems spontaneous in the moment when I think [...] at that moment, I get ideas, they might seem, so spontaneous, but I think it's a conditioning that comes out of a way of experiencing the creation of activities.

In the reported excerpt, there are several *units of analysis*, also extrapolated from different questions, which made it possible to focus attention and highlight various factors. Nunzia, while referring to a spontaneity that characterises her teaching design activity, speaks of "conditioned spontaneity" referring to an external conditioning as the source of the spontaneous emergence of ideas.

> Nunzia (NDM1): [...] Some conditions have been created, even with the association I belong to ["Matematici per la città"], to design together, to reflect together, so when I design an activity on my own, I don't know how much it is spontaneous or how much it is actually the result of this continuous process of sharing and building together and so, sometimes, I say: "Is it mine? Did it come to me spontaneously?" […] So, all those inputs, I feel I carry them inside even when I design an activity on my own, as if I were reliving that collective moment even being alone... it's the result of this interaction and it's certainly the result of a sensitivity that has increased over the years in relation to reflection […] thinking back to the previous activities I do, I condition the new ones in some way.

The excerpt reported allows a deeper insight into the nature of the "conditioned spontaneity" Nunzia speaks of, bringing to light possible motivations underlying the creative process. In contrast to the occurrences related to spontaneity, in NDM the frequency analysis had recorded a total of 12 occurrences related to the aspect of reflection and investigation. By carefully reading the extrapolated part of the narrative, one can see that the *word reflect/ -ion* is used to refer to moments of collective confrontation and reflection, which stimulate and influence the spontaneity Nunzia speaks of. In this sense, the reference to reflection is not to be attributed to a methodical and structured creative process, but represents an element, of a social nature, that conditions and encourages the spontaneity of the teaching design processes. The frequency analysis alone, together with the assumptions with which the *units* had been selected from question Q1, had not been able to capture this nuance of meaning. This second phase of analysis, on the other hand, shed light on certain aspects, helping to finalise a more comprehensive result, in which the quantitative part communicates with the qualitative one.

> Nunzia (NDM1): And then, surely, also the educational training or, in any case, the sensitivity that the different environments, also of education, that I have frequented have developed. […] Certainly some didactic courses focused on certain aspects, but then I cultivated these aspects by making specific choices [...] I am just thinking about some workshop experiences of education in which I have participated […] reading magazines, articles, books, participating in conferences where the reflection is on these aspects, certainly this conditions you a bit because then, you think about it when you are in the classroom [...].

Nunzia's project choices are strongly motivated not only by experiences of collective reflection but also by post-graduate educational training. In particular, in her account,

Nunzia refers to reading journals, articles, books and to training courses or conferences on mathematics education that, over the years, she carefully selected because they gave space for collective design activities and because they promised first-hand experience of laboratories (Lo Sapio, et al., 2022).

The access to the significance with which the *units of analysis* were involved in the discourse made it possible to enrich the results obtained through the exploratory phase. In particular, the analysis of the frequencies of occurrence of the *units* extrapolated from Q2 had highlighted the centrality of the theme of experience, training and conditioning in Nunzia's (NDM) narrative. This result also matched the analysis of meaning. The "conditioned spontaneity" of which Nunzia speaks in fact appears to be associated with two main factors. The first relates to the confrontation with other teachers that brings back to the experiences of collective reflection and shared project design, widely developed within the association "Matematici per la città", the second factor instead, refers to the teaching and training experiences collected over the years.

Still referring to the aspect related to spontaneity, the frequency analysis of the *units* extrapolated from Q1, showed that immediately after the NDM transcription, the highest frequency of occurrence of references related to the idea of spontaneity was associated with the PL transcription. Again, by analysing the context in which the *units* appear inside the transcript, the quantitative results obtained from a first phase of analysis were contextualised and better understood.

> Paola (PL1): So look, I'll tell you that I usually get ideas all of a sudden, in the sense that... how do they come to me? The ideas come to me with an association, which is often instinctive, of elements that are apparently very far away. [...]

Paola attributes her "sudden" arising of ideas to an association that is quite easy and instinctive for her. This process of association was perceived as a kind of conditioning from the perspective of "conditioned spontaneity". This feeling prompted a search for meaning.

> Paola (PL1): [...] I am quite convinced that this fact, which is quite easy for me, more than anything else I find it instinctive to put together apparently distant worlds, is a consequence of my mathematical education, in the sense that it is a bit like when you have the data of a problem or the hypothesis of a theorem and you have to put them together and arrive at the final result or proof of the theorem. I have the feeling that when I come up with a project idea, in the end, I actually think in this way [...] My feeling is that

> this comes to me in an instinctive way because I am used to thinking this way from the studies I have done, I recognise that the mental path is exactly the same.

In arguing her answer, the idea of a reference to the concept of "conditioned spontaneity" seemed well-founded. In fact, Paola links the instinctiveness of the associations that guide the process of creating design ideas to her mathematical education, which taught her to put seemingly distant elements into communication with each other. In particular, according to Paola, in the association of ideas, the mental path is the same as when from the data of a problem she seeks a solution or from the hypotheses of a theorem she arrives at the thesis.

Already previously, with the analysis of the NDM transcript, it became clear the importance of this second phase of analysis, which made it possible to give value to the *units* researched, taking into account the general meaning of the discourse. The exploration of the context in which the Q1 *units* appear made it possible in PL to confirm the result of the frequency analysis and also to access new information, further underlining this relevance. The frequency analysis of the Q2 *units* had not revealed any main topic in the PL discourse: with reference to educational training and to the presence of possible conditioning, the occurrence recorded for each *unit* was equal to 1. Looking at the PL transcript in detail, the idea of a "conditioned spontaneity" emerged, associated, by Paola, with her mathematical education. From the count of occurrences such relevance had not emerged as the *words education/ -al* and *condition/ -ing*, *influence/ -d* and *inspir-ed/ -ation* were not always directly involved in the narrative.

Considering the frequency analysis of the *units* extrapolated from Q1, immediately after PL, the highest frequency of occurrence of *units* related to the idea of spontaneity was associated with the GA transcript, where the reference to spontaneity occurred a total of 6 times, while the reference to the aspect of investigation and reflection occurred a total of 5 times. Again, *units* of analysis were identified within the narrative in order to conduct a more qualitative and interpretative investigation.

> Giancarlo (GA1): […] I think these two components: improvisation and project design are consubstantial, they are like to say... they are part of the same nature, they work together. So let's say, there cannot be structured project design without a part of improvisation and creativity and vice versa. In my opinion, creativity is also cultivated within a structured and organised project design.

For Giancarlo, the spontaneous arising of ideas and the activation of a more methodical process of investigation seem to coexist. Creativity is associated with the "part of improvisation" that, again, appears to be fuelled in the sense of "conditioned spontaneity".

> Giancarlo (GA1): […] These two things, for me, are part of two broader categories which are spontaneous capacity and capacity for reflection [...]. Training both the capacity for reflection, of how I think I can propose something, and the spontaneous capacity, how I read a situation in an improvised and let's say, instantaneous manner [...] opens up the possibility for me in the construction of more and more articulated, creative and well-structured and organised paths.

The frequency analysis for the *units* extrapolated from Q1, although it revealed in GA a higher frequency of references to the idea of spontaneity (6), did not show a clear difference compared to the occurrence of references to the aspect of investigation and reflection (5). This result was also confirmed from the point of view of the interpretation. As we read in the excerpt reported, Giancarlo identifies two types of capacity, one spontaneous and the other reflective, which together contribute to the design of educational paths and activities. Thus, the two aspects are both present and seem to be placed at the same level. In the vision proposed by Giancarlo in fact, spontaneity seems to be conditioned by a more structured and organised type of design, which in turn appears to be conditioned by moments of improvisation and creativity, generating a sort of cyclicity.

Taking into account the frequency analysis of the *units* extrapolated from Q1, immediately after GA, the highest frequency of occurrence of *units* referring to the idea of spontaneity was associated with the FL, MM and MP transcripts. As with the GA transcript, also for the FL transcript, the frequency of occurrence of the Q1 *units* did not show a clear difference between the frequencies associated with references to spontaneity (5) and the frequencies associated with references to research and reflection (3). This result is confirmed in Franco's narrative, a brief extract of which is given below, from which it emerges that the spontaneous arising of ideas and the activation of a more methodical process of investigation seem to coexist.

> Franco (FL1): When I design... It's not a simple question, I think both things happen. [...] In certain cases, there is a project that comes from an experience or a memory of something. Quite honestly, when I design for example all geometry activities, the distant inspiration is always the memory of Emma Castelnuovo. I think that for me, those three years of middle school done with her were such a wealth of experience that this

> reference is like a memory I have deep inside. So, it is very easy for me to draw inspiration from her.

In his reflection on what inspires a design, Franco refers to a personal experience concerning the memory of Emma Castelnuovo. His experience as a pupil of Emma is perceived as a source of inspiration for the design of mathematics teaching activities.

> Franco (FL1): Then, there are other cases where something else happens. We, in Cenci, at the Casa Laboratorio di Cenci, have been experimenting for many many years, for forty years, since 1980, with activities that we do either outdoors or in school camps, both with children and with adults. So, there is a heritage of experience, of research carried out in the group... It was also a group linked to the cooperative educational movement and that heritage is certainly at the origin of many of my proposals. [...] I am a great believer in cooperative education, [...] in the fact that there are ideas that I certainly experienced at school, with the girls and boys I worked with, that I don't really know whose they are. They are ideas born in years of research, from meetings, from suggestions, from maybe, someone thought one thing, someone added another [...].

Continuing his reflection, Franco identifies another aspect that conditions his teaching design. These are professional and co-designing experiences that find common ground with the collective reflection present in NDM. In particular, in Franco, as in Nunzia, it is possible to identify a reference to ideas whose origin is not easy to recognise, since they are stimulated by research, experiences and the encounter of many voices.

> Franco (FL3): I'm not systematic at all, so... I know people who plan and follow well-defined phases and this is very useful for them. I definitely don't think my way is the best way though, my way is very intuitive and it also follows very much the moment and the educational relationship at that moment.

In the excerpt reported, a clear reference emerges to an intuitive and thus more spontaneous aspect in teaching design. From this perspective, personal, professional and social experiences - as a pupil of Emma, of experimentation at the Casa-Laboratorio in Cenci and of collaborative design - can be identified as elements that influence the arising of design ideas in the sense of "conditioned spontaneity". Here too, the analysis of meaning made it possible to add information to the results derived from the exploratory phase, finding points of connection with it. In particular, the frequency analysis of the *units* extrapolated from Q2 revealed the centrality of experiences in the FL narrative. Following a

focus on the context in which the *units of analysis* appear, the consistency of the experiential aspect in relation to the instructional design process was indeed detected.

From the frequency analysis of the *units* associated with question Q1, there were 5 total occurrences of the references to spontaneity in the MM transcript, with a significant difference to the recorded occurrences of the references to the research and reflection aspect (10). By finding the *units* inside the transcript and exploring their context, it was possible to investigate the meaning attributed to them.

> Marco (MM1): So, let's say that at the beginning of my work at school, of my work as a teacher, it was all a bit more improvised, in the sense that in certain situations one put in place didactic activities that he had planned in the classes, but they were a bit of a tests. Gradually, the work done in the classes has become much more organic because it has helped me a lot to follow the work of Emma Castelnuovo as well [...] because I know that the things I propose follow a certain order and lead me to create learning. [...] The fact that I follow the texts with the class, as you know I have adopted Emma Castelnuovo's texts, reassures me and makes me more organised.

From the brief excerpt, it appears that initially Marco seems to make a distinction between the initial phase of his work as a teacher and his current approach. Initially, he notes a more spontaneous component in his classroom practice that led to proposing teaching designs perceived as tests. With the reference of Emma Castelnuovo's work and the adoption of her texts, the classroom practice is felt to be more ordered and organised. Continuing with his argument, Marco elaborates on this aspect.

> Marco (MM1): So, in my opinion, compared to the question, it's more, how could I say it... a work that arises spontaneously... But gradually, the more you gain confidence and awareness of what you do, the more you tend to structure it. […] Taking into account the texts, the things you have done before, there is also an ease in being able to manage and plan, to know what you have to do, to get the materials [...]. Also because, by the way, sometimes in the classroom [...] there are things that also arise at the moment, spontaneously.

It was possible to observe in Marco's narrative a change of narrative regime: from the biographical one, which made it possible to analyse the situation through a more distant view - the initial phase of his work as a teacher and the current phase - to the micro-phenomenological one - focused on the teaching design and implementation activity.

From Marco's words, the distinction identified in his path as a teacher also seems to be found in the process of designing a teaching activity, which begins spontaneously and gradually acquires a more structured and organised form. In this case, too, it was possible to recognise, in the spontaneity, a kind of conditioning. In particular, from the analysis of Marco's account, spontaneity, which characterises not only the design phase but also the teaching implementation phase, is conditioned by the reading of texts and previous experiences, which help both to make the work more organic and to acquire confidence and awareness in the potential of the proposed teaching actions.

The frequency analysis of the *units* extrapolated from question Q1 showed for MP a situation similar to that of MM, with an occurrence frequency of references to spontaneity equal to 5 and an occurrence frequency of references to the aspect of investigation and reflection equal to 14. Through this second phase of analysis, the aim was to access the meaning given to the *units* within the discourse.

> Maria (MP1): So, I would say that there is a basis or at least an investigation, in the sense that [...] for example, if I know that it is necessary at this moment to deal with multiplication, fractions or symmetry, a subject in the curriculum, the first thing I do is to go and research, to see what the research says, to see what other teachers who are in contact with research are doing [...]. Then, another thing I do, sometimes, if I have doubts, I write to my researcher friends: "Ah you know, I would like to do this thing, what do you suggest?". [...] Then, there are the conferences, maybe one gets inspiration [...].

Maria begins her narrative by attributing the origin of her creative acts to a more structured process of investigation, confirming the result of the frequency analysis of Q1 units. This process involves the identification of key points to be explored during the teaching activity and is guided by the consultation of books, research articles and online groups. Analysing all transcripts in parallel revealed an aspect already present in NDM and FL, related to the importance of collective reflection for instructional design. From the transcript excerpt reported, one point on which Maria's investigation focuses is precisely characterised by the direct dialogue with colleagues and friends who are researchers in mathematics education.

> Maria (MP1): […] However, one thing that I always do is to create narrative frames... often and on those maybe there's more an idea, even a sudden one, that comes to me, that sometimes I don't quite understand where I got it from. I create stories a lot, even fairy tales.

In arguing her answer, Maria also seems to trace a more spontaneous aspect that characterises her creative and didactic design process. Although the causes of the sudden emergence of ideas are still unclear to her at this point, the discourse seems to point in the direction of a "conditioned spontaneity", the conditionings of which, at this stage of the narrative, are not yet explicit.

> Maria (MP1): Sometimes the stories can be very short, it can be just a little story or a dramatisation. [...] One time, I suddenly took a puppet we had at school, drew a big number line and decided that there was his house, there was his school, there was where he had his snack and we made up with the children, on the spot, a series of stages he had to do. [...] Other times, instead, there are longer stories [...] for example there was one that I used several times [...] about the sea monster in Camogli, to introduce the concept of multiplication. There the idea was structured in this way: through the research work and other experiences we had done before, we had focused on the idea. [...] I chose Camogli, because I had gone to the sea there at that time, near Genoa, because I was in Lombardy and it was the closest seaside... and also the children, so if they went to the sea, for them the sea was there.

Deepening the discourse on the creation of narrative contexts for the design of mathematics teaching activities, some significant experiences were evoked. Through Maria's words it was possible to recognise a reference to spontaneity and improvisation also in the implementation phase of a teaching activity, finding common ground with what had already emerged from the MM transcript. The story of the puppet, born on the spot and through an entirely spontaneous creative process, was enriched and developed by the participation of the pupils, through a teaching collaborative activity in which several voices came together.

Another example mentioned by Maria refers to a story set in Camogli used to introduce the multiplication. In the analysis phase, the recollection of this second experience was recognised as a decisive point in the narrative because it set in motion a reflection effort aimed at tracing an origin to the conditioning. The choice of a monster from Camogli comes from Maria's personal experience - her summer holiday in Camogli - but also from the consideration and desire to involve a context familiar and close to her pupils. Both aspects intertwined with the teaching design and conditioned Maria's design ideas.

> Maria (MP1): Ah, look! I just remembered this thing that I had forgotten...I think I grew up, from a very young age, very involved in words, stories and fairy tales in particular. I used to invent them, I used to create them, I also used to listen to them all the time, either because my parents read them to me or invented them, either because I had the tapes with the stories being listened to, the sound fairy tales. [...] And I used to spend hours just listening. In short, I think that then, interacting with children, having then moved on to mathematics and interacting with scientific culture [...] I always try... I realise, it comes spontaneously to me, to relate even mathematics or the sciences always a bit to my mental form. And for example, now with this Camogli thing, I remember that when I was a little girl, I had this book "Il polipo e i pirati" and it was a Ligurian legend, in which there was this giant octopus [...] it came to me now! It must have been part of this project! It probably brings back memories, sometimes very old ones, when I make things up for children...

During the narration, Maria once again tries to reflect on the origins of her spontaneous recourse to fairy tales, stories and the creation of narrative frames for the introduction of teaching activities. In this passage, it was possible to fully grasp the potential of the Explicitation Interview (Vermersch, 1994), understood as a form of guided introspection that gave Maria the opportunity, on the one hand, to evoke past experiences and, on the other hand, to access knowledge of which she was not yet aware but which, in arguing her answer, she tried to explore and make explicit. From the biographical regime (Baudouin, 2010), which made it possible to capture the lived experience over time by blurring the details in its description, we moved on to a micro-phenomenological regime (Depraz, 2012) in which, with a zoom effect, the description of events became more detailed, allowing us to retrace precise sequences of experiences - the passion for books, the stories narrated and invented, the memory of the book "Il polipo e i pirati ".

The analysis of the frequencies of the *units* extrapolated from question Q2, did not reveal any significant results in relation to the MP transcript, with respect to which, a slightly higher occurrence than the others was recorded only in reference to the *words condition/ -ed/ -ing*, *influence/ -d* and *inspir-ed/ -ation*. The reference to inspirations and conditionings was indeed found to be a nodal point in the MP narrative. Maria, through introspective investigation, was able to identify in her personal experience and memories as a child a possible conditioning to her teaching design process. In this aspect, points in common with the idea of "conditioned spontaneity" were traced: Maria realises that she "spontaneously" designs activities in a certain way - bringing mathematics or science back to her "mental

form" - and in doing so, she recalls her past experience, connecting her childhood memories and personal interests with the process of creating and designing teaching activities.

Considering the frequency analysis of the *units* extrapolated from Q1, next, the highest frequency of occurrence of units related to the idea of spontaneity was associated with the RC transcript, where the reference to spontaneity occurred a total of 4 times and the reference to the aspect of investigation and reflection occurred a total of 5 times.

> Raffaele (RC1): Most of the ideas that come to me come suddenly while I'm doing something else. [...] while I'm shopping, while I'm jogging, while I'm reading a book that has nothing to do with anything, I get ideas and then I try to develop them later. So, I can't tell you whether actually, I mean whether to classify them as ideas that come out suddenly or that are the result of a process. They are certainly, in some way, the result of a process of which, however, I am not always aware here. It seems to me that they happen suddenly, but I also wonder how much this "suddenly" is due to the stimulations I had, to the thoughts I had [...].

Raffaele attributes his creative act to a spontaneous emergence of ideas, which however also seems to involve the aspect of reflection and investigation. While this, on the one hand, deviates from the result of the frequency analysis of Q1 *units*, on the other hand, it motivates the occurrences found. Indeed, Raffaele relates the spontaneous arising of ideas, during the preparation of the lesson or the meeting with the students, with a process of which he is not always aware, but which he recognises as being conditioned by external stimuli and previous reflections (Lo Sapio, et al., 2022). Thus, the concept of "conditioned spontaneity" also seems to be present in Raffaele's account.

> Raffaele (RC2): Certainly the personal background comes up a lot in the project planning of the activities. [...] It's something that I somehow relate back to my personal background, my life experience, my school experience as a student, [...] but maybe also to the very motivation why I decided that I would be a teacher in life. More than teaching mathematics, it is really about having the opportunity to interact with students to help them bring out their potential to interact with others.

Arguing his answer, Raffaele finds a possible source for the conditioning. In particular, he refers to a series of personal experiences that he considers significant in relation to the process of teaching design. From the excerpt reported, it emerges that Raffaele believes that his personal experience as a student and his motivation, also social, to want to become

a teacher - characterised by the desire to educate students to dialogue with others and to live in a collective way - greatly influence his design choices. In this case, it was possible to detect an affinity with the results returned by the frequency analysis of the Q2 units, from which emerged, in the RC transcript, an inclination towards the experiential and social component.

Still considering the results provided by the frequency analysis of the *units* extrapolated from Q1, the lowest frequency of occurrence of references linked to the idea of spontaneity was associated with the CP, MS and TG. Here again, the analysis was deepened by means of a study of meanings and a comparison between the narrative materials available.

> Chiara (CP1): It depends! Maybe we need to distinguish between the project of the mathematical activity itself, a longer process of investigation, which leads me to search among books, websites, ideas from notes taken during conferences, and the methodology to be used, more improvised creative ideas.

In CP, the frequency analysis of Q1 *units* showed an overall occurrence of references to spontaneity equal to 1 and an overall occurrence of references to the aspect of investigation and reflection equal to 5. Accessing the meaning of the discourse, however, it appears that for Chiara, the spontaneous arising of ideas and the activation of a more methodical process of investigation seem to coexist. In her answer, she refers to the spontaneity of some ideas, which initiate the planning and to the work of defining the activities, which she considers a work of investigation for which she uses different sources (Lo Sapio, et al., 2022).

> Chiara (CP2): I have been a researcher in the history of mathematics and most of the activities I have created from scratch are inspired by my knowledge in this discipline and my passions.

> Chiara (CP3): Classroom activities usually [...] are influenced by objects, or simply ideas, that I can physically bring and work on [...]. Once I have the artefact or mathematical content to work on, a more or less long and laborious process of investigation begins to figure out how to develop the activity [...]. At this phase the creative process is strongly influenced by activities read in books, websites, notes from lectures attended, or ideas born from conversations with colleagues [...].

During the interview, Chiara reconsiders, at different times, the possible elements that condition the ideas behind her teaching design. Although not explicitly, the concept of

"conditioned spontaneity" returns, but in this case it appears connected, on the one hand, to Chiara's passion for the history of mathematics and her experience as a researcher, and on the other to her choice of artefacts and consultation of different sources. The aspect of collective reflection also emerges in Chiara: ideas also seem to arise from the stimuli derived from conversations with colleagues.

Considering again the results provided by the analysis of the frequencies of the units extrapolated from Q1, we moved on to analyse the MS narrative, in respect of which there was no notable difference between the overall frequency of references to the idea of spontaneity (1) and the overall frequency of references to the aspect of investigation and reflection (3).

> Marina (MS1): Ideas come to me and then develop spontaneously [...]. This is the result of experience, not only that, but also the result of a particular way to view things.

Marina attributes the origin of her teaching design to the emergence of ideas that arise and develop spontaneously. Again, it is possible to refer to "conditioned spontaneity". Indeed, in Marina's words, the spontaneity that characterises her teaching design process stems from a series of past experiences and her own personal view of things.

> Marina (MS1): When I design, I certainly make a canvas of what can be done, but in reality, when the activity is over, I always realise that I have done different things and much more than I had planned to do.

> Marina (MS2): We are constantly evolving subjects, every new experience... from every new experience you come out modified and so, surely then... then over time you acquire a way to view things. [...] I also had for about ten years the double role of being both teacher and tutor of students. [...] I think I have a computer full of activities that have been documented and accurately annotated, both in relation to didactic-methodological aspects, and in relation to the "Indicazioni Nazionali", social contexts.

During the interview, Marina elaborates on her discourse. The spontaneity that leads to the arising of ideas in the design phase is also found in the implementation of teaching activities that sometimes evolve in unexpected ways. Moreover, her role as a kindergarten teacher and her role as a tutor for future primary teachers have allowed Marina to collect numerous experiences of teaching activities, on which she has been able to reflect and work, taking into account the learning objectives, but also the socio-cultural contexts of reference. From a more general vision, in line with the biographical narrative regime, the Explication

Interview methodology conducted to a more specific and granular vision, which involved the micro-phenomenological regime in the narrative. The access to a more detailed reflection made it possible to focus attention on two particular professional experiences - the one as a teacher and the one as a tutor - considered, for the purposes of the analysis conducted, significant from the perspective of "conditioned spontaneity".

Also in TG, the frequency analysis of Q1 *units* recorded an overall occurrence of references to spontaneity equal to 1 and an overall occurrence of references to the aspect of investigation and reflection equal to 5.

> Teresa (TG1): So, it depends. First of all [...] I have a very clear goal to achieve in terms of learning. But then, sometimes an idea comes to me due to, I don't know, some past experience or some reading I've done over the years or recently, from a book, from a magazine, whatever... Other times I think: "OK! I want to deal with this subject" and I start to wander around... first of all on the internet and then, I consult books, magazines, I research everything I have available [...] activities, ideas, materials to propose to my students […]. So, it's a bit of a sudden and a bit of an investigation process, it depends.

Starting from the assumption that the Q1 unit with the highest frequency determines the central aspect of the discourse, in the sense of Content Analysis, it is expected that, in the TG transcript, the aspect related to investigation and research is the most significant. It can be seen from the transcript that indeed Teresa attributes a particular value to the research, documentation and consultation activity she is involved in on a daily basis. If, on the one hand, the origin of teaching design is attributed to research and investigation work, on the other hand, Teresa recognises the presence of a more spontaneous element that seems to be conditioned precisely by the reading of books and magazines, by the teaching materials collected, but also by past experiences. Thus, the idea of "conditioned spontaneity" is also present in TG.

> Teresa (TG2): I certainly have a clear picture of what was passed on to me when I was a young girl, that is, the model of teacher, of teachers [...] that I had, from primary school to high school. [...] I have models that I refer to. I hope I am not always, sometimes I am, like my mathematics teacher in high school. She didn't consider us at all [...] she would write with one hand and erase with the other...She was so anxious to finish the program, to say everything she had to say, without worrying at all if we understood, if there was some passage that wasn't clear. She would go on her way and then collect marks from the written and oral examinations. In this sense certainly the example I had as a child is

> foundational for me. On the contrary, in primary school I had a teacher [...] who had a very laboratory approach. So, I have always, in my career, tried to emulate that kind of approach. Of course, much of the work I do in the classroom is the result of the study I had at university [...] of the research I did afterwards with the Association ["Matematici per la città"].

In the processes put in place for the creation of teaching activities, Teresa also finds connections with her experience as a student. Alternating the biographical register with the micro-phenomenological one, she re-elaborated particular personal experiences, recalling examples and teaching models that have oriented and still orient her design choices. The experience as a student has allowed Teresa to appreciate the potential of a laboratory approach that seems to continue to be preferred for her teaching actions and her practice in the classroom. From the excerpt reported, also emerged the importance of university and post-university education, the latter strongly characterized by the numerous opportunities for discussion and co-designing within the cultural association "Matematici per la città" of which Teresa is a part.

In all the transcripts analysed it was possible to find ***references to improvisation and spontaneity in the teaching design phase***, identifying in this aspect a first *recurring theme - context unit*. The ***references to external stimuli and conditioning as sources of the spontaneous arising of ideas*** has instead been recognized as a second *recurring theme - context unit*. Through this second phase of analysis, it was possible to note the appearance of *units* Q1 also in correlation with *units* extrapolated from other questions, which allowed to enrich the results obtained from the analysis of the frequencies of occurrence.

6.5.2 Context analysis of Q2 *units*

The question Q2 was formulated precisely for the purpose of investigating the origin of the creativity of the interviewed teacher, but the frequency analysis alone did not in all cases succeed in extrapolating relevant results concerning the propensity of the interviewed teachers towards particular topics. Focusing on Q2 *units*, we chose to deepen the analysis by investigating the reference context in which these *units* appear in the interviewed teachers' narrative, in order to make explicit the value with which they were understood in the discourse.

Following the alphabetical order of the transcripts, in CP, some of the Q2 *units* recur exactly in the answer to this question.

> Chiara (CP2): I am also influenced by the social and cultural context in which I worked: a school in the Aurora area, a poor neighbourhood with a strong presence of recently immigrated families. I worked there for four years and I fell in love with that context and how much richness these students can give you, who come from the most varied places in the world, who have family experiences behind them, sometimes terrible, sometimes very sweet. [...] Once I understood how to "catch" these students, each in his own way, of course, my creative vein increased, because I saw how much my effort paid off. Generally, this effort in design choices consisted of creating laboratory activities with artefacts of various kinds [...]. The presence of foreigners in the classroom, from different countries around the world, also inspired me for activities that involved the students and their different cultures.

In the excerpt quoted, it is possible to find the *words experience/ -s; cultur-e/ -al, social, inspir-ed/ -ation* identified from question Q2 and used by Chiara to draw attention to fundamental aspects she takes into account in her teaching practice. Chiara recalls her experience as a teacher in a particularly difficult neighbourhood of Turin from a socio-cultural point of view and reflects on the influences that this experience brought to her design work. The "effort" to which Chiara refers in her account is translated into the creation of laboratory activities involving the use of artefacts. The appearance of *units* extrapolated from question Q2 and the reference to the aspects that had just emerged were then also found in response to question Q3.

> Chiara (CP3): […] First I try to identify the target audience for the activity, prior knowledge and skills, cultural context, more or less effective teaching methods they are used to, and the mathematical skills [...] I would like them to work on. For example, with the Next-Land project, we defined that the activities should be aimed at students in the first classes of middle schools, located in social and cultural disadvantaged contexts. We also decided to focus on activities that would develop a positive attitude towards mathematics through laboratory activities in extra-curricular contexts.

Chiara, outlining the work phases that characterise her teaching design process, identifies the analysis of the context as a first point to consider, regarding which she takes into account the students in terms of their mathematical competences - previous and upcoming - the effectiveness of the teaching methodologies to be proposed and the social

and cultural context they belong to. In arguing her answer, Chiara recalled specific examples at various times during the interview: the collective design and subsequent realisation of scientific posters by the students, the discovery of numbering systems in use in ancient civilisations through Chinese chopsticks, the Roman abacus, the Egyptian symbols, and the activities planned for the Next-Land project. All examples where the focus on context, students and on the different cultures involved was perceived to be central. The creation of teaching activities based on the laboratory aspect and the use of artefacts thus made it possible to associate purely mathematical objectives with moments of confrontation, encounter and dialogue between students, finding possible links with other disciplines as well.

The frequency analysis of the Q2 *units* had shown a slightly higher occurrence than the others in correspondence with the *words culture-e/ -al* and *social*. Through this second phase of analysis it was indeed possible to identify in Chiara's transcript a particular sensitivity towards the cultural and social aspect, confirming the result obtained from the quantitative analysis.In addition, it was also possible to find an initial confirmation of the results obtained from the frequency analysis of the Q4 units, in which a correspondence with the *word context/ -s* was found among the highest occurrence frequencies. From the perspective of Content Analysis and with reference to the question explored, this word delimited the boundaries of the dominant topic of the text analysed. Indeed, in the CP, a particular attention to context emerged, both in terms of social and cultural context, and in terms of the educational context to be constructed through classroom practice.

Considering the FL transcript, the centrality of the theme of experiences emerged from the results provided by the frequency analysis of the Q2 *units*. This result was also confirmed by an initial exploration of the context. As already done for the CP transcript, it was chosen to deepen the analysis also for FL, considering the *units* extrapolated from question Q2, with the aim of identifying other interesting elements for the characterisation of *mathematics education creativity*.

> Franco (FL1): [...] when you turn didactics into a laboratory, which is my fixation, [...] the interesting thing is precisely to try to grasp the many intelligences of the group of children or the group of adults, put them together and discover one of the main things in life which is: together we think better! And this is really fundamental because one is always enriched by the sensitivity, by the <u>experiences</u> of another, and if one cooperates and does not

> compete, in saying "I am better than you", there are many, many things that one discovers.

From Franco's transcript emerges the potential of laboratory activity that allows each student to cooperate with the others, in the absence of competition, and to enrich their own experience through the experience of the other and together with the other. This further investigation made it possible to identify a link between Chiara's (CP) interest in connecting different cultures and Franco's idea of trying to grasp the multiple intelligences of students in order to understand that "together we think better". From this perspective, the desire to turn teaching into a laboratory seems to be understood as a way to "bring closer" distances and put different worlds together.

> Franco (FL2): I think that the culture of the teacher counts a lot, culture in the anthropological sense, that is, not only what he studied, but the city where he lived, the family where he was born, the places he frequented. For example, there was a good friend of mine who [...] used to say that the best teachers are those who have not only been teachers, but have also done other jobs. And I think he was saying something very meaningful. Because if you have other work experiences, you have a more open perspective anyway, a bit different [...]. This is to say that we should always enrich ourselves with skills that are not only school-related, in short. I think that a good teacher should be very curious... First of all he or she should read a lot, watch movies, go to the theatre, listen to music, have a rich, varied cultural life. […] I have taken great advantage of the other professions I have done, of this double profession I have done all my life, because on the one hand I was a teacher at school, and on the other I was an educator at the Casa Laboratorio di Cenci, a trainer of teachers. [...] Certainly the theatre also taught me a lot about the method.

Other *units* extrapolated from question Q2 were identified in the reported extract. In particular, what emerged from Franco's words was an idea of culture understood in a broader, "anthropological" sense, incarnating various aspects and experiences. In fact, by the word culture, Franco seems to mean a series of personal experiences - the city where he lived, the family where he was born, the places he frequented - and of education - what he studied, the jobs he has done - that also encompass the social aspect, in the sense of living a "varied cultural life" through meeting and confronting others.

> Franco (FL2): […] It was all very much influenced by the theatre experience, by living in the countryside [...] and certainly by art. Art has been a great source of nourishment for me, both the art of theatre and the love of painting.

The sharing of a series of experiences, situations and actions that Franco considered culturally meaningful, steered the reflection in a precise direction that made it possible to perceive the relevance attributed by Franco to his "double profession", his interests and passions. Recalling the idea of a friend of his, Franco emphasised the importance for a teacher to collect different experiences.

> Franco (FL2): […] Let's say that at school we neglect some fundamental elements, that is, concentration is an extraordinary ally of experience, of the depth of experience [as a student]. Sometimes, at school, [...] we neglect some fundamental elements of attention which are for example space, reciprocal positions, body position. For instance, I am convinced that you do geometry better standing than sitting... building, moving... certain things are better learned outdoors than indoors. [...] The body is somewhat disheartened by school, segregated... the living, vibrant body is always seen as the enemy and that is wrong.

This second phase of analysis also brought out a new connotation of experience which, in accordance with the results of the frequency analysis and with what has just been observed, appears to be a central point in the FL discourse. In fact, in Franco's transcript, the value of the experiential aspect is not only recognised in reference to itself - in terms of personal experience and educational training, or more generally, from a cultural point of view - but also in relation to the school experience offered to the students through teaching design. In the excerpt reported, in fact, Franco puts the spotlight on the students' school experience and on the value attributed to the space and the corporeal aspect for learning purposes. What emerges, then, is a focus on the students, on individuality and on the class group, which seems to be expressed through the design of laboratory teaching activities, in which movement and participation with the whole body is encouraged, and in which space is designed and organised to foster attention and concentration, with a view to making the learning experience meaningful.

On the other hand, the frequency analysis of the Q2 *units* had not provided a clear result in relation to the GA transcript. It was therefore decided to investigate the context in which the selected *units* appear within the text.

> Giancarlo (GA2): So, I have no hesitation in answering that, in my opinion, one hundred per cent, we are involved from the point of view of our personal knowledge and skills and that precisely, in the project design, in addition to our personal influence, our ideas, there is also precisely, the context and the culture that surrounds us. I would have great difficulty in setting up an activity where I do not have the opportunity to put my ideas and my knowledge and I would have great difficulty in any case in not having the social and cultural context in mind. In my opinion, you cannot plan and it is very difficult to do so by keeping out our personal, professional, social and cultural components.

Answering question Q2, Giancarlo focuses on the influence that personal, professional, but also social and cultural components have on his teaching design. In fact, Giancarlo believes that his personal, professional, social and cultural background originates and influences his design choices, but adds that in order to design he also needs to know the social and cultural context of the other, towards which his designs are oriented.

> Giancarlo (GA2): […] I remembered when I started doing this work, that I was a physics student. They asked us to organise science dissemination activities. As well as being a student, I was also a basketball coach, so the dissemination activity I tried to design, and which we then proposed at a science festival, was a laboratory on the physics of sport. [...] So I proposed this laboratory, which was the result of my personal experience... of my two personal experiences at that time... which were the fact that I taught children basketball and the fact that I was a university student.

In the argumentation of his answer, two of the three references to the *word experience/ -s*, extrapolated from question Q2, appear. In particular, the use of the laboratory as a teaching methodology is also present in GA. In fact, Giancarlo recalls a particular moment in which his design was intertwined with his experience as a university student and basketball coach, which led him to design and propose a laboratory-type activity that involved physics and sport at the same time.

In the course of the interview, Giancarlo returns to some points that have already been discussed, taking up and clarifying certain aspects.

> Giancarlo (GA3): When I design teaching activities there are phases, points that I follow. First of all: try to understand [...] for whom this activity is intended. To get an idea of the social and cultural context in which the intervention is taking place. Have an idea of the starting level of knowledge of the target group. […] What happens to me most often is

> that when I'm designing I do this exercise of creativity [...] (which) consists of putting myself in the shoes of the child or student who has to receive the proposal I'm structuring. [...] If, putting myself in their shoes, what I've designed and we've designed I like and it works then, let's say, it's something we propose, otherwise we're constantly trying to go back, to take ourselves out of the child's shoes, to go again to do... to design.

The excerpt shown has been chosen with the intention of focusing on two particular aspects that are related to some observations made earlier. In the first part of the narration, Giancarlo seems to focus on the socio-cultural context of the students to whom the educational interventions are addressed, which seems to motivate and orientate his choices in the educational sphere. In this aspect, similarities were found with Chiara's reflections (CP) who identified context analysis as the first stage of her teaching design process. The centrality of the students and the attention to their socio-cultural context seemed to be able to be understood as an opportunity for confrontation and encounter, themes already encountered in CP and FL.

In the second part of his narrative, Giancarlo refers to an "exercise of creativity" that he uses for the design of his activities: it involves putting himself in the shoes of his students, trying to imagine the potential of the proposed teaching experience. In this aspect, a meeting point has been identified with reflection on the student's lived experience, already present in FL. The teacher, in order to make the experience meaningful for the student, imagines the activity through his eyes, "puts himself or herself in the shoes of the student" and invents, plans and possibly rethinks his teaching proposal. The frequency analysis of the Q2 *units* had shown a slightly higher occurrence for GA than for the others in correspondence with the words *experience/ -s*; *culture-e/ -al* and *social*. This second phase of analysis revealed a real relevance of these aspects in Giancarlo's narrative. Recalling a particular anecdote, he emphasised the importance of his personal experience and educational training in his project choices. Moreover, the centrality of the social context also emerged, understood with a double valence and perceived as a necessary element closely intertwined with his teaching plans.

Considering the MM transcript, the analysis of the frequencies of the Q2 units revealed the centrality of the theme of experiences, the social and educational aspect. By exploring the context in which the Q2 units appear, it was possible to confirm the result obtained from the quantitative analysis.

> Marco (MM2): Surely one's personal history, the experiences one has had, the ways and places one has worked, in and out of school, shape you, absolutely. What I notice is that [...] the things I come up with are always collaborative situations. [...] I always come up with ideas of collaborative games where the team has to work together [...] and this necessarily comes a little bit from my somewhat personal history, where all the most beautiful things I have done personally come from collaborative situations between people. When we were working in Scampia with friends... we were designing together. So, there is really a propensity to think of learning, of work, as work done in a group [...] So, when I design I think of [...] situations where there is little competition, where there is time, where we reflect, where we work together, where we think together. [...] I always tend to propose things that respect and look at my education, my experience, and also situations where I have been as a pupil and then also as a teacher, where I am comfortable.

From the excerpt reported, the experiential, social and educational aspect indeed appears to be a key point in the MM narrative. The situations Marco proposes to his students and the classroom climate he tries to build seem to be strongly influenced by all those personal, social and professional experiences in which Marco finds himself or has found himself comfortable and which he perceives or has perceived as formative and didactically significant. During the interview, Marco recalled occasions of collaborative and cooperative work that seem to have characterised his path as a student, his university period, but also his educational training and teaching experience in Scampia, where there are many situations of economic, social and cultural disadvantage. Sensitivity to the social dimension was thus perceived both in relation to teaching design and its implementation with students. Marco seems to prefer and value moments of co-designing, with respect to which points in common were identified with the issues highlighted above. In particular, a connection with the idea of collective reflection was evident, which also emerged in relation to the concept of "conditioned spontaneity". Learning also seems to be perceived as a social activity and a "work done in a group" that necessarily guides the teaching implementation in this same sense. In fact, it seemed that a collaborative and cooperative teaching atmosphere - also understood in the sense of a laboratory - was preferred and designed to incorporate different individualities and create moments of encounter. Taking into account the individual student but also the group, the situations proposed by Marco are designed, as in FL, to allow students to experience collective reflection, in the absence of competition.

The explicit reference to the social and cultural aspect is also taken up in MM in the answer to question Q4.

> Marco (MM4): […] If you put the social in the things you do, things make sense. Every time you put an element that has to do with reality, especially the reality close to the students, [...] that are in the time of the students, that are in the space of the students, that have the social and cultural question in them, in my opinion these are the things that work the most. [...]

As well as the focus on the individual student and the group, the interest in the social and cultural context of the students, already present in the transcripts analysed, also returns in Marco. In particular, taking into account the socio-cultural situation that characterises the students' everyday life, Marco emphasises the potential of mathematics education. During the interview, through a number of examples such as analysing the issue of transport from a mathematical point of view, making maps, redesigning the neighbourhood football pitch, Marco focused attention on the mathematical discourse which, when close to the students' reality, particularly involves and motivates them, making it possible to reflect even on issues that are not strictly mathematical.

Returning to the theme of educational training, in the course of the interview, Marco expressed his views on professional training, highlighting its strengths and weaknesses.

> Marco (MM2): One tends to teach, for better or for worse, according to one's own experience [...] because educational training is what you do to yourself. It's not that all this educational training allows you... I also see my colleagues, educational training is very cold, it's very distant, it's not that it affects the way you think, or plan [...]. Who you have been as a pupil, or as a citizen, or as a person, then in one way or another it has a weight. In me, my story has weight, my experiences, in others, who have followed different paths, their story has weight. The problem is that then, what should change is the educational training, the work one does to learn new techniques, new ways. This is somewhat lacking. It always remains superficial and then, in the end, when one finds oneself in a corner, within a class, if the things learnt, the information has not become a deep baggage, something that is really deep, then one tends, in those situations in the corner, to come up with a model of how one has learnt. [...] initial educational training should train you a little bit on everything, it should give you more working tools and the ability to entertain a varied audience that at one moment needs a frontal lecture, at

> another moment needs a workshop... [...] Because anyway, educational training, in one sense or another, is not there.

Through Marco's words, on the one hand, the strong value he attributes to educational training and the incidence it can have on design and classroom practice seems evident. On the other hand, the need for an educational training that can offer a variety of useful experiences to enter the classroom and that can be common to all teachers and not particularised to personal choices seems to emerge. The point made by Marco was considered relevant and certainly stimulated reflection on a number of issues that will be addressed in detail later.

The frequency analysis of the Q2 *units* had been recorded for the MP transcription, with reference to the *word condition/ -ed/ -ing*, *influence/ -d* and *inspir-ed/ -ation*, a slightly higher frequency of occurrence than the others. This result was then confirmed by an initial analysis, through the emergence of an idea of "conditioned spontaneity". As in the other cases, it was decided to proceed with the analysis by finding the Q2 *units* in the MP transcript in order to contextualise them within the sense of the discourse.

> Maria (MP2): In my opinion, probably one important thing for a teacher is to be a person who has a good relationship with culture and the pleasure of studying, which for teachers working with young children is not always a given. [...] It is not a matter of qualification. It would be important, in my opinion, regardless of a person's background, that when they come to do this job they are clear that it should be, in some way, an intellectual job, as well as very practical, because you also have to do a lot of concrete things. [...] You need to study all the time and in my opinion, it's also a beautiful part of the job, because you are always learning things that you would never have learnt on your own, if it wasn't for your pupils. And I have to say that, even talking to some colleagues, many bring back this idea: "Ah I saw something I liked, I watched a show, I read a book, I went to a nice place..." and you find the hook to share it with the pupils and sometimes even turn it into a didactical activity.

In Maria, the same idea of culture that is present in FL and explored through Franco's words seems to emerge. Again, culture is understood in a broader sense: it does not seem to be necessarily linked to the "qualification", but instead appears motivated by the awareness that being a teacher is also an "intellectual job". In this sense, the teacher's cultural experiences - study and research, theatre, books, travel - are perceived as stimuli for

teaching design and classroom practice. Through this closer analysis, the depth of the role also tacitly played by the pupils became evident. As in the narration of the puppet story or the choice of the Camogli monster (MP1), the ideas that arose spontaneously or that took their inspiration from the teacher's personal, social or educational experience are then enriched, expanded, developed, even following different paths from those imagined, through the stimuli, needs and voice of the pupils. Sharing with the students was thus interpreted as an opportunity for several voices to encounter each other, which can also lead the teacher to discover and learn new things.

Even in the MS transcript, attention is pointed to the importance of the meeting.

> Marina (MS4): I create the group, I create effective and efficient horizontal dynamics because otherwise it doesn't work. Because learning, we all know that it is social, it is always the consequence of an encounter, of a connection.

Searching within the MS text for the Q2 *units*, the *word social* appeared in response to the last question of the interview. The social aspect for Marina seems to characterise the type of learning she proposes. In particular, considering learning as the result of an encounter, it was possible to identify commonalities with ideas that had already emerged and been explored in relation to other transcripts. The attention to the group and to the dynamics that can be created, the idea of learning that is social, find connections with Franco's commitment in the desire to grasp the many intelligences of the students, with the references to the opportunities for confrontation and encounter also present in CP, GA and MP and with Marco's (MM) sensitivity to the social dimension.

The frequency analysis of Q2 *units* had shown that NDM's transcript seemed to focus around the theme of experience, training and conditioning that inspires and influences design choices and classroom practice. This result had already been confirmed through the meaning analysis conducted in relation to the Q1 *units*, with reference to the concept of "conditioned spontaneity". It was decided, as in the other cases, to research the context in which the Q2 units are set, in order to focus attention on other aspects and possibly deepen issues that had already emerged.

> Nunzia (NDM2): I think that [...] also social relations, the type of social relations, strongly condition your being in relation with the pupils and the context that is created in the classroom. [...] Living the dimension, the commitment, the educational responsibility in a way that is not solitary, but already thinking of it as a collective process where it is

> important to share it with others, to build together with others is already a way of living even in a society [...] And this you inevitably bring into the classroom.

The experience of collective reflection, which Nunzia already mentioned in response to Q1 and found in the stories of other interviewed teachers, is taken up in greater depth in the excerpt reported, and highlights, once again, the value of the social dimension in her teaching practice. In particular, it emerges from Nunzia's words how the experience of sharing "educational responsibility" and the possibility of experiencing it as a "collective process" influence her way of relating to students and the dialogue she tries to create in the classroom.

> Nunzia (NDM2): If I think of my own life, I went to school in a town [...] where I was not really stimulated, then I went to University in Naples and then I moved to Naples. Moving to Naples has meant for me starting to frequent, I don't know, places of cultural confrontation [...] a whole series of realities that are in the historic centre, [...] meeting people from different cultures, with different experiences. [...] If I think about this part of my life, how it influences not only my private life [...] but also my professional life: it sensitises you to some issues because you get to know about them, you relate to those situations, [...] also issues connected to what it means to be a citizen, how we can, through education, contribute to a society attentive to certain values.

The move to Naples, where Nunzia currently lives and works, represented an opportunity for her to relate to a broader cultural context and to meet people from different cultures, drawing her attention to issues such as migrants' rights, gender equality or the meaning of being a citizen (Lo Sapio, et al., 2022). In this case too, it was possible to note the shift from the biographical to the micro-phenomenological regime, which allowed for a more granular narration of events, enabling situations and facts to be framed and attributed a value within the discourse. During the interview, in fact, Nunzia focused on a particular moment in her life, perceived as a real turning point that opened up new horizons and to a broader cultural context that stimulated her interest in different themes. This narrative allowed Nunzia to connect her sensitivity to the social and cultural dimension with her teaching action.

> Nunzia (NDM2): All the activities that we carry out with the association ["Matematici per la città"] always try to work on "breaking down distances" in some way: breaking distances within the city, within a territory, within a classroom, this is a climate that I feel I

> always carry inside. I live it externally in the places I frequent and it also influences the type of activity. [...] When I enter a classroom, that distance [...] with mathematics, between those who think they are not good at mathematics and therefore distance themselves, the distances between those who consider themselves "good" and those who think they are not... that's where I try to intervene: to break these distances.

Sensitivity to the social and cultural dimension also leads Nunzia to conceive the moment in the classroom in a collective and never solitary manner. Through projects that bring together different places and areas of the city, different cultures, different points of view, through collaborative, inclusive and never competitive activities, Nunzia aims to work with a view to "breaking distances". This desire also finds points in common with the intentions of other teachers, with respect to whom attention to the many intelligences of the students, to their social and cultural contexts, and to the proposal of laboratory, collaborative and non-competitive situations, as a way of bringing distances closer, creating moments of encounter and putting different realities together, have already emerged.

With the research in the text of the Q1 units, the idea of a "conditioned spontaneity" had appeared in NDM, which in the case of Nunzia, had initially been associated with two different factors: her post-university education and a series of educational training experiences recognised as significant, and the moments of collective confrontation and reflection and of shared design, widely developed within the association " Matematici per la città". This further analysis, concentrating on the context in which the Q2 units appear, has made it possible to investigate the personal, social and cultural aspect, also perceived as a source of the spontaneous arising of ideas. The personal, social and cultural experiences Nunzia refers to in her narrative influence her way of relating to and conceiving teaching practice. This influence is also perceived in her teaching design and implementation actions that start from and take into account the subjective social and cultural reality experienced by others and result in a desire to educate, through mathematics, to "break distances".

With reference to the units extrapolated from question Q2, Paola's (PL) transcript returned interesting reflections on the connection between her personal and professional experience and the emergence of design ideas.

> Paola (PL2): I am quite sure that what I propose to my pupils, as well as my way of teaching, includes everything that [...] is part of my life, everything that I can discover,

> that I am curious about, the books I read, the films I watch, the people I meet, the things that I feel are most mine [...]. My dimension of life has a profound influence on the creation of the activities, that is, in those activities there is really me! This is a certain fact and often, I try to put in what I think is important for children to discover, to know, that is, that can form them in some way as citizens.

In answering the second question, Paola states that what she proposes to her students is strongly influenced by personal interests and curiosities: books, articles, films, places and people she frequents. Paola's answer allowed for a broadening of the discourse regarding the concept of "conditioned spontaneity". Until now, Paola had attributed the "sudden" emergence of ideas to her mathematical education that allowed her to instinctively associate apparently distant ideas. Through the analysis conducted, a further conditioning was identified that encompasses the personal sphere: experiences, interests, passions. All these factors contribute to the design of teaching activities that hold a deeper motivation: to educate citizenship through mathematical discourse. Personal interests and experiences of various kinds influence the design choices: starting from the elements that characterise her own life, Paola reflects on the lives of her pupils to arrive at the realisation of a common path.

Previously, with the aim of identifying *recurring themes*, Raffaele's (RC) answers had already revealed his social motivation in being a teacher and his desire to educate students in confrontation with others and in collective living. The relevance attributed, in the RC discourse, to the social aspect confirmed the result of the frequency analysis of the Q2 *units*. Deepening the analysis by researching the *units* extrapolated from Q2, this aspect was continued to be explored.

> Raffaele (RC2): I take into account the social and cultural dimension of the class group. So, in my teaching experience one year I was a substitute teacher in a school in a disadvantaged neighbourhood of Turin, with a high prevalence of second-generation immigrants, in the Porta Palazzo district. I had classes in which most of the students were of either Chinese or North African origin [...] Until I made sure that I was recognised by the students as an adult interested in participating and understanding their social and cultural context, the teaching activity was not successful. So, what did I do [...] but listen to the students and try to understand what their interests and demands were.

Once again, the results returned by the frequency of occurrence of the Q2 units were confirmed, from which a propensity towards the experiential and social component emerged in RC. Answering question Q2, Raffaele recalls a particular episode that involved him not only in design work, but also in a relational type of work. The RC discourse returns to the focus on the experience that, in this case, stimulated Raffaele to reflect on the efficacy of his teaching design work. The period of teaching in a difficult neighbourhood in Turin seems to have enabled Raffaele to appreciate the importance of a design that considered and responded to the socio-cultural context of his students. It was only by considering the students' interests and, more generally, their context of reference that Raffaele was able to be recognised by the students as an adult genuinely interested in the needs and requirements of the class and to propose meaningful and engaging mathematics teaching activities.

Continuing with the analysis, the Q2 *units* were also located in the TG transcript in order to extrapolate their meaning.

> Teresa (TG2): My teaching approach is my personal, social and cultural dimension. So, they influence, I think, one hundred per cent [...] Obviously, I think about the students in front of me. Even in the design of a teaching activity, the basic component is what I am, which is the outcome of my personal, cultural, social experiences and then, there is the other component which is the dimension of the students I have in front of me. Of course, it's not that I go my own way regardless of the students. I propose activities, but I also recalibrate them or think about them in terms of my students.

In the excerpt reported, Teresa once again focuses attention on experience, but looking at it in terms of a personal, social and cultural dimension. This was intended as a confirmation of the result of the frequency analysis of the Q2 *units* which, although not highlighting a clearly predominant theme in TG discourse, nevertheless reported slightly higher frequency values in correspondence with the *words experience/ -s; cultur-e/ -al* and *social*. In addition to emphasising the importance of her own personal, social and cultural experience, Teresa reflects on the presence of another component that characterises her teaching approach: "the dimension of the students". In fact, her designs do not seem to prescind from the students, but are designed from them and for them.

At this point in the analysis, it was possible to identify the emergence of a new *recurring theme*, found with different shades in all the transcripts. It is the ***attention to students and their context*** as a characterising element of teaching design.

6.5.3 Context analysis of Q3 *units*

The frequency of occurrence of the Q3 *units* had revealed the presence of references to activities and projects in each teacher's transcript. However, the lack of access to the context in which the *units* are embedded has not allowed us to understand their meaning within the discourse. As was done for the units of analysis extrapolated from questions Q1 and Q2, it was decided to continue by analysing the context in which the *units* extrapolated from question Q3 appear, in order to understand the meaning conveyed and identify the presence of other possible recurring themes. By searching for the *word phase/ -s* within each transcript, attention was focused on the work of teaching design, trying to distinguish possible phases of work followed by teachers. Instead, analysing the context of the occurrence of the *units project/ -s*, *activit-y/ -ies* and *When I design*, it was attempted to understand in what terms the recalled projects and activities and the reference to the teaching design process were included in the narrative of the interviewed teachers.

In the analysis conducted, reference to work phases in the teaching design process had already emerged in correspondence to CP. In particular, in response to question Q3, Chiara had identified a first working phase, which had already been explored previously, consisting of "context analysis".

> Chiara (CP3): Phase 2: Identification of the mathematical content to convey. I research ideas for a mathematical laboratory activity, looking for inspiration from the location or the materials I have available. [...]

In the excerpt shown, Chiara identifies a second phase of work that characterises her teaching design. In particular, the choice of laboratory activities involving the use of materials and artefacts returns. In this case, however, a specific reference to the space is also added, which seems to inspire the teaching design. In the course of the interview, Chiara recalled several examples of designs in which she was inspired by the space or the materials at her disposal. One example concerned the activities planned for the Next-land project: the spiral-shaped decorations at Palazzo Madama and the Cavour cipher at the Museo del Risorgimento guided choices and objectives in the design phase. For the "Le

Strade della Matematica" project, the monument dedicated to Peano and his curve inspired an activity on fractals. In the examples concerning classroom activities, inspiration was instead attributed to artefacts, mathematical machines but also to the use of floor tiles for tessellation activities or to introduce Pythagoras, or to walls, doors and windows for reasoning about angles.

> Chiara (CP3): Phase 3: Identification of the best teaching methodology. [...] Usually the methodology I use involves the mathematical laboratory if the students are in the classroom, with the use of games and movement if they are outside the school or in the courtyard or corridor.

The third phase is attributed to the identification of a teaching methodology, represented by laboratory activities - when in the classroom - and activities involving play and movement - when in larger spaces. The examples given by Chiara during the interview recalled situations of cooperation and collaboration between pupils, re-proposing a theme already encountered previously.

> Chiara (CP3): Phase 4: Experience and redesign. After experiencing the activity in the classroom or outside the school, I keep the materials aside and write down some notes on what succeeded and what did not. If I can, I also include some feedback from students or colleagues who were able to see the activity, for example, support teachers or departmental colleagues I talked to, or fellow co-designers, or museum guides.... I then reuse the revised and corrected activity in subsequent years.

In the last phase, Chiara identifies not only the implementation of the designed activity but also a moment of confrontation and redesign. Through the collection of feedback from students, colleagues or observers, Chiara revises the designed activity, modifying it appropriately, and then reusing it later in a new version. The use of numerous examples of realised and implemented teaching activities confirmed the result obtained from the frequency analysis of Q3 *units*, which had identified CP as the predominant topic of discourse in the examples of teaching design. Analysing in parallel the frequencies obtained, with reference to all the transcriptions, in CP, the frequency associated with the *word phase/ -s* was among the highest recorded frequencies. This result was linked to the presence of clear and well-defined phases.

Compared to the FL transcription, a clear reference to a much more intuitive teaching design and free from defined work phases had already emerged in the analysis conducted.

The frequencies of occurrence recorded regarding the Q3 *units* did not show any particular predominance in Franco's transcript and some of these units have already been encountered in connection with the concept of "conditioned spontaneity". By searching in the FL transcript for other references to examples of activities and planning, it was possible to deepen discourses that had already begun regarding the centrality of movement and the involvement of the whole body in teaching activities.

> Franco (FL2): [...] I experience them with the body. I know that if I experience them with the body, I understand better. That's why I think we need teachers who move, who do motor activities, especially those who teach with young children, but with everyone. With young children it is really mandatory because the body is an extraordinary source of knowledge. The body sitting too many hours is the worst enemy of attention […].

The use of non-competitive laboratory activities in which dynamism, movement, attention to body positions and reciprocal positions are favoured, had already emerged previously in FL. In the excerpt reported, the importance of experimenting with the whole body is again emphasised, especially in relation to younger learners, in order to make learning more meaningful.

Although in FL there were no explicit references to the *units* extrapolated from Q3, in the course of the interview Franco recalled numerous examples of planning, born from reflections, research and group experiences, experimented at school, at the Casa-Laboratorio in Cenci or on other occasions. From the recounting of such experiences, aspects related to teaching design and implementation processes emerged that it was possible to interweave with deeper themes and reflections also found in other transcripts. For instance, the recounting of particular experiences in the classroom, together with his pupils, made it possible to grasp, on the one hand, the value that Franco seems to attribute to the corporeal aspect of knowledge and learning, and on the other, the importance of giving space and voice to students, appreciating their ideas, which could turn out to be "brilliant intuitions". In the analysis of the FL transcript, there were many moments that made it possible to recover experiences and reflections relevant to the study conducted. I have chosen to report below some of the most explanatory and significant considerations in the sense previously stated.

> Franco (FL2): […] I had a pupil, he was very very vivacious, [...] one day he jumped on a bench, took the plumb line I had, [...] he looked at this line from above and said: "A line

seen from here is a point", which I think is a brilliant intuition. But he did it precisely because of his bodily ability, that different way of looking was very much related to his different motor skills. He always put his head upside down and I think that helped him to think.

Franco (FL3): We can put the globe in many positions, the worst being that of the perch that usually holds it [...]. If you take it off that perch that holds it and you have the globe in your hand, you can have many interesting experiences. For example, something we discovered in Cenci many years ago: we put the globe in a position where [...] the terrestrial axes of my globe and the terrestrial one are parallel. This is an action that I always propose to the young students because I like to have this globe, I like that we mark time by saying: "Let's finish this activity when the sun rises in Bahia, or let's start breakfast when the sun rises in New York". So, they start to learn a bit about the countries of the world, but then they realise this rotation [...] and they discover that the sky goes at 15° per hour and this is a beautiful thing. [...] One time a little girl brought a melon and we drew these 15° slices on a melon. [...] Then this little girl pulls out a melon slice and says: "This is a melon hour!" and the melon hour was a 15° slice. '"The melon hour" nobody forgets!

The memory of some experiences with children, but also with adults, made it possible to recover from the FL discourse a testimony of the effectiveness of the use of artefacts and the implementation of tools - new or already existing - in teaching practice. With reference to this point as well, I have chosen to report below just a few fragments extrapolated from the FL transcript, selected because they made it possible to grasp a double aspect at the same time: the usefulness and potential of teaching activities that use tools, constructions and artefacts, and the developments, reflections and capacity to "make sense" that they hold within themselves.

Franco (FL1): We invented a lot of instruments, some of those instruments we then discovered already existed [...]. And it was also very nice, very rewarding, to discover that then one of our intuitions had already been thought up by others many centuries before. [...] There were some intuitions that came from the students themselves.

Franco (FL2): We frequently create, with both children and adults, something called the *volvella* of the Sun and the Moon. [...] This *volvella* is very interesting because it has a way of reasoning about angles from the observation of the Sun and Moon. [...] it helped me to think of a way of presenting angles to children [...] that I think is effective. [...] angles

> contain infinity, so if you draw them on a paper it is obvious that you don't have the idea of the angle, but if you take two sticks in your hand and move them, they evoke the infinity of the angle which is very important as a concept.
>
> Franco (FL3): […] Since in the didactics we try to apply at Cenci, sense is fundamental...making sense of things...then, making sense of geometry, discovering its efficacy to count nature I think is very impactful.

As in the case of Chiara (CP), the use of working phases for the teaching design had already appeared for Giancarlo. In particular, with the analysis conducted for the search for *recurring themes*, the identification of an initial work phase characterised by a reflection on the activity's target audience had emerged, in order to have a clear idea of the socio-cultural context of reference.

> Giancarlo (GA3): In the project design phase, in my opinion, one has to take into account how much available time one has to intervene in a given context, with a given group. And then, I believe, there is an important phase which is the co-construction of the activity, in the sense that it is much more difficult to design an activity alone than to share it with other people who may be other researchers, other teachers. [...] In the project design there has to be this phase: the research of sharing and co-designing the activity, comparing with other adults because then, from there, in my opinion, the most interesting things come out.

By distinguishing work phases, Giancarlo highlights the time for the activity to be proposed and the co-design work. In the search for moments of sharing and co-designing, the sense of "collective reflection" and confrontation, related to the social aspect, already recognised as relevant in the GA discourse as a result of the previous considerations, was rediscovered.

Looking for references to educational activities, during the interview, Giancarlo recalled the example of an educational path about fractions, addressed to primary school pupils. The design, born in the context of his Phd research work, has developed in concert with the mathematics teacher of the class involved and another doctoral student.

> Giancarlo (GA3): […] It was an activity very rich in stimuli, in inspirations, precisely because everyone made a contribution. In my opinion it is fundamental [...] to always try to share with someone what you want to try to do, both in the phase before the start of the activity, and during... and in itinere. Even in itinere it is important to try to understand

> if the initial project can be kept as it is or if it must be adapted, if it must converge on other paths...on other trajectories.

The process that led to the design of the teaching path was characterized by a continuous exchange, stimuli and ideas for reflection of which Giancarlo seems to have been able to recognise and appreciate the importance of both educational training and teaching.

The frequency analysis of the Q3 *units* showed in the transcript MM a predominance of references to designs and teaching activities. It was decided to deepen the analysis by researching within the text the *units* involved and exploring the context in which they are inserted. From the analysis of the MM transcript, no well-defined work phases emerged, but it was possible to recognize an usual practice that characterises Marco's didactic design. It is a phase dedicated to reorganisation, in which Marco revises online, or through texts, the material he needs, he rethinks activities already carried out and then re-propose them or get inspiration for new designs, consult texts or repeats certain topics to fix well the mathematical dimension that he intends to involve in his activity.

> Marco (MM3): In the project phase, even though it is an activity I have done several times, I always go over what the mathematical dimension is. [...] Reviewing things, repeating them, rethinking them, gives me confidence [...] it's as if one doesn't take anything for granted, one always tries to do things very carefully.

This moment of collection, reorganization and review, seems to be fundamental for Marco because it allows him to focus on the proposed activity, to look at it carefully, making sure that nothing is taken for granted.

The reference to the type of activity that Marco usually proposes to his students has already emerged during the analysis carried out, in relation to the occurrence in the text of other *units of analysis*. In finding elements that could confirm the centrality of the social aspect in the MM transcript, in fact, characteristic aspects of Marco's educational activities and designs emerged, in which it seems to be customary to use group work, with the aim of building a collaborative and cooperative educational environment. Researching in the text MM specific examples of teaching designs have found tales of activities in which it was possible to distinguish three phases. A first phase of laboratory activity, organized to allow students to explore, experiment, observe and understand what they are working on. A second phase of classroom work more structured and supported by the text and the

exercises it proposes, aimed at reorganising and systematising the discoveries made. A final phase of the final laboratory in which the findings are applied.

The frequency analysis of *units* extrapolated from question Q3 showed, in the MP transcript, a predominance of references to designs and teaching activities and only 2 occurrences in relation to the *word phase/ -s*. It was decided to continue the analysis by researching the Q2 *units* in the MP text, with the aim of understanding in what terms and with what reasons they were included in the speech.

> Maria (MP2): Now, I am reminded of when I went to Vienna, to visit a friend of mine who lives there, in the summer, and I went to see the houses designed by Hundertwasser. [...] I was intrigued by this thing that he had waged a battle against the straight line, he said that the straight line on the one hand is boring because it has no unexpected events, and on the other hand it is not natural because it does not exist in nature. [...] On the one hand, we had already done enough work with the pupils on this tree thing, so I wanted to continue it, to go deeper, and on the other hand, this straight line thing seemed very interesting to me to recover some geometry concepts and so, for example, to tell them: "But what do you think it means that the straight line has no unexpected events?". So, I came to the school and proposed to my colleagues and we started with something inspired by this trip I had taken. First the children recounted their journeys of the summer, after which I recounted mine. I showed them some pictures on the interactive whiteboard of these houses, we did some painting activities inspired by this artist, some reflections. [...] This activity had started from my trip, but then by chance I found that [...] some teachers in Milan had done this [an activity similar to the one proposed] and in 2020, for distance learning, they had uploaded a video on YouTube. Here the idea, born by chance, due to my personal experiences, came together with a research that ended up on didactic activities that others had already prepared and that others had already experimented with.

With the frequency analysis of the Q2 *units* and the first results presented, the reference to inspirations and conditionings had already proved to be a nodal point in the MP narrative. In particular, already previously (MP1) Maria, had identified as possible motivations to "conditioned spontaneity", a series of experiences related to childhood memories, interests and passions and its "mental form". Reading the excerpt, Maria's personal experience - her trip to Vienna and the visit to the houses of Hundertwasser (Figure 32) - seem to have conditioned and oriented her teaching design.

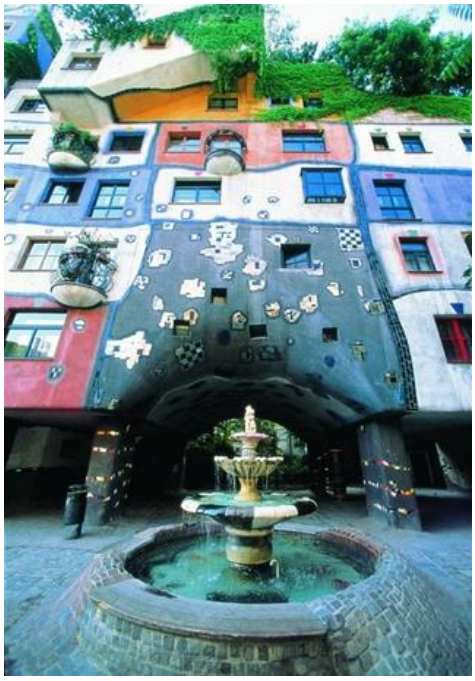


Figure 32.
In the image, taken from the site (House of Hundertwasser - Design Miss, s.d.) is shown the building designed by architect and artist Friedensreich Hundertwasser, the "House of Hundertwasser". It is a complex of fifty housing projects, born between 1983 and 1985. The facades of the houses are painted in bright colors and decorated with recycled materials, the windows are different from each other and, in general, all the structures are characterised by soft and wavy lines. Each terrace has hanging gardens, designed to bring greenery to every home, with the aim of enriching the city of trees and shrubs (House of Hundertwasser - Design Miss, s.d.).

The Hundertwasser's houses seem to be for Maria a good starting point for a geometry activity, for the design of which she chooses to confront herself with her colleagues, also allowing the importance of the co-design aspect to emerge. In the excerpt quoted, Maria also seems to attribute the origins of the design to another factor, related to a more research aspect. Interested in wanting to share an activity with her students that could be inspired by her journey and by Hundertwasser's project, she researched information about the artist on the web and discovered that other teachers in Milan had also planned and experimented with an activity centred precisely on the idea of the "tenant tree". His personal experience, the inspiration taken from other designs and the comparison with colleagues activated the teaching design.

> Maria (MP2): [...] On the one hand [the activity] provoked various reflections, then they [the students] themselves came to say: "Ah, but in this activity there is a little bit of science, a little bit of art, a little bit of geometry, a little bit of geography"... because then we found these houses where they were... which way, orienting ourselves with the sun, we were going to go to Vienna. We put a little bit of everything in and overall it looked very interesting. Yes, this happens a lot, it happens that they ask what subject we are doing. Or connections come up unexpectedly with other things we've done before, like we were just talking about these kinds of wavy lines and one little girl, among those in

difficulty, said: "Hundertwasser's houses move like a worm. Last year I found a worm in the vegetable garden".

Continuing the narrative on the planned activity, a strong multidisciplinary character emerges: the geometric aspects involved and the theme discussed also find links and connections with other disciplines. The boundaries of the various school subjects appear blurred to the point that the students themselves are not sure which subject they are dealing with. From Maria's words, it emerges that this aspect often characterises her teaching proposals which, on the one hand, seem to combine mathematical objectives with objectives of a different nature and, on the other hand, stimulate non-obvious reflections and connections, in a strongly inclusive perspective. With these assumptions, the planned activity was perceived to be impregnated, not only with a purely mathematical motivation, but also with a more social motivation that led to confrontation and the emergence of reflections on various aspects.

Maria (MP2): [...] I was very impressed by these houses here because they also seemed to me a bit like the realisation of a utopian, political dimension. In the meantime, these are social houses that I have seen, which are where some of my pupils live. For example, in Milan and Turin they have done some interesting things, the "Bosco Verticale", the "25 Verde", for the integration of trees in buildings, but here they made houses for the rich, instead he thought of making them for people who did not have the money and who had to have a house from the municipality. And then, beyond that, also an idea of community that he had, of freedom [...]. There were a whole series of political dimensions, which are the ones that belong to me, I saw them realised in some way, and so that's also why I wanted to share them, because it wasn't just an aesthetic fact, but a fact that one could reflect on in many ways. [...] You can't always make a connection with certain values, certain slightly more complex discourses. In some situations, I don't even want to impose my thoughts on children who don't have the tools to criticise or question them.

In the last part of the narrative, it becomes even more evident that Hundertwasser's project is interesting for Maria, not only from a geometric point of view, because it aims to reproduce a natural abstraction, but also for its social relevance and ecological contribution. Hundertwasser's houses are part of flats for the city's less affluent and were built with the aim of integrating trees into the urban context, points that Maria perceives as topical and urgent also with reference to contemporary society and the socio-political and cultural situation experienced by her students who, for the most part, live in social housing

and far from green spaces. The activity allowed Maria, on the one hand, to deepen a mathematical content, in respect to which it has been possible to work in a multidisciplinary perspective, on the other hand, it seems to have provided an opportunity for a collective reflection on social and political issues. Therefore, it was possible to find a social, political and cultural motivation oriented, through mathematical discourse and classroom practice, towards educating students in active citizenship.

Continuing the analysis, references to possible work phases were sought in MP.

> Maria (MP3): So, not really, in the sense that it changes a lot from activity to activity. Sometimes I modify a lot with respect to what the pupils do, maybe I notice that they are struggling with something and so I invent, even on the spot, a different way of using a material [...]. First phase: I identify the topic, [...] then I go and look at some research literature and then, I try to find a framework that is either a narrative that we then dramatise, or games to propose to the children. [...] In other cases, maybe instead of a narrative frame or a game there can be a material, and I was also very inspired by the Montessori work [...]. Then clearly, with respect to the phases, there is also that I realise and try to bring something into the classroom and then I check the feedback from the pupils and then clearly I modify it, I interrupt it or vice versa, I deepen it, I attach many pieces to it if I see that it is working. [...] I always try to be careful about that, not to push things that they don't like.

With reference to the possible distinction of design work into phases, Maria states that there are no predetermined ones that she strictly follows for each activity, but that they often change from activity to activity. However, she does manage to distinguish three moments that seem to usually characterise her design work, regardless of the activity. A first phase in which the curriculum topic to be addressed is chosen. This is followed by a second phase of documentation and research, in which Maria seems to be concerned with the creation or choice of a narrative framework or game to propose to her pupils. In this regard, reference is made to the use, construction and choice of materials, often inspired by Montessori's work. A final phase of experimentation, in which the planned activity takes shape and through the pupils' impressions and interests is modified, implemented, deepened, sometimes even with completely spontaneous and unplanned on-going interventions. The examples of activities and projects made it possible to collect new reflections and enrich situations already explored, outlining numerous commonalities between motions, choices, methodologies and objectives.

With reference to the MS transcript, the frequency analysis of the Q3 *units* did not reveal any occurrences for the *word phase/ -s*, whereas a higher frequency (9) emerged for the *word project/ -s*, *activit-y/ -ies* with respect to the other units. As in the case of the other transcripts, it was decided to deepen the analysis by searching the MS text for the Q3 *units* in order to interpret their meaning in relation to the context in which they occur.

> Marina (MS3): I don't have a pre-established package to organise my activities, but there are some things I definitely do all the time like circle-times for the decision of whatever we have to do or are doing. If we have to make a choice, if we have to solve a problem...I always do circle-times and I prefer circle-times to brain-storming because they are more inclusive. [...] I notice that by organising circle-times at the beginning the same people always speak, it's true, but then, little by little, the contributions increase. [...] They are voluntary speeches, linked to no form of obligation... that's for sure. Certainly one thing I always do and do more and more is to limit my interventions.

Marina, confirming the result of the frequency analysis of Q3 *units*, does not seem to identify any precise stages in her teaching design work. As in other cases, however, she identifies a practice that seems to be always considered and quite usual in her didactic action: the use of circle-time to make decisions and solve problems. As Marina explains in her narrative, circle-time moments are understood as moments of confrontation and dialogue in which, over time, she has chosen to leave more and more space for the students' voices, limiting her interventions. It seemed possible to interpret this attention as a willingness to create an encounter of several voices and a propensity to construct moments of sharing in the classroom, in which the students' free interventions are intertwined with the teacher's mediation. Such moments were perceived as being oriented in an inclusive direction, probably motivated by Marina's idea of social learning and close to the desire to educate students also in collective living.

With respect to the Q3 *units*, frequency analysis showed that the NDM transcript seemed to focus, in the sense of Content Analysis, around examples of project planning and teaching activities. By searching for such references within the NDM text, an attempt was made to deepen the analysis conducted also from a more qualitative and interpretative point of view, through an exploration of meanings.

> Nunzia (NDM3): Reflecting on how I usually prepare to enter the classroom, I realise that I always do: to think, when I design an activity, of what situation this activity is going to create in the classroom, that is, precisely what context, what relationships could, let's say, be triggered within the group [...]. I project myself right into the classroom thinking about the context that is created. Thinking about this, I realise that a creative process is really activated, [...] I try to create a situation... I imagine creating different spaces in the classroom where perhaps there are topics posted on the walls and where the students have to choose which topics they feel most uncertain about or which they want to return to, and which ones they feel particularly safe, calm and which they want to discuss with others. So, already in relation to an activity of this type, of recapitulation, of picking up the thread of the discourse, I, projecting myself into the classroom situation, always try to modify the space, to understand how that activity can create dynamics and relationships in the class. [...] I realise that sometimes it's enough to really unsettle them from the point of view of expectation, [...] they expect to have something already done, instead I try to build it together with them. […] I often use objects that I bring into the classroom, with which we then build activities.

Nunzia tries to reflect on her usual way of preparing for entering the classroom and, as already happened in the analysis of other transcripts, she does not seem to identify precise and habitual work phases. She does, however, manage to distinguish a moment that seems to characterise all her teaching plans, exclusively devoted to reflection on the situation to be created in the classroom with the class group. The numerous references to activities and plans within the NDM transcript, also revealed by the frequency analysis of the Q3 units, made it possible to explore themes that had already emerged and also shed light on new aspects. Nunzia imagines the context she could build, the situation and the class atmosphere that her activity could give rise to. Nunzia's interest in conceiving the moment in the classroom in a collective manner, paying attention to the "relationships could be triggered within the group" and to the possibility of building knowledge together, thus combining an educational objective with a social objective, was found in this intention. From the reported excerpt, Nunzia seems to identify in this imaginative effort of projection in the classroom, the activation of a creative process that leads her to include in her attention to the context also an attention to the space and objects to be used in the classroom, which appears to be fundamental to orient the activities in the chosen direction. In the course of the interview, Nunzia explored this aspect in depth, which seemed to profoundly motivate the teaching design: the dynamics that Nunzia seeks to foster surprise

the students, but at the same time make them active participants in the construction of knowledge.

Continuing, the analysis of meanings went on to search for the Q3 *units* in the PL transcript.

> Paola (PL2): I give you the example of the project on the enlargement of the map of the municipalities, where did that come from? It came from the fact that over the last ten years, for personal reasons, I have interfaced a great deal with the administrative bodies of my city [...] and I have realised how important it is for a citizen to know how the administrative machine works [...]. And so, to me, the idea of transferring this into a project in which the young people had to enlarge the map of the municipalities and also find out more about how the city is made, came from that. If I had not experienced a certain phase of my life, I would not have come up with that idea.

In answering the second question, Paola refers to a particular project, which appears to be strongly influenced by a series of social and cultural experiences she has been able to collect over time. In the excerpt quoted, Paola recalls a particular experience and recounts how her political commitment led her to design an activity in which mathematical objectives were intertwined with objectives related to citizenship education, which she considered crucial for her students. Dialogue and discussion on topics involving fields other than just mathematics, and a commitment to also focus on aspects related to being a good citizen were perceived as related to the desire to "breaking distances" and creating an encounter.

> Paola (PL3): So, let's say that when I design an activity there is usually first an introductory phase that I use to explain to the students where I want to take them. [...] Very often they are quite rich and broad projects [...] interdisciplinary or multidisciplinary, depending on the situation. [...] When we go a little more specific instead of the mathematical activities, there is always a phase in which I give them a problem, something concrete to solve that they solve intuitively, using mathematics in an intuitive way [...] and then a formalisation phase.

Answering question Q3, Paola seems to distinguish several phases in the design process, each dedicated to a precise aspect of the work with her students. The first phase, which she defines as the "introductory phase", is aimed at identifying the problem to be presented to her students. The second phase concerns a reflection on the mathematical and more

practical activity, always followed by a final phase aimed at formalising and generalising what has been discovered.

During the interview, Paola recalled several examples of designed and experimented activities, many also as part of the association "Mathematici per la città". In the "Indovina chi viene a scuola" ("Guess who's coming to school") project, for example, mathematical logic activities were combined with moments of discussion and dialogue with politicians, journalists, actors, directors and exponents of different cultural spheres whom Paola's students had the opportunity to interview. In the "Le regole del gioco" ("The Rules of the Game") project, designed to investigate the relationship between mathematics and football, the mathematical dimension was flanked by the playful and sporting aspect, which involved the students in dynamic and moving activities, often carried out in the open air. In general, it emerged from Paola's transcript that the mathematical problems posed to the students are approached in a laboratory mode and in a spirit of collaboration and cooperation.

Continuing, the analysis moved on to consider the RC transcript, in respect of which the Q3 *units* were searched in the text.

> Raffaele (RC3): Usually, I realise that I follow phases: the first thing I do is try to fix what the objective of my activity is. [...] I always try to make sure that there is a dynamic: both an educational content goal and a social goal with respect to the classroom context. I work with secondary school students so, especially in the initial stages of the year, with the first ones, there are social goals: creation of the class group, construction of the learning mode... which I always try to work on [...]. Then, after the goal definition, I move on to the preparation of the laboratory and initial activity. Well, I realised, thanks to the questions you asked me, that my project planning is always rather detailed in the initial phase, that is, when I design an activity, I have the initial phases very clear but I don't define very well from a certain point on what will happen because I first want to see what happens in class and how, with the discussion with the students, I can, I don't want to say adjust, but perhaps a bit bend on the actual interests of the students, on the things that come out of the discussion in the laboratory that I had prepared. [...] Let's say that they are rather cyclical phases, in which, from time to time, after each meeting I reflect briefly on what came out and try to make adjustments, and this redesign actually, I realised, happens quite often during the course of the activity.

Raffaele outlines precise working phases that structured his teaching design process. In the first phase, with the definition of a social objective, it was possible to recognise a meeting point with Nunzia's (NDM) reflections concerning the creation of the classroom context. It also emerged, an explicit reference to the proposal of laboratory activities, which leave ample space for the students to reflect and express themselves, identifying paths that do not appear as already predetermined. In fact, in reflecting on the work phases, Raffaele notes that the activities are well defined initially, but not very detailed at the end, precisely to allow the activity itself to take shape through collective discussion in the classroom and the contribution of each student. In the last phase, a cyclical aspect is highlighted, which has already emerged in reference to other transcripts, in which Raffaele experiments and reflects on what has emerged in order to adjust, expand or redefine the initial design.

Finally, the TG transcript was considered, with respect to which the analysis was deepened by searching for the Q3 *units*, in order to appreciate its meaning in a broader context.

> Teresa (TG3): So... about the phases that I adopt when I design an activity... [...] certainly there is a continuous research phase [...]. For example, I often [...] prepare a schedule that is inclusive of my ideas, of what I see others proposing [...], I prepare a schedule and if I don't write it down I have it clear in my mind.

Thinking of her teaching design activity, Teresa identifies a phase dedicated to research which, from what has also emerged from the answers already analysed, seems to occupy a prominent role in her professional practice. In this phase, in fact, Teresa documents, researches activities and materials to propose to her students, consults texts and websites from which she draws inspiration for her teaching design. This phase seems to be propaedeutic for the creation of a schedule, in which Teresa combines her project ideas with ideas from external stimuli. Through the recounting of anecdotes and examples of activities, it was possible to find important elements that characterise Teresa's designs and in which it was possible to recognise points in common with what had already emerged from other transcripts. The experimental, laboratory and visual approach seemed to be favoured. The use of various types of resources emerged, such as tools, manipulable materials and technological means, which would valorise the sensitive experience, the active exploration and the collaboration of the class group, with the aim of achieving a more organic and precise formalisation.

6.5.4 Context analysis of Q4 *units*

The analysis was concluded by searching within the collected transcripts for the Q4 *units*. The results are presented below. With the aim of identifying the presence of other recurring themes, it was investigated a possible reference to the "game of the people", understood as a reaction to an attention to context and a search for engaging themes or activities for the students.

For the FL transcript, the frequency analysis of the Q4 *units* had recorded a higher occurrence with reference to the *word game/ -s*. This unit was therefore searched for within the FL transcript, in order to detect a possible interest of the interviewed teacher in the "game of the people" and to shed light on the interpretative aspect underlying this interest.

> Franco (FL4): Painting comes to mind right now... drawing, painting... Children are generally fascinated by drawing, especially when they are young [...]. Children like to draw well and sometimes it fascinates them to draw nature, to draw houses and of course they always stumble into the question of perspective. So, in my experience, perspective is always something that fascinates... now, I don't know if it's a "game of the people", but I am convinced that it is something that fascinates. So, for example, Emma Castelnuovo, who is always in my memory, used to say that painters came to encounter perspective in an intelligent way two centuries before mathematicians and she used to say: "They got there before mathematicians because they needed it, they had a concrete need to paint" [...]. So, for example, I know that if we enter into a cultural content, as perspective can be, as shadows can be in painting, children are more attentive, because they are inside a game, a perceptive game... they observe... you look better if you have a question in your head. I always remember a wonderful visit we made to the Uffizi, with a class of mine, in which we were chasing shadows. That is, they went around all the Uffizi looking only at the shadows, how the shadows were represented.

In Franco's answer to question Q4, it was possible to find the units of analysis referring to the game and involvement. In particular, as can be read in the excerpt shown, Franco starts by considering painting and drawing, and then links up with the idea of perspective, through a process that also recalls the memory of Emma Castelnuovo. In arguing his response, Franco focuses on a particular experience he had with his students, in which he emphasises the importance, from the point of view of involvement, of dealing with cultural content. The

idea of painting that seems to emerge is that of a practice that habitually manages to engage and fascinate students, especially younger ones. Similarly, perspective and the search for shadows also appear as themes that are able to capture the students' attention.

> Franco (FL4): Children also come to school with some knowledge and the interesting thing is precisely that from that knowledge they build their own knowledge, articulated, complex... The more you value what they already know, the more they understand that their knowledge is important. The more you tack on knowledge that you want to pass on to them, neglecting what they already knew, the more you risk that the school's knowledge becomes a detached thing, something that does not concern their life, their experience, something that concerns the school. [...] That's the problem with school: if the problem of the exercise book doesn't somehow, sometimes, intertwine with my problem, why do I have to think that mathematics is useful to me? I do maths because I go to school, but I forget it as soon as possible, because it bores me. Or, some people get passionate about maths because they are the ones who get passionate about maths, like the ones who get passionate about music, sports, literature... However, the school should have the ambition to get all and sundry as passionate as possible about something.

In the conclusion of his argument, Franco emphasises the importance of starting from knowledge already possessed by the students, in order to build a new knowledge that enhances what they already know and propels them towards something new and more articulated.

From the frequency analysis of the Q4 *units*, in CP, a higher frequency of occurrence was recorded with reference to the *units fascinat-es/ -ing*, *engag-e/ -ing/ -ement* and *context/ -s* which, from the perspective of Content Analysis and with reference to the question explored, were understood as central topics of CP discourse. This result was partly confirmed in the discussed exploration of meaning, through which it was possible to find interesting reflections in CP that trace a focus on the context of its students, both in terms of the social and cultural context and in terms of the teaching context to be constructed through classroom practice. With the desire to investigate a possible reflection on the "game of the people" in the teaching design phase, the *units of analysis* extrapolated from question Q4, which had not yet been analysed from the point of view of their meaning and the context in which they appear, were searched within the CP transcript.

> Chiara (CP4): With the new generations, it is not easy to really understand what really fascinates and engages students, and it often depends very much on the student. It sounds trivial, but certainly engaging activities are those that require the use of devices, such as mobile phones or tablets. [...] I'm still not sure, however, that from these experiences there was really an engagement on the mathematical content, but only on the methodology and the tool itself.

Answering question Q4, in reflecting on possible "games of the people" for her students, Chiara traces an involvement in the technological aspect. Experiences with educational and interactive sites seem to capture the students' attention, but they also raise new questions in Chiara: how much of this involvement is also associated with the mathematical content?

> Chiara (CP4): Certainly activities with artefacts and where there is movement are engaging. I am thinking for example of Next-Land activities, even there, the artefact generates curiosity and engagement, in this case also about the mathematical contents itself [...].

The use of artefacts, which has already emerged in relation to Chiara's design choices, and the mathematical activities involving movement and the use of the whole body, seem to resolve the question that has arisen. These aspects are in fact perceived by Chiara as elements that manage to capture the students' attention, in the sense intended by Freire, while also allowing them to be involved with the mathematical content.

> Chiara (CP4): I have to say that the activities where I have seen some of my students light up, get really passionate [...] are the maths competitions [...]. In groups they took part in these competitions, which are held nationwide at the same time, and which in real time return the partial ranking. It is true that the students are "selected" and so you would think that they already love maths and it would be easy for them to get hooked, but many of them were not particularly brilliant at maths, yet you would see their eyes shine when they managed to solve a problem correctly, joining forces all together. The team game and the stimulating context of the challenge, [...] allowed them to be involved and focused, and to be able to solve particularly tricky problems that are not usually tackled in class. Here, students [...] can give vent to their imagination and creativity, use and compare their own arguments and methods of solving, without worrying about remembering formulas or definitions by heart or answering according to the teacher's

> expectations. Another relevant aspect of such competitions is the collaboration, within the team, between students from different classes. Each student makes his or her knowledge and skills available to the group. Almost every answer is the result of the work of several students and discovering that it has given the team points brings an exciting wave of enthusiasm […].

The transition to the biographical regime allowed Chiara to recall and share a particular anecdote. From the excerpt reported, teamwork and collaboration among students seem to trigger particular group dynamics that enable them to solve mathematical problems and questions through the work, knowledge and skills of all. Chiara's rich argumentation in response to question Q4 seemed to reveal two important aspects: the game, which fascinated and involved her students in mathematical activities, and the cooperation, which initiated discussions and reflections, also of a mathematical nature. In Chiara's narrative, the importance of conceiving the teaching moment as a collective moment of cooperation and exchange, in which the game seems to find a place, also appeared to re-emerge. In the interview, Chiara also emphasised how valuable such a teaching approach could be in schools located in socio-culturally disadvantaged contexts, where the playful approach and team play could be useful in recovering pupils' motivation, overcoming the linguistic limitations typical of the school's users and identifying or cultivating excellence in mathematics.

With respect to the GA transcript, a focus on the context of the students in the design phase has already emerged during this second analysis step. By searching the GA text for the *unit game/ -s*, an attempt was made to investigate the presence of references to the "game of the people" as well.

> Giancarlo (GA4): In my opinion, the playful component must never be lacking and... however, the playful component is one that somewhat reflects the society of the time, in the sense that today, there are games and practices that allow you to resonate with the students much more than practices that were used in other times. The first work to be done is to try to understand what the games are... the game as a concept. [...] You have to, in my opinion, have a very critical eye on what are the trending themes of students in our times and certainly, let's say, one major area is the use of technology... and everything that orbits around it. The use of social, the use of technology in education has always been a theme but, from my experience and what I think, it is very difficult to use technology as a tool that gives you something extra. [...] Today, in my opinion, there is a

> need to do a lot of self-criticism on the part of teachers and researchers on what are the ways and strategies to use technology in a more effective way and not as a tool for its own sake [...]. Every didactic tool has its potential, and it must be made clear that in certain situations one can use pen and paper, in others, perhaps it is better to use technology. In this way you give more meaning and value to the objects and proposals that are proposed.

In his answer to question Q4, Giancarlo refers to the game, understood as playful practice, and to the importance of proposing games that reflect the society of the time and that are perceived by students as akin to their interests. Like Chiara (CP), Giancarlo, in reflecting on possible "games of the people" for his students, identifies an answer in the technological aspect, considered as one of "the trending themes of students in our times". Previously, with Chiara (CP), there seemed to be an involvement related more to the technological and interactive aspect and less to the mathematical content, even with Giancarlo, limits to the use of technology seem to emerge. In fact, in the excerpt reported, Giancarlo emphasises the need to use technology in a more conscious and effective way, designing activities that can include it on a different level from the simple substitute role of paper, pen or book. From this perspective, according to Giancarlo, only by trying to extrapolate new, different and creative uses of the available technologies can they become relevant for teaching purposes. Taking a measurement through a technological instrument, observing a phenomenon through a camera or using a smartphone, interacting with people at very great distances, are just some of the examples proposed by Giancarlo in the course of the interview, through which he suggests the use of technology in the educational field. In this scenario then, technology, considered interesting and captivating for students, seemed to be able to be included in a teaching activity with a singular role that cannot be replaced by other tools, being necessary for the purpose of the activity.

In the MM transcript, the frequency analysis of the Q4 *units* showed a predominance of occurrences in relation to the *game/ -s unit*. The reference to the game has already emerged in the MM discourse, connected to the kind of teaching activities proposed and the desire to make the pupils participate in a collaborative and non-competitive game. On the basis of the reflections made, the analysis was continued in order to obtain new important considerations.

> Marco (MM4): […] Talking about what's going on in the neighbourhood, drugs, social situations, social isolation [...] how many cultural centres there are, how many spaces...

> talking about their world is definitely the part they actually follow the most. [...] Because they open up a direct perception, you are telling them: "There is a way to understand what is around you, how society is going around you, your neighbourhood, your city. If you understand what it means to analyse data, you can understand what's going on". When you start to touch on social issues and also stimulate them... the fact of saying that this group can change, can reflect on their neighbourhood, on their social condition, when you talk for example about shoes, brands, the market, advertising, you make them perceive that actually knowing things, they can transform them.

In Marco's speech, in response to question Q4, the importance of interweaving educational design with social and cultural issues seems to emerge. Thus, the mathematical dimension is introduced to students to enable them to reflect on the reality in which they live and with which they are confronted everyday. In this sense, mathematics seems to be perceived as a useful tool "to analyse data" and "understand what's going on".

> Marco (MM4): I have always understood this political action to be very strong. The fact that the class begins to think with this phase of statistics, or probability [...] A little bit when you talk about young people and having the tools to know the world they live in, youth problems. That's a bit... I think I'm doing a bit of that operation there, of making them aware of their condition as kids who live in difficult neighbourhoods, who if they don't get up they go like statistics, they become part of the statistics, part of the numbers. In this sense I mean the question. [...] A nice game, I call it a game because I like the term, but it could be a didactic activity that leads precisely to an awareness. To use the neighbourhood to investigate it, to investigate the condition of young people, to propose a transformation, even with technology, of a space, of a square [...]. This is a multidisciplinary activity that I think should be done, that schools, with Italian, with mathematics, with technology […].

Marco traces in this educational action also a political action: students, through the analysis of their own reality, understand it and become aware of it, and determine the ways in which they can transform it.

For the MP transcript, the frequency analysis of Q4 *units* indicated a prevalence of references to the *game*. It was decided to continue the analysis by finding the *units* in the text and then taking into account the context in which they appear.

> Maria (MP4): I also thank you for this question because it stimulated me a lot to reflect, in a perspective I had not thought of. [...] I realised that after all, yes, it's a type of reflection that I do and that I think is also very common among us primary teachers. I also remember talking about it with a colleague a few days ago. He had made a similar point about technology, in the sense that he said: "What fascinates them (the students), what fundamentally engages them now is this, is the screen. If we can find a way to use these screens, in a way that is useful for teaching, in my opinion it can be a first hook. Then it is clear that we cannot do it for everything, because they also need the body dimension, the manipulation". I also agree with him, he has put forward a reflection in this vein similar to Freire's, as I interpreted it. [...] I use it (technology) of course, to work, for what is necessary, [...] but on the one hand, I think there is also a lot of urgency in this age range, in which they are very young, precisely because they are so much on screens outside, at least we can have this role of making them recover what they do not do outside of school, that of making them recover the body and the hands.

Question Q4 allows Maria to reflect in a perspective she had not thought of, enabling her to recognise in her teaching design an attention to the "game of the people". In particular, in arguing her answer, Maria refers to a moment of confrontation with one of her colleagues who seems to trace technology as a subject that fascinates and engages pupils. As already emerged in CP and GA, the aspect related to technology reappeared, again, recognised as a captivating element, but again questioned in reference to the educational sphere. In the words of Maria, who states that she agrees with her colleague, her idea of the double role of the teacher seemed to resurface: on the one hand, the teacher is called to perform a "practical" job, on the other hand, the teacher is called to perform an "intellectual job". So, from this point of view, while recognising the practical and involving aspect of technology, Maria probably seems to perceive this duality, which leads her to propose in the classroom also, and above all, new and meaningful learning dimensions and situations in which her pupils can "recover what they do not do outside of school". With respect to this last point, the reference to laboratory activities involving movement, exploration and the participation of pupils with the whole body comes back to MP.

> Maria (MP4): I observe a lot of what they do, like the games at school when they are free, which can be at break time but then, also, I have to say... I leave enough moments where they can do an activity of their choice [...]. So, it's important for me to see what they choose to do and one thing I've noticed a lot that they do is to build, to create with paper, which is the material they have most at their disposal. [...] Or, there was a time

> when an origami game was all the rage in class that someone calls "Fortune teller" [...] Another thing, for example, that they do a lot is to exchange notes, messages, things like that. Sometimes they have little newspapers in which there are secret codes or they invent them [...]. Then there are the card games that are always very successful, for example, they are great for starting to do a bit of arithmetic, a bit of addition... the "game of the goose", things of that kind, even if some that were more "games of the people" for my generation, for some of them are no longer so, partly because some students come from other countries, partly because time has passed and things have changed [...].

In arguing her response, Maria tries to reflect on possible evidence of an attention to the "game of the people" in her usual teaching practice. To this purpose, she shares some examples of situations that have inspired and oriented her teaching design. These are activities and games chosen and practised by the students in their free time at school. Maria's interest in understanding what fascinates and engages her students allows her to identify particular practices, playful and otherwise, and to create teaching activities through them, in the sense of the "game of the people" as understood by Freire. During the interview, Maria recalled some examples of activities carried out in this sense: geometric constructions inspired by origami, activities on measurement through the manipulation of paper, work on symmetry from secret codes found in magazines.

> Maria (MP4): Another thing is the market game. [...] Children from families in economical difficulties, they are with the thought of money because they probably perceive that it is a problem and it is what would allow them to solve the problems. [...] We are also in a neighbourhood where there are students who are children of those who sell at the neighbourhood market stalls [...]. There are some children who have a lot of mental calculation because they grew up in that context. [...] One year we were able to do the school newspaper, sell it, do a charity market for Christmas, the second grade children were at the till. They were there having to calculate the remainders and all these things here. We trained beforehand to do this and this was something they were very involved in. [...] This started from the idea that... I could see them there among themselves, even at break time, playing, selling and buying, you try to do it a little bit with fake money, you see that they start, they get excited. [...] Then we found a way to bring it back to real life as well, and that was one thing that worked.

In this last example proposed by Maria, attention to the context of reference also emerges: Maria, observing and taking note of the reality in which her students live and with which they are confronted, designs an activity that takes this into account and contextually succeeds in involving mathematical discourse as well.

The frequency analysis of Q4 units had recorded zero occurrences of references to games for MS, and in general, for this transcript, no interesting results had emerged. In this second stage of analysis, it has been possible to explore the meaning of the *units* researched, and sometimes it has been noted that although it was not possible to find a specific *unit* in the text, it was still possible to access content that was significant with respect to the research purpose of the present study.

It was also chosen to deepen the analysis by MS, trying to trace useful elements for the conducted study.

> Marina (MS4): I try to work...get things started by the children, get things out of the children or the students. [...] We started the maths laboratory with a song, [...] it's a technique devised by Giordano Bruno but used a lot in schools, but also in groups, where simple songs are sung and repeated verse by verse: I say the verse then the others repeat it. We had to do a maths laboratory and I arrived with this rhyme to repeat. Everyone was a bit like that. Then I finished the rhyme and read two passages from Plato's Timaeus. "Professor, at one point, I thought we were doing it all wrong" [a student says]. But, in the end, what happened... when we went on with the activities, actually there was total engagement.

Starting from an initial consideration regarding her usual way of designing, Marina emphasises the importance of starting from the students, giving them space and a way to "get things out", through the dialogue and the laboratory, which seems to be a teaching methodology also used by Marina. The excerpt quoted was also chosen for the evidence of a change of register, from biographical to micro-phenomenological, through which Marina shifted from a broader and more general discourse to a more meticulous and detailed narration of a specific moment: a mathematics laboratory for teachers and university students. In response to question Q4, it was not possible to trace explicit references to the game, but the proposed laboratory activity, the involvement of the audience through song, and the reading of two passages from Plato's Timaeus, made it possible to interpret the activity in terms of a "game of the people". From the excerpt reported, it was

also possible to note that in this activity cited by Marina, as well as in the examples reported and analysed above (MS3), Marina appeared to be attentive to the group and its creation.

Analysing Nunzia's (NDM), Paola's (PL) and Teresa's (TG) transcripts, it was often possible to identify references to co-designing and experimentation experiences carried out within the association "Matematici per la città". Indeed, the connections and interweavings between their narrations were evident, which also underlined the recognition of certain moments as significant on the part of all three. Through the contribution of all of them, it was possible to achieve interesting results in terms of attention to the "game of the people". It is therefore for this reason that it was decided to present the analysis conducted on the NDM, PL and TG transcripts consequentially, which was done both by considering the individual transcript and through a global view that included all three.

With reference to the NDM transcript, the frequency analysis of the Q4 *units* had recorded 40 occurrences of the *word game/ -s*, indicating the highest frequency recorded among all the transcripts considered. The analysis was therefore continued by searching for Q4 units within the NDM transcription in order to identify possible references to "game of the people".

> Nunzia (NDM4): The theme of the game [...] has always been a very central theme in our activities: we ["Matematici per la città"] have worked on many types of games such as sports or even, the last activity we built on the tombola, which is also a game of our Neapolitan tradition, revising it, reworking it, reorganising it, associating it with other things [...] also considering this game to be a truly formidable tool for creating that context I was telling you about at the beginning. [...] I'll tell you the truth, I keep myself a bit distant, distant up to a certain point from digital tools, that is, we do use digital tools, but if I had to think about the games that the generations I'm dealing with now use, and so these are on digital systems, well, I haven't worked much on those with the students. [...] I prefer games that create dynamics within the classroom and not that isolate, in some way, the individual with the digital tool [...]. Role-playing games, games in the classroom, games with artefacts, games like tombola, doing maths from the game of football, sport, volleyball, we had fun thinking about things like that. When I was trying to describe to you at the beginning this mood that had to be created in the classroom, it was really a game mood, which I always try to bring out, to live in [...]. Playing around the city was also another expedient we used in our teaching activities. [...] So, when I design, the games I think of are games where there is the possibility of relating and making contact.

The extract from the answer to question Q4 contains several references to the game, which motivates the result of the frequency analysis of Q4 *units*. In NDM, the game seems to be perceived as a tool capable of activating a series of dynamics that Nunzia considers fundamental for the construction of a peaceful, stimulating and collaborative classroom context. In the reflection on possible "games of the people" there appears a reference to digital tools and technology, with respect to which Nunzia seems to choose to remain distant, preferring group situations that are more collaborative and inclusive.

This aspect also emerged in Paola's transcript, following the search in the PL text of the *units* extrapolated from question Q4.

> Paola (PL4): First of all, the game element is often present in the projects that I, together with the association ["Matematici per la città"], propose. [...] I asked myself about the "game of the people" [...] in the case of "Le regole del gioco", for football, which is not my game, but it is certainly one of the games of the people for excellence [...]. It has to be said that usually, when we start from a game with the association, we often start from a game that in some way is a classic, traditional game, such as football from a certain point of view, but also "Indovina chi?", but then it is revised and remodelled according to what we want to do. So there is a revisitation, if you like in a mathematical key, or at any rate in a didactic key, of a certain game. And this is something that we do very often and almost never, we have done it starting from online games because the idea is to take them off the screens and this year, let's say, more than ever[...]. The game that we proposed and are continuing to propose to a group of students was just that: the "Tombola infernale" which puts together Dante, mathematics, the Inferno, drawing, coding... and is a real game that we are building.

As in NDM, Paola's answer to question Q4 also contains several references to the game. In her transcript, Paola refers to the activities designed and implemented as part of the association "Matematici per la città", resuming and enriching some discourses already started with the analysis of the NDM transcript. In particular, the decision not to consider digital games is once again emphasised since, as also emerged in MP, "the idea is to take them (students) off the screens" and to propose them laboratory, manipulative activities in which group dynamics can be established. Paola also reflects on the choice to involve the game of football in the "Le regole del gioco" project. This choice, far from Paola's passions, stems from a reflection on the students' lives and interests and seems to be framed within the perspective of the "game of the people". Paola also discusses the work done within the

association "Matematici per la città" in relation to mathematics teaching activities involving or starting from a game. In these cases, the choice of a game often seems to fall on classic or traditional games that are then reshaped and revisited through a didactic-mathematical lens. A clear example reported by Paola is that of the "Tombola Internale" which is a revisited version of the classic tombola game, dedicated to Dante's Inferno and to mathematics. The project to which Paola refers was entirely designed and realised during the Covid-19 pandemic period, by the association in collaboration with a group of students. It too seems to be considered in the context of the "game of the people".

Continuing, the analysis moved on to the TG transcript, searching the text for the *units* extrapolated from question Q4.

> Teresa (TG2): I am thinking for example of this year, a recent example... I had to talk about the parabola, about conics [...] and I asked myself how I could do it [...]. And so, I remembered that when we studied conics with the association ["Matematici per la città"], and therefore with young students, we did it starting with light. So, I told my students to bring torches, we darkened the room, we created, based on the angle of the torch, we saw the conic sections.
>
> Teresa (TG4): […] When we did the activity on conics, the class got enthusiastic!

Teresa, like Nunzia (NDM) and Paola (PL) also remembers a particular experience she had with the association " Matematici per la città", which she then used to design a teaching activity with her students at school. Starting with a game, visual and perceptual, it was possible to involve the students in a mathematical discourse.

> Teresa (TG4): When I design I always ask myself the question of how to work while also entertaining them, because I think that if the students feel involved, if their spirit, if their soul is smiling inside, they remember the experience... they are more relaxed, less frightened by what could be... they also release them from certain anxieties that accompany them almost constantly. I don't always manage to find activities that have this kind of approach, partly because of time, partly because of a personal limitation of mine that I hope to improve over time... but yes, always, I wonder if I can get them to play, to get them interested with activities that are not always the usual ones.

The examples given by Teresa in the course of the interview - the study of conics through the light produced by torches, experiments with Geogebra, the transition from graphic

register to the algebraic one - made it possible to perceive, even in the description of the practical performance of the activities, a focus on the experience she intends to provide for her students.

The analysis concluded with the search for Q4 *units* in the RC transcript, in order to deepen certain aspects that had already emerged and possibly arrive at new considerations.

> Raffaele (RC4): I remembered an anecdote from the beginning of last year [...]. I thought of the various practices, things, games, activities that interest the students that I have observed during break-time or when they are not doing the structured activity but are given the chance to express themselves and do what they want to do, and I thought of the various trends of the moment... I'm thinking of when it was the trend to throw the water bottle to make it spin and make it stand up or the various spinners that were used... In short, the things that interested them. [...] Last year there was this situation where, the way the school was set up, there were stairs to climb, two classes had to go up to the second floor after break-time, first one and then the other, and at a certain point, these two classes, my first-year class and my second-year class, started competing over who had to go up first and then, they started spending the whole break playing various games to see who would win and who would lose. I kind of jumped on the bandwagon a little bit when they started playing "odd or even" to see who was the winner and so, with the first-year students we worked on the game of odd or even, with various variants, [...] all issues that are part of the maths curriculum of a first year middle school, which I was planning to work on later in the year, but it was good for me at that moment there to take inspiration from their game to work on it. [...] We also thought of another game [...], of doing odd or even with multiplication [...]. They then convinced the second-year students, who had not thought about it, to challenge themselves to this other game, which was more advantageous for them, and they won continuously for several days [...]. At the end of this [...], actually the learning they achieved was significant and consolidated because, my reinterpretation is... because they actually associated it with a theme that interested them, with a game that could actually be what Freire calls the "game of the people".

Answering question Q4, Raffaele evokes a particular experience he had with his students in which a reference to Freire's "game of the people" seems to emerge. The activity originated, as in MP, as a result of an observation of the students in free and non-study moments at school. Raffaele notices some quite usual games and practices that manage to involve

and interest the students. Specifically, in the challenge to win the flights of stairs first, Raffaele identified a good pretext to stimulate mathematical reflection, strongly motivated by the desire to win. The activity, which arose from observing common practices among the students, then developed spontaneously, dragging with it a series of mathematical contents that Raffaele planned to explore with his students.

A more accurate analysis of all the transcripts allowed to identify the presence of new *recurring themes - context units*. In particular, in all the narrations analyzed it was possible to note the ***references to laboratory and collaborative activities, in which the sensitive experience, movement and involvement of students with the whole body are emphasised***. Despite the singularity of the experiences shared by the teachers interviewed, analyzing the transcripts, it was possible to identify another *recurring theme* in the ***realisation of opportunities for encounters and desire to educate, through mathematics, to bring together different "worlds" and to "break the distances"***.

6.6 Coding into *content categories*

As extensively discussed in the previous section, the analysis conducted made it possible to recover from the stories of the teachers interviewed the existence of common reflections, recognised in terms of *recurring themes*:

- ***References to improvisation and spontaneity in the teaching design phase;***
- ***References to external stimuli and conditioning as sources of the spontaneous arising of ideas;***
- ***Attention to students and their context;***
- ***References to laboratory and collaborative activities, in which the sensitive experience, movement and involvement of students with the whole body are emphasised;***
- ***Realisation of opportunities for encounters and desire to educate, through mathematics, to bring together different "worlds" and to "break the distances".***

These *recurring themes* were considered as new *units of analysis*, specifically as *context units*, to be organised in the previously defined *content categories*. Through the coding of the *context units*, the *recurring themes* "References to improvisation and spontaneity in the

teaching design phase" and "References to external stimuli and conditioning as sources of the spontaneous arising of ideas" were both attributed to the *category* "conditioned spontaneity". In the transcripts analysed, there were clear references to more spontaneous, sudden and instinctive ideas in the teaching design and implementation phase, for which all the teachers tried to find an explanation. In fact, in arguing their answers, the teachers interviewed attributed various motivations to the spontaneous arising of ideas in their teaching design activity: personal experiences and memories, moments of shared reflection and design, university and post-university training, research and documentation through a variety of sources. These elements were interpreted from the perspective of the studies presented that enabled the definition of the *category*, in other words, as factors that, interacting with each other, seem to influence the thought processes involved in the creative act. In all the cases analysed, it was possible to recognise in the accounts of the teachers interviewed references to the idea of "conditioned spontaneity", explicitly stated in Nunzia's transcript (NDM) and found with different reasons and associations in all the transcripts analysed.

On the other hand, the recurring theme "Attention to students and their context" was attributed to the category "Attention to the context". With Teresa (TG) emerged the reference to a "student dimension" that was recognised, through different routes, by all the teachers interviewed. In fact, in arguing their answers, the teachers declared an attention to the students both in terms of their interests and needs, and in terms of the socio-cultural context of reference. With Maria (MP) and Paola (PL), what emerged was the desire to involve a context in their teaching designs which, starting from a reflection on their pupils' lives, was perceived by them as familiar, a desire also found in Marina (MS) with her attention to learning objectives and socio-cultural contexts of reference. The centrality of these aspects was also found in Chiara's (CP) account, in terms of attention to her students' context, both as a context of being part of, and as a context being constructed through didactic practice. Interest in the "student dimension" also emerged in Franco's (FL) transcript as attention to individuality and the class group, in order to make the learning experience meaningful. In this perspective, it was possible to find connections with Giancarlo's (GA) idea of designing by putting himself in his students' shoes. Also in the transcripts of Marco (MM), Nunzia (NDM) and Raffaele (RC), emphasis was placed on the attention to the social and cultural context of the class group with the aim to achieve mathematically meaningful designs that are also responsive to the context of reference. The analysis conducted thus revealed that teachers, in different ways and with different

motivations, found the "student dimension" to be a key element that oriented their design choices. The *context unit* "References to laboratory and collaborative activities, in which sensitive experience, movement and involvement of students with the whole body are emphasised" has also been incorporated in the "Attention to the context" *category*. In this case, the meaning of context corresponds to an attention to the class group and to the educational and didactic context that each teacher constructs and proposes to his or her students through definite design and methodological choices. In some cases, regarding the proposal of a collaborative and cooperative teaching environment, it was made explicit the use of materials and artefacts, explored and used with the students, but also chosen, conceived and constructed together with them. Thinking of the moment in the classroom in a collective manner seemed to be a choice shared by all the participants who, in the course of the interview, highlighted, each in their own way, the multiple potentialities of this didactic methodology. In this sense, the design of collaborative and non-competitive activities was intended as the teachers' focus on the individual student, the class group and the learning experience that they intend to offer through teaching action and classroom practice.

Finally, the coding of the *context units* made it possible to assign the *recurring theme* "Realisation of opportunities for encounters and desire to educate, through mathematics, to bring together different "worlds" and to "break the distances" to the *category* "Mathematical education objectives intertwined with citizenship education objectives". This assignment arose from the fact that, when analysing the stories of the teachers interviewed, it was perceived that there was a willingness to incorporate individualities and different worlds into teaching designs and classroom practice, in order to create moments of encounter through mathematical discourse. In particular, there seemed to emerge an attitude of thinking of the teaching moment as a social and collective moment, oriented towards opening up inclusive, democratic and stimulating reflection spaces, capable of welcoming the ideas and voices of each student. The multidisciplinary nature of the activities and the commitment to bringing together, through mathematics, different cultures, realities and points of view have made it possible to identify in the proposed projects the presence of mathematical objectives intertwined with objectives of a different nature, oriented towards "breaking the distances" and developing a critical conscience in the students.

CHAPTER 7

DISCUSSION

"I cannot understand educational practice except as a complex and contradictory totality. I think about the importance of all the components of this practice, but I recognise, because educational practice is uniquely human, that the importance of the educator is extraordinary. And this is why a creative educator, a liberated educator or educator in the process of liberation, an educator who puts him or herself at stake, who ventures out, who is not afraid of freedom, an educator who is capable of loving, of loving even the process of education itself, of loving his or her own practice in which he or she finds himself or herself, an educator who invents and reinvents methods, techniques every day, is capable of creating where apparently nothing existed. This is the kind of educator we should help to exist!"

(Freire, 1989, January 23)

With the aim of drawing general conclusions that enable the answer to the research question delineated, in the present chapter, crucial points of the research study conducted are taken up and critically reviewed. In particular, the analysis and interpretation of the emerging findings highlight significant aspects, through which we achieve the focal point of the discussion: the structural definition of three *dimensions*. These *dimensions* make it possible to characterise the *mathematics education creativity*, representing a significant arrival point for this research work. It should be clarified, however, that the conclusions that are proposed are not intended as definitive points of arrival in a broader research perspective. In this regard, in light of the analysis conducted and the considerations that have emerged, the last paragraph is dedicated to a suggestion of possible future research developments.

7.1 Key points of the research path

The analysis conducted made it possible to investigate the way in which the teachers interviewed recount their experience of creating mathematical activities, making it possible to identify elements to which they seem to attribute the origin of their creative acts. In particular, the frequency analysis of the *recording units* represented a first stage of analysis

that yielded an interesting result, but only of a quantitative nature. The frequency analysis of the *units* extrapolated from Q1 counted, in the transcripts analysed, references to the aspect of research and reflection and references attributable to the idea of spontaneity in the interviewed teachers' teaching design and implementation processes. The information returned delineated a varied picture but only related to the occurrence of specific *words* in each teacher's discourse and still considered unclear with respect to the subjective experience of the teachers interviewed. The frequency analysis of the *units* extrapolated from Q2 provided quantitative information on the presence of references to personal, social and cultural experiences in each teacher's transcript. Again, the results were considered provisional because the occurrence of the recording *units* certainly provided interesting data on the presence of certain topics in each teacher's discourse, but on its own, it did not allow us to understand the role that personal, social and cultural experiences occupy in the teachers' creative process. The frequency analysis of the *units* extrapolated from Q3 made it possible to gain an insight into the references to examples of design and teaching activities in each teacher's narrative, which also made it possible to identify the use of potential work phases that characterise the process of teaching design. Again, in order to obtain a more comprehensive result, the need emerged to contextualise the *units* researched in the sense attributed to them by each teacher in the course of the interview. Finally, the frequency analysis of the *units* extrapolated from Q4 returned quantitative information about the interviewed teachers' references to game, context and involvement, revealing an interest in these aspects, but without understanding their underlying intentions. These initial results provided interesting information, outlining in some cases, the presence of commonalities between the transcripts analysed. They were considered as partial and provisional results since the quantitative data alone did not allow the *units of analysis* to be framed in the sense attributed to them by each teacher during his or her interview. This aspect strongly motivated the passage to a second phase of analysis which concerned the exploration of the context in which the *recording units of analysis* appear within each teacher's transcript, allowing the identification of *context units of analysis*. The quantitative results obtained as a consequence of the frequency analysis were contextualised through the exploration of the meanings attributed to the *units of analysis* researched. The excerpts of text extrapolated and reported were chosen precisely to guide the observation of what emerged from the analysis conducted, focusing attention on the most significant aspects, but also allowing us to appreciate the value of the Explication Interview. In some cases, in fact, the shift from the biographical regime to the micro-phenomenological one made it

possible to grasp a deeper reflection, which often enabled the teachers interviewed to understand things that did not seem to be explicit yet.

Hence, the second phase of analysis, with the search for *recurring themes*, made it possible to attribute a contextual meaning to the *recording units* researched and to arrive at a more complete result, referring to the subjective experience of the teachers interviewed and in which the quantitative part is in communication with the qualitative and interpretative one. The *recurring themes* that emerged were subsequently organised into the three defined *content categories* and coded accordingly:

- The recurring themes "***References to improvisation and spontaneity in the teaching design phase***" and "***References to external stimuli and conditioning as sources of the spontaneous arising of ideas***" belong to the *content category* "**Conditioned spontaneity**";
- The recurring themes "***Attention to students and their context***" and "***References to laboratory and collaborative activities, in which the sensitive experience, movement and involvement of students with the whole body are emphasised***" belong to the *content category* "**Attention to the context**";
- The recurring theme "***Realisation of opportunities for encounters and desire to educate, through mathematics, to bring together different "worlds" and to "break the distances***"" belongs to the *content category* "**Mathematical education objectives intertwined with citizenship education objectives**".

From the coding phase, it emerges that no *category* was left empty and reflection on the *context units*, contained in each of them, could lead to tracing significant aspects of the interviewed teachers' teaching design and implementation with a view to characterising *mathematics education creativity*.

7.2 Findings

The operational definition of the category "conditioned spontaneity" allowed for the inclusion, on the one hand, of references to more spontaneous and instinctive ideas and, on the other, references to possible conditioning. Taking into account the coding into content categories, the recurring themes attributed to this category include references in

this sense. The teachers interviewed seem to recognise the presence of illuminating moments that characterise not only the teaching design phase, but also the implementation phase. The spontaneity to which they refer does not, however, appear to be completely free. The teachers, in different ways and at different times, in the course of the interview, seem to trace the presence of a range of factors that stimulate and influence the spontaneous arising of ideas, conditioning the creative processes underlying their teaching design. In this effort of reflection and research, oriented towards identifying possible causes to the processes involved in the creation of teaching activities, the Explicitation Interview appears to be fundamental, assuming a key role in the teachers' re-elaboration of re-evoked and shared experiences. All the teachers reflected on their own design work, identifying, in factors of a different nature, possible influences in the realisation of the innovative mathematical activities they propose, both in relation to the preparation of the lessons and in relation to the encounter with the students (Freire, 1970). Chiara (CP), for instance, refers to her personal interests and research experience in the field of the history of mathematics. She also mentions the importance she attributes to confrontation with colleagues and to her professional experience in a challenging context that required her to make a great effort to involve students. Franco (FL) recalls the memory of Emma Castelnuovo and his school experience as a student. He refers to personal interests and experiences that he considers culturally significant, to the importance attributed to her "double profession" and draws attention to particular experiences of co-design and collective reflection that seem to influence his teaching design and implementation. Giancarlo (GA) refers to the influence of personal, professional, but also social and cultural components on his educational choices. In particular, he recounts experiences of sharing and co-designing and evokes a particular episode in which his teaching design derived from the intertwining of his experience as a university student and as a basketball coach. Again, Marco (MM) refers to the reading of texts and to a series of previous work and educational experiences that he considers particularly significant. He recalls moments of collaboration and cooperation that characterised his experience as a student and as a teacher, even in socio-culturally disadvantaged contexts, and that still profoundly influence the type of teaching activities he proposes to his students. Maria (MP) recounts particular personal experiences, involving childhood memories and travel and, in more general terms, experiences recognised as culturally relevant, which have conditioned and still condition her design choices and classroom practice. She also refers to her educational training, to her experience as a teacher in a disadvantaged context and to moments of confrontation with colleagues and researcher friends, and thus to the importance of collective reflection

for didactic design. Marina (MS) refers to the social aspect and to the importance of previous experiences that seem to characterise the kind of learning she proposes and her personal view of things. As Franco, Marina also tells of a "double profession", a double role, which seems to condition her teaching design and implementation process. Nunzia (NDM) shares particular personal and cultural experiences that also seem to influence her social commitment through her militant activity in the association "Matematici per la città". She also narrates about moments of confrontation and collective reflection and her post-university education that seem to stimulate and condition the spontaneity she speaks of. Paola (PL), on the other hand, refers to her personal interests, her mathematical education that allows her to connect apparently distant elements and her experience of collaboration in the association "Matematici per la città" and in the administrative organisations of her city. Again, Raffaele (RC) refers to his schooling experience as a student and to a series of personal, social and educational experiences that have enabled him to trace in his educational activity a social motivation characterised by the desire to educate students in confrontation and collective living, which he considers a priority. Lastly, Teresa (TG), shares her experience as a student that allowed her to acquire greater awareness of her role as a teacher and refers to her educational training experiences, both undergraduate and postgraduate, which she continues to enrich through her work of research and consultation of teaching materials. Teresa also recounts the several opportunities for discussion and co-design that she experienced within the association "Matematici per la città", which were perceived as stimuli for teaching design and classroom practice.

At first glance, the picture presented appears to be extremely varied, probably because the teachers' narratives are developed around, of course, subjective and different experiences which, in the perspective of the guided introspection of the Explicitation Interview, seem to acquire a profound significance in the eyes of the teachers. However, several points of encounter emerge, the first of which can be traced in the teachers' pedagogical objectives. In fact, they all combine purely mathematical objectives with democratic citizenship education objectives. The reference *content category* contains the *recurring theme* "*Realisation of opportunities for encounters and desire to educate, through mathematics, to bring together different "worlds" and to "break the distances*"". The content *category* "*Mathematical education objectives intertwined with citizenship education objectives*" was developed to encompass precisely the references to teaching activities in which mathematics education is recognised as an inclusive activity, of confrontation and dialogue between pupils, closely connected with the development of a critical awareness

of citizenship. From the teachers' accounts, their experiences enter and mingle with teaching design and classroom practice. In this sense, the moment of planning and the moment of meeting with the students appear to be oriented towards creating situations that are never pushed towards competitiveness or dynamics that create distance, but always projected towards encounter. The didactic moment seems to be perceived as a social moment, capable of encompassing different individualities, each with their own personal, social and cultural background and their own learning mode. Below, I quote an emotional and significant passage from the book "Diario di scuola" (Pennac, 2010) in which the singularity of each pupil is emphasised, who, inevitably, on entering the classroom, brings with him his own world and entrusts to the school, and therefore, to the teachers, the possibility of making it a reason for encounter or separation.

> Our students [...] never come to school alone. An onion enters the classroom: several layers of grief, fear, worry, resentment, anger, unfulfilled desires, furious renunciations accumulated on a substratum of dishonourable past, threatening present, foreclosed future. Look at them, here they come, the body in the making and the family in the backpack. The lesson can only begin after they have laid down the burden and peeled the onion. Difficult to explain, but often just a glance, a benevolent sentence, the word of an adult, confident, clear and balanced is enough to dissolve those magons, ease those spirits, place them in a strictly indicative present. Of course, the benefit will be temporary, the onion will reassemble on the way out, and perhaps tomorrow we will have to start all over again. But teaching is just that: starting over until we disappear as teachers. If we do not manage to place our students in the present indicative of our lesson, if our knowledge and the pleasure of using it do not take root on those little boys and girls [...], their existence will waver over infinite voids. Of course, we will not be the only ones digging those holes or failing to fill them, but those women and men will still have spent one or more years of their youth sitting in front of us. And that's no small thing: that's eternity in a jar. (Pennac, 2010, p. 55)

In the narrations of the teachers interviewed, it is possible to perceive this idea of the student "never alone" who, like Pennac's "onion", participates in the educational moment with a series of factors that constitute a unique and original voice. The inclusive moments of collaboration, motivated by the idea of learning as a social and democratic activity, leave space for students' voices and allow them to encounter each other, through the mediation of the teacher. In the Chiara's transcript (CP) for instance, it is possible to recognise a focus on the students and the different cultures involved that lead her to conceive the moment in the classroom as a moment of encounter and dialogue. Franco (FL) also recognizes an attention to individual students and the many intelligences, through the proposal of

teaching activities which have the transversal objective of making pupils understand that "together we think better!". Giancarlo (GA), in his narration, sees the moment in the classroom as a precious opportunity for discussion and encounter. Marco (MM) also refers to moments of encounter in which each student actively and democratically participates in mathematical discourse. Maria (MP), on the other hand, emphasizes how educational moments that allow for democratic and inclusive encounters of multiple voices can lead to non-obvious reflections and connections. Marina (MS) also tells of inclusive moments of confrontation and dialogue oriented to educate students for collective living. With Nunzia (NDM), the idea of working from the perspective of "breaking distances" is made explicit. In fact, Nunzia tells of her commitment to combine an educational objective with a social objective, designing and implementing activities that connect different places, cultures and points of view. Paola (PL) also aims to educate citizenship through mathematical discourse, with the goal of preparing future citizens and "breaking distances" through encounter and dialogue in teaching practice. Raffaele (RC) recognises in his didactic action a strong social motivation that drives him to encourage moments of dialogue and confrontation that can prepare students for collective living. Finally, Teresa (TG) proposes to combine didactic objectives with those of education to dialogue and active citizenship. In trying to grasp the multiple and singular intelligences of the students, the moment in the classroom seems to be understood by all the teachers interviewed as a moment which, through mathematics, allows one to "peel the onion" and thus, to open up inclusive and democratic spaces for reflection, capable of bringing together different worlds and "breaking distances". In this exchange of ideas in classrooms, conducted in an active manner and understood as a meeting of several voices, it seems possible to trace the presence of an individual voice, a voice of others and a historical voice (Boero, Pedemonte & Robotti, 1997). The student voice mixes with the voice of others, the opinions and ideas coming from the rest of the class, and with the historical voices, which embody the scientific revolutions whose meaning needs to be passed to the new generations and which the teacher mediates, through his or her own experiences and conceptions. In this perspective of encounter, all teachers aim to educate democratic citizenship through mathematics, whose boundaries often appear blurred. Indeed, from the teachers' accounts, a multidisciplinary aspect of the activities often emerges which, through defined designs or spontaneous reflections by the students, allow different themes and disciplines to be involved. Operationally, this intention seems to be pursued through attention to context. The operational definition of the *category "Attention to context"* made it possible to incorporate in it references to an attention to the class group, to the individual student - ideas, interests,

needs - and to the context to be built, in terms of the learning experience offered. But also references to an attention to the social, cultural and economic context in which students live and with which they relate everyday. The *recurring themes* contained in the *category* actually reflect this dual attention, on the one hand, to the context to which they belong, and on the other, to the context to be created. In the interviews conducted, these two intentions can be interpreted in a "past or present" and a "future" sense, respectively. Taking into account the *recurring themes* belonging to the category "Attention to context", this aspect appears, explicitly or less explicitly, in all the transcripts analysed. In order to clarify this interpretation, the NDM transcription can be taken into consideration, in which this passage seems to be more evident, making it easier to grasp the double intention linked to the word context. Nunzia identifies her social and cultural experiences as an element that conditions her teaching design process. In particular, in her teaching designs, Nunzia seems to take particular account of her socio-cultural context, related to past and present life experiences. But it is also important for her to pay attention to the socio-cultural context of her students, understood as past and contemporary experiences that characterise the reality in which they are embedded. In this sense, the socio-cultural aspect appears to be perceived as a self-awareness and as an awareness of the other and recognised, in this duality, as a factor that influences, motivates and orientates choices in the educational sphere. From this point of view, the references to the context can be understood as connected to a "past or present" sense. On the other hand, in the designing phase of a teaching activity, Nunzia also imagines the situation and the classroom atmosphere that her activity could originate, projecting herself into the context that could be built. In this reference, not to an objectivity, but to a potentiality in being, it is possible to perceive the context as connected to a "future" intention.

This dual aspect, which can be understood as a further meeting point, seems to emerge especially from the teachers' reflections on the "game of the people". In particular, Franco's (FL) considerations, in response to question Q4, offer the possibility of reflecting on the "game of the people", recognising the need felt by Freire (2018) to search the programmatic content of an educational proposal starting from dimensions of reality, perceived by the students as meaningful, to be proposed to them through an engaging modality that also allows for discussion and critical reflection. In his answer, Franco (FL4) shares a particular experience he had with his students, searching for shadows at the Uffizi Galleries in Florence. Through this anecdote, Franco emphasises the importance, from the point of view of involvement, of dealing with cultural content. In Franco's shared experience, although the mathematical field of interest in this study is not specifically involved, it is possible to grasp

interesting aspects that are close to Freire's idea of "game of the people". Starting from the anecdote shared, an attempt is now made to develop possible indicators, capable of identifying a teaching activity designed with a focus on the "game of the people". In this perspective, it is chosen to associate:

- the involvement of interesting cultural content that starts from or links up with knowledge and skills already possessed by the students (Castelnuovo, 1946), to Freire's idea - also echoed by Gutstein - of involving significant dimensions of reality and therefore of dealing with themes that are interesting and close to the target audience;
- the recourse to what pupils habitually practise, to Freire's desire to use playful practices, or other kinds of practices perceived as usual and traditional;
- the observation with a "question in the head" (FL4), to the explicitation of the request or objective that characterises the planned activity, which can be understood as an essential element in the perspective of the "game of the people", since it is capable of motivating the students and guiding the educational process.

Thus, the reference to cultural contents or questions that connect to knowledge and skills already possessed by the students, the use of games - or particularly engaging activities - and the observation with a "question in the head", can be understood as three elements, possible indicators of a teaching activity designed with a focus on the "game of the people".

Answering question Q4, Chiara (CP) refers to a mathematics competition that, through team play and collaboration, particularly involved her students in mathematical activities. In the recounting of this anecdote, it seems possible to detect three elements, possible indicators of teaching activities designed with a focus on the "game of the people". In particular, the playful approach and the team game is perceived as an element that succeeded in involving and enthusing the students, asking them to apply logical-mathematical skills and abilities they already possessed. The stimulating context of the challenge and the willingness to compete together appear as particularly motivating for the students, allowing them to observe with a "question in their head". Still in response to question Q4, Giancarlo (GA) refers to the technology and the need to use it, in the educational context, in a more conscious way. In this regard, he suggests some practical examples: measurement by a technological instrument, observation of a phenomenon

through a camera or again, use of smartphones to interact with people at a great distance. Technology is understood by Giancarlo as an interesting and engaging theme for students, both in terms of content and practical aspect. In each of these examples, a strong motivation is present, both for the use of the technological tool and for the proposed activity - an aspect that can be traced back to the observation with a "question in the head". From this perspective, therefore, it is also possible to recognise in GA the presence of the three elements, which are possible indicators of a teaching activity designed with a focus on the "game of the people". In response to question Q4, the importance of intertwining mathematics education with social, cultural and even political issues emerges from Marco's (MM) transcript. In fact, Marco refers to particular mathematical contents, such as statistics and probability, which allow him to talk with his students about what is happening in the neighbourhood: for example, drug problems, social isolation. Through mathematics, real and contemporary issues are analysed - the issue of transport, map-making, redesigning the neighbourhood football pitch - which Marco perceives as particularly engaging and captivating for his students. Mathematics seems to be understood as a tool to investigate and critique social injustices (Gutstein, 2007) that enables students to reflect on the reality in which they live and with which they are confronted everyday. In this scenario, it seems possible to recognise traces of the three indicator elements of a teaching activity designed with a focus on the "game of the people". In particular, the socio-cultural context of reference that calls the children to reflect on the reality that surrounds them can be associated with the need to involve an interesting cultural content. The demand to become aware of reality through mathematics and then be able to transform it, seems to motivate and direct the development of the activity in terms of observing with a "question in the head". Finally, the possibility of a practice that engages these two elements and simultaneously involves the learners can be recognised in dialogue and confrontation. In arguing her answer to question Q4, Maria (MP) shares some situations that inspired her teaching design from the perspective of the "game of the people". She refers to pupils' activities and games that she was able to observe at times when pupils were free and not engaged in teaching activities. This observation allowed Maria to identify particular practices, both playful and not, that she used in the construction of teaching activities. Among the examples recalled - geometric constructions inspired by origami, activities on measurement through the manipulation of paper, work on symmetry from secret codes found in magazines - Maria tells of the organisation of the charity Christmas market. In this last example, in which the attention to the pupils' context and reality is evident, it seems possible to identify the three elements intended as possible indicators of teaching activities

designed with a focus on the "game of the people". In the activity recalled by Maria, it is possible to recognise the involvement of an interesting cultural content for the pupils, which links to skills they already possess - the pupils in fact seem to be particularly good at calculations due to their family experience in the neighbourhood market. In addition, an activity habitually practised by the students or with which they are familiar is involved - a market game. Finally, in the purpose of charity and the need to be able to do maths to be able to stand at the market checkout, it is possible to identify the objective of the designed activity, understood in the sense of observing with a "question in the head", which seemed to motivate and orientate the entire educational path. Also in Marina's (MS) account, in response to question Q4, it is possible to identify the three indicator elements of a didactic activity designed with a focus on the "game of the people". Marina in fact, through the mathematics laboratory and the singing, introduced and articulated a teaching activity that, on the one hand, surprised the participants and, on the other, seemed to prove very engaging. Therefore, the proposed laboratory activity, the involvement of the audience through singing, and the reading of two passages from Plato's Timaeus, allow the activity to be interpreted in terms of a "game of the people". In the examples recalled by Nunzia, Paola and Teresa, which can often be traced back to their experience in the association "Matematici per la città", the game, from which a teaching activity starts or around which it develops, seems almost always to be understood as a playful practice. The pursuit of the game is attributed to a consideration of the students' interests and preferences and, in some cases among those mentioned, there also seems to be an involvement of cultural content. For example, the project exploring the city's administrative bodies with students, the mathematical walks to discover the artistic and cultural heritage of the city of Naples, the popular game of football reinterpreted in a mathematical key, the traditional Neapolitan tombola meeting Dante's Inferno. In all the examples mentioned, the activities, often involving different disciplines and fields, appear motivated by a well-defined common thread. The learning objective, which is directly linked to the game - or possibly can be achieved through it - is combined with objectives of education to confrontation and citizenship. On the basis of the considerations made, it seems possible to identify the presence of the three indicator elements of a teaching activity designed with a focus on the "game of the people". In response to question Q4, Raffaele (RC) refers to an anecdote that leads back to the idea of the "game of the people". As in the case of MP, the activity derives from an observation of the students in a free moment in which they are engaged in challenging each other to be the first to go upstairs. This example seems to be interpreted from the point of view of the "game of the people", being able to identify in it the presence

of the three indicator elements of a teaching activity designed in this sense. The activity, arising from the observation of common practices among the students and motivated by the desire to win, developed spontaneously, dragging with it a series of mathematical contents that Raffaele planned to discuss with his students. The teaching experiences shared by the teachers and interpreted from the perspective of the "game of the people" reveal a strong point of contact with Freire's ideas and the concept of a human as a *being-in-situation*. For the design of the activities referred to, all the teachers seem to have a clear understanding of the context of their students - context understood in the double meaning interpreted - and seem to recognise that each of them, necessarily, lives in his or her own time, space and in relation to the world around him or her. The themes connected with the activities can be interpreted as possible *generative themes*, from the perspective of Freire's pedagogy (1970). In particular, in the light of the considerations made, themes involving social, cultural and political aspects of the reality experienced by the students can be perceived as the starting point of an educational process of "conscientization", understood as the practice of freedom.

Attention to context, understood in its "future" sense, can be found in the proposal of laboratory and collaborative activities, in which sensitive experience, movement and the involvement of students with their whole bodies appear to be emphasised. This aspect also represents one of the recurring themes belonging to the category "Attention to context" that seems to be deeply intertwined with the idea of "conditioned spontaneity". As pointed out at the beginning of the paragraph, the spontaneity mentioned by the teachers appears to be conditioned by several factors, including the social aspect. This aspect, which represents another meeting point in the narratives collected, seems to characterise the teaching action of the interviewed teachers - both the planning moment and the encounter with the students (Freire, 1970). With reference to the teaching design phase, collective reflection and shared design experiences return in the narratives of each teacher interviewed. The emergence of ideas is often explicitly attributed to the stimuli derived from discussions and conversations with colleagues or other experts. The possibility of experiencing "educational responsibility" (NDM2) in a collective manner also seems to influence the learning experience that teachers propose to their students - context-"future" - and the way they position themselves in the educational relationship. As has already emerged, learning seems to be perceived as a social and democratic activity, characterised by the encounter of several voices. This vision seems to be pursued by teachers through the design and proposal of collaborative and cooperative learning

contexts, through which students also experience collective reflection, in the absence of competition. Activities often turn out to be structured in laboratory form, in which the game also seems to find space. Sensitive experience, the manipulation of artefacts, movement and the involvement of students with their whole bodies are perceived by teachers as central elements in the construction and implementation of teaching activities. The democratic and inspiring principles of *active schools* seem to emerge in the proposal of this type of activity (Chapter 1 - 2). In particular, it is possible to find Dewey's (1954) idea of proposing teaching laboratories intertwined with pupils' concrete, everyday experiences, in order to allow the development of an operational intelligence and the construction of knowledge. Or Freinet's (1969) idea of pupils' learning based on their experience with the surrounding reality and cooperation with peers and teachers. Moreover, the laboratory does not seem to be understood as a physical place, rather as a space for collective reflection, encouraging critical thinking (Gutstein, 2007) and dedicated to group work with a common goal, also finding references to Paulo Freire's pedagogical vision and Emma Castelnuovo's educational practice. In particular, the idea of considering the material and sensitive experiences at the centre of the work between pupils and teacher - a guide in the process of discovery and learning - can be interpreted as a meeting point with Paulo Freire's educational proposal (1970). Furthermore, in the observation that starts from something concrete and develops through the exploration of artefacts and the involvement of pupils with the whole body, a strong connection emerges with Emma Castelnuovo's (1963) idea of "doing mathematics with dirty hands". In the teachers' answers, attention also recurs to the space, dedicated to the implementation of activities, which is often reorganised to foster learning or if necessary designed outside the school environment - courtyard, vegetable garden, mathematical walks, iconic places in the city, wide open spaces. The connections with active and popular pedagogy (Chapter 1) are also evident in this aspect. In particular, the focus on space seems to echo Montessori's (2016) intention in designing and organising classrooms to foster pupils' learning. The involvement of outdoor spaces, on the other hand, seems to reflect Freinet's (1969) idea of using walks in the countryside or village visits to enable the introduction of topics, also from a multidisciplinary perspective.

The design in this sense seems to imply a reflection on the classroom atmosphere that is intended to be created and thus, on the "future"-context. This point, and more generally, an attention to the context in its double sense, together with the search for moments of confrontation - in the phase of organising, planning or re-designing a teaching activity - emerge in the narrative of all the teachers. In the course of the interview, with question Q3,

the teachers were asked to reflect on possible working phases in the educational design process which, however, were not recognised by all of them. However, through the analysis conducted, these aspects emerged that seem to characterise, implicitly or explicitly, work phases in the process of designing teaching activities.

7.3 The three *dimensions*: a characterisation of *mathematics education creativity*

The complexity and uniqueness of the experiences shared by the interviewed teachers were analysed by the Content Analysis methodology, which made it possible to shed light on the data collected, firstly, from a quantitative point of view and, subsequently, also from a qualitative and interpretative point of view. The aspects that emerged seem to reveal the motivations behind the effectiveness, even in challenging contexts, of each interviewed teacher's educational proposals. At the same time, however, when compared to the current Italian school context, they seem to highlight the strengths and weaknesses of the most diffused teaching practices and of the most common idea of school, probably also affected by political choices over the years. Suffice it to think of the theme of teacher educational training, which emerged from all the interviews conducted and which was declined according to a different but interesting aspect by Marco (MM) - a middle school teacher. The educational training, for the secondary school, appears to be highly customised and therefore, very diversified in relation to the choices that each teacher has made and continues to make. This aspect, however, could represent a double-edged sword because, if on the one hand this "autonomy" allows teachers to deepen their own direction of interest, on the other hand, it could lead to not creating a baggage of knowledge and skills truly significant for teaching.

In the Italian context, the teacher educational training, especially for the secondary school teachers, is a delicate and important theme. Currently, in fact, in Italy, for secondary school teachers, there is not a real initial training path, common to all, that can "prepare" them for entry into the classroom. Training is instead entrusted to the personal choices of the teachers and to the experience they acquire in the field. Marco himself, during his interview, perceives current training as lacking, emphasising the need for useful and common training for all teachers, oriented towards broadening the horizon towards new teaching techniques and methodologies. Certainly, this research work does not allow this complex issue to be resolved, but it could fit into the discourse by shedding light on some interesting aspects also

related to teacher training. From the first results obtained, it seems legitimate to believe that the school and, with it, the teaching of mathematics, play a critical and fundamental role in the education of students, not only with respect to purely didactic contents. The idea that seems to emerge is that of a school oriented to forming students from an educational, social and cultural point of view or, with respect to Freire's vision, from the point of view of conscientização. In this sense, the teaching action, stimulated and conditioned by personal, social and educational experiences that imply attention to the context of the students - "past or present" and "future" - and the purpose to intertwine mathematical objectives with citizenship objectives, appears aimed at promoting a process of awareness that can allow students to acquire critical tools necessary to understand the world and recognise situations of social injustice (Freire, 2018). The teaching practices shared by the interviewed teachers and their way of positioning themselves in the educational relationship, seem to be valuable, especially in the disadvantaged socio-cultural contexts in which the teachers work or have worked, and also applicable in more fortunate contexts. In both cases, their teaching practice seems to contribute to students' recognition of situations of oppression and unequal treatment, access and justice, providing them with openings for understanding the potential of their actions in the world - in accordance with Gutstein's perspective (Gutstein, 2007). In this sense, it appears to foster a development of knowledge through a teaching practice that interweaves "the problem of the exercise book" (FL4) with the students' personal and real-life problems. In this vision, the teacher assumes a crucial role: he or she designs and implements mathematically meaningful teaching paths, activating relational dynamics that contribute to the construction of a virtuous context in the classroom, open to the democratic encounter of different worlds and voices.

In this action, of designing and meeting with students (Freire, 1970), the teachers' creative process is recognised, by the teachers themselves, as being "conditioned" by different elements that, explored, analysed and dissected, it appears possible to place in three different *dimensions*, which we choose to indicate as: ***personal dimension***, ***social dimension*** and ***professional dimension***.

Below, for each *dimension*, the elements of which it is composed that characterise its structural definition are stated.

Personal dimension

The *personal dimension* includes life experiences lived outside the school or work context - for example, family experiences, travel, in the community where the teacher lives - and childhood memories. The teacher's personal interests, curiosities and passions also belong to the *personal dimension*. In addition, the cultural and political aspect of the teacher is also included in this *dimension* and thus, the different experiences recognised as culturally relevant, the principles and political ideals that characterise each teacher.

Social Dimension

The *social dimension* includes experiences of confrontation and collective reflection, among colleagues but also with researchers or other educational figures. Also included in the *social dimension* are opportunities for collaborative and cooperative work, developed in the sense of didactic co-design, and moments of meeting and exchanging stimuli and suggestions for organising or re-designing didactic activities. The increasingly widespread on-line meeting with the educational community - for example, on-line groups of teachers, blogs or specific forums dedicated to discussion on the mathematics education system - is also chosen to be part of this *dimension*.

Professional dimension

The *professional dimension* includes all the teacher's experiences in educational training and work. The study career, university and post-graduate education belong to this *dimension*. It also includes all the training experiences sought and constructed ad hoc by each teacher, such as, for example, courses, conferences and seminars on mathematics education. But also, moments dedicated to consulting research articles, books, journals and, more generally, didactic material concerning mathematics education. The *professional dimension* also contains the teaching experiences accumulated over the years and all the possible professional experiences referring to areas and contexts other than education.

The three *dimensions* just defined are intended as aspects that characterise mathematics education creativity, thus enabling the research question to be answered. The *personal dimension* is in fact recognised by the interviewed teachers as particularly significant. Its experiences are perceived as a source of great inspiration for teaching design and implementation. From the interviews conducted, it also emerges that teachers seek moments of confrontation and collective reflection, part of the *social dimension*, which they identify as one of the motivations behind their creative process. The confrontation with other colleagues or researchers is transformed into a meeting of several voices that orients the construction of teaching activities in this same sense and in an inclusive direction that leads to perceiving learning as a social and democratic activity. Moreover, the teachers also consider all the training and self-training experiences that contribute to conditioning the spontaneity of their creative processes at the basis of their teaching design and implementation to be significant. All the teachers are or have been teachers and are involved in mathematics education on different levels that seem to provide as many different stimuli. The variety of professional experiences, not necessarily related to the field of education - research in the history of mathematics, participation in informal mathematics education projects, experience in disadvantaged contexts, but also experience as a sports coach, in city administrative organisations, the "double professions" and others - also seem to influence the creative processes, allowing them to broaden their perspectives. Referring back to Daniel Pennac's passage quoted earlier (Pennac, 2010, p. 55), teachers also seem to experience school "never alone", and just like the students, they appear as an "onion", with "several layers" that they necessarily carry with them, both when planning an activity and when meeting with their students. The "layers", which represent the three *dimensions* described, characterise the teachers' creativity, leading them to design and implement activities in the sense outlined in the previous paragraph. In this view, the teacher designs and enters the classroom bringing with him, necessarily, his "layers", the three *dimensions*, which mingle and interact with the students' dimension, to arrive at the realisation of a common path. The *personal*, *social* and *professional dimensions* depend heavily on the subjectivity of the teacher to whom they refer, who, in turn, with respect to the kind of experience gained, may or may not develop a sensitivity to a specific *dimension*. For instance, a teacher who chooses not to live the educational experience as an experience that is also social, and therefore, avoids moments of collaboration and sharing, may fail to develop a due sensitivity to the *social dimension* and not be creative in the sense of *mathematics education creativity*. The analysis of the interviews conducted allows for the recognition, in the interviewed teachers, of a particular sensitivity to all three defined

dimensions, which, in addition to characterising the teachers in the sense of *mathematics education creativity*, also allows them to grasp their formative role, in reference to the mathematics education and the intellectual and moral development of the pupils. More generally, the sensitivity to all three *dimensions* makes it possible to detect in teachers their role as part of a broader social movement (Gutstein, 2006). In this sense, through their classroom practice, the interviewed teachers seem to be oriented to interweaving mathematical discourse with critical thinking, through the proposal of activities and situations that sometimes surprise pupils, finding in this a point of contact with Freire. In particular, with reference to the relationship in educational practice between educational techniques and methods and the figure of the educator, his or her personal capacity and subjectivity, Freire (1989) characterises education as a relationship between the figure of the teacher and the pupil. This relationship involves the mediation of a content - of an object on which the teacher has performed his/her act of knowledge, making it an instrument of experience (Freire, 1970) - to be given to the pupil who may or may not be scared of it (Freire, 1989). In this sense, the teacher, in his or her educational practice, is called upon to epistemologically surprise and shake up his or her students, with a view to involving and stimulating in them reflective critical capacities.

To conclude, I quote below the Freire citation that opened this chapter.

> *"I cannot understand educational practice except as a complex and contradictory totality. I think about the importance of all the components of this practice, but I recognise, because educational practice is uniquely human, that the importance of the educator is extraordinary. And this is why a creative educator, a liberated educator or educator in the process of liberation, an educator who puts him or herself at stake, who ventures out, who is not afraid of freedom, an educator who is capable of loving, of loving even the process of education itself, of loving his or her own practice in which he or she finds himself or herself, an educator who invents and reinvents methods, techniques every day, is capable of creating where apparently nothing existed. This is the kind of educator we should help to exist!"*
> (Freire, 1989, January 23)

The *personal*, *social* and *professional dimensions*, considered as aspects that characterise *mathematics education creativity*, by their structural definition, turn out to be extremely subjective and linked to the individuality of the teacher to whom they refer. This aspect emphasises the humanity that Freire attributes to educational practice, which therefore also seems to depend on the teacher and his or her way of posing in the educational relationship. In the interviewed teachers, sensitivity to the three *dimensions* seems to have

allowed the development of an awareness that makes them "creative" teachers, who are not afraid to get involved, to "venture out" and, in this sense, "liberated". Teachers capable, through a creative process fed by the *personal*, *social* and *professional dimensions*, of "creating where apparently nothing existed". Like Freire, I profoundly agree that this is truly "the kind of educator we should help to exist" and it seems natural, in the light of the considerations made, to try to think of acting with a view to making teachers sensitive to the three dimensions. It is in this, then, that new reflections arise, opening up the possibility of future research directions that can identify the key to help this kind of educator continue to exist and "liberate" those in the "liberation process".

7.4 Possible future developments

The three *dimensions* outlined have made it possible to characterise *mathematics education creativity*, representing a meaningful point of arrival for the present research work. In a broader research perspective, the conclusions reached open up possible future developments, concerning which a suggestion is made in this section.

A first direction, which takes into account the implicit and inevitable limitations of the study, concerns the possibility of expanding the sample of teachers to be interviewed. This would make it possible to have access to more data to analyse and to continue working, across *dimensions*, in a perspective of refinement.

Another direction concerns the possibility of investigating the *dimensions* of creativity in more detail. In particular, this research articulation could involve the teachers who took part in the PoY project's educational training and then conducted the mathematics teaching activities in their classroom. Reflection could be oriented towards capturing, in the narratives of the teachers involved, the resonance that the moments of collective reflection and educational training had on their current teaching action. From the analysis conducted, it emerged that the professional and educational experiences of the teachers interviewed have influenced and still influence their respective choices in the didactic sphere, contributing to developing sensitivity to the *professional dimension* and thus affecting their creative process at the basis of the mathematics teaching design and implementation. With regard to the *professional dimension*, and to the emerging need to encourage meaningful educational training courses, the involvement of PoY teachers could be oriented towards investigating whether, and in what terms, the educational

training course offered has influenced their teaching design and implementation process. Moreover, since the analysis carried out showed that the comparison with colleagues and other figures in the educational sphere and the sharing of the moment of designing and revising a teaching activity were also found to be key elements for the creativity of the teachers interviewed, contributing to developing sensitivity to their *social dimension*, the involvement of PoY teachers could be oriented towards investigating how the moments of co-designing influenced their current teaching choices. With regard to this aspect, the research perspective could involve a reflection on the creation of "communities of teachers", in which collaboration seems fundamental (Borko & Potari, 2020). The development of a sensitivity to the *social dimension* could start from encouraging the creation of "communities of practice" in which teachers can collaborate, work together, sharing suggestions, goals and objectives.

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C'è chi insegna
guidando gli altri come cavalli
passo per passo:
forse c'è chi si sente soddisfatto
così guidato.

C'è chi insegna lodando
quanto trova di buono e divertendo:
c'è pure chi si sente soddisfatto
essendo incoraggiato.

C'è pure chi educa, senza nascondere
l'assurdo ch'è nel mondo, aperto ad ogni
sviluppo ma cercando
d'essere franco all'altro come a sé,
sognando gli altri come ora non sono:
ciascuno cresce solo se sognato.

"Ciascuno cresce solo se sognato", così Danilo Dolci scrive a conclusione di un suo verso. Ma quanto forte bisogna sognare per permettere a qualcos'altro di crescere? Con quanta fiducia occorre crederci? Me lo sono chiesta più volte e da resistente sognatrice ho sempre trovato nelle parole di Dolci autenticità e conforto. Provando a sognare ciò che ancora non è, può almeno essere data una possibilità a ciò che potrebbe essere. Nello scrivere i ringraziamenti, a conclusione di questo intenso e meraviglioso percorso, inevitabilmente mi accorgo di guardare a quello che è stato da una prospettiva nuova e diversa. Ogni tassello sorprendentemente trova il suo posto, ogni scelta la sua ragione e ogni sogno il suo valore. Innamorata da sempre della matematica e sognando di diventare insegnante, mi ritrovo cresciuta, consapevole dell'assurdo che vi è nel mondo, aperta all'imprevedibile e profondamente grata a chi non ha mai smesso di credere in me, sognando ciò che ancora non era, dando una possibilità a ciò che poteva essere. Tutto a un tratto, le parole di Dolci, lette e rilette tantissime volte, risuonano come nuove. Mi ci ritrovo dentro non più da sognatrice, ma da chi è cresciuta accolta, ascoltata e sognata. Da chi è cresciuta con la consapevolezza di non essere mai stata sola, con accanto, attorno e nel cuore instancabili

sognatori, capaci di sognare gli altri come ora non sono. È a loro che devo la magia di questo percorso, è a loro che, con un assoluto e profondo senso di riconoscenza, esprimo la mia gratitudine, provando a trovare le parole giuste.

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