Abstract

In the study of several elliptic problems with solutions in a Sobolev space $S(\Omega)$ (with or without weight) on an open set Ω of $\mathbb{R}^n$, not necessarily bounded or regular, it is sometimes necessary to estabilish regularity results and a priori estimates for the solutions. These results often rely on the boundedness and possibly on the compactness of the multiplication operator

$$u \longrightarrow g\,u \qquad (i)$$

which is defined in $S(\Omega)$ and which takes values in a suitable Lebesgue space $L^p(\Omega)$, where g is a given function in a normed space V. Hence, it's necessary to obtain an estimate of the following type :

$$\|g\,u\|_{L^p(\Omega)} \leq c \cdot \|g\|_V \cdot \|u\|_{S(\Omega)}\,, \qquad (ii)$$

where $c \in \mathbb{R}_+$ depends on the regularity properties of Ω and on the summability exponents, and g satisfies suitable conditions.

If L is the differential operator associated to the corresponding elliptic problem, the estimate (ii) allows us, for instance, to prove the boundedness of the operator L, when

g is a coefficient of L. In some particular cases, it's not possible to obtain certain regularity results for the operator L itself, because of its non regular coefficients. Hence, there is the need to introduce a suitable class of operators L_h, whose coefficients, more regular, approximate the ones of L. This " deviation " of the coefficients of L_h from the ones of L needs to be done controlling the norms of the approximating coefficients with the norms of the given ones. Hence, it is necessary to obtain estimates where the dependence on the coefficients is expressed just in terms of their norms (in this case, for instance, there are no problems when passing to the limit). In other words, if g is a coefficient of operator L and g_h is a coefficient of L_h more regular, it's necessary to have a " good control" of the difference $g - g_h$. The introduction of decompositions for functions in suitable function spaces (where the coefficients of differential operator L belong) plays an important role in this approximation process.

Having this in mind, our purpose is to construct suitable decompositions for functions belonging to some specific functional spaces whose introduction is related to the solvability of certain elliptic problems of above mentioned type. As application, we want to study the boundeness and the compactness of an operator in Sobolev spaces with or without weight.

Let F be a Banach space and F_0 be a subset of F, then we can consider the closure C of F_0 in F. The idea of our decomposition is to split a function $g \in C$ in the sum of a "good" part g_h, which is more regular, and of a " bad " part $g - g_h$ whose norm can be controlled by means of a continuity modulus of the function g itself.

In the first part of work we deepen the study of some weighted functional spaces whose introduction is related to the solvability of Dirichlet problems for linear second order elliptic equations in non regular domains and in weighted Sobolev spaces. As application, using some decomposition results for functions belonging to such weighted spaces, we prove some weighted norm inequalities on certain irregular domains of $\mathbb{R}^n$ and the boundedness and the compactness of the operator (i) defined in a weighted Sobolev space.

The structure of Chapter 1 and of Chapter 2 reflects the above purposes.

In Chapter 1 we describe some properties and applications of certain weighted Sobolev spaces which represent the setting of our main results.

If k is non - negative integer, p is a real number, $1 \leq p < +\infty$, Ω is a domain in $\mathbb{R}^n$ with boundary $\partial\Omega$, σ is a vector of non - negative (positive almost everywhere) measurable functions on Ω, which will be called weight, the weighted Sobolev space, usually denoted by $W^{k,p}(\Omega;\sigma)$, is defined as the set of all functions $u = u(x)$ which are defined $a.e.$ on Ω and whose generalized (in the sense of distributions) derivatives $\partial^\alpha u$ of orders $|\alpha| \leq k$ satisfy

$$\int_\Omega |\partial^\alpha u(x)|^p \sigma_\alpha(x)\, dx < +\infty \,.$$

In Chapter 2, we consider a class of weight functions denoted by $\mathcal{A}(\Omega)$ and the corresponding weighted Sobolev spaces $W^{k,p}_s(\Omega)$ defined on open subsets Ω of $\mathbb{R}^n$. More precisely, a measurable weight function $\rho : \Omega \to \mathbb{R}_+$ belongs to the class $\mathcal{A}(\Omega)$ if and only if there exists a costant $\gamma \in \mathbb{R}_+$, independent on x and y, such that

$$\gamma^{-1}\, \rho(y) \leq \rho(x) \leq \gamma\, \rho(y)\,, \qquad \forall\, y \in \Omega, \quad \forall\, x \in \Omega \cap B(y, \rho(y)),$$

For $k \in \mathbb{N}_0$, $s \in \mathbb{R}$ and $1 \leq p \leq +\infty$, we denote by $W^{k,p}_s(\Omega)$ the space of distributions u on Ω such that $\rho^{s+|\alpha|-k}\,\partial^\alpha u \in L^p(\Omega)$ for $|\alpha| \leq k$ and equipped with the norm

$$\|u\|_{W^{k,p}_s(\Omega)} = \sum_{|\alpha|\leq k} \|\rho^{s+|\alpha|-k}\,\partial^\alpha u\|_{L^p(\Omega)}\,,$$

where ρ is a weight function belonging to the class $\mathcal{A}(\Omega)$.

Moreover, in this Chapter we deepen the study of weighted functional spaces K^r_t and their properties.

Let $r \in [1, +\infty[$ and $t \in \mathbb{R}$, we denote by $K^r_t(\Omega)$ the class of all functions g, locally belonging to $L^r(\Omega)$, such that

$$\sup_{\Omega} \left(\rho^{t-\frac{n}{r}}(x)\,\|g\|_{L^r(\Omega\cap B(x,\rho(x)))}\right) < +\infty\,,$$

where the weight function ρ belongs to the class $\mathcal{A}(\Omega)$. It is easy to prove that the spaces $L^\infty_t(\Omega)$ and $C^\infty_o(\Omega)$ are subsets of $K^r_t(\Omega)$ (the space $L^\infty_t(\Omega)$ is the space of all functions g such that $\rho^t\,g \in L^\infty(\Omega)$). Therefore we can define two new spaces of functions $\widetilde{K}^r_t(\Omega)$ and $\overset{\circ}{K}{}^r_t(\Omega)$ as the closures of $L^\infty_t(\Omega)$ and $C^\infty_o(\Omega)$ in $K^r_t(\Omega)$.

We construct suitable decompositions of functions $g \in \widetilde{K}^r_t(\Omega)$ and of functions $g \in \overset{\circ}{K}{}^r_t(\Omega)$. Moreover, in the framework of spaces $K^r_t(\Omega)$, we study the operator (i) defined on weighted Sobolev space $W^{k,p}_s(\Omega)$ and taking values in $L^q(\Omega)$ with appropriate $q \in [p, r[$. We give suitable conditions on p, q, s, r, ρ, Ω and on the function $g \in K^r_t(\Omega)$ so that the following estimate holds

$$\|g\,u\|_{L^q(\Omega)} \leq c \cdot \|g\|_{K^r_t(\Omega)} \cdot \|u\|_{W^{k,p}_s(\Omega)}\,, \qquad (iii)$$

If $g \in \widetilde{K}^r_t(\Omega)$ or $g \in \overset{\circ}{K}{}^r_t(\Omega)$, from (iii) we deduce boundedness and compactness results for the considered operator. The use of our decompositions in these results allows us to put in evidence how the bad part $(g - g_h)$ of the function g in $\widetilde{K}^r_t(\Omega)$ or in $\overset{\circ}{K}{}^r_t(\Omega)$, affects the estimate.

In the study of the above mentioned Dirichlet problems on irregular or unbounded domains, there is the need to put some conditions at the infinity on the lower order coefficients of the elliptic differential operator. Such conditions are ensured, for instance, by the assumption that the coefficients belong to space $\overset{\circ}{K}{}^r_t(\Omega)$. This also gives the compactness of the operator (i).

In view of these last considerations, we put in evidence a new characterization of the spaces $\overset{\circ}{K}{}^r_t(\Omega)$ by means of the introduction a new subspace of $K^r_t(\Omega)$, denoted by $\overset{*}{K}{}^r_t(\Omega)$. We state that under suitable conditions on the weight function $\rho \in \mathcal{A}(\Omega)$ the space $\overset{*}{K}{}^r_t(\Omega)$ is settled between $\overset{\circ}{K}{}^r_t(\Omega)$ and $\widetilde{K}^r_t(\Omega)$. In particular we give a condition on the weight function in order to obtain that $\overset{*}{K}{}^r_t(\Omega) = \overset{\circ}{K}{}^r_t(\Omega)$.

In the last part of work we want to deepen the study of spaces of Morrey type. Also in this case, using some decomposition results for functions belonging to a suitable subspace of a space of Morrey type we want to deduce a compactness result for the operator (i) defined on Sobolev spaces without weight.

In Chapter 3 we analyze this aspect. Let Ω be an unbounded open subset of $\mathbb{R}^n$, $n \geq 2$. For $p \in [1, +\infty[$ and $\lambda \in [0, n[$, we consider the space $M^{p,\lambda}(\Omega)$ of the functions g in $L^p_{loc}(\overline{\Omega})$ such that

$$\|g\|^p_{M^{p,\lambda}(\Omega)} = \sup_{\substack{\tau \in]0,1] \\ x \in \Omega}} \tau^{-\lambda} \int_{\Omega \cap B(x,\tau)} |g(y)|^p \, dy < +\infty,$$

where $B(x,\tau)$ is the open ball with center x and radius τ.

This space of Morrey type is a generalization of the classical Morrey space $L^{p,\lambda}$ and strictly contains $L^{p,\lambda}(\mathbb{R}^n)$ when $\Omega = \mathbb{R}^n$. Its introduction is related to the solvability of certain elliptic problems with discontinuous coefficients in the case of unbounded domains.

In the first part of Chapter 3, we turn our attention to the density property of Morrey type spaces. The example $|x|^{(\lambda-n+1)/p}$ shows, the space $L^\infty(\Omega)$ is not dense in the space $M^{p,\lambda}(\Omega)$. So, it's important and useful to give a new characterization of functions in the closure of $L^\infty(\Omega)$ and $C_o^\infty(\Omega)$ in $M^{p,\lambda}(\Omega)$ (which are respectively denoted with $\widetilde{M}^{p,\lambda}(\Omega)$ and $M_0^{p,\lambda}(\Omega)$). By means of such chacterization lemmas we are allowed to construct suitable decompositions of functions in $\widetilde{M}^{p,\lambda}(\Omega)$ and $M_0^{p,\lambda}(\Omega)$.

The second part of Chapter 3 is devoted to the analysis of the following multiplication operator

$$u \in W^{k,p}(\Omega) \to g\,u \in L^q(\Omega)$$

with a suitable q greater than p and g belonging to a space of Morrey type $M^{p,\lambda}(\Omega)$. By means of our decomposition results we are allowed to deduce a compactness result for the above mentioned operator.

The deeper examination of the structure of $M^{p,\lambda}(\Omega)$ and of its subspaces lead us to the definition of a new functional space, that is a weighted Morrey type space, denoted by $M_\rho^{p,\lambda}(\Omega)$. In literature several authors have considered different kinds of weighted spaces of Morrey type and their applications to the study of elliptic equations, both in the degenerate case and in the non-degenerate one.

In Chapter 3 we consider another class of weight functions, denoted by $\mathcal{G}(\Omega)$, and we define the corresponding weighted space $M^{p,\lambda}_{\rho}(\Omega)$. More precisely, let $d \in \mathbb{R}_+$, a measurable weight function $\rho : \Omega \to \mathbb{R}_+$ belongs to the class $\mathcal{G}(\Omega,\, d)$ if and only if there exists $\gamma \in \mathbb{R}_+$, independent on x and y, such that

$$\gamma^{-1}\,\rho(y) \leq \rho(x) \leq \gamma\,\rho(y)\,, \qquad \forall\, y \in \Omega, \quad \forall\, x \in \Omega(y,d).$$

We put

$$\mathcal{G}(\Omega) = \bigcup_{d>0} \mathcal{G}(\Omega, d).$$

Let $\rho \in \mathcal{G}(\Omega) \cap L^{\infty}(\Omega)$ and let d be the positive real number such that $\rho \in \mathcal{G}(\Omega, d)$. Fix a Lebesgue measurable subset E of Ω, for $p \in [1, +\infty[$, $\lambda \in [0, n[$ we denote by $M^{p,\lambda}_{\rho}(\Omega)$ the space of all functions $g \in M^{p,\lambda}(\Omega)$ such that

$$\lim_{h \to +\infty} \Bigg(\sup_{\substack{E \in \Sigma(\Omega) \\ \sup\limits_{\substack{x \in \Omega \\ \tau \in]0,d]}} \tau^{-\lambda}\rho(x)|E(x,\tau)| \leq \frac{1}{h}}} \| g\, \chi_E \|_{M^{p,\lambda}(\Omega)} \Bigg) = 0,$$

We prove that the space $M^{p,\lambda}_{\rho}(\Omega)$ is settled between $M^{p,\lambda}_{0}(\Omega)$ and $\widetilde{M}^{p,\lambda}(\Omega)$. In particular, we provide some conditions on ρ that entail $M^{p,\lambda}_{0}(\Omega) = M^{p,\lambda}_{\rho}(\Omega)$.

We remark that the results of this work can be used in the study of elliptic problems. More precisely, the estimates obtained in Chapter 2 can be used, for instance, in the study of some elliptic problems on irregular domains (i.e. domains with singular boundary) and in weighted Sobolev spaces $W^{k,p}_{s}$ to prove that the considered operators (whose lower order coefficients belong to weighted functional spaces K^{r}_{t}) have closed range or are semi-Fredholm. The estimates obtained in Chapter 3 can be useful, for instance, in the study of Dirichlet problems concerning elliptic equations in

unbounded domains (whose boundary is sufficiently smooth) and in classical Sobolev spaces to estabilish a priori estimates for differential operator whose lower order coefficients belong to spaces of Morrey type.
Moreover we put in evidence that the introduction of spaces $\overset{*}{K}{}^r_t(\Omega)$ and $M^{p,\lambda}_\rho(\Omega)$ offers new points of views in the approach to the study of some classes of elliptic problems with discontinuos coefficients.