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Existence, Regularity and Testability Results in Economic Models with Externalities

Coordinatore

Prof. Sergio Destefanis

Relatore

Prof. Elena L. del Mercato

Candidato

Dott. Vincenzo Platino

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“Existence, Regularity and Testability Results in Economic Models with Externalities”

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Doctoral Student : *Vincenzo Platino* [1]

DISES, Universitá degli Studi di Salerno – Italy

Thesis Advisor : *Elena del Mercato* [2]

Paris School of Economics (PSE) - University Paris 1 Panthon-Sorbonne, France

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1 Introduction

This thesis deals with economic models in the presence of *externalities*. Following Laffont (1988), we provide below the definition of externality.

> *An externality is any “indirect effect” that a consumption or a production activity has on individual preferences and on consumption or production possibilities.*

“Indirect effect” means that the effect is created by an economic agent different from the one who is affected, and the effect is not produced through the price system. In this case the price system only plays the role of matching demand and supply. [3] The definition above shows that the presence of externalities

[1] Dipartimento di Scienze Economiche e Statistiche (DISES), Universitá degli Studi di Salerno, Via Ponte don Melillo, 84084 Fisciano (SA), Italy. E-mail: vincenzo.platino@gmail.com.

[2] Centre d’Economie de la Sorbonne (CES), 106-112 Boulevard de l’Hopital 75647 Paris Cedex 13, France. E-mail: Elena.delMercato@univ-paris1.fr.

[3] On the other hand, the external effects that directly pass through the price system are called *pecuniary externalities*.

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requires a new description of agents' characteristics (individual preferences, consumption sets and production technologies).

The thesis consists of three chapters. Chapter 1 deals with the existence of competitive equilibria in a general production economy with externalities. In Chapter 2, we provide some regularity results in production economies with externalities. In Chapter 3 we study some testable restrictions in a specific model with externalities and public goods. One finds below an introduction of the three chapters.

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I am very grateful to Elena del Mercato for encouraging me to work on general equilibrium models and externalities from a differentiable viewpoint. Chapter 1 is based on del Mercato and Platino (2010) which has been presented at the "Public Economic Theory (PET 10) and Public Goods, Public Projects, Externalities (PGPPE) Closing Conference (2010)", Bogazici University, Istanbul, Turkey, and at the "Fifth Economic Behavior and Interaction Models (EBIM) Doctoral Workshop on Economic Theory", Bielefeld University, Germany. So, Chapter 1 has also benefited from the comments of these audiences. I thank Jean-Marc Bonnisseau for helpful comments and suggestions which have improved Chapter 2. I am very grateful to Bram De Rock for encouraging me to work on collective choice models and for helping me in the realization of Chapter 3. I thank also Laurens Cherchye and Frederic Vermeulen for helpful comments and suggestions which have improved Chapter 3. Chapter 3 is based on Cherchye, De Rock, and Platino (2010) which has been presented at the "Dauphine Workshop in Economic Theory – Recent Advances in Revealed Preference Theory: testable restrictions in markets and games", Université Paris-Dauphine, France. So, Chapter 3 has also benefited from the comments of these audiences.

1.1 Chapter 1 – "Existence of Equilibria in a General Equilibrium model with Production and Externalities: A Differentiable Approach"

In Chapter 1, we consider a general model of production economy with consumption and production externalities. In a differentiable framework, our purpose is to prove the non-emptiness and the compactness of the set of competitive equilibria with consumptions and prices strictly positive.

Why do we care about the existence of equilibria from a differentiable viewpoint? The starting point of studying the set of *regular* economies is the non-empty and compact set of equilibria in a differentiable setting. The relevance of *regular* economies and issues related to the global approach of the equilibrium analysis are discussed in the following subsection.

Our model of externalities is based on the seminal works by Laffont and Laroque (1972), Laffont (1977,1978,1988), where individual consumption sets, individual preferences and production technologies depend on the choices of all households and firms. We provide below some economic examples of this dependency.

- **(Externalities on preferences).** Building of a mall in a residential area is an example of positive (or negative) externalities created by a firm on the preferences of people living in that area.
- **(Externalities on consumption sets).** As in general equilibrium models à la Arrow–Debreu, each individual has to choose a consumption in his consumption set which describes the set of all consumption alternatives which are *a priori possible* for the individual. In the following examples, externalities affect individual consumption sets and do not directly affect preferences. *i*) In the case of internet or electricity, the congestion due to the global consumption limits the physically possible individual consumption, *ii*) an increase in the production of transport services decreases the minimal threshold of consumption of fuel, (*iii*) an increase of polluting production makes worse the individual health, and consequently it increases the survival threshold of consumption of medicines.
- **(Externalities on production technologies).** In counterpart, externalities may be created by consumers on firms. For instance, an over consumption of air-conditioner and consequently of electricity, might produce an electrical breakdown, decreasing all the production activities. Finally, externalities may be created by firms on firms. For example, *(i)* the polluting production of a firm that damages the land used by an agricultural firm might cause a reduction of the production of the agricultural firm, *(ii)* a firm that extracts oil from a land can affect another firm that extracts oil from a nearby land whenever the oil comes from an joint underground reservoir.

In Chapter 1, we consider a private ownership economy with a finite number of commodities, households and firms. Each firm is characterized by a technology described by an inequality on a differentiable function called the *transformation function.*[4] Each household is characterized by a consumption set, preferences and an initial endowment of commodities. Each consumption set is described by an inequality on a differentiable function called the *possibility function.* The same idea is used in recent works on restricted participation in financial markets where portfolio sets are described by linear or differen-

[4] For production technologies described by transformation functions, see for instance Mas-Colell et al. (1995).

tiable functions.[5] Individual preferences are represented by a utility function. Firms are owned by households. Utility, possibility and transformation functions depend on the consumption of all households and on the production activity of all firms.

Facing a price system, each firm chooses in his production set a production plan which solves his profit maximization problem taking as given the choices of the others, i.e. given the level of externality created by the other firms and households. Each household chooses in his consumption set a consumption bundle which solves his utility maximization problem under the budget constraint taking as given the choices of the others, i.e. given the level of externality created by the other households and firms. The associated concept of equilibrium is nothing else than an equilibrium *à la Nash*, the resulting allocation being feasible with the initial resources of agents. This notion includes as a particular case the classical equilibrium definition without externalities at all.

The main result of Chapter 1 is Theorem 12 which states that for all initial endowments which satisfy appropriate survival conditions, the set of competitive equilibria with consumptions and prices strictly positive is non-empty and compact.

Following the seminal work by Smale (1974), and more recent works by Villanacci and Zenginobuz (2005) and Bonnisseau and del Mercato (2010), we prove Theorem 12 using Smale's extended approach and homotopy arguments.[6] The homotopy idea is that any economy with externalities is connected by an arc to some economy without externalities at all. Along this arc, equilibria move in a continuous way without sliding off the boundary.

Smale's extended approach differs from the one based on the aggregate excess demand function by the feature that equilibria are described in terms of first order conditions and market clearing conditions. In the presence of externalities, this approach overcomes the following difficulty: the individual demand functions depend on the individual demand functions of the others, which depend on the individual demand functions of the others, and so on. So, it would be impossible to define an aggregate excess demand function which depends only on prices and initial endowments.

We now compare our contribution with previous works. Villanacci and Zenginobuz (2005) focus on a specific kind of externalities, namely pubic goods. Bonnisseau and del Mercato (2010) consider a pure exchange economy where

[5] See for instance, Siconolfi (1986,1988), Balasko, Cass and Siconolfi (1990), Polemarchakis and Siconolfi (1997), Cass, Siconolfi and Villanacci (2001), Carosi and Villanacci (2005), Aouani and Cornet (2009).

[6] The reader can find a survey of this approach in Villanacci et al. (2002).

only consumption externalities are studied. So, Chapter 1 extends the latter one to the case of externalities in a production economy.

In Kung (2008) and Mandel (2008), each consumption set coincides with the positive orthant of the commodity space. So, concerning the consumption side, our result is more general since it also allows externalities on general consumption sets. Furthermore, in Kung (2008) firms produce private and public goods but there are no private externalities on the production side. Concerning the existence proof, differently from our contribution, Mandel (2008) uses an approach based on the aggregate excess demand and the *Brouwer degree*. But, in order to use aggregate excess demand's approach the author has to *enlarge* the commodity space treating externalities as additional variables. Furthermore, following Chapter 4 of Milnor (1965), our proof is based on the theory of *degree modulo 2*. The degree theory modulo 2 is simpler than the Brouwer degree that requires the concept of oriented manifold in order to deduce the existence result from regularity properties and from the Index Theorem.

Finally, the result by Bonnisseau and Médecin (2001) is more general than ours since in that work individual consumption sets and firms technologies are represented by correspondences, and the existence proof is based on fixed point arguments. Moreover, in Bonnisseau and Médecin (2001) non-convexities are allowed on the production side. For this reason, their existence result involves more sophisticated techniques than those we use. Since we are interested in a model where one can perform comparative static analysis, at the cost of loosing generality, to describe individual consumption sets and firms technologies we choose to use an inequality on differentiable functions instead of more general correspondences. Furthermore, in order to use Smale's extended approach and standard first order conditions, fixing the externalities we require classical convexity assumptions to be satisfied. In this mild context, we provide an existence proof simpler than that of Bonnisseau and Médecin (2001).

1.2 Chapter 2 – "Externalities in Production Economies: Regularity results"

In Chapter 2, we consider a production economy with consumption and production externalities. Our propose is to provide sufficient conditions for the generic regularity of such economies.

Why do we care about regular economies? We recall that an economy is *regular* if it has a finite set of equilibria and if every equilibrium locally depends in a continuous or differentiable manner on the parameters describing the economy. Therefore, at a regular economy, it is possible to perform comparative static analysis. The relevance of regular economies and issues related to the global approach of the equilibrium analysis can be found in Smale (1981), Mas-Colell

(1985), Balasko (1988).

Regular economies are also important for two key aspects listed below.

(1) *Pareto improving policies*. It is well known that several sources of market failures such as incomplete financial markets, public goods and externalities prevent competitive equilibrium allocations to be Pareto optimal. In recent works, the achievement of Pareto improving policies is based on the set of regular economies. In different settings, see for instance Geanakoplos and Polemarchakis (1986, 2008), Citanna, Kajii and Villanacci (1998), Citanna, Polemarchakis and Tirelli (2006), Villanacci and Zenginobuz (2006, 2010).

(2) *Testable restrictions*. An economic model is testable if it generates restrictions that must be satisfied by the observable data. It is well known that there are two ways to construct testable restrictions. The "non-parametric" approach is based on the general revealed preferences axiom (*GARP*) or related axioms. On the other hand, the "parametric" approach is based on differentiable techniques which give rise to conditions remindful Slutsky conditions. This approach focuses on the *local structure* of the equilibrium manifold, that is, on regular economies, see for instance Chiappori, Ekeland, Kübler and Polemarchakis (2004).

It is an important and still open issue to study Pareto improving policies in the presence of externalities. Furthermore, before implementing any public policy one should verify whether the observed data are consistent with the economic model. So, Chapter 2 is a first step to study testable restrictions and Pareto improving policies in production economies with externalities.

As in Chapter 1, we consider a private ownership economy with consumption and production externalities. But, we restrict our attention to the case in which all the consumption sets coincide with the positive orthant of the commodity space. So, concerning the consumption side the model discussed in Chapter 2 is less general than the one considered in Chapter 1.

Now we describe our contributions. We provide an example of a production economy with externalities and an infinite set of equilibria for all the initial endowments. The example shows that regularly fails because of the *first order external effect* on transformation functions. So, in order to avoid situations such as the one shown by the example, we consider a displacement of the boundary of the production sets, that is, *simple perturbations* of the transformation functions. But, as shown by Bonnisseau and del Mercato (2010) in the case of only consumption externalities, regularity may fail whenever the *second-order external effects* are too strong. So, the basic assumptions and the perturbations mentioned above may be not sufficient to control the second-order external effects thereby preventing the regularity result. Thus, we also introduce two additional assumptions on the second-order external effects.

Our main result is Theorem14 which states that almost all *perturbed economies* are regular, where the term *almost all* means in a open and full measure.[7] As a consequence of Theorem 14, we get Corollary 19 which states the non-emptiness and the openness of the set of regular economies in the space of endowments and transformation functions.

Finally, we compare our contribution with previous contributions. As in Chapter 1, we follow Smale's extended approach. Concerning recent works on public goods and externalities, Villanacci and Zenginobuz (2005), Kung (2008) and Bonnisseau and del Mercato (2010) also use Smale's extended approach. Villanacci and Zenginobuz (2005) focus on a specific kind of externalities, namely public goods. In Kung (2008), differently from our model, there are no externalities on the production side. Furthermore, in order to get a regularity result, the author does not make any additional assumptions on utility functions, but perturbations of utility functions are also needed. In Bonnisseau and del Mercato (2010), only consumption externalities are considered. So, our regularity result extends the latter one to the case of production economy.

The model in Mandel (2008) is more general than ours since the author allows for non-convexity on the production side. But, as stressed in Chapter 1, differently from our contribution, the author has to *enlarge* the commodity space treating externalities as additional variables. Moreover, the author assumes that a small change in the externalities created by all the agents on a agent does not generate changes in the choices of the latter agent which would in turn involve the exact same change on the behavior of the others, see Assumption TR2 in Mandel (2008). But, differently from our assumptions, Assumption TR2 involves endogenous variables, more precisely the derivatives of households' demands and firms' supplies. So, this assumption implicitly involves the Lagrange multipliers, that is the equilibrium prices.

1.3 Chapter 3 – "Testable restrictions in a specific model with externalities and public goods: The collective consumption model"

As we emphasized in the previous subsection, it is important to study testable restrictions in general equilibrium models in the presence of externalities.

Testable restrictions on the classical general equilibrium model have been widely studied in literature, see for example the seminal paper of Brown and Matzkin (1996), and Chiappori, Ekeland, Kübler and Polemarchakis (2004).

[7] See Smale (1981).

The first testable restrictions in a model that involves externalities and public goods are provided by Browning and Chiappori (1998) for a *collective consumption model*. More precisely, the authors consider a non-unitary household model in which the decisions taken by the two intra-household members are Pareto efficient. In the last decades, the collective consumption model for the analysis of household decisions has become increasingly popular. The reasons for this interest stand in that individuals within a household are heterogeneous (i.e. they have different preferences) and an intra-household decision process takes place within a household. The standard *unitary model* considers a household as a single decision maker who maximizes his preferences subject to his budget constraint. But, there exists empirical evidence showing that the unitary model does not hold for household decisions. So, the unitary model is obviously too restrictive, since it implicitly endows households, rather than individuals, with preferences over consumption goods. In particular, the well-known properties of the classical demand function and especially the symmetry of the Slutsky matrix are often rejected. [8]

In Browning and Chiappori (1998), one does not observe what goods are privately consumed and what goods are publicly consumed within the household. The authors assume that only prices and aggregate demand with respect to some power distribution between the two intra-household members are observed. Using a "parametric" approach based on differentiable techniques, the authors prove that the aggregate demand is compatible with the Pareto optimal decision behavior if it satisfies some restrictions on a "Pseudo-Slutsky" matrix. The "Pseudo-Slutsky" matrix is the sum of the classical Slutsky matrix which measures the change in demand induced by the variation of prices and income, and another matrix which measures the change in demand induced by the variation of power distribution. Furthermore, the authors show that a collective model with two intra-household members can be rejected if at least five goods are present in the economy.

Successively, Chiappori and Ekeland (2006) generalize the previous model considering a group with many individuals and production, and provide necessary and sufficient restrictions in terms of "Pseudo-Slutsky" matrix. Importantly, using the "parametric" approach, the authors show that **the private and public nature of consumption is not testable**. More precisely, the authors show that the collective consumption model has exactly the same testability implications as two more specific collective models, i.e. a first benchmark case where all goods are publicly consumed within the household and a second benchmark case, without externalities at all, where all goods are privately consumed within the household.

[8] See for example, Browning and Meghir (1991) and Browning and Chiappori (1998).

Differently from Browning and Chiappori (1998), Cherchye, De Rock, Vermeulen (2007) provide a "non-parametric" characterization of the collective consumption model. The "non-parametric" approach in the tradition of Afriat (1967) and Varian (1982) contributions. This approach does not rely on any functional specification regarding the group consumption process, and it typically focuses on revealed preference axioms (i.e. GARP or related axioms). In Cherchye, De Rock, Vermeulen (2007), assuming positive externalities the authors derive necessary and sufficient conditions for a rationalization of a data set consistent with the collective consumption model. Furthermore, the authors show that it is sufficient to have a data set with three observations and three goods to reject collective rationality for a household with two members.

In Chapter 3, using the "non-parametric" restrictions found by Cherchye, De Rock, Vermeulen (2007), we provide examples showing that **the private and public nature of consumption have testable implications**.

So, in contrast with the previous literature, we find that the "non-parametric" approach does imply testability of privateness versus publicness of consumption, even if one only observes the aggregate group consumption. Furthermore, we obtain that the case where all the goods are publicly consumed within the household is independent from the case where all the goods are privately consumed within the household. More precisely, a data set that satisfies the restrictions for the first case does not necessarily satisfy the restrictions for the second case, and vice versa.

How can we interpret this difference between the testability conclusions of our approach and the ones of the "parametric" approach? Our explanation is that, unlike Chiappori and Ekeland's approach, our "non-parametric" restrictions involve personalized prices à la Lindahl and personalized consumptions, although we do not require personalized prices and personalized consumptions to be observable. Under this view, the nature of the "non-parametric" approach seems to imply stronger testability restrictions.

References

Afriat, S., 1967. The construction of utility functions from expenditure data. *International Economic Review* 8, 67-77.

Aouani, Z., Cornet, B., 2009. Existence of financial equilibria with restricted participation, *Journal of Mathematical Economics* 45, 772-786.

Balasko, Y., 1988. Foundations of the Theory of General Equilibrium. Academic Press, Boston.

Balasko, Y., Cass, D., Siconolfi, P., 1990. The structure of financial equilibrium with exogenous yields: the case of restricted participation. *Journal of Mathematical Economics* 19, 195-216.

Bonnisseau, J.-M., del Mercato, E. L., 2010. Externalities, consumption constraints and regular economies. *Economic Theory* 44, 123-147.

Bonnisseau, J.-M., Médecin, J.-P., 2001. Existence of marginal pricing equilibria in economies with externalities and non-convexities, *Journal of Mathematical Economics* 36, 271–294.

Brown, D., Matzkin, R., 1996. Testable restrictions on the equilibrium manifold. *Econometrica* 64, 1249-1262.

Browning, M., Chiappori, P.-A., 1998. Efficient Intra-Household Allocations: A General Characterization and Empirical Tests. *Econometrica* 66, 1241-1278.

Browning, M., Meghir, C., 1991. The Effects of Male and Female Labor Supply on Commodity demands. *Econometrica* 59, 925-951.

Carosi, L., Villanacci, A., 2005. Relative wealth dependent restricted participation on financial markets. Report 271, University of Pisa, Italy.

Cass, D., Siconolfi, P., Villanacci, A., 2001. Generic regularity of competitive equilibria with restricted participation. *Journal of Mathematical Economics* 36, 61-76.

Cherchye, L., De Rock, B., Vermeulen, F., 2007. The Collective Model of Household Consumption: A Nonparametric Characterization. *Econometrica* 75, 553-574.

Cherchye, L., De Rock, B., Platino, V., 2010. Private versus public consumption within groups: testing the nature of goods from aggregate data. CES Working Paper 2010.90.

Chiappori, P.-A., 1988. Rational Household Labor Supply. *Econometrica* 56, 63-90.

Chiappori, P.-A., 1992. Collective Labor Supply and Welfare. *The Journal of Political Economy* 100, 437-467.

Chiappori P.-A., Ekeland, I., 2006. The Micro Economics of Group Behavior: General Characterization. *Journal of Economic Theory* 130, 1-26.

Chiappori, P.-A., Ekeland, I., Kübler, F., Polemarchakis, H., 2004. Testable implications of general equilibrium theory: a differentiable approach. *Journal of Mathematical Economics* 40, 105-119.

Citanna, A., Kajii, A., Villanacci, A., 1998. Constrained suboptimality in incomplete markets: a general approach and two applications, *Economic Theory* 11, 495-521.

Citanna, A., Polemarchakis, H. M., Tirelli, M., 2006. The taxation of trades in assets, *Journal of Economic Theory* 126, 299-313.

del Mercato, E. L., Platino, V., 2010. Externalities in production economies and existence of competitive equilibria: A differentiable approach, Working Paper.

Geanakoplos, J. D., Polemarchakis, H. M., 1986. Existence, regularity and constrained suboptimality of competitive allocations when the asset market is incomplete. In: Starrett, D., Heller, W. P., Starr, R. M. (eds.), Uncertainty, Information and Communication: Essays in Honor of K. J. Arrow, vol. III, 65-96. Cambridge University Press.

Geanakoplos, J. D., Polemarchakis, H. M., 2008. Pareto improving taxes, *Journal of Mathematical Economics* 44, 682-696.

Kung, F.-C., 2008. Voluntary contributions to multiple public goods in a production economy with widespread externalities. *Journal of Mathematical Economics* 44, 1364-1378.

Laffont, J.-J., 1977. Effets externes et théorie économique. Monographies du Séminaire d'Econométrie. Editions du CNRS, Paris, France.

Laffont, J.-J., 1978. Decentralization with externalities. *European Economic Review* 7, 359-375.

Laffont, J.-J., 1988. Fundamentals of Public Economics. The MIT Press Cambridge.

Laffont, J.-J., Laroque, G., 1972. Effets externes et théorie de l'équilibre général, Cahiers du Séminaire d'Econom étrie 14. CNRS, Paris, France.

Mandel, A., 2008. An index formula for production economies with externalities, *Journal of Mathematical Economics* 44, 1385–1397.

Mas-Colell, A., 1985. The theory of general economic equilibrium. A differentiable approach. Cambridge University Press.

Mas-Colell, A., Whinston, M. D., Green, J. R., 1995. Microeconomic Theory. Oxford University Press.

Milnor, J. W., 1965. Topology from the Differentiable Viewpoint. The University Press of Virginia.

Polemarchakis, H. M., Siconolfi, P., 1997. Generic existence of competitive equilibria with restricted participation. *Journal of Mathematical Economics* 28, 289-311.

Siconolfi, P., 1986. Equilibrium with restricted participation on incomplete financial markets. CARESS Working Paper 86-21, University of Pennsylvania.

Siconolfi, P., 1988. Equilibrium with asymmetric constraints on portfolio holdings and incomplete financial markets. In: Non-Linear Dynamics in Economics and Social Sciences - Proceeding of the Meeting held in Certosa di Pontignano, January 7-9, 1988, Società Pitagora, Italy.

Smale, S., 1974. Global analysis and economics. II A: Extension of a theorem of Debreu, *Journal of Mathematical Economics* 1, 1–14.

Smale, S., 1981. Global analysis and economics. In: Arrow, K., Intriligator, M. (Eds.), Handbook of Mathematical Economics, vol. I. North- Holland, Amsterdam, Chapter 8, 331-370.

Varian, H., 1982. Nonparametric Approach to Demand Analysis. *Econometrica* 50, 945-973.

Villanacci, A., Carosi, L., Benevieri, P., Battinelli, A., 2002. Differential Topology and General Equilibrium with Complete and Incomplete Markets. Kluwer Academic Publishers.

Villanacci, A., Zenginobuz, U., 2005. Existence and Regularity of Equilibria in a General Equilibrium Model With Private Provision of a Public Good, *Journal of Mathematical Economics* 41, 617–636.

Villanacci, A., Zenginobuz, U., 2006. Pareto Improving Interventions in a General Equilibrium Model with Private Provision of Public Goods, *Review of Economic Design* 10, 249–271.

Villanacci, A., Zenginobuz, U., 2010. Subscription Equilibrium with Production: Non-Neutrality and Constrained Suboptimality, *Journal of Economic Theory*, available online from September 7, 2010.

Chapter 1

Existence of equilibria in a general equilibrium model with production and externalities: A differentiable approach [1]

Vincenzo Platino [2]

DISES, Universitá degli Studi di Salerno, Italy

Key words: externalities, production economies, competitive equilibrium, Smale's extended approach, homotopy approach.

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1 Long abstract

We consider a general model of a private ownership economy with consumption and production externalities. Each firm is characterized by a technology described by an inequality on a differentiable function called the *transformation function*. Each household is characterized by a consumption set, preferences and an initial endowment of commodities. Each consumption set is described by an inequality on a differentiable function called the *possibility function*. Individual preferences are represented by a utility function. Firms are owned by households. Utility, possibility and transformation functions depend on the consumption of all households and on the production activities of all firms.

Using Smale's extended approach and homotopy arguments, under differentiability and boundary conditions, we prove the non-emptiness and the compactness of the set of competitive equilibria with consumptions and prices strictly positive.

[1] This chapter is based on del Mercato and Platino (2010).

[2] Dipartimento di Scienze Economiche e Statistiche (DISES), Universitá degli Studi di Salerno, Via Ponte don Melillo, 84084 Fisciano (SA), Italy. E-mail: vincenzo.platino@gmail.com.

The homotopy idea is that any economy with externalities is connected by an arc to some economy without externalities at all. Along this arc, equilibria move in a continuous way without sliding off the boundary. Smale's extended approach differs from the one based on the aggregate excess demand function by the feature that equilibria are described in terms of first order conditions and market clearing conditions. In the presence of externalities, this approach overcomes the following difficulty: the individual demand functions depend on the individual demand functions of the others, which depend on the individual demand functions of the others, and so on. So, it would be impossible to define an aggregate excess demand function which depends only on prices and initial endowments.

Chapter 1 is organized as follows. In Section 2, we present the model and the assumptions. In Section 3, the concept of competitive equilibrium is adapted to our economy. Then, we focus on the equilibrium function which is built on first order conditions associated with households and and firms maximization problems. In Section 4, we present our main result, Theorem 12.

In Appendix A we provide some technical details. In Appendix B, the reader can find the characterization of Pareto optimal allocation without externalities.

2 The model and the assumptions

There is a finite number C of physical commodities or goods labeled by the superscript $c \in \mathcal{C} := \{1, \ldots, C\}$. The commodity space is $\mathbb{R}^C$. There is a finite number H of households or consumers labeled by the subscript $h \in \mathcal{H} := \{1, \ldots, H\}$. Each household h is characterized by an endowment of commodities, a possibility function and preferences described by a utility function. There is a finite number J of firms labeled by the subscript $j \in \mathcal{J} := \{1, \ldots, J\}$. Each firm j is owned by the households and it is characterized by a technology described by a transformation function. Individual utility, possibility and transformation functions are affected by the consumption choices of all households and the production activities of all firms which represent the *externalities* created on individual agents (households and firms) by all the other agents. The notations are summarized below.

- $y_j := (y_j^1, .., y_j^c, .., y_j^C)$ is the production plan of firm j. As usual, the output components are positive and the input components are negative; $y_{-j} := (y_z)_{z \neq j}$ denotes the production plan of firms other than j and $y := (y_j)_{j \in \mathcal{J}}$ denotes the production of all the firms.
- x_h^c is the consumption of commodity c by household h; $x_h := (x_h^1, .., x_h^c, .., x_h^C)$ denotes household h's consumption; $x_{-h} := (x_k)_{k \neq h}$

denotes the consumption of households other than h and $x := (x_h)_{h\in\mathcal{H}}$ denotes the consumption of all the households.

- For each $j \in \mathcal{J}$, the technology of firm j is described by an inequality on a function t_j called the *transformation function*. This description is usual for smooth production economies, see for instance Mas-Colell et al. (1995). An innovation of this chapter comes from the dependency of the production set with respect to the production activities of other firms and the consumption of households. That is, given y_{-j} and x, the production set of the firm j is described by the following set,

$$Y_j(y_{-j}, x) := \Big\{ y_j \in \mathbb{R}^C \ : \ t_j(y_j, y_{-j}, x) \geq 0 \Big\}$$

  where the transformation function t_j is a function from $\mathbb{R}^C \times \mathbb{R}^{C(J-1)} \times \mathbb{R}^{CH}_{++}$ to $\mathbb{R}$. So, t_j describes the way firm j's technology is affected by the actions of the other agents.

- As in general equilibrium models à la Arrow–Debreu, each household h has to chose a consumption in his consumption set X_h. Analogously to the production side, each consumption set X_h is described in terms of an inequality on a function χ_h.[3] We call χ_h the *possibility function* of households h. The main innovation of this chapter comes from the dependency of the consumption set with respect to the consumptions of the other households and the production activities of firms. So, given x_{-h} and y the consumption set of household h is given by

$$X_h(x_{-h}, y) := \Big\{ x_h \in \mathbb{R}^C_{++} \ : \ \chi_h(x_h, x_{-h}, y) \geq 0 \Big\}$$

  where the possibility function χ_h is a function from $\mathbb{R}^C_{++} \times \mathbb{R}^{C(H-1)}_{+} \times \mathbb{R}^{CJ}$ to $\mathbb{R}$. Thus, χ_h describes the way in which the set of all consumption alternatives which are *a priori possible* for household h is affected by the actions of the other agents.

- Each household $h \in \mathcal{H}$ has preferences described by a utility function,

$$u_h : (x_h, x_{-h}, y) \in \mathbb{R}^C_{++} \times \mathbb{R}^{C(H-1)}_{+} \times \mathbb{R}^{CJ} \longrightarrow u_h(x_h, x_{-h}, y) \in \mathbb{R}$$

  $u_h(x_h, x_{-h}, y)$ is the utility level of household h associated with (x_h, x_{-h}, y). So, u_h describes the way household h's preferences are affected by the consumption and the production of the other agents.

- $s_{jh} \in [0,1]$ is the share of firm j owned by household h; $s_h := (s_{jh})_{j\in\mathcal{J}} \in [0,1]^J$ denotes the vector of the shares of all firms owed by household h; $s := (s_h)_{h\in\mathcal{H}} \in [0,1]^{JH}$. The set of all shares is

$$S := \{ s \in [0,1]^{JH} \ : \ \forall\, j \in \mathcal{J}, \ \sum_{h\in\mathcal{H}} s_{jh} = 1 \}$$

[3] In same spirit, see Smale (1974), and Bonnisseau and del Mercato (2010).

- e_h^c is the endowment of commodity c owned by household h;
  $e_h := (e_h^1, .., e_h^c, .., e_h^C)$ denotes household h's endowment; $e := (e_h)_{h\in\mathcal{H}}$.
- $\mathcal{E} := ((u_h, \chi_h, e_h, s_h)_{h\in\mathcal{H}}, (t_j)_{j\in\mathcal{J}})$ is an *economy*.
- p^c is the price of one unit of commodity c; $p := (p^1, .., p^c, .., p^C) \in \mathbb{R}^C_{++}$; prices of goods are expressed in units of account.
- Given $w = (w^1, .., w^c, .., w^C) \in \mathbb{R}^C$, we denote

$$w^{\setminus} := (w^1, .., w^c, .., w^{C-1}) \in \mathbb{R}^{C-1}$$

We make the following assumptions on the transformation functions $(t_j)_{j\in\mathcal{J}}$.

Assumption 1 *For all $j \in \mathcal{J}$,*

(1) The function t_j is a C^1 function.
(2) For each $(y_{-j}, x) \in \mathbb{R}^{C(J-1)} \times \mathbb{R}^{CH}_{++}$, $t_j(0, y_{-j}, x) \geq 0$.
(3) For each $(y_{-j}, x) \in \mathbb{R}^{C(J-1)} \times \mathbb{R}^{CH}_{++}$, the function $t_j(\cdot, y_{-j}, x)$ is differentiably strictly decreasing, i.e.

$$\forall\, (y_{-j}, x) \in \mathbb{R}^{C(J-1)} \times \mathbb{R}^{CH}_{++} \text{ and } \forall\, y'_j \in \mathbb{R}^C, D_{y_j} t_j(y'_j, y_{-j}, x) \ll 0$$

(4) For each $(y_{-j}, x) \in \mathbb{R}^{C(J-1)} \times \mathbb{R}^{CH}_{++}$, the function $t_j(\cdot, y_{-j}, x)$ is C^2 and it is differentiably strictly quasi-concave, i.e. for every $y_j \in \mathbb{R}^C$, $D^2_{y_j} t_j(y_j, y_{-j}, x)$ is negative definite on $\ker D_{y_j} t_j(y_j, y_{-j}, x)$. [4]

We remark that, given the externalities, the assumptions on t_j are standard in "smooth" general equilibrium models. Indeed, from Point 1 of Assumption 1 the production set is closed and from Point 4 of Assumption 1 it is convex. Point 2 of Assumption 1 states that inactivity is possible. As usual Point 2 of Assumption 1 implies that for any price system $p \in \mathbb{R}^C_{++}$ and for any given externalities, the optimal profit of firm j is non-negative. [5] This property ensures that every consumer has a positive wealth, since the aggregate profit is non-negative. Consequently the individual budget constraint is non-empty for any given externality and price system. Point 3 of Assumption 1 represents the "free disposal" property.

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[4] Let v and v' be two vectors in $\mathbb{R}^n$, $v \cdot v'$ denotes the *inner product* of v and v'. Let A be a real matrix with m rows and n columns, and B be a real matrix with n rows and l columns, AB denotes the *matrix product* of A and B. Without loss of generality, vectors are treated as row matrices and A denotes both the matrix and the following linear application $A : v \in \mathbb{R}^n \to A(v) := Av^T \in \mathbb{R}^{[m]}$ where v^T denotes the transpose of v and $\mathbb{R}^{[m]} := \{w^T : w \in \mathbb{R}^m\}$. When $m = 1$, $A(v)$ coincides with the inner product $A \cdot v$, treating A and v as vectors in $\mathbb{R}^n$.

[5] Indeed, by Point 2 of Assumption 1 the production plan 0 is in the production set of form j whatever are the externalities.

Define the set Y of all production plans which are in the production sets whatever are the externalities, that is

$$Y := \left\{ y' \in \mathbb{R}^{CJ} | \, \exists \, (x, y) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ} \; : \; t_j(y'_j, y_{-j}, x) \geq 0, \; \forall \, j \in \mathcal{J} \right\} \quad (1)$$

The following assumption on Y can be interpreted as the asymptotic irreversibility and "no free lunch" assumption at the aggregate level.

Assumption 2 *If $y' \in CY$ and $\sum_{j \in \mathcal{J}} y'_j \geq 0$, then $y'_j = 0$ for every $j \in \mathcal{J}$.*[6]

The assumption above ensures that the set of feasible allocation of the economy $\mathcal{E}$ is bounded. Furthermore, Assumption 2 guarantees that the set of feasible allocation is bounded for any fixed externalities. As a consequence of this assumption, one gets the boundedness of the set of feasible allocations along all the arc associated with the homotopy defined in Subsection 4.2 (see Lemma 16 and Step 2.1 of Lemma 17). One should notice that Assumption 2 is in the same spirit as Assumption UB (Uniform Boundedness) of Bonnisseau and Médecin (2001) and Assumption P3 of Mandel (2008).

We make the following assumptions on the utilities functions $(u_h)_{h \in \mathcal{H}}$.

Assumption 3 *For all $h \in \mathcal{H}$,*

(1) The function u_h is continuous in its domain and it is C^1 in the interior of its domain.

(2) For each $(x_{-h}, y) \in \mathbb{R}^{C(H-1)}_{++} \times \mathbb{R}^{CJ}$, the function $u_h(\cdot, x_{-h}, y)$ is differentiably strictly increasing, i.e.

$$\forall \, (x_{-h}, y) \in \mathbb{R}^{C(H-1)}_{++} \times \mathbb{R}^{CJ} \text{ and } \forall \, x'_h \in \mathbb{R}^{C}_{++}, D_{x_h} u_h(x'_h, x_{-h}, y) \gg 0$$

(3) For each $(x_{-h}, y) \in \mathbb{R}^{C(H-1)}_{++} \times \mathbb{R}^{CJ}$, the function $u_h(\cdot, x_{-h}, y)$ is C^2 and it is differentiably strictly quasi-concave, i.e., for every $x_h \in \mathbb{R}^{C}_{++}$, $D^2_{x_h} u_h(x_h, x_{-h}, y)$ is negative definite on $\ker D_{x_h} u_h(x_h, x_{-h})$.

(4) For each $(x_{-h}, y) \in \mathbb{R}^{C(H-1)}_{+} \times \mathbb{R}^{CJ}$ and for each $u \in \operatorname{Im} u_h(\cdot, x_{-h}, y)$,

$$\operatorname{cl}_{\mathbb{R}^C} \{ x_h \in \mathbb{R}^{C}_{++} : u_h(x_h, x_{-h}, y) \geq u \} \subseteq \mathbb{R}^{C}_{++}$$

So, fixing the externalities, the assumptions on u_h are standard in "smooth" general equilibrium models. In Points 1 and 4 of Assumption 3 we consider consumption x_{-h} in the closure of $\mathbb{R}^{C(H-1)}_{++}$, just to look at the limit of a behavior (see Steps 1.2 and 2.2 of Lemma 17).

We make the following assumptions on the possibility functions $(\chi_h)_{h \in \mathcal{H}}$.

[6] CY denotes the asymptotic cone of Y.

Assumption 4 *For all $h \in \mathcal{H}$,*

(1) χ_h is continuous in its domain and it is C^1 in the interior of its domain.
(2) (Convexity of the consumption set) For each $(x_{-h}, y) \in \mathbb{R}_+^{C(H-1)} \times \mathbb{R}^{CJ}$, the function $\chi_h(\cdot, x_{-h}, y)$ is quasi-concave.[7]
(3) (Survival consition) There exists $\overline{x}_h \in \mathbb{R}_{++}^C$ such that $\chi_h(\overline{x}_h, x_{-h}, y) \geq 0$ for every $x_{-h} \in \mathbb{R}_+^{C(H-1)}$ and for every $y \in \mathbb{R}^{CJ}$.
(4) (Individual desiderability) (a) For each $(x_{-h}, y) \in \mathbb{R}_+^{C(H-1)} \times \mathbb{R}^{CJ}$ the function $\chi_h(\cdot, x_{-h}, y)$ is differentiable and for every $x_h \in \mathbb{R}_{++}^C$, $D_{x_h}\chi_h(x_h, x_{-h}, y) \neq 0$; (b) for each $(x_{-h}, y) \in \mathbb{R}_{++}^{C(H-1)} \times \mathbb{R}^{CJ}$ and for every $x_h \in \mathbb{R}_{++}^C$, $D_{x_h}\chi_h(x_h, x_{-h}, y) \notin -\mathbb{R}_{++}^C$.

We remark that, fixing the externalities, from Points 1 and 2 of Assumption 4, one gets the usual assumptions on the closedness and on the convexity of the consumption set. Point 3 of Assumption 4, is called the "survival condition" since it guarantees that there exists at least a consumption bundle which belongs to the consumption set $X_h(x_{-h}, y)$, whatever are the externalities. Point 4(a) in Assumption 4 means that the consumption set is "smooth" while Point 4(b) implies that household h can increase the consumption of at least one commodity remaining in his consumption set. Consequently, according to Point 2 of Assumption 3 (that is stricly increasing utility functions), Point 4(b) of Assumption 4 means each household can increase his utility remaining in his consumption set, from which one classically deduces that the individual budget constraint is binding. In Assumption 4, we consider consumption bundles x_{-h} in the closure of $\mathbb{R}_{++}^{C(H-1)}$, just to look at limit of a behavior (see Proposition 7 which is used in Step 2.2 of Lemma 17).

$\mathcal{T}$ denotes the set of $t := (t_j)_{j \in \mathcal{J}}$ satisfying Assumption 1 and Assumption 2, $\mathcal{U}$ denotes the set of $u := (u_h)_{h \in \mathcal{H}}$ satisfying Assumption 3, and $\mathcal{X}$ denotes the set of $\chi := (\chi_h)_{h \in \mathcal{H}}$ satisfying Assumption 4.

Remark 5 *From now on, we take $u \in \mathcal{U}$, $\chi \in \mathcal{X}$, $t \in \mathcal{T}$ and $s \in S$ as given. So, an economy is complete characterized by the individual endowments $e = (e_h)_{h \in \mathcal{H}}$.*

We define now the set of endowments which satisfy the *Survival Assumption* for given possibility functions. As it is well known, the Survival Assumption states that each household can dispose of a strictly positive quantity of every commodity from his initial endowment still remaining in the interior of his consumption set.

[7] Since χ_h is C^1 in the interior of its domain, then for each $(x_{-h}, y) \in \mathbb{R}_{++}^{C(H-1)} \times \mathbb{R}^{CJ}$, the function $\chi_h(\cdot, x_{-h}, y)$ is differentiably quasi-concave.

Definition 6 *Let $\chi \in \mathcal{X}$. Define the set $E_\chi := \prod_{h\in\mathcal{H}} E_{\chi_h} \subseteq \mathbb{R}^{CH}_{++}$ where*

$$E_{\chi_h} := \left\{x_h \in \mathbb{R}^C_{++} \;:\; \chi_h(x_h, x_{-h}, y) \geq 0,\ \forall\ (x_{-h}, y) \in \mathbb{R}^{C(H-1)}_+ \times \mathbb{R}^{CJ}\right\} + \mathbb{R}^C_{++}$$

From Point 3 of Assumption 4, E_χ is nonempty and it is open by definition. From Points 3 and 4(a) of Assumption 4, the Survival Assumption is satisfied on the set E_χ since for all $e \in E_\chi$ the following property holds true for every $h \in \mathcal{H}$.[8]

$$\forall (x_{-h}, y) \in \mathbb{R}^{C(H-1)}_+ \times \mathbb{R}^{CJ},\ \exists \widehat{x}_h \in \mathbb{R}^C_{++} : \chi_h(\widehat{x}_h, x_{-h}, y) > 0 \text{ and } \widehat{x}_h \ll e_h \quad (2)$$

As a direct consequence of Points 1 and 2 of Assumptions 4 and (2) we get the following proposition. The continuous selection functions given by Proposition 7 will play a fundamental role in the construction of the continuous homotopy used to show our main result (see Theorem 12). Specifically, we use Proposition 7 to define the homotopies given by (16) and (17).

Proposition 7 *For all $h \in \mathcal{H}$, there exists a continuous selection function $\widehat{x}_h : \mathbb{R}^{C(H-1)}_+ \times \mathbb{R}^{CJ} \times E_{\chi_h} \to \mathbb{R}^C_{++}$ such that for each $(x_{-h}, y, e_h) \in \mathbb{R}^{C(H-1)}_+ \times \mathbb{R}^{CJ} \times E_{\chi_h}$, $\chi_h(\widehat{x}_h(x_{-h}, y, e_h), x_{-h}, y) > 0$ and $\widehat{x}_h(x_{-h}, y, e_h) \ll e_h$.*

3 Competitive equilibrium with externalities

In this section, we first provide the notion of competitive equilibrium associated with our economy. Second, we define the equilibrium function using the first order conditions associated with firms and households maximization problems.

Without loss of generality, commodity C is the *numeraire good.* So, given $p^\backslash \in \mathbb{R}^{C-1}_{++}$ with innocuous abuse of notation, we denote $p := (p^\backslash, 1) \in \mathbb{R}^C_{++}$.

Definition 8 $(x^*, y^*, p^{*\backslash}) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ} \times \mathbb{R}^{C-1}_{++}$ *is a competitive equilibrium for the economy $e = (e_h)_{h\in\mathcal{H}}$ if*

[8] Let $e_h \in E_{\chi_h}$. Thus, $e_h = \overline{x}_h + v$ with $\overline{x}_h$ given by Point 3 of Assumption 4 and $v \gg 0$. Fix the externalities and consider $\widehat{x}_h := \overline{x}_h + \varepsilon D_{x_h}\chi_h(\overline{x}_h, x_{-h}, y)$. By Point 4($a$) of Assumption 4 and the definition of differentiable function, property (2) is satisfied for some $\varepsilon > 0$.

(1) for all $j \in \mathcal{J}$, y_j^* solves the following problem

$$\begin{aligned} &\max_{y_j \in \mathbb{R}^C} \; p^* \cdot y_j \\ &\text{subject to } \; t_j(y_j, y_{-j}^*, x^*) \geq 0 \end{aligned} \tag{3}$$

(2) For all $h \in \mathcal{H}$, x_h^* solves the following problem

$$\begin{aligned} &\max_{x_h \in \mathbb{R}^C_{++}} \; u_h(x_h, x_{-h}^*, y^*) \\ &\text{subject to } \; \chi_h(x_h, x_{-h}^*, y^*) \geq 0 \\ &\qquad\qquad\quad p^* \cdot x_h \leq p^* \cdot (e_h + \sum_{j \in \mathcal{J}} s_{jh} y_j^*) \end{aligned} \tag{4}$$

(3) $(x^*, y^*) \in \mathbb{R}^{CH}_{++} \times \in \mathbb{R}^{CJ}$ satisfies market clearing conditions, that is

$$\sum_{h \in \mathcal{H}} x_h = \sum_{h \in \mathcal{H}} e_h + \sum_{j \in \mathcal{J}} y_j \tag{5}$$

In the following propositions, using Karush–Kuhn–Tucker's conditions we characterize the solutions of firms and households maximization problems.

Proposition 9 *Given $y_{-j}^* \in \mathbb{R}^{C(J-1)}$, $x^* \in \mathbb{R}^{CH}_{++}$ and $p^{*\setminus} \in \mathbb{R}^{C-1}_{++}$,*

(1) if y_j^* is a solution to problem (3), then it is the unique solution.

(2) $y_j^* \in \mathbb{R}^C$ is the solution to problem (3) if and only if there exists $\alpha_j^* \in \mathbb{R}_{++}$ such that (y_j^*, α_j^*) is the unique solution to the following system

$$\begin{cases} p^* + \alpha_j D_{y_j} t_j(y_j, y_{-j}^*, x^*) = 0 \\ t_j(y_j, y_{-j}^*, x^*) = 0 \end{cases} \tag{6}$$

Proposition 10 *Given $e_h \in E_{\chi_h}$, $x_{-h}^* \in \mathbb{R}^{C(H-1)}_{++}$, $y^* \in \mathbb{R}^{CJ}$ and $p^{*\setminus} \in \mathbb{R}^{C-1}_{++}$,*

(1) if $p^* \cdot \sum_{j \in \mathcal{J}} s_{jh} y_j^* \geq 0$, then there exists a unique solution to problem (4).

(2) $x_h^* \in \mathbb{R}^C_{++}$ is the solution to problem (4) if and only if there exists $(\lambda_h^*, \mu_h^*) \in \mathbb{R}_{++} \times \mathbb{R}$ such that $(x_h^*, \lambda_h^*, \mu_h^*)$ is the unique solution to the following system

$$\begin{cases} D_{x_h} u_h(x_h, x_{-h}^*, y^*) - \lambda_h p^* + \mu_h D_{x_h} \chi_h(x_h, x_{-h}^*, y^*) = 0 \\ -p^* \cdot (x_h - e_h - \sum_{j \in \mathcal{J}} s_{jh} y_j^*) = 0 \\ \min \Big\{ \mu_h, \chi_h(x_h, x_{-h}^*, y^*) \Big\} = 0 \end{cases} \tag{7}$$

Define the set of endogenous variables as

$$\Xi := \left(\mathbb{R}^C_{++} \times \mathbb{R}_{++} \times \mathbb{R}\right)^H \times \left(\mathbb{R}^C \times \mathbb{R}_{++}\right)^J \times \mathbb{R}^{C-1}_{++}$$

with generic element $\xi := (x, \lambda, \mu, y, \alpha, p^\backslash) := ((x_h, \lambda_h, \mu_h)_{h\in\mathcal{H}}, (y_j, \alpha_j)_{j\in\mathcal{J}}, p^\backslash)$.

We can now describe equilibria using the propositions above and market clearing conditions (5). One should notice that, due to the Walras law and the second equation in (7), the market clearing condition for commodity C is "redundant". Therefore, in the following remark we omit in (5) the condition for commodity C.

Remark 11 $\xi^* = (x^*, \lambda^*, \mu^*, y^*, \alpha^*, p^{*\backslash}) \in \Xi$ *is an extended competitive equilibrium for the economy* $e \in E_\chi$ *if and only if*

(1) $(x^*_h, \lambda^*_h, \mu^*_h,)$ *solves system (7) for all* $h \in \mathcal{H}$*,*
(2) (y^*_j, α^*_j) *solves system (6) for all* $j \in \mathcal{J}$*,*
(3) $(x^{*\backslash}, y^{*\backslash})$ *satisfies the following market clearing conditions*

$$\sum_{h\in\mathcal{H}} x^\backslash_h - \sum_{h\in\mathcal{H}} e^\backslash_h - \sum_{j\in\mathcal{J}} y^\backslash_j = 0$$

For a given economy $e \in E_\chi$, the *equilibrium function* $F_e : \Xi \to \mathbb{R}^{\dim \Xi}$,

$$F_e(\xi) := ((F^{h.1}_e(\xi), F^{h.2}_e(\xi), F^{h.3}_e(\xi))_{h\in\mathcal{H}}(F^{j.1}_e(\xi), F^{j.2}_e(\xi))_{j\in\mathcal{J}}, F^M_e(\xi)) \quad (8)$$

is defined by $F^{h.1}_e(\xi) := D_{x_h}u_h(x_h, x_{-h}, y) - \lambda_h p + \mu_h D_{x_h}\chi_h(x_h, x_{-h}, y)$, $F^{h.2}_e(\xi) := -p \cdot (x_h - e_h - \sum_{j\in\mathcal{J}} s_{jh}y_j)$, $F^{h.3}_e(\xi) := \min\{\mu_h, \chi_h(x_h, x_{-h}, y)\}$, $F^{j.1}_e(\xi) := p + \alpha_j D_{y_j}t_j(y_j, y_{-j}, x)$, $F^{j.2}_e(\xi) := t_j(y_j, y_{-j}, x)$, and $F^M_e(\xi) := \sum_{h\in\mathcal{H}} x^\backslash_h - \sum_{j\in\mathcal{J}} y^\backslash_j - \sum_{h\in\mathcal{H}} e^\backslash_h$.

By Remark 11, $\xi^* \in \Xi$ is an extended equilibrium for the economy $e \in E_\chi$ if and only if $F_e(\xi^*) = 0$. With innocuous abuse of terminology, we call ξ^* simply an equilibrium.

4 Existence of competitive equilibria

In this section we prove our main result, that is competitive equilibria with consumptions and prices strictly positive exist, and the set of equilibria is compact.

Theorem 12 (Existence and compactness) *Given* $(u, \chi, t) \in \mathcal{U} \times \mathcal{X} \times \mathcal{T}$ *and* $s \in S$*, for each economy* $e \in E_\chi$*, the set of equilibria is non-empty and compact.*

In order to prove Theorem 12, following the seminal paper by Smale (1974), we use homotopy arguments, namely Theorem 13 which is a consequence of the homotopy invariance of the topological degree. More specifically, following Chapter 4 of Milnor (1965), Theorem 13 is based on the topological degree theory of *degree mod 2*. The reader can find a survey of this approach in Villanacci et al. (2002). The theory of *degree mod 2* is simpler than the one used in Mas-Colell (1985) that requires the concepts of oriented manifold and the associated topological degree – the *Brouwer degree* – in order to deduce the existence result from regularity properties of equilibria and from the Index Formula.

Theorem 13 (Homotopy Theorem) *Let* M *and* N *be two* C^2 *boundaryless manifolds of the same dimension,* $y \in N$ *and* $f, g : M \to N$ *be such that:* 1. f *and* g *are* C^0*;* 2. $\#g^{-1}(y)$ *is odd, and* g *is* C^1 *in an open neighborhood of* $g^{-1}(y)$*;* 3. y *is a regular value for* g*;* 4. *there exists a continuous homotopy* L *from* g *to* f *such that* $L^{-1}(y)$ *is compact. Then,* $f^{-1}(y)$ *is compact and* $f^{-1}(y) \neq \emptyset$*.*

To apply Theorem 13, we consider the equilibrium function F_e defined in Section 3 which plays the role of the function f. In order to construct the required homotopy and the function that will play the role of the function g, we proceed as follows. First, we construct the so called "test economy". The test economy will be built using a Pareto optimal allocation of an appropriate production economy à la Arrow–Debreu *without externalities at all*. Second, we construct the equilibrium function G associated to the test economy playing the role of the function g. Finally, we provide the required homotopy H_e from G to F_e playing the role of L.

The test economy and the equilibrium function G are defined in Subsection 4.1. The homotopy H_e is given in Subsection 4.2. In Subsection 4.3, we verify that all the assumptions of Theorem 13 are satisfied. More specifically, $G^{-1}(0)$ is a singleton, G is C^1 in an open neighborhood of $G^{-1}(0)$, 0 is a regular value of G and $H_e^{-1}(0)$ is compact.

One should notice that, as a consequence of the properties above, one gets that *degree mod 2* of G is equal to 1. Since the homotopy H_e is continuous and $H_e^{-1}(0)$ is compact, the homotopy invariance of the topological degree implies that the *degree mod 2* of the equilibrium function F_e is equal to 1.

4.1 Test economy

In order to construct the test economy, fixing the externalities, we first consider a Pareto optimal allocation of a standard production economy à la Arrow–Debreu. Second, we construct the equilibrium function G using the Second Welfare Theorem in such a way that, at the test economy, the equilibrium exists and it is unique.

Let $\overline{x} := (\overline{x}_h)_{h\in\mathcal{H}} \in \mathbb{R}^{CH}_{++}$ be an arbitrary consumption and $\overline{y} := (\overline{y}_j)_{j\in\mathcal{J}} \in \mathbb{R}^{CJ}$ be an arbitrary production. Fixing the externalities at $(\overline{x}, \overline{y})$, define $\overline{u}_h(x_h) := u_h(x_h, \overline{x}_{-h}, \overline{y})$, $\overline{t}_j(y_j) := t_j(y_j, \overline{y}_{-j}, \overline{x})$, and the corresponding production economy à la Arrow–Debreu, namely

$$\overline{\mathcal{E}} := ((\mathbb{R}^C_{++}, \overline{u}_h)_{h\in\mathcal{H}}, (\overline{t}_j)_{j\in\mathcal{J}}, \sum_{h\in\mathcal{H}} e_h) \tag{9}$$

where all the consumption sets coincide with the strictly positive orthant of the commodity space, that is $X_h = \mathbb{R}^C_{++}$. Since there are no externalities at all, the notions of feasibility and Pareto optimality are standard. It is well known that, under Assumptions 1, 2 and 3, there exists a Pareto optimal allocation of the economy $\overline{\mathcal{E}}$, denoted by

$$(\widetilde{x}, \widetilde{y}) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$$

and there exist Lagrange multipliers $(\widetilde{\theta}, \widetilde{\gamma}, \widetilde{\beta}) = ((\widetilde{\theta}_h)_{h\neq 1}, \widetilde{\gamma}, (\widetilde{\beta}_j)_{j\in\mathcal{J}}) \in \mathbb{R}^{H-1}_{++} \times \mathbb{R}^C_{++} \times \mathbb{R}^J_{++}$ such that $(\widetilde{x}, \widetilde{y}, \widetilde{\theta}, \widetilde{\gamma}, \widetilde{\beta})$ is the unique solution to the following system.[9]

$$\begin{cases} D_{x_1} u_1(x_1, \overline{x}_{-1}, \overline{y}) - \gamma = 0 \\ \theta_h D_{x_h} u_h(x_h, \overline{x}_{-h}, \overline{y}) - \gamma = 0, \ \forall\, h \neq 1 \\ u_h(x_h, \overline{x}_{-h}, \overline{y}) - u_h(\widetilde{x}_h, \overline{x}_{-h}, \overline{y}) = 0, \ \forall\, h \neq 1 \\ \gamma + \beta_j D_{y_j} t_j(y_j, \overline{y}_{-j}, \overline{x}) = 0, \forall\, j \in \mathcal{J} \\ t_j(y_j, \overline{y}_{-j}, \overline{x}) = 0, \forall\, j \in \mathcal{J} \\ \sum_{h\in\mathcal{H}} x_h - \sum_{j\in\mathcal{J}} y_j - \sum_{h\in\mathcal{H}} e_h = 0 \end{cases} \tag{10}$$

It is well known that the Pareto optimal allocation $(\widetilde{x}, \widetilde{y})$ can be supported by some price system $\widetilde{p}$. That is, using Debreu's vocabulary, $(\widetilde{x}, \widetilde{y})$ is an *equilibrium relative to some price system* $\widetilde{p}$.[10] From system (10), one easily deduces

[9] For a formal proof, see Appendix B.

[10] See Section 6.4 of Debreu (1959).

below the supporting price $\widetilde{p}$ and the equilibrium equations satisfied by $(\widetilde{x}, \widetilde{y})$ for appropriate Lagrange multipliers.

More precisely, there exists $(\widetilde{\lambda}, \widetilde{\mu}, \widetilde{\alpha}) := ((\widetilde{\lambda}_h, \widetilde{\mu}_h)_{h\in\mathcal{H}}, (\widetilde{\alpha}_j)_{j\in\mathcal{J}}) \in (\mathbb{R}_{++} \times \mathbb{R})^H \times \mathbb{R}^J_{++}$ such that

$$\begin{cases} D_{x_h} u_h(\widetilde{x}_h, \overline{x}_{-h}, \overline{y}) - \widetilde{\lambda}_h \widetilde{p} = 0, \ \forall\, h \in \mathcal{H} \\ \widetilde{p} \cdot \widetilde{x}_h = \widetilde{p} \cdot (\widetilde{e}_h + \sum\limits_{j\in\mathcal{J}} s_{jh} \widetilde{y}_j), \ \forall\, h \in \mathcal{H} \\ \widetilde{p} + \widetilde{\alpha}_j D_{y_j} t_j(\widetilde{y}_j, \overline{y}_{-j}, \overline{x}) = 0, \ \forall\, j \in \mathcal{J} \\ t_j(\widetilde{y}_j, \overline{y}_{-j}, \overline{x}) = 0, \ \forall\, j \in \mathcal{J} \\ \sum\limits_{h\in\mathcal{H}} \widetilde{x}_h - \sum\limits_{j\in\mathcal{J}} \widetilde{y}_j - \sum\limits_{h\in\mathcal{H}} e_h = 0 \end{cases} \tag{11}$$

where

$$\widetilde{\lambda}_1 := \widetilde{\gamma}^C, \ \widetilde{\lambda}_h := \frac{\widetilde{\gamma}^C}{\widetilde{\theta}_h} \ \forall\, h \neq 1, \ \widetilde{\alpha}_j := \frac{\widetilde{\beta}_j}{\widetilde{\gamma}^C}, \ \widetilde{p}^{\backslash} := \frac{\widetilde{\gamma}^{\backslash}}{\widetilde{\gamma}^C}$$

and

$$\widetilde{e}_h := \widetilde{x}_h - \sum_{j\in\mathcal{J}} s_{jh} \widetilde{y}_j \tag{12}$$

We call **test economy** the economy defined below

$$\widetilde{\mathcal{E}} := ((\mathbb{R}^C_{++}, \overline{u}_h, \widetilde{e}_h, s_h)_{h\in\mathcal{H}}, (\overline{t}_j)_{j\in\mathcal{J}})$$

The test economy $\widetilde{\mathcal{E}}$ is a standard *private ownership economy* à la Arrow–Debreu with no externalities at all. By system (11), since Karush–Kuhn–Tucker conditions are sufficient to solve the classical firms and households maximization problems, $(\widetilde{x}, \widetilde{\lambda}, \widetilde{y}, \widetilde{\alpha}, \widetilde{p}^{\backslash})$ is a competitive equilibrium for the economy $\widetilde{\mathcal{E}}$. Importantly, as will be shown in Lemma 14 of Subsection 4.3, the economy $\widetilde{\mathcal{E}}$ has a unique equilibrium. [11]

Using the Pareto optimal allocation $(\widetilde{x}, \widetilde{y})$ and the Lagrange multipliers defined above, consider the vector

$$\widetilde{\xi} := (\widetilde{x}, \widetilde{\lambda}, \widetilde{\mu}, \widetilde{y}, \widetilde{\alpha}, \widetilde{p}^{\backslash}) \in \Xi \tag{13}$$

[11] One should notice that the endowments given by (12) are not necessarily positive. There are different redistributions that give rise to positive endowments. For example, $\widetilde{s}_{jh} = \dfrac{\widetilde{p} \cdot \widetilde{x}_h}{\widetilde{p} \cdot \sum_{h\in\mathcal{H}} \widetilde{x}_h}$ and $\widetilde{e}_h = \widetilde{s}_{jh} \sum_{h\in\mathcal{H}} e_h$. But, in the latter case, the uniqueness of the equilibrium is not so obvious since the redistribution depends on the supporting price and on the Pareto optimal consumption allocation. For details, see the proof of Lemma 14.

where $\widetilde{\mu}_h = 0$ for all $h \in \mathcal{H}$. In a natural way, from system (11), one deduces the equilibrium function G satisfying $G(\widetilde{\xi}) = 0$. The function $G : \Xi \to \mathbb{R}^{\dim \Xi}$,

$$G(\xi) := ((G^{h.1}(\xi), G^{h.2}(\xi), G^{h.3}(\xi))_{h\in\mathcal{H}}, (G^{j.1}(\xi), G^{j.2}(\xi))_{j\in\mathcal{J}}, G^M(\xi)) \tag{14}$$

is defined by $G^{h.1}(\xi) := D_{x_h}u_h(x_h, \overline{x}_{-h}, \overline{y}) - \lambda_h p$, $G^{h.2}(\xi) := -p \cdot (x_h - \widetilde{e}_h - \sum_{j\in\mathcal{J}} s_{jh}y_j)$, $G^{h.3}(\xi) := \min\{\mu_h, \chi_h(\widehat{x}_h(x_{-h}, y, e_h), x_{-h}, y)\}$, $G^{j.1}(\xi) := p + \alpha_j D_{y_j}t_j(y_j, \overline{y}_{-j}, \overline{x})$, $G^{j.2}(\xi) := t_j(y_j, \overline{y}_{-j}, \overline{x})$, and $G^M(\xi) := \sum_{h\in\mathcal{H}} x_h^{\setminus} - \sum_{j\in\mathcal{J}} y_j^{\setminus} - \sum_{h\in\mathcal{H}} e_h^{\setminus}$. We remark that the continuous function $\widehat{x}_h$ is given by Proposition 7 and $G^{h.3}(\widetilde{\xi}) = \widetilde{\mu}_h = 0$ since $\chi_h(\widehat{x}_h(x_{-h}, y, e_h), x_{-h}, y) > 0$.

4.2 The homotopy

The basic idea is to homotopize endowments and externalities by an arc from the equilibrium conditions given in the economy $\widetilde{\mathcal{E}}$ to the ones associated to our economy $\mathcal{E}$. But, one finds the following difficulty.

At equilibrium, the individual wealth is positive at the beginning and at the end of the arc. Indeed, in the economy $\widetilde{\mathcal{E}}$, the budget constraint in system (11) and the endowment defined by (12) imply that the individual wealth $\widetilde{p} \cdot \widetilde{x}_h$ is positive. In the economy $\mathcal{E}$, the individual wealth is also positive by inactivity assumption and standard arguments from profit maximization. But, the individual wealth might not be positive along the homotopy arc, and consequently the individual budget constraint might be empty. We illustrate the reason below.

If one homotopizes the endowments, then the individual wealth is given by $p \cdot [\tau e_h + (1-\tau)\widetilde{e}_h] + p \cdot \sum_{j\in\mathcal{J}} s_{jh}y_j$ which is by (12) equal to

$$p \cdot [\tau e_h + (1-\tau)\widetilde{x}_h] + p \cdot \sum_{j\in\mathcal{J}} s_{jh}[y_j - (1-\tau)\widetilde{y}_j] \tag{15}$$

So, the individual wealth is positive if $p \cdot y_j \geq p \cdot (1-\tau)\widetilde{y}_j$ for every $j \in \mathcal{J}$. Using standard arguments from profit maximization, this condition is satisfied if the production plan $(1-\tau)\widetilde{y}_j$ belongs to the production set of firm j. On the other hand, if, at the same time, one homotopizes the externalities, then the production set along the arc is given by

$$Y_j(\tau y_{-j} + (1-\tau)\overline{y}_{-j}, \tau x + (1-\tau)\overline{x})$$

But, one cannot be sure that the production plan $(1-\tau)\widetilde{y}_j$ belongs to the production set above. Consequently, the individual wealth given by (15) might not be positive.

Thus, to overcome this difficulty, we will define the homotopy $H_e$in two times by two homotopies. Namely, in the first homotopy Φ_e, we homotopize the initial endowments, and in the second homotopy Γ_e we homotopize the externalities in the production sets. Finally, we remark that if one assume strong convexity assumption on the production set Y_j, i.e. the function t_j is quasi-concave also with respect to externalities, one does not need two homotopies since initial endowments and externalities can be homotopized at the same time.

Define the following convex combinations

$$
\begin{aligned}
x(\tau) &:= \tau x + (1-\tau)\overline{x} \\
y(\tau) &:= \tau y + (1-\tau)\overline{y} \\
e_h(\tau) &:= \tau e_h + (1-\tau)\widetilde{e}_h
\end{aligned}
$$

and the following two homotopies, $\Phi_e : \Xi \times [0,1] \to \mathbb{R}^{\dim \Xi}$ defined by

$$
\Phi_e(\xi,\tau) := ((\Phi_e^{h.1}(\xi,\tau), \Phi_e^{h.2}(\xi,\tau), \Phi_e^{h.3}(\xi,\tau))_{h\in\mathcal{H}}, (\Phi_e^{j.1}(\xi,\tau), \Phi_e^{j.2}(\xi,\tau))_{j\in\mathcal{J}}, \Phi_e^{M}(\xi,\tau))
$$

$$
\begin{aligned}
\Phi_e^{h.1}(\xi,\tau) &:= D_{x_h}u_h(x_h, x_{-h}(\tau), y(\tau)) - \lambda_h p \\
\Phi_e^{h.2}(\xi,\tau) &:= -p\cdot[x_h - e_h(\tau) - \sum_{j\in\mathcal{J}} s_{jh}y_j] \\
\Phi_e^{h.3}(\xi,\tau) &:= \min\{\mu_h, \chi_h(\widehat{x}_h(x_{-h}, y, e_h), x_{-h}, y)\} \\
\Phi_e^{j.1}(\xi,\tau) &:= p + \alpha_j D_{y_j} t_j(y_j, \overline{y}_{-j}, \overline{x}) \\
\Phi_e^{j.2}(\xi,\tau) &:= t_j(y_j, \overline{y}_{-j}, \overline{x}) \\
\Phi_e^{M}(\xi,\tau) &:= \sum_{h\in\mathcal{H}} x_h^{\setminus} - \sum_{j\in\mathcal{J}} y_j^{\setminus} - \sum_{h\in\mathcal{H}} e_h^{\setminus}
\end{aligned}
\tag{16}
$$

and $\Gamma_e : \Xi \times [0,1] \to \mathbb{R}^{\dim \Xi}$ defined by

$$\Gamma_e(\xi,\tau) := ((\Gamma_e^{h.1}(\xi,\tau), \Gamma_e^{h.2}(\xi,\tau), \Gamma_e^{h.3}(\xi,\tau))_{h\in\mathcal{H}}, (\Gamma_e^{j.2}(\xi,\tau), \Gamma_e^{j.2}(\xi,\tau))_{j\in\mathcal{J}}, \Gamma_e^M(\xi,\tau))$$

$$\begin{aligned}
\Gamma_e^{h.1}(\xi,\tau) &:= D_{x_h}u_h(x_h,x_{-h},y) - \lambda_h p + \\
&\quad \tau\mu_h D_{x_h}\chi_h(\tau x_h + (1-\tau)\widehat{x}_h(x_{-h},y,e_h), x_{-h}, y) \\
\Gamma_e^{h.2}(\xi,\tau) &:= -p\cdot[x_h - e_h - \sum_{j\in\mathcal{J}} s_{jh}y_j] \\
\Gamma_e^{h.3}(\xi,\tau) &:= \min\{\mu_h, \chi_h(\tau x_h + (1-\tau)\widehat{x}_h(x_{-h},y,e_h), x_{-h}, y)\} \\
\Gamma_e^{j.1}(\xi,\tau) &:= p + \alpha_j D_{y_j}t_j(y_j, y_{-j}(\tau), x(\tau)) \\
\Gamma_e^{j.2}(\xi,\tau) &:= t_j(y_j, y_{-j}(\tau), x(\tau)) \\
\Gamma_e^{M}(\xi,\tau) &:= \sum_{h\in\mathcal{H}} x_h^{\setminus} - \sum_{j\in\mathcal{J}} y_j^{\setminus} - \sum_{h\in\mathcal{H}} e_h^{\setminus}
\end{aligned} \tag{17}$$

where the continuous function $\widehat{x}_h$ is given by Proposition 7.

Now, define the homotopy $H_e : \Xi\times[0,1]\to\mathbb{R}^{\dim\Xi}$,

$$H_e(\xi,\psi) := \begin{cases} \Phi_e(\xi,2\psi) \text{ if } 0\le\psi\le\frac{1}{2} \\ \\ \Gamma_e(\xi,2\psi-1) \text{ if } \frac{1}{2}\le\psi\le 1 \end{cases} \tag{18}$$

Observe that H_e is a continuous function. Indeed, Φ_e and Γ_e are continuous because they are composed by continuous functions (see Point 1 of Assumptions 1, 3 and 4, and Proposition 7). Moreover, $H_e\left(\xi,\frac{1}{2}\right)$ is well defined since

$$\Phi_e(\xi,1) = \Gamma_e(\xi,0)$$

Finally, observe that

$$H_e(\xi,0) = \Phi_e(\xi,0) = G(\xi) \text{ and } H_e(\xi,1) = \Gamma_e(\xi,1) = F_e(\xi)$$

4.3 Properties of the homotopy

To verify the assumptions of Theorem 13, we provide the three following lemmas, namely Lemmas 14, 15 and 17.

Lemma 14 *$G^{-1}(0) = \{\widetilde{\xi}\}$ where $\widetilde{\xi}$ is given by (13), and G is C^1 in an open neighborhood of $\widetilde{\xi}$.*

Proof. By (11) and (14), $\widetilde{\xi}\in G^{-1}(0)$. Let $\xi'\in\Xi$ be such that $G(\xi')=0$, we show that $\xi'=\widetilde{\xi}$.

Claim 1. $(x', y') = (\widetilde{x}, \widetilde{y})$. Otherwise, suppose that $(x', y') \neq (\widetilde{x}, \widetilde{y})$. Consider the convex combination

$$(x^{**}, y^{**}) := \frac{1}{2}(x', y') + \frac{1}{2}(\widetilde{x}, \widetilde{y})$$

We first prove that (x^{**}, y^{**}) is a feasible allocation of the economy $\overline{\mathcal{E}}$ defined by (9). Indeed, since $G^{j.2}(\xi') = G^{j.2}(\widetilde{\xi}) = 0$ and the function $t_j(\cdot, \overline{y}_{-j}, \overline{x})$ is strictly quasi-concave (see Point 4 of Assumption 1), we get

$$t_j(y_j^{**}, \overline{y}_{-j}, \overline{x}) > 0, \ \forall \, j \in \mathcal{J} \tag{19}$$

From (12) and $G^{h.2}(\xi') = 0$, summing over h we get

$$\sum_{h \in \mathcal{H}} x_h'^{\,C} - \sum_{j \in \mathcal{J}} y_j'^{\,C} - (\sum_{h \in \mathcal{H}} \widetilde{x}_h^C - \sum_{j \in \mathcal{J}} \widetilde{y}_j^C) = -p'^{\backslash} \cdot [\sum_{h \in \mathcal{H}} x_h'^{\backslash} - \sum_{j \in \mathcal{J}} y_j'^{\backslash} - (\sum_{h \in \mathcal{H}} \widetilde{x}_h^{\backslash} - \sum_{j \in \mathcal{J}} \widetilde{y}_j^{\nu\backslash})]$$

Since $(\widetilde{x}, \widetilde{y})$ is a Pareto optimal allocation, from the last equation of system (10) we have that $\sum_{h \in \mathcal{H}} \widetilde{x}_h - \sum_{j \in \mathcal{J}} \widetilde{y}_j = \sum_{h \in \mathcal{H}} e_h$. Therefore, we obtain $\sum_{h \in \mathcal{H}} x_h'^{\,C} - \sum_{j \in \mathcal{J}} y_j'^{\,C} - \sum_{h \in \mathcal{H}} e_h^C = -p'^{\backslash} \cdot [\sum_{h \in \mathcal{H}} x_h'^{\backslash} - \sum_{j \in \mathcal{J}} y_j'^{\backslash} - \sum_{h \in \mathcal{H}} e_h^{\backslash}]$ which is equal to zero by $G^M(\xi') = 0$. Thus, $\sum_{h \in \mathcal{H}} x_h' - \sum_{j \in \mathcal{J}} y_j' = \sum_{h \in \mathcal{H}} e_h$, and consequently

$$\sum_{h \in \mathcal{H}} x_h^{**} - \sum_{j \in \mathcal{J}} y_j^{**} = \sum_{h \in \mathcal{H}} e_h \tag{20}$$

Thus, (19) and (20) imply that (x^{**}, y^{**}) is a feasible allocation of $\overline{\mathcal{E}}$.

Second, we show that for all $h \in \mathcal{H}$,

$$u_h(x_h', \overline{x}_{-h}, \overline{y}) \geq u_h(\widetilde{x}_h, \overline{x}_{-h}, \overline{y}) \tag{21}$$

Indeed, by (12) $G^{h.1}(\xi') = G^{h.2}(\xi') = 0$ and Karush-Kuhn-Tucker sufficient conditions, x_h' solves the following maximization problem

$$\begin{aligned} &\max_{x_h \in \mathbb{R}^C_{++}} \ u_h(x_h, \overline{x}_{-h}, \overline{y}) \\ &\text{subject to } \ p' \cdot x_h \leq p' \cdot \widetilde{x}_h + \sum_{j \in \mathcal{J}} \widetilde{s}_{jh} \, p' \cdot (y_j' - \widetilde{y}_j) \end{aligned}$$

Notice that $\widetilde{x}_h$ belongs to the budget constraint of the problem above since $\sum_{j \in \mathcal{J}} s_{jh} p' \cdot (y_j' - \widetilde{y}_j) \geq 0$. Indeed, from $G^{j.1}(\xi') = G^{j.2}(\xi') = 0$ and Karush-Kuhn-Tucker sufficient conditions, y_j' solves the following optimization problem

$$\begin{aligned} &\max_{y_j \in \mathbb{R}^C} \ p' \cdot y_j \\ &\text{subject to } \ t_j(y_j, \overline{y}_{-j}, \overline{x}) \geq 0 \end{aligned}$$

From $G^{j.2}(\widetilde{\xi}) = 0$, $\widetilde{y}_j$ belongs to constraint set of the problem above, and so $p' \cdot (y'_j - \widetilde{y}_j) \geq 0$. Therefore, (21) is completely proved.

Finally, (21) and the strict quasi-concavity of $u_h(\cdot, \overline{x}_{-h}, \overline{y})$ (see Point 3 of Assumption 3) imply that for all $h \in \mathcal{H}$

$$u_h(x_h^{**}, \overline{x}_{-h}, \overline{y}) > \min\{u_h(x'_h, \overline{x}_{-h}, \overline{y}), u_h(\widetilde{x}_h, \overline{x}_{-h}, \overline{y})\} = u_h(\widetilde{x}_h, \overline{x}_{-h}, \overline{y})$$

which contradicts the Pareto optimality of $(\widetilde{x}, \widetilde{y})$.

Claim 2. $(\lambda', \mu', \alpha', p'^{\backslash}) = (\widetilde{\lambda}, \widetilde{\mu}, \widetilde{\alpha}, \widetilde{p}^{\backslash})$. By $G^{h.1}(\xi') = G^{h.1}(\widetilde{\xi}) = 0$, we get

$$\lambda'_h = D_{x_h^C} u_h(x'_h, \overline{x}_{-h}, \overline{y}) = D_{x_h^C} u_h(\widetilde{x}_h, \overline{x}_{-h}, \overline{y}) = \widetilde{\lambda}_h$$

So, for every commodity $c \neq C$,

$$p'^{\,c} = \frac{D_{x_h^c} u_h\left(x'_h, \overline{x}_h, \overline{y}\right)}{\lambda'_h} = \frac{D_{x_h^c} u_h(\widetilde{x}_h, \overline{x}_h, \overline{y})}{\widetilde{\lambda}_h} = \widetilde{p}^c$$

Proposition 7 and $G^{h.3}(\xi') = G^{h.3}(\widetilde{\xi}) = 0$ imply $\mu'_h = 0 = \widetilde{\mu}_h$. By $G^{j.1}(\xi') = G^{j.1}(\widetilde{\xi}) = 0$, we get

$$\alpha'_j = -\frac{p'^{\,c}}{D_{y_j^c} t_j(y'_j, \overline{y}_{-j}, \overline{x})} = -\frac{\widetilde{p}^c}{D_{y_j^c} t_j(\widetilde{y}_j, \overline{y}_{-j}, \overline{x})} = \widetilde{\alpha}_j$$

Claim 3. *G is C^1 in a open neighborhood of* $G^{-1}(0) = \widetilde{\xi}$. Since χ_h and $\widehat{x}$ are continuous functions, the function g_h defined below is continuous

$$g_h : \xi \in \Xi \to g_h : (\xi) := (\chi_h(\widehat{x}_h(x_{-h}, y, e_h), x_{-h}, y) - \mu_h) \in \mathbb{R}$$

For all $h \in \mathcal{H}$, $g_h(\widetilde{\xi}) > 0$ since $\chi_h(\widehat{x}_h(\widetilde{x}_{-h}, \widetilde{y}, e_h), \widetilde{x}_{-h}, \widetilde{y}) > 0$ and $\widetilde{\mu}_h = 0$. Thus, in some open neighborhood $\mathcal{I}(\widetilde{\xi}) \subseteq \Xi$ of $\widetilde{\xi}$ we get $g_h(\xi) > 0$ for all $h \in \mathcal{H}$. Therefore, in the open neighborhood $\mathcal{I}(\widetilde{\xi})$, the component $G^{h.3}(\xi) = \mu_h$ for all $h \in \mathcal{H}$ while the components $G^{h.1}(\xi), G^{h.2}(\xi), G^{j.1}(\xi), G^{j.2}(\xi)$ and $G^M(\xi)$ are given by (14). So, $G(\xi)$ is obviously a C^1 function in $\mathcal{I}(\widetilde{\xi})$. ■

Lemma 15 *$D_\xi G(\widetilde{\xi})$ has rank* $\dim \Xi$.

Proof. The computation of $D_\xi G(\widetilde{\xi})$ is described below,

where $\overline{u}_h$ and $\overline{t}_j$ are given in (9) and $\widehat{I} := [I_{C-1}|0]_{(C-1)\times C}$. Define

$$\Delta := \Big((\Delta x_h, \Delta\lambda_h, \Delta\mu_h)_{h\in\mathcal{H}}, (\Delta y_j, \Delta\alpha_j)_{j\in\mathcal{J}}, \Delta p^{\backslash}\Big) \in \mathbb{R}^{H(C+2)} \times \mathbb{R}^{J(C+1)} \times \mathbb{R}^{C-1}$$

In order to prove that $D_\xi G(\widetilde{\xi})$ has full rank, we show that if $D_\xi G(\widetilde{\xi})(\Delta) = 0$ then $\Delta = 0$. $D_\xi G(\widetilde{\xi})(\Delta) = 0$ is given by the following system.

$$\begin{cases} (h.1)\ \ D^2_{x_h} u_h(\widetilde{x}_h, \overline{x}_{-h}, \overline{y})(\Delta x_h) - \Delta\lambda_h \widetilde{p} - \widetilde{\lambda}_h \widehat{I}^T(\Delta p^\setminus) = 0\ \forall\, h \in \mathcal{H} \\ (h.2)\ \ \widetilde{p} \cdot (\sum_{j\in\mathcal{J}} s_{jh}\Delta y_j) - \widetilde{p}\cdot \Delta\widetilde{x}_h = 0\ \forall\, h \in \mathcal{H} \\ (h.3)\ \ \Delta\mu_h = 0\ \forall\, h\in\mathcal{H} \\ (j.1)\ \ \widetilde{\alpha}_j D^2_{y_j} t_j(\widetilde{y}_j, \overline{y}_{-j}, \overline{x})(\Delta y_j) + \Delta\alpha_j D_{y_j} t_j(\widetilde{y}_j, \overline{y}_{-j}, \overline{x}) + \widehat{I}^T(\Delta p^\setminus) = 0\ \forall\, j \in \mathcal{J} \\ (j.2)\ \ D_{y_j} t_j(\widetilde{y}_j, \overline{y}_{-j}, \overline{x}) \cdot \Delta y_j = 0\ \forall\, j\in\mathcal{J} \\ (M)\ \ \sum_{h\in\mathcal{H}} \Delta x_h^\setminus - \sum_{j\in\mathcal{J}} \Delta y_j^\setminus = 0 \end{cases} \tag{22}$$

We first prove that if $\Delta x_h = 0$ for every $h \in \mathcal{H}$, then $\Delta = 0$. In this case, Equation $(h.1)$ in system (22) becomes $-\widetilde{p}\cdot\Delta\lambda_h - \lambda_h \widehat{I}^T(\Delta p^\setminus) = 0$. Considering commodity C, we obtain $\Delta\lambda_h = 0$ for all $h \in \mathcal{H}$. So, $-\widetilde{\lambda}_h \widehat{I}^T(\Delta p^\setminus) = 0$ implies $\Delta p^\setminus = 0$ since $\widetilde{\lambda}_h > 0$. Equations $(j.1)$ and $(j.2)$ in system (22) can be now written as follows

$$\begin{pmatrix} \widetilde{\alpha}_j D^2_{y_j} t_j(\widetilde{y}_j, \overline{y}_{-j}, \overline{x}) & D_{y_j} t_j(\widetilde{y}_j, \overline{y}_{-j}, \overline{x})^T \\ D_{y_j} t_j(\widetilde{y}_j, \overline{y}_{-j}, \overline{x}) & 0 \end{pmatrix} \begin{pmatrix} \Delta y_j \\ \Delta\alpha_j \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{23}$$

Since $t_j(\cdot, \overline{y}_{-j}, \overline{x})$ is strictly quasi-concave, (see Point 4 of Assumption 1), and $D_{y_j} t_j(y_j, \overline{y}_{-j}, \overline{x}) \neq 0$, (Point 3 of Assumption 1), the following matrix has full rank. [12]

$$\begin{pmatrix} \widetilde{\alpha}_j D^2_{y_j} t_j(\widetilde{y}_j, \overline{y}_{-j}, \overline{x}) & D_{y_j} t_j(\widetilde{y}_j, \overline{y}_{-j}, \overline{x})^T \\ D_{y_j} t_j(\widetilde{y}_j, \overline{y}_{-j}, \overline{x}) & 0 \end{pmatrix}$$

[12] A differentiably strictly quasi-concave function with gradient different from zero has the bordered Hessian with determinant different from zero.

	x_h	λ_h	μ_h	y_j	α_j	$p^\setminus$
$D_{x_h}\overline{u}_h(x_h) - \lambda_h p$	$D^2_{x_h}\overline{u}_h(\widetilde{x}_h)$	$-\widetilde{p}^T$				$-\widetilde{\lambda}_h \widehat{I}^T$
$-p\cdot[x_h - \widetilde{e}_h - \sum_{j\in\mathcal{J}} s_{jh}(y_j - \widetilde{y}_j)]$	$-\widetilde{p}$			$\widetilde{p}s_{jh}$		
μ_h			1			
$p + \alpha_h D_{y_j}\overline{t}_j(y_j)$				$\widetilde{\alpha}_j D^2_{y_j}\overline{t}_j(\widetilde{y}_j)$	$D_{y_j}\overline{t}_j(\widetilde{y}_j)^T$	$\widehat{I}^T$
$\overline{t}_j(y_j)$				$D_{y_j}\overline{t}_j(\widetilde{y}_j)$		
$\sum_{h\in\mathcal{H}} x_h^\setminus - \sum_{j\in\mathcal{J}} y_j^\setminus - \sum_{h\in\mathcal{H}} e_h^\setminus$	$\widehat{I}$			$-\widehat{I}$		

Therefore, $(\Delta y_j, \Delta \alpha_j) = (0, 0)$ by (23). So, one gets $\Delta = 0$.

Second, we prove that $\Delta x_h = 0$ for every $h \in \mathcal{H}$.

Suppose by contradiction that there is $\overline{h} \in \mathcal{H}$ such that $\Delta x_{\overline{h}} \neq 0$. We first claim that

$$\sum_{h \in \mathcal{H}} \Delta x_h \frac{D^2_{x_h} u_h(\widetilde{x}_h, \overline{x}_{-h}, \overline{y})}{\widetilde{\lambda}_h} (\Delta x_h) < 0 \tag{24}$$

Multiplying both sides of $G^{j.1}(\widetilde{\xi}) = 0$ by $s_{jh} \Delta y_j$, we get

$$\widetilde{p} \cdot s_{jh} \Delta y_j + D_{y_j} t_j(\widetilde{y}_j, \overline{y}_{-j}, \overline{x}) \cdot s_{jh} \Delta y_j = 0 \tag{25}$$

So, $\widetilde{p} \cdot s_{jh} \Delta y_j = 0$ by equation $(j.2)$ in system (22). Summing over j, for each $h \in \mathcal{H}$, one gets $\widetilde{p} \cdot (\sum_{j \in \mathcal{J}} s_{jh} \Delta y_j) = 0$ which implies

$$\widetilde{p} \cdot \Delta x_h = 0 \tag{26}$$

by equation $(h.2)$ in system (22). Multiplying $G^{h.1}(\widetilde{\xi}) = 0$ by Δx_h, one obtains

$$D_{x_h} u_h(\widetilde{x}_h, \overline{x}_{-h}, \overline{y}) \cdot \Delta x_h = 0$$

From Point 3 of Assumption 3 and $\widetilde{\lambda}_h > 0$, one gets

$$\Delta x_h \frac{D^2_{x_h} u_h(\widetilde{x}_h, \overline{x}_{-h}, \overline{y})}{\widetilde{\lambda}_h} (\Delta x_h) \leq 0, \quad \forall\, h \in \mathcal{H}$$

with a strict inequality for $\overline{h}$, since $\Delta x_{\overline{h}} \neq 0$. Summing over h, one obtains (24).

Second, we claim that

$$\Delta p^{\setminus} \cdot (\sum_{h \in \mathcal{H}} \Delta x_h^{\setminus}) < 0 \tag{27}$$

From (26), multiplying $(h.1)$ in system (22) by $\dfrac{\Delta x_h}{\widetilde{\lambda}_h}$ and summing over h, one gets

$$\sum_{h \in \mathcal{H}} \Delta x_h \frac{D^2_{x_h} u_h(\widetilde{x}_h, \overline{x}_{-h}, \overline{y})}{\widetilde{\lambda}_h} (\Delta x_h) = \Delta p^{\setminus} \cdot (\sum_{h \in \mathcal{H}} \Delta x_h^{\setminus})$$

Thus, (27) follows from (24).

Finally, we show below that $\Delta p^{\setminus} \cdot (\sum_{h \in \mathcal{H}} \Delta x_h^{\setminus}) \geq 0$ which leads to a contradiction taking into account (27), and consequently $\Delta x_h = 0$ for all $h \in \mathcal{H}$ which complete the proof of the lemma.

By (M) in system (22), one has

$$\Delta p^{\setminus} \cdot (\sum_{h\in\mathcal{H}} \Delta x_h^{\setminus}) = \Delta p^{\setminus} \cdot (\sum_{j\in\mathcal{J}} \Delta y_j^{\setminus})$$

By $(j.2)$ in system (22), multiplying $(j.1)$ in system (22) by Δy_j and summing over j one has $\Delta p^{\setminus} \cdot (\sum_{j\in\mathcal{J}} \Delta y_j^{\setminus}) = -\sum_{j\in\mathcal{J}} \widetilde{\alpha}_j \Delta y_j D^2_{y_j} t_j(\widetilde{y}_j, \overline{y}_{-j}, \overline{x})(\Delta y_j)$. Thus,

$$\Delta p^{\setminus} \cdot (\sum_{h\in\mathcal{H}} \Delta x_h^{\setminus}) = -\sum_{j\in\mathcal{J}} \widetilde{\alpha}_j \Delta y_j D^2_{y_j} t_j(\widetilde{y}_j, \overline{y}_{-j}, \overline{x})(\Delta y_j)$$

Therefore, $\Delta p^{\setminus} \cdot (\sum_{h\in\mathcal{H}} \Delta x_h^{\setminus}) \geq 0$ since $t_j(\cdot, \overline{y}_{-j}, \overline{x})$ is strictly quasi-concave (see Point 4 of Assumption 1). ■

Lemma 16 *For each $r \in \mathbb{R}^C_{++}$, the following sets are bounded.*

$$\begin{aligned} A_{t,r} :=& \{(x', y') \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ} \mid \exists\, (x, y) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ} : t_j(y'_j, y_{-j}, x) \geq 0 \\ & \forall\, j \in \mathcal{J} \text{ and } \sum_{h\in\mathcal{H}} x'_h - \sum_{j\in\mathcal{J}} y'_j \leq r\} \end{aligned} \tag{28}$$

$$\bar{A}_{t,r} := \{(x', y') \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ} : t_j(y'_j, \overline{y}_{-j}, \overline{x}) \geq 0,\ \forall j \in \mathcal{J} \text{ and } \sum_{h\in\mathcal{H}} x'_h - \sum_{j\in\mathcal{J}} y'_j \leq r\} \tag{29}$$

Proof. We prove that $A_{t,r}$ is bounded. Consequently, $\bar{A}_{t,r}$ is bounded since it is included in $A_{t,r}$. Let $(x', y') \in A_{t,r}$. Since $x'_h \gg 0$ for every $h \in \mathcal{H}$, we have that x' is bounded from below by zero. Therefore for every $h \in \mathcal{H}$

$$0 \ll x'_h \ll \sum_{h\in\mathcal{H}} x'_h \leq r + \sum_{j\in\mathcal{J}} y'_j$$

Thus, to show that $A_{t,r}$ is bounded it is enough to prove that the set $Y \cap M_r$ is bounded where the set Y is defined by (1) and

$$M_r := \{y' \in \mathbb{R}^{CJ} \ : \ \sum_{j\in\mathcal{J}} y'_j + r \geq 0\}$$

Since a subset of $\mathbb{R}^n$ is bounded if and only if its asymptotic cone is reduced to zero, we show now that $C(Y \cap M_r) = \{0\}$. One should notice that $C(Y \cap M_r) \subseteq CY \cap CM_r$.[13] Since the asymptotic cone of a set is immune to translation, we get $CM_r = CM_0$, where $M_0 := \{y' \in \mathbb{R}^{CJ} \ : \ \sum_{j\in\mathcal{J}} y'_j \geq 0\}$. M_0 is a closed cone with vertex 0, thus $CM_0 = M_0$. So, in order to prove that $CY \cap CM_r$ we have to show that $CY \cap M_0 = \{0\}$ which directly follows from Assumption 2. ■

[13] Let $(B_i)_{i\in\mathcal{I}} \subseteq \mathbb{R}^n$ be a family of subsets of $\mathbb{R}^n$, $C(\cap_{i\in\mathcal{I}} B_i) \subseteq \cap_{i\in\mathcal{I}} CB_i$.

Lemma 17 *For each $e \in E_\chi$, $H_e^{-1}(0)$ is compact.*

Proof. Let $e \in E_\chi$. Observe that $H_e^{-1}(0) = \Phi_e^{-1}(0) \cup \Gamma_e^{-1}(0)$. Since the union of a finite number of compact sets is compact, it is enough to show that $\Phi_e^{-1}(0)$ and $\Gamma_e^{-1}(0)$ are compact.

Claim 1. *$\Phi_e^{-1}(0)$ is compact.*

We prove that, up to a subsequence, every sequence $(\xi^\nu, \tau^\nu)_{\nu\in\mathbb{N}} \subseteq \Phi_e^{-1}(0)$ converges to an element of $\Phi_e^{-1}(0)$, where $\xi^\nu := (x^\nu, \lambda^\nu, \mu^\nu, y^\nu, \alpha^\nu, p^{\nu\setminus})_{\nu\in\mathbb{N}}$. First observe that, since $\{\tau^\nu : \nu \in \mathbb{N}\} \subseteq [0,1]$, up to a subsequence, $(\tau^\nu)_{\nu\in\mathbb{N}}$ converges to some $\tau^* \in [0,1]$. From Steps 1.1, 1.2, 1.3 and 1.4 below, we have that up to a subsequence, $(\xi^\nu)_{\nu\in\mathbb{N}}$ converges to some $\xi^* := (x^*, \lambda^*, \mu^*, y^*, \alpha^*, p^{*\setminus}) \in \Xi$. Since the homotopy Φ_e is continuous, taking the limit, we get the desired result, that is $(\xi^*, \tau^*) \in \Phi_e^{-1}(0)$.

Step 1.1. *Up to a subsequence, $(x^\nu, y^\nu)_{\nu\in\mathbb{N}}$ converges to some $(x^*, y^*) \in \mathbb{R}_+^{CH} \times \mathbb{R}^{CJ}$.* We first prove that the sequence $(x^\nu, y^\nu)_{\nu\in\mathbb{N}}$ belongs to the set $\bar{A}_{t,r}$ given by (29) of Lemma 16. Using a similar strategy as in Claim 1 of Lemma 14, by (12), $\Phi_e^{h.2}(\xi^\nu, \tau^\nu) = 0$ and $\Phi_e^M(\xi^\nu, \tau^\nu) = 0$ one easily gets

$$\sum_{h\in\mathcal{H}} x_h^\nu - \sum_{j\in\mathcal{J}} y_j^\nu = \sum_{h\in\mathcal{H}} e_h, \quad \forall\, \nu \in \mathbb{N}$$

So, $(x^\nu, y^\nu)_{\nu\in\mathbb{N}} \subseteq \bar{A}_{t,r}$ by $\Phi_e^{j.2}(\xi^\nu, \tau^\nu) = 0$. Consequently, the sequence $(x^\nu, y^\nu)_{\nu\in\mathbb{N}}$ belongs to $\operatorname{cl} \bar{A}_{t,r}$ which is compact by Lemma 16. Up to a subsequence, $(x^\nu, y^\nu)_{\nu\in\mathbb{N}}$ converges to some $(x^*, y^*) \in \operatorname{cl} \bar{A}_{t,r} \subseteq \mathbb{R}_+^{CH} \times \mathbb{R}^{CJ}$, and thus $(x^*, y^*) \in \mathbb{R}_+^{CH} \times \mathbb{R}^{CJ}$.

Step 1.2. *The consumption allocation x^* is strictly positive, i.e. $x^* \gg 0$.* The proof is based on Point 4 of Assumption 3. By (12) and $\Phi_e^{h.1}(\xi^\nu, \tau^\nu) = \Phi_e^{h.2}(\xi^\nu, \tau^\nu) = 0$, x_h^ν solves the following problem for every $\nu \in \mathbb{N}$.[14]

$$\begin{aligned} &\max_{x_h \in \mathbb{R}_{++}^C} \; u_h(x_h, x_{-h}^\nu(\tau^\nu), y^\nu(\tau^\nu)) \\ &\text{subject to } p^\nu \cdot x_h \le p^\nu \cdot [\tau^\nu e_h + (1-\tau^\nu)\widetilde{x}_h] + p^\nu \cdot \sum_{j\in\mathcal{J}} s_{jh}(y_j^\nu - (1-\tau^\nu)\widetilde{y}_j) \end{aligned} \tag{30}$$

We claim first that for every $\nu \in \mathbb{N}$, the point

$$\tau^\nu e_h + (1-\tau^\nu)\widetilde{x}_h \tag{31}$$

belongs to the budget constraint of the problem above. By $\Phi_e^{j.1}(\xi^\nu, \tau^\nu) = \Phi_e^{j.2}(\xi^\nu, \tau^\nu) = 0$ and Karush–Kuhn–Tucker sufficient conditions, y_j^ν solves the

[14] Karush–Kuhn–Tucker conditions are sufficient to solve problem (30).

following problem for every $\nu \in \mathbb{N}$.

$$\begin{aligned}&\max_{y_j \in \mathbb{R}^C} p^\nu \cdot y_j \\ &\text{subject to } t_j(y_j, \overline{y}_{-j}, \overline{x}) \geq 0\end{aligned} \tag{32}$$

Since inactivity is possible, $t_j(0, \overline{y}_{-j}, \overline{x}) \geq 0$ by Point 2 of Assumption 1. Since $(\widetilde{x}, \widetilde{y})$ is a Pareto optimal allocation, $t_j(\widetilde{y}_j, \overline{y}_{-j}, \overline{x}) = 0$ by system (10). Since $t_j(\cdot, \overline{y}_{-j}, \overline{x})$ is strictly quasi-concave, we get

$$t_j(\tau^\nu 0 + (1-\tau^\nu)\widetilde{y}_j, \overline{y}_{-j}, \overline{x}) = t_j((1-\tau^\nu)\widetilde{y}_j, \overline{y}_{-j}, \overline{x}) \geq 0$$

So, the production plan $(1-\tau^\nu)\widetilde{y}_j$ belongs to the constraint set of problem (32), and thus $p^\nu \cdot (y_j^\nu - (1-\tau^\nu)\widetilde{y}_j) \geq 0$. Therefore,

$$p^\nu \cdot \sum_{j \in \mathcal{J}} s_{jh}(y_j^\nu - (1-\tau^\nu)\widetilde{y}_j) \geq 0$$

which completes the proof of the claim.

We claim now that x_h^* belongs to the closure of some upper contour set. Obviously, for every $\nu \in \mathbb{N}$

$$u_h(x_h^\nu, x_{-h}^\nu(\tau^\nu), y^\nu(\tau^\nu)) \geq u_h(\tau^\nu e_h + (1-\tau^\nu)\widetilde{x}_h, x_{-h}^\nu(\tau^\nu), y^\nu(\tau^\nu))$$

By Point 2 of Assumption 3, for every $\varepsilon > 0$ we have that

$$u_h(x_h^\nu + \varepsilon \mathbf{1}, x_{-h}^\nu(\tau^\nu), y^\nu(\tau^\nu)) > u_h(\tau^\nu e_h + (1-\tau^\nu)\widetilde{x}_h, x_{-h}^\nu(\tau^\nu), y^\nu(\tau^\nu))$$

where $\mathbf{1} := (1, \ldots, 1) \in \mathbb{R}^C_{++}$. So, taking the limit for $\nu \to +\infty$ and using the continuity of u_h given by Point 1 of Assumption 3, we get

$$u_h(x_h^* + \varepsilon \mathbf{1}, x_{-h}^*(\tau^*), y^*(\tau^*)) \geq u_h(\tau^* e_h + (1-\tau^*)\widetilde{x}_h, x_{-h}^*(\tau^*), y^*(\tau^*)) := u$$

That is, for every $\varepsilon > 0$ the point $(x_h^* + \varepsilon \mathbf{1})$ belongs to the following set

$$\{x_h \in \mathbb{R}^C_{++} : u_h(x_h, x_{-h}^*(\tau^*), y^*(\tau^*)) \geq u\}$$

So, the point x_h^* belongs to the closure of set above which is included in $\mathbb{R}^C_{++}$ by Point 4 of Assumption 3. Therefore, $x_h^* \in \mathbb{R}^{CH}_{++}$. One should notice that, since $\tau^* \in [0,1]$, $x_{-h}^*(\tau^*)$ is not necessarily strictly positive. For that reason, in Point 4 of Assumption 3 we consider x_{-h} in $\mathbb{R}^{C(H-1)}_+$.

Step 1.3. *Up to a subsequence,* $(\alpha^\nu, p^{\nu\setminus})_{\nu \in \mathbb{N}}$ *converges to some* $(\alpha^*, p^{*\setminus}) \in \mathbb{R}^J_{++} \times \mathbb{R}^{C-1}_{++}$. By $\Phi_e^{j.1}(\xi^\nu, \tau^\nu) = 0$, considering commodity C, we get

$$\alpha_j^\nu = -\frac{1}{D_{y_j^C} t_j(y_j^\nu, \overline{y}_{-j}, \overline{x})}, \;\; \forall\, \nu \in \mathbb{N}$$

Taking the limit for $\nu \to +\infty$ and using the continuity of Dt_j and the "free disposal" property (see Points 1 and 3 of Assumption 1), the sequence $(\alpha_j^\nu)_{\nu\in\mathbb{N}}$ converges to

$$\alpha_j^* := -\frac{1}{D_{y_j^C} t_j(y_j^*, \overline{y}_{-j}, \overline{x})} > 0$$

By $\Phi_e^{j.1}(\xi^\nu, \tau^\nu) = 0$, for every commodity $c \neq C$ and for all $\nu \in \mathbb{N}$ we have

$$p^{\nu\, c} = -\alpha_j^\nu D_{y_j^c} t_j(y_j^\nu, \overline{y}_{-j}, \overline{x})$$

Taking the limit and using Points 1 and 3 of Assumption 1, for all $c \neq C$ we get

$$p^{*\, c} = -\alpha_j^* D_{y_j^c} t_j(y_j^*, \overline{y}_{-j}, \overline{x}) > 0$$

Therefore, $p^{*\setminus} \in \mathbb{R}_{++}^{C-1}$.

Step 1.4. *Up to a subsequence, $(\lambda^\nu, \mu^\nu)_{\nu\in\mathbb{N}}$ converges to some $(\lambda^*, \mu^*) \in \mathbb{R}_{++}^H \times \mathbb{R}_+^H$.* By $\Phi_e^{h.3}(\xi^\nu, \tau^\nu) = 0$ and Proposition 7, we have $\mu_h^\nu = 0$ for every $\nu \in \mathbb{N}$. Taking the limit, we get $\mu_h^* = 0$.

By $\Phi_e^{h.1}(\xi^\nu, \tau^\nu) = 0$, considering commodity C, for every $\nu \in \mathbb{N}$ we get

$$\lambda_h^\nu = D_{x_h^C} u_h(x_h^\nu, x_{-h}^\nu(\tau^\nu), y^\nu(\tau^\nu))$$

Taking the limit and using the continuity of Du_h (see Point 1 of Assumption 3) we have

$$\lambda_h^* = D_{x_h^C} u_h(x_h^*, x_{-h}^*(\tau^*), y^*(\tau^*))$$

which is strictly positive since fixing the externalities the function u_h is strictly increasing (see Point 2 of Assumption 3).

Claim 2. *$\Gamma_e^{-1}(0)$ is compact.*

Let $(\xi^\nu, \tau^\nu)_{\nu\in\mathbb{N}}$ be a sequences in $\Gamma_e^{-1}(0)$. As in Claim 1, $(\tau^\nu)_{\nu\in\mathbb{N}}$ converges to $\tau^* \in [0,1]$. From Seps 2.1, 2.2, 2.3 and 2.4 below, we have that, up to a subsequence, $(\xi^\nu)_{\nu\in\mathbb{N}}$ converges to an element $\xi^* := (x^*, \lambda^*, \mu^*, y^*, \alpha^*, p^{*\setminus}) \in \Xi$. Since Γ_e is a continuous function, taking limit one gets $(\xi^*, \tau^*) \in \Gamma_e^{-1}(0)$.

Step 2.1. *Up to a subsequence, $(x^\nu, y^\nu)_{\nu\in\mathbb{N}}$ converges to some $(x^*, y^*) \in \mathbb{R}_+^{CH} \times \mathbb{R}^{CJ}$.* By $\Gamma_e^{j.2}(\xi^\nu, \tau^\nu) = 0$, we have that for every $\nu \in \mathbb{N}$ and for every j

$$t_j(y_j^\nu, y_{-j}^\nu(\tau^\nu), x^\nu(\tau^\nu)) = 0$$

Summing $\Gamma_e^{h.2}(\xi^\nu, \tau^\nu) = 0$ over h, by $\Gamma_e^M(\xi^\nu, \tau^\nu) = 0$ we get $\sum\limits_{h\in\mathcal{H}} x_h^\nu - \sum\limits_{j\in\mathcal{J}} y_j^\nu = \sum\limits_{h\in\mathcal{H}} e_h$ for all $\nu \in \mathbb{N}$. Therefore, $(x^\nu, y^\nu)_{\nu\in\mathbb{N}}$ belongs to the set $A_{t,r}$ given by (28). Consequently, the sequence $(x^\nu, y^\nu)_{\nu\in\mathbb{N}}$ belongs to $\operatorname{cl} A_{t,r}$ which is compact by

Lemma 16. So, up to a subsequence, $(x^\nu, y^\nu)_{\nu\in\mathbb{N}}$ converges to some $(x^*, y^*) \in \operatorname{cl} A_{t,r} \subseteq \mathbb{R}_+^{CH} \times \mathbb{R}^{CJ}$, and thus $(x^*, y^*) \in \mathbb{R}_+^{CH} \times \mathbb{R}^{CJ}$.

Step 2.2. *The consumption allocation x^* is strictly positive, i.e. $x^* \gg 0$.* The argument is similar to the one used in Step 1.2 of Claim 1. It suffices to replace

(1) the problem (30) with the following problem

$$\begin{aligned}&\max_{x_h\in\mathbb{R}^C_{++}} \; u_h(x_h, x^\nu_{-h}, y^\nu)\\&\text{subject to } \chi_h(\tau^\nu x_h + (1-\tau^\nu)\widehat{x}_h(x^\nu_{-h}, y^\nu, e_h), x^\nu_{-h}, y^\nu) \geq 0\\&\qquad\qquad p^\nu \cdot x_h \leq p^\nu \cdot e_h + p^\nu \cdot \sum_{j\in\mathcal{J}} s_{jh} y^\nu_j\end{aligned}$$

according to $\Gamma^{h.1}_e(\xi^\nu, \tau^\nu) = \Gamma^{h.2}_e(\xi^\nu, \tau^\nu) = \Gamma^{h.3}_e(\xi^\nu, \tau^\nu) = 0$,

(2) the point given by (31) with $\widehat{x}_h(x^\nu_{-h}, y^\nu, e_h)$ given by Proposition 7,

(3) the problem (32) with the following problem

$$\begin{aligned}&\max_{y_j\in\mathbb{R}^C} p^\nu \cdot y_j\\&\text{subject to } t_j(y_j, y^\nu_{-j}(\tau^\nu), x^\nu(\tau^\nu)) \geq 0\end{aligned} \tag{33}$$

according to $\Gamma^{j.1}_e(\xi^\nu, \tau^\nu) = \Gamma^{j.2}_e(\xi^\nu, \tau^\nu) = 0$.

Next, as in Step 1.2 of Claim 1 one easily shows that x^*_h belongs to the closure of $\{x_h \in \mathbb{R}^C_{++} : u_h(x_h, x^*_{-h}, y^*) \geq u := u_h(\widehat{x}_h(x^*_{-h}, y^*, e_h), x^*_{-h}, y^*)\}$. One should notice that although x^*_{-h} may not be positive, Assumption 4 and consequently Proposition 7 ensure that $\widehat{x}_h(x^*_{-h}, y^*, e_h)$ is well defined and strictly positive.

Step 2.3. *Up to a subsequence, $(\alpha^\nu, p^{\nu\,\backslash})_{\nu\in\mathbb{N}}$ converges to some $(\alpha^*, p^{*\,\backslash}) \in \mathbb{R}^J_{++} \times \mathbb{R}^{C-1}_{++}$.* Using Points 1 and 3 of Assumption 1, the proof is similar to the one of Step 1.3 in Claim 1.

Step 2.4. *Up to a subsequence, $(\lambda^\nu, \mu^\nu)_{\nu\in\mathbb{N}}$ converges to some $(\lambda^*, \mu^*) \in \mathbb{R}^H_{++} \times \mathbb{R}^H_+$.* We have two possible cases, in Case a), $\tau^* = 0$, and in Case b), $\tau^* \in (0, 1]$.

Case a). $\tau^* = 0$. Using $\Gamma^{h.3}_e(\xi^\nu, \tau^\nu) = 0$, we first claim that there exists $\nu^* \in \mathbb{N}$ such that for every $\nu \geq \nu^*$,

$$\mu^\nu_h = 0$$

Since $\tau^* = 0$, the sequence

$$(\tau^\nu x^\nu_h + (1-\tau^\nu)\widehat{x}_h(x^\nu_{-h}, y^\nu, e_h), x^\nu_{-h}, y^\nu)_{\nu\in\mathbb{N}}$$

converges to $(\widehat{x}_h(x^*_{-h}, y^*, e_h), x^*_{-h}, y^*)$. By Proposition 7,

$$\chi_h(\widehat{x}_h(x^*_{-h}, y^*, e_h), x^*_{-h}, y^*) > 0$$

So, the continuity of the functions $\widehat{x}_h$ and χ_h (see Proposition 7 and Point 1 of Assumption 4) imply that there is $\nu^* \in \mathbb{N}$ such that for every $\nu \geq \nu^*$,

$$\chi_h(\tau^\nu x_h^\nu + (1-\tau^\nu)\widehat{x}_h(x_{-h}^\nu, y^\nu, e_h), x_{-h}^\nu, y^\nu) > 0$$

which proves the claim. Thus, the sequence $(\mu_h^\nu)_{\nu\in\mathbb{N}}$ converges to $\mu_h^* = 0$.

By $\Gamma_e^{h.1}(\xi,\tau) = 0$, considering commodity C, we get $\lambda_h^\nu = D_{x_h^C} u_h(x_h^\nu, x_{-h}^\nu, y^\nu)$ for every $\nu \geq \nu^*$. Taking the limit and using the continuity of Du_h (Point 1 of Assumption 3), we get

$$\lambda_h^* = D_{x_h^C} u_h(x_h^*, x_{-h}^*, y^*)$$

which is strictly positive by Point 2 of Assumption 3.

Case b). $\tau^* \in (0,1]$. We first claim that up to a subsequence, $(\lambda^\nu, \mu^\nu)_{\nu\in\mathbb{N}} \subseteq \mathbb{R}_{++}^H \times \mathbb{R}_+^H$ converges to some $(\lambda^*, \mu^*) \in \mathbb{R}_+^H \times \mathbb{R}_+^H$. Second, we show that $\lambda^* \gg 0$.

In order to prove the claim above, it is enough to show that $(\lambda_h^\nu, \mu_h^\nu)_{\nu\in\mathbb{N}}$ is bounded for every $h \in \mathcal{H}$. Otherwise, suppose that there is a subsequence that without loss of generality we continue to denote with $(\lambda_h^\nu, \mu_h^\nu)_{\nu\in\mathbb{N}}$ such that $\|(\lambda_h^\nu, \mu_h^\nu)\|$ diverges to $+\infty$. Consider the following sequence in the sphere which is a compact set. [15]

$$\left(\frac{(\lambda_h^\nu, \mu_h^\nu)}{\|(\lambda_h^\nu, \mu_h^\nu)\|}\right)_{\nu\in\mathbb{N}}$$

Up to a subsequence, $\left(\frac{(\lambda_h^\nu, \mu_h^\nu)}{\|(\lambda_h^\nu, \mu_h^\nu)\|}\right)_{\nu\in\mathbb{N}}$ converges to some $(\lambda_h, \mu_h) \neq (0,0)$. [16] Obviously, $\lambda_h \geq 0$ and $\mu_h \geq 0$, since $\lambda_h^\nu > 0$ and $\mu_h^\nu \geq 0$ for all $\nu \in \mathbb{N}$.

Dividing both sides of $\Gamma_e^{h.1}(\xi^\nu, \tau^\nu) = 0$ by $\|(\lambda_h^\nu, \mu_h^\nu)\|$, and taking the limit, we get

$$\lambda_h p^* = \tau^* \mu_h D_{x_h} \chi_h(\tau^* x_h^* + (1-\tau^*)\widehat{x}_h(x_{-h}^*, y^*, e_h), x_{-h}^*, y^*) \tag{34}$$

Notice that $\mu_h > 0$ and $\lambda_h > 0$. Indeed from Point 4(a) of Assumption 4, we know that

$$D_{x_h} \chi_h(\tau^* x_h^* + (1-\tau^*)\widehat{x}_h(x_{-h}^*, y^*, e_h), x_{-h}^*, y^*) \neq 0$$

Thus, $\mu_h > 0$ because if $\mu_h = 0$, from (34) we get $\lambda_h = 0$ which contradicts the fact that $(\lambda_h, \mu_h) \neq (0,0)$. Finally, $\mu_h > 0$, $\tau^* > 0$, $p^* \in \mathbb{R}_{++}^C$ and (34) imply $\lambda_h > 0$.

[15] Since $\|(\lambda_h^\nu, \mu_h^\nu)_{\nu\in\mathbb{N}}\|$ diverges to $+\infty$, without loosing of generality, we suppose that $\|(\lambda_h^\nu, \mu_h^\nu)\| > 0$ for every ν.

[16] Observe that $(\lambda_h, \mu_h) \neq (0,0)$ since $\|(\lambda_h, \mu_h)\| = 1$.

We prove now that

$$\lambda_h p^* \cdot \widehat{x}_h(x^*_{-h}, y^*, e_h) < \lambda_h p^* \cdot x^*_h \tag{35}$$

Since $\lambda_h > 0$, Proposition 7 implies that

$$\lambda_h p^* \cdot \widehat{x}_h(x^*_{-h}, y^*, e_h) < \lambda_h p^* \cdot e_h \tag{36}$$

Multiplying $\Gamma_e^{h.2}(\xi^\nu, \tau^\nu) = 0$ by λ^ν_h, for every $\nu \in \mathbb{N}$ we get $\lambda^\nu_h p^\nu \cdot e_h + \lambda^\nu_h p^\nu \cdot \sum_{j\in\mathcal{J}} s_{jh} y^\nu = \lambda^\nu_h p^\nu \cdot x^\nu_h$. Thus, dividing both sides by $\|(\lambda^\nu_h, \mu^\nu_h)\|$ and taking the limit, we get

$$\lambda_h p^* \cdot e_h + \lambda_h p^* \cdot \sum_{j\in\mathcal{J}} s_{jh} y^*_j = \lambda_h p^* \cdot x^*_h \tag{37}$$

Therefore, (35) follows from (36) and (37) since

$$\lambda_h p^* \cdot \sum_{j\in\mathcal{J}} s_{jh} y^*_j \geq 0$$

The inequality above follows by $\Gamma_e^{j.1}(\xi^\nu, \tau^\nu) = \Gamma_e^{j.2}(\xi^\nu, \tau^\nu) = 0$ and the possibility of inactivity (Point 2 of Assumption 1). Indeed, Karush-Kuhn-Tucker sufficient conditions imply that y^ν_j solves problem (33), and consequently $p^\nu \cdot y^\nu_j \geq 0$ for every $\nu \in \mathbb{N}$. Multiplying both sides by λ^ν_h, dividing by $\|(\lambda^\nu_h, \mu^\nu_h)\|$ and taking the limit, we get $\lambda_h p^* \cdot y^*_j \geq 0$ for every $j \in \mathcal{J}$.

Finally, we show that

$$\lambda_h p^* \cdot \widehat{x}_h(x^*_{-h}, y^*, e_h) \geq \lambda_h p^* \cdot x^*_h \tag{38}$$

which combined with (35) leads to a contradiction. Therefore, our claim is completely proved.

Since $\mu_h > 0$, there exists $n \in \mathbb{N}$ such that $\mu^\nu_h > 0$ for every $\nu \geq n$. From $\Gamma_e^{h.3}(\xi^\nu, \tau^\nu) = 0$, we get $\chi_h(\tau^\nu x^\nu_h + (1-\tau^\nu)\widehat{x}_h(x^\nu_{-h}, y^\nu, e_h), x^\nu_{-h}, y^\nu) = 0$ for every $\nu \geq n$. Taking the limit, one gets

$$\chi_h(\tau^* x_h + (1-\tau^*)\widehat{x}_h(x^*_{-h}, y^*, e_h), x^*_{-h}, y^*) = 0$$

Therefore, (34) and the Karush-Kuhn-Tucker sufficient conditions imply that x^*_h solves the following problem.

$$\begin{aligned} &\min_{x_h \in \mathbb{R}^C_{++}} \lambda_h p^* \cdot x_h \\ &\text{subject to } \chi_h(\tau^* x_h + (1-\tau^*)\widehat{x}_h(x^*_{-h}, y^*, e_h), x^*_{-h}, y^*) \geq 0 \end{aligned} \tag{39}$$

By Proposition 7, $\widehat{x}_h(x^*_{-h}, y^*, e_h)$ belongs to the constraint of this problem, and so (38) holds true.

Therefore, one concludes that the sequence $(\lambda_h^\nu, \mu_h^\nu)_{\nu\in\mathbb{N}}$ is bounded, and consequently it admits a subsequence converging to some $(\lambda^*, \mu^*) \in \mathbb{R}_+^H \times \mathbb{R}_+^H$.

Now we show that $\lambda^* \gg 0$. From $\Gamma_e^{h.1}(\xi^\nu, \tau^\nu) = 0$, taking the limit, we get

$$\lambda_h^* p^* = D_{x_h} u_h(x_h^*, x_{-h}^*, y^*) + \tau^* \mu_h^* D_{x_h} \chi_h(\tau^* x_h^* + (1-\tau^*)\widehat{x}_h(x_{-h}^*, y^*, e_h), x_{-h}^*, y^*)$$

Since $\mu_h^* \geq 0$, by Point 2 of Assumption 3 and Point 4 of Assumption 4, we have

$$\lambda_h^* p^{*\,c} = D_{x_h^c} u_h(x_h^*, x_{-h}^*, y^*) + \tau^* \mu_h^* D_{x_h^c} \chi_h(\tau^* x_h^* + (1-\tau^*)\widehat{x}_h(x_{-h}^*, y^*, e_h), x_{-h}^*, y^*) > 0$$

for some commodity c. Since $p^{*\,c} > 0$, $\lambda_h^* > 0$ which completes the proof of the step. ■

Appendix A

Proof of Proposition 7. For all $h \in \mathcal{H}$, the correspondence $\phi_h : \mathbb{R}_+^{C(H-1)} \times \mathbb{R}^{CJ} \times E_{\chi_h} \rightrightarrows \mathbb{R}_{++}^C$ defined by

$$\phi_h(x_{-h}, y, , e_h) := \{x_h \in \mathbb{R}_{++}^C : \chi_h(x_h, x_{-h}, y) > 0 \text{ and } x_h \ll e_h\}$$

is non-empty convex valued by (2) and by Point 2 of Assumption 4. From Point 1 of Assumption 4, for all $x_h \in \mathbb{R}_{++}^C$, the following set

$$\phi_h^{-1}(x_h) := \{(x_{-h}, y, e_h) \in \mathbb{R}_+^{C(H-1)} \times \mathbb{R}^{CJ} \times E_{\chi_h} : \chi_h(x_h, x_{-h}, y,) > 0 \text{ and } x_h \ll e_h\}$$

is open in $\mathbb{R}_+^{C(H-1)} \times \mathbb{R}^{CJ} \times E_{\chi_h}$. Moreover, $\mathbb{R}_+^{C(H-1)} \times \mathbb{R}^{CJ} \times E_{\chi_h}$ equipped with the metric induced by the Euclidean distance is metrizable, thus paracompact. Then, we have the desired result since the correspondence ϕ_h satisfies all the assumptions of Michael's Selection Theorem. [17] ■

Proof of Proposition 9. Since the externalities are fixed, the proof is standard as in the case without externalities. ■

Proof of Proposition 10. By (2), that is the Survival Assumption, there exists $\widehat{x}_h$ such that $\chi_h(\widehat{x}_h, x_{-h}^*, y^*) \geq 0$ and $\widehat{x}_h \ll e_h$. If $p^* \cdot \sum_{j\in\mathcal{J}} s_{jh} y_j^* \geq 0$, then $\widehat{x}_h$ belongs to the constraint set of problem (4). In order to apply Weierstrass's

[17] See Florenzano (2003), Proposition 1.5.1, page 29.

Theorem, one replaces problem (4) with the following problem.

$$\begin{aligned}
&\max_{x_h \in \mathbb{R}^C_{++}} \; u_h(x_h, x^*_{-h}, y^*) \\
&\text{subject to } \; \chi_h(x_h, x^*_{-h}, y^*) \geq 0 \\
&p^* \cdot x_h \leq p^* \cdot (e_h + \sum_{j \in \mathcal{J}} s_{jh} y^*_j) \\
&u_h(x_h, x^*_{-h}, y^*) \geq u_h(\widehat{x}_h, x^*_{-h}, y^*)
\end{aligned} \tag{40}$$

It is an easy matter to show that problems (4) and (40) are equivalent. Furthermore, by the continuity of the functions u_h and χ_h (see Point 1 of Assumptions 3 and 4) and by Point 4 of Assumption 3, the constraint set associated with problem (40) is a compact set included in $\mathbb{R}^C_{++}$. Since $u_h(\cdot, x^*_{-h}, y^*)$ is continuous, from Weierstrass's theorem, a solution of problem (40) exists, and it also a solution to problem (4).

The solution to problem (4) is unique, since the objective function is strictly quasi-concave (see Point 3 of Assumption 3), $\chi_h(\cdot, x^*_{-h}, y^*)$ is quasi-concave (see Point 2 of Assumption 4) and the budget set is convex. So, point (1) is completely proved.

Point (2) follows showing that problem (4) satisfies the Karush-Kuhn-Tucker necessary and sufficient conditions. We define $g^1(x_h) := -p^* \cdot (x_h - e_h - \sum_{j \in \mathcal{J}} s_{jh} y^*_j)$ and $g^2(x_h) := \chi_h(x_h, x^*_{-h}, y^*)$. Karush-Kuhn-Tucker necessary conditions are satisfied. Indeed, Slater's condition holds true since $g^1(\widehat{x}_h) > 0$, $g^2(\widehat{x}_h) > 0$, and g^1 and g^2 are pseudo-concave.[18] By Point 2 of Assumption 3 and Point 4(b) of Assumption 4, the Lagrange multiplier λ^*_h associated to the budget constraint is strictly positive. Karush-Kuhn-Tucker sufficient conditions are satisfied. Indeed, $u_h(\cdot, x^*_{-h}, y^*)$ is pseudo-concave, and g^1 and g^2 are quasi-concave.[19]

Finally, using Point 4 of Assumption 4 and Proposition 7, one shows the uniqueness of the Lagrange multipliers.[20] ■

Appendix B

Characterization of Pareto optimality without externalities

[18] Since g^1 is linear, it is pseudo-concave. From Points 2 and 4(a) of Assumption 4, g^2 is quasi-concave with gradient different from zero, then it is pseudo-concave.
[19] From Point 3 of Assumption 3, $u_h(\cdot, x^*_{-h}, y^*)$ is differentiable strictly quasi-concave, then it is pseudo-concave. Since g^1 is linear, it is quasi-concave. Finally, from Point 2 of Assumption 4, g^2 is quasi-concave.
[20] For additional details, we refer to the proof of Proposition 5 in del Mercato (2006).

Let $\overline{\mathcal{E}} := ((\mathbb{R}^C_{++}, \overline{u}_h)_{h\in\mathcal{H}}, (\bar{t}_j)_{j\in\mathcal{J}}, r)$ be the economy defined by (9) where $r := \sum_{h\in\mathcal{H}} e_h$. Define the following sets

$$U_r := \{(u_h)_{h\in\mathcal{H}} \in \prod_{h\in\mathcal{H}} \operatorname{Im} \overline{u}_h : \exists\, (x,y) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}, \bar{t}_j(y_j) \geq 0 \;\forall\, j \in \mathcal{J}, \\ \sum_{h\in\mathcal{H}} x_h - \sum_{j\in\mathcal{J}} y_j \leq r \text{ and } \overline{u}_h(x_h) \geq u_h \;\forall\, h \in \mathcal{H}\}$$

$$\widehat{U}_r := \{(u_h)_{h\neq 1} \in \prod_{h\neq 1} \operatorname{Im} \overline{u}_h : \exists\, u_1 \in \operatorname{Im} \overline{u}_1, (u_1, (u_h)_{h\neq 1}) \in U_r\}$$

By Point 2 of Assumption 1, the set $\bar{A}_{t,r}$ defined by (29) is non-empty. Thus, the sets U_r and $\widehat{U}_r$ are non-empty. Let $(u'_h)_{h\neq 1} \in \widehat{U}_r$. Consider the following optimization problem

$$\begin{array}{ll} \displaystyle\max_{(x,y)\in\mathbb{R}^{CH}_{++}\times\mathbb{R}^{CJ}} & \overline{u}_1(x_1) \\ \text{subject to} & \bar{t}_j(y_j) \geq 0 \text{ for every } j \in \mathcal{J} \\ & \overline{u}_h(x_h) \geq u'_h \text{ for every } h \neq 1 \\ & \displaystyle\sum_{h\in\mathcal{H}} x_h - \sum_{j\in\mathcal{J}} y_j \leq r \end{array} \tag{41}$$

Proposition 18 *There exists a unique solution $(\widetilde{x}, \widetilde{y})$ to problem (41).*

Proof. In order to apply Weierstrass' Theorem, one replaces problem (41) with the following problem

$$\begin{array}{ll} \displaystyle\max_{(x,y)\in\mathbb{R}^{CH}_{++}\times\mathbb{R}^{CJ}} & \overline{u}_1(x_1) \\ \text{subject to} & \bar{t}_j(y_j) \geq 0 \text{ for every } j \in \mathcal{J} \\ & \overline{u}_h(x_h) \geq u'_h \text{ for every } h \neq 1 \\ & \overline{u}_1(x_1) \geq u'_1 \\ & \displaystyle\sum_{h\in\mathcal{H}} x_h - \sum_{j\in\mathcal{J}} y_j \leq r \end{array} \tag{42}$$

where $u'_1 \in \operatorname{Im} \overline{u}_1$ is given by the definition of $\widehat{U}_r$. Since $(u'_h)_{h\in\mathcal{H}} \in U_r$, there exists $(x', y') \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$ such that $\bar{t}_j(y'_j) \geq 0$ for all $j \in \mathcal{J}$, $\sum_{h\in\mathcal{H}} x'_h - \sum_{j\in\mathcal{J}} y'_j \leq r$ and $\overline{u}_h(x'_h) = u'_h$, $\forall\, h \in \mathcal{H}$.

Denote K_1 the constraints set associated with problem (42). K_1 is non-empty since $(u'_h)_{h\in\mathcal{H}} \in U_r$. We first claim that K_1 is a compact set included in $\mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$. We notice that $K_1 = N \cap \bar{A}_{t,r}$ where

$$N := \{(x,y) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ} : \overline{u}_h(x_h) \geq u'_h \;\forall\, h \in \mathcal{H}\}$$

and $\bar{A}_{t,r}$ is defined by (29). So, from Lemma 16 we have that K_1 is bounded. Furthermore, K_1 is closed. Indeed, take a sequence $(x^\nu, y^\nu)_{\nu\in\mathbb{N}}$ in K_1 converging to some (x, y). Since $(x^\nu, y^\nu)_{\nu\in\mathbb{N}} \subseteq N$, (x, y) belongs to the set $\text{cl}_{\mathbb{R}^{CH}\times\mathbb{R}^{CJ}} N$ which is included in $\mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$ by Point 4 of Assumption 3. So, $(x, y) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$. Since the function $\overline{u}_h$ are continuous (see Point 1 of Assumption 3), $(x, y) \in N$. Since the functions $\overline{t}_j$ are continuous (see Point 1 of Assumption 1), $(x, y) \in \bar{A}_{t,r}$ and so $(x, y) \in K_1$ which completes the proof of the claim.

By Weierstrass' Theorem, there exists a solution $(\widetilde{x}, \widetilde{y})$ to problem (42). The solution to problem (42) is unique since the objective function is strictly quasi-concave (see Point 3 of Assumption 3) and the constraints set is convex (see Point 4 of Assumption 1 and Point 3 of Assumption 3).

We complete the proof showing that problems (41) and (42) are equivalent. Denote with K the constraints set associated with problem (41). Let $(\widetilde{x}, \widetilde{y})$ be a solution to problem (41), one gets $\overline{u}_1(\widetilde{x}_1) \geq \overline{u}_1(x'_1) = u'_1$ since $(x', y') \in K$. Thus, $(\widetilde{x}, \widetilde{y}) \in K_1$ and $(\widetilde{x}, \widetilde{y})$ solves problem (42) since since $K_1 \subseteq K$. Viceversa, suppose that $(\widetilde{x}, \widetilde{y})$ is a solution to problem (42), obviously $(\widetilde{x}, \widetilde{y}) \in K$. Let $(x, y) \in K$. If $\overline{u}_1(x_1) \geq u'_1$, then $(x, y) \in K_1$ and so $\overline{u}_1(\widetilde{x}_1) \geq \overline{u}_1(x_1)$. If $\overline{u}_1(x_1) < u'_1$, then $\overline{u}_1(\widetilde{x}_1) \geq u'_1 > \overline{u}_1(x_1)$. Thus, $(\widetilde{x}, \widetilde{y})$ solves problem (41), which complete the proof of the lemma. ∎

Proposition 19 *Let $(\widetilde{x}, \widetilde{y})$ be the allocation given by Lemma 18. There exists $(\widetilde{\theta}, \widetilde{\gamma}, \widetilde{\beta}) := ((\widetilde{\theta}_h)_{h\neq 1}, \widetilde{\gamma}, (\widetilde{\beta}_j)_{j\in\mathcal{J}}) \in \mathbb{R}^{H-1}_{++} \times \mathbb{R}^{C}_{++} \times \mathbb{R}^{J}_{++}$ such that $(\widetilde{x}, \widetilde{y}, \widetilde{\theta}, \widetilde{\gamma}, \widetilde{\beta})$ is the unique solution to the following system.*

$$\begin{cases} D_{x_1}\overline{u}_1(x_1) - \gamma = 0 \\ \theta_h D_{x_h}\overline{u}_h(x_h) - \gamma = 0, \ \forall\ h \neq 1 \\ \overline{u}_h(x_h) - \overline{u}_h(\widetilde{x}_h) = 0, \ \forall\ h \neq 1 \\ \gamma + \beta_j D_{y_j}\overline{t}_j(y_j) = 0, \forall\ j \in \mathcal{J} \\ \overline{t}_j(y_j) = 0, \forall\ j \in \mathcal{J} \\ r - \sum_{h\in\mathcal{H}} x_h + \sum_{j\in\mathcal{J}} y_j = 0 \end{cases} \tag{43}$$

Proof. The result follows showing that Karush-Kuhn-Tuker' conditions are necessary conditions to solve problem (41). The Lagrangean function associated with problem (41) is given by

$$\mathcal{L}(x, y, \theta, \gamma, \beta) = \overline{u}_1(x_1) + \sum_{h\neq 1} \theta_h(\overline{u}_h(x_h) - u'_h) + \sum_{j\in\mathcal{J}} \beta_j \overline{t}_j(y_j) + \gamma(r - \sum_{h\in\mathcal{H}} x_h + \sum_{j\in\mathcal{J}} y_j)$$

where $(\theta, \gamma, \beta) := ((\theta_h)_{h\neq 1}, \gamma, (\beta_j)_{j\in\mathcal{J}}) \in \mathbb{R}^{H-1}_{+} \times \mathbb{R}^{C}_{+} \times \mathbb{R}^{J}_{+}$ is the vector of the Lagrange multipliers associated with the constraints set of problem (41). So, the Karush-Kuhn-Tuker conditions are given by

$$\begin{cases} (1)\ D_{x_1}\overline{u}_1(x_1) - \gamma = 0 \\ (2)\ \theta_h D_{x_h}\overline{u}_h(x_h) - \gamma = 0,\ \forall\ h \neq 1 \\ (3)\ \gamma + \beta_j D_{y_j}\overline{t}_j(y_j) = 0, \forall\ j \in \mathcal{J} \\ (4)\ \min\{\theta_h, \overline{u}_h(x_h) - u'_h\} = 0,\ \forall\ h \neq 1 \\ (5)\ \min\{\beta_j, \overline{t}_j(y_j)\} = 0, \forall\ j \in \mathcal{J} \\ (6)\ \min\{\gamma, r - \sum_{h\in\mathcal{H}} x_h + \sum_{j\in\mathcal{J}} y_j\} = 0 \end{cases} \tag{44}$$

It is enough to show that the Jacobian matrix associated with the constraints functions of problem (41) has full row rank. The Jacobian matrix is described below.

$$\begin{array}{c} \\ \overline{u}_2(x_2) - u'_2 \\ \vdots \\ \overline{u}_H(x_H) - u'_H \\ r - \sum_{h\in\mathcal{H}} x_h + \sum_{j\in\mathcal{J}} y_j \\ \overline{t}_1(y_1) \\ \vdots \\ \overline{t}_J(y_J) \end{array} \begin{array}{c} \begin{array}{ccccccc} x_2 & \dots & x_I & x_1 & y_1 & \dots & y_J \end{array} \\ \begin{pmatrix} D_{x_2}\overline{u}_2(x_2) & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & D_{x_H}\overline{u}_H(x_H) & 0 & 0 & \dots & 0 \\ -I_C & \dots & -I_C & -I_C & I_C & \dots & I_C \\ 0 & \dots & 0 & 0 & D_{y_1}\overline{t}_1(y_1) & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & D_{y_J}\overline{t}_J(y_J) \end{pmatrix} \end{array}$$

The matrix above has full row rank since $D_{x_h}\overline{u}_h(x_h) \gg 0$ and $D_{y_j}\overline{t}_j(y_j) \ll 0$ (see Point 3 of Assumption 1 and Point 2 of Assumption 3) imply that the determinant of the square sub-matrix D defined below is different from zero.

$$
\begin{array}{cccccccc}
 & x_2^1 & \dots & x_H^1 & x_1 & y_J^1 & \dots & y_J^1 \\
D := & \multicolumn{7}{c}{\left(\begin{array}{ccccccc}
D_{x_2^1}\overline{u}_2(x_2) & \dots & 0 & 0 & 0 & \dots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \dots & D_{x_H^1}\overline{u}_H(x_H) & 0 & 0 & \dots & 0 \\
\begin{smallmatrix}-1\\0\end{smallmatrix} & \dots & \begin{smallmatrix}-1\\0\end{smallmatrix} & I_C & \begin{smallmatrix}-1\\0\end{smallmatrix} & \dots & \begin{smallmatrix}-1\\0\end{smallmatrix} \\
0 & \dots & 0 & 0 & D_{y_1^1}\overline{t}_1(y_1) & \dots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \dots & 0 & 0 & 0 & \dots & D_{y_J^1}\overline{t}_J(y_J)
\end{array}\right)}
\end{array}
$$

Therefore, the conditions given by (44) are necessary to solve problem (41). By Lemma 18 and equations (1), (2) and (3) in system (44), the Lagrange multipliers are unique. Furthermore, Point 3 of Assumption 1 and Point 2 of Assumption 3 imply that the Lagrange multipliers satisfying system (44) are strictly positive, and consequently, all the constraints in problem (41) are binding. So, in particular one gets

$$\overline{u}_h(\widetilde{x}_h) = u'_h \ \forall\ h \neq 1$$

Therefore from system (44) one deduces system (43) and the lemma is completely proved. ■ Using Proposition 19, one easily proves the following proposition.

Proposition 20 *If $(\widetilde{x}, \widetilde{y})$ is a solution of problem (41), then $(\widetilde{x}, \widetilde{y})$ solves the problem below*

$$
\begin{array}{rl}
\max\limits_{(x,y)\in\mathbb{R}^{CH}_{++}\times\mathbb{R}^{CJ}} & \overline{u}_1(x_1) \\
\textit{subject to} & \overline{t}_j(y_j) \geq 0 \ \textit{for each}\ j \in \mathcal{J} \\
 & \overline{u}_h(x_h) \geq \overline{u}_h(\widetilde{x}_h) \ \textit{for}\ h \neq 1 \\
 & \sum\limits_{h\in\mathcal{H}} x_h - \sum\limits_{j\in\mathcal{J}} y_j \leq r
\end{array}
\tag{45}
$$

Proof. It follows from system (43) in Lemma 19 and the fact that Karush-Kuhn-Tucker conditions are sufficient to solve problem (45). Indeed, by Points 2 and 3 of Assumption 3 the function $\overline{u}_1$ is quasi-concave with gradient different from zero, and by Point 4 of Assumption 1 and Point 3 of Assumption 3, the constraint functions associated with problem (45) are quasi-concave. ■

We remind that $(\widetilde{x},\widetilde{y}) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$ is Pareto optimal allocation of the production economy $\overline{\mathcal{E}}$ if there is no other allocation $(\widehat{x},\widehat{y}) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$ such that

(1) $\overline{t}_j(\widehat{y}_j) \geq 0$ for all $j \in \mathcal{J}$ and $\sum\limits_{h\in\mathcal{H}} \widehat{x}_h \leq r + \sum\limits_{j\in\mathcal{J}} \widehat{y}_j$
(2) $\overline{u}_h(\widehat{x}_h) \geq \overline{u}_h(\widetilde{x}_h)$ for all $h \in \mathcal{H}$ and $\overline{u}_k(\widehat{x}_k) > \overline{u}_k(\widetilde{x}_k)$ for some $k \in \mathcal{H}$.

Proposition 21 $(\widetilde{x},\widetilde{y})$ *solves problem (45) if and only if it is a Pareto optimal allocation of* $\overline{\mathcal{E}}$.

Proof. By definition of Pareto optimal allocation, if $(\widetilde{x},\widetilde{y})$ is a Pareto optimal allocation then $(\widetilde{x},\widetilde{y})$ solves problem (45). Suppose now that $(\widetilde{x},\widetilde{y})$ solves problem (45), we prove that $(\widetilde{x},\widetilde{y})$ is a Pareto optimal allocation. By contradiction, suppose that there is an allocation $(\widehat{x},\widehat{y}) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$ such that $\overline{t}_j(\widehat{y}_j) \geq 0$ for all $j \in \mathcal{J}$, $\sum\limits_{h\in\mathcal{H}} \widehat{x}_h \leq r + \sum\limits_{j\in\mathcal{J}} \widehat{y}_j$, $\overline{u}_h(\widehat{x}_h) \geq \overline{u}_h(\widetilde{x}_h)$ for all $h \in \mathcal{H}$ and $\overline{u}_k(\widehat{x}_k) > \overline{u}_k(\widetilde{x}_k)$ for some $k \in \mathcal{H}$. If $k = 1$, then we get a contradiction since $(\widetilde{x},\widetilde{y})$ solves problem (45). If $k \neq 1$, by the continuity of $\overline{u}_k$ (see Point 1 of Assumption 3), there exists $\varepsilon > 0$ such that $\overline{u}_k(\widehat{x}_k - \varepsilon\mathbf{1}^c) > \overline{u}_k(\widetilde{x}_k)$ where $\mathbf{1}^c \in \mathbb{R}^C_+$ has all the components equal to 0 except the component c which is equal to 1. Thus, the allocation $(x,y) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$ defined below

$$\begin{aligned} & x_1 := \widehat{x}_1 + \varepsilon\mathbf{1}^c \\ & x_k := \widehat{x}_k - \varepsilon\mathbf{1}^c \\ & x_h := \widehat{x}_h \ \forall\, h \in \mathcal{H}\setminus\{1,k\} \\ & y_j := \widehat{y}_j \ \forall\, j \in \mathcal{J} \end{aligned} \tag{46}$$

satisfies the constraints of problem (45) and $\overline{u}_1(x_1) > \overline{u}_1(\widetilde{x}_1)$ since $\overline{u}_1$ is strictly increasing (see Point 2 of Assumption 3). So, once again we get a contradiction since $(\widetilde{x},\widetilde{y})$ solves problem (45). ∎

References

Bonnisseau, J.-M., del Mercato, E. L., 2010. Externalities, consumption constraints and regular economies. *Economic Theory* 44, 123-147.
Bonnisseau, J.-M., Médecin, J.-P., 2001. Existence of marginal pricing equilibria in economies with externalities and non-convexities, *Journal of Mathematical Economics* 36, 271–294.
Debreu, G., 1959. Theory of Value. Yale University Press.
del Mercato, E. L., 2006. Existence of competitive equilibria with externalities: a differential viewpoint, *Journal of Mathematical Economics* 42, 525–543.
del Mercato, E. L., Platino, V., 2010. Externalities in production economies

and existence of competitive equilibria: A differentiable approach, Working Paper.

Florenzano, M., 2003. General Equilibrium Analysis, Existence and Optimality Properties of Equilibria (Kluwer Academic Publishers).

Mandel, A., 2008. An index formula for production economies with externalities, *Journal of Mathematical Economics* 44, 1385–1397.

Mas-Colell, A., 1985. The theory of general economic equilibrium. A differentiable approach. Cambridge University Press.

Mas-Colell, A., Whinston, M. D., Green, J. R., 1995. Microeconomic Theory. Oxford University Press.

Milnor, J. W., 1965. Topology from the Differentiable Viewpoint. The University Press of Virginia.

Smale, S., 1974. Global analysis and economics. II A: Extension of a theorem of Debreu, *Journal of Mathematical Economics* 1, 1–14.

Villanacci, A., Carosi, L., Benevieri, P., Battinelli, A., 2002. Differential Topology and General Equilibrium with Complete and Incomplete Markets. Kluwer Academic Publishers.

Chapter 2

Externalities in production economies: Regularity results [1]

Vincenzo Platino [2]

DISES, Universitá degli Studi di Salerno, Italy

Key words: externalities, production economies, competitive equilibrium, regular economies.

1 Long abstract

We consider a general model of a private ownership economy with consumption and production externalities. Each firm is characterized by a technology described by an inequality on a differentiable function called the *transformation function*. Each household is characterized by a consumption set, preferences and an initial endowment of commodities. In this chapter, we assume that all the consumption sets coincide with the positive orthant of the commodity space. Individual preferences are represented by a utility function. Firms are owned by households. Utility and transformation functions depend on the consumption of all households and on the production activities of all firms.

As in Chapter 1, we follow Smale's extended approach. Our purpose is to provide sufficient conditions for the regularity of such economies. Showing by an example that basic assumptions are not enough to guarantee a regularity result in the space of initial endowments, we provide sufficient conditions for the regularity in the space of endowments and transformation functions.

Chapter 2 is organized as follows. Section 2 is devoted to the model and basic assumptions. In Section 3, we briefly resume the definitions of competitive

[1] This chapter is based on del Mercato and Platino (2011).

[2] Dipartimento di Scienze Economiche e Statistiche (DISES), Universitá degli Studi di Salerno, Via Ponte don Melillo, 84084 Fisciano (SA), Italy. E-mail: vincenzo.platino@gmail.com.

equilibria and of the equilibrium function. In Section 4, we state the definition of a regular economy and we recall its properties. In Section 5, we provide our example and the notion of a *perturbed economy.* In Section 6, we introduce two additional assumptions. In Section 7, we present our main results, namely Theorem 14 which states the regularity result for *for almost all perturbed economies.* In Section 8, we provide an important consequence of Theorem 14, namely Corollary 19 which states the regularity result in the space of endowments and transformation functions.

In Appendix A, one finds classical results from differential topology used in our analysis. In Appendix B, the reader can find a comparison of our results with those of Mandel (2008).

2 The model and the assumptions

There is a finite number C of physical commodities or goods labeled by the superscript $c \in \mathcal{C} := \{1, \ldots, C\}$. The commodity space is $\mathbb{R}^C$. There are a finite number J of firms labeled by the subscript $j \in \mathcal{J} := \{1, \ldots, J\}$ and a finite number H of households or consumers labeled by the subscript $h \in \mathcal{H} := \{1, \ldots, H\}$. Each firm j is owned by the households and it is characterized by a technology described by a transformation function. Each household h is characterized by preferences described by a utility function, shares on the firms profits and an endowment of commodities. In Chapter 2, we assume that all consumption sets coincide with the positive orthant of the commodity space. Utility and transformation functions are affected by the consumption choices of all households and by the production activities of all firms. The notations are summarized below.

- $y_j := (y_j^1, .., y_j^c, .., y_j^C)$ is the production plan of firm j. As usual, the output components are positive and the input components are negative, $y_{-j} := (y_z)_{z \neq j}$ denotes the production plan of firms other than j and $y := (y_j)_{j \in \mathcal{J}}$ denotes the production of all the firms.
- x_h^c is the consumption of commodity c by household h,
  $x_h := (x_h^1, .., x_h^c, .., x_h^C)$ denotes household h's consumption, $x_{-h} := (x_k)_{k \neq h}$ denotes the consumption of households other than h and $x := (x_h)_{h \in \mathcal{H}}$ denotes the consumption of all the households.
- For each $j \in \mathcal{J}$, the technology of firm j is described by an inequality on a function t_j called the *transformation function.* An innovation of this chapter comes from the dependency of the production set with respect to the production activities of other firms and the consumption of households. That is, given y_{-j} and x, the production set of the firm j is described by

the following set,

$$Y_j(y_{-j}, x) := \left\{ y_j \in \mathbb{R}^C \ : \ t_j(y_j, y_{-j}, x) \geq 0 \right\}$$

where the transformation function t_j is a function from $\mathbb{R}^C \times \mathbb{R}^{C(J-1)} \times \mathbb{R}^{CH}_{++}$ to $\mathbb{R}$, $t := (t_j)_{j \in \mathcal{J}}$. So, t_j describes the way firm j's technology is affected by the actions of the other agents.

- Each household $h \in \mathcal{H}$ has preferences described by a utility function,

$$u_h : (x_h, x_{-h}, y) \in \mathbb{R}^C_{++} \times \mathbb{R}^{C(H-1)}_{+} \times \mathbb{R}^{CJ} \longrightarrow u_h(x_h, x_{-h}, y) \in \mathbb{R}$$

  $u_h(x_h, x_{-h}, y)$ is the utility level of household h associated with (x_h, x_{-h}, y), $u := (u_h)_{h \in \mathcal{H}}$. So, u_h describes the way household h's preferences are affected by the actions of the other agents.

- $s_{jh} \in [0, 1]$ is the share of firm j owned by household h; $s_h := (s_{jh})_{j \in \mathcal{J}} \in [0, 1]^J$ denotes the vector of the shares of all firms owed by household h; $s := (s_h)_{h \in \mathcal{H}} \in [0, 1]^{JH}$. The set of all shares is given by

$$S := \{ s \in [0, 1]^{JH} \ : \ \forall \, j \in \mathcal{J}, \ \sum_{h \in \mathcal{H}} s_{jh} = 1 \}$$

- e^c_h is the endowment of commodity c owned by household h; $e_h := (e^1_h, .., e^c_h, .., e^C_h)$ denotes household h's endowment; $e := (e_h)_{h \in \mathcal{H}}$.
- $\mathcal{E} := ((u_h, e_h, s_h)_{h \in \mathcal{H}}, (t_j)_{j \in \mathcal{J}})$ is an *economy*.
- p^c is the price of one unit of commodity c, prices are expressed in units of account, $p := (p^1, .., p^c, .., p^C) \in \mathbb{R}^C_{++}$.
- Given $w = (w^1, .., w^c, .., w^C) \in \mathbb{R}^C$, we denote

$$w^{\setminus} := (w^1, .., w^c, .., w^{C-1}) \in \mathbb{R}^{C-1}$$

We make the following assumptions on the transformation functions $t = (t_j)_{j \in \mathcal{J}}$.

Assumption 1 *For all $j \in \mathcal{J}$,*

(1) The function t_j is a C^2 function.
(2) For each $(y_{-j}, x) \in \mathbb{R}^{C(J-1)} \times \mathbb{R}^{CH}_{++}$, $t_j(0, y_{-j}, x) \geq 0$.
(3) For each $(y_{-j}, x) \in \mathbb{R}^{C(J-1)} \times \mathbb{R}^{CH}_{++}$, the function $t_j(\cdot, y_{-j}, x)$ is differentiably strictly decreasing, i.e.

$$\forall \, (y_{-j}, x) \in \mathbb{R}^{C(J-1)} \times \mathbb{R}^{CH}_{++} \text{ and } \forall \, y'_j \in \mathbb{R}^C, D_{y_j} t_j(y'_j, y_{-j}, x) \ll 0$$

(4) For each $(y_{-j}, x) \in \mathbb{R}^{C(J-1)} \times \mathbb{R}^{CH}_{++}$, the function $t_j(\cdot, y_{-j}, x)$ is C^2 and it is differentiably strictly quasi-concave, i.e. for every $y_j \in \mathbb{R}^C$, $D^2_{y_j} t_j(y_j, y_{-j}, x)$

is negative definite on $\ker D_{y_j} t_j(y_j, y_{-j}, x)$. [3]

We remark that, given the externalities, the assumptions on t_j are standard in "smooth" general equilibrium models. Indeed, from Point 1 of Assumption 1 the production set is closed and smooth, from Point 4 of Assumption 1 it is convex. Point 2 of Assumption 1 states that inactivity is possible. Point 3 of Assumption 1 represents the "free disposal" property.

Define the set Y_t of all production plans which are on the production sets whatever are the externalities, that is

$$Y_t := \left\{ y' \in \mathbb{R}^{CJ} | \; \exists \; (y, x) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ} \; : \; t_j(y'_j, y_{-j}, x) \geq 0, \; \forall \, j \in \mathcal{J} \right\} \quad (1)$$

The following assumption can be interpreted in a similar way as Assumption 2 in Chapter 1, that is the asymptotic irreversibility and "no free lunch" assumptions at the aggregate level for any possible displacement of the boundary of the production sets.

Assumption 2 *Let* $b := (b_j)_{j \in \mathcal{J}} \in \mathbb{R}^J$ *and* $t + b := (t_j + b_j)_{j \in \mathcal{J}}$. *For all* $b \geq 0$, *if* $y' \in CY_{t+b}$ *and* $\sum_{j \in \mathcal{J}} y'_j \geq 0$, *then* $y'_j = 0$ *for every* $j \in \mathcal{J}$. [4]

The assumption above ensures that the for all possible displacements of the boundary of the production sets, the set of feasible allocation is bounded, see Lemma 15.

We make the following assumptions on the utilities functions $u = (u_h)_{h \in \mathcal{H}}$.

Assumption 3 *For all* $h \in \mathcal{H}$,

(1) The function u_h *is continuous in its domain and it is* C^2 *in the interior of its domain.*

(2) For each $(x_{-h}, y) \in \mathbb{R}^{C(H-1)}_{++} \times \mathbb{R}^{CJ}$, *the function* $u_h(\cdot, x_{-h}, y)$ *is differentiably strictly increasing, i.e.*

$$\forall \; (x_{-h}, y) \in \mathbb{R}^{C(H-1)}_{++} \times \mathbb{R}^{CJ} \; \text{and} \; \forall \; x'_h \in \mathbb{R}^C_{++}, D_{x_h} u_h(x'_h, x_{-h}, y) \gg 0$$

(3) For each $(x_{-h}, y) \in \mathbb{R}^{C(H-1)}_{++} \times \mathbb{R}^{CJ}$, *the function* $u_h(\cdot, x_{-h}, y)$ *is* C^2 *and it is differentiably strictly quasi-concave, i.e., for every* $x_h \in \mathbb{R}^C_{++}$, $D^2_{x_h} u_h(x_h, x_{-h}, y)$

[3] Let v and v' be two vectors in $\mathbb{R}^n$, $v \cdot v'$ denotes the *inner product* of v and v'. Let A be a real matrix with m rows and n columns, and B be a real matrix with n rows and l columns, AB denotes the *matrix product* of A and B. Without loss of generality, vectors are treated as row matrices and A denotes both the matrix and the following linear application $A : v \in \mathbb{R}^n \to A(v) := Av^T \in \mathbb{R}^{[m]}$ where v^T denotes the transpose of v and $\mathbb{R}^{[m]} := \{w^T : w \in \mathbb{R}^m\}$. When $m = 1$, $A(v)$ coincides with the inner product $A \cdot v$, treating A and v as vectors in $\mathbb{R}^n$.

[4] CY_{t+b} denotes the asymptotic cone of Y_{t+b}.

is negative definite on $\ker D_{x_h} u_h(x_h, x_{-h})$.

(4) For each $(x_{-h}, y) \in \mathbb{R}_+^{C(H-1)} \times \mathbb{R}^{CJ}$ *and for each* $u \in \operatorname{Im} u_h(\cdot, x_{-h}, y)$,

$$\mathrm{cl}_{\mathbb{R}^C}\{x_h \in \mathbb{R}_{++}^C : u_h(x_h, x_{-h}, y) \geq u\} \subseteq \mathbb{R}_{++}^C$$

Fixing the externalities, the assumptions on u_h are standard in "smooth" general equilibrium models.

$\mathcal{T}$ denotes the set of $t = (t_j)_{j \in \mathcal{J}}$ satisfying Assumption 1 and Assumption 2, and $\mathcal{U}$ denotes the set of $u = (u_h)_{h \in \mathcal{H}}$ satisfying Assumption 3.

Remark 4 *From now on,* $u \in \mathcal{U}$ *and* $s \in S$ *are kept fixed and an economy is parameterized by transformation functions and initial endowments* (t, e) *taken in the following set* $\mathcal{T} \times \mathbb{R}_{++}^{CH}$.

3 Competitive equilibrium with externalities

This section summarizes the notions and the main result of Chapter 1.

Without loss of generality, commodity C is the *numeraire good*. So, given $p^{\setminus} \in \mathbb{R}_{++}^{C-1}$ with innocuous abuse of notation, we denote $p := (p^{\setminus}, 1) \in \mathbb{R}_{++}^C$.

Definition 5 $(x^*, y^*, p^{*\setminus}) \in \mathbb{R}_{++}^{CH} \times \mathbb{R}^{CJ} \times \mathbb{R}_{++}^{C-1}$ *is a competitive equilibrium for the economy* $(t, e) \in \mathcal{T} \times \mathbb{R}_{++}^{CH}$ *if*

(1) for all $j \in \mathcal{J}$, y_j^* *solves the following problem*

$$\begin{aligned} &\max_{y_j \in \mathbb{R}^C} \; p^* \cdot y_j \\ &\text{subject to } \; t_j(y_j, y_{-j}^*, x^*) \geq 0 \end{aligned} \tag{2}$$

(2) For all $h \in \mathcal{H}$, x_h^* *solves the following problem*

$$\begin{aligned} &\max_{x_h \in \mathbb{R}_{++}^C} \; u_h(x_h, x_{-h}^*, y^*) \\ &\text{subject to } \; p^* \cdot x_h \leq p^* \cdot (e_h + \sum_{j \in \mathcal{J}} s_{jh} y_j^*) \end{aligned} \tag{3}$$

(3) $(x^*, y^*) \in \mathbb{R}_{++}^{CH} \times \in \mathbb{R}^{CJ}$ *satisfies market clearing conditions, that is*

$$\sum_{h \in \mathcal{H}} x_h = \sum_{h \in \mathcal{H}} e_h + \sum_{j \in \mathcal{J}} y_j \tag{4}$$

In the following propositions, using the Karush–Kuhn–Tucker necessary and sufficient conditions, we characterize the solutions of firms and households maximization problems.

Proposition 6 *Given* $y_{-j}^* \in \mathbb{R}^{C(J-1)}$, $x^* \in \mathbb{R}_{++}^{CH}$ *and* $p^{*\backslash} \in \mathbb{R}_{++}^{C-1}$,

(1) if y_j^* *is a solution to problem (5), then it is the unique solution.*

(2) $y_j^* \in \mathbb{R}^C$ *is the solution to problem (5) if and only if there exists* $\alpha_j^* \in \mathbb{R}_{++}$ *such that* (y_j^*, α_j^*) *is the unique solution to the following system*

$$\begin{cases} p^* + \alpha_j D_{y_j} t_j(y_j, y_{-j}^*, x^*) = 0 \\ t_j(y_j, y_{-j}^*, x^*) = 0 \end{cases} \tag{5}$$

Proposition 7 *Given* $x_{-h}^* \in \mathbb{R}_{++}^{C(H-1)}$, $y^* \in \mathbb{R}^{CJ}$ *and* $p^{*\backslash} \in \mathbb{R}_{++}^{C-1}$,

(1) if $p^* \cdot \sum_{j \in \mathcal{J}} s_{jh} y_j^* \geq 0$, *then there exists a unique solution to problem (5).*

(2) $x_h^* \in \mathbb{R}_{++}^C$ *is the solution to problem (5) if and only if there exists* $\lambda_h^* \in \mathbb{R}_{++}$ *such that* (x_h^*, λ_h^*) *is the unique solution to the following system*

$$\begin{cases} D_{x_h} u_h(x_h, x_{-h}^*, y^*) - \lambda_h p^* = 0 \\ -p^* \cdot (x_h - e_h - \sum_{j \in \mathcal{J}} s_{jh} y_j^*) = 0 \end{cases} \tag{6}$$

Let $\Xi := (\mathbb{R}_{++}^C \times \mathbb{R}_{++})^H \times (\mathbb{R}^C \times \mathbb{R}_{++})^J \times \mathbb{R}_{++}^{C-1}$ be the set of endogenous variables with generic element $\xi := (x, \lambda, y, \alpha, p^\backslash) := ((x_h, \lambda_h)_{h \in \mathcal{H}}, (y_j, \alpha_j)_{j \in \mathcal{J}}, p^\backslash)$.

We can now describe equilibria using the propositions above and the market clearing conditions (4). One should notice that, due to the Walras law and the second equation in (6), the market clearing condition for commodity C is "redundant".

For a given economy $(t, e) \in \mathcal{T} \times \mathbb{R}_{++}^{CH}$, the *equilibrium function* $F_{t,e} : \Xi \to \mathbb{R}^{\dim \Xi}$,

$$F_{t,e}(\xi) := ((F_{t,e}^{h.1}(\xi), F_{t,e}^{h.2}(\xi))_{h \in \mathcal{H}} (F_{t,e}^{j.1}(\xi), F_{t,e}^{j.2}(\xi))_{j \in \mathcal{J}}, F_{t,e}^M(\xi)) \tag{7}$$

is defined by $F_{t,e}^{h.1}(\xi) := D_{x_h} u_h(x_h, x_{-h}, y) - \lambda_h p$, $F_{t,e}^{h.2}(\xi) := -p \cdot (x_h - e_h - \sum_{j \in \mathcal{J}} s_{jh} y_j)$, $F_{t,e}^{j.1}(\xi) := p + \alpha_j D_{y_j} t_j(y_j, y_{-j}, x)$, $F_{t,e}^{j.2}(\xi) := t_j(y_j, y_{-j}, x)$, and $F_{t,e}^M(\xi) := \sum_{h \in \mathcal{H}} x_h^\backslash - \sum_{j \in \mathcal{J}} y_j^\backslash - \sum_{h \in \mathcal{H}} e_h^\backslash$.

$\xi^* \in \Xi$ is an extended equilibrium for the economy $(t, e) \in \mathcal{T} \times \mathbb{R}_{++}^{CH}$ if and only if $F_{t,e}(\xi^*) = 0$. We call ξ^* simply an equilibrium.

Theorem 8 (Existence and compactness) *For every economy* $(t, e) \in \mathcal{T} \times \mathbb{R}_{++}^{CH}$, *the equilibrium set* $F_{t,e}^{-1}(0)$ *is non-empty and compact.*

4 Regular economy and its properties

In this section, first we recall the notion of a regular economy. Second, we provide the main properties of a regular economy, see Proposition 10 below.

Definition 9 *$(t,e) \in \mathcal{T} \times \mathbb{R}^{CH}_{++}$ is a regular economy if for each $\xi^* \in F^{-1}_{t,e}(0)$,*

(1) $F_{t,e}$ is a C^1 function around ξ^*. [5]
(2) The differential mapping $D_\xi F_{t,e}(\xi^*)$ is onto.

$\mathcal{R}$ denotes the set of regular economies.

Our main result is Theorem 14 in Section 7 which states the regularity result for almost all perturbed economies.

Now, define $B := \mathbb{R}^{CJ} \times \mathbb{R}^{CH}_{++}$, and endow the set $C^2(B,\mathbb{R})$ with the C^2 Whitney topology (see Definition 21 in Appendix A), the set $\mathbb{R}^{CH}_{++}$ with the Euclidean topology, and the set $\mathcal{T} \times \mathbb{R}^{CH}_{++}$ with the topology induced by the product topology on $C^2(B,\mathbb{R})^J \times \mathbb{R}^{CH}_{++}$.

As a consequence of Theorem 14, the set $\mathcal{R}$ is a non-empty open subset of $\mathcal{T} \times \mathbb{R}^{CH}_{++}$, see Corollary 19 in Section 8. So, one easily deduces the following proposition from Theorem 8, Corollary 19, Lemma 20 in Section 8, a consequence of the Regular Value Theorem and the Implicit Function Theorem (see Corollary 23 and Theorem 26 in Appendix A).

Proposition 10 (Properties of regular economies) *For each $(t,e) \in \mathcal{R}$,*

(1) the equilibrium set associated with the economy (t,e) is a non-empty finite set, i.e.,

$$\exists\, r \in \mathbb{N} \setminus \{0\} : \; F^{-1}_{t,e}(0) = \{\xi^1, ..., \xi^r\}$$

(2) there exist an open neighborhood I of (t,e) in $\mathcal{T} \times \mathbb{R}^{CH}_{++}$, and for each $i = 1, \ldots, r$ an open neighborhood U_i of ξ^i in Ξ and a continuous function $g_i : I \to U_i$ such that

(a) $U_j \cap U_k = \emptyset$ if $j \neq k$,
(b) $g_i(t,e) = \xi^i$,
(c) for all $(t',e') \in I$, $F^{-1}_{t',e'}(0) = \{g_i(t',e') : \; i = 1, \ldots, r\}$,
(d) the economies $(t',e') \in I$ are regular.

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[5] $F_{t,e}$ is a C^1 function around ξ^* means that there exists an open neighborhood $I(\xi^*)$ of ξ^* in Ξ such that the restriction of $F_{t,e}$ to $I(\xi^*)$ is a C^1 function.

5 An example and perturbations of production sets

In this section, first we provide an example of a production economy with externalities and an infinite set of equilibria for all initial endowments. Second, in order to avoid situations such as the one shown by the example, we consider displacements of the boundaries of the production sets, that is, *simple perturbations* of the transformation functions.

Consider one household, two firms and two commodities. Let $x = (x^1, x^2)$ be the consumption of the household and $y_j = (y_j^1, y_j^2)$ be the production plan of firm $j = 1, 2$. We denote with $e = (e^1, e^2)$ the initial endowment. The utility function is given by

$$u(x^1, x^1) = \frac{1}{2}\ln x^1 + \frac{1}{2}\ln x^2$$

In this example, each firm uses commodity 2 to produce commodity 1. Moreover, the production set of firm j is affected by the output of the other firm. The production set of firm j is the following set.

$$Y_j(y_{-j}^1) = \{(y_j^1, y_j^2) \in \mathbb{R}^2 : y_j^2 \leq 0 \text{ and } t_j(y_j^1, y_j^2, y_{-j}^1) := 2\sqrt{-y_j^2} - y_{-j}^1 - y_j^1 \geq 0\}$$

We remark that all the basic assumptions to get the existence of equilibria are satisfied except Point 2 of Assumption 1. Although Point 2 of Assumption 1 is not satisfied, the existence result holds true since, at equilibrium, the aggregate profit is non-negative. See the aggregate profit given by condition (10) below, and related comments at page 7 of Chapter 1.

Normalize the price of commodity 2. At equilibrium, firm j solves the following maximization problem

$$\max_{y_j^1>0,\, y_j^2<0} \quad p^* y_j^1 + y_j^2$$
$$\text{subject to } \; 2\sqrt{-y_j^2} - y_{-j}^{*1} - y_j^1 \geq 0$$

For each firm $j = 1, 2$, the associated Karush-Kuhn-Thucker conditions are given by

$$p^* = \alpha_j, \;\; 1 = \alpha_j \frac{1}{\sqrt{-y_j^2}}, \;\; 2\sqrt{-y_j^2} - y_{-j}^{*1} - y_j^1 = 0$$

Thus, at equilibrium, one gets

$$y_1^{*1} = 2p^* - y_2^{*1} \text{ and } y_1^{*2} = -(p^*)^2 \tag{8}$$

and

$$y_2^{*1} = 2p^* - y_1^{*1} \text{ and } y_2^{*2} = -(p^*)^2 \tag{9}$$

By (8), at equilibrium, the aggregate profit is given by

$$\sum_{j=1}^{2}(p^* y_j^{*1} + y_j^{*2}) = p^*(2p^* - y_2^{*1}) - (p^*)^2 + p^* y_2^{*1} - (p^*)^2 = 0 \tag{10}$$

So, household's maximization problem is given by

$$\max_{x \in \mathbb{R}^2_{++}} \ \tfrac{1}{2}\ln x^1 + \tfrac{1}{2}\ln x^2$$
$$\text{subject to } \ p^* x^1 + x^2 \leq p^* e^1 + e^2$$

The associated Karush-Kuhn-Thucker conditions are given by

$$\frac{1}{2x^1} = \lambda p^*, \ \ \frac{1}{2x^2} = \lambda, \ \ p^* x^1 + x^2 = p^* e^1 + e^2$$

Thus, at equilibrium, one gets

$$x^{*1} = \frac{1}{2p^*}(p^* e^1 + e^2) \text{ and } x^{*2} = \frac{1}{2}(p^* e^1 + e^2) \tag{11}$$

Using market clearing condition for commodity 1, one finds the equilibrium price

$$p^* = \frac{1}{8}\left(\sqrt{(e^1)^2 + 16e^2} - e^1\right) \tag{12}$$

Finally, using (8), (9), (11) and (12), any bundle

$$((p^*, 1), x^*, y_1^*, y_2^*) \in \mathbb{R}^2_{++} \times \mathbb{R}^2_{++} \times \mathbb{R}^2 \times \mathbb{R}^2 \text{ such that } y_2^{*1} \in [0, 2p^*]$$

is a competitive equilibrium. Thus, we have an infinite set of equilibria which are parametrized by $y_2^{*1} \in [0, 2p^*]$.

One should notice that without externalities at all, if the output price increases then the output supply of both firms increases too.[6] So, equilibria are completely determined. In our example, we have an infinite set of equilibria since, for given y_2^{*1}, if the output price p^* increases by k units then the output supply y_1^{*1} of firm 1 increases by $2k$ units, and consequently the output supply y_2^{*1} of firm 2 does not change since the price increase is compensated by firm 1's output increase. Therefore, the output supply of firm 2 is indeterminate since the two effects offset each others.

So, in order to overcome the effects described above, we consider *simple perturbations* of the transformation functions. The definition of a perturbed economy for a given $t \in \mathcal{T}$ is provided below.

[6] In this case, the transformation function of firm j is given by $t_j(y_j^1, y_j^2) := 2\sqrt{-y_j^2} - y_j^1$.

Definition 11 (Perturbed economies) *Let $t \in \mathcal{T}$, a perturbed production economy $(t+b, e)$ is parametrized by transformation levels $b := (b_j)_{j\in\mathcal{J}} \in \mathbb{R}^J_{++}$ and endowment $e \in \mathbb{R}^{CH}_{++}$ where*

$$t + b := (t_j + b_j)_{j\in\mathcal{J}}$$

$\Lambda_t := \mathbb{R}^J_{++} \times \mathbb{R}^{CH}_{++}$ denotes the set of perturbed production economies.

It an easy matter to show that for every $(b, e) \in \Lambda_t$, the perturbed production economy $(t + b, e) \in \mathcal{T} \times \mathbb{R}^{CH}_{++}$ since $t + b$ satisfies Assumptions 1 and 2.

6 Two additional assumptions

One should notice that in the previous example,

(1) the perturbations of the production sets are sufficient to control the first-order external effects,
(2) there are no second-order external effects since the derivatives of the marginal productions with respect to the choices of the others are equal to zero.

But, as shown in Bonnisseau and del Mercato (2010), in the case of only consumption externalities, regularity may fail when the second-order external effects are too strong. So, the basic assumptions and the perturbations introduced in the previous paragraph may be not sufficient to control the second-order external effects thereby preventing the regularity result. Thus, we introduce the following two additional assumptions.

Assumption 12 *Let $(x, y, z) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ} \times \mathbb{R}^{CJ}$ such that $z \in \prod_{j\in\mathcal{J}} \ker D_{y_j} t_j(y_j, y_{-j}, x)$ and $\sum_{j\in\mathcal{J}} z_j = 0$. Then, $z_j \sum_{f\in\mathcal{J}} D^2_{y_f y_j} t_j(y_j, y_{-j}, x)(z_f) < 0$ whenever $z_j \neq 0$.*

Assumption 13 *Let $(x, v, y, z) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CH} \times \mathbb{R}^{CJ} \times \mathbb{R}^{CJ}$ such that $v \in \prod_{h\in\mathcal{H}} \ker D_{x_h} u_h(x_h, x_{-h}, y)$, $z \in \prod_{j\in\mathcal{J}} \ker D_{y_j} t_j(y_j, y_{-j}, x)$ and $\sum_{h\in\mathcal{H}} v_h = \sum_{j\in\mathcal{J}} z_j$, then*

(1) $v_h \sum_{k\in\mathcal{H}} D^2_{x_k x_h} u_h(x_h, x_{-h}, y)(v_k) < 0$ whenever $v_h \neq 0$,
(2) $z_j \sum_{k\in\mathcal{H}} D^2_{x_k y_j} t_j(y_j, y_{-j}, x)(v_k) \leq 0$ whenever $v_k \in \bigcap_{j\in\mathcal{J}} \ker D_{y_j} t_j(y_j, y_{-j}, x)$ for every $k \in \mathcal{H}$.

Assumptions 12 and 13 can be interpreted in a similar way as Assumption 9 in Bonnisseau and del Mercato (2010). More specifically,

- Assumption 12 means that the effect of changes in the production plans $(y_f)_{f \neq j}$ of firms other than j on the marginal production $D_{y_j} t_j(y_j, y_{-j}, x)$ is "dominated" by the effect of changes in the production plan y_j of firm j. Indeed, under Point 4 of Assumption 1, Assumption 12 states that the absolute value of $z_j D^2_{y_j} t_j(y_j, y_{-j}, x)(z_j)$ is larger than the remaining term $z_j \sum_{f \neq j} D^2_{y_f y_j} t_j(y_j, y_{-j}, x)(z_f)$.
- Points 1 of Assumption 13 means that the effect of changes in the consumptions $(x_k)_{k \in h}$ of the households other than h on the marginal utility $D_{x_h} u_h(x_h, x_{-h}, y)$ is "dominated" by the effect of changes in the consumption x_h of household h. Under Point 3 of Assumption 3, Points 1 of Assumption 13 means that the absolute value of $v_h D^2_{x_j} u_h(x_h, x_{-h}, y)(v_h)$ is larger than the remaining term $v_h \sum_{k \neq h} D^2_{x_k x_h} u_h(x_h, x_{-h}, y)(v_k)$.

We provide below an example of transformation functions which satisfy Assumption 12. In the example, there are two firms and two commodities. Let $y_j = (y_j^1, y_j^2)$ be the production plan of firm $j = 1, 2$. Each firm uses commodity 2 to produce commodity 1, so $y_j^1 > 0$ and $y_j^2 < 0$ for every $j = 1, 2$. Each production technology is affected by the output of the other firm in the following way.

$$t_1(y_1, y_2) := 2\sqrt{(-y_1^2)\rho y_2^1} - y_1^1 \text{ and } t_2(y_2, y_1) := 2\sqrt{(-y_2^2)\delta y_1^1} - y_2^1$$

with $\rho > 0$ and $\delta > 0$. An example of utility function which satisfies Point 1 of Assumption 13 is provided in Section 4 of Bonnisseau and del Mercato (2010).

7 Regularity for almost all perturbed economies

In this section, we prove the following theorem which is our main result. Let $t \in \mathcal{T}$, consider the set of perturbed economies Λ_t given by Definition 11.

Theorem 14 (Regularity for almost all perturbed economies) *The set Λ_t^r of $(b, e) \in \Lambda_t$ such that $(t + b, e)$ is a regular economy is an open and full measure subset of Λ_t.*

In order to prove the theorem above, we introduce the following notations and we provide three auxiliary lemmas, namely Lemmas 15, 16 and 17.

For given $(b, e) \in \Lambda_t$, by Point 1 of Assumptions 1 and 3 the equilibrium function $F_{t+b,e}$ is C^1 everywhere. So, by Definition 9 the economy $(t + b, e)$ is regular if

$$\forall\, \xi^* \in F^{-1}_{t+b,e}(0),\ \operatorname{rank} D_\xi F_{t+b,e}(\xi^*) = \dim \Xi$$

Define the following set

$$\widetilde{C} := \left\{(\xi, b, e) \in \widetilde{F}^{-1}(0) : \text{ rank } D_\xi \widetilde{F}(\xi, b, e) < \dim \Xi \right\}$$

where the function $\widetilde{F} : \Xi \times \Lambda_t \to \mathbb{R}^{\dim \Xi}$ is defined by

$$\widetilde{F}(t, b, e) := F_{t+b,e}(\xi)$$

and denote with Π the restriction to $\widetilde{F}^{-1}(0)$ of the projection of $\Xi \times \Lambda_t$ onto Λ_t, that is

$$\Pi : (\xi, b, e) \in \widetilde{F}^{-1}(0) \to \Pi(\xi, b, e) := (b, e) \in \Lambda_t$$

We can now express the set Λ_t^r given in Theorem 14 as

$$\Lambda_t^r = \Lambda_t \setminus \Pi(\widetilde{C})$$

So, in order to prove Theorem 14, it is enough to show that $\Pi(\widetilde{C})$ is a closed set in Λ_t and $\Pi(\widetilde{C})$ is of measure zero.

We first claim that $\Pi(\widetilde{C})$ is a closed set in Λ_t. From Point 1 of Assumptions 1 and 3, $\widetilde{F}$ is a continuous function on $\Xi \times \Lambda_t$ and $D_\xi \widetilde{F}$ is a continuous function on $\widetilde{F}^{-1}(0)$. The set $\widetilde{C}$ is characterized by the fact that the determinant of all the square submatrices of $D_\xi \widetilde{F}(\xi, b, e)$ of dimension $\dim \Xi$ is equal to zero. Since the determinant is a continuous function and $D_\xi \widetilde{F}$ is continuous on $\widetilde{F}^{-1}(0)$, the set $\widetilde{C}$ is closed in $\widetilde{F}^{-1}(0)$. Thus, $\Pi(\widetilde{C})$ is closed since the projection Π is proper.[7] The properness of the projection Π is provided in Lemma 16 given below.

Furthermore, we also provide Lemma 15 which states that the set of feasible allocations is bounded and it is used to prove Step 1 in the proof of Lemma 16.

Lemma 15 *For every $(t, r) \in \mathcal{T} \times \mathbb{R}^C_{++}$, the following set is bounded.*

$$\mathcal{F}_{t,r} := \{(x, y) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ} \mid t_j(y_j, y_{-j}, x) \geq 0, \ \forall\, j \in \mathcal{J} \text{ and } \sum_{h \in \mathcal{H}} x_h - \sum_{j \in \mathcal{J}} y_j \leq r\} \tag{13}$$

Proof. The set $\mathcal{F}_{t,r}$ is bounded since it is included in the set $A_{t,r}$ defined in Lemma 16 of Chapter 1 which is bounded by Assumption 2. ■

Lemma 16 *The projection $\Pi : \widetilde{F}^{-1}(0) \to \Lambda_t$ is a proper function.*

Proof. We show that any sequence $(\xi^\nu, b^\nu, e^\nu)_{\nu \in \mathbb{N}} \subseteq \widetilde{F}^{-1}(0)$, up to a subsequence, converges to an element of $\widetilde{F}^{-1}(0)$, knowing that the sequence $\Pi(\xi^\nu, b^\nu, e^\nu)_{\nu \in \mathbb{N}} = (b^\nu, e^\nu)_{\nu \in \mathbb{N}} \subseteq \Lambda_t$ converges to some $(b^*, e^*) \in \Lambda_t$.

[7] See Definition 25 in Appendix A.

We recall that $\xi^\nu = (x^\nu, \lambda^\nu, y^\nu, \alpha^\nu, p^{\nu\backslash})$. In order to simplify the notation, define

$$t^\nu(y_j, y_{-j}, x) := t(y_j, y_{-j}, x) + b^\nu$$

Step 1. *Up to a subsequence, $(x^\nu, y^\nu)_{\nu\in\mathbb{N}}$ converges to $(x^*, y^*) \in \mathbb{R}^{CH}_+ \times \mathbb{R}^{CJ}$.*

The idea of the proof is to show that for an appropriate $(\bar{t}, \bar{r}) \in \mathcal{T} \times \mathbb{R}^C_{++}$, the sequence $(x^\nu, y^\nu)_{\nu\in\mathbb{N}}$ belongs to the set $\mathcal{F}_{\bar{t},\bar{r}}$ defined by Lemma 15.

Consider the following set $\{b^\nu : \nu \in \mathbb{N}\} \cup \{b^*\}$ which is obviously compact, and define

$$\bar{b} := \max \{b^\nu : \nu \in \mathbb{N}\} \cup \{b^*\} \quad \text{and} \quad \bar{t} := t + \bar{b}$$

By definition, we get $\bar{t}(y_j, y_{-j}, x) \geq t^\nu(y_j, y_{-j}, x)$ for every $(y_j, y_{-j}, x) \in \mathbb{R}^C \times \mathbb{R}^{C(J-1)} \times \mathbb{R}^{CH}_{++}$ and for every $\nu \in \mathbb{N}$. Since $\widetilde{F}^{j.2}(\xi^\nu, b^\nu, e^\nu) = 0$, for every $\nu \in \mathbb{N}$ we get

$$\bar{t}_j(y^\nu_j, y^\nu_{-j}, x^\nu) \geq 0$$

Now, for every commodity c consider the following compact set $\{e^{\nu c} : \nu \in \mathbb{N}\} \cup \{e^{*c}\}$, and define

$$\bar{r}^c := \max_{e^c \in \{e^{\nu c}:\nu\in\mathbb{N}\}\cup\{e^{*c}\}} \sum_{h\in\mathcal{H}} e^c_h \quad \text{and} \quad \bar{r} := (\bar{r}^c)_{c\in\mathcal{C}}$$

Summing $\widetilde{F}^{h.2}(\xi^\nu, b^\nu, e^\nu) = 0$ over h, by $\widetilde{F}^M(\xi^\nu, b^\nu, e^\nu) = 0$ we have that $\sum_{h\in\mathcal{H}} x^\nu_h - \sum_{j\in\mathcal{J}} y^\nu_j = \sum_{h\in\mathcal{H}} e^\nu_h$ for all $\nu \in \mathbb{N}$. So, by definition for all $\nu \in \mathbb{N}$, we get

$$\sum_{h\in\mathcal{H}} x^\nu_h - \sum_{j\in\mathcal{J}} y^\nu_j \leq \bar{r}$$

Thus, $(x^\nu, y^\nu)_{\nu\in\mathbb{N}} \subseteq \mathcal{F}_{\bar{t},\bar{r}}$. Consequently, the sequence $(x^\nu, y^\nu)_{\nu\in\mathbb{N}}$ belongs to $\operatorname{cl} \mathcal{F}_{\bar{t},\bar{r}}$ which is compact since it is bounded by Lemma 15. So, up to a subsequence, $(x^\nu, y^\nu)_{\nu\in\mathbb{N}}$ converges to some $(x^*, y^*) \in \operatorname{cl} \mathcal{F}_{\bar{t},\bar{r}} \subseteq \mathbb{R}^{CH}_+ \times \mathbb{R}^{CJ}$, and thus $(x^*, y^*) \in \mathbb{R}^{CH}_+ \times \mathbb{R}^{CJ}$.

Step 2. *The consumption allocation x^* is strictly positive, i.e. $x^* \gg 0$.*

The proof is based on Point 4 of Assumption 3. By $\widetilde{F}^{h.1}(\xi^\nu, b^\nu, e^\nu) = \widetilde{F}^{h.2}(\xi^\nu, b^\nu, e^\nu) = 0$ and Karush–Kuhn–Tucker sufficient conditions, x^ν_h solves the following problem for every $\nu \in \mathbb{N}$.

$$\begin{aligned} &\max_{x_h\in\mathbb{R}^C_{++}} \; u_h(x_h, x^\nu_{-h}, y^\nu) \\ &\text{subject to } p^\nu \cdot x_h \leq p^\nu \cdot e^\nu_h + p^\nu \cdot \sum_{j\in\mathcal{J}} s_{jh} y^\nu_j \end{aligned}$$

We first claim that for every $\nu \in \mathbb{N}$, the point e_h^ν belongs to the budget constraint of the problem above. By $\widetilde{F}^{j.1}(\xi^\nu, b^\nu, e^\nu) = \widetilde{F}^{j.2}(\xi^\nu, b^\nu, e^\nu) = 0$ and Karush–Kuhn–Tucker sufficient conditions, y_j^ν solves the following problem for every $\nu \in \mathbb{N}$.

$$\begin{aligned} &\max_{y_j \in \mathbb{R}^C} p^\nu \cdot y_j \\ &\text{subject to } t_j^\nu(y_j, y_{-j}^\nu, x^\nu) \geq 0 \end{aligned} \tag{14}$$

Since inactivity is possible, $t_j^\nu(0, y_{-j}^\nu, x^\nu) \geq 0$ by Point 2 of Assumption 1. So, $p^\nu \cdot y_j^\nu \geq p^\nu \cdot 0 = 0$. Therefore,

$$p^\nu \cdot \sum_{j \in \mathcal{J}} s_{jh} y_j^\nu \geq 0$$

which completes the proof of the claim.

We claim now that x_h^* belongs to the closure of some upper contour set. Obviously, for every $\nu \in \mathbb{N}$

$$u_h(x_h^\nu, x_{-h}^\nu, y^\nu) \geq u_h(e_h^\nu, x_{-h}^\nu, y^\nu)$$

By Point 2 of Assumption 3, for every $\varepsilon > 0$ we have that $u_h(x_h^\nu + \varepsilon\mathbf{1}, x_{-h}^\nu, y^\nu) > u_h(e_h^\nu, x_{-h}^\nu, y^\nu)$ where $\mathbf{1} := (1, \dots, 1) \in \mathbb{R}_{++}^C$. So, taking the limit for $\nu \to +\infty$ and using the continuity of u_h given by Point 1 of Assumption 3, since $(e_h^\nu)_{\nu \in \mathbb{N}}$ converges to $e_h^* \in \mathbb{R}_{++}^C$ we get $u_h(x_h^* + \varepsilon\mathbf{1}, x_{-h}^*, y^*) \geq u_h(e_h^*, x_{-h}^*, y^*) := u$. That is, for every $\varepsilon > 0$ the point $(x_h^* + \varepsilon\mathbf{1})$ belongs to the following set

$$\{x_h \in \mathbb{R}_{++}^C : u_h(x_h, x_{-h}^*, y^*) \geq u\}$$

So, the point x_h^* belongs to the closure of set above which is included in $\mathbb{R}_{++}^C$ by Point 4 of Assumption 3. Therefore, $x_h^* \in \mathbb{R}_{++}^{CH}$.

Step 3. *Up to a subsequence,* $(\alpha^\nu, p^{\nu\,\backslash})_{\nu \in \mathbb{N}}$ *converges to some* $(\alpha^*, p^{*\,\backslash}) \in \mathbb{R}_{++}^J \times \mathbb{R}_{++}^{C-1}$.

By $\widetilde{F}^{j.1}(\xi^\nu, b^\nu, e^\nu) = 0$, considering commodity C, we get

$$\alpha_j^\nu = -\frac{1}{D_{y_j^C} t_j^\nu(y_j^\nu, y_{-j}^\nu, x^\nu)}, \quad \forall\, \nu \in \mathbb{N}$$

Define $t^* := t + b^*$, where b^* is the limit of the sequence $(b^\nu)_{\nu \in \mathbb{N}}$. Taking the limit for $\nu \to +\infty$, by Points 1 and 3 of Assumption 1, the sequence $(\alpha_j^\nu)_{\nu \in \mathbb{N}}$ converges to

$$\alpha_j^* := -\frac{1}{D_{y_j^C} t_j^*(y_j^*, y_{-j}^*, x^*)} > 0$$

By $\widetilde{F}^{j.1}(\xi^\nu, b^\nu, e^\nu) = 0$, for every commodity $c \neq C$ and for all $\nu \in \mathbb{N}$ we have

$$p^{\nu\, c} = -\alpha_j^\nu D_{y_j^c} t_j^\nu(y_j^\nu, y_{-j}^\nu, x^\nu)$$

Taking the limit, by Points 1 and 3 of Assumption 1, for all $c \neq C$ we get

$$p^{*\,c} = -\alpha_j^* D_{y_j^c} t_j^*(y_j^*, y_{-j}^*, x^*) > 0$$

Therefore, $p^{*\,\backslash} \in \mathbb{R}_{++}^{C-1}$.

Step 4. *Up to a subsequence, $(\lambda^\nu)_{\nu\in\mathbb{N}}$ converges to some $\lambda^* \in \mathbb{R}_{++}^H$.*

By $\widetilde{F}^{h.1}(\xi^\nu, b^\nu, e^\nu) = 0$, considering commodity C for every $\nu \in \mathbb{N}$ we get

$$\lambda_h^\nu = D_{x_h^C} u_h(x_h^\nu, x_{-h}^\nu, y^\nu)$$

Taking the limit and using the continuity of Du_h (see Point 1 of Assumption 3) we have

$$\lambda_h^* = D_{x_h^C} u_h(x_h^*, x_{-h}^*, y^*)$$

which is strictly positive since fixing the externalities the function u_h is differentiably strictly increasing (see Point 2 of Assumption 3). ■

To complete the proof of Theorem 14, we claim now that $\Pi(\widetilde{C})$ is of measure zero in Λ_t. The result follows by Lemma 17 given below and a consequence of Sard's Theorem (see Theorem 24 in Appendix A). Indeed, Lemma 17 and Theorem 24 imply that there exists a full measure subset Ω of Λ_t such that for each $(b, e) \in \Omega$ and for each ξ^* such that $\widetilde{F}(\xi^*, b, e) = 0$, $\operatorname{rank} D_\xi \widetilde{F}(\xi^*, b, e) = \dim \Xi$. Now, let $(b, e) \in \Pi(\widetilde{C})$, then there exists $\xi \in \Xi$ such that $\widetilde{F}(\xi, b, e) = 0$ and $\operatorname{rank} D_\xi \widetilde{F}(\xi, b, e) < \dim \Xi$. So, $(b, e) \notin \Omega$. This prove that $\Pi(\widetilde{C})$ is included in the complementary of Ω, that is in $\Omega^C := \Lambda_t \setminus \Omega$. Since Ω^C has zero measure, so too does $\Pi(\widetilde{C})$. Thus, the set of regular perturbed economies Λ_t^r is of full measure since $\Omega \subseteq \Lambda_t^r$ which completes the proof of Theorem 14.

Lemma 17 *0 is a regular value for $\widetilde{F}$.*

Proof. It is enough to prove that for each $(\xi^*, b^*, e^*) \in \widetilde{F}^{-1}(0)$, the Jacobian matrix $D_{\xi,b,e}\widetilde{F}(\xi^*, b^*, e^*)$ has full row rank.

Let $\Delta := ((\Delta x_h, \Delta\lambda_h)_{h\in\mathcal{H}}, (\Delta y_j, \Delta\alpha_j)_{j\in\mathcal{J}}, \Delta p^\backslash) \in \mathbb{R}^{H(C+1)} \times \mathbb{R}^{J(C+1)} \times \mathbb{R}^{C-1}$. We need to show that $\Delta D_{\xi,b,e}\widetilde{F}(\xi^*, b^*, e^*) = 0$ implies $\Delta = 0$. Consider the computation of the partial Jacobian matrix with respect to the following variables.[8]

$$((x_h, \lambda_h, e_h)_{h\in\mathcal{H}}, (y_j, \alpha_j, b_j)_{j\in\mathcal{J}}, p^\backslash)$$

[8] The computation of $D_{\xi,b,e}\widetilde{F}(\xi^*, b^*, e^*)$ is described in Appendix B.

The partial system $\Delta D_{\xi,b,e}\widetilde{F}(\xi^*, b^*, e^*) = 0$ is written in detail below.

$$\begin{cases} \sum_{h\in\mathcal{H}} \Delta x_h D^2_{x_k x_h} u_h(x_h^*, x_{-h}^*, y^*) - \Delta\lambda_k p^* + \sum_{j\in\mathcal{J}} \alpha_j^* \Delta y_j D^2_{x_k y_j} t_j(y_j^*, y_{-j}^*, x^*) + \\ \sum_{j\in\mathcal{J}} \Delta\alpha_j D_{x_k} t_j(y_j^*, y_{-j}^*, x^*) + \Delta p^{\setminus}[I_{C-1}|0] = 0, \ \forall\, k \in \mathcal{H} \\ -\Delta x_h \cdot p^* = 0, \ \forall\, h \in \mathcal{H} \\ \sum_{h\in\mathcal{H}} \Delta x_h D^2_{y_f x_h} u_h(x_h^*, x_{-h}^*, y^*) + \sum_{h\in\mathcal{H}} \Delta\lambda_h s_{fh} p^* + \sum_{j\in\mathcal{J}} \alpha_j^* \Delta y_j D^2_{y_f y_j} t_j(y_j^*, y_{-j}^*, x^*) + \\ \sum_{j\in\mathcal{J}} \Delta\alpha_j D_{y_f} t_j(y_j^*, y_{-j}^*, x^*) + \Delta p^{\setminus}[I_{C-1}|0] = 0, \ \forall\, f \in \mathcal{J} \\ \Delta y_j \cdot D_{y_j} t_j(y_j^*, y_{-j}^*, x^*) = 0, \ \forall\, j \in \mathcal{J} \\ \Delta\lambda_h p^* - \Delta p^{\setminus}[I_{C-1}|0] = 0, \ \forall\, h \in \mathcal{H} \\ -\sum_{h\in\mathcal{H}} \lambda_h^* \Delta x_h^{\setminus} - \sum_{h\in\mathcal{H}} \Delta\lambda_h (x_h^{*\setminus} - e_h^{*\setminus} - \sum_{j\in\mathcal{J}} s_{jh} y_j^{*\setminus}) + \sum_{j\in\mathcal{J}} \Delta y_j^{\setminus} = 0 \\ \Delta\alpha_j = 0, \ \forall\, j \in \mathcal{J} \end{cases}$$

Since $p^{*C} = 1$, we get

$$\Delta\lambda_h = 0 \text{ for each } h \in \mathcal{H} \text{ and } \Delta p^{\setminus} = 0$$

So, the above system becomes

$$\begin{cases} (1) \ \ \sum_{h\in\mathcal{H}} \Delta x_h D^2_{x_k x_h} u_h(x_h^*, x_{-h}^*, y^*) + \sum_{j\in\mathcal{J}} \alpha_j^* \Delta y_j D^2_{x_k y_j} t_j(y_j^*, y_{-j}^*, x^*) = 0, \ \forall\, k \in \mathcal{H} \\ (2) \ \ -\Delta x_h \cdot p^* = 0, \ \forall\, h \in \mathcal{H} \\ (3) \ \ \sum_{h\in\mathcal{H}} \Delta x_h D^2_{y_f x_h} u_h(x_h^*, x_{-h}^*, y^*) + \sum_{j\in\mathcal{J}} \alpha_j^* \Delta y_j D^2_{y_f y_j} t_j(y_j^*, y_{-j}^*, x^*) = 0, \ \forall\, f \in \mathcal{J} \\ (4) \ \ \Delta y_j \cdot D_{y_j} t_j(y_j^*, y_{-j}^*, x^*) = 0, \ \forall\, j \in \mathcal{J} \\ (5) \ \ -\sum_{h\in\mathcal{H}} \lambda_h^* \Delta x_h^{\setminus} + \sum_{j\in\mathcal{J}} \Delta y_j^{\setminus} = 0 \end{cases} \tag{15}$$

Multiplying both sides of equation $\widetilde{F}^{j.1}(\xi^*, b^*, e^*) = 0$ by Δy_j and using equation (4) in system (15), we get $\Delta y_j \cdot p^* = -\alpha_j^* \Delta y_j \cdot D_{y_j} t_j(y_j^*, y_{-j}^*, x^*) = 0$. Summing over j and considering commodity C, we obtain $-\sum_{j\in\mathcal{J}} \Delta y_j^{\setminus} \cdot p^{*\setminus} = \sum_{j\in\mathcal{J}} \Delta y_j^C$. Multiplying equation (2) in system (15) by λ_h^*, summing over h and considering commodity C, we obtain $\sum_{h\in\mathcal{H}} \lambda_h^* \Delta x_h^C = -\sum_{h\in\mathcal{H}} \lambda_h^* \Delta x_h^{\setminus} \cdot p^{*\setminus}$. Finally using equation (5) in system (15), we get $\sum_{h\in\mathcal{H}} \lambda_h^* \Delta x_h^C = -\sum_{j\in\mathcal{J}} \Delta y_j^{\setminus} \cdot p^{*\setminus} =$

$\sum_{j\in\mathcal{J}} \Delta y_j^C$. From the previous result and equation (5) in system (15), we obtain

$$\sum_{h\in\mathcal{H}} \lambda_h^* \Delta x_h = \sum_{j\in\mathcal{J}} \Delta y_j$$

Observe that from $\widetilde{F}^{h.1}(\xi^*, b^*, e^*) = 0$ and equation (2) in system (15), we get

$$(\Delta x_h)_{h\in\mathcal{H}} \in \prod_{h\in\mathcal{H}} \ker D_{x_h} u_h(x_h^*, x_{-h}^*, y^*)$$

From equation (4) in system (15) , we obtain

$$(\Delta y_j)_{j\in\mathcal{J}} \in \prod_{j\in\mathcal{J}} \ker D_{y_j} t_j(y_j^*, y_{-j}^*, x^*)$$

Multiplying both sides of equation $\widetilde{F}^{j.1}(\xi^*, b^*, e^*) = 0$ by Δx_k and using equation (2) in system (15), one obtains

$$\Delta x_k \in \bigcap_{j\in\mathcal{J}} \ker D_{y_j} t_j(y_j^*, y_{-j}^*, x^*),\ \forall\, k \in \mathcal{H}$$

Now, for every $h \in \mathcal{H}$ and for every $j \in \mathcal{J}$ define

$$v_h := \lambda_h^* \Delta x_h \text{ and } z_j := \Delta y_j \tag{16}$$

From the previous arguments it follows that the vector $((x_h^*, v_h)_{h\in\mathcal{H}}, (y_j^*, z_j)_{j\in\mathcal{J}})$ satisfies the following conditions.

$$\sum_{h\in\mathcal{H}} v_h = \sum_{j\in\mathcal{J}} z_j \tag{17}$$

$$(v_h)_{h\in\mathcal{H}} \in \prod_{h\in\mathcal{H}} \ker D_{x_h} u_h(x_h^*, x_{-h}^*, y^*) \tag{18}$$

$$(z_j)_{j\in\mathcal{J}} \in \prod_{j\in\mathcal{J}} \ker D_{y_j} t_j(y_j^*, y_{-j}^*, x^*) \tag{19}$$

$$v_k \in \bigcap_{j\in\mathcal{J}} \ker D_{y_j} t_j(y_j^*, y_{-j}^*, x^*),\ \forall\, k \in \mathcal{H} \tag{20}$$

Multiplying both sides of equation (1) in system (15) by v_k, we get

$$\sum_{h\in\mathcal{H}} \Delta x_h D^2_{x_k x_h} u_h(x_h^*, x_{-h}^*, y^*)(v_k) = -\sum_{j\in\mathcal{J}} \alpha_j^* \Delta y_j D^2_{x_k y_j} t_j(y_j^*, y_{-j}^*, x^*)(v_k)$$

Since $\lambda_h^* \neq 0$ for all $h \in \mathcal{H}$, then it follows by (16) that for each $k \in \mathcal{H}$

$$\sum_{h\in\mathcal{H}} \frac{v_h}{\lambda_h^*} D^2_{x_k x_h} u_h(x_h^*, x_{-h}^*, y^*)(v_k) = -\sum_{j\in\mathcal{J}} \alpha_j^* z_j D^2_{x_k y_j} t_j(y_j^*, y_{-j}^*, x^*)(v_k)$$

Summing over $k \in \mathcal{H}$, we get

$$\sum_{h\in\mathcal{H}} \frac{v_h}{\lambda_h^*} \sum_{k\in\mathcal{H}} D^2_{x_k x_h} u_h(x_h^*, x_{-h}^*, y^*)(v_k) = -\sum_{j\in\mathcal{J}} \alpha_j^* z_j \sum_{k\in\mathcal{H}} D^2_{x_k y_j} t_j(y_j^*, y_{-j}^*, x^*)(v_k)$$

From (17), (18) and (19), all the conditions of Assumption 13 are satisfied. Since $\alpha_j^* > 0$ for each $j \in \mathcal{J}$, the equality above, (20) and Point 2 of Assumption 13 imply that

$$\sum_{h\in\mathcal{H}} \frac{1}{\lambda_h^*} v_h \sum_{k\in\mathcal{H}} D^2_{x_k x_h} u_h(x_h^*, x_{-h}^*, y^*)(v_k) \geq 0$$

Since $\lambda_h^* > 0$ for all $h \in \mathcal{H}$, Point 1 of Assumption 13 implies that $v_h = 0$ for all $h \in \mathcal{H}$, and by (16) we get

$$\Delta x_h = 0, \ \forall \ h \in \mathcal{H}$$

So, condition (17) becomes

$$\sum_{j\in\mathcal{J}} z_j = 0 \tag{21}$$

and equation (3) in system (15) becomes

$$\sum_{j\in\mathcal{J}} \alpha_j^* z_j D^2_{y_f y_j} t_j(y_j^*, y_{-j}^*, x^*) = 0, \ \forall \ f \in \mathcal{J}$$

Multiplying both sides by z_f and summing up $f \in \mathcal{J}$, we obtain

$$\sum_{j\in\mathcal{J}} \alpha_j^* z_j \sum_{f\in\mathcal{J}} D^2_{y_f y_j} t_j(y_j^*, y_{-j}^*, x^*)(z_f) = 0$$

By (19) and (21), all the conditions of Assumption 12 are satisfied. Since $\alpha_j^* > 0$ for each $j \in \mathcal{J}$, Assumption 12 implies that $z_j = 0$ for each $j \in \mathcal{J}$, and so by (16), we get

$$\Delta y_j = 0, \ \forall \ j \in \mathcal{J}$$

Thus, $\Delta = 0$ which completes the proof. ■

The following remark is an easy consequence of Theorem 14.

Remark 18 *Since Λ_t^r is a full measure subset of $\Lambda_t = \mathbb{R}^J_{++} \times \mathbb{R}^{CH}_{++}$, then it is dense in Λ_t. Thus, one easily deduces that Λ_t^r is dense in $\mathbb{R}^J_+ \times \mathbb{R}^{CH}_{++}$.*

8 Regularity in the space of endowments and transformation functions

In the following corollary, we provide an important consequence of Theorem 14, that is the set of regular economies is a non-empty open subset of the space of endowments and transformation functions.

As in Section 4, we recall that $B = \mathbb{R}^{CJ} \times \mathbb{R}^{CH}_{++}$, $C^2(B, \mathbb{R})$ is endowed with the C^2 Whitney topology (see Definition 21 in Appendix A) and the set $\mathcal{T} \times \mathbb{R}^{CH}_{++}$

is endowed with the topology induced by the product topology on $C^2(B,\mathbb{R})^J \times \mathbb{R}^{CH}_{++}$.

Corollary 19 *The set $\mathcal{R}$ of regular economies is a open subset of $\mathcal{T} \times \mathbb{R}^{CH}_{++}$ and it contains the following set $\bigcup_{t\in\mathcal{T}}\{(t+b,e)\colon (b,e)\in\Lambda^r_t\}$.* [9]

In order to prove the corollary above, we introduce the following notation and we provide an auxiliary lemma, namely Lemma 20. Describe the set of regular economies as

$$\mathcal{R} = \{(t,e) \in \mathcal{T} \times \mathbb{R}^{CH}_{++} \ : \ \forall\, \xi \in F(\cdot,t,e)^{-1}(0),\ \operatorname{rank} D_\xi F(\xi,t,e) = \dim \Xi\}$$

where the global equilibrium function $F : \Xi \times \mathcal{T} \times \mathbb{R}^{CH}_{++} \to \mathbb{R}^{\dim \Xi}$ given by $F(\xi,t,e) := F_{t,e}(\xi)$. Using the definition of open set and the following lemma, one easily checks that the set $\mathcal{R}$ is a open subset of $\mathcal{T} \times \mathbb{R}^{CH}_{++}$.

Lemma 20 *The functions F and $D_\xi F$ defined on $\Xi \times C^2(B,\mathbb{R})^J \times \mathbb{R}^{CH}_{++}$ are continuous.*

Proof. We prove that F is continuous. In a analogous way one easily shows that $D_\xi F$ is a continuous function. Since $C^2(B,\mathbb{R})$ is a linear space and F is a linear function with respect to $t \in C^2(B,\mathbb{R})^J$, it is enough to prove that F is continuous at any point $(\overline{\xi},0,\overline{e})$ with $t = 0 \in C^2(B,\mathbb{R})^J$ and $(\overline{\xi},\overline{e}) \in \Xi \times \mathbb{R}^{CH}_{++}$. Since F is continuous at $(\overline{\xi},0,\overline{e})$ if and only if all its components are continuous, we show that the component $F^{j.1}(\xi,t,e) = p + \alpha_j D_{y_j} t_j(y_j,y_{-j},x)$ is continuous at $(\overline{\xi},0,\overline{e})$. Using the same strategy, one easily proves that all the other components of F are continuous.

Fix a commodity $c \in \mathcal{C}$ and $\varepsilon > 0$, we claim that there exists an open neighborhood I of $(\overline{y},\overline{x},\overline{\alpha}_j,\overline{p}^c,0)$ in $B \times \mathbb{R}^2_{++} \times C^2(B,\mathbb{R})$ such that

$$\forall\ (y,x,\alpha_j,p^c,t) \in I,\ |p^c + \alpha_j D_{y^c_j} t_j(y_j,y_{-j},x) - \overline{p}^c| < \varepsilon$$

Fix $\varepsilon_2 > 0$ and $\varepsilon_3 > 0$ such that $\varepsilon' := \dfrac{\varepsilon - \varepsilon_3}{\overline{\alpha}_j + \varepsilon_2} > 0$, and a continuous and strictly positive function δ defined on B such that

$$\delta(\overline{y},\overline{x}) < \varepsilon'$$

Since δ is continuous at $(\overline{y},\overline{x})$, for given $\eta := \varepsilon' - \delta(\overline{y},\overline{x}) > 0$ there exists an open neighborhood $I(\overline{y},\overline{x},\varepsilon_1)$ of $(\overline{y},\overline{x})$ in B such that

$$\forall\ (y,x) \in I(\overline{y},\overline{x},\varepsilon_1),\ \delta(y,x) < \eta + \delta(\overline{y},\overline{x}) = \varepsilon' \tag{22}$$

[9] The set Λ^r_t is given by Theorem 14.

Consider now the open neighborhood $N(0,\delta)$ of the function 0 determined by δ in the C^2 Whitney topology, that is

$$N(0,\delta) = \{t \in C^2(B,\mathbb{R}) \;:\; |t(y,x)| < \delta(y,x) \text{ and}$$
$$\|D^k t(y,x)\| < \delta(y,x),\ \forall\ (y,x) \in B \text{ and } \forall\ k = 1,2\}$$

By (22), for every $t \in N(0,\delta)$ and for every $(y,x) \in I(\overline{y},\overline{x},\varepsilon_1)$ we get

$$|D_{y_j^c} t_j(y_j, y_{-j}, x)| \leq \| D_{y_j} t_j(y_j, y_{-j}, x)\| < \varepsilon'$$

Define now the following open neighborhood of $(\overline{y},\overline{x},\overline{\alpha}_j,\overline{p}^c,0)$ in $B \times \mathbb{R}^2_{++} \times C^2(B,\mathbb{R})$

$$I := I(\overline{y},\overline{x},\varepsilon_1) \times I(\overline{\alpha}_j,\varepsilon_2) \times I(\overline{p}^c,\varepsilon_3) \times N(0,\delta)$$

For every $(x,y,\alpha_j,p^c,t) \in I$ we get

$$|p^c + \alpha_j D_{y_j^c} t_j(y_j, y_{-j}, x) - \overline{p}^c \leq |p^c - \overline{p}^c| + \alpha_j |D_{y_j^c} t_j(y_j, y_{-j}, x)| <$$
$$\varepsilon_3 + (\overline{\alpha}_j + \varepsilon_2)|D_{y_j^c} t_j(y_j, y_{-j}, x)| \leq \varepsilon_3 + (\overline{\alpha}_j + \varepsilon_2)\|D_{y_j} t_j(y_j, y_{-j}, x)\| <$$
$$\varepsilon_3 + (\overline{\alpha}_j + \varepsilon_2)\varepsilon' = \varepsilon_3 + (\overline{\alpha}_j + \varepsilon_2)\frac{\varepsilon - \varepsilon_3}{\overline{\alpha}_j + \varepsilon_2} = \varepsilon$$

So, the claim is completely proved. ■

Appendix A

Whitney topology

Let $B := \mathbb{R}^{CJ} \times \mathbb{R}^{CH}_{++}$. We are interested on C^2 functions defined on B (see Point 1 of Assumption 1), $C^2(B,\mathbb{R})$ denotes the set of C^2 functions from B to $\mathbb{R}$. We provide below the definition of the C^2 Whitney topology on $C^2(B,\mathbb{R})$. Since $C^2(B,\mathbb{R})$ is a linear space, in order to define the C^2 Whitney topology on $C^2(B,\mathbb{R})$, it is sufficient to define neighborhood basis of the function zero. For additional details, see for instance Allen (1981, p. 284), Smale (1974, p. 4) and Golubitsky and Guillemin (1973, p. 42).

Definition 21 *Let $\delta : B \to \mathbb{R}$ be a continuous and strictly positive function. The open neighborhood $N(0,\delta)$ of the function $0 \in C^2(B,\mathbb{R})$ is defined as*

$$N(0,\delta) := \{g \in C^2(B,\mathbb{R}) \;:\; |g(z)| < \delta(z)\ \forall\ z \in B \textit{ and}$$
$$\|D^k g(z)\| < \delta(z)\ \forall\ z \in B \textit{ and } \forall\ k = 1,2\}$$

Neighborhoods of $f \in C^2(B,\mathbb{R})$, for $f \neq 0$ can be constructed for translation, that is the neighborhood of f determined by δ is given by $N(f,\delta) = f + N(0,\delta)$. For every $f \in C^2(B,\mathbb{R})$, the collection of $\{N(f,\delta)\}_{\delta \in C^0(B,\mathbb{R}_{++})}$ forms *a neighborhood basis of the function f in the C^2 Whitney topology, where δ varies in the space of all continuous and strictly positive functions.*

The C^2 Whitney topology on $C^2(B, \mathbb{R})$ is not necessarily metrizable. However, if the set B is compact, the C^2 Whitney topology coincides with the topology of the C^2 uniform convergence on compacta. [10]

Regular values and transversality

The theory of general economic equilibrium from a differentiable prospective is based on results from differential topology. Following are the ones used in our analysis. These results, as well as generalizations on these issues, can be found for instance in Guillemin and Pollack (1974), Hirsch (1976), Mas-Colell (1985) and Villanacci et al. (2002).

Theorem 22 *(Regular Value Theorem) Let M, N be C^r manifolds of dimensions m and n, respectively. Let $f : M \to N$ be a C^r function. Assume $r > \max\{m - n, 0\}$. If $y \in N$ is a regular value for f, then*

(1) if $m < n$, $f^{-1}(y) = \emptyset$,
(2) if $m \geq n$, either $f^{-1}(y) = \emptyset$, or $f^{-1}(y)$ is an $(m - n)$-dimensional submanifold of M.

Corollary 23 *Let M, N be C^r manifolds of the same dimension. Let $f : M \to N$ be a C^r function. Assume $r \geq 1$. Let $y \in N$ a regular value for f such that $f^{-1}(y)$ is non-empty and compact. Then, $f^{-1}(y)$ is a finite subset of M.*

The following results is a consequence of Sard's Theorem for manifolds.

Theorem 24 *(Transversality Theorem) Let M, Ω and N be C^r manifolds of dimensions m, p and n, respectively. Let $f : M \times \Omega \to N$ be a C^r function. Assume $r > \max\{m-n, 0\}$. If $y \in N$ is a regular value for f, then there exists a full measure subset Ω^* of Ω such that for any $\omega \in \Omega^*$, $y \in N$ is a regular value for f_ω, where*

$$f_\omega : \xi \in M \to f_\omega(\xi) := f(\xi, \omega) \in N$$

Definition 25 *Let (X, d) and (Y, d') be two metric spaces. A function $\pi : X \to Y$ is proper if it is continuous and one among the following conditions holds true.*

(1) π is closed and $\pi^{-1}(y)$ is compact for each $y \in Y$,

[10] We recall that by definition, $f^n \to f$ in the topology of C^2 uniform convergence on compacta if and only if $(f^n)_{n\in\mathbb{N}}$, $(Df^n)_{n\in\mathbb{N}}$ and $(D^2 f^n)_{n\in\mathbb{N}}$ converge uniformly to f, Df and $D^2 f$ respectively on any compact set included in B.

(2) if K is a compact subset of Y, then $\pi^{-1}(K)$ is a compact subset of X,

(3) if $(x^n)_{n\in\mathbb{N}}$ is a sequence in X such that $(\pi(x^n))_{n\in\mathbb{N}}$ converges in Y, then $(x^n)_{n\in\mathbb{N}}$ has a converging subsequence in X.

The above conditions are equivalent.

Theorem 26 *(Implicit Function Theorem) Let M, N be C^r manifolds of the same dimension. Assume $r \geq 1$. Let (X, τ) be a topological space, and $f : M \times X \to N$ be a continuous function such that $D_\xi f(\xi, x)$ exists and it is continuous on $M \times X$. If $f(\xi, x) = 0$ and $D_\xi f(\xi, x)$ is onto, then there exist an open neighborhood I of x in X, an open neighborhood U of ξ in M and a continuous function $g : I \to U$ such that $g(x) = \xi$, $f(\xi', x') = 0$ holds for $(\xi', x') \in U \times I$ if and only if $\xi' = g(x')$, and $D_\xi f(\xi', x')$ is onto for every $(\xi', x') \in U \times I$ such that $f(\xi', x') = 0$.*

Appendix B

The computation of $D_{\xi,b,e}\widetilde{F}(\xi^*, b^*, e^*)$ is described below, where $\mathbf{0}_{C\times(C-1)}$ is a zero matrix and $\widehat{I} := [I_{C-1}|0]_{(C-1)\times C}$

	x_1	λ_1	$\dots$	x_h	λ_h	$\dots$	x_H	λ_H	y_1	α_1	$\dots$	y_j	α_j	$\dots$	y_J	α_J	$e_1^{\backslash} e_1^C \dots e_h e_h^C \dots e_H e_H^C$	$p^{\backslash}$	$b_1 \dots b_j \dots b_J$
$D_{x_1} u_1(x_1, x_{-1}, y) - \lambda_1 p$	$D^2_{x_1} u_1$	$-p^{*T}$	$\dots$	$D^2_{x_h x_1} u_1$	0^T	$\dots$	$D^2_{x_H x_1} u_1$	0^T	$D^2_{y_1 x_1} u_1$	0^T	$\dots$	$D^2_{y_j x_1} u_1$	0^T	$\dots$	$D^2_{y_J x_1} u_1$	0^T	$\mathbf{0}\, 0^T \dots \mathbf{0}\, 0^T \dots \mathbf{0}\, 0^T$	$\lambda_1^* \widehat{I}^T$	$0^T \dots 0^T \dots 0^T$
$-p \cdot (x_1 - e_1 - \sum_{j \in \mathcal{J}} s_{j1} y_j)$	$-p^*$	0	$\dots$	0	0	$\dots$	0	0	$s_{11} p^*$	0	$\dots$	$s_{j1} p^*$	0	$\dots$	$s_{J1} p^*$	0	$p^{*\backslash}{}_1 \dots 0 \; 0 \dots 0 \; 0$	$(x_1^{*\backslash} - e_1^{*\backslash} - \sum_{j \in \mathcal{J}} s_{j1} y_j^{*\backslash})$	$0 \dots 0 \dots 0$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots \; \vdots \ddots \vdots \; \vdots \ddots \vdots \; \vdots$	$\vdots$	$\vdots \ddots \vdots \ddots \vdots$
$D_{x_h} u_h(x_h, x_{-h}, y) - \lambda_h p$	$D^2_{x_1 x_h} u_h$	0^T	$\dots$	$D^2_{x_h} u_h$	$-p^{*T}$	$\dots$	$D^2_{x_H x_h} u_h$	0^T	$D^2_{y_1 x_h} u_h$	0^T	$\dots$	$D^2_{y_j x_h} u_h$	0^T	$\dots$	$D^2_{y_J x_h} u_h$	0^T	$\mathbf{0}\; 0 \dots \mathbf{0}\, 0^T \dots \mathbf{0}\; 0$	$\lambda_h^* \widehat{I}^T$	$0 \dots 0 \dots 0$
$-p \cdot (x_h - e_h - \sum_{j \in \mathcal{J}} s_{jh} y_j)$	0	0	$\dots$	$-p^*$	0	$\dots$	0	0	$s_{1h} p^*$	0	$\dots$	$s_{jh} p^*$	0	$\dots$	$s_{Jh} p^*$	0	$0 \; 0 \dots p^{*\backslash}{}_1 \dots 0 \; 0$	$(x_h^{*\backslash} - e_h^{*\backslash} - \sum_{j \in \mathcal{J}} s_{jh} y_j^{*\backslash})$	$0 \dots 0 \dots 0$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots \; \vdots \ddots \vdots \; \vdots \ddots \vdots \; \vdots$	$\vdots$	$\vdots \ddots \vdots \ddots \vdots$
$D_{x_H} u_H(x_H, x_{-H}, y) - \lambda_H p$	$D^2_{x_1 x_H} u_H$	0^T	$\dots$	$D^2_{x_h x_H} u_H$	$-p^{*T}$	$\dots$	$D^2_{x_H} u_H$	0^T	$D^2_{y_1 x_H} u_H$	0^T	$\dots$	$D^2_{y_j x_H} u_H$	0^T	$\dots$	$D^2_{y_J x_H} u_H$	0^T	$\mathbf{0}\, 0^T \dots \mathbf{0}\, 0^T \dots \mathbf{0}\, 0^T$	$\lambda_H^* \widehat{I}^T$	$0^T \dots 0^T \dots 0^T$
$-p \cdot (x_H - e_H - \sum_{j \in \mathcal{J}} s_{jH} y_j)$	0	0	$\dots$	0	0	$\dots$	$-p^*$	0	$s_{1H} p^*$	0	$\dots$	$s_{jH} p^*$	0	$\dots$	$s_{JH} p^*$	0	$0 \; 0 \dots 0 \; 0 \dots p^{*\backslash}{}_1$	$(x_h^{*\backslash} - e_h^{*\backslash} - \sum_{j \in \mathcal{J}} s_{jh} y_j^{*\backslash})$	$0 \dots 0 \dots 0$
$p + \alpha_1 D_{y_1} t_1(y_1, y_{-1}, x)$	$\alpha_1 D^2_{x_1 y_1} t_1$	0^T	$\dots$	$\alpha_1 D^2_{x_h y_1} t_1$	0^T	$\dots$	$\alpha_1 D^2_{x_H y_1} t_1$	0^T	$\alpha_1 D^2_{y_1} t_1$	$D_{y_1} t_1^T$	$\dots$	$\alpha_1 D^2_{y_j y_1} t_1$	0^T	$\dots$	$\alpha_1 D^2_{y_J y_1} t_1$	0^T	$\mathbf{0}\, 0^T \dots \mathbf{0}\, 0^T \dots \mathbf{0}\, 0^T$	$\widehat{I}^T$	$0^T \dots 0^T \dots 0^T$
$t_1(y_1, y_{-1}, x) + b_1$	$D_{x_1} t_1$	0	$\dots$	$D_{x_h} t_1$	0	$\dots$	$D_{x_H} t_1$	0	$D_{y_1} t_1$	0	$\dots$	$D_{y_j} t_1$	0	$\dots$	$D_{y_J} t_1$	0	$0 \; 0 \dots 0 \; 0 \dots 0 \; 0$	0	$1 \dots 0 \dots 0$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots \; \vdots \ddots \vdots \; \vdots \ddots \vdots \; \vdots$	$\vdots$	$\vdots \ddots \vdots \ddots \vdots$
$p + \alpha_j D_{y_j} t_j(y_j, y_{-j}, x)$	$\alpha_j D^2_{x_1 y_j} t_j$	0^T	$\dots$	$\alpha_j D^2_{x_h y_j} t_j$	0^T	$\dots$	$\alpha_j D^2_{x_H y_j} t_j$	0^T	$\alpha_j D^2_{y_1 y_j} t_j$	0^T	$\dots$	$\alpha_j D^2_{y_j} t_j$	$D_{y_j} t_j^T$	$\dots$	$\alpha_1 D^2_{y_J y_j} t_j$	0^T	$\mathbf{0}\, 0^T \dots \mathbf{0}\, 0^T \dots \mathbf{0}\, 0^T$	$\widehat{I}^T$	$0^T \dots 0^T \dots 0^T$
$t_j(y_j, y_{-j}, x) + b_j$	$D_{x_1} t_j$	0	$\dots$	$D_{x_h} t_j$	0	$\dots$	$D_{x_H} t_j$	0	$D_{y_1} t_j$	0	$\dots$	$D_{y_j} t_j$	0	$\dots$	$D_{y_J} t_j$	0	$0 \; 0 \dots 0 \; 0 \dots 0 \; 0$	0	$0 \dots 1 \dots 0$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots \; \vdots \ddots \vdots \; \vdots \ddots \vdots \; \vdots$	$\vdots$	$\vdots \ddots \vdots \ddots \vdots$
$p + \alpha_J D_{y_J} t_J(y_J, y_{-J}, x)$	$\alpha_J D^2_{x_1 y_J} t_J$	0^T	$\dots$	$\alpha_J D^2_{x_h y_J} t_J$	0^T	$\dots$	$\alpha_J D^2_{x_H y_J} t_J$	0^T	$\alpha_J D^2_{y_1 y_J} t_J$	0^T	$\dots$	$\alpha_J D^2_{y_j y_J} t_J$	0^T	$\dots$	$\alpha_J D^2_{y_J} t_J$	$D_{y_J} t_J^T$	$\mathbf{0}\, 0^T \dots \mathbf{0}\, 0^T \dots \mathbf{0}\, 0^T$	$\widehat{I}^T$	$0^T \dots 0^T \dots 0^T$
$t_J(y_J, y_{-J}, x) + b_J$	$D_{x_1} t_J$	0	$\dots$	$D_{x_h} t_J$	0	$\dots$	$D_{x_H} t_J$	0	$D_{y_1} t_J$	0	$\dots$	$D_{y_j} t_J$	0	$\dots$	$D_{y_J} t_J$	0	$0 \; 0 \dots 0 \; 0 \dots 0 \; 0$	0	$0 \dots 0 \dots 1$
$\sum_{h \in \mathcal{H}} x_h^{\backslash} - \sum_{j \in \mathcal{J}} y_j^{\backslash} - \sum_{h \in \mathcal{H}} e_h^{\backslash}$	$\widehat{I}$	0^T	$\dots$	$\widehat{I}$	0^T	$\dots$	$\widehat{I}$	0^T	$-\widehat{I}$	0^T	$\dots$	$-\widehat{I}$	0^T	$\dots$	$-\widehat{I}$	0^T	$-I0^T \dots -I0^T \dots -I0^T$	$\mathbf{0}$	$0^T \dots 0^T \dots 0^T$

Comparison with Mandel (2008)

In this section we compare the results obtained in Chapter 2 with the regularity result in Mandel (2008).

As discussed in Introduction, Mandel (2008) has to enlarge the commodity space treating externalities as additional variables. Moreover, fixing the production technologies, in order to provide a regularity result for almost all initial endowments, the author assumes that a small change in the externalities created by all the agents on an agent does not generate changes in the choices of the latter agent which would in turn involve the exact same change on the behavior of the others, see Assumption TR2 at page 1395. As stressed in Introduction, Assumption TR2 involves endogenous variables, more precisely the derivatives of consumers' demands and firms' supplies.

We now show that Assumption TR2 implicitly involves the Lagrange multipliers, that is the equilibrium prices. Furthermore, we show that this assumption is equivalent to assume that the partial Jacobian matrix of the following components

$$((F_{t,e}^{h.1}(\xi), F_{t,e}^{h.2}(\xi))_{h\in\mathcal{H}}, (F_{t,e}^{j.1}(\xi), F_{t,e}^{j.2}(\xi))_{j\in\mathcal{J}})$$

of the equilibrium function defined in (7), with respect to the variables $((x_h, \lambda_h)_{h\in\mathcal{H}}$, $(y_j, \alpha_j)_{j\in\mathcal{J}})$ has full rank, which implies that the Jacobian matrix $D_\xi F_{t,e}(\xi^*)$ given in Definition 9 is onto.

For simplicity, we prove this equivalence by considering the simple case of one household, two firms and production externalities among firms. In order to state Assumption TR2 of Mandel (2008) in this context, one needs to enlarge the commodity space introducing the additional variables $\eta_j \in \mathbb{R}^2$ for $j = 1, 2$ which represent the externalities created by firm j. Furthermore, one requires that at equilibrium the supply $y_j(p, \eta_{-j})$ of firm j must be equal to the externalities η_j created by firm j, that is

$$\eta_j - y_j(p, \eta_{-j}) = 0$$

A simple case: Assumption TR2, Mandel (2008). The square matrix **C** given below has full rank.

$$\begin{array}{c} \\ \eta_1^1 - y_1^1(p,\eta_2) \\ \eta_1^2 - y_1^2(p,\eta_2) \\ \eta_2^1 - y_2^1(p,\eta_1) \\ \eta_2^2 - y_2^2(p,\eta_1) \end{array} \begin{array}{c} \begin{array}{cccc} \eta_1^1 & \eta_1^2 & \eta_2^1 & \eta_2^2 \end{array} \\ \begin{pmatrix} 1 & 0 & -D_{\eta_2^1} y_1^1(p,\eta_2) & -D_{\eta_2^2} y_1^1(p,\eta_2) \\ 0 & 1 & -D_{\eta_2^1} y_1^2(p,\eta_2) & -D_{\eta_2^2} y_1^2(p,\eta_2) \\ -D_{\eta_1^1} y_2^1(p,\eta_1) & -D_{\eta_1^2} y_2^1(p,\eta_1) & 1 & 0 \\ -D_{\eta_1^1} y_2^2(p,\eta_1) & -D_{\eta_1^2} y_2^2(p,\eta_1) & 0 & 1 \end{pmatrix} \end{array}$$

We claim that the assumption above is equivalent to assume that the matrix **A** defined below has full rank. The matrix **A** is nothing else that the partial

Jacobian matrix of the components $(F_{t,e}^{j.1}(\xi), F_{t,e}^{j.2}(\xi))_{j=1,2}$ of the equilibrium function defined in (7), with respect to the variables $(y_j, \alpha_j)_{j=1,2}$.

$$
\begin{array}{cccccc}
y_1^1 & y_1^2 & \alpha_1 & y_2^1 & y_2^2 & \alpha_2
\end{array}
$$

$$
\begin{array}{c}
p^1 + \alpha_1 D_{y_1^1} t_1(y_1, y_2) \\
p^2 + \alpha_1 D_{y_1^2} t_1(y_1, y_2) \\
t_1(y_1, \eta_2) \\
p^1 + \alpha_2 D_{y_2^1} t_2(y_2, y_1) \\
p^2 + \alpha_2 D_{y_2^2} t_2(y_2, y_1) \\
t_2(y_2, \eta_1)
\end{array}
\begin{pmatrix}
\alpha_1 D^2_{y_1^1} t_1 & \alpha_1 D^2_{y_1^2 y_1^1} t_1 & D_{y_1^1} t_1 & \alpha_1 D^2_{y_2^1 y_1^1} t_1 & \alpha_1 D^2_{y_2^2 y_1^1} t_1 & 0 \\
\alpha_1 D^2_{y_1^1 y_1^2} t_1 & \alpha_1 D^2_{y_1^2} t_1 & D_{y_1^2} t_1 & \alpha_1 D^2_{ya_2^1 y_1^2} t_1 & \alpha_1 D^2_{y_2^2 y_1^2} t_1 & 0 \\
D_{y_1^1} t_1 & D_{y_1^2} t_1 & 0 & D_{y_2^1} t_1 & D_{y_2^2} t_1 & 0 \\
\alpha_2 D^2_{y_2^1} t_2 & \alpha_2 D^2_{y_2^2 y_2^1} t_2 & 0 & D_{y_2^1} t_2 & \alpha_2 D^2_{y_1^1 y_2^1} t_2 & \alpha_2 D^2_{y_1^2 y_2^1} t_2 \\
\alpha_2 D^2_{y_2^1 y_2^2} t_2 & \alpha_2 D^2_{y_2^2} t_2 & 0 & D_{y_2^2} t_2 & \alpha_2 D^2_{\eta_1^1 y_2^2} t_2 & \alpha_2 D^2_{\eta_1^2 y_2^2} t_2 \\
D_{y_1^1} t_2 & D_{y_1^2} t_2 & 0 & D_{y_2^1} t_2 & D_{y_2^2} t_2 & 0
\end{pmatrix}
$$

In order to prove that matrix **A** has full rank if and only if matrix **C** has full rank, we consider an auxiliary matrix **B** which is the partial Jacobian matrix of the following system.

$$
\begin{cases}
(h.1) \quad D_{x_h} u_h(x_h) - \lambda_h p = 0 \\
(h.2) \quad -p \cdot (x_h - e_h - \sum_{j=1}^{2} y_j) = 0 \\
(j.1) \quad p + \alpha_j D_{y_j} t_j(y_j, y_{-j}) = 0, \quad j = 1, 2 \\
(j.2) \quad t_j(y_j, y_{-j}) = 0, \quad j = 1, 2 \\
(M) \quad x_h^{\setminus} - \sum_{j=1,2} y_j^{\setminus} - e_h^{\setminus} = 0 \\
(E) \quad \eta_j - y_j = 0, \quad j = 1, 2
\end{cases}
$$

We recall that equation (E) means that at equilibrium the supply y_j of firm j must be equal to the externalities η_j created by firm j. The matrix **B** given below is the partial Jacobian matrix of the left side of equations $(j.1)$, $(j.2)$, (M) and (E) with respect to the variables $(y_j, \alpha_j, \eta_j)_{j=1,2}$.

$$
\begin{array}{c}
\\
p^1 + \alpha_1 D_{y_1^1} t_1(y_1, \eta_2) \\
p^2 + \alpha_1 D_{y_1^2} t_1(y_1, \eta_2) \\
t_1(y_1, \eta_2) \\
p^1 + \alpha_2 D_{y_2^1} t_2(y_2, \eta_1) \\
p^2 + \alpha_2 D_{y_2^2} t_2(y_2, \eta_1) \\
t_2(y_2, \eta_1) \\
\eta_1^1 - y_1^1 \\
\eta_1^2 - y_1^2 \\
\eta_2^1 - y_2^1 \\
\eta_2^2 - y_2^2
\end{array}
\quad
\begin{array}{cccccccccc}
y_1^1 & y_1^2 & \alpha_1 & y_2^1 & y_2^2 & \alpha_2 & \eta_1^1 & \eta_1^2 & \eta_2^1 & \eta_2^2 \\
\end{array}
$$

$$
\begin{pmatrix}
\alpha_1 D^2_{y_1^1} t_1 & \alpha_1 D^2_{y_1^2 y_1^1} t_1 & D_{y_1^1} t_1 & 0 & 0 & 0 & 0 & 0 & \alpha_1 D^2_{\eta_2^1 y_1^1} t_1 & \alpha_1 D^2_{\eta_2^2 y_1^1} t_1 \\
\alpha_1 D^2_{y_1^1 y_1^2} t_1 & \alpha_1 D^2_{y_1^2} t_1 & D_{y_1^2} t_1 & 0 & 0 & 0 & 0 & 0 & \alpha_1 D^2_{\eta_2^1 y_1^2} t_1 & \alpha_1 D^2_{\eta_2^2 y_1^2} t_1 \\
D_{y_1^1} t_1 & D_{y_1^2} t_1 & 0 & 0 & 0 & 0 & 0 & 0 & D_{\eta_2^1} t_1 & D_{\eta_2^2} t_1 \\
0 & 0 & 0 & \alpha_2 D^2_{y_2^1} t_2 & \alpha_2 D^2_{y_2^2 y_2^1} t_2 & D_{y_2^1} t_2 & \alpha_2 D^2_{\eta_1^1 y_2^1} t_2 & \alpha_2 D^2_{\eta_1^2 y_2^1} t_2 & 0 & 0 \\
0 & 0 & 0 & \alpha_2 D^2_{y_2^1 y_2^2} t_2 & \alpha_2 D^2_{y_2^2} t_2 & D_{y_2^2} t_2 & \alpha_2 D^2_{\eta_1^1 y_2^2} t_2 & \alpha_2 D^2_{\eta_1^2 y_2^2} t_2 & 0 & 0 \\
0 & 0 & 0 & D_{y_2^1} t_2 & D_{y_2^2} t_2 & 0 & D_{\eta_1^1} t_2 & D_{\eta_1^2} t_2 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

It is easy to check that matrix **A** has full rank if and only if matrix **B** has full rank. So, in order to prove our claim it is enough to show that matrix **B** has full rank if and only if matrix **C** has full rank.

First of all, one should notice that $D_{\eta_{-j}} y_j(p, \eta_{-j})$ given in matrix **C** can be obtained differentiating the Karush-Kuhn-Tucker necessary and sufficient conditions associated with the profit maximization problem of firm j and using the Cramer rule. Now, consider the square submatrix B_1 obtained by taking the first three rows and the first three columns of matrix **B**. The submatrix B_1 has full rank.[11] So, multiplying the first three rows of matrix **B** by the inverse matrix of B_1 and summing row 1 to row 7 and row 2 to row 8 one gets the following matrix.[12]

---

[11] We recall that a differentiably strictly quasi-concave function with gradient different from zero has a bordered Hessian with determinant different from zero (see Points 3 and 4 of Assumption 1).

[12] One should notice that also $D_{\eta_{-j}} \alpha_j(p, \eta_{-j})$ is obtained differentiating the Karush-Kuhn-Tucker necessary and sufficient conditions associated with the profit maximization problem of firm j and using the Cramer rule.

$$
\begin{array}{cccccccccc}
y_1^1 & y_1^2 & \alpha_1 & y_2^1 & y_2^2 & \alpha_2 & \eta_1^1 & \eta_1^2 & \eta_2^1 & \eta_2^2
\end{array}
$$
$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -D_{\eta_2^1} y_1^1(p,\eta_2) & -D_{\eta_2^2} y_1^1(p,\eta_2) \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -D_{\eta_2^1} y_1^2(p,\eta_2) & -D_{\eta_2^2} y_1^2(p,\eta_2) \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -D_{\eta_2^1} \alpha_1^1(p,\eta_2) & D_{\eta_2^2} \alpha_1^1(p,\eta_2) \\
0 & 0 & 0 & \alpha_2 D^2_{y_2^1} t_2 & \alpha_2 D^2_{y_2^2 y_2^1} t_2 & D_{y_2^1} t_2 & \alpha_2 D^2_{\eta_1^1 y_2^1} t_2 & \alpha_2 D^2_{\eta_1^2 y_2^1} t_2 & 0 & 0 \\
0 & 0 & 0 & \alpha_2 D^2_{y_2^1 y_2^2} t_2 & \alpha_2 D^2_{y_2^2} t_2 & D_{y_2^2} t_2 & \alpha_2 D^2_{\eta_1^1 y_2^2} t_2 & \alpha_2 D^2_{\eta_1^2 y_2^2} t_2 & 0 & 0 \\
0 & 0 & 0 & D_{y_2^1} t_2 & D_{y_2^2} t_2 & 0 & D_{\eta_1^1} t_2 & D_{\eta_1^2} t_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -D_{\eta_2^1} y_1^1(p,\eta_2) & -D_{\eta_2^2} y_1^1(p,\eta_2) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -D_{\eta_2^1} y_1^2(p,\eta_2) & -D_{\eta_2^2} y_1^2(p,\eta_2) \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

Similarly, consider the square submatrix B_2 obtained by taking the 4th, the 5th and the 6th rows and the 4th, the 5th and the 6th columns. The submatrix B_2 has full rank. So, multiplying the 4th, the 5th and the 6th rows of the previous matrix by the inverse matrix of B_2 and summing row 4 to row 9 and row 5 to row 10 one gets the following matrix.

$$
\begin{array}{cccccccccc}
y_1^1 & y_1^2 & \alpha_1 & y_2^1 & y_2^2 & \alpha_2 & \eta_1^1 & \eta_1^2 & \eta_2^1 & \eta_2^2
\end{array}
$$
$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -D_{\eta_2^1} y_1^1(p,\eta_2) & -D_{\eta_2^2} y_1^1(p,\eta_2) \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -D_{\eta_2^1} y_1^2(p,\eta_2) & -D_{\eta_2^2} y_1^2(p,\eta_2) \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -D_{\eta_2^1} \alpha_1^1(p,\eta_2) & D_{\eta_2^2} \alpha_1^1(p,\eta_2) \\
0 & 0 & 0 & 1 & 0 & 0 & -D_{\eta_1^1} y_2^1(p,\eta_1) & -D_{\eta_1^2} y_2^1(p,\eta_1) & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -D_{\eta_1^1} y_2^2(p,\eta_1) & -D_{\eta_1^2} y_2^2(p,\eta_1) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -D_{\eta_1^1} \alpha_2(p,\eta_1) & -D_{\eta_1^2} \alpha_2(p,\eta_1) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -D_{\eta_2^1} y_1^1(p,\eta_2) & -D_{\eta_2^2} y_1^1(p,\eta_2) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -D_{\eta_2^1} y_1^2(p,\eta_2) & -D_{\eta_2^2} y_1^2(p,\eta_2) \\
0 & 0 & 0 & 0 & 0 & 0 & -D_{\eta_1^1} y_2^1(p,\eta_1) & -D_{\eta_1^2} y_2^1(p,\eta_1) & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -D_{\eta_1^1} y_2^2(p,\eta_1) & -D_{\eta_1^2} y_2^2(p,\eta_1) & 0 & 1
\end{pmatrix}
$$

Finally, it is easy to check that the matrix above has full rank if and only if matrix **C** has full rank. Consequently, **B** has full rank if and only if **C** has full rank which completes the proof of our claim.

References

Allen, B., 1981. Utility perturbations and the equilibrium price set. *Journal of Mathematical Economics* 8, 277-307.

Equilibrium Theory. Academic Press, Boston.

Bonnisseau, J.-M., del Mercato, E. L., 2010. Externalities, consumption constraints and regular economies. *Economic Theory* 44, 123-147.

Citanna, A., Kajii, A., Villanacci, A., 1998. Constrained suboptimality in incomplete markets: a general approach and two applications, *Economic Theory* 11, 495-522.

del Mercato, E. L., Platino, V., 2010. Externalities in production economies and existence of competitive equilibria: A differentiable approach, Working Paper.

del Mercato, E. L., Platino, V., 2011. Regularity of competitive equilibria in a production economy with externalities. Working Paper.

Golubitsky, M., Guillemin, V., 1973. Stable mappings and their singularities. Springer-Verlag, New York.

Guillemin, V., Pollack, A., 1974. Differential topology. Prentice-Hall, Englewood Cliffs, NJ.

Hirsch, M. W., 1976. Differential topology. Springer-Verlag, New York.

goods in a production economy with widespread externalities. *Journal of Mathematical Economics* 44, 1364-1378.

Mandel, A., 2008. An index formula for production economies with externalities, *Journal of Mathematical Economics* 44, 1385–1397.

Mas-Colell, A., 1985. The theory of general economic equilibrium. A differentiable approach. Cambridge University Press.

Platino, V., 2010. Existence of equilibria in a general equilibrium model with production and externalities: A differentiable approach, Chapter 1 in: *Existence, regularity and testability results in economic models with externalities*, Doctoral Thesis, Universitá degli Studi di Salerno.

Smale, S., 1974. Global analysis and economics. II A: Extension of a theorem of Debreu, *Journal of Mathematical Economics* 1, 1–14.

Villanacci, A., Carosi, L., Benevieri, P., Battinelli, A., 2002. Differential Topology and General Equilibrium with Complete and Incomplete Markets. Kluwer Academic Publishers.

Chapter 3

Testable restrictions in a specific model with externalities and public goods: The collective consumption model [1]

Vincenzo Platino [2]

DISES, Universitá degli Studi di Salerno, Italy

Key words: Collective household models, intra-household allocation, revealed preferences, nonparametric analysis.

1 Long abstract

We consider the *collective consumption model* with the two intra-household members. We present the "non-parametric" methodology to test two benchmark cases of the collective consumption model, that is

- the case where all goods are publicly consumed within the household and
- the case where all goods are privately consumed within the household and the individual preferences are egoistic.

Differently from the the previous literature, we find that the private and public nature of consumption does have testable implications, even if one only observes the aggregate group consumption. We believe that such "non-parametric" approach is able to obtain stronger testability conclusions since it focuses on conditions which involves personalized prices à la Lindahl and personalized consumptions. Importantly, we do not require personalized prices and personalized consumptions to be observable. Also, we derive the minimum

[1] This Chapter is based on Cherchye, De Rock and Platino (2010).

[2] Dipartimento di Scienze Economiche e Statistiche (DISES), Universitá degli Studi di Salerno, Via Ponte don Melillo, 84084 Fisciano (SA), Italy. E-mail: vincenzo.platino@gmail.com.

number of observations that enables us to distinguish between the collective model and the two benchmark cases.

Chapter 3 is organized as follows. Section2 sets the stage by briefly recapturing "non-parametric" tests of the unitary model. Section 3 introduces the necessary and sufficient conditions for the collective rationalization. Section 4 focuses on the two benchmark cases. In Section 5 we describe our main results and Section 6 contains some concluding remarks.

In Appendix one finds the proofs of the results.

2 The Unitary Model

The unitary approach consider a household as a single decision maker that maximizes its utility function subject to the budget constraint. Suppose to observe T choices of n-valued bundles. For each observation t, $q_t \in \mathbb{R}^n_+$ denotes the consumption bundle and $p_t \in \mathbb{R}^n_{++}$ the associate price vector. Let $S := \{(p_t, q_t); t = 1, \ldots, T\}$ be the set of observations.

In this Section, we want to investigate if the data set has been generated by a concave, continuous and monotonically increasing utility function U. We start defining the concept of unitary rationalization of a data set as follows:

Definition 1 *Let $S = \{(p_t, q_t); t = 1, \ldots, T\}$ be a set of observations. A utility function U provides a unitary rationalization of S if for each q_t*

$$U(q_t) \geq U(q)$$

for all $q \in \mathbb{R}^n_+$ such that $p_t \cdot q \leq p_t \cdot q_t$.

A unitary rationalization of the data set S requires that, for each observation t, quantity q_t maximizes the utility function subject to the budget constraint $p_t \cdot q_t$.

Varian (1982) shows that there exists a concave, continuous and monotonically increasing utility function that rationalizes the data set S if and only if the the data set S satisfies the *Generalize Axiom of Revealed Preference* (*GARP*).

Definition 2 (*GARP*) *Let $S = \{(p_t, q_t); t = 1, \ldots, T\}$ be a set of observations. The set S satisfies the Generalize Axiom of Revealed Preference (*GARP*) if there exist relations R_0, R such that*

(i) if $p_s \cdot q_s \geq p_s \cdot q_t$ then $q_s R_0 q_t$;

(ii) if $q_s R_0 q_u, q_u R_0 q_v, \ldots, q_z R_0 q_t$ *for some (possibly empty) sequence* (u, v, ..., z) *then* $q_s R q_t$*;*

(iii) if $q_s R q_t$ *then* $p_t \cdot q_t \leq p_t \cdot q_s$.

Rule *(i)* states that the consumption vector q_s is 'direct revealed' over q_t (i.e. $q_s R_0 q_t$) if q_s was chosen when the consumption vector q_t was available (i.e. $p_s \cdot q_s \geq p_s \cdot q_t$). R is the transitive closure of R_0 and it is known as the 'revealed preference' relation; see rule *(ii)*. Finally rule *(iii)* states that each consumption vector q_t is expenditure minimizing (i.e. $p_t \cdot q_t \leq p_t \cdot q_s$) with respect to all the revealed preferred vectors q_s (i.e. $q_s R q_t$).

This leads to the following proposition:

Proposition 3 *Let* $S = \{(p_t, q_t); t = 1, \ldots, T\}$ *be a set of observations. The following conditions are equivalent:*

(i) there exists a concave, continuous and monotonically increasing utility function U *that provide a unitary rationalization of* S;

(ii) The data set S *satisfies* GARP.

Finally, one should observe that unitary rationality can be tested if we have a data set with at least two observations and two goods.

A data set S with only one observation and/or one good always satisfies *GARP*. More precisely, if $T = 1$ it is not possible to specify the set of the revealed preferred bundles, that is the 'better than' set. Therefore it is not possible to reject *GARP*. Moreover, if $q_t \in \mathbb{R}_+$ all the scalar products in Definition 2 are scalar multiplications. This implies that $q_s R_0 q_t$ if and only if $q_s \geq q_t$ and obviously also $q_s R q_t$ if and only if $q_s \geq q_t$. Thus, the quantity q_t is always cost minimizing. It follows that it is not possible to reject *GARP*.

3 General Collective Consumption Model

The collective approach assumes that members within a household are heterogeneous and have own preferences. Browing and Chiappori (1998) consider a non-unitary household model in which the decisions taken by the two intra-household members are Pareto efficient, without specifying a particular point on the Pareto frontier. The authors assume that all the goods can be consumed privately, publicly or both, yet only prices and aggregate demand with respect to some power distribution between the two intra-household members are observed. Using a parametric approach, Browning and Chiappori (1998) prove that the aggregate demand is compatible with the Pareto optimal decision behavior if it satisfies some restrictions on a Pseudo-Slutsky matrix. The Pseudo-Slutsky matrix is the sum of the classical Slutsky matrix which measures the change in demand induced by the variation of prices and income, and another matrix which measures the change in demand induced by the variation of power distribution. In addition, Browning and Chiappori (1998) show that a collective model with two intra-household members can be tested if we have a data set with at least five goods.

This section recaptures the principal results of Cherchye, De Rock, Vermeulen (2007) . More precisely, the authors, using a nonparametric approach, provide testable conditions involving personalized prices à la Lindahl and personalized consumptions. However, the authors do not require that personalized prices and personalized consumptions are observable data.

3.1 General Collective Rationality

We consider one household with two members (A and B) that purchases a vector of goods $q \in \mathbb{R}^n_+$ with corresponding prices $p \in \mathbb{R}^n_{++}$. All goods can be consumed privately, publicly or both.
We assume that the empirical analyst has no information on the decomposition of the observed quantities q into the bundles of private and public consumptions. Therefore, we need to introduce (unobserved) *feasible personalized quantities* x that comply with the (observed) aggregate quantities q. More formally, we define:

$$x = (x^A, x^B, x^G) \text{ with } x^A, x^B, x^G \in \mathbb{R}^n_+, \quad \text{and } x^A + x^B + x^G = q \qquad (1)$$

The feasible personalized quantities x capture a possible feasible decomposition of the (observed) aggregate consumption q in the (unobserved) private quantities x^A and x^B and in the public consumption x^G.

Suppose to observe T choices of n-valued bundles. For each observation t the vector $q_t \in \mathbb{R}^n_+$ records the quantities chosen by the group under the prices $p_t \in \mathbb{R}^n_{++}$. We denote with $S = \{(p_t, q_t); t = 1, \dots, T\}$ the corresponding set of T observations (i.e. the data set). Collective rationality in terms of the general collective model (general-CR) of a set of observations S requires the existence of utility functions U^A and U^B such that each observed quantity bundle can be characterized as Pareto efficient. Thus, we get the following definition:

Definition 4 (general-CR) *Let $S = \{(p_t, q_t); t = 1, \dots, T\}$ be a set of observations. A pair of utility functions U^A and U^B provides a general-CR (i.e. a collective rationalization in terms of the general collective model) of S if for each observation t there exist feasible personalized quantities $x_t = (x^A_t, x^B_t, x^G_t)$*

and $\mu_t \in \mathbb{R}_{++}$ *such that:*

$$U^A(x_t) + \mu_t U^B(x_t) \geq U^A(x) + \mu_t U^B(x)$$

for all $x = (x^A, x^B, x^G)$ *with* $x^c \in \mathbb{R}^n_+$, $c = A, B, G$ *and* $p_t \cdot (x^A + x^B + x^G) \leq p_t \cdot q_t$.

The weight μ_t represents the relative bargaining power of member B with respect to member A. It reflects the Pareto efficient characterization of the optimal intra-household allocation. A general-CR of the data set S requires the existence, for each observation t, of feasible personalized quantities x_t that maximize a weighted sum of the intra-household member utilities subject to the household budget constraint $p_t \cdot q_t$.[3]

3.2 Revealed preference characterization

Following Cherchye, De Rock, Vermeulen (2007), we define *feasible personalized prices* (p_t^A, p_t^B) for the (observed) prices p_t as follows

$$\begin{gathered} p_t^A = (p_t^{AA}, p_t^{AB}, p_t^{AG}) \text{ and } p_t^B = (p_t - p_t^{AA}, p_t - p_t^{AB}, p_t - p_t^{AG}) \\ \text{with } p_t^{AA}, p_t^{AB}, p_t^{AG} \in \mathbb{R}^n_+ \text{ and } p_t^c \leq p_t, \;\; c = A, B, G \end{gathered} \tag{2}$$

where p_t^A and p_t^B captures the fractions of the price for the feasible personalized quantities x_t paid respectively by members A and B. More precisely, p_t^{AA} and p_t^{AB} are respectively the price paid by member A for the own private consumption and for the private consumption of member B, and p_t^{AG} is the price paid by member A for the public consumption. The interpretation of p_t^B is similar. One should notice that p_t^A and p_t^B can be interpreted as the Lindahl prices of member A and B respectively.

Proposition 6 states that collective rationality requires *GARP* consistency for each individual member:

Definition 5 *Consider feasible personalized prices and quantities for a set of observations* $S = \{(p_t, q_t); t = 1, \ldots, T\}$. *For* $m = A, B$, *the set* $\{(p_t^m, x_t);$ $t = 1, ..., T\}$ *satisfies* GARP *if there exist relations* R_0^m, R^m *such that*

(i) if $p_s^m \cdot x_s \geq p_s^m \cdot x_t$ *then* $x_s R_0^m x_t$;

[3] It is immediate to note that collective rationalization of a data set S is more general that unitary rationalization. In fact, if $\mu_t = 0$ and $x_t^A = q_t$, for each observation t, we get back to unitary rationalization. However, following Cherchye, De Rock, Vermeulen (2007), we did not allow for this possibility. Therefore, we assume $\mu_t \in \mathbb{R}_{++}$ for each observation t.

(ii) if $x_s R_0^m x_u$, $x_u R_0^m x_v$, $\ldots$, $x_z R_0^m x_t$ *for some (possibly empty) sequence* (u, v, ..., z) *then* $x_s R^m x_t$;

(iii) if $x_s R^m x_t$, *then* $p_t^m \cdot x_t \leq p_t^m \cdot x_s$.

The following Proposition is due to Cherchye, De Rock, Vermeulen (2007). It provides the necessary and sufficient conditions for a collective rationalization of the data set S in terms of feasible personalized prices and quantities.

Proposition 6 *Let* $S = \{(p_t, q_t); t = 1, \ldots, T\}$ *be a set of observations. The following conditions are equivalent:*

(i) there exists a combination of concave, continuous and monotonically increasing utility functions U^A *and* U^B *that provide a general-CR of* S;

(ii) there exist feasible personalized prices and quantities such that for each member $m = 1, 2$, *the set* $\{(p_t^m, x_t); t = 1, ..., T\}$ *satisfies* GARP.

Proof. See Cherchye, De Rock, Vermeulen (2007), Proposition 1, page 557. ■

Collective rationality (i.e. Proposition 6) differs from unitary rationality (i.e. Proposition 1), since it requires *GARP* consistency for each intra-household member m in terms of the (unobserved) feasible personalized prices and quantities (i.e. p_t^m and x_t, $m = A, B$) and not at the (observed) aggregate level S.

One should notice that, for each observation t, it is possible to construct infinite feasible personalized prices and quantities, (p_t^A, p_t^B, x_t). Therefore, collective rationality requires that there exists at least one *feasible personalized data set* $\widehat{S} := \{(p_t^A, p_t^B, x_t); t = 1, \ldots, T\}$ such that the the necessary and sufficient conditions given in Proposition 6 are satisfied.

Since the necessary and sufficient conditions given in Proposition 6 are expressed in terms of the (unobservable) variables, it is however difficult to use them in empirical work. Cherchye, De Rock, Vermeulen (2007) construct necessary and sufficient conditions expressed in terms of the (observed) aggregate prices and quantities S.[4]

Finally, one should observe that collective rationality can be tested if we have a data set with at least three observations and three goods.

A data set S with two observations and/or two goods always fulfils the con-

[4] See Cherchye, De Rock, Vermeulen (2007), Proposition 2, page 561 and Proposition 4, page 564.

ditions given in Proposition 2. Suppose to have $T = 2$ and $n \geq 2$. In this case, one can always assign one observation to each individual, say for example $x_1^A = q_1$, $x_2^B = q_2$, and consider the following personalized prices $p_t^{AA} = p_t$, $p_t^{AB} = p_t^{AG} = 0$ for each observation $t = 1, 2$. Using this setting, *GARP* cannot be rejected. In fact one always has $x_1 R_0^A x_2$ (or equivalently $p_1^A \cdot q_1 = p_1 \cdot q_1 \geq 0$) and $0 \leq p_2^A \cdot q_1 = p_2 \cdot q_1$. Similar arguments hold for member B. Suppose now to have $T \geq 2$ and $n = 2$. In this case, we can always assign one good to each individual, say for example $(x_t^A)_1 = (q_t)_1$, $(x_t^B)_2 = (q_t)_2$, and define $p_t^{AA} = p_t$, $p_t^{AB} = p_t^{AG} = 0$ for each observation $t = 1, \ldots, T$. Since all scalar products are scalar multiplications, it is easy to show that it is not possible to reject *GARP*.

4 Two benchmark models

In the previous Section, we have considered a model that takes into account intra-household externalities and public consumption. In this section we will focus on two benchmark cases. Specifically, we will consider a model in which all the the goods are publicly consumed and another model in which all the goods are privately consumed and individuals have egoistic preferences (i.e. egoistic model). Due to complexity of the general model, these two special cases are mostly used in standard economic theory.

Chiappori and Ekeland (2006), using a parametric approach, show that the public or private nature of household consumption does not have testable implications.

4.1 All Goods are Publicly Consumed

In the first benchmark we assume that all the consumption is public. We formalize this by assuming individuals preferences that are represented by a concave, continuous and monotonically increasing utility functions $U^m_{pub}(x^G) := U^m(0, 0, x^G)$. Clearly, in this case we have $x^G = q$ (or $x^A + x^B = 0$). So, the actual personalized quantities are effectively observed. Thereby, we define collective rationality in terms of the collective model with only public consumption (public-CR) as follows:

Definition 7 (public-CR) *Let $S = \{(p_t, q_t); t = 1, \ldots, T\}$ be a set of observations. A pair of utility functions U^A_{pub} and U^B_{pub} provides a public-CR of S (i.e. a collective rationalization in terms of the collective model with only public consumption), if for each observation t there exists $\mu_t \in \mathbb{R}_{++}$ such that*

$$U^A_{pub}(q_t) + \mu_t U^B_{pub}(q_t) \geq U^A_{pub}(q) + \mu_t U^B_{pub}(q)$$

for all $q \in \mathbb{R}^n_+$ *and* $p_t \cdot q \leq p_t \cdot q_t$.

A public rationalization of the data set S requires that each consumption vector q_t maximize a weighted sum of the intra-household member utilities subject to the household budget constraint $p_t \cdot q_t$.

As for the collective model, the analyst does not observe the fraction of the price paid by the two members for the quantities q_t. To characterize non-parametric conditions for public rationalization, we need to define feasible personalized prices (p_t^A, p_t^B). Obviously, the Lindahl prices p_t^{AG} and $p_t - p_t^{AG}$ are the only relevant components to get the desired characterization.

Proposition 8 *Let* $S = \{(p_t, q_t); t = 1, \ldots, T\}$ *be a set of observations. The following conditions are equivalent:*

(i) there exists a combination of concave, continuous and monotonically increasing utility functions U^A_{pub} *and* U^B_{pub} *that provides a* public-CR *of* S;

(ii) there exist feasible personalized prices and quantities, with $x_t^A = x_t^B = 0$, *such that for each member* $m = A, B$, *the set* $\{(p_t^m, x_t);\ t = 1, ..., T\}$ *satisfies* GARP.

The above proposition follows directly from Proposition 6. A public-CR requires *GARP* consistency for each intra-household member m in terms of (unobserved) feasible personalized prices and (observed) quantities (i.e. p_t^m and q_t, $m = A, B$).

Proposition 8 is different from Proposition 6. More precisely, the conditions for general-CR are nonlinear since all scalar multiplications are expressed in terms of (unobserved) feasible personalized prices (p_t^A, p_t^B) and quantities, q_t. Differently, the conditions for public-CR are linear since all scalar multiplications are in terms of (unobserved) feasible personalized prices (p_t^A, p_t^B) and (observed) quantities q_t. This difference suggests that these two models have different testable implications.

4.2 Egoistic Model

In the second benchmark case, all the goods are privately consumed i.e. $x^A + x^B = q$. In addition, the individuals have egoistic preferences, which implies that they only care for their own consumption (i.e. no consumption externalities). We formalize this by assuming that individual preferences are represented by concave, continuous and monotonically increasing utility functions $U^A_{ego}(x^A) := U^A(x^A, 0, 0)$ and $U^B_{ego}(x^B) := U^B(0, x^B, 0)$. The corresponding concept of collective rationality in terms of the collective model with all private consumption is the following:

Definition 9 *A pair of utility functions* U^A_{ego} *and* U^B_{ego} *provides an egoistic-CR of* S *(i.e. a collective rationalization in terms of the collective model with all consumption private and egoistic preferences), if for each observation* t *there exist feasible personalized quantities* x_t*, with* $x^G_t = 0$ *and* $\mu_t \in \mathbb{R}_{++}$ *such that*

$$U^A_{ego}(x^A_t) + \mu_t U^B_{ego}(x^B_t) \geq U^A_{ego}(x^A) + \mu_t U^B_{ego}(x^B)$$

for all $x = (x^A, x^B, 0)$ *with* $x^m \in \mathbb{R}^n_+$*,* $m = A, B$ *and* $p_t \cdot (x^A + x^B) \leq p_t \cdot q_t$*.*

Egoistic-CR of the data set S requires that there exist, for each observation t, 'private' quantities x^A_t and x^B_t that maximize a weighted sum of the intra-household member utilities subject to the household budget constraint $p_t \cdot q_t$.

The econometrician does not observe the true private quantities (q^A_t, q^B_t). Differently from the previous case, the (observed) prices p_t are exactly the prices paid for each individual for own consumption. More precisely, for each observation t, $p^{AA}_t = p_t$, and $p^{AB}_t = 0$. Proposition 10 gives the necessary and sufficient conditions for a egoistic-CR of the data set S:

Proposition 10 *Let* $S = \{(p_t, q_t); t = 1, \ldots, T\}$ *be a set of observations. The following conditions are equivalent:*

(i) there exists a combination of concave, continuous and monotonically increasig utility functions U^A_{ego} *and* U^B_{ego} *that provide an* egoistic-CR *of* S;

(ii) there exist feasible personalized prices, with $p^{AA}_t = p_t$ *and* $p^{AB}_t = 0$*, and feasible personalized quantities, with* $x^G_t = 0$*, such that for each member* $m = A, B$*, the set* $\{(p^m_t, x_t);\ t = 1, ..., T\}$ *satisfies* GARP.

Egoistic-CR (Proposition 10) requires *GARP* consistency at individual level in terms of the (observed) prices p_t and the (unobserved) feasible personalized quantities x_t. The necessary and sufficient conditions are different with respect to those of general-CR and public-CR. In particular these conditions are linear in terms of the (unobserved) feasible personalized quantities x_t. Therefore, this difference suggests that the three models have different testable implications.

5 Testing the nature of goods

In this section we first provide two examples that show that the nature of goods is testable even if one observes only aggregate data. Our results thus imply that consistency with the general model does not necessarily imply consistency with the two benchmark models. Secondly, our examples also show that the two benchmark models are independent from each other. More precisely, if the aggregate data set is consistent with one of the two benchmark models, it need not be consistent with the other benchmark model.

5.1 Collective Rationality does not imply public-CR

In this subsection we provide an example that contains a data set that satisfies general-CR but not public-CR.

Example 1 *Suppose that the data set S contains the following 3 observations of bundles consisting of 3 quantities:*

$$q_1 = (5, 2, 2),\ q_2 = (2, 5, 2),\ q_3 = (2, 2, 5)$$
$$p_1 = (4, 1, 1),\ p_2 = (1, 4, 1),\ p_3 = (1, 1, 4)$$

This data set S satisfies the conditions in Proposition 6 (i.e. there exists a general-CR), but it fails to satisy the conditions in Proposition 8 (i.e. there does not exist a public-CR).

See the Appendix for the explanation of the example.

This example leads to two remarkable results. Firstly, as discussed in the introduction, it contrasts with the results of Chiappori and Ekeland (2006). These authors, following a parametric approach, show that the general collective model and the collective model with only public consumption are indistinguishable if one only observes aggregate data. More precisely, the authors show that, when only aggregate data are available, the general collective consumption model has exactly the same testability implications. Example 1 shows that this is no longer the case if one adopts our revealed preference approach.

Secondly, this example shows that a data set with only three goods and three observations is enough to distinguish between the general collective model and the collective model with only public consumption. Moreover, one should notice that, as the general collective consumption model, it is not possible to reject public-CR if the number of observations or the number of goods is smaller than three.

Suppose $T = 2$ and $n \geq 2$. One can always consider the following personalized prices: $p_1^{AG} = p_1$ and $p_2^{AG} = 0$. It is easy to verify that, using these feasible personalized prices, *GARP* conditions in Proposition 8 cannot be rejected. Next, if $n = 2$ and $T \geq 2$, one can always suppose that each individual pay for one good, say for example $(p_t^{AG})_1 = (p_1)_1$ and $(p_t^{AG})_2 = 0$, for each observation $t = 1, ..., T$. Again, it is immediate to verify that it is a solution for the *GARP* conditions in Proposition 8.

Thus, three observations and three goods represent the lower bounds for the collective models to have testable implications.

Finally, it should be noted that the parametric approach needs a data set with at least five goods to test the collective consumption model characterized in Propositions 8; see Browning and Chiappori (1998) and Chiappori and Ekeland (2006). Thus, our revealed preference approach requires a smaller number of goods than the parametric approach.

5.2 General-CR does not imply egoistic-CR

In this section, we provide an example with contains a data set that satisfies general-CR but not an egoistic-CR.

Example 2 *Suppose that the data set S contains the following 4 observations of bundles consisting of 4 quantities:*

$$q_1 = (1,0,0,0),\ q_2 = (0,1,0,0),\ q_3 = (0,0,1,0),\ q_4 = (0,0,0,1)$$
$$p_1 = (7,4,4,4),\ p_2 = (4,7,4,4),\ p_3 = (4,4,7,4),\ p_4 = (4,4,4,7)$$

This data set S satisfies the conditions in Proposition 6 (i.e. there exists a general-CR), but it rejects the conditions in Proposition 10 (i.e. there does not exist an egoistic-CR).

See the Appendix for the explanation of the example.

As for the previous examples, we have two remarks. First, in contrast to the differentiable approach, our 'revealed preference' methodology makes it possible to distinguish between the general collective model and the egoistic model. Thus, we can conclude that the private nature of the goods is testable. Secondly, in our example we considered a data set with four observations and four goods. However, we conjecture that it is possible to figure out examples for data set for with four observations and three goods. For mathematical elegance we have used many zeros in our data set.

Finally, one should notice that we considered a data set with four observations. We prove in Proposition 11 that this is the minimum number of observations that we need to test egoistic-CR.

Proposition 11 *Let $S = \{(p_t, q_t); t = 1,2,3\}$ be a set of three observations. Suppose that there exists a general-CR of S, then there always exists a combination of concave, continuous and monotonically increasig utility functions U^A_{ego} and U^B_{ego} that provides an egoistic-CR of S.*

5.3 Independence of egoistic-CR and public-CR

In the previous subsection, we have shown that the general collective model is distinguishable from the two specific benchmark models. In the Appendix we argue that it is possible to distinguish between the two benchmark models. More precisely, in the Appendix we show that the data set in Example 1 satisfies the testable conditions for a egoistic-CR, and the data set in Example 2 satisfies the testable conditions for a public-CR. Thus, we can conclude that if the data set is consistent with one benchmark model, this does not necessarily imply that it is also consistent with the other benchmark model.

It is enough have a data set with four observations and four goods [5] to be able to test the nature of the goods. Moreover, this result could directly carry over to 'intermediate' collective models that stand between the two benchmark cases, i.e models which assume that part of the goods is privately consumed (without externalities) while all other goods are publicly consumed. See Cherchye, De Rock, Vermeulen (2010) for a detailed discussion.

6 Conclusion

Chapter 3 has adopted the nonparametric 'revealed preference' methodology due to Cherchye, De Rock, Vermeulen (2007) for analyzing the testable restrictions for the two benchmark cases of the collective consumption model. That is, the case in which all the goods are publicly consumed and the case in which all the goods are privately and individuals have egoistic preferences. These two polar cases were analyzed previously by Chiappori and Ekeland (2006). Using a parametric approach, the authors showed that these two benchmark cases have the same testable restrictions than the general model (i.e. all goods can be consumed privately, publicly or both). So, their main result is that it is not possible to test the nature of the goods from aggregate data on group behavior.

Differently from Chiappori and Ekeland (2006), using a nonparametric characterization which involve personalized prices à la Lindahl and feasible personalized consumptions, we show that the nature of goods is testable. More precisely, we obtain different testable restrictions as soon as we have a data set with four observations and four goods. Importantly, in our approach, we do not require that Lindahl prices and personalized quantities are observable. Therefore, this approach could be useful for empirical applications.

[5] We think that it should be enough a data set with four observations and three goods.

We want to conclude this chapter considering a possible extension. This basic framework could be extended considering many group members. Of course, this generalization will not affect the core of our results.

Appendix

Example 1

There exists a general-CR of S. Consider the following personalized quantities and prices:

$$\begin{aligned} x_1 &= (q_1, 0, 0), & p_1^A &= (p_1, 0, p_1), & p_1^B &= (0, p_1, 0) \\ x_2 &= (\frac{1}{2}q_2, \frac{1}{2}q_2, 0), & p_2^A &= (p_2, 0, p_2), & p_2^B &= (0, p_2, 0) \\ x_3 &= (0, q_3, 0), & p_3^A &= (p_3, 0, p_3), & p_3^B &= (0, p_3, 0) \end{aligned}$$

It is easy to show that the *GARP* conditions in Proposition 6 are satisfied for both members. So, one can conclude that the data set at hand satisfies general-CR.

There exists an egoistic-CR of S. It is immediate to see that these personalized feasible prices and quantities satisfy the conditions in Proposition 10. So, we can conclude that the data set in this example is consistent with an egoistic-CR.

There does not exist a public-CR of S. Let us prove this ad absurdum and assume that we have a construction of feasible prices that satisfies condition *(ii)* in Proposition 8.

One should notice that for any $t, s = 1, 2, 3$, with $t \neq s$, the structure of the data set in this example implies that $p_t \cdot q_t > p_t \cdot q_s$. Therefore we must have feasible prices such that either $p_t^{AG} \cdot q_t > p_t^{AG} \cdot q_s$ or $(p_t - p_t^{AG}) \cdot q_t > (p_t - p_t^{AG}) \cdot q_s$. *GARP* conditions in Proposition 8 require that if $p_t^{AG} \cdot q_t \geq p_t^{AG} \cdot q_s$, then $p_s^{AG} \cdot q_s \leq p_s^{AG} \cdot q_t$. So, since for any $t, s = 1, 2, 3$ with $t \neq s$, $p_s \cdot q_s > p_s \cdot q_t$ and $p_s^{AG} \cdot q_s \leq p_s^{AG} \cdot q_t$ one gets $(p_s - p_s^{AG}) \cdot q_s > (p_s - p_s^{AG}) \cdot q_t$. Therefore, if $x_t R_0^A x_s$, we must have $x_s R_0^B x_t$. Given that this holds for any $t, s = 1, 2, 3$, with $t \neq s$, we conclude that *(i)* $x_1 R_0^A x_2$ and $x_2 R_0^A x_3$ for member A, and *(ii)* $x_3 R_0^B x_2$ and $x_2 R_0^B x_1$ for member B is a possible solution of public-CR.

Assume that $p_2^{AG} = (\pi_1, \pi_2, \pi_3)$. The *GARP* condition for member A in Proposition 8 requires that

$$p_2^{AG} \cdot q_2 \leq p_2^{AG} \cdot q_1 \Leftrightarrow 2\pi_1 + 5\pi_2 + 2\pi_3 \leq 5\pi_1 + 2\pi_2 + 2\pi_3$$
$$\Leftrightarrow 0 \leq \pi_1 - \pi_2.$$

The *GARP* condition for member B in Proposition 8 requires that

$$\begin{aligned}(p_2 - p_2^{AG}) \cdot q_2 \leq (p_2 - p_2^{AG}) \cdot q_3 &\Leftrightarrow 2(1-\pi_1) + 5(4-\pi_2) + 2(1-\pi_3) \\ &\leq 2(1-\pi_1) + 2(4-\pi_2) + 5(1-\pi_3) \\ &\Leftrightarrow 3 \leq \pi_2 - \pi_3.\end{aligned}$$

Overall, this implies that $3 \leq \pi_2 \leq \pi_1$, which gives us the desired contradiction since by construction $\pi_1 \leq 1$. Of course, all the other possible solutions of public-CR lead to the same contradictions. Therefore, we conclude that there cannot exist a public-CR of the data set in Example 1.

Example 2

There exists a general-CR of S. Consider the following personalized quantities and prices with $p_2^{AG} = (4, 3.5, 0, 0)$ and $p_3^{AG} = (4, 4, 3.5, 0)$.

$$\begin{array}{lll} x_1 = (0,0,q_1), & p_1^A = (p_1, p_1, p_1), & p_1^B = (0,0,0); \\ x_2 = (0,0,q_2), & p_2^A = (p_2, p_2, p_2^{AG}), & p_2^B = (0,0,p_2 - p_2^{AG}); \\ x_3 = (0,0,q_3), & p_3^A = (p_3, p_3, p_3^G), & p_3^B = (0,0,p_3 - p_3^{AG}); \\ x_4 = (0,0,q_4), & p_4^A = (0,0,0), & p_4^B = (0,0,p_4). \end{array}$$

It is easy to show that the *GARP* conditions in Proposition 6 are satisfied for both members. So, one can conclude that the data set at hand satisfies general-CR.

There exists a public-CR of S. It is immediate to see that these personalized feasible prices and quantities satisfy the conditions in Proposition 8. So, we can conclude that the data set in this example is consistent with a public-CR.

There does not exist an egoistic-CR of S. Let us prove this ad absurdum and assume that we have a construction of feasible prices that satisfies condition *(ii)* in Proposition 10.

Again, one should notice that for any $t, s = 1, 2, 3$, with $t \neq s$, the structure of the data set in this example implies that $p_t \cdot q_t > p_t \cdot q_s$. Therefore, with no loss of generality, we can assume that the solution of feasible prices leads to *(i)* $x_1 R_0^A x_2, x_2 R_0^A x_3$ and $x_3 R_0^A x_4$ for member A, and *(ii)* $x_4 R_0^B x_3, x_3 R_0^A x_2$ and $x_2 R_0^B x_1$ for member B.

Assume that $x_2^A = (0, \alpha, 0, 0)$ and $x_3^A = (0, 0, \beta, 0)$. The *GARP* conditions for the two members in Proposition 10 require that the following holds:

$$
\begin{aligned}
p_2^A \cdot x_2 \leq p_2^A \cdot x_1 &\Leftrightarrow 7\alpha \leq 4; \\
p_3^A \cdot x_3 \leq p_3^A \cdot x_2 &\Leftrightarrow 7\beta \leq 4\alpha \leq 4; \\
p_2^B \cdot x_2 \leq p_2^B \cdot x &\Leftrightarrow 7(1-\alpha) \leq 4(1-\beta) \leq 4; \\
p_3^B \cdot x_3 \leq p_3^B \cdot x_4 &\Leftrightarrow 7(1-\beta) \leq 4.
\end{aligned}
$$

This implies that $\frac{3}{7} \leq \alpha \leq \frac{4}{7}$, $\frac{3}{7} \leq \beta \leq \frac{4}{7}$ and $\frac{7\beta}{4} \leq \alpha$ and thus also that $\alpha \geq \frac{3}{4}$. Thereby we obtain the desired contradiction and we conclude that there cannot exist an egoistic-CR of the data set in Example 2.

Proof of Proposition 4. Example 1 of Cherchye, De Rock, Vermeulen (2007) shows that we cannot have a general-CR if we have a data set with the following structure: $p_1 \cdot q_1 \geq p_1 \cdot (q_2 + q_3)$, $p_2 \cdot q_2 \geq p_2 \cdot (q_1 + q_3)$ and $p_3 \cdot q_3 \geq p_3 \cdot (q_1 + q_2)$ hold simultaneously. With no loss of generality, we assume that $p_2 \cdot q_2 < p_2 \cdot (q_1 + q_3)$.

Consider the following personalized quantities and prices for an $\alpha \in [0, 1]$:

$$
\begin{aligned}
x_1 &= (q_1, 0, 0), & p_1^A &= (p_1, 0, p_1), & p_1^B &= (0, p_1, 0) \\
x_2 &= (\alpha q_2, (1-\alpha) q_2, 0), & p_2^A &= (p_2, 0, p_2), & p_2^B &= (0, p_2, 0) \\
x_3 &= (0, q_3, 0), & p_3^A &= (p_3, 0, p_3), & p_3^B &= (0, p_3, 0)
\end{aligned}
$$

These feasible prices and quantities are consistent with the collective model with only private goods (i.e. $x_t^G = 0$) and egoistic preferences (i.e. $p_t^{AA} = p_t$ and $p_t^{AB} = 0$).

In order to prove that this is a solution for egoistic-CR, we need to check *GARP* for the sets $\{(p_t^A, x_t); t = 1, 2, 3\}$ and $\{(p_t^B, x_t); t = 1, 2, 3\}$.
We start considering member A. We need to verify that condition *(iii)* in Definition 5 is satisfied. First, for each observation t, we construct the set of the revealed preferred bundles (i.e. Opening Conditions) and after we check if every observation t is cost minimizing over the revealed preferred set (Closing Conditions).

<u>Opening Conditions</u>:

(i.1) $p_1^{AA} \cdot x_1^A \geq p_1^{AA} \cdot x_2^A \Rightarrow p_1 \cdot q_1 \geq \alpha p_1 \cdot q_2$;

(ii.1) $p_1^{AA} \cdot x_1^A \geq p_1^{AA} \cdot x_3^A \Rightarrow p_1 \cdot q_1 \geq 0$;

(iii.1) $p_2^{AA} \cdot x_2^A \geq p_2^{AA} \cdot x_3^A \Rightarrow \alpha p_2 \cdot q_2 \geq 0$.

Closing Conditions:

(iv.1) $p_2^{AA} \cdot x_2^A \leq p_2^{AA} \cdot x_1^A \Rightarrow \alpha p_2 \cdot q_2 \leq p_2 \cdot q_1$;

(v.1) $p_3^{AA} \cdot x_3^A \leq p_3^{AA} \cdot x_1^A \Rightarrow 0 \leq p_3 \cdot q_1$;

(vi.1) $p_3^{AA} \cdot x_3^A \leq p_3^{AA} \cdot x_2^A \Rightarrow 0 \leq \alpha p_3 \cdot q_2$.

We consider now member B. The conditions to satisfy the $GARP$ are given by:

Opening Conditions:

(i.2) $(p_3 - p_3^{AB}) \cdot x_3^B \geq (p_3 - p_3^{AB}) \cdot x_2^B \Rightarrow p_3 \cdot q_3 \geq (1-\alpha) p_3 \cdot q_2$;

(ii.2) $(p_3 - p_3^{AB}) \cdot x_3^B \geq (p_3 - p_3^{AB}) \cdot x_1^B \Rightarrow p_3 \cdot q_3 \geq 0$;

(iii.1) $(p_2 - p_2^{AB}) \cdot x_2^B \geq (p_2 - p_2^{AB}) \cdot x_1^B \Rightarrow (1-\alpha) p_2 \cdot q_2 \geq 0$.

Closing Conditions:

(iv.2) $(p_2 - p_2^{AB}) \cdot x_2^B \leq (p_2 - p_2^{AB}) \cdot x_3 \Rightarrow (1-\alpha) p_2 \cdot q_2 \leq p_2 \cdot q_3$;

(v.2) $(p_1 - p_1^{AB}) \cdot x_1^B \leq (p_1 - p_1^{AB}) \cdot x_3^B \Rightarrow 0 \leq p_1 \cdot q_3$;

(vi.2) $(p_1 - p_1^{AB}) \cdot x_1^B \leq (p_1 - p_1^{AB}) \cdot x_2^B \Rightarrow 0 \leq (1-\alpha) p_1 \cdot q_2$.

Conditions *(ii.m)*, *(iii.m)*, $m = A, B$ are trivially satisfied. So, $x_1 R_0^A x_3$, $x_2 R_0^A x_3$, and $x_3 R_0^B x_1$, $x_2 R_0^B x_1$. One should notice that also the corresponding closing conditions, i.e. *(v.m)*, *(vi.m)*, $m = A, B$ are trivially satisfied.
We do not know if x_1 belongs to the set of the revealed preferred bundles of x_2 (i.e. if $x_1 R_0^A x_2$), and if x_3 belongs to the set of the revealed preferred bundles of x_2 (i.e. if $x_3 R_0^A x_2$), or equivalently if conditions *(i.m)*, $m = A, B$ hold true. However, given that $p_2 \cdot q_2 < p_2 \cdot (q_1 + q_3)$, there must exists an $\alpha \in [0, 1]$ such that $\alpha p_2 \cdot q_2 \leq p_2 \cdot q_1$ and $(1-\alpha) p_2 \cdot q_2 \leq p_2 \cdot q_3$. This implies that conditions *(iv.m)*, $m = A, B$ are satisfied. So, all closing conditions are satisfied.
Therefore $GARP$ is satisfied and we cannot reject egoistic-CR. ■

References

Browning, M., Chiappori, P.-A., 1998. Efficient Intra-Household Allocations: A General Characterization and Empirical Tests. *Econometrica* 66, 1241-1278.

Cherchye, L., De Rock, B., Platino, V., 2010. Private versus public consumption within groups: testing the nature of goods from aggregate data. CES Working Paper 2010.90.

Cherchye, L., De Rock, B., Vermeulen, F., 2007. The Collective Model of Household Consumption: A Nonparametric Characterization. *Econometrica* 75, 553-574.

Cherchye, L., De Rock, B., Vermeulen, F., 2010. The revealed preference approach to collective consumption behavior: nonparametric testing and sharing rule recovery. Accepted for publication on *Review of Economic Studies*.

Chiappori P.-A., Ekeland, I., 2006. The Micro Economics of Group Behavior: General Characterization. *Journal of Economic Theory* 130, 1-26.

Varian, H., 1982. Nonparametric Approach to Demand Analysis. *Econometrica* 50, 945-973.