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FLAVOR MIXING IN QUANTUM FIELD THEORY AND QUANTUM INFORMATION

Tutor:
Prof. Silvio De Siena

Candidato:
Dott. Marco Di Mauro

Coordinatore:
Prof. Giuseppe Grella

ἐν οἶδα ὅτι οὐδὲν οἶδα

(Socrates, paraphrased from Plato's Apology)

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FLAVOR MIXING
IN QUANTUM FIELD THEORY
AND QUANTUM INFORMATION

Marco Di Mauro

Anno Accademico 2009/2010

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Introduction

The recent experimental observation of neutrino oscillations has set a milestone in high energy physics, being the first-ever evidence of a phenomenon not described by the now nearly half a century old Standard Model. Its theoretical description is, like the analog mixing of quarks (which is instead included in the Standard Model to take into account the observed neutral meson oscillations and the associated CP violation), implemented by writing down a mixing transformation which, as the name suggests, mixes fields whose quanta have different masses. It is this mass difference which gives rise to the phenomenon of oscillations; we shall indeed see that it can be made to disappear by taking equal masses¹.

A full theoretical understanding of the phenomenon of mixing can be possible only in the framework of Quantum Field Theory, standard Quantum Mechanics being unsuitable to describe superpositions of states of different masses due to the Bargmann superselection rule². This observation led to much work in this direction in the last 16 years, resulting most notably in the discovery of a nontrivial nonperturbative vacuum structure associated with mixing and of well defined flavor states describing mixed particles. These states have been shown to be compatible with flavor conservation in the production and detection vertices, consistently with the Standard Model. This implies that the flavor vacuum has to be taken as the physical one if lepton conservation in the Standard Model vertices is to be preserved.

One of the most striking properties of this vacuum lies in the fact that it has nonzero energy density due to the presence of a condensate. This has been shown to be a potential candidate to give contributions to the cosmological constant. The observation that this vacuum energy density is always positive definite leads to the observation that the same will be true also in supersymmetric (SUSY) theories which contain particle mixing. Such theories, even if mixing is preserving supersymmetry (SUSY), will then show a (nonperturbative) spontaneous supersymmetry breaking. That this is the case has recently been proven in a simple model, but there are quite strong reasons to believe that it be a general phenomenon. Being mixing a fundamental component of supersymmetric extensions of the Standard Model, this fact is of obvious phenomenological relevance.

¹Of course this refers to the renormalized masses; it could well be possible to start with fields having equal bare masses, which are then rendered different by radiative corrections. This will not concern us in this work as we are going to treat only effective Lagrangians just suited for describing the mixing, not investigating possible mechanisms from which mass differences and mixing itself originate.

²Nevertheless, an approximated description (which actually predates the field theoretical one) is possible in Quantum Mechanics; it will be shown that the full treatment reduces to the quantum mechanical one in the ultrarelativistic limit. Unfortunately, the fact that neutrinos in fact are always ultrarelativistic in practice makes an experimental observation of the field theoretical corrections very hard.

There still are many points to be clarified concerning flavor states, such as their interpretation as representation of the Poincaré group and their entanglement properties. The study of the latter properties constitutes a new point of view on the mixing of particles and has already led to some intriguing results which in some cases constitute a genuine generalization of known results in Quantum Information Theory. This points out the relevance of mixed particles to this subject, at least on the theoretical side, the actual use of neutrinos in place of photons in quantum experiments or as quantum computation devices being not convenient, essentially because of the difficulty of handling neutrinos as opposed to photons.

Some additional work in this direction has led to the observation that time evolution of flavor neutrino states dynamically generates entanglement. This is referred to as dynamical entanglement, as opposed to the static entanglement referred to in the previous paragraph. Current work is focusing on the use of the so-called dynamic symmetry approach to quantum entanglement, pioneered by A. Klyachko, which has the nice feature of giving a neat description of quantum entanglement in terms of a very compelling physical picture, valid for pure states of general multipartite systems³ (known criteria of characterization of multipartite entanglement are somewhat artificial). Most of the referred work is done in the simpler framework of Quantum Mechanics, the extension to Quantum Field Theory being somewhat tricky due to the nontrivial vacuum structure uncovered in that context. Nevertheless, some progress along this direction has been achieved recently thanks to the dynamic symmetry approach, and the development of these results is under current investigation.

On the Poincaré front, the problem reduces to the fact that mixed particles cannot be considered on-shell, thus they cannot be classified as representations of the Poincaré group, leading in this way to a clash with relativistic symmetry. Recently a big step towards the resolution of this problem has been made by implementing mixing as the interaction of a multiplet of fields with some external field. Preliminary work in this direction points at a vector field as the responsible of mixing, at the same time unraveling an approximate non abelian gauge structure hidden in mixing (which may be connected to another hidden gauge structure recently discovered in particle mixing, related to a geometric phase); this gauge symmetry becomes exact in the maximal mixing limit. We have been able to show that by rewriting mixing in this way, it is possible to regard mixed particles as on-shell like any other particle, thus recovering a Poincaré structure. In the process one ambiguity which is inherent to the formulation of quantum field mixing is lifted. In this framework, the Lorentz violation associated to mixing re-emerges as the preferred frame singled out by the external vector field, thus losing its fundamental character.

Concerning the nature of this vector field, there is much room for speculation. For example, it may be seen (at least in the two flavor case) as a kind of “birefringent” medium with respect to neutrinos, which acts on the two flavors in the same way an ordinary birefringent medium acts on the two polarization states of a photon. Another possible interpretation (not necessarily incompatible with the previous one) sees the vector field as being a component of dark matter. This picture opens the possibility of a position dependent oscillation length, due to inhomogeneities of the distribution of the vector field over intergalactic distances, and this may be put under observational test by looking for example at neutrinos produced by an

³with the restriction that the Hilbert space be finite dimensional

extragalactic supernova. Work in this direction is in progress.

A third possible interpretation suggested by the formalism, which goes in the opposite direction consists in viewing the vector field as a fictitious one, which would be employed to write down mixing as a kind of Aharonov–Bohm interaction, much in the same way as fractional statistics in $2 + 1$ dimensions can be written in terms of a fictitious non dynamical gauge field described by a Chern–Simons Lagrangian. This interpretation, while very appealing theoretically, presents nontrivial technical difficulties and has not been pursued yet.

As an added bonus, this formalism allows us to give a fairly clear thermodynamical interpretation of mixing. While such an interpretation has been already suggested in chapters 2 and 3, these results seem to be a consistent step forward.

The present work consists of five chapters.

Chapters 1 and 2 review the necessary background material concerning the formulation of mixing, both in the quantum mechanical and in the field theoretical instances. Especially in the quantum mechanical case, this is by now standard material and has been included for completeness.

In chapters 3–5 work based on the formalism exposed in the first two chapters is exposed. These chapters are independent on each other and can be read in any order.

In Chapter 3 we use the formalism developed in Chapter 2 to show how the vacuum structure associated to mixing can lead to dynamical SUSY breaking. Some speculations concerning the role of mixing as a viable mechanism for SUSY breaking from the phenomenological point of view are outlined.

In Chapter 4 we study mixing from the point of view of Quantum Information Theory. A lightning introduction to the necessary quantum information theoretical tools, such as multipartite quantum entanglement, single particle entanglement and the dynamical symmetry approach is provided, with the only purpose of preparing the ground and without any claim of completeness. Most of this chapter will make use of the approximate quantum mechanical approach to mixing, but also the preliminary results obtained in the field theoretical case are outlined.

In Chapter 5 we attempt the formulation of mixing as the effect of a background vector field and show how this new point of view can be used to address the problems with Poincaré invariance raised by the usual formulation. Some phenomenological consequences are outlined. The thermodynamical interpretation suggested by this formalism will be deferred to an appendix in which the simpler quantum mechanical instance is studied.

The author's own contribution is exposed in chapters 3–5, and can be embedded in three largely independent research lines.

In particular, chapter 3 is based on Ref.[49], which is currently under the process of peer review. The possibility of spontaneous SUSY breaking induced by flavor mixing is pointed out, a general conjecture is proposed and a proof of it is provided in a simple case. Hints are given concerning the generalization of the proof to more complicated situations.

Chapter 4 and appendix A are mainly based on Refs.[28] and [25]. In these papers the bases of static entanglement in flavor mixing in quantum mechanics are given, and the properties of flavor states as entangled states are studied in the two, three and four flavor cases. The effect of decoherence on the amount of entanglement in these states is studied as well by using a more realistic wave packet approach. Sections 4.3.2, 4.9, 4.10, 4.11 have been included as they constitute the background to some work in progress in which the author is involved.

Chapter 5 and appendix B are based on Refs.[31] and [32]. It is shown how viewing fermion mixing as the effect of the interaction with an external field helps to solve some problems with the standard approach and in addition leads to lift an ambiguity present in it. The actual implementation of this program is shown to unveil an approximate hidden gauge structure in fermion mixing. Possible phenomenological consequences of this approach are given. Some links between this approach and optical analogues of neutrino oscillations, as well as to some current work in Quantum Gravity Phenomenology, are pointed out. Also, a possible thermodynamical interpretation is outlined in the simpler case of Quantum Mechanics. The extension to the bosonic instance, which still constitutes work in progress, is outlined. Some loosely related work, which however has not been included in the present thesis, and concerning the Lorentz invariance properties of the exact oscillation formulas, is contained in [30] and has been reviewed in [28].

Publication list

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5. M. Blasone, M. Di Mauro, G. Vitiello, “Flavor mixing and gauge structure,” to appear in the proceedings of the eleventh international symposium *Frontiers of Fundamental Physics* [FFP11], Paris 2010. [32]
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Chapter 1

Introduction to particle mixing

The $SU(3) \times SU(2) \times U(1)$ Standard Model of Particle Physics [53] is probably the most successful theoretical construction for what concerns the matching with experimental results. Besides matching these results with unprecedented accuracy, it also allowed to predict some new phenomena, such as neutral current weak processes, new quarks and leptons and corrections to deep inelastic scattering of electrons on hadrons. All experimental observations have been found in agreement with it, apart from one, *neutrino oscillations*. This one fact is the main subject of this work. In fact, in the last years experimenters have finally established that neutrinos can change their flavors in their route from the source to the detector. Despite having been suggested by B. Pontecorvo back in the fifties (see e.g [15]), this phenomenon is not included in the Standard Model. In fact it depends crucially on the neutrinos (or at least two of them) to have nonzero masses, while in the Standard Model they are treated as massless. While it is possible to incorporate neutrino masses in the Standard Model without difficulty, this raises some theoretical issues such as the explanation of the smallness of neutrino masses with respect to the masses of the other particles. Some explanations such as the seesaw mechanism have been proposed, but this issue is not completely settled and will probably involve some nontrivial physics beyond the Standard Model (see e.g. [114]). Also the origin of the oscillations, which are the manifestation of mixing in the neutrino sector i.e. of a mismatch between flavor and mass of these particles, is not known.

The phenomenon of mixing and oscillations is not exclusive of neutrinos, but also shows in the hadronic sector of the Standard Model. In fact, the d , s and b quarks are mixed (quark mixing, unlike neutrino mixing, has been included in the formulation of the Standard Model since the seventies [44, 96]), and the same is true for some neutral meson like the $K^0 - \bar{K}^0$ couple. How the meson mixing parameters emerge from the quark ones is a major unsolved problem of contemporary Particle Physics, being related to the strong coupling behavior of quarks confined in mesons.

In this work we will not try to explain such problems as the origin of mixing or the derivation of meson mixing parameters (although we attempt to explain mixing as the effect of some background field in chapter 5, this merely shifts the problem of its origin to the problem of the origin of this background field). Instead, we will take particle mixing (both bosonic and fermionic) as given and try to see which consequences arise from this, in particular in connection

with the definition of the flavor states of mixed particles.

1.1 Mixing in Quantum Mechanics

In this section we review the standard formalism of particle mixing in Quantum Mechanics, pioneered by B. Pontecorvo in 1958 (described for example in [15] to which we refer for details) in view of the applications to be described in chapter 4. The starting point are the mixing relations among the states:

$$|\nu_e\rangle = |\nu_1\rangle \cos \theta + |\nu_2\rangle \sin \theta \quad (1.1)$$

$$|\nu_\mu\rangle = -|\nu_1\rangle \sin \theta + |\nu_2\rangle \cos \theta, \quad (1.2)$$

(where θ is the mixing angle) assuming that the neutrinos are ultrarelativistic (which is always verified in practice). Starting from these we can derive the Bilenky–Pontecorvo oscillation formulae, which give the probability of transition between one state and the other as a function of the time elapsed since the production:

$$P_{\nu_e \rightarrow \nu_e}(t) = P_{\nu_\mu \rightarrow \nu_\mu}(t) = 1 - \sin^2 2\theta \sin^2 \left(\frac{\Delta\omega}{2} t \right), \quad (1.3)$$

$$P_{\nu_e \rightarrow \nu_\mu}(t) = P_{\nu_\mu \rightarrow \nu_e}(t) = \sin^2 2\theta \sin^2 \left(\frac{\Delta\omega}{2} t \right), \quad (1.4)$$

where $\Delta\omega = \omega_1 - \omega_2 \equiv E_1 - E_2$ is the energy difference.

Written in this way, these formulae are unsuited for the description of the experimental situation, as the elapsed time is not directly accessible to measurements. It is thus preferable to express them in terms of the spatial distance between the production and the detection events. In the ultrarelativistic approximation one has $L = ct$ and

$$P_{\nu_e \rightarrow \nu_e}(R) = P_{\nu_\mu \rightarrow \nu_\mu}(R) = 1 - \frac{1}{2} \sin^2 2\theta \left(1 - \cos \frac{2\pi R}{L} \right), \quad (1.5)$$

$$P_{\nu_e \rightarrow \nu_\mu}(R) = P_{\nu_\mu \rightarrow \nu_e}(R) = \frac{1}{2} \sin^2 2\theta \left(1 - \cos \frac{2\pi R}{L} \right), \quad (1.6)$$

where p is the momentum of the neutrinos in the beam and the quantity

$$L \equiv \frac{4\pi p}{|m_1^2 - m_2^2|} \quad (1.7)$$

is called oscillation length, and gives the length scale at which the oscillation probability from one flavor to the other is appreciably different from zero. This is the relevant scale in the description of the oscillation phenomenon, and as a consequence of the small values of the neutrino masses it is a macroscopic quantity.

The oscillation formulae are straightforwardly generalized to the case of N flavor mixing, described by a unitary $N \times N$ matrix U :

$$P_{\nu_\alpha \rightarrow \nu_\beta}(E, R) = \sum_k |U_{\alpha k}|^2 |U_{\beta k}|^2 + 2\text{Re} \sum_{k>j} U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^* e^{-i\Delta\phi_{kj}(E, R)}, \quad (1.8)$$

with the phase differences

$$\Delta\phi_{kj}(E, R) = \frac{\Delta m_{kj}^2 R}{2E}, \quad (1.9)$$

Even if this is the approach we will be mostly using in the following, it should be borne in mind that, since it uses plane waves, it's not very realistic and has to be thought at best as an approximation.

The situation can be made more realistic by using wave packets. For example one can use a Gaussian wave packet:

$$\langle \nu_\alpha; \mathbf{p} | \nu_\alpha \rangle \equiv \psi_\alpha(\mathbf{p}; \mathbf{p}_\alpha, \sigma_{pP}) = \frac{1}{[\sqrt{2\pi}\sigma_{pP}]^{\frac{3}{2}}} \exp\left[-\frac{(\mathbf{p} - \mathbf{p}_\alpha)^2}{4\sigma_{pP}^2}\right], \quad (1.10)$$

where one is assuming that the width σ_{pP} is the same for all the species of massive neutrinos we are considering in the production (P) process, and it is the same along all three directions. In this approach the ultrarelativistic neutrino approximation is made in the following way [157, 74, 73]:

$$E_\alpha \approx E + \xi \frac{m_\alpha^2}{2E}, \quad p \approx E - (1 - \xi) \frac{m_\alpha^2}{2E}, \quad (1.11)$$

from which:

$$v_\alpha \approx 1 - \frac{m_\alpha^2}{2E^2}, \quad (1.12)$$

where E is the energy determined by the kinematics of the production process for a massless neutrino and ξ is an order one dimensionless parameter which is determined by the energy momentum conservation in the production process.

The probability to detect a flavor l' neutrino in the point \mathbf{x} is given by [157]:

$$P_{\nu_l \rightarrow \nu_{l'}}(x) = \sum_{\alpha=\beta=1}^2 |U_{l'\beta}|^2 |U_{l\alpha}|^2 + 2 \sum_{\alpha>\beta} |U_{l'\alpha}^* U_{l\alpha} U_{l'\beta} U_{l\beta}^*| \cos\left(2\pi \frac{x}{L_{\alpha\beta}^{osc}} - \varphi_{\alpha\beta l l'}\right) \times e^{-\left(\frac{x}{L_{\alpha\beta}^{coh}}\right)^2} e^{-2\pi^2 \xi^2 \left(\frac{\sigma_x}{L_{\alpha\beta}^{osc}}\right)^2}. \quad (1.13)$$

By comparison with (1.8) we see the appearance of new length parameters besides the oscillation lengths:

$$L_{\alpha\beta}^{coh} \doteq \frac{4\sqrt{2}\sigma_x E^2}{|\Delta m_{\alpha\beta}^2|}. \quad (1.14)$$

called coherence length. We also notice the presence of two additional factors, the second of which is a damping factor

$$F_{\alpha\beta} \doteq e^{-2\pi^2 \xi^2 \left(\frac{\sigma_x}{L_{\alpha\beta}^{osc}}\right)^2}. \quad (1.15)$$

This factor is equal to one if $\sigma_x \ll |L_{\alpha\beta}^{osc}|$, otherwise it goes to zero. So this is a necessary condition to detect oscillations, otherwise the oscillation term gets erased and one remains with a constant transition probability $P_{\nu_l \rightarrow \nu_{l'}}(x) = \sum_{\alpha=1}^2 |U_{l'\beta}|^2 |U_{l\alpha}|^2$. The presence of this term reflects the fact that in order for the oscillation phenomenon to be observed, the localization

of both the source and the detector has to be in regions much smaller than the scale set by the oscillation lengths.

The other factor

$$e^{-\left(\frac{x}{L_{\alpha\beta}^{coh}}\right)^2} \quad (1.16)$$

is due to the fact that wave packets with different momenta have different group velocities. When $x \simeq L^{coh}$ grows this factor gets damped. This means that after some time the mass eigenstate wave packets do not overlap significantly any more, so they cannot interfere to produce the oscillations. This observation allow to predict the coherence length without having to compute the probability (1.13). If we consider two wave packets whose centres initially coincide and having the same width σ_{xP} , whose group velocity difference is $\Delta v = |v_1 - v_2|$, we expect that after a distance

$$R = \frac{2\sigma_{xP}}{\Delta v} \frac{v_1 + v_2}{2}, \quad (1.17)$$

their overlap will be negligible. Of course this is just a rough estimate. In fact (1.14) shows that the coherence length is proportional to $\sigma_x = \sqrt{\sigma_{xP}^2 + \sigma_{xD}^2}$, not just to σ_{xP} as in (1.17). This means that it is possible to increase the coherence length by accurately measuring the particle momenta, as such measurements increase σ_{xD} . This is a beautiful example of the possibility to restore coherence through a measurement. Indeed, the influence of the detector allows the interference of two wavepackets which have negligible overlap. It is not anyway possible to increase indefinitely the coherence length by increasing σ_x as the latter has to be much smaller than the coherence length for the oscillations to be detectable.

We furthermore notice that in the plane wave framework one has $\sigma_x \rightarrow \infty$ which would erase the oscillations.

Chapter 2

Mixing in Quantum Field Theory

In this chapter we will develop the quantum field theoretical formalism for describing the mixing of quantum fields which have different masses. For definiteness we shall mostly talk about two Dirac neutrinos, but the results can be extended to the case of charged bosons, of three flavors with CP violation and to neutral fields. The general theory of mixing, valid for fields of arbitrary spin, has been developed in [89], giving qualitatively similar results albeit with increased technical complexity. A more detailed review of this material can be found in [45]

Although the quantum mechanical treatment described in the previous chapter is a very useful approximation, it is very important to notice that it is not rigorous. In fact, the very superposition of states describing different mass particles is not allowed in ordinary quantum mechanics due to the well known Bargmann superselection rule. In general the use of quantum mechanics in a typically relativistic situation such as the description of neutrinos is questionable. The only justification for its use appears to be its accuracy, which as far as we know is due to an accident and not to some fundamental reason. Another issue concerns the proper definition of the flavor states and of their Fock space in this framework. It has been shown in fact that the use of the quantum mechanical formalism leads to flavor violation in the production and detection vertices when oscillating particles are involved.

Even though the formulation of the problem in the framework of QFT has some phenomenological consequences as it predicts a correction in the oscillation formulae (unfortunately, at least in the case of neutrinos these correction are probably too small to be detectable in the near future), its main advantages concern the theoretical point of view. Besides being intrinsically relativistic and thus allowing to overcome the Bargmann rule, this framework allows to give a proper definition of the flavor states which avoids conflict with tree level flavor conservation in the Standard Model processes. Moreover, it has been possible to put this framework on a mathematically rigorous basis [81].

The field theoretical formalism of particle mixing is a very nice example of the existence of unitarily inequivalent representations of the canonical commutator relations in QFT. In fact, it was first shown in [39] that the Fock space of flavor states is unitarily inequivalent to the Fock space of mass eigenstates in the infinite volume limit¹. Flavor states and in particular

¹This situation is analogous to that found in many different situations of physical interest, such as supercon-

the flavor vacuum have the structure of condensates of pairs of massive particles and have the structure of generalized coherent states [124].

In the following we will treat the two Dirac fermion (neutrino) case rather in detail and just briefly outline the extension to other situations of interest, referring to the cited literature for details. A pivotal role is played by the flavor charges, which we will study in the next section. Flavor neutrino states will be defined as the eigenstates of these charges.

2.1 Flavor charges of mixed neutrinos

2.1.1 Massive neutrinos

Let us consider for the moment the situation in which we have just one lepton generation. Obviously in this situation there is no mixing.

Consider the decay process $W^+ \rightarrow e^+ + \nu_e$. Below the spontaneous symmetry breaking scale the relevant terms in the Lagrangian density for weak interactions are

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int} \quad (2.1)$$

where \mathcal{L}_0 is the free lepton Lagrangian:

$$\bar{\nu}_e(x)(i\gamma_\mu\partial^\mu - m_{\nu_e})\nu_e(x) + \bar{e}(x)(i\gamma_\mu\partial^\mu - m_e)e(x), \quad (2.2)$$

where $\bar{\nu}_e(x) = \nu_e^\dagger\gamma_0$, m_e and m_{ν_e} are the electron and electron neutrino mass respectively. \mathcal{L}_{int} is the charged current interaction term:

$$\mathcal{L}_{int} = \frac{g}{2\sqrt{2}} [W_\mu^+(x)\bar{\nu}_e(x)\gamma_\mu(1 - \gamma^5)e(x) + h.c.]. \quad (2.3)$$

The Lagrangian (3.13) is obviously invariant under $U(1)$ phase transformations:

$$e(x) \rightarrow e^{i\alpha}e(x), \quad \nu_e(x) \rightarrow e^{i\alpha}\nu_e(x), \quad \alpha \in \mathbb{R}, \quad (2.4)$$

so the Noether charges :

$$Q_e^{tot} = Q_{\nu_e} + Q_e, \quad (2.5)$$

$$Q_{\nu_e}(t) \equiv \int d^3\mathbf{x}\nu_e^\dagger(x)\nu_e(x), \quad (2.6)$$

$$Q_e(t) \equiv \int d^3\mathbf{x}e^\dagger(x)e(x), \quad (2.7)$$

are conserved (we use the notation $x_0 = t$ ($c = 1$)). By using the equal time commutation relations:

$$[Q_{\nu_e}(t), \mathcal{L}_{int}(x)] = -[Q_e(t), \mathcal{L}_{int}(x)] \quad (2.8)$$

ductivity, thermal field theory, quantum field theory on curved spacetimes and black hole physics, spontaneous symmetry breakdown and phase transitions, quantum dissipative systems and so on (see [35] for a recent comprehensive review), so it is really ubiquitous and at the root of many physical phenomena.

we find

$$[Q_e^{tot}, \mathcal{L}_{int}(x)] = 0 \quad (2.9)$$

which expresses the conservation of lepton charge in weak charged current processes. We also have:

$$[Q_e^{tot}, \mathcal{L}_0(x)] = 0. \quad (2.10)$$

which means that in any process described by (3.13) the electron neutrino state $|\nu_e\rangle$ is an eigenstate of the flavor charge Q_{ν_e} , provided of course its energy is under the threshold necessary for it to decay through the vertex (2.3).

Let us now switch to the situation in which there are two flavors and mixing. The Lagrangian keeps the form (3.13), where now the kinetic terms are:

$$\mathcal{L}_0 = (\bar{\nu}_e, \bar{\nu}_\mu) (i\gamma_\mu \partial^\mu - M_\nu) \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} + (\bar{e}, \bar{\mu}) (i\gamma_\mu \partial^\mu - M_l) \begin{pmatrix} e \\ \mu \end{pmatrix}, \quad (2.11)$$

in which the mass matrices of neutrinos and charged leptons appear:

$$M_\nu = \begin{pmatrix} m_{\nu_e} & m_{\nu_{e\mu}} \\ m_{\nu_{e\mu}} & m_{\nu_\mu} \end{pmatrix}; \quad M_l = \begin{pmatrix} m_e & 0 \\ 0 & m_\mu \end{pmatrix}. \quad (2.12)$$

Notice that the neutrino mass matrix is non diagonal.

The interaction Lagrangian is now given by:

$$\mathcal{L}_{int} = \frac{g}{2\sqrt{2}} [W_\mu^+(x) \bar{\nu}_e(x) \gamma^\mu (1 - \gamma^5) e(x) + W_\mu^+(x) \bar{\nu}_\mu(x) \gamma^\mu (1 - \gamma^5) \mu(x) + h.c.] \quad (2.13)$$

and, unlike \mathcal{L}_0 , is diagonal in the neutrino fields ν_e and ν_μ . The total Lagrangian \mathcal{L} is invariant under the combined phase rotations:

$$e(x) \rightarrow e^{i\alpha} e(x), \quad \nu_e(x) \rightarrow e^{i\alpha} \nu_e(x), \quad (2.14)$$

$$\mu(x) \rightarrow e^{i\alpha} \mu(x), \quad \nu_\mu(x) \rightarrow e^{i\alpha} \nu_\mu(x). \quad (2.15)$$

which are generated by:

$$Q_e(t) = \int d^3\mathbf{x} e^\dagger(x) e(x), \quad Q_{\nu_e}(t) = \int d^3\mathbf{x} \nu_e^\dagger(x) \nu_e(x), \quad (2.16)$$

$$Q_\mu(t) = \int d^3\mathbf{x} \mu^\dagger(x) \mu(x), \quad Q_{\nu_\mu}(t) = \int d^3\mathbf{x} \nu_\mu^\dagger(x) \nu_\mu(x), \quad (2.17)$$

respectively. As a consequence of this invariance we have:

$$[Q_l^{tot}, \mathcal{L}(x)] = 0, \quad (2.18)$$

where Q_l^{tot} is the total flavor Noether charge:

$$Q_l^{tot} = Q_{\nu_e}(t) + Q_{\nu_\mu}(t) + Q_e(t) + Q_\mu(t) = Q_e^{tot}(t) + Q_\mu^{tot}(t), \quad (2.19)$$

$$Q_e^{tot}(t) = Q_{\nu_e}(t) + Q_e(t), \quad Q_\mu^{tot}(t) = Q_{\nu_\mu}(t) + Q_\mu(t). \quad (2.20)$$

This invariance guarantees that the total lepton number is conserved. The flavor charges (2.17) have the same structure with and without mixing. The free Lagrangian \mathcal{L}_0 is not separately invariant under the phase rotations (2.14) and (2.15), due to the fact that the neutrino mass matrix is nondiagonal. In fact:

$$[Q_e^{tot}(t), \mathcal{L}_0(x)] \neq 0, \quad (2.21)$$

$$[Q_\mu^{tot}(t), \mathcal{L}_0(x)] \neq 0. \quad (2.22)$$

The same is not true for the interaction Lagrangian, which commutes with both the electron and muon lepton charges with or without mixing:

$$[Q_e^{tot}(t), \mathcal{L}_{int}(x)] = 0, \quad (2.23)$$

$$[Q_\mu^{tot}(t), \mathcal{L}_{int}(x)] = 0. \quad (2.24)$$

This means that the considerations made in the preceding case to define flavor states of neutrinos can be repeated in this case: even when there is mixing, a flavor neutrino state is defined in the production vertex as an eigenstate of the corresponding flavor charge, namely Q_{ν_e} for electron neutrinos and Q_{ν_μ} for muon neutrinos. For this situation to be physically realized the spatial extension of the neutrino source must be much smaller than the oscillation length. Being the oscillation length very big, this is the typical experimental situation.

2.1.2 Mixed neutrinos and definition of flavor charges

Let us now concentrate on the neutrino sector of the above Lagrangian. Being the interaction Lagrangian diagonal in the flavor fields, it will be sufficient to consider only \mathcal{L}_0 . Let us suppose for now that there is no mixing i.e. that also the free Lagrangian is diagonal. It will then describe two free Dirac fields of masses m_1 and m_2 :

$$\mathcal{L}_\nu(x) = \bar{\nu}_m(x) (i \not{\partial} - M_\nu^d) \nu_m(x), \quad (2.25)$$

where $\nu_m^T = (\nu_1, \nu_2)$ and $M_\nu^d = \text{diag}(m_1, m_2)$. This Lagrangian is obviously invariant under global $U(1)$ transformations of the type $\nu'_m(x) = e^{i\alpha} \nu_m(x)$, to which is associated the conserved charge $Q_\nu = \int I^0(x) d^3\mathbf{x}$ (con $I^\mu(x) = \bar{\nu}_m(x) \gamma^\mu \nu_m(x)$), which is the total lepton number of the neutrinos.

Consider now the global $SU(2)$ transformation:

$$\nu'_m(x) = e^{i\alpha_j \cdot \tau_j} \nu_m(x) \quad j = 1, 2, 3. \quad (2.26)$$

where α_j are real parameters and $\tau_j = \sigma_j/2$, where σ_j are the Pauli matrices. Being $m_1 \neq m_2$ this is not a symmetry of \mathcal{L}_ν ; its on-shell variation is:

$$\delta \mathcal{L}_\nu = i\alpha_j \bar{\nu}_m(x) [\tau_j, M_\nu^d] \nu_m(x) = -\alpha_j \partial_\mu J_{m,j}^\mu(x), \quad (2.27)$$

where the currents are given by:

$$J_{m,j}^\mu(x) = \bar{\nu}_m(x) \gamma^\mu \tau_j \nu_m(x), \quad j = 1, 2, 3. \quad (2.28)$$

The corresponding Noether charges are in principle not conserved:

$$Q_{m,j}(t) = \int d^3\mathbf{x} J_{m,j}^0(x), \quad (2.29)$$

and they form a representation of the $su(2)$ algebra:

$$[Q_{m,i}(t), Q_{m,j}(t)] = i\varepsilon_{ijk} Q_{m,k}(t). \quad (2.30)$$

The Casimir operator of this algebra is proportional to the above defined total conserved charge:

$$Q_{m,0} = \frac{1}{2} Q_\nu \quad (2.31)$$

Actually, being $[M_\nu^d, \tau_3] = 0$, $Q_{m,3}$ is also conserved. By looking at the explicit expression for this charge we see that the charges Q_{ν_1} and Q_{ν_2} of the ν_1 and ν_2 fields respectively (which are just the flavor charges in the no mixing case) are separately conserved, which is obvious being these fields free. These charges can be obtained by suitably combining the above defined $SU(2)$ charges:

$$Q_{\nu_1} \equiv \frac{1}{2} Q_\nu + Q_{m,3}; \quad Q_{\nu_2} \equiv \frac{1}{2} Q_\nu - Q_{m,3}. \quad (2.32)$$

$$Q_{\nu_i} = \int d^3\mathbf{x} \nu_i^\dagger(x) \nu_i(x), \quad (2.33)$$

where $i = 1, 2$.

Let us now switch to the non diagonal case:

$$\mathcal{L}_\nu(x) = \bar{\nu}_f(x) (i \not{\partial} - M_\nu) \nu_f(x), \quad (2.34)$$

where $\nu_f^T = (\nu_e, \nu_\mu)$. Let us repeat the same steps. The on-shell variation of this Lagrangian under the global $SU(2)$ transformation

$$\nu'_f(x) = e^{i\alpha_j \tau_j} \nu_f(x) \quad j = 1, 2, 3, \quad (2.35)$$

is given by:

$$\delta \mathcal{L}_\nu(x) = i\alpha_j \bar{\nu}_f(x) [\tau_j, M_\nu] \nu_f(x) = -\alpha_j \partial_\mu J_{f,j}^\mu(x), \quad (2.36)$$

where

$$J_{f,j}^\mu(x) = \bar{\nu}_f(x) \gamma^\mu \tau_j \nu_f(x), \quad j = 1, 2, 3. \quad (2.37)$$

The resulting charges

$$Q_{f,j}(t) = \int d^3\mathbf{x} J_{f,j}^0(x) \quad (2.38)$$

close again the $su(2)$ algebra but now $Q_{f,3}(t)$ is time-dependent. From the physical point of view this corresponds to an exchange of charge between ν_e and ν_μ which of course is just the neutrino oscillation phenomenon. The time dependent flavor charges for the mixed fields are given by [34]:

$$Q_{\nu_e}(t) = \frac{1}{2}Q_\nu + Q_{f,3}(t), \quad Q_{\nu_\mu}(t) = \frac{1}{2}Q_\nu - Q_{f,3}(t), \quad (2.39)$$

$$Q_{\nu_\sigma}(t) = \int d^3\mathbf{x} \nu_\sigma^\dagger(x) \nu_\sigma(x), \quad (2.40)$$

where $\sigma = e, \mu$ and $Q_{\nu_e}(t) + Q_{\nu_\mu}(t) = Q_\nu$. The flavor charges (2.40) reduce to (2.17) in the diagonal case.

The $SU(2)$ structure just discussed is very important as it links the flavor charges to the mixing generator we shall encounter in the next section.

2.2 Mixing transformations of fermion fields and vacuum structure

In this section we shall study the mixing transformations of fermions and the corresponding vacuum structure in the context of QFT. For simplicity we shall consider only the two flavor case.

The Lagrangian (2.34) is diagonalizable to (2.25) by means of the mixing transformations

$$\nu_e = \nu_1 \cos \theta + \nu_2 \sin \theta \quad (2.41)$$

$$\nu_\mu = -\nu_1 \sin \theta + \nu_2 \cos \theta, \quad (2.42)$$

where ν_e and ν_μ are definite flavor Dirac fields and ν_1 and ν_2 are Dirac fields with definite masses, m_1 and m_2 respectively. The latter admit the usual Fourier expansion:

$$\nu_i(x) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}, r} [u_{\mathbf{k}, i}^r \alpha_{\mathbf{k}, i}^r(t) + v_{-\mathbf{k}, i}^r \beta_{-\mathbf{k}, i}^{r\dagger}(t)], \quad i = 1, 2, \quad (2.43)$$

where V is the volume. In the following we shall restrict our system in a box of finite volume, and the infinite volume limit will be taken in the end. We have

$$\alpha_{\mathbf{k}, i}^r(t) = \alpha_{\mathbf{k}, i}^r e^{ik \cdot x}, \quad \beta_{-\mathbf{k}, i}^{r\dagger}(t) = \beta_{-\mathbf{k}, i}^{r\dagger} e^{-ik \cdot x} \quad (2.44)$$

$$u_{\mathbf{k}, i}^1 = \left(\frac{\omega_{k, i} + m_i}{2\omega_{k, i}} \right)^{\frac{1}{2}} \begin{pmatrix} 1 \\ 0 \\ \frac{k_3}{\omega_{k, i} + m_i} \\ \frac{k_1 + ik_2}{\omega_{k, i} + m_i} \end{pmatrix}; \quad u_{\mathbf{k}, i}^2 = \left(\frac{\omega_{k, i} + m_i}{2\omega_{k, i}} \right)^{\frac{1}{2}} \begin{pmatrix} 0 \\ 1 \\ \frac{k_1 - ik_2}{\omega_{k, i} + m_i} \\ \frac{-k_3}{\omega_{k, i} + m_i} \end{pmatrix} \quad (2.45)$$

$$v_{-\mathbf{k}, i}^1 = \left(\frac{\omega_{k, i} + m_i}{2\omega_{k, i}} \right)^{\frac{1}{2}} \begin{pmatrix} \frac{-k_3}{\omega_{k, i} + m_i} \\ \frac{k_1 - ik_2}{\omega_{k, i} + m_i} \\ 1 \\ 0 \end{pmatrix}; \quad v_{-\mathbf{k}, i}^2 = \left(\frac{\omega_{k, i} + m_i}{2\omega_{k, i}} \right)^{\frac{1}{2}} \begin{pmatrix} \frac{-k_1 + ik_2}{\omega_{k, i} + m_i} \\ \frac{k_3}{\omega_{k, i} + m_i} \\ 0 \\ 1 \end{pmatrix} \quad (2.46)$$

and

$$\omega_{k,i} = \sqrt{\mathbf{k}^2 + m_i^2}. \quad (2.47)$$

The ladder operators $\alpha_{\mathbf{k},i}^r$ and $\beta_{\mathbf{k},i}^r$, $i = 1, 2$, $r = 1, 2$ annihilate the vacuum $|0\rangle_{1,2} \equiv |0\rangle_1 \otimes |0\rangle_2$: $\alpha_{\mathbf{k},i}^r|0\rangle_{12} = \beta_{\mathbf{k},i}^r|0\rangle_{12} = 0$. The canonical anticommutation relations are

$$\left\{ \nu_i^\alpha(x), \nu_j^{\beta\dagger}(y) \right\}_{t=t'} = \delta^3(\mathbf{x} - \mathbf{y}) \delta_{\alpha\beta} \delta_{ij}, \quad \alpha, \beta = 1, \dots, 4, \quad (2.48)$$

or equivalently:

$$\left\{ \alpha_{\mathbf{k},i}^r, \alpha_{\mathbf{q},j}^{s\dagger} \right\} = \delta_{\mathbf{kq}} \delta_{rs} \delta_{ij}; \quad \left\{ \beta_{\mathbf{k},i}^r, \beta_{\mathbf{q},j}^{s\dagger} \right\} = \delta_{\mathbf{kq}} \delta_{rs} \delta_{ij}, \quad i, j = 1, 2. \quad (2.49)$$

and all other anticommutators vanish. The orthonormality and completeness relations for the spinors are:

$$u_{\mathbf{k},i}^{r\dagger} u_{\mathbf{k},i}^s = v_{\mathbf{k},i}^{r\dagger} v_{\mathbf{k},i}^s = \delta_{rs}, \quad (2.50)$$

$$u_{\mathbf{k},i}^{r\dagger} v_{-\mathbf{k},i}^s = v_{-\mathbf{k},i}^{r\dagger} u_{\mathbf{k},i}^s = 0, \quad (2.51)$$

$$\sum_r (u_{\mathbf{k},i}^r u_{\mathbf{k},i}^{r\dagger} + v_{-\mathbf{k},i}^r v_{-\mathbf{k},i}^{r\dagger}) = 1. \quad (2.52)$$

Eqs. (2.41) relate the two mass terms

$$H_{1,2} = m_1 \nu_1^\dagger \nu_1 + m_1 \nu_2^\dagger \nu_2 \quad (2.53)$$

$$H_{e,\mu} = m_{ee} \nu_e^\dagger \nu_e + m_{\mu\mu} \nu_\mu^\dagger \nu_\mu + m_{e\mu} (\nu_e^\dagger \nu_\mu + \nu_\mu^\dagger \nu_e), \quad (2.54)$$

where

$$m_{ee} = m_1 \cos^2 \theta + m_2 \sin^2 \theta, \quad (2.55)$$

$$m_{\mu\mu} = m_1 \sin^2 \theta + m_2 \cos^2 \theta, \quad (2.56)$$

$$m_{e\mu} = (m_2 - m_1) \sin \theta \cos \theta. \quad (2.57)$$

This shows that mixed fields, unlike massive fields, are interacting, which is a fundamental difference from the point of view of QFT, Being the interaction terms quadratic in the fields, it's possible to treat the system nonperturbatively.

QFT lives on two levels. The underlying dynamics (i.e. the Lagrangian and the corresponding equations of motion) is given in terms of the Heisenberg (interacting) fields. On the other hand the physical observables are expressed in terms of asymptotic IN (or OUT) fields, also called physical or free fields. In the LSZ formalism IN fields (resp. OUT fields) are obtained by making the weak limit of the Heisenberg fields $t \rightarrow -\infty$ (resp $t \rightarrow \infty$). The meaning of the weak limit is that the realization of the underlying dynamics in terms of the free fields is representation dependent. This representation dependence takes roots in the fact that in QFT the canonical commutation relations have an infinite number of unitarily inequivalent representations (UIRs), which is a basic feature of QFT ²[136, 135]. Since physical observables are

²This happens because the hypotheses of the Stone-von Neumann theorem of QM break down in the case of systems with an infinite number of degrees of freedom.

described in terms of asymptotic fields, different UIRs describe different phases of the system. The typical example are theories exhibiting spontaneous symmetry breaking (SSB), where the same Heisenberg fields describe both the symmetric and the ordered phase. To obtain physically relevant results a careful study of the Haag expansion (also known as dynamical map) which links Heisenberg and free fields. In particular, the assumption that free and interacting fields share the same vacuum and the same Fock representation turns out to be wrong (this latter result is the celebrated Haag's theorem [80]).

For the reasons briefly outlined above the mixing relations (2.41) need a careful analysis, which is what we shall do in the following. We shall study the relation between the Fock spaces $\mathcal{H}_{1,2}$ and $\mathcal{H}_{e,\mu}$, which refer to the fields ν_1, ν_2 and ν_e, ν_μ respectively, in the infinite volume limit. From what we said above we see that these two spaces describe a free and an interacting theory respectively, so we expect that they will become orthogonal in this limit.

First of all, let us rewrite the relations (2.41) in the form:

$$\nu_e^\alpha(x) = G_\theta^{-1}(t)\nu_1^\alpha G_\theta(t) \quad (2.58)$$

$$\nu_\mu^\alpha(x) = G_\theta^{-1}(t)\nu_2^\alpha G_\theta(t) \quad (2.59)$$

where $G(\theta)$ is the generator of the mixing transformations and it is given by:

$$G_\theta(t) = \exp \left[\theta \int d^3\mathbf{x} \left(\nu_1^\dagger(x)\nu_2(x) - \nu_2^\dagger(x)\nu_1(x) \right) \right], \quad (2.60)$$

and it is a unitary operator at finite volume: $G_\theta^{-1}(t) = G_{-\theta}(t) = G_\theta^\dagger(t)$. This object generates a canonical transformation since the canonical anticommutation relations (2.48) are preserved. Eq. (2.60) can be verified by double differentiation with respect to θ :

$$d^2\nu_e^\alpha/d\theta^2 = -\nu_e^\alpha, \quad d^2\nu_\mu^\alpha/d\theta^2 = -\nu_\mu^\alpha, \quad (2.61)$$

and solving these two differential equations with the initial conditions:

$$\nu_e^\alpha|_{\theta=0} = \nu_1^\alpha, \quad d\nu_e^\alpha/d\theta|_{\theta=0} = \nu_2^\alpha \quad \text{and} \quad \nu_\mu^\alpha|_{\theta=0} = \nu_2^\alpha, \quad d\nu_\mu^\alpha/d\theta|_{\theta=0} = -\nu_1^\alpha, \quad (2.62)$$

From its very definition $G_\theta(t)$ is an element of the group $SU(2)$. Indeed, by defining the operators:

$$S_+(t) \equiv \int d^3\mathbf{x} \nu_1^\dagger(x)\nu_2(x) \quad , \quad S_-(t) \equiv \int d^3\mathbf{x} \nu_2^\dagger(x)\nu_1(x) = (S_+)^\dagger, \quad (2.63)$$

it is possible to write:

$$G_\theta(t) = \exp[\theta(S_+ - S_-)]. \quad (2.64)$$

By introducing the operator:

$$S_3 \equiv \frac{1}{2} \int d^3\mathbf{x} \left(\nu_1^\dagger(x)\nu_1(x) - \nu_2^\dagger(x)\nu_2(x) \right), \quad (2.65)$$

and the Casimir operator

$$S_0 \equiv \frac{1}{2} \int d^3\mathbf{x} \left(\nu_1^\dagger(x)\nu_1(x) + \nu_2^\dagger(x)\nu_2(x) \right), \quad (2.66)$$

we see that the $su(2)$ algebra is closed:

$$[S_+(t), S_-(t)] = 2S_3 \quad , \quad [S_3, S_\pm(t)] = \pm S_\pm(t) \quad , \quad [S_0, S_3] = [S_0, S_\pm(t)] = 0 . \quad (2.67)$$

By using the Fourier expansions Fourier (2.43) we get:

$$\begin{aligned} S_+(t) \equiv \sum_{\mathbf{k}} S_+^{\mathbf{k}}(t) &= \sum_{\mathbf{k}} \sum_{r,s} \left(u_{\mathbf{k},1}^{r\dagger}(t) u_{\mathbf{k},2}^s(t) \alpha_{\mathbf{k},1}^{r\dagger} \alpha_{\mathbf{k},2}^s + v_{-\mathbf{k},1}^{r\dagger}(t) u_{\mathbf{k},2}^s(t) \beta_{-\mathbf{k},1}^r \alpha_{\mathbf{k},2}^s + \right. \\ &\left. + u_{\mathbf{k},1}^{r\dagger}(t) v_{-\mathbf{k},2}^s(t) \alpha_{\mathbf{k},1}^{r\dagger} \beta_{-\mathbf{k},2}^{s\dagger} + v_{-\mathbf{k},1}^{r\dagger}(t) v_{-\mathbf{k},2}^s(t) \beta_{-\mathbf{k},1}^r \beta_{-\mathbf{k},2}^{s\dagger} \right) , \end{aligned} \quad (2.68)$$

$$\begin{aligned} S_-(t) \equiv \sum_{\mathbf{k}} S_-^{\mathbf{k}}(t) &= \sum_{\mathbf{k}} \sum_{r,s} \left(u_{\mathbf{k},2}^{r\dagger}(t) u_{\mathbf{k},1}^s(t) \alpha_{\mathbf{k},2}^{r\dagger} \alpha_{\mathbf{k},1}^s + v_{-\mathbf{k},2}^{r\dagger}(t) u_{\mathbf{k},1}^s(t) \beta_{-\mathbf{k},2}^r \alpha_{\mathbf{k},1}^s + \right. \\ &\left. + u_{\mathbf{k},2}^{r\dagger}(t) v_{-\mathbf{k},1}^s(t) \alpha_{\mathbf{k},2}^{r\dagger} \beta_{-\mathbf{k},1}^{s\dagger} + v_{-\mathbf{k},2}^{r\dagger}(t) v_{-\mathbf{k},1}^s(t) \beta_{-\mathbf{k},2}^r \beta_{-\mathbf{k},1}^{s\dagger} \right) , \end{aligned} \quad (2.69)$$

$$S_3 \equiv \sum_{\mathbf{k}} S_3^{\mathbf{k}} = \frac{1}{2} \sum_{\mathbf{k},r} \left(\alpha_{\mathbf{k},1}^{r\dagger} \alpha_{\mathbf{k},1}^r - \beta_{-\mathbf{k},1}^{r\dagger} \beta_{-\mathbf{k},1}^r - \alpha_{\mathbf{k},2}^{r\dagger} \alpha_{\mathbf{k},2}^r + \beta_{-\mathbf{k},2}^{r\dagger} \beta_{-\mathbf{k},2}^r \right) , \quad (2.70)$$

$$S_0 \equiv \sum_{\mathbf{k}} S_0^{\mathbf{k}} = \frac{1}{2} \sum_{\mathbf{k},r} \left(\alpha_{\mathbf{k},1}^{r\dagger} \alpha_{\mathbf{k},1}^r - \beta_{-\mathbf{k},1}^{r\dagger} \beta_{-\mathbf{k},1}^r + \alpha_{\mathbf{k},2}^{r\dagger} \alpha_{\mathbf{k},2}^r - \beta_{-\mathbf{k},2}^{r\dagger} \beta_{-\mathbf{k},2}^r \right) . \quad (2.71)$$

The $su(2)$ algebra is closed for every \mathbf{k} component:

$$[S_+^{\mathbf{k}}(t), S_-^{\mathbf{k}}(t)] = 2S_3^{\mathbf{k}} , \quad [S_3^{\mathbf{k}}(t), S_\pm^{\mathbf{k}}(t)] = \pm S_\pm^{\mathbf{k}}(t) , \quad [S_0^{\mathbf{k}}, S_3^{\mathbf{k}}] = [S_0^{\mathbf{k}}, S_\pm^{\mathbf{k}}] = 0 , \quad (2.72)$$

$$[S_\pm^{\mathbf{k}}(t), S_\pm^{\mathbf{p}}(t)] = [S_3^{\mathbf{k}}(t), S_\pm^{\mathbf{p}}(t)] = [S_3^{\mathbf{k}}, S_3^{\mathbf{p}}] = 0 , \quad \mathbf{k} \neq \mathbf{p} \quad (2.73)$$

which means that the original $su(2)$ algebra splits in mutually commuting $su_{\mathbf{k}}(2)$ algebras, given by (2.72). The group structure is then $\bigotimes_{\mathbf{k}} SU_{\mathbf{k}}(2)$.

The relation between the Hilbert spaces $\mathcal{H}_{1,2}$ and $\mathcal{H}_{e,\mu}$ let us consider a generic matrix element of say $\nu_1^\alpha(x)$, ${}_{1,2}\langle a | \nu_1^\alpha(x) | b \rangle_{1,2}$, where $|a\rangle_{1,2}, |b\rangle_{1,2} \in \mathcal{H}_{1,2}$. By using the inverse of the first of the (2.58) we get:

$${}_{1,2}\langle a | G_\theta(t) \nu_e^\alpha(x) G_\theta^{-1}(t) | b \rangle_{1,2} \equiv {}_{1,2}\langle a | \nu_1^\alpha(x) | b \rangle_{1,2} . \quad (2.74)$$

Being ν_e defined on $\mathcal{H}_{e,\mu}$, this equation shows that $G_\theta(t)^{-1} |a\rangle_{1,2} \in \mathcal{H}_{e,\mu}$, so we see that $G_\theta^{-1}(t)$ maps $\mathcal{H}_{1,2}$ in $\mathcal{H}_{e,\mu}$:

$$G_\theta^{-1}(t) : \mathcal{H}_{1,2} \mapsto \mathcal{H}_{e,\mu} . \quad (2.75)$$

In particular this is true for the vacuum $|0\rangle_{1,2}$:

$$|0(t)\rangle_{e,\mu} = G_\theta^{-1}(t) |0\rangle_{1,2} , \quad (2.76)$$

where $|0(t)\rangle_{e,\mu}$ is the vacuum of $\mathcal{H}_{e,\mu}$ which will be referred to as the *flavor vacuum*. Let us remark that all these manipulations are done *at finite volume* V , so that they are mathematically well defined.

Being $G_\theta(t)$ a linear operator, we can define the ladder operators of the fields $\nu_e(x)$ and $\nu_\mu(x)$ at any given time by:

$$\begin{aligned}
\alpha_{\mathbf{k},e}^r(t) |0(t)\rangle_{e,\mu} &= G_\theta^{-1}(t) \alpha_{\mathbf{k},1}^r |0\rangle_{1,2} = 0, \\
\alpha_{\mathbf{k},\mu}^r(t) |0(t)\rangle_{e,\mu} &= G_\theta^{-1}(t) \alpha_{\mathbf{k},2}^r |0\rangle_{1,2} = 0, \\
\beta_{\mathbf{k},e}^r(t) |0(t)\rangle_{e,\mu} &= G_\theta^{-1}(t) \beta_{\mathbf{k},1}^r |0\rangle_{1,2} = 0, \\
\beta_{\mathbf{k},\mu}^r(t) |0(t)\rangle_{e,\mu} &= G_\theta^{-1}(t) \beta_{\mathbf{k},2}^r |0\rangle_{1,2} = 0,
\end{aligned} \tag{2.77}$$

from which follows the dynamical map:

$$\begin{aligned}
\alpha_{\mathbf{k},e}^r(t) &\equiv G_\theta^{-1}(t) \alpha_{\mathbf{k},1}^r G_\theta(t), \\
\alpha_{\mathbf{k},\mu}^r(t) &\equiv G_\theta^{-1}(t) \alpha_{\mathbf{k},2}^r G_\theta(t), \\
\beta_{\mathbf{k},e}^r(t) &\equiv G_\theta^{-1}(t) \beta_{\mathbf{k},1}^r G_\theta(t), \\
\beta_{\mathbf{k},\mu}^r(t) &\equiv G_\theta^{-1}(t) \beta_{\mathbf{k},2}^r G_\theta(t).
\end{aligned} \tag{2.78}$$

This allows us to expand the flavor fields in the same spinor bases used for ν_1 and ν_2 (we shall see later that this is not the only possibility, which gives rise to an ambiguity):

$$\nu_e(\mathbf{x}, t) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k},r} e^{i\mathbf{k}\cdot\mathbf{x}} \left[u_{\mathbf{k},1}^r \alpha_{\mathbf{k},e}^r(t) + v_{-\mathbf{k},1}^r \beta_{-\mathbf{k},e}^{r\dagger}(t) \right], \tag{2.79}$$

$$\nu_\mu(\mathbf{x}, t) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k},r} e^{i\mathbf{k}\cdot\mathbf{x}} \left[u_{\mathbf{k},2}^r \alpha_{\mathbf{k},\mu}^r(t) + v_{-\mathbf{k},2}^r \beta_{-\mathbf{k},\mu}^{r\dagger}(t) \right]. \tag{2.80}$$

The explicit expression of the dynamical map (2.78) is:

$$\begin{aligned}
\alpha_{\mathbf{k},e}^r(t) &= \cos \theta \alpha_{\mathbf{k},1}^r + \sin \theta \sum_s \left[u_{\mathbf{k},1}^{r\dagger}(t) u_{\mathbf{k},2}^s(t) \alpha_{\mathbf{k},2}^s + u_{\mathbf{k},1}^{r\dagger}(t) v_{-\mathbf{k},2}^s(t) \beta_{-\mathbf{k},2}^{s\dagger} \right] \\
\alpha_{\mathbf{k},\mu}^r(t) &= \cos \theta \alpha_{\mathbf{k},2}^r - \sin \theta \sum_s \left[u_{\mathbf{k},2}^{r\dagger}(t) u_{\mathbf{k},1}^s(t) \alpha_{\mathbf{k},1}^s + u_{\mathbf{k},2}^{r\dagger}(t) v_{-\mathbf{k},1}^s(t) \beta_{-\mathbf{k},1}^{s\dagger} \right] \\
\beta_{-\mathbf{k},e}^r(t) &= \cos \theta \beta_{-\mathbf{k},1}^r + \sin \theta \sum_s \left[v_{-\mathbf{k},2}^{s\dagger}(t) v_{-\mathbf{k},1}^r(t) \beta_{-\mathbf{k},2}^s + u_{\mathbf{k},2}^{s\dagger}(t) v_{-\mathbf{k},1}^r(t) \alpha_{\mathbf{k},2}^{s\dagger} \right] \\
\beta_{-\mathbf{k},\mu}^r(t) &= \cos \theta \beta_{-\mathbf{k},2}^r - \sin \theta \sum_s \left[v_{-\mathbf{k},1}^{s\dagger}(t) v_{-\mathbf{k},2}^r(t) \beta_{-\mathbf{k},1}^s + u_{\mathbf{k},1}^{s\dagger}(t) v_{-\mathbf{k},2}^r(t) \alpha_{\mathbf{k},1}^{s\dagger} \right].
\end{aligned} \tag{2.81}$$

which in the reference frame such that $\mathbf{k} = (0, 0, |\mathbf{k}|)$ simplifies in:

$$\begin{aligned}
\alpha_{\mathbf{k},e}^r(t) &= \cos \theta \alpha_{\mathbf{k},1}^r + \sin \theta \left(U_{\mathbf{k}}^*(t) \alpha_{\mathbf{k},2}^r + \epsilon^r V_{\mathbf{k}}(t) \beta_{-\mathbf{k},2}^{r\dagger} \right) \\
\alpha_{\mathbf{k},\mu}^r(t) &= \cos \theta \alpha_{\mathbf{k},2}^r - \sin \theta \left(U_{\mathbf{k}}(t) \alpha_{\mathbf{k},1}^r - \epsilon^r V_{\mathbf{k}}(t) \beta_{-\mathbf{k},1}^{r\dagger} \right) \\
\beta_{-\mathbf{k},e}^r(t) &= \cos \theta \beta_{-\mathbf{k},1}^r + \sin \theta \left(U_{\mathbf{k}}^*(t) \beta_{-\mathbf{k},2}^r - \epsilon^r V_{\mathbf{k}}(t) \alpha_{\mathbf{k},2}^{r\dagger} \right) \\
\beta_{-\mathbf{k},\mu}^r(t) &= \cos \theta \beta_{-\mathbf{k},2}^r - \sin \theta \left(U_{\mathbf{k}}(t) \beta_{-\mathbf{k},1}^r + \epsilon^r V_{\mathbf{k}}(t) \alpha_{\mathbf{k},1}^{r\dagger} \right),
\end{aligned} \tag{2.82}$$

where $\epsilon^r = (-1)^r$ and we defined:

$$\begin{aligned} U_{\mathbf{k}}(t) &\equiv u_{\mathbf{k},2}^{r\dagger}(t)u_{\mathbf{k},1}^r(t) = v_{-\mathbf{k},1}^{r\dagger}(t)v_{-\mathbf{k},2}^r(t) \\ V_{\mathbf{k}}(t) &\equiv \epsilon^r u_{\mathbf{k},1}^{r\dagger}(t)v_{-\mathbf{k},2}^r(t) = -\epsilon^r u_{\mathbf{k},2}^{r\dagger}(t)v_{-\mathbf{k},1}^r(t) \end{aligned} \quad (2.83)$$

These coefficients have the structure:

$$V_{\mathbf{k}}(t) = |V_{\mathbf{k}}| e^{i(\omega_{k,2} + \omega_{k,1})t}, \quad U_{\mathbf{k}}(t) = |U_{\mathbf{k}}| e^{i(\omega_{k,2} - \omega_{k,1})t}; \quad (2.84)$$

and

$$|U_{\mathbf{k}}| = \left(\frac{\omega_{k,1} + m_1}{2\omega_{k,1}} \right)^{\frac{1}{2}} \left(\frac{\omega_{k,2} + m_2}{2\omega_{k,2}} \right)^{\frac{1}{2}} \left(1 + \frac{\mathbf{k}^2}{(\omega_{k,1} + m_1)(\omega_{k,2} + m_2)} \right) \quad (2.85)$$

$$|V_{\mathbf{k}}| = \left(\frac{\omega_{k,1} + m_1}{2\omega_{k,1}} \right)^{\frac{1}{2}} \left(\frac{\omega_{k,2} + m_2}{2\omega_{k,2}} \right)^{\frac{1}{2}} \left(\frac{k}{(\omega_{k,2} + m_2)} - \frac{k}{(\omega_{k,1} + m_1)} \right) \quad (2.86)$$

$$|U_{\mathbf{k}}|^2 + |V_{\mathbf{k}}|^2 = 1. \quad (2.87)$$

We have thus unraveled the structure of the mixing transformation at the level of the ladder operators. It has the structure of a Bogoliubov transformation with coefficients $U_{\mathbf{k}}$ and $V_{\mathbf{k}}$ combined with a rotation with angle θ . It is possible to disentangle the Bogoliubov transformation and the rotation only partially [39], so these transformations are more complicated than the usual Bogoliubov transformations which appear in many physical situations.

The operator $G^{-1}(\theta) = \exp[\theta(S_- - S_+)]$ is the generator of the generalized coherent states of $SU(2)$ [124]. Thus Eq.(2.76) shows that the flavor vacuum is such a coherent state.

Let us give the explicit expression of the flavor vacuum at $t = 0$ in the chosen reference frame. By using the Gaussian decomposition, $G^{-1}(\theta)$ can be written as:

$$\exp[\theta(S_- - S_+)] = \exp(-\tan \theta S_+) \exp(-2 \ln \cos \theta S_3) \exp(\tan \theta S_-) \quad (2.88)$$

where $0 \leq \theta < \frac{\pi}{2}$. Then (2.76) can be rewritten as

$$|0\rangle_{e,\mu} = \prod_{\mathbf{k}} |0\rangle_{e,\mu}^{\mathbf{k}} = \prod_{\mathbf{k}} \exp(-\tan \theta S_+^{\mathbf{k}}) \exp(-2 \ln \cos \theta S_3^{\mathbf{k}}) \exp(\tan \theta S_-^{\mathbf{k}}) |0\rangle_{1,2}. \quad (2.89)$$

After some algebra [39] we get the following expression for $|0\rangle_{e,\mu}$ in terms of $S_{\pm}^{\mathbf{k}}$ and $S_3^{\mathbf{k}}$:

$$\begin{aligned} |0\rangle_{e,\mu} &= \prod_{\mathbf{k}} |0\rangle_{e,\mu}^{\mathbf{k}} = \prod_{\mathbf{k}} \left[1 + \sin \theta \cos \theta (S_-^{\mathbf{k}} - S_+^{\mathbf{k}}) + \frac{1}{2} \sin^2 \theta \cos^2 \theta ((S_-^{\mathbf{k}})^2 + (S_+^{\mathbf{k}})^2) + \right. \\ &\quad \left. - \sin^2 \theta S_+^{\mathbf{k}} S_-^{\mathbf{k}} + \frac{1}{2} \sin^3 \theta \cos \theta (S_-^{\mathbf{k}} (S_+^{\mathbf{k}})^2 - S_+^{\mathbf{k}} (S_-^{\mathbf{k}})^2) + \frac{1}{4} \sin^4 \theta (S_+^{\mathbf{k}})^2 (S_-^{\mathbf{k}})^2 \right] |0\rangle_{1,2}. \end{aligned} \quad (2.90)$$

Being $G_{\theta}^{-1}(t)$ a unitary operator, $|0\rangle_{e,\mu}$ is normalized. Before expanding this expression in terms of the ladder operators let us compute the scalar product ${}_{1,2}\langle 0|0\rangle_{e,\mu}$. We obtain

$${}_{1,2}\langle 0|0\rangle_{e,\mu} = \prod_{\mathbf{k}} \left(1 - \sin^2 \theta {}_{1,2}\langle 0|S_+^{\mathbf{k}} S_-^{\mathbf{k}}|0\rangle_{1,2} + \frac{1}{4} \sin^4 \theta {}_{1,2}\langle 0|(S_+^{\mathbf{k}})^2 (S_-^{\mathbf{k}})^2|0\rangle_{1,2} \right), \quad (2.91)$$

where

$$\begin{aligned}
& {}_{1,2}\langle 0|S_+^{\mathbf{k}}S_-^{\mathbf{k}}|0\rangle_{1,2} = \\
& = {}_{1,2}\langle 0|\left(\sum_{\sigma,\tau}\sum_{r,s}\left[v_{-\mathbf{k},1}^{\sigma\dagger}(t)u_{\mathbf{k},2}^\tau(t)\right]\left[u_{\mathbf{k},2}^{s\dagger}(t)v_{-\mathbf{k},1}^r(t)\right]\beta_{-\mathbf{k},1}^\sigma\alpha_{\mathbf{k},2}^\tau\alpha_{\mathbf{k},2}^{s\dagger}\beta_{-\mathbf{k},1}^{r\dagger}\right)|0\rangle_{1,2} = \\
& = \sum_{r,s}|v_{-\mathbf{k},1}^{r\dagger}(t)u_{\mathbf{k},2}^s(t)|^2 \equiv 2|V_{\mathbf{k}}|^2. \tag{2.92}
\end{aligned}$$

and

$${}_{1,2}\langle 0|(S_+^{\mathbf{k}})^2(S_-^{\mathbf{k}})^2|0\rangle_{1,2} = 2|V_{\mathbf{k}}|^4. \tag{2.93}$$

The function $V_{\mathbf{k}}$, defined in (2.83), is plotted in Fig. 2.1. We will see shortly that it is proportional to the density of the flavor vacuum condensate. $|V_{\mathbf{k}}|^2$ depends on \mathbf{k} only through its modulus k , it is limited in the interval $[0, 1]$, has a maximum in $|\mathbf{k}| = \sqrt{m_1 m_2}$ which defines the characteristic scale of the condensate. Moreover, $|V_{\mathbf{k}}|^2 = 0$ when $m_1 = m_2$ in accordance with the fact that in that case there is no mixing. We also notice that $|V_{\mathbf{k}}|^2 \rightarrow 0$ in the limit $k \rightarrow \infty$.

We have:

$$\begin{aligned}
{}_{1,2}\langle 0|0\rangle_{e,\mu} & = \prod_{\mathbf{k}} (1 - \sin^2 \theta |V_{\mathbf{k}}|^2)^2 \equiv \prod_k \Gamma(\mathbf{k}) = \\
& = \prod_{\mathbf{k}} e^{\ln \Gamma(\mathbf{k})} = e^{\sum_{\mathbf{k}} \ln \Gamma(\mathbf{k})}. \tag{2.94}
\end{aligned}$$

The properties of $|V_{\mathbf{k}}|^2$ imply that $\Gamma(\mathbf{k}) < 1$ for any \mathbf{k} , m_1, m_2 . By exploiting the usual relation $\sum_{\mathbf{k}} \rightarrow \frac{V}{(2\pi)^3} \int d^3\mathbf{k}$, we take the infinite volume limit:

$$\lim_{V \rightarrow \infty} {}_{1,2}\langle 0|0\rangle_{e,\mu} = \lim_{V \rightarrow \infty} e^{\frac{V}{(2\pi)^3} \int d^3\mathbf{k} \ln \Gamma(k)} = 0 \tag{2.95}$$

which shows the orthogonality of the two vacua in the limit. Being the orthogonality due to the IR contributions as it is an infinite volume effect, the result is not dependent on large momenta, so it is not necessary to consider the problem of the regularization of the UV divergence in the integral of $\ln \Gamma(k)$. Of course the two vacua are not orthogonal any more when $\theta = 0$ or $m_1 = m_2$.

Eq.(2.95) is an expression of the unitary inequivalence of the two representations in the infinite volume limit and shows the nontrivial nature of the mixing transformations. These induce a nontrivial structure in the flavor vacuum, which turns out to be a generalized $SU(2)$ coherent state. We thus understand how rough an approximation is to identify the flavor vacuum with the vacuum of the definite mass fields, since it leads to the loss of the coherent state structure of the vacuum and of its physical consequences. This is an approximation even in the case of finite volume.

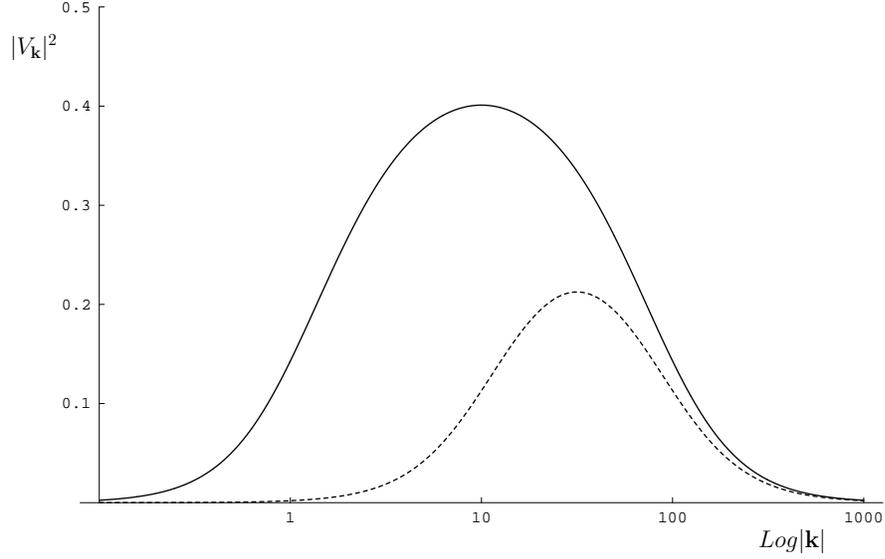


Figure 2.1: Fermionic condensation density $|V_k|^2$ as a function of k with sample values of the mass parameters m_1 and m_2 .

Continuous line: $m_1 = 1$, $m_2 = 100$

Dashed line: $m_1 = 10$, $m_2 = 100$

The complete expression of the flavor vacuum is:

$$\begin{aligned}
|0\rangle_{e,\mu} = & \prod_{\mathbf{k}} \prod_r \left[(1 - \sin^2 \theta |V_{\mathbf{k}}|^2) - \epsilon^r \sin \theta \cos \theta |V_{\mathbf{k}}| (\alpha_{\mathbf{k},1}^{r\dagger} \beta_{-\mathbf{k},2}^{r\dagger} + \alpha_{\mathbf{k},2}^{r\dagger} \beta_{-\mathbf{k},1}^{r\dagger}) + \right. \\
& \left. + \epsilon^r \sin^2 \theta |V_{\mathbf{k}}| |U_{\mathbf{k}}| (\alpha_{\mathbf{k},1}^{r\dagger} \beta_{-\mathbf{k},1}^{r\dagger} - \alpha_{\mathbf{k},2}^{r\dagger} \beta_{-\mathbf{k},2}^{r\dagger}) + \sin^2 \theta |V_{\mathbf{k}}|^2 \alpha_{\mathbf{k},1}^{r\dagger} \beta_{-\mathbf{k},2}^{r\dagger} \alpha_{\mathbf{k},2}^{r\dagger} \beta_{-\mathbf{k},1}^{r\dagger} \right] |0\rangle_{1,2}.
\end{aligned} \tag{2.96}$$

It appears that this expression involves four different particle–antiparticle pairs having zero total momentum and spin. This vacuum structure is more complex than the one found in systems which involve the usual Bogoliubov transformation e.g. the BCS theory. This is due to the more complicated structure of the transformations involved.

The condensation density of the flavor vacuum is given by:

$${}_{e,\mu} \langle 0 | \alpha_{\mathbf{k},i}^{r\dagger} \alpha_{\mathbf{k},i}^r | 0 \rangle_{e,\mu} = {}_{e,\mu} \langle 0 | \beta_{\mathbf{k},i}^{r\dagger} \beta_{\mathbf{k},i}^r | 0 \rangle_{e,\mu} = \sin^2 \theta |V_{\mathbf{k}}|^2, \quad i = 1, 2. \tag{2.97}$$

It is instructive to see how it is possible to partially disentangle the rotation from the Bogoliubov transformation [39].

Let us do the following ansätze:

$$|U_{\mathbf{k}}| \equiv \cos \Theta_{\mathbf{k}} \quad , \quad |V_{\mathbf{k}}| \equiv \sin \Theta_{\mathbf{k}} \quad , \quad 0 \leq \Theta_{\mathbf{k}} < \frac{\pi}{4} \tag{2.98}$$

$$e^{i(\omega_1 - \omega_2)t} \equiv e^{i\psi} \quad , \quad e^{2i\omega_1 t} \equiv e^{i\phi_1} \quad , \quad e^{2i\omega_2 t} \equiv e^{i\phi_2} \tag{2.99}$$

so that the (2.82) become

$$\alpha_{\mathbf{k},e}^r = B_2^{-1} R^{-1} \alpha_{\mathbf{k},1}^r R B_2 \quad (2.100)$$

$$\beta_{-\mathbf{k},e}^r = B_2^{-1} R^{-1} \beta_{-\mathbf{k},1}^r R B_2$$

$$\alpha_{\mathbf{k},\mu}^r = B_1^{-1} R^{-1} \alpha_{\mathbf{k},2}^r R B_1 \quad (2.101)$$

$$\beta_{-\mathbf{k},\mu}^r = B_1^{-1} R^{-1} \beta_{-\mathbf{k},2}^r R B_1$$

where

$$R = \exp \left\{ \theta \sum_{\mathbf{k},r} \left[\left(\alpha_{\mathbf{k},1}^{r\dagger} \alpha_{\mathbf{k},2}^r + \beta_{-\mathbf{k},1}^{r\dagger} \beta_{-\mathbf{k},2}^r \right) e^{i\psi} - \left(\alpha_{\mathbf{k},2}^{r\dagger} \alpha_{\mathbf{k},1}^r + \beta_{-\mathbf{k},2}^{r\dagger} \beta_{-\mathbf{k},1}^r \right) e^{-i\psi} \right] \right\} \quad (2.102)$$

$$B_1 = \exp \left\{ - \sum_{\mathbf{k},r} \Theta_{\mathbf{k}} \epsilon^r \left[\alpha_{\mathbf{k},1}^r \beta_{-\mathbf{k},1}^r e^{-i\phi_1} - \beta_{-\mathbf{k},1}^{r\dagger} \alpha_{\mathbf{k},1}^{r\dagger} e^{i\phi_1} \right] \right\} \quad (2.103)$$

$$B_2 = \exp \left\{ \sum_{\mathbf{k},r} \Theta_{\mathbf{k}} \epsilon^r \left[\alpha_{\mathbf{k},2}^r \beta_{-\mathbf{k},2}^r e^{-i\phi_2} - \beta_{-\mathbf{k},2}^{r\dagger} \alpha_{\mathbf{k},2}^{r\dagger} e^{i\phi_2} \right] \right\} \quad (2.104)$$

Using these relations and noting the invariance property $R|0\rangle_{1,2} = |0\rangle_{1,2}$, we can split the sectors $\{|0(\Theta)\rangle_1\}$ and $\{|0(\Theta)\rangle_2\}$ from the complete representation space $\{|0\rangle_{e,\mu}\} : \{|0(\Theta)\rangle_1\} \otimes \{|0(\Theta)\rangle_2\} \subset \{|0\rangle_{e,\mu}\}$.

The two states $|0(\Theta)\rangle_1$ and $|0(\Theta)\rangle_2$ are obtained as follows:

$$|0(\Theta)\rangle_1 \equiv B_1^{-1}(\Theta)|0\rangle_1 = \prod_{\mathbf{k},r} \left(\cos \Theta_{\mathbf{k}} + \epsilon^r e^{i\phi_1} \sin \Theta_{\mathbf{k}} \beta_{-\mathbf{k},1}^{r\dagger} \alpha_{\mathbf{k},1}^{r\dagger} \right) |0\rangle_1, \quad (2.105)$$

$$|0(\Theta)\rangle_2 \equiv B_2^{-1}(\Theta)|0\rangle_2 = \prod_{\mathbf{k},r} \left(\cos \Theta_{\mathbf{k}} - \epsilon^r e^{i\phi_2} \sin \Theta_{\mathbf{k}} \beta_{-\mathbf{k},2}^{r\dagger} \alpha_{\mathbf{k},2}^{r\dagger} \right) |0\rangle_2. \quad (2.106)$$

which have the same structure of the thermal fundamental state of Thermo Field Dynamics (TFD) for fermions³. We thus get a thermal type structure of the vacuum state for mixed particles, which gives us the possibility to give a thermodynamic like treatment of the mixing phenomenon. This connection between mixing and temperature keeps popping out in different investigations, as for example the ones described in chapters 3 and 5. This seems to be a hint of something very deep concerning mixing and maybe the structure of QFT itself.

2.3 Flavor states of mixed neutrinos and exact oscillation formulae

After discussing the properties of the flavor vacuum, let us turn to the flavor states of the neutrinos. They are defined as the eigenstates of the flavor charges defined in section 2.1. We will see that they are also eigenstates of the momentum operators, but they are not eigenstates

³This is also the form of the BCS fundamental state.

of the Hamiltonian, causing problems with the usual interpretation of mixed particles in terms of representations of the Poincaré group. We will postpone a possible solution of this problem until chapter 5, where we will see that it can be overcome by considering mixing as due to the interaction with an external field. We will concentrate on single particle states, the extension to multiparticle states being straightforward.

Let us briefly discuss the no mixing case first. The normal ordered charge operators for the fields ν_1 and ν_2 are:

$$: Q_{\nu_i} : \equiv \int d^3 \mathbf{x} : \nu_i^\dagger(x) \nu_i(x) = \sum_r \int d^3 \mathbf{k} \left(\alpha_{\mathbf{k},i}^{r\dagger} \alpha_{\mathbf{k},i}^r - \beta_{-\mathbf{k},i}^{r\dagger} \beta_{-\mathbf{k},i}^r \right), \quad (2.107)$$

where $i = 1, 2$ and $: \dots :$ denotes normal ordering with respect to the vacuum $|0\rangle_{1,2}$. The single particle states of definite mass neutrinos are defined as:

$$|\nu_{\mathbf{k},i}^r\rangle = \alpha_{\mathbf{k},i}^{r\dagger} |0\rangle_{1,2}, \quad i = 1, 2, \quad (2.108)$$

and they are eigenstates of Q_{ν_1} and Q_{ν_2} . These can be identified as the leptonic charges of the neutrinos in the no mixing case.

In the mixed case the situation is subtler. Flavor neutrino single particle states are defined as the eigenstates of the charges $Q_{\nu_\sigma}(t)$ at a given time t . These charges are related to the ones of unmixed neutrinos by the relations:

$$Q_{\nu_e}(t) = \cos^2 \theta Q_{\nu_1} + \sin^2 \theta Q_{\nu_2} + \sin \theta \cos \theta \int d^3 \mathbf{x} \left[\nu_1^\dagger(x) \nu_2(x) + \nu_2^\dagger(x) \nu_1(x) \right], \quad (2.109)$$

$$Q_{\nu_\mu}(t) = \sin^2 \theta Q_{\nu_1} + \cos^2 \theta Q_{\nu_2} - \sin \theta \cos \theta \int d^3 \mathbf{x} \left[\nu_1^\dagger(x) \nu_2(x) + \nu_2^\dagger(x) \nu_1(x) \right]. \quad (2.110)$$

We notice that the last term of these expressions coincides with the charge $Q_{m,1}$ defined in (2.29). This term prevents from constructing eigenstates of the charges $Q_{\nu_\sigma}(t)$ in the Hilbert space $\mathcal{H}_{1,2}$. This fact, together with the unitary inequivalence of the mass and flavor vacua, is a necessary and rigorous consequence of the fact that neutrinos are described by quantum relativistic fields. We just have to learn to live with it.

The normal ordered flavor charge operators are written as:

$$: Q_{\nu_\sigma}(t) : \equiv \int d^3 \mathbf{x} : \nu_\sigma^\dagger(x) \nu_\sigma(x) : = \sum_r \int d^3 \mathbf{k} \left(\alpha_{\mathbf{k},\nu_\sigma}^{r\dagger}(t) \alpha_{\mathbf{k},\nu_\sigma}^r(t) - \beta_{-\mathbf{k},\nu_\sigma}^{r\dagger}(t) \beta_{-\mathbf{k},\nu_\sigma}^r(t) \right), \quad (2.111)$$

where $\sigma = e, \mu$, and $: \dots :$ denotes normal ordering with respect to the vacuum $|0\rangle_{e,\mu}$. These operators are diagonal in the flavor ladder operators constructed in the previous section through the mixing generator. The normal ordering $: \dots :$ of a given operator A is defined in the usual way:

$$: A : \equiv A - {}_{e,\mu} \langle 0 | A | 0 \rangle_{e,\mu}. \quad (2.112)$$

Observe that $: Q_{\nu_\sigma}(t) : = G_\theta^{-1}(t) : Q_{\nu_j} : G_\theta(t)$, with $(\sigma, j) = (e, 1), (\mu, 2)$, and

$$: Q_\nu : = : Q_{\nu_e}(t) : + : Q_{\nu_\mu}(t) : = : Q_{\nu_1} : + : Q_{\nu_2} : = : Q_\nu :. \quad (2.113)$$

As anticipated, flavor neutrino states are defined as the eigenstates of Q_{ν_σ} at reference time $t = 0$:

$$|\nu_{\mathbf{k},\sigma}^r\rangle \equiv \alpha_{\mathbf{k},\nu_\sigma}^{r\dagger}(0)|0(0)\rangle_{e,\mu}, \quad \sigma = e, \mu \quad (2.114)$$

and analogous for the antiparticles. We have:

$$\begin{aligned} \because Q_{\nu_e}(0) \because |\nu_{\mathbf{k},e}^r\rangle &= |\nu_{\mathbf{k},e}^r\rangle, & \because Q_{\nu_\mu}(0) \because |\nu_{\mathbf{k},\mu}^r\rangle &= |\nu_{\mathbf{k},\mu}^r\rangle, \\ \because Q_{\nu_e}(0) \because |\nu_{\mathbf{k},\mu}^r\rangle &= 0 = \because Q_{\nu_\mu}(0) \because |\nu_{\mathbf{k},e}^r\rangle, & \because Q_{\nu_\sigma}(0) \because |0\rangle_{e,\mu} &= 0. \end{aligned} \quad (2.115)$$

The flavor states $|\nu_{\mathbf{k},e}^r\rangle$ and $|\nu_{\mathbf{k},\mu}^r\rangle$ at time $t = 0$ in the reference frame such that $\mathbf{k} = (0, 0, |\mathbf{k}|)$ have the following explicit expressions:

$$\begin{aligned} |\nu_{\mathbf{k},e}^r\rangle &\equiv \alpha_{\mathbf{k},e}^{r\dagger}(0)|0\rangle_{e,\mu} = \\ &= \left[\cos\theta \alpha_{\mathbf{k},1}^{r\dagger} + |U_{\mathbf{k}}| \sin\theta \alpha_{\mathbf{k},2}^{r\dagger} - \epsilon^r |V_{\mathbf{k}}| \sin\theta \alpha_{\mathbf{k},1}^{r\dagger} \alpha_{\mathbf{k},2}^{r\dagger} \beta_{-\mathbf{k},1}^{r\dagger} \right] G_{\mathbf{k},s \neq r}^{-1}(\theta) \prod_{\mathbf{p} \neq \mathbf{k}} G_{\mathbf{p}}^{-1}(\theta) |0\rangle_{1,2}, \end{aligned} \quad (2.116)$$

$$\begin{aligned} |\nu_{\mathbf{k},\mu}^r\rangle &\equiv \alpha_{\mathbf{k},\mu}^{r\dagger}(0)|0\rangle_{e,\mu} = \\ &= \left[\cos\theta \alpha_{\mathbf{k},2}^{r\dagger} - |U_{\mathbf{k}}| \sin\theta \alpha_{\mathbf{k},1}^{r\dagger} + \epsilon^r |V_{\mathbf{k}}| \sin\theta \alpha_{\mathbf{k},1}^{r\dagger} \alpha_{\mathbf{k},2}^{r\dagger} \beta_{-\mathbf{k},2}^{r\dagger} \right] G_{\mathbf{k},s \neq r}^{-1}(\theta) \prod_{\mathbf{p} \neq \mathbf{k}} G_{\mathbf{p}}^{-1}(\theta) |0\rangle_{1,2}, \end{aligned} \quad (2.117)$$

where $G(\theta, t) = \prod_{\mathbf{p}} \prod_{s=1}^2 G_{\mathbf{p},s}(\theta, t)$. The third term of these states represents a many particle component, which is suppressed in the ultrarelativistic limit $|\mathbf{k}| \gg \sqrt{m_1 m_2}$.

At this point we can give the exact oscillation formulae by computing the expectation value of the flavor charges on the flavor states in the Heisenberg representation. The result is [41]:

$$\begin{aligned} \mathcal{Q}_{\nu_e \rightarrow \nu_e}^{\mathbf{k}}(t) &= \langle \nu_{\mathbf{k},e}^r | \because Q_{\nu_e}(t) \because | \nu_{\mathbf{k},e}^r \rangle = \left| \left\{ \alpha_{\mathbf{k},e}^r(t), \alpha_{\mathbf{k},e}^{r\dagger}(0) \right\} \right|^2 + \left| \left\{ \beta_{-\mathbf{k},e}^{r\dagger}(t), \alpha_{\mathbf{k},e}^{r\dagger}(0) \right\} \right|^2 \\ &= 1 - \sin^2(2\theta) \left[|U_{\mathbf{k}}|^2 \sin^2 \left(\frac{\omega_{k,2} - \omega_{k,1}}{2} t \right) + |V_{\mathbf{k}}|^2 \sin^2 \left(\frac{\omega_{k,2} + \omega_{k,1}}{2} t \right) \right], \end{aligned} \quad (2.118)$$

$$\begin{aligned} \mathcal{Q}_{\nu_e \rightarrow \nu_\mu}^{\mathbf{k}}(t) &= \langle \nu_{\mathbf{k},e}^r | \because Q_{\nu_\mu}(t) \because | \nu_{\mathbf{k},e}^r \rangle = \left| \left\{ \alpha_{\mathbf{k},\mu}^r(t), \alpha_{\mathbf{k},e}^{r\dagger}(0) \right\} \right|^2 + \left| \left\{ \beta_{-\mathbf{k},\mu}^{r\dagger}(t), \alpha_{\mathbf{k},e}^{r\dagger}(0) \right\} \right|^2 \\ &= \sin^2(2\theta) \left[|U_{\mathbf{k}}|^2 \sin^2 \left(\frac{\omega_{k,2} - \omega_{k,1}}{2} t \right) + |V_{\mathbf{k}}|^2 \sin^2 \left(\frac{\omega_{k,2} + \omega_{k,1}}{2} t \right) \right]. \end{aligned} \quad (2.119)$$

The conservation of the total flavor charge is guaranteed by:

$$\mathcal{Q}_{\nu_e \rightarrow \nu_e}^{\mathbf{k}}(t) + \mathcal{Q}_{\nu_e \rightarrow \nu_\mu}^{\mathbf{k}}(t) = 1. \quad (2.120)$$

The differences of these formulae with respect to the Pontecorvo ones are the momentum dependence (through the Bogoliubov coefficients) of the amplitudes and the presence of an additional oscillating term, containing the sum of the frequencies in addition to the term containing their difference. In the ultrarelativistic limit we have $|U_{\mathbf{k}}|^2 \rightarrow 1$ and $|V_{\mathbf{k}}|^2 \rightarrow 0$, so that the additional term gets suppressed and the usual oscillation formulae are recovered.

2.4 Exact flavor states and flavor conservation

In this Section, following [17] to which we refer for the details, we sketch the computation of the amplitudes of the following two decays at tree level:

$$W^+ \rightarrow e^+ + \nu_e , \quad (2.121)$$

$$W^+ \rightarrow e^+ + \nu_\mu , \quad (2.122)$$

where neutrinos are produced through charged current processes. Despite being obtained in a particular case, the results are valid for any neutrino production process. Our purpose is to show that the flavor states defined in section 2.3 do not lead to a violation of lepton number conservation in the three level production vertices, consistently with the Standard Model. This is to be contrasted with the case of the Pontecorvo states defined in Chapter 1, which instead do produce a relevant lepton number violation in the vertices [17] (see also [19, 116]).

Remember the definition of the flavor states:

$$|\nu_{\mathbf{k},\sigma}^r\rangle \equiv \alpha_{\mathbf{k},\nu_\sigma}^{r\dagger} |0\rangle_f, \quad \sigma = e, \mu . \quad (2.123)$$

Moreover we have, at time t , $|\nu_{\mathbf{k},\sigma}^r(t)\rangle = e^{iH_0 t} \alpha_{\mathbf{k},\nu_\sigma}^{r\dagger} |0\rangle_f$.

In the scattering theory for finite range potentials, it is assumed [88] that the interaction Hamiltonian $H_{int}(x)$ can be switched off adiabatically as $x_{in}^0 \rightarrow -\infty$ and $x_{out}^0 \rightarrow +\infty$ so that the initial and final states can be represented by the eigenstates of the free Hamiltonian. However, in the present case and more generally in the decay processes where the mixed neutrinos are produced, the application of the adiabatic hypothesis leads to erroneous conclusions (as made in [102]). Indeed, the flavor neutrino field operators do not have the mathematical characterization necessary to be defined as asymptotic field operators acting on the massive neutrino vacuum. Moreover, the flavor states $|\nu_{\mathbf{k},\sigma}^r\rangle$ are not eigenstates of the free Hamiltonian. Therefore, the integration limits in the amplitudes of decay processes where mixed neutrinos are produced must be chosen so that the time interval $\Delta t = x_{out}^0 - x_{in}^0$ is much shorter than the characteristic neutrino oscillation time t_{osc} : $\Delta t \ll t_{osc}$.

The calculation is performed considering at the first order of the perturbation theory the amplitudes of the decays (2.121) and (2.122).

In general, if $|\psi_i\rangle$ and $|\psi_f\rangle$ denote initial and final states, the probability amplitude $\langle \psi_f | e^{-iHt} | \psi_i \rangle$ is given by

$$\langle \psi_f | e^{-iHt} | \psi_i \rangle = \langle e^{iH_0 t} \psi_f | e^{iH_0 t} e^{-iHt} | \psi_i \rangle = \langle e^{iH_0 t} \psi_f | U_I(t) | \psi_i \rangle . \quad (2.124)$$

Here the time evolution operator $U_I(t)$ in the interaction picture is given approximatively by

$$U_I(t) \simeq 1 - i \int_0^t dt' H_{int}(t') , \quad (2.125)$$

with $H_{int}(t) = e^{iH_0 t} H_{int} e^{-iH_0 t}$ interaction hamiltonian in the interaction picture. In the following H_0 is the free part of the Hamiltonian for the fields involved in the decays (2.121) and (2.122) and the relevant interaction Hamiltonian is given by [53]:

$$H_{int}(x) = -\frac{g}{\sqrt{2}} W_\mu^+(x) J_W^{\mu+}(x) + \text{h.c.} = -\frac{g}{2\sqrt{2}} W_\mu^+(x) \bar{\nu}_e(x) \gamma^\mu (1 - \gamma^5) e(x) + \text{h.c.}, \quad (2.126)$$

where $W^+(x)$, $e(x)$ and $\nu_e(x)$ are the fields of the boson W^+ , the electron and the flavor (electron) neutrino, respectively.

Let us consider the process $W^+ \rightarrow e^+ + \nu_e$ and the states defined in Eq.(2.123). The amplitude of the decay at first order in perturbation theory is given by⁴

$$\begin{aligned}
A_{W^+ \rightarrow e^+ + \nu_e} &= \langle \nu_{\mathbf{k},e}^r, e_{\mathbf{q}}^s | \left[-i \int_{x_{in}^0}^{x_{out}^0} d^4x H_{int}(x) \right] | W_{\mathbf{p},\lambda}^+ \rangle \\
&= W \langle 0 | \langle \nu_{\mathbf{k},e}^r(x_{in}^0) | \langle e_{\mathbf{q}}^s | \left\{ \frac{i g}{2\sqrt{2}} \int_{x_{in}^0}^{x_{out}^0} d^4x \right. \\
&\quad \left. \left[W_{\mu}^+(x) \bar{\nu}_e(x) \gamma^{\mu} (1 - \gamma^5) e(x) \right] \right\} | W_{\mathbf{p},\lambda}^+ \rangle | 0 \rangle_e | 0(x_{in}^0) \rangle_f.
\end{aligned} \tag{2.127}$$

The final result is [17]:

$$\begin{aligned}
A_{W^+ \rightarrow e^+ + \nu_e} &= \frac{i g}{2\sqrt{2}(2\pi)^{3/2}} \delta^3(\mathbf{p} - \mathbf{q} - \mathbf{k}) \int_{x_{in}^0}^{x_{out}^0} dx^0 \frac{\varepsilon_{\mathbf{p},\mu,\lambda}}{\sqrt{2E_p^W}} \\
&\times \left\{ \cos^2 \theta e^{-i\omega_{k,1}x_{in}^0} \bar{u}_{\mathbf{k},1}^r \gamma^{\mu} (1 - \gamma^5) v_{\mathbf{q},e}^s e^{-i(E_p^W - E_q^e - \omega_{k,1})x^0} \right. \\
&+ \sin^2 \theta \left[e^{-i\omega_{k,2}x_{in}^0} |U_{\mathbf{k}}| \bar{u}_{\mathbf{k},2}^r \gamma^{\mu} (1 - \gamma^5) v_{\mathbf{q},e}^s e^{-i(E_p^W - E_q^e - \omega_{k,2})x^0} \right. \\
&+ \left. \left. e^{i\omega_{k,2}x_{in}^0} \varepsilon^r |V_{\mathbf{k}}| \bar{v}_{-\mathbf{k},2}^r \gamma^{\mu} (1 - \gamma^5) v_{\mathbf{q},e}^s e^{-i(E_p^W - E_q^e + \omega_{k,2})x^0} \right] \right\}. \tag{2.128}
\end{aligned}$$

Next we consider the process $W^+ \rightarrow e^+ + \nu_{\mu}$. By using the Hamiltonian (2.126), we have now

$$\begin{aligned}
A_{W^+ \rightarrow e^+ + \nu_{\mu}} &= \langle \nu_{\mathbf{k},\mu}^r, e_{\mathbf{q}}^s | \left[-i \int_{x_{in}^0}^{x_{out}^0} d^4x H_{int}(x) \right] | W_{\mathbf{p},\lambda}^+ \rangle \\
&= W \langle 0 | \langle \nu_{\mathbf{k},\mu}^r(x_{in}^0) | \langle e_{\mathbf{q}}^s | \left\{ \frac{i g}{2\sqrt{2}} \right. \\
&\times \left. \int_{x_{in}^0}^{x_{out}^0} d^4x \left[W_{\mu}^+(x) \bar{\nu}_e(x) \gamma^{\mu} (1 - \gamma^5) e(x) \right] \right\} | W_{\mathbf{p},\lambda}^+ \rangle | 0 \rangle_e | 0(x_{in}^0) \rangle_f.
\end{aligned} \tag{2.129}$$

and the final result is:

$$\begin{aligned}
A_{W^+ \rightarrow e^+ + \nu_{\mu}} &= \frac{i g}{2\sqrt{2}(2\pi)^{3/2}} \delta^3(\mathbf{p} - \mathbf{q} - \mathbf{k}) \sin \theta \cos \theta \int_{x_{in}^0}^{x_{out}^0} dx^0 \frac{\varepsilon_{\mathbf{p},\mu,\lambda}}{\sqrt{2E_p^W}} \\
&\times \left[e^{-i\omega_{k,2}x_{in}^0} \bar{u}_{\mathbf{k},2}^r \gamma^{\mu} (1 - \gamma^5) v_{\mathbf{q},e}^s e^{-i(E_p^W - E_q^e - \omega_{k,2})x^0} \right.
\end{aligned}$$

⁴Note that in the case of the flavor states, because of the orthogonality of the Hilbert spaces at different times [39], instead of Eq.(2.124) the amplitude should be defined as $\langle \psi_{\sigma}(x_{out}^0) | e^{-iH(x_{out}^0 - x_{in}^0)} | \psi_{\sigma}(x_{in}^0) \rangle = \langle \psi_{\sigma}(x_{in}^0) | U_I(x_{out}^0, x_{in}^0) | \psi_{\sigma}(x_{in}^0) \rangle$.

$$\begin{aligned}
& - e^{-i\omega_{k,1}x_{in}^0} |U_{\mathbf{k}}| \bar{u}_{\mathbf{k},1}^r \gamma^\mu (1 - \gamma^5) v_{\mathbf{q},e}^s e^{-i(E_p^W - E_q^e - \omega_{k,1})} \\
& + e^{i\omega_{k,1}x_{in}^0} \varepsilon^r |V_{\mathbf{k}}| \bar{v}_{-\mathbf{k},1}^r \gamma^\mu (1 - \gamma^5) v_{\mathbf{q},e}^s e^{-i(E_p^W - E_q^e + \omega_{k,1})} \Big]. \quad (2.130)
\end{aligned}$$

Let us now consider the amplitudes given by (2.128), (2.130) for short time intervals Δt . The physical meaning of such a time scale Δt is represented by the relation $\frac{1}{\Gamma} \ll \Delta t \ll L_{osc}$, where Γ is the W^+ decay width and L_{osc} is the typical flavor oscillation length. Given the experimental values of Γ and L_{osc} , this interval is well defined. In the following, when we use the expression ‘‘short time limit’’, we refer to the time scale defined above. Of course, energy fluctuations are constrained by the Heisenberg uncertainty relation, where Δt is the one given above. We will see that the use of the exact flavor states gives results which agree with lepton charge conservation in the production vertex. On the other hand, the same computation gives a clear violation of the lepton charge when the Pontecorvo states are used [17]. The origin of such a violation is due to the fact that the Pontecorvo flavor states are defined by use of the vacuum state $|0\rangle_{1,2}$ for the massive neutrino states.

Let us first consider the decay $W^+ \rightarrow e^+ + \nu_e$. We obtain the following result at first order in Δt :

$$\begin{aligned}
A_{W^+ \rightarrow e^+ + \nu_e} & \simeq \frac{ig}{2\sqrt{2}(2\pi)^{3/2}} \frac{\varepsilon_{\mathbf{p},\mu,\lambda}}{\sqrt{2E_p^W}} \delta^3(\mathbf{p} - \mathbf{q} - \mathbf{k}) \Delta t \times \\
& \times \left\{ \cos^2 \theta \bar{u}_{\mathbf{k},1}^r + \sin^2 \theta \left[|U_{\mathbf{k}}| \bar{u}_{\mathbf{k},2}^r + \varepsilon^r |V_{\mathbf{k}}| \bar{v}_{-\mathbf{k},2}^r \right] \right\} \gamma^\mu (1 - \gamma^5) v_{\mathbf{q},e}^s. \quad (2.131)
\end{aligned}$$

The final result is

$$A_{W^+ \rightarrow e^+ + \nu_e} \simeq \frac{ig}{2\sqrt{2}(2\pi)^{3/2}} \frac{\varepsilon_{\mathbf{p},\mu,\lambda}}{\sqrt{2E_p^W}} \delta^3(\mathbf{p} - \mathbf{q} - \mathbf{k}) \Delta t \bar{u}_{\mathbf{k},1}^r \gamma^\mu (1 - \gamma^5) v_{\mathbf{q},e}^s. \quad (2.132)$$

This amplitude resembles the one for the production of a free neutrino with mass m_1 .

Let us now turn to the process $W^+ \rightarrow e^+ + \nu_\mu$. Proceeding in a similar way as above, we obtain

$$\begin{aligned}
A_{W^+ \rightarrow e^+ + \nu_\mu} & \simeq \frac{ig}{4\sqrt{2}(2\pi)^{3/2}} \frac{\varepsilon_{\mathbf{p},\mu,\lambda}}{\sqrt{2E_p^W}} \delta^3(\mathbf{p} - \mathbf{q} - \mathbf{k}) \Delta t \sin 2\theta \\
& \times \left[\bar{u}_{\mathbf{k},2}^r - |U_{\mathbf{k}}| \bar{u}_{\mathbf{k},1}^r + \varepsilon^r |V_{\mathbf{k}}| \bar{v}_{-\mathbf{k},1}^r \right] \gamma^\mu (1 - \gamma^5) v_{\mathbf{q},e}^s, \quad (2.133)
\end{aligned}$$

in the short time limit.

It is possible to show [17] that the quantity in square brackets vanishes identically i.e.

$$A_{W^+ \rightarrow e^+ + \nu_\mu} \simeq 0. \quad (2.134)$$

This proves that, in the short time limit, the use of the exact flavor states leads to the conservation of lepton charge in the production vertex in agreement with what we expected from the Standard Model.

2.5 Generalization of the mixing transformations

In Section 2.2 we have expressed the flavor fields ν_e and ν_μ in the same bases as the (free) fields with definite masses ν_1 and ν_2 , respectively. As we have anticipated, this is actually a special choice, and that a more general possibility exists, as noticed first in [67, 68]. This ambiguity has no physical consequences (at least in the two flavor case) since the observable quantities turn out not to be affected. Anyway, as we will show in chapter 5, a different point of view will allow us to lift this ambiguity.

Let us introduce the notation. As we saw, the fields ν_e and ν_μ can be written in the following form:

$$\nu_\sigma(x) = G_\theta^{-1}(t) \nu_j(x) G_\theta(t) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}, r} \left[u_{\mathbf{k}, j}^r \alpha_{\mathbf{k}, \sigma}^r(t) + v_{-\mathbf{k}, j}^r \beta_{-\mathbf{k}, \sigma}^{r\dagger}(t) \right] e^{i\mathbf{k}\cdot\mathbf{x}}, \quad (2.135)$$

where $(\sigma, j) = (e, 1), (\mu, 2)$ and

$$\begin{pmatrix} \alpha_{\mathbf{k}, \sigma}^r(t) \\ \beta_{-\mathbf{k}, \sigma}^{r\dagger}(t) \end{pmatrix} = G_\theta^{-1}(t) \begin{pmatrix} \alpha_{\mathbf{k}, j}^r(t) \\ \beta_{-\mathbf{k}, j}^{r\dagger}(t) \end{pmatrix} G_\theta(t) \quad (2.136)$$

The explicit expression of the flavor annihilation operators is (in the reference frame $\mathbf{k} = (0, 0, |\mathbf{k}|)$):

$$\begin{pmatrix} \alpha_{\mathbf{k}, e}^r(t) \\ \alpha_{\mathbf{k}, \mu}^r(t) \\ \beta_{-\mathbf{k}, e}^{r\dagger}(t) \\ \beta_{-\mathbf{k}, \mu}^{r\dagger}(t) \end{pmatrix} = \begin{pmatrix} c_\theta & s_\theta |U_{\mathbf{k}}| & 0 & s_\theta \epsilon^r |V_{\mathbf{k}}| \\ -s_\theta |U_{\mathbf{k}}| & c_\theta & s_\theta \epsilon^r |V_{\mathbf{k}}| & 0 \\ 0 & -s_\theta \epsilon^r |V_{\mathbf{k}}| & c_\theta & s_\theta |U_{\mathbf{k}}| \\ -s_\theta \epsilon^r |V_{\mathbf{k}}| & 0 & -s_\theta |U_{\mathbf{k}}| & c_\theta \end{pmatrix} \begin{pmatrix} \alpha_{\mathbf{k}, 1}^r(t) \\ \alpha_{\mathbf{k}, 2}^r(t) \\ \beta_{-\mathbf{k}, 1}^{r\dagger}(t) \\ \beta_{-\mathbf{k}, 2}^{r\dagger}(t) \end{pmatrix} \quad (2.137)$$

where $c_\theta \equiv \cos \theta$, $s_\theta \equiv \sin \theta$.

The point is that in the expansion Eq.(2.135) one could use eigenfunctions associated with arbitrary masses μ_σ , and therefore not necessarily the same as the masses which appear in the Lagrangian. Indeed, the transformation Eq.(2.136) can be generalized [67, 68] by writing the flavor fields as

$$\nu_\sigma(x) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}, r} \left[u_{\mathbf{k}, \sigma}^r \tilde{\alpha}_{\mathbf{k}, \sigma}^r(t) + v_{-\mathbf{k}, \sigma}^r \tilde{\beta}_{-\mathbf{k}, \sigma}^{r\dagger}(t) \right] e^{i\mathbf{k}\cdot\mathbf{x}}, \quad (2.138)$$

where u_σ and v_σ are spinors with a given mass μ_σ . We denote by a tilde the generalized flavor operators [67, 68] in order to distinguish them from the ones defined in Eq.(2.136). The expansion (2.135) corresponds to the particular choice $\mu_e \equiv m_1$, $\mu_\mu \equiv m_2$.

The relation between the general flavor and the mass operators is now:

$$\begin{pmatrix} \tilde{\alpha}_{\mathbf{k}, \sigma}^r(t) \\ \tilde{\beta}_{-\mathbf{k}, \sigma}^{r\dagger}(t) \end{pmatrix} = K_{\theta, \mu}^{-1}(t) \begin{pmatrix} \alpha_{\mathbf{k}, j}^r(t) \\ \beta_{-\mathbf{k}, j}^{r\dagger}(t) \end{pmatrix} K_{\theta, \mu}(t) , \quad (2.139)$$

with $(\sigma, j) = (e, 1), (\mu, 2)$, where $K_{\theta, \mu}(t)$ is the generator of the transformations (2.41) and can be expressed as

$$K_{\theta, \mu}(t) = I_\mu(t) G_\theta(t) \quad (2.140)$$

$$I_\mu(t) = \prod_{\mathbf{k}, r} \exp \left\{ i \sum_{(\sigma, j)} \xi_{\sigma, j}^{\mathbf{k}} \left[\alpha_{\mathbf{k}, j}^{r\dagger}(t) \beta_{-\mathbf{k}, j}^{r\dagger}(t) + \beta_{-\mathbf{k}, j}^r(t) \alpha_{\mathbf{k}, j}^r(t) \right] \right\} \quad (2.141)$$

with

$$\xi_{\sigma, j}^{\mathbf{k}} \equiv (\chi_\sigma - \chi_j)/2, \quad \cot \chi_\sigma = |\mathbf{k}|/\mu_\sigma, \quad \cot \chi_j = |\mathbf{k}|/m_j. \quad (2.142)$$

For $\mu_e \equiv m_1$, $\mu_\mu \equiv m_2$, we have $I_\mu(t) = 1$.

The explicit matrix form of the flavor operators is [67, 68]:

$$\begin{pmatrix} \tilde{\alpha}_{\mathbf{k}, e}^r(t) \\ \tilde{\alpha}_{\mathbf{k}, \mu}^r(t) \\ \tilde{\beta}_{-\mathbf{k}, e}^{r\dagger}(t) \\ \tilde{\beta}_{-\mathbf{k}, \mu}^{r\dagger}(t) \end{pmatrix} = \begin{pmatrix} c_\theta \rho_{e1}^{\mathbf{k}} & s_\theta \rho_{e2}^{\mathbf{k}} & ic_\theta \lambda_{e1}^{\mathbf{k}} & is_\theta \lambda_{e2}^{\mathbf{k}} \\ -s_\theta \rho_{\mu1}^{\mathbf{k}} & c_\theta \rho_{\mu2}^{\mathbf{k}} & -is_\theta \lambda_{\mu1}^{\mathbf{k}} & ic_\theta \lambda_{\mu2}^{\mathbf{k}} \\ ic_\theta \lambda_{e1}^{\mathbf{k}} & is_\theta \lambda_{e2}^{\mathbf{k}} & c_\theta \rho_{e1}^{\mathbf{k}} & s_\theta \rho_{e2}^{\mathbf{k}} \\ -is_\theta \lambda_{\mu1}^{\mathbf{k}} & ic_\theta \lambda_{\mu2}^{\mathbf{k}} & -s_\theta \rho_{\mu1}^{\mathbf{k}} & c_\theta \rho_{\mu2}^{\mathbf{k}} \end{pmatrix} \begin{pmatrix} \alpha_{\mathbf{k}, 1}^r(t) \\ \alpha_{\mathbf{k}, 2}^r(t) \\ \beta_{-\mathbf{k}, 1}^{r\dagger}(t) \\ \beta_{-\mathbf{k}, 2}^{r\dagger}(t) \end{pmatrix} \quad (2.143)$$

where $c_\theta \equiv \cos \theta$, $s_\theta \equiv \sin \theta$ and

$$\rho_{ab}^{\mathbf{k}} \delta_{rs} \equiv \cos \left(\frac{\chi_a - \chi_b}{2} \right) \delta_{rs} = u_{\mathbf{k}, a}^{r\dagger} u_{\mathbf{k}, b}^s = v_{-\mathbf{k}, a}^{r\dagger} v_{-\mathbf{k}, b}^s \quad (2.144)$$

$$i\lambda_{ab}^{\mathbf{k}} \delta_{rs} \equiv i \sin \left(\frac{\chi_a - \chi_b}{2} \right) \delta_{rs} = u_{\mathbf{k}, a}^{r\dagger} v_{-\mathbf{k}, b}^s = v_{-\mathbf{k}, a}^{r\dagger} u_{\mathbf{k}, b}^s \quad (2.145)$$

with $a, b = 1, 2, e, \mu$.

Since $\rho_{12}^{\mathbf{k}} = |U_{\mathbf{k}}|$ and $i\lambda_{12}^{\mathbf{k}} = \epsilon^r |V_{\mathbf{k}}|$, etc., the operators (2.143) reduce to the ones in (2.137) when $\mu_e \equiv m_1$ and $\mu_\mu \equiv m_2$ ⁵.

The generalization of the flavor vacuum, which is annihilated by the general flavor operators given by Eq.(2.139), is [67, 68]:

$$|\tilde{0}(t)\rangle_{e, \mu} \equiv K_{\theta, \mu}^{-1}(t) |0\rangle_{1, 2} \quad (2.146)$$

Of course, when $\mu_e \equiv m_1$ and $\mu_\mu \equiv m_2$, this state reduces to the flavor vacuum $|0(t)\rangle_{e, \mu}$ above defined.

The relation between the general flavor operators of Eq.(2.139) and the flavor operators of Eq.(2.136) is [67, 68]:

$$\begin{pmatrix} \tilde{\alpha}_{\mathbf{k}, \sigma}^r(t) \\ \tilde{\beta}_{-\mathbf{k}, \sigma}^{r\dagger}(t) \end{pmatrix} = J_\mu^{-1}(t) \begin{pmatrix} \alpha_{\mathbf{k}, \sigma}^r(t) \\ \beta_{-\mathbf{k}, \sigma}^{r\dagger}(t) \end{pmatrix} J_\mu(t) \quad (2.147)$$

$$J_\mu(t) = \prod_{\mathbf{k}, r} \exp \left\{ i \sum_{(\sigma, j)} \xi_{\sigma, j}^{\mathbf{k}} \left[\alpha_{\mathbf{k}, \sigma}^{r\dagger}(t) \beta_{-\mathbf{k}, \sigma}^{r\dagger}(t) + \beta_{-\mathbf{k}, \sigma}^r(t) \alpha_{\mathbf{k}, \sigma}^r(t) \right] \right\} \quad (2.148)$$

⁵In performing such an identification, one should take into account that the operators for antiparticles differ for a minus sign, related to the different spinor bases used in the expansions (2.135) and (2.138). Such a sign difference is however irrelevant in what follows.

We have thus shown that the Hilbert space for the flavor fields is not unique: an infinite number of vacua (and consequently infinitely many Hilbert spaces) can be generated by introducing the arbitrary mass parameters μ_e, μ_μ . It is obvious that physical quantities must not depend on these parameters. Indeed the exact oscillation formulae are independent of the arbitrary mass parameters [40]. We have in fact the equalities

$$\begin{aligned} & \left| \left\{ \tilde{\alpha}_{\mathbf{k},e}^r(t), \tilde{\alpha}_{\mathbf{k},e}^{r\dagger}(0) \right\} \right|^2 + \left| \left\{ \tilde{\beta}_{\mathbf{k},e}^{r\dagger}(t), \tilde{\alpha}_{\mathbf{k},e}^{r\dagger}(0) \right\} \right|^2 = \\ & = \left| \left\{ \alpha_{\mathbf{k},e}^r(t), \alpha_{\mathbf{k},e}^{r\dagger}(0) \right\} \right|^2 + \left| \left\{ \beta_{-\mathbf{k},e}^{r\dagger}(t), \alpha_{\mathbf{k},e}^{r\dagger}(0) \right\} \right|^2, \end{aligned} \quad (2.149)$$

$$\begin{aligned} & \left| \left\{ \tilde{\alpha}_{\mathbf{k},\mu}^r(t), \tilde{\alpha}_{\mathbf{k},e}^{r\dagger}(0) \right\} \right|^2 + \left| \left\{ \tilde{\beta}_{-\mathbf{k},\mu}^{r\dagger}(t), \tilde{\alpha}_{\mathbf{k},e}^{r\dagger}(0) \right\} \right|^2 = \\ & = \left| \left\{ \alpha_{\mathbf{k},\mu}^r(t), \alpha_{\mathbf{k},e}^{r\dagger}(0) \right\} \right|^2 + \left| \left\{ \beta_{-\mathbf{k},\mu}^{r\dagger}(t), \alpha_{\mathbf{k},e}^{r\dagger}(0) \right\} \right|^2, \end{aligned} \quad (2.150)$$

which ensure the cancellation of the arbitrary mass parameters.

The important point for the full understanding of the result (2.149), (2.150) is that the charge operators Q_{ν_σ} are the Casimir operators with respect to the Bogoliubov transformation (2.147) and so they are *invariant* under the action of the Bogoliubov generator (2.148), i.e. $\tilde{Q}_{\nu_\sigma} = Q_{\nu_\sigma}$, where $\tilde{Q}_{\nu_\sigma} \equiv \tilde{\alpha}_\sigma^\dagger \tilde{\alpha}_\sigma - \tilde{\beta}_\sigma^\dagger \tilde{\beta}_\sigma$. Besides the direct computations leading to Eqs. (2.149), (2.150), such an invariance provides a strong and immediate proof of the independence of the oscillation formula from the μ_σ parameters. Thus, the expectation values of the flavor charge operators are the only physical relevant quantities in the context of the above theory, all other operators having expectation values depending on the arbitrary parameters above introduced.

2.6 Boson mixing

The nontrivial nature of the mixing transformations manifests itself also in the case of boson field mixing (e.g. neutral kaons). The main difference with respect to the fermionic case is in the condensate structure of the flavor vacuum. Until we are restricted to considerations in the framework of the Standard Model, this case is to be considered as an *effective* theory, since the mixed bosons are composite objects, and their mixing is to be traced back to the underlying quark mixing⁶. In the wider framework of a supersymmetric theory it is possible to have mixing of fundamental scalars. This is the situation we will consider in the next chapter. For generality, let us consider the mixing of two generic complex scalar fields [18], leaving the case of real fields to a later section.

We can easily extend the analysis of the currents and charges of fermions to the case of bosons. Let us consider the Lagrangian density

$$\mathcal{L}(x) = \partial_\mu \Phi_m^\dagger(x) \partial^\mu \Phi_m(x) - \Phi_m^\dagger(x) M_d \Phi_m(x) \quad (2.151)$$

⁶The problem of deriving the meson mixing parameters from the quark mixing parameters is an unsolved one since it involves nontrivial QCD strong coupling effects, see e.g.[130, 131]

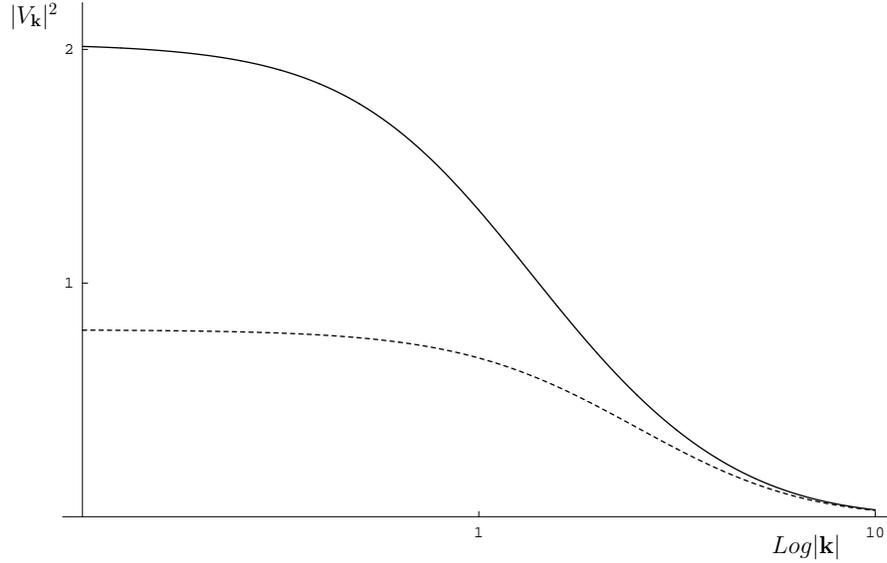


Figure 2.2: Boson condensation density $|V_{\mathbf{k}}|^2$ as a function of k and for sample values of the mass parameters m_1 e m_2 .

Continuous line: $m_1 = 1$, $m_2 = 10$

Dashed line: $m_1 = 2$, $m_2 = 10$

where $\Phi_m^T = (\phi_1, \phi_2)$ are complex and $M_d = \text{diag}(m_1^2, m_2^2)$. We have:

$$\Phi'_m(x) = e^{i\alpha_j \tau_j} \Phi_m(x) \quad (2.152)$$

$$\delta\mathcal{L}(x) = i\alpha_j \Phi_m^\dagger(x) [\tau_j, M_d] \Phi_m(x) = -\alpha_j \partial_\mu J_{m,j}^\mu(x), \quad (2.153)$$

$$J_{m,j}^\mu(x) = i \Phi_m^\dagger(x) \tau_j \overleftrightarrow{\partial}^\mu \Phi_m(x), \quad j = 1, 2, 3. \quad (2.154)$$

As in the fermion case the corresponding charges $Q_{m,j}(t)$ close the $su(2)$ algebra and the boson mixing generator is proportional to $Q_{m,2}(t)$.

The vacuum structure and the exact flavor oscillation formulae is determined in the same way as in the fermion case. The mixing relations

$$\begin{aligned} \phi_A(x) &= \cos \theta \phi_1(x) + \sin \theta \phi_2(x) \\ \phi_B(x) &= -\sin \theta \phi_1(x) + \cos \theta \phi_2(x) \end{aligned} \quad (2.155)$$

(where the suffixes A and B denote generically the mixed fields) imply the existence of a nontrivial vacuum which once again has the structure of a $SU(2)$ generalized coherent state.

The main difference is in the form of the Bogoliubov coefficients:

$$U_{\mathbf{k}}(t) \equiv |U_{\mathbf{k}}| e^{i(\omega_{k,2} - \omega_{k,1})t}, \quad V_{\mathbf{k}}(t) \equiv |V_{\mathbf{k}}| e^{i(\omega_{k,1} + \omega_{k,2})t} \quad (2.156)$$

$$|U_{\mathbf{k}}| \equiv \frac{1}{2} \left(\sqrt{\frac{\omega_{k,1}}{\omega_{k,2}}} + \sqrt{\frac{\omega_{k,2}}{\omega_{k,1}}} \right), \quad |V_{\mathbf{k}}| \equiv \frac{1}{2} \left(\sqrt{\frac{\omega_{k,1}}{\omega_{k,2}}} - \sqrt{\frac{\omega_{k,2}}{\omega_{k,1}}} \right) \quad (2.157)$$

$$|U_{\mathbf{k}}|^2 - |V_{\mathbf{k}}|^2 = 1, \quad (2.158)$$

which results in a different expression of the condensation density

$${}_{A,B} \langle 0(t) | a_{\mathbf{k},i}^\dagger a_{\mathbf{k},i} | 0(t) \rangle_{A,B} = \sin^2 \theta |V_{\mathbf{k}}|^2, \quad i = 1, 2, \quad (2.159)$$

The function $|V_{\mathbf{k}}|^2$ is plotted in Fig. 2.2 with sample values for the masses. It has its maximum in $|\mathbf{k}| = 0$ ($|V_{max}|^2 = \frac{(m_1 - m_2)^2}{4m_1 m_2}$ and $|V_{\mathbf{k}}|^2 \simeq \left(\frac{\Delta m^2}{4|\mathbf{k}|^2}\right)^2$ for $|\mathbf{k}|^2 \gg \frac{m_1^2 + m_2^2}{2}$).

The single particle mixed boson state with flavor A is:

$$|a_{\mathbf{k},A}\rangle \equiv a_{\mathbf{k},A}^\dagger(0) |0\rangle_{A,B} \quad (2.160)$$

which is an eigenstate of the flavor charges ($\sigma = A, B$)

$$Q_\sigma(t) = \int d^3\mathbf{k} \left(a_{\mathbf{k},\sigma}^\dagger(t) a_{\mathbf{k},\sigma}(t) - b_{-\mathbf{k},\sigma}^\dagger(t) b_{-\mathbf{k},\sigma}(t) \right), \quad (2.161)$$

with eigenvalues one and zero respectively. An analogous definition is of course given for the state with flavor B .

Moreover ${}_{A,B} \langle 0 | Q_\sigma(t) | 0 \rangle_{A,B} = 0$ and

$$\mathcal{Q}_{\mathbf{k},\sigma}(t) \equiv \langle a_{\mathbf{k},A} | Q_\sigma(t) | a_{\mathbf{k},A} \rangle = \left| \left[a_{\mathbf{k},\sigma}(t), a_{\mathbf{k},A}^\dagger(0) \right] \right|^2 - \left| \left[b_{-\mathbf{k},\sigma}^\dagger(t), a_{\mathbf{k},A}^\dagger(0) \right] \right|^2. \quad (2.162)$$

The exact oscillation formulae are:

$$\begin{aligned} \mathcal{Q}_{\mathbf{k},A}(t) &= 1 - \sin^2(2\theta) |U_{\mathbf{k}}|^2 \sin^2 \left(\frac{\omega_{k,2} - \omega_{k,1}}{2} t \right) \\ &\quad + \sin^2(2\theta) |V_{\mathbf{k}}|^2 \sin^2 \left(\frac{\omega_{k,2} + \omega_{k,1}}{2} t \right), \end{aligned} \quad (2.163)$$

$$\begin{aligned} \mathcal{Q}_{\mathbf{k},B}(t) &= \sin^2(2\theta) |U_{\mathbf{k}}|^2 \sin^2 \left(\frac{\omega_{k,2} - \omega_{k,1}}{2} t \right) \\ &\quad - \sin^2(2\theta) |V_{\mathbf{k}}|^2 \sin^2 \left(\frac{\omega_{k,2} + \omega_{k,1}}{2} t \right). \end{aligned} \quad (2.164)$$

2.7 Mixing and oscillations of Majorana fermions

Now let us consider situations in which the mixed fields are neutral i.e. Majorana spinors and real scalars [38].

As we have seen, in the derivation of the oscillation formulae by use of the flavor Hilbert space a central role is played by the flavor charges [34]. However, these charges vanish identically in the case of neutral fields. We will see that in this case their role is taken by the momentum operators, and the oscillation formulae we will obtain match those obtained in the charged field case. Since the results obtained are analogous to the one found in the charged case exists, it is customary to use the terms “flavor fields”, “flavor states” and “flavor vacuum” also in the neutral case.

Let us start with the Majorana spinor case. Besides being useful in the discussion of the supersymmetric case in the next chapter, this case has an intrinsic importance since it is still not known whether the neutrinos we observe are Dirac or Majorana particles.

Let us briefly recall the definition of Majorana spinors. The charge-conjugation operator C is defined as satisfying the relations

$$C^{-1}\gamma_\mu C = -\gamma_\mu^T \quad , \quad C^\dagger = C^{-1} \quad , \quad C^T = -C \quad . \quad (2.165)$$

from which we define the charge conjugate ψ^c of the spinor ψ as

$$\psi^c(x) \equiv \gamma_0 C \psi^*(x) \quad . \quad (2.166)$$

A Majorana fermion is defined as a field that satisfies the Dirac equation

$$(i \not{\partial} - m)\psi = 0 \quad (2.167)$$

and the self-conjugation relation

$$\psi = \psi^c \quad . \quad (2.168)$$

The two equations (2.167) and (2.168) ensure that the Majorana field is a neutral fermion field.

The dynamics is ruled by the Lagrangian:

$$\mathcal{L}(x) = \bar{\psi}_f(x)(i \not{\partial} - M)\psi_f(x) = \bar{\psi}_m(x)(i \not{\partial} - M_d)\psi_m(x) \quad , \quad (2.169)$$

with $\psi_f^T = (\nu_e, \nu_\mu)$ being the flavor fields and $M = \begin{pmatrix} m_e & m_{e\mu} \\ m_{e\mu} & m_\mu \end{pmatrix}$. The flavor fields are connected to the free fields $\psi_m^T = (\nu_1, \nu_2)$ with $M_d = \text{diag}(m_1, m_2)$ by the mixing rotation:

$$\nu_e(x) = \nu_1(x) \cos \theta + \nu_2(x) \sin \theta \quad , \quad (2.170)$$

$$\nu_\mu(x) = -\nu_1(x) \sin \theta + \nu_2(x) \cos \theta \quad . \quad (2.171)$$

The Fourier expansion of the free fields is given by [114]

$$\nu_i(x) = \sum_{r=1,2} \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} e^{i\mathbf{k}\cdot\mathbf{x}} \left[u_{\mathbf{k},i}^r(t) \alpha_{\mathbf{k},i}^r + v_{-\mathbf{k},i}^r(t) \alpha_{-\mathbf{k},i}^{r\dagger} \right] \quad , \quad i = 1, 2. \quad (2.172)$$

where $u_{\mathbf{k},i}^r(t) = e^{-i\omega_{k,i}t} u_{\mathbf{k},i}^r$, $v_{\mathbf{k},i}^r(t) = e^{i\omega_{k,i}t} v_{\mathbf{k},i}^r$, with $\omega_{k,i} = \sqrt{\mathbf{k}^2 + m_i^2}$. In order for the Majorana condition (2.168) to be satisfied, the four spinors must also satisfy the following condition:

$$u_{\mathbf{k},i}^s = \gamma_0 C (u_{\mathbf{k},i}^s)^* \quad ; \quad v_{\mathbf{k},i}^s = \gamma_0 C (v_{\mathbf{k},i}^s)^* \quad . \quad (2.173)$$

By proceeding in the by now standard way we get the same result we got in the Dirac case. In particular the Bogoliubov coefficients are:

$$U_{\mathbf{k}}(t) \equiv |U_{\mathbf{k}}| e^{i(\omega_{k,2} - \omega_{k,1})t} \quad , \quad V_{\mathbf{k}}(t) \equiv |V_{\mathbf{k}}| e^{i(\omega_{k,2} + \omega_{k,1})t} \quad , \quad (2.174)$$

$$|U_{\mathbf{k}}| \equiv \left(\frac{\omega_{k,1} + m_1}{2\omega_{k,1}} \right)^{\frac{1}{2}} \left(\frac{\omega_{k,2} + m_2}{2\omega_{k,2}} \right)^{\frac{1}{2}} \left(1 + \frac{|\mathbf{k}|^2}{(\omega_{k,1} + m_1)(\omega_{k,2} + m_2)} \right), \quad (2.175)$$

$$|V_{\mathbf{k}}| \equiv \left(\frac{\omega_{k,1} + m_1}{2\omega_{k,1}} \right)^{\frac{1}{2}} \left(\frac{\omega_{k,2} + m_2}{2\omega_{k,2}} \right)^{\frac{1}{2}} \left(\frac{|\mathbf{k}|}{(\omega_{k,2} + m_2)} - \frac{|\mathbf{k}|}{(\omega_{k,1} + m_1)} \right), \quad (2.176)$$

$$|U_{\mathbf{k}}|^2 + |V_{\mathbf{k}}|^2 = 1. \quad (2.177)$$

The flavor fields can be expanded as:

$$\nu_{\sigma}(x) = \sum_{r=1,2} \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} e^{i\mathbf{k}\cdot\mathbf{x}} \left[u_{\mathbf{k},j}^r(t) \alpha_{\mathbf{k},\sigma}^r(t) + v_{-\mathbf{k},j}^r(t) \alpha_{-\mathbf{k},\sigma}^{r\dagger}(t) \right], \quad (2.178)$$

with $\sigma, j = (e, 1), (\mu, 2)$ and the flavor annihilation operators given by (for $\mathbf{k} = (0, 0, |\mathbf{k}|)$):

$$\begin{aligned} \alpha_{\mathbf{k},e}^r(t) &\equiv G_{\theta}^{-1}(t) \alpha_{\mathbf{k},1}^r G_{\theta}(t) = \cos \theta \alpha_{\mathbf{k},1}^r + \sin \theta \left(U_{\mathbf{k}}^*(t) \alpha_{\mathbf{k},2}^r + \epsilon^r V_{\mathbf{k}}(t) \alpha_{-\mathbf{k},2}^{r\dagger} \right), \\ \alpha_{\mathbf{k},\mu}^r(t) &\equiv G_{\theta}^{-1}(t) \alpha_{\mathbf{k},2}^r G_{\theta}(t) = \cos \theta \alpha_{\mathbf{k},2}^r - \sin \theta \left(U_{\mathbf{k}}(t) \alpha_{\mathbf{k},1}^r - \epsilon^r V_{\mathbf{k}}(t) \alpha_{-\mathbf{k},1}^{r\dagger} \right). \end{aligned} \quad (2.179)$$

We define the state for a mixed Majorana particle with definite flavor, spin and momentum as:

$$|\alpha_{\mathbf{k},e}^r(t)\rangle \equiv \alpha_{\mathbf{k},e}^{r\dagger}(t) |0(t)\rangle_{e,\mu}. \quad (2.180)$$

We notice that the following quantity is constant in time:

$$\begin{aligned} &\left| \left\{ \alpha_{\mathbf{k},e}^r(t), \alpha_{\mathbf{k},e}^{r\dagger}(t') \right\} \right|^2 + \left| \left\{ \alpha_{-\mathbf{k},e}^{r\dagger}(t), \alpha_{\mathbf{k},e}^{r\dagger}(t') \right\} \right|^2 + \\ &+ \left| \left\{ \alpha_{\mathbf{k},\mu}^r(t), \alpha_{\mathbf{k},e}^{r\dagger}(t') \right\} \right|^2 + \left| \left\{ \alpha_{-\mathbf{k},\mu}^{r\dagger}(t), \alpha_{\mathbf{k},e}^{r\dagger}(t') \right\} \right|^2 = 1. \end{aligned} \quad (2.181)$$

The corresponding of Eq.(2.181) for Dirac fields, was consistently interpreted as expressing the conservation of total charge. In the present case we are dealing with a neutral field and thus the charge operator vanishes identically. Nevertheless the quantities in Eq.(2.181) are well defined and are the Majorana field counterpart of the corresponding ones for the case of Dirac fields. Thus we look for a physical interpretation of such oscillating quantities.

Let us consider the momentum operator defined as

$$P^j \equiv \int d^3\mathbf{x} T^{0j}(x), \quad (2.182)$$

where the energy–momentum tensor for the fermion field, $T^{\mu\nu}$, is defined by

$$T^{\mu\nu} \equiv i\bar{\psi}\gamma^{\nu}\partial_{\mu}\psi. \quad (2.183)$$

For the free fields ψ_i we have:

$$\mathbf{P}_i = \int d^3\mathbf{x} \psi_i^{\dagger}(x) (-i\nabla) \psi_i(x) = \int d^3\mathbf{k} \sum_{r=1,2} \mathbf{k} \left(\alpha_{\mathbf{k},i}^{r\dagger} \alpha_{\mathbf{k},i}^r - \alpha_{-\mathbf{k},i}^{r\dagger} \alpha_{-\mathbf{k},i}^r \right), \quad i = 1, 2. \quad (2.184)$$

We then define the momentum operator for mixed fields:

$$\mathbf{P}_\sigma(t) = \int d^3\mathbf{x} \psi_\sigma^\dagger(x) (-i\nabla) \psi_\sigma(x) = \int d^3\mathbf{k} \sum_{r=1,2} \mathbf{k} \left(\alpha_{\mathbf{k},\sigma}^{r\dagger}(t) \alpha_{\mathbf{k},\sigma}^r(t) - \alpha_{-\mathbf{k},\sigma}^{r\dagger}(t) \alpha_{-\mathbf{k},\sigma}^r(t) \right), \quad (2.185)$$

with $\sigma = e, \mu$. We have

$$\mathbf{P}_\sigma(t) = G_\theta^{-1}(t) \mathbf{P}_i G_\theta(t) \quad (2.186)$$

and the conservation of total momentum as a consequence of

$$\mathbf{P}_e(t) + \mathbf{P}_\mu(t) = \mathbf{P}_1 + \mathbf{P}_2 \equiv \mathbf{P} \quad , \quad [\mathbf{P}, G_\theta(t)] = 0 \quad , \quad [\mathbf{P}, H] = 0. \quad (2.187)$$

We now consider the expectation values on the flavor state $|\alpha_{\mathbf{k},e}^r\rangle \equiv |\alpha_{\mathbf{k},e}^r(0)\rangle$. At time $t = 0$, this state is an eigenstate of the momentum operator $\mathbf{P}_e(0)$:

$$\mathbf{P}_e(0) |\alpha_{\mathbf{k},e}^r\rangle = \mathbf{k} |\alpha_{\mathbf{k},e}^r\rangle. \quad (2.188)$$

At $t \neq 0$ the expectation value for the momentum (normalized to initial value) gives:

$$\mathcal{P}_{\mathbf{k},\sigma}^e(t) \equiv \frac{\langle \alpha_{\mathbf{k},e}^r | \mathbf{P}_\sigma(t) | \alpha_{\mathbf{k},e}^r \rangle}{\langle \alpha_{\mathbf{k},e}^r | \mathbf{P}_\sigma(0) | \alpha_{\mathbf{k},e}^r \rangle} = \left| \left\{ \alpha_{\mathbf{k},\sigma}^r(t), \alpha_{\mathbf{k},e}^{r\dagger}(0) \right\} \right|^2 + \left| \left\{ \alpha_{-\mathbf{k},\sigma}^{r\dagger}(t), \alpha_{\mathbf{k},e}^{r\dagger}(0) \right\} \right|^2, \quad (2.189)$$

with $\sigma = e, \mu$, which is the same form of the expression one obtains for the expectation values of the flavor charges in the case of Dirac fields [41]. The flavor vacuum expectation value of the momentum operator $\mathbf{P}_\sigma(t)$ vanishes at all times:

$${}_{e,\mu}\langle 0 | \mathbf{P}_\sigma(t) | 0 \rangle_{e,\mu} = 0 \quad , \quad \sigma = e, \mu. \quad (2.190)$$

The explicit calculation of the oscillating quantities $\mathcal{P}_{\mathbf{k},\sigma}^e(t)$ gives:

$$\mathcal{P}_{\mathbf{k},e}^e(t) = 1 - \sin^2 2\theta \left[|U_{\mathbf{k}}|^2 \sin^2 \left(\frac{\omega_{k,2} - \omega_{k,1}}{2} t \right) + |V_{\mathbf{k}}|^2 \sin^2 \left(\frac{\omega_{k,2} + \omega_{k,1}}{2} t \right) \right], \quad (2.191)$$

$$\mathcal{P}_{\mathbf{k},\mu}^e(t) = \sin^2 2\theta \left[|U_{\mathbf{k}}|^2 \sin^2 \left(\frac{\omega_{k,2} - \omega_{k,1}}{2} t \right) + |V_{\mathbf{k}}|^2 \sin^2 \left(\frac{\omega_{k,2} + \omega_{k,1}}{2} t \right) \right], \quad (2.192)$$

in complete agreement with the Dirac field case [41].

2.8 Mixing of neutral bosons

Let us now consider the mixing of two real scalar fields [38]. The situation is analogous to the Majorana fermion case.

The Lagrangian is:

$$\begin{aligned}\mathcal{L}(x) &= \frac{1}{2} [\partial_\mu \phi_A(x)]^2 - \frac{1}{2} m_A^2 \phi_A^2(x) + \frac{1}{2} [\partial_\mu \phi_B(x)]^2 \\ &\quad - \frac{1}{2} m_B^2 \phi_B^2(x) - m_{AB}^2 \phi_A(x) \phi_B(x)\end{aligned}\tag{2.193}$$

$$= \frac{1}{2} [\partial_\mu \phi_1(x)]^2 - \frac{1}{2} m_1^2 \phi_1^2(x) + \frac{1}{2} [\partial_\mu \phi_2(x)]^2 - \frac{1}{2} m_2^2 \phi_2^2(x)\tag{2.194}$$

where ϕ_A, ϕ_B are flavor fields and ϕ_1, ϕ_2 are massive fields, with masses m_1 and m_2 respectively. The mixing transformations are:

$$\begin{aligned}\phi_A(x) &= \cos \theta \phi_1(x) + \sin \theta \phi_2(x) \\ \phi_B(x) &= -\sin \theta \phi_1(x) + \cos \theta \phi_2(x),\end{aligned}\tag{2.195}$$

We have $m_A^2 = m_1^2 \cos^2 \theta + m_2^2 \sin^2 \theta$, $m_B^2 = m_2^2 \cos^2 \theta + m_1^2 \sin^2 \theta$, $m_{AB}^2 = (m_2^2 - m_1^2) \sin \theta \cos \theta$. The Fourier expansions of the fields ϕ_1, ϕ_2 are ($x_0 \equiv t$):

$$\phi_j(x) = \int \frac{d^3 \mathbf{k}}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{2\omega_{k,j}}} \left(a_{\mathbf{k},j} e^{-i\omega_{k,j}t} + a_{-\mathbf{k},j}^\dagger e^{i\omega_{k,j}t} \right) e^{i\mathbf{k}\cdot\mathbf{x}}, \quad j = 1, 2,\tag{2.196}$$

while the ones for the flavor fields are:

$$\phi_\sigma(x) = \int \frac{d^3 \mathbf{k}}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{2\omega_{k,j}}} \left(a_{\mathbf{k},\sigma}(t) e^{-i\omega_{k,j}t} + a_{-\mathbf{k},\sigma}^\dagger(t) e^{i\omega_{k,j}t} \right) e^{i\mathbf{k}\cdot\mathbf{x}},\tag{2.197}$$

where we use the notation $\sigma, j = (A, 1), (B, 2)$ and we have

$$a_{\mathbf{k},A}(t) = \cos \theta a_{\mathbf{k},1} + \sin \theta \left(\hat{U}_{\mathbf{k}}^*(t) a_{\mathbf{k},2} + \hat{V}_{\mathbf{k}}(t) a_{-\mathbf{k},2}^\dagger \right),\tag{2.198}$$

$$a_{\mathbf{k},B}(t) = \cos \theta a_{\mathbf{k},2} - \sin \theta \left(\hat{U}_{\mathbf{k}}(t) a_{\mathbf{k},1} - \hat{V}_{\mathbf{k}}(t) a_{-\mathbf{k},1}^\dagger \right),\tag{2.199}$$

where the Bogoliubov coefficients are the same as in the charged scalar case:

$$\hat{U}_{\mathbf{k}}(t) \equiv |\hat{U}_{\mathbf{k}}| e^{i(\omega_{k,2} - \omega_{k,1})t}, \quad \hat{V}_{\mathbf{k}}(t) \equiv |\hat{V}_{\mathbf{k}}| e^{i(\omega_{k,1} + \omega_{k,2})t},\tag{2.200}$$

$$|\hat{U}_{\mathbf{k}}| \equiv \frac{1}{2} \left(\sqrt{\frac{\omega_{k,1}}{\omega_{k,2}}} + \sqrt{\frac{\omega_{k,2}}{\omega_{k,1}}} \right), \quad |\hat{V}_{\mathbf{k}}| \equiv \frac{1}{2} \left(\sqrt{\frac{\omega_{k,1}}{\omega_{k,2}}} - \sqrt{\frac{\omega_{k,2}}{\omega_{k,1}}} \right),\tag{2.201}$$

$$|\hat{U}_{\mathbf{k}}|^2 - |\hat{V}_{\mathbf{k}}|^2 = 1\tag{2.202}$$

We observe that the following quantity is constant in time:

$$\begin{aligned}& \left| \left[a_{\mathbf{k},A}(t), a_{\mathbf{k},A}^\dagger(t') \right] \right|^2 - \left| \left[a_{-\mathbf{k},A}^\dagger(t), a_{\mathbf{k},A}^\dagger(t') \right] \right|^2 \\ &+ \left| \left[a_{\mathbf{k},B}(t), a_{\mathbf{k},A}^\dagger(t') \right] \right|^2 - \left| \left[a_{-\mathbf{k},B}^\dagger(t), a_{\mathbf{k},A}^\dagger(t') \right] \right|^2 = 1.\end{aligned}\tag{2.203}$$

As in the Majorana field case, we define the momentum operator. defined as usual as [88]: $P^j \equiv \int d^3\mathbf{x} \Theta^{0j}(x)$, with $\Theta^{\mu\nu} \equiv \partial^\mu \phi \partial^\nu \phi - g^{\mu\nu} [\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2]$. For the free fields ϕ_i we have:

$$\mathbf{P}_i = \int d^3\mathbf{x} \pi_i(x) \nabla \phi_i(x) = \int d^3\mathbf{k} \frac{\mathbf{k}}{2} \left(a_{\mathbf{k},i}^\dagger a_{\mathbf{k},i} - a_{-\mathbf{k},i}^\dagger a_{-\mathbf{k},i} \right) \quad , \quad i = 1, 2. \quad (2.204)$$

where $\pi_i(x) = \partial_0 \phi_i(x)$ are the conjugate momenta. The momentum operators for mixed fields are:

$$\mathbf{P}_\sigma(t) = \int d^3\mathbf{x} \pi_\sigma(x) \nabla \phi_\sigma(x) = \int d^3\mathbf{k} \frac{\mathbf{k}}{2} \left(a_{\mathbf{k},\sigma}^\dagger(t) a_{\mathbf{k},\sigma}(t) - a_{-\mathbf{k},\sigma}^\dagger(t) a_{-\mathbf{k},\sigma}(t) \right), \quad (2.205)$$

where $\sigma = A, B$. and we have:

$$\mathbf{P}_\sigma(t) = G_\theta^{-1}(t) \mathbf{P}_i G_\theta(t). \quad (2.206)$$

The total momentum is conserved:

$$\mathbf{P}_A(t) + \mathbf{P}_B(t) = \mathbf{P}_1 + \mathbf{P}_2 \equiv \mathbf{P} \quad , \quad [\mathbf{P}, G_\theta(t)] = 0 \quad , \quad [\mathbf{P}, H] = 0. \quad (2.207)$$

We have:

$$\mathbf{P}_A(0) |a_{\mathbf{k},A}\rangle = \mathbf{k} |a_{\mathbf{k},A}\rangle, \quad (2.208)$$

and we can define the quantities:

$$\mathcal{P}_\sigma^A(t) \equiv \frac{\langle a_{\mathbf{k},A} | \mathbf{P}_\sigma(t) | a_{\mathbf{k},A} \rangle}{\langle a_{\mathbf{k},A} | \mathbf{P}_\sigma(0) | a_{\mathbf{k},A} \rangle} = \left| [a_{\mathbf{k},\sigma}(t), a_{\mathbf{k},A}^\dagger(0)] \right|^2 - \left| [a_{-\mathbf{k},\sigma}^\dagger(t), a_{\mathbf{k},A}^\dagger(0)] \right|^2. \quad (2.209)$$

In the end we get the oscillation formulae:

$$\mathcal{P}_{\mathbf{k},A}^A(t) = 1 - \sin^2(2\theta) \left[|\hat{U}_{\mathbf{k}}|^2 \sin^2 \left(\frac{\omega_{k,2} - \omega_{k,1}}{2} t \right) - |\hat{V}_{\mathbf{k}}|^2 \sin^2 \left(\frac{\omega_{k,2} + \omega_{k,1}}{2} t \right) \right] \quad (2.210)$$

$$\mathcal{P}_{\mathbf{k},B}^A(t) = \sin^2(2\theta) \left[|\hat{U}_{\mathbf{k}}|^2 \sin^2 \left(\frac{\omega_{k,2} - \omega_{k,1}}{2} t \right) - |\hat{V}_{\mathbf{k}}|^2 \sin^2 \left(\frac{\omega_{k,2} + \omega_{k,1}}{2} t \right) \right]. \quad (2.211)$$

which agree with the analogues for charged bosons (2.164-2.165).

Chapter 3

Flavor mixing induced spontaneous Supersymmetry breaking

The flavor vacuum, which is the physical vacuum for describing the mixing consistently with lepton charge conservation in the Standard Model, has a striking properties which will be the subject of this chapter. In fact, as a consequence of its condensate structure, it has a nonvanishing energy density, which is positive definite both for fermions and bosons. This suggests that a supersymmetric model containing mixed fields may display a spontaneous breaking of supersymmetry (SUSY) [49] (see also[112]). It is indeed well known that the order parameter for SUSY is just the vacuum energy density, which has to be non negative as an immediate consequence of the SUSY algebra. Moreover, this spontaneous breaking would be nonperturbative in nature, being triggered by a nonperturbative phenomenon such as mixing. As pointed out by Witten [154], if SUSY is to have some chances to be relevant from the phenomenological point of view, it has to be broken by a nonperturbative effect. This observation, together with the fact that mixing is definitely a fact of Nature, implies that mixing could be considered a viable candidate to explain why SUSY is not seen at our scales and open the way to the construction of some phenomenologically relevant model.

In the following we shall state precisely the conjecture and prove it for a simple model [49]. Some inherent aspects, such as the nature of the Goldstone spinor associated to the breaking, will not be addressed since at the time of writing they are the subject of ongoing work. Before embarking in our task, we will give a short introduction to SUSY breaking, with the purpose of preparing the ground to our discussion and without any claim of completeness. Also, we will not attempt to give an introduction to SUSY since this would lead us too far away. For these topics we refer to the excellent existing literature (see e.g. [147, 145, 133]).

3.1 Spontaneous and dynamical supersymmetry breaking: general aspects

It is a fact that our world is not supersymmetric. Were it not so, every known particle would have an equal mass superpartner which would have been observed. In view of the many ad-

vantages that SUSY would offer if it played a rôle in our world, such as providing solutions or at least improvements of fine tuning problems such as the hierarchy problem [143] and the cosmological constant problem [144, 117], it is desirable to have some mechanism of breaking of this symmetry such that all these advantages be kept, while at the same time explaining the failure in its direct observation. Over the last three decades, many such mechanisms have been proposed and intensively studied (see e.g. [129] or [133] for excellent reviews), but to the author’s knowledge none of these has lead to some compelling phenomenological model.

The first mechanisms for spontaneous SUSY breaking have been proposed by Fayet–Ilipoulos [65] and by O’ Raifeartaigh [122], respectively for theories involving vector (gauge) supermultiplets and chiral supermultiplets. These mechanisms work at the tree (i.e. classical level). The peculiar nature of SUSY, in particular the perturbative nonrenormalization theorems valid for supersymmetric theories [78] imply that if SUSY is unbroken at the tree level, it will be unbroken at any order of perturbation theory, thus preventing a radiative correction induced breaking like the Coleman–Weinberg mechanism [55] for ordinary symmetries. On the other hand, it is possible to have a breaking of SUSY triggered by nonperturbative effects [154] such as instantons¹ [1].

A necessary and sufficient condition for the spontaneous breaking of SUSY is the nonvanishing vacuum energy. This can be understood intuitively by noting that the vacuum quantum fluctuations of fermion and bosons quantum fields have opposite signs, so that in a theory which enjoys a symmetry between these fields these fluctuations should cancel. This also intuitively explains the nonrenormalization theorems since bosonic and fermionic loops again have opposite signs, thus ensuring the vanishing of radiative corrections.

From a more rigorous point of view, this condition is a direct consequence of the SUSY algebra. In the simplest instance (which is the only of our concern) this algebra contains four Hermitian fermionic charges (usually called supercharges) Q_α , $\alpha = 1, \dots, 4$, which form a Majorana spinor Q (in this chapter we adopt the 4–component notation and we also adopt all the conventions of [149, 148] for convenience) and which satisfy the anticommutation relations

$$\{Q, Q\} = 2(\gamma_\mu C)P^\mu \quad (3.1)$$

where C is the charge conjugation matrix and P^μ is the 4–momentum operator, with $P^0 = H$. This implies that

$$H = \frac{1}{8}\text{Tr}(C\gamma^0 \{Q, Q\}), \quad (3.2)$$

i.e. the Hamiltonian is expressed as the sum of the squares of Hermitian operators. This means that the energy of any state is positive or zero. A state can have zero energy only if it is annihilated by all of the Q ’s, i.e.

$$Q_\alpha|\psi\rangle = 0 \quad \Rightarrow \quad \langle\psi|H|\psi\rangle = \frac{1}{8}\langle\psi|\text{Tr}(C\gamma^0 \{Q, Q\})|\psi\rangle = 0. \quad (3.3)$$

Such a state will be the minimum energy state of the theory i.e. its vacuum, and will be SUSY–preserving. If on the other hand the vacuum of the theory does not have vanishing energy

¹In fact, if SUSY is to be of some help in solving the hierarchy problem it would be desirable that it be broken dynamically (see [154] for a discussion).

density, it is not annihilated by all the supercharges, meaning that SUSY is spontaneously broken. This means that vacuum energy can be consistently considered as the order parameter of SUSY, being zero for unbroken symmetry and nonzero for broken symmetry.

In the following sections we will show in a simple case that the flavor vacuum in a classically supersymmetric theory with mixing has a nonvanishing vacuum energy density, thus meaning that the flavor vacuum spontaneously breaks SUSY. This means that mixing constitutes a new possible mechanism to attain SUSY breaking. Since the existence of the flavor vacuum is a nonperturbative effect and the theory is supersymmetric at the classical level, this should be classified as a dynamical SUSY breaking mechanism. An interesting direction to pursue is the construction of a realistic phenomenological model including this mechanism.

3.2 Spontaneous supersymmetry breaking induced by flavor mixing

3.2.1 General conjecture

In the following we shall limit ourselves to the two flavor case. The generalization to the three flavor case is straightforward and does not add anything. Consider for definiteness a supersymmetric Lagrangian $\mathcal{L}(\psi_1, B_1; \psi_2, B_2)$ describing two massive supermultiplets Φ_i with fermionic and bosonic components generically denoted by ψ_i and B_i respectively, with $i = 1, 2$ and with $m_1 \neq m_2$. Since the model is supersymmetric we have:

$$\langle 0|H|0\rangle = 0, \tag{3.4}$$

which implies $Q_\alpha|0\rangle = 0$. Here $|0\rangle = |0\rangle_1 \otimes |0\rangle_2$ is the vacuum state for H , the Hamiltonian corresponding to \mathcal{L} . As already discussed Eq.(3.4) is a necessary signature of SUSY. Suppose that the two supermultiplets are mixed in the usual way:

$$\begin{aligned} \Phi_a &= \cos\theta \Phi_1 + \sin\theta \Phi_2 = G^{-1}(\theta)\Phi_1G(\theta); \\ \Phi_b &= -\sin\theta \Phi_1 + \cos\theta \Phi_2 = G^{-1}(\theta)\Phi_2G(\theta), \end{aligned} \tag{3.5}$$

where Φ_a and Φ_b are fields with definite flavor and $G(\theta)$ denotes the appropriate mixing generator². We assume that both fermions and bosons mix with the same angle, so we should not spoil the supersymmetry of the Lagrangian; we shall see that this is the case in our simple model and argue that this is the case in general. The mixing should also not induce spontaneously breaking of SUSY at tree level, hence, due to nonrenormalization theorems [78], SUSY is preserved at any perturbative order [129]. Nonetheless, as we have seen many times by now, mixed fields live in the flavor vacuum, which we here generically denote with

$$|0\rangle_f = G^{-1}(\theta)|0\rangle, \tag{3.6}$$

²Since this generator depends on the fields, the rhs of (3.5) is just a bookkeeping notation; it is not known yet whether it is possible to write down a mixing generator which universally acts in the required way on all the components of a given supermultiplet. This would be another interesting point to develop.

and the vacua $|0\rangle_f$ and $|0\rangle$ are unitarily inequivalent in the infinite volume limit. They are orthogonal and $|0\rangle_f$ shows its condensate of quanta of fields with definite mass. We conjecture that the expectation value of the Hamiltonian on such vacuum is always of the form:

$${}_f\langle 0|H|0\rangle_f = h(\theta, m_1 - m_2) \geq 0 \quad (3.7)$$

with $h = 0$ only if $\theta = 0$ or $m_1 = m_2$, hence SUSY is dynamically broken by the nontrivial mixing, the effect shown by (3.7) relying entirely on the nonperturbative nature of the quantum corrections.

3.2.2 Mixing of two free chiral supermultiplets

Let us now prove our conjecture explicitly for the simple case of two free massive chiral supermultiplets, described by the Wess–Zumino Lagrangian. The field content is: two free Majorana fermions ψ_i , two free real scalars S_i , two free real pseudoscalars P_i , two scalar dummy fields F_i and two pseudoscalar dummy fields G_i , ($i = 1, 2$) and the Lagrangian is:

$$\begin{aligned} \mathcal{L} = & -\frac{i}{2} \bar{\psi}_1 \not{\partial} \psi_1 - \frac{1}{2} \partial_\mu S_1 \partial^\mu S_1 - \frac{1}{2} \partial_\mu P_1 \partial^\mu P_1 + \frac{1}{2} F_1^2 + \frac{1}{2} G_1^2 \\ & + m_1 \left(F_1 S_1 + G_1 P_1 - \frac{i}{2} \bar{\psi}_1 \psi_1 \right) + (1 \rightarrow 2) . \end{aligned} \quad (3.8)$$

Although the field content of the theory is the same as that of a $\mathcal{N} = 2$ hypermultiplet, which is made out of two $\mathcal{N} = 1$ chiral supermultiplets, the $SU(2)_R$ symmetry for us is explicitly broken by the request of having $m_1 \neq m_2$, hence we are not dealing with a $\mathcal{N} = 2$ SUSY theory but rather with two copies of a free $\mathcal{N} = 1$ Wess–Zumino theory³. For this reason we consider the following SUSY transformations that leave invariant (up to a surface term, as customary) the off-shell Lagrangian (3.8):

$$\delta S_i = i\bar{\alpha}\psi_i, \quad \delta P_i = i\bar{\alpha}\gamma_5\psi_i; \quad (3.9)$$

$$\delta\psi_i = \partial_\mu(S_i - \gamma_5 P_i)\gamma^\mu\alpha + (F_i + \gamma_5 G_i)\alpha; \quad (3.10)$$

$$\delta F_i = i\bar{\alpha}\gamma^\mu\partial_\mu\psi_i, \quad \delta G_i = i\bar{\alpha}\gamma_5\gamma^\mu\partial_\mu\psi_i \quad (3.11)$$

where $i = 1, 2$, and the parameter α is a Majorana spinor.

The off-shell formulation is crucial for showing that SUSY is untouched by the mixing at the Lagrangian level. Of course, the dynamical information resides in the on-shell expressions that are recovered by using the Euler-Lagrange equations for the dummy fields,

$$F_i^{on} = -m_i S_i, \quad G_i^{on} = -m_i P_i, \quad i = 1, 2, \quad (3.12)$$

in the Lagrangian (3.8) to obtain

$$\begin{aligned} \mathcal{L} = & -\frac{i}{2} \bar{\psi}_1 (\not{\partial} + m_1) \psi_1 - \frac{1}{2} \partial_\mu S_1 \partial^\mu S_1 - \frac{1}{2} m_1^2 S_1^2 - \frac{1}{2} \partial_\mu P_1 \partial^\mu P_1 - \frac{1}{2} m_1^2 P_1^2 + (1 \rightarrow 2) \\ = & -\frac{i}{2} \bar{\psi} (\not{\partial} + M_d) \psi - \frac{1}{2} \partial_\mu S \partial^\mu S - \frac{1}{2} S^T M_d^2 S - \frac{1}{2} \partial_\mu P \partial^\mu P - \frac{1}{2} P^T M_d^2 P, \end{aligned} \quad (3.13)$$

³This observation suggests that maybe the origin of mixing can be looked for in some $\mathcal{N} = 2$ supersymmetric model with spontaneous breaking of the $SU(2)_R$ symmetry. We will not pursue this point any further.

(here $\psi = (\psi_1, \psi_2)^T$, $S = (S_1, S_2)^T$, $P = (P_1, P_2)^T$, and $M_d = \text{diag}(m_1, m_2)$) and in the expressions for the transformations (3.10) and (3.11). Note that the transformations (3.11) become identities satisfied when the fermions are on-shell, while (3.10) becomes

$$\delta\psi_i = \partial_\mu(S_i - \gamma_5 P_i)\gamma^\mu\alpha - m_i(S_i + \gamma_5 P_i)\alpha. \quad (3.14)$$

The transformations (3.9) are untouched by this procedure. The on-shell Lagrangian (3.13) is invariant under (3.9) and (3.14).

$$\psi_f = U\psi, \quad S_f = US, \quad P_f = UP, \quad F_f = UF, \quad G_f = UG, \quad (3.15)$$

where $\psi_f = (\psi_a, \psi_b)^T$, etc., $F = (F_1, F_2)^T$, $G = (G_1, G_2)^T$, and $U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. With these the Lagrangian (3.8) can be written as

$$\begin{aligned} \mathcal{L} &= -\frac{i}{2}\bar{\psi}_a \not{\partial}\psi_a - \frac{1}{2}\partial_\mu S_a \partial^\mu S_a - \frac{1}{2}\partial_\mu P_a \partial^\mu P_a + \frac{1}{2}F_a^2 + \frac{1}{2}G_a^2 \\ &+ m_a \left(F_a S_a + G_a P_a - \frac{i}{2}\bar{\psi}_a \psi_a \right) + (a \rightarrow b) \\ &+ m_{ab} \left(F_a S_b + F_b S_a + G_a P_b + G_b P_a - \frac{i}{2}\bar{\psi}_a \psi_b - \frac{i}{2}\bar{\psi}_b \psi_a \right) = \mathcal{L}_0 + \mathcal{L}_{mix}, \end{aligned} \quad (3.16)$$

where $m_a = m_1 \cos^2\theta + m_2 \sin^2\theta$, $m_b = m_1 \sin^2\theta + m_2 \cos^2\theta$, and $m_{ab} = (m_2 - m_1) \sin\theta \cos\theta$, while, due to linearity of both the SUSY transformations (3.9)-(3.11) and the mixing transformations (3.15) we have ($\sigma = a, b$):

$$\delta S_\sigma = i\bar{\alpha}\psi_\sigma, \quad \delta P_\sigma = i\bar{\alpha}\gamma_5\psi_\sigma; \quad (3.17)$$

$$\delta\psi_\sigma = \partial_\mu(S_\sigma - \gamma_5 P_\sigma)\gamma^\mu\alpha + (F_\sigma + \gamma_5 G_\sigma)\alpha; \quad (3.18)$$

$$\delta F_\sigma = i\bar{\alpha}\gamma^\mu\partial_\mu\psi_\sigma, \quad \delta G_\sigma = i\bar{\alpha}\gamma_5\gamma^\mu\partial_\mu\psi_\sigma \quad (3.19)$$

i.e. off-shell the mixed fields transform just like the unmixed ones under SUSY. The first terms of \mathcal{L} in (3.16) (first line), denoted with \mathcal{L}_0 have the same functional form of the unmixed Lagrangian (3.8), where all functions are evaluated in the new variables S_σ , etc.. This is as it must be, because the mixing transformations are not a symmetry of \mathcal{L} but rather a *redefinition* of the fields, which in general changes the functional form of the Lagrangian.

On the other hand, it might appear that the form invariance of the SUSY transformations (3.17)-(3.19) only guarantees the invariance of the first line in (3.16) descending directly from the invariance of (3.8) under (3.9)-(3.11). In other words, it appears that \mathcal{L}_{mix} explicitly breaks SUSY. In fact, it is not so. It is straightforward to see that \mathcal{L}_{mix} is invariant on its own under (3.17) - (3.19), $\delta\mathcal{L}_{mix} = \partial_\mu(i\bar{\alpha}m_{ab}[(S_b + \gamma_5 P_b)\gamma^\mu\psi_a + (a \leftrightarrow b)])$.

The on-shell expressions for the dummy fields in the mixed case are (compare with (3.12))

$$F_\sigma^{on} = -m_\sigma S_\sigma - m_{\sigma\tau} S_\tau, \quad G_\sigma^{on} = -m_\sigma P_\sigma - m_{\sigma\tau} P_\tau, \quad (3.20)$$

where $\sigma, \tau = a, b$ and $\tau \neq \sigma$ and the on-shell expression for the fermionic transformation (3.18) is now (compare with (3.14))

$$\delta\psi_\sigma = \partial_\mu(S_\sigma - \gamma_5 P_\sigma)\gamma^\mu\alpha - m_\sigma(S_\sigma + \gamma_5 P_\sigma)\alpha - m_{\sigma\tau}(S_\tau + \gamma_5 P_\tau)\alpha, \quad (3.21)$$

with $\sigma, \tau = a, b$ and $\tau \neq \sigma$ (the other transformations (3.17) and (3.19) have exactly the same fate as the corresponding ones in the unmixed case (3.9) and (3.11), respectively). From this we conclude that the on-shell Lagrangian obtained by using (3.20) in (3.16) has the following form

$$\mathcal{L} = -\frac{i}{2}\bar{\psi}_f(\not{\partial} + M)\psi_f - \frac{1}{2}\partial_\mu S_f\partial^\mu S_f - \frac{1}{2}S_f^T M^2 S_f - \frac{1}{2}\partial_\mu P_f\partial^\mu P_f - \frac{1}{2}P_f^T M^2 P_f, \quad (3.22)$$

with $M = \begin{pmatrix} m_a & m_{ab} \\ m_{ab} & m_b \end{pmatrix}$, is left invariant by the on-shell transformations (3.17) and (3.21).

Notice that (3.22) coincides with the Lagrangian obtained by implementing the mixing transformations directly on the on-shell unmixed Lagrangian (3.13), as it should be.

We emphasize again that a crucial role here is played here by the linearity of SUSY transformations. That is the reason why the mixing transformations (3.15) commute with the SUSY transformations (3.9)-(3.11) allowing for SUSY to be preserved in the mixed Lagrangian (3.22). This is a very robust result and can be easily applied to more general situations. For instance, interaction terms can be accommodated with no changes in this formalism without spoiling such a linearity. Indeed, we just need to work with dummy fields both in the Lagrangian and in the transformation rules, as we did here for the massive free Wess–Zumino case, and only later move to the dynamical fields.

3.2.3 Consequences for vacuum energy and SUSY phenomenology

Having established the SUSY of the mixed Lagrangian, in what follows we shall not need the off-shell expressions and we shall refer only to the on-shell Lagrangians, fields and transformations. We shall now quantize the fields and compute the expectation value of the Hamiltonian on the flavor vacuum to prove that SUSY is spontaneously broken.

The Fourier expansions of the fields are ($i = 1, 2$):

$$\psi_i(x) = \sum_{r=1}^2 \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} e^{i\mathbf{k}\mathbf{x}} \left[u_{\mathbf{k},i}^r(t)\alpha_{\mathbf{k},i}^r + v_{-\mathbf{k},i}^r(t)\alpha_{-\mathbf{k},i}^{\dagger r} \right], \quad (3.23)$$

$$S_i(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{2\omega_{k,i}}} e^{i\mathbf{k}\mathbf{x}} \left[b_{\mathbf{k},i} e^{-i\omega_{k,i}t} + b_{-\mathbf{k},i}^\dagger e^{i\omega_{k,i}t} \right], \quad (3.24)$$

$$P_i(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{2\omega_{k,i}}} e^{i\mathbf{k}\mathbf{x}} \left[c_{\mathbf{k},i} e^{-i\omega_{k,i}t} + c_{-\mathbf{k},i}^\dagger e^{i\omega_{k,i}t} \right], \quad (3.25)$$

where $v_{\mathbf{k},i}^r = \gamma_0 C(u_{\mathbf{k},i}^r)^*$ and $u_{\mathbf{k},i}^r = \gamma_0 C(v_{\mathbf{k},i}^r)^*$ by the Majorana condition and the operators $\alpha_{\mathbf{k},i}^r$, $b_{\mathbf{k},i}$ and $c_{\mathbf{k},i}$ annihilate the vacuum $|0\rangle = |0\rangle^\psi \otimes |0\rangle^S \otimes |0\rangle^P$. Clearly the expectation value of the Hamiltonian on this vacuum is zero, as proven by a straightforward computation:

$$\langle 0|H_\psi|0\rangle = - \int d^3\mathbf{k} (\omega_{k,1} + \omega_{k,2}), \quad (3.26)$$

and

$$\langle 0|H_B|0\rangle = \int d^3\mathbf{k} (\omega_{k,1} + \omega_{k,2}), \quad (3.27)$$

where $H_B = H_S + H_P$, so that:

$$\langle 0|(H_\psi + H_B)|0\rangle = 0. \quad (3.28)$$

The mixing transformations (3.15) can be written as:

$$\psi_\sigma(x) \equiv G_\psi^{-1}(\theta) \psi_i(x) G_\psi(\theta), \quad (3.29)$$

$$S_\sigma(x) \equiv G_S^{-1}(\theta) S_i(x) G_S(\theta), \quad (3.30)$$

$$P_\sigma(x) \equiv G_P^{-1}(\theta) P_i(x) G_P(\theta), \quad (3.31)$$

respectively, where $(\sigma, i) = (a, 1), (b, 2)$, and $G_\psi^{-1}(\theta), G_S^{-1}(\theta), G_P^{-1}(\theta)$, are the generators of the mixing transformations [39]–[18],[50], which are given by:

$$G_\psi(\theta) = \exp \left[\frac{\theta}{2} \int d^3\mathbf{x} \left(\psi_1^\dagger(x) \psi_2(x) - \psi_2^\dagger(x) \psi_1(x) \right) \right]; \quad (3.32)$$

$$G_S(\theta) = \exp \left[-i\theta \int d^3\mathbf{x} (\pi_1^S(x) S_2(x) - \pi_2^S(x) S_1(x)) \right]; \quad (3.33)$$

$$G_P(\theta) = \exp \left[-i\theta \int d^3\mathbf{x} (\pi_1^P(x) P_2(x) - \pi_2^P(x) P_1(x)) \right], \quad (3.34)$$

where with $\pi_i^S(x)$ and $\pi_i^P(x)$ we have denoted the conjugate momenta of the fields $S_i(x)$ and $P_i(x)$, respectively.

The flavor annihilation operators are defined as $\alpha_{\mathbf{k},\sigma}^r \equiv G_\psi^{-1}(\theta) \alpha_{\mathbf{k},i}^r G_\psi(\theta)$, $b_{\mathbf{k},\sigma} \equiv G_S^{-1}(\theta) b_{\mathbf{k},i} G_S(\theta)$, and $c_{\mathbf{k},\sigma} \equiv G_P^{-1}(\theta) c_{\mathbf{k},i} G_P(\theta)$. They annihilate the flavor vacuum $|0\rangle_f \equiv |0\rangle_f^\psi \otimes |0\rangle_f^S \otimes |0\rangle_f^P$, where:

$$|0\rangle_f^\psi \equiv G_\psi^{-1}(\theta) |0\rangle^\psi, \quad |0\rangle_f^S \equiv G_S^{-1}(\theta) |0\rangle^S, \quad |0\rangle_f^P \equiv G_P^{-1}(\theta) |0\rangle^P, \quad (3.35)$$

are the flavor vacua of the fields $\psi_\sigma(x), S_\sigma(x), P_\sigma(x)$, respectively.

As by now is well known, the vacuum $|0\rangle_f$ has a condensate structure [16] and we have the following condensation densities:

$${}_f\langle 0|\alpha_{\mathbf{k},i}^{r\dagger} \alpha_{\mathbf{k},i}^r|0\rangle_f = \sin^2 \theta |V_{\mathbf{k}}^\psi|^2, \quad (3.36)$$

$${}_f\langle 0|b_{\mathbf{k},i}^\dagger b_{\mathbf{k},i}|0\rangle_f = {}_f\langle 0|c_{\mathbf{k},i}^\dagger c_{\mathbf{k},i}|0\rangle_f = \sin^2 \theta |V_{\mathbf{k}}^B|^2, \quad (3.37)$$

where $i = 1, 2$ and the reference frame in which $\mathbf{k} = (0, 0, |\mathbf{k}|)$ has been adopted for convenience.

Let us rewrite the moduli of the two Bogoliubov coefficients $V_{\mathbf{k}}^\psi$ and $V_{\mathbf{k}}^B$ appearing in Eqs.(3.36) and (3.37):

$$|V_{\mathbf{k}}^\psi| = \frac{(\omega_{k,1} + m_1) - (\omega_{k,2} + m_2)}{2\sqrt{\omega_{k,1}\omega_{k,2}(\omega_{k,1} + m_1)(\omega_{k,2} + m_2)}} |\mathbf{k}|, \quad |V_{\mathbf{k}}^B| = \frac{1}{2} \left(\sqrt{\frac{\omega_{k,1}}{\omega_{k,2}}} - \sqrt{\frac{\omega_{k,2}}{\omega_{k,1}}} \right). \quad (3.38)$$

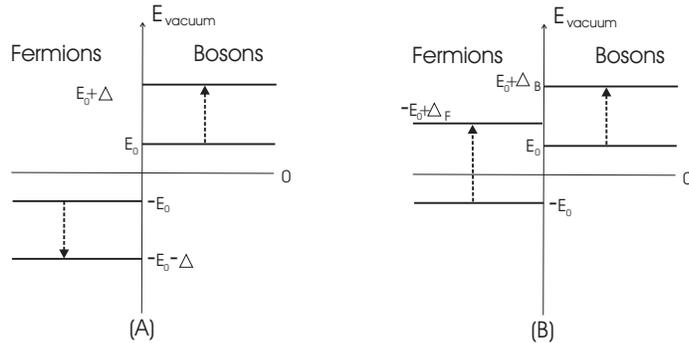


Figure 3.1: (A) SUSY preserving shifts of the zero-point energies. (B) Schematic behaviors of the mixing-induced shifts that break SUSY in the vacuum. Notice that the breaking is present also in the case that $\Delta_F = \Delta_B$.

The expectation value of the fermionic part of H is given by [16]:

$${}_f\langle 0|H_\psi|0\rangle_f = - \int d^3\mathbf{k} (\omega_{k,1} + \omega_{k,2}) (1 - 2 |V_{\mathbf{k}}^\psi|^2 \sin^2 \theta), \quad (3.39)$$

while for the bosonic part we obtain:

$${}_f\langle 0|H_B|0\rangle_f = \int d^3\mathbf{k} (\omega_{k,1} + \omega_{k,2}) (1 + 2 |V_{\mathbf{k}}^B|^2 \sin^2 \theta). \quad (3.40)$$

Combining Eqs.(3.39) and (3.40) we have the result:

$${}_f\langle 0|(H_\psi + H_B)|0\rangle_f = 2 \sin^2 \theta \int d^3\mathbf{k} (\omega_{k,1} + \omega_{k,2}) (|V_{\mathbf{k}}^\psi|^2 + |V_{\mathbf{k}}^B|^2), \quad (3.41)$$

which is different from zero and positive when $\theta \neq 0$ and $m_1 \neq m_2$.

This proves our conjecture for the case in point. As clear from the above, the reason of this spontaneous SUSY breaking are the condensates that both, the fermionic and the bosonic, lift the zero point energy (not respecting SUSY as shown schematically in Fig. 3.1). This effect is unreachable by a perturbative analysis and is thus entirely nonperturbative. This is as it must be: the mixed Lagrangian (3.16) is incapable of giving spontaneous SUSY breaking at the perturbative level, being $F_\sigma = 0 = G_\sigma$ [122], so it is necessary that nonperturbative corrections dynamically induce the breaking.

3.3 Discussion

Being based on a specific model, this is only a very partial proof of the conjecture but it is reasonable to argue that the mechanism shown here for the Wess–Zumino model is actually model–independent at least when SUSY can be implemented linearly hence the mixing transformations commute with it (which is the case for most of the known models). Indeed, the qualitative features of the mixing of higher spin fields are the same as those presented here [89]:

in the flavor vacuum those fields condense and shift the zero point energy always of a positive amount, whether they are bosons or fermions. Having shown here that the breaking mechanism induced by mixing is based precisely on this “SUSY asymmetric” vacuum condensate effect of Fig. 3.1 and it does not depend on the explicit form of the functions in (3.38), since in (3.39)–(3.41) only their squares appear, we can attempt to say that the presence of supermultiplets containing higher spin fields should not alter the final outcome. Another possible issue with the general case is whether in the presence of interactions (as it is compulsory in a realistic model), fields condense in the flavor vacuum in a way that spoils the spontaneous breaking. Unfortunately, the explicit expressions of the functions corresponding to those in (3.38) in this case are not easy to obtain but it seems very unlikely that interaction could restore SUSY in the vacuum by modifying the condensates from the free case of Fig. 3.1 (B) to that of Fig. 3.1 (A).

Having clarified that, a more urgent task than a complete proof of the conjecture (which, nonetheless, is surely one direction worth investigating) would be to probe this conjecture within phenomenologically relevant models. This latter program is the way to test whether it is realistic to consider mixing as the actual responsible for SUSY breaking.

Other directions to investigate are the connection of the SUSY breaking illustrated here with the well known SUSY breaking induced by a nonzero temperature, discussed for example in [56] and in [43] (and to compare this connection to the analogous connection between mixing and temperature made in chapter 5 [31]), and the possible cosmological implications of Eq.(3.41) (in a non supersymmetric context it has been shown that the flavor vacuum energy can be interpreted as a new dark energy component of the universe [46], [48]). Further analysis of these aspects will be done elsewhere.

The results we have just exposed seem to be in contrast with the well known Seiberg non-renormalization theorem [128], which for our purposes states that a supersymmetric model which just contains chiral supermultiplets does not display SUSY breaking even at nonperturbative level. We remark that the derivation of this result is based on the concept of Wilsonian effective action (see e.g. [153]) which is defined through the use of the functional integral. A proper discussion of this issue would need a way to include in the functional integral the unitarily inequivalent vacua such as the flavor vacuum. At the time of writing this question has not been answered yet, even if some partial work in this direction has been done [7] (see also [108] for a discussion in the context of spontaneous breaking of ordinary symmetries). More work in this direction is surely needed, due to the intrinsic importance of this issue.

Chapter 4

Entanglement properties of flavor states

In this chapter we will mostly describe mixing in the Quantum Mechanics context, in order to avoid the subtleties due to the unitary inequivalence of the flavor and mass Hilbert spaces. Our aim is to give a different point of view on flavor neutrino states. In fact, also in the quantum mechanical case they have the structure of non separable states, that is, they are *entangled* states. It is then natural to try to use entanglement to characterize flavor states. They represent an instance of single particle entangled states.

The fact that flavor states of mixed particles are entangled states has been already pointed out several times in the literature, see e.g. [87, 57, 103] (for a review see [14]). However, in the context of particle physics entanglement has been mainly discussed in relation to the Bell inequalities and to decoherence, in particular for the $K^0\bar{K}^0$ meson system.

Before starting the description of flavor states as entangled states we will briefly review multipartite entanglement and single partite entanglement, which are the main tools we use in our treatment. Another approach to entanglement, based on uncertainties, will be outlined and applied to the mixing case. Our treatment of entanglement will be necessarily sketchy due to the enormous amount of literature, to which we refer for details.

4.1 Multipartite entanglement measures

While entanglement of pure quantum states of bipartite systems is very well characterized by the von Neumann entropy $E_{vN} = -\text{Tr}[\rho_1 \log_2 \rho_1]$, where $\rho_1 = \text{Tr}_2[\rho_{12}]$ is the reduced density matrix, or any monotone of this entropy as linear entropy, the same is not true for mixed states of bipartite systems or for states of multipartite systems (in this case it is difficult to characterize entanglement already for pure states). This last case is of particular interest for us since flavor states in the case of mixing of three or more particles are multipartite entangled states.

For bipartite mixed states, several entanglement measures have been proposed [9, 138, 141]. Although providing interesting operative definitions, the entanglement of formation and of distillation [9] are very hard to compute. A celebrated result is the Wootters formula for the

entanglement of formation for two-qubit mixed states [155]. An alternative measure, closely related to the entanglement of formation, is the concurrence (the entanglement of formation is a monotonically increasing function of the concurrence) [54]. The same difficulties of computation are encountered with the relative entropy of entanglement [138]. At present a computable entanglement monotone is the logarithmic negativity $E_{\mathcal{N}}$, based on the requirement of positivity of the density operator under partial transposition $E_{\mathcal{N}} = \log_2 \|\tilde{\rho}_{12}\|_1$, where $\|\cdot\|_1$ denotes the trace norm, i.e. $\|\mathcal{O}\|_1 = \text{Tr}[\sqrt{\mathcal{O}^\dagger \mathcal{O}}]$ for any Hermitian operator \mathcal{O} [141]. The so-called *bona fide* density matrix $\tilde{\rho}_{12}$ is obtained through the partial transposition with respect to one mode, say mode 2, of ρ_{12} , i.e. $\tilde{\rho}_{12} \equiv \rho_{12}^{PT2}$. Given an arbitrary orthonormal product basis $|i_1, j_2\rangle$, the matrix elements of $\tilde{\rho}_{12}$ are determined by the relation $\langle i_1, j_2 | \tilde{\rho}_{12} | k_1, l_2 \rangle = \langle i_1, l_2 | \rho_{12} | k_1, j_2 \rangle$. Obviously, for pure states such a measure provides the same results as the von Neumann entropy.

The quantification of entanglement in the multipartite case also turns out to be a difficult task, and at the time of writing no clear definition for it exists, though much progress has been achieved and many possible measures have been proposed. Also at the qualitative level interesting results have been obtained concerning the different inequivalent ways a multipartite system can be entangled, especially in the case where the subsystems are qubits, which is of direct interest to us as flavor states can be seen as multiqubit states. It has been shown [61, 139] that in the three qubit case two classes of genuine tripartite entanglement (as opposed to bipartite entanglement between pairs of qubits) exist, called *GHZ* and *W* classes from the names of their maximally entangled representatives. In the four qubit case the number of classes is nine.

Some of the proposed entanglement measures for multipartite systems are natural generalizations of the ones used for bipartite systems, while others are specific for the multipartite case (see e.g. [125, 86] for a review). In the following we shall outline the definitions of the ones we will be using. In section 4.10 we shall outline a very promising characterization, known as dynamical symmetry approach, which besides having a clear physical interpretation (unlike many other proposed measures), seems to be quite general, in fact general enough to allow its utilization in some simple field theoretical situations, as we shall see in the case of two flavor mixing. In this case the application of the dynamical symmetry approach provides a very beautiful example of characterization of entanglement in a quantum field theoretical situation.

Average von Neumann entropy In our analysis of flavor states we will consider pure finite dimensional states. Following [113, 42, 127, 119, 64] we will characterize their entanglement through measures performed on all the possible bipartitions of the system. Since we shall limit ourselves to pure states, we will define as a proper measure of multipartite entanglement a functional of the von Neumann entropy¹ on a given bipartition of the system. Let $\rho = |\psi\rangle\langle\psi|$ be the density matrix corresponding to the pure state $|\psi\rangle$ describing the quantum system S divided in N parts i.e. $S = \{S_1, S_2, \dots, S_N\}$. Let us consider the bipartition of this system in two subsystems $S_{A_n} = \{S_{i_1}, S_{i_2}, \dots, S_{i_n}\}$, with $1 \leq i_1 < i_2 < \dots < i_n \leq N$ ($1 \leq n < N$), and

¹As the von Neumann entropy cannot distinguish classical correlations from quantum ones, it is unsuitable to quantify entanglement of mixed states.

$S_{B_{N-n}} = \{S_{j_1}, S_{j_2}, \dots, S_{j_{N-n}}\}$, with $1 \leq j_1 < j_2 < \dots < j_{N-n} \leq N$, and $i_q \neq j_p$, and let

$$\rho_{A_n} \equiv \rho_{i_1, i_2, \dots, i_n} = Tr_{B_{N-n}}[\rho] = Tr_{j_1, j_2, \dots, j_{N-n}}[\rho] \quad (4.1)$$

be the reduced density matrix with respect to the subsystem $S_{B_{N-n}}$. We can compute the von Neumann entropy associated to this bipartition:

$$E_{vN}^{(A_n; B_{N-n})} = -Tr_{A_n}[\rho_{A_n} \log_2 \rho_{A_n}] \quad (4.2)$$

The averaged von Neumann entropy is thus defined as:

$$\langle E_{vN}^{(n; N-n)} \rangle = \binom{N}{n}^{-1} \sum_{A_n} E_{vN}^{(A_n; B_{N-n})}, \quad (4.3)$$

where the sum is performed over all possible bipartitions of the systems in two subsystems having respectively n and $N - n$ elements, with $1 \leq n < N$.

Average linear entropy A very important entanglement measure, also given in terms of bipartite entanglement measures, is the so-called global entanglement defined in [113], which is defined as the sum of the concurrences between one qubit and the others, and can be expressed as the linear entropy averaged on subsystems [42]. This measure was generalized in [126] where a set of average linear entropies of all possible bipartitions of the system was considered. Another approach [64] is based on the distribution of purity of a subsystem on all the possible bipartitions of the total system.

Let us now define linear entropy and average linear entropy. Linear entropy can be considered as a first order expansion of von Neumann entropy. Let as before $S = \{S_1, S_2, \dots, S_N\}$ be our N -partite system and $S_{A_n} = \{S_{i_1}, S_{i_2}, \dots, S_{i_n}\}$, with $1 \leq i_1 < i_2 < \dots < i_n \leq N$ ($1 \leq n < N$), and $S_{B_{N-n}} = \{S_{j_1}, S_{j_2}, \dots, S_{j_{N-n}}\}$, with $1 \leq j_1 < j_2 < \dots < j_{N-n} \leq N$, and $i_q \neq j_p$ a given bipartition, and $\rho = |\psi\rangle\langle\psi|$ a given pure density matrix. Let

$$\rho_{A_n} \equiv \rho_{i_1, i_2, \dots, i_n} = Tr_{B_{N-n}}[\rho] = Tr_{j_1, j_2, \dots, j_{N-n}}[\rho] \quad (4.4)$$

be the reduced density matrix of the subsystem S_{A_n} obtained by tracing over $S_{B_{N-n}}$. The linear entropy associated to this bipartition is defined as

$$S_L^{(A_n; B_{N-n})}(\rho) = \frac{d}{d-1}(1 - Tr_{A_n}[\rho_{A_n}^2]), \quad (4.5)$$

where $d = \min\{\dim S_{A_n}, \dim S_{B_{N-n}}\} = \min\{2^n, 2^{N-n}\}$ is the dimension of the Hilbert space. Linear entropy is related to von Neumann entropy E_{vN} by the relation $E_{vN} = -x \log_2 x - (1-x) \log_2 (1-x)$, with $x = \frac{1+\sqrt{1-S_L}}{2}$.

For the generic reduced state ρ of a two level system we have $S_L = 2[1 - Tr(\rho^2)] = 4Det\rho = 4\lambda_1(1 - \lambda_1)$, where λ_1 is one of the two non negative eigenvalues of ρ and we used the relation $\lambda_1 + \lambda_2 = 1$.

Now the average linear entropy can be defined as:

$$\langle S_L^{(n; N-n)}(\rho) \rangle = \binom{N}{n}^{-1} \sum_{A_n} S_L^{(A_n; B_{N-n})}(\rho), \quad (4.6)$$

where the sum is performed over all possible bipartitions of the system having n and $N - n$ elements.

Average logarithmic negativity As well known, the entropic measures cannot be used to quantify the entanglement of mixed states. In order to measure the multipartite entanglement of mixed states, and following the same procedure as in the previous subsection, we introduce a generalized version of the logarithmic negativity [141]. Let ρ be a multipartite mixed state associated with a system S , partitioned into N parties. Again we consider the bipartition of the N -partite system S into two subsystems S_{A_n} and $S_{B_{N-n}}$. We denote by

$$\tilde{\rho}_{A_n} \equiv \rho^{PT B_{N-n}} = \rho^{PT j_1, j_2, \dots, j_{N-n}} \quad (4.7)$$

the *bona fide* density matrix, obtained by the partial transposition of ρ with respect to the parties belonging to the subsystem $S_{B_{N-n}}$. The logarithmic negativity associated with the fixed bipartition will be given by

$$E_{\mathcal{N}}^{(A_n; B_{N-n})} = \log_2 \|\rho_{A_n} \log_2 \tilde{\rho}_{A_n}\|_1. \quad (4.8)$$

Finally, we define the average logarithmic negativity

$$\langle E_{\mathcal{N}}^{(n; N-n)} \rangle = \binom{N}{n}^{-1} \sum_{A_n} E_{\mathcal{N}}^{(A_n; B_{N-n})}, \quad (4.9)$$

where the sum is intended over all the possible bipartitions of the system.

4.1.1 Three qubit system

As we will see, flavor neutrino states in Quantum Mechanic can be seen as multiqubit states. Let us briefly review the classification of entangled states of a three qubit system. It has been shown [54] that the three qubit states showing genuine tripartite entanglement fall into two classes which are inequivalent under Stochastic Local Operations and Classical Communication (SLOCC) (see e.g [115]). These two classes are known as *GHZ* and *W* class respectively, from the names of their maximally entangled representatives.

The *GHZ* (Greenberger–Horne–Zeilinger, [77]) state is given by:

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle). \quad (4.10)$$

It is a remarkable state, as it has many interesting properties. It is maximally entangled in various senses. For example, it maximally violates Bell-type inequalities, the mutual information of the results of the measurements is maximal and it is maximally stable against white noise. Moreover it is possible, starting from this state, to obtain locally and with probability one an *EPR* state shared by any of the three parts. Anyway the entanglement of this state is very fragile against the loss of one of the parties. Upon performing a partial trace over one of the parts, the remaining two will be in a separable state.

The *W* state is given by:

$$|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle); \quad (4.11)$$

Also this state has some interesting properties. A partial trace with respect to any of the parties gives a mixed bipartite state. Corresponding to the variety of measured that have been proposed to quantify the entanglement of mixed states, there are various ways to quantify the bipartite entanglement of this resulting state. Many of these criteria agree on the fact that the state $|W\rangle$ contains the highest possible bipartite entanglement with respect to any other state (pure or mixed) of three qubits, so it is more robust than the GHZ state against loss of one of the parties. All the reduced density matrices of $|W\rangle$ are equal, and given by:

$$\rho_{AB} = \frac{2}{3}|\Psi^+\rangle\langle\Psi^+| + \frac{1}{3}|00\rangle\langle 00|, \quad (4.12)$$

where $|\Psi^+\rangle = (1/\sqrt{2})(|01\rangle + |10\rangle)$ is a Bell state i.e. is a maximally two qubit entangled state.

Let us compute the average von Neumann entropy on these states. Being $N = 3$ odd, it is only possible to consider unbalanced bipartitions. We have:

$$E_{21}^{(3)} \equiv E_{vN}^{(A_2;B_1)}(\rho_{W^{(3)}}) = \langle E_{vN}^{(2;1)}(\rho_{W^{(3)}}) \rangle = \log_2 3 - \frac{2}{3} \simeq 0.918296, \quad (4.13)$$

$$E_{vN}^{(A_2;B_1)}(\rho_{GHZ^{(3)}}) = \langle E_{vN}^{(2;1)}(\rho_{GHZ^{(3)}}) \rangle = 1. \quad (4.14)$$

For the classification of states of four or more qubits we refer to the literature. See e.g. [139] for the four qubit case.

The generalizations to the case of N qubits of the GHZ and W states are given by:

$$|GHZ^{(N)}\rangle = \frac{1}{\sqrt{2}}(|000\dots 0\rangle + |111\dots 1\rangle), \quad (4.15)$$

$$|W^{(N)}\rangle = \frac{1}{\sqrt{N}}(|100\dots 0\rangle + |010\dots 0\rangle + |001\dots 0\rangle + \dots + |000\dots 1\rangle). \quad (4.16)$$

They share the same properties of their $N = 3$ counterparts.

4.2 Single particle entanglement

While the usual discussions of entanglement concern composite systems with spatially separated components, in the context of flavor mixing we are considering states which describe single particles.

A common mistake which used to be made in the literature is the statement that to have entangled states it is necessary to have more than one particle. This is due to misunderstandings in the entanglement concept. As by now it has been established, it is possible [137, 134] to have entanglement in single particle states. Mode entanglement defined for single-photon states of the radiation field or associated with systems of identical particles has been discussed in Ref. [156]. The concept of mode entanglement in single-particle states has been widely discussed and is by now well established [156, 137, 134]. Successful experimental realizations using single-photon states have been reported as well [104]. Moreover, remarkably, the nonlocality of single-photon states has been experimentally demonstrated [82], verifying a theoretical prediction

[132]. Furthermore, the existing schemes to probe nonlocality in single-particle states have been generalized to include massive particles of arbitrary type [60].

The point is that entanglement is not an *absolute* property of quantum states, but rather a property *relative* to a given set of subsystems, i.e. to a tensor product structure of the Hilbert space. Even a very simple system like a couple of qubits has an infinite number of tensor product structures, so that a generic quantum state of this system can be entangled or not depending on which structure we choose to use (this is related to the choice of the Lie algebra in the dynamic symmetry approach which will be discussed in section 4.10). In fact the following general result holds:

Theorem 1 *Let $|\psi\rangle$ be a quantum state in the finite dimensional space \mathcal{H} of non prime dimension $d = mn$. Then there exist a tensor product structure $\mathcal{H} \equiv \mathcal{V}^m \otimes \mathcal{W}^n$ such that $|\psi\rangle$ is factorizable*

Of course, the tensor product structure should be chosen so to correspond to physically measurable degrees of freedom, between which there can be entanglement. Examples of such degrees of freedom can be the positions of two particles, but an equally valid example is given by the occupation numbers of some quantum field. In this case there is entanglement not between particles, but between the (more fundamental) modes of the field, of which particles are the manifestation. Another possibility is given by the position and the spin of an electron. Between them it is possible to have entanglement.

Let us consider for example a photon which hits a beam splitter. If one of the spatial modes passes through the beam splitter while the second one gets reflexed, the quantum state of the system is (in the occupation number notation)

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B), \quad (4.17)$$

which is nothing but a Bell state for a two mode system. In this example the relevant subsystems for considering entanglement are the two spatial modes: their state is not factorizable. It is natural to ask whether it is possible to perform measures of Bell inequalities on this system. The answer to this question turns out to be affirmative. The state (4.17) can be used [134] to create locally an entangled two particle state, which can then be used to perform measurements on Bell inequalities. All this can be performed without involving additional photons.

A possible physical realization of this situation can be obtained by putting in the locations where the two modes A and B are two cavities, each of which contains an atom (the two atoms can be different) prepared in its ground state $|g\rangle$. Experimental techniques are available such that the photon enters the cavity and excites the atom to a state $|e\rangle$ with a 100% efficiency. If both atoms are initially in the state $|g\rangle$ we can construct the composite state

$$|\phi\rangle_{AB} = \frac{1}{\sqrt{2}}(|g\rangle_A|e\rangle_B + |e\rangle_A|g\rangle_B), \quad (4.18)$$

where now A and B refer to the locations of the atoms. This is a clearly entangled state. The composite state of the two photonic modes is not relevant any more since now both modes will be in the state $|0\rangle$.

Since this transfer can be seen as part of the measuring apparatus, the Bell inequality experiment is directly performed on the two mode photon state. The final result is that the state (4.17) contains entanglement if the modes A and B are spatially separated (i.e. the photon is delocalized). The entanglement is between the modes A and B , not between the photon and the vacuum as we could think.

The previous example, although makes evident that it is possible to have entanglement in single particle states, is based on the nonlocality of the initial photon. Actually the identification of entanglement and nonlocality is another common misconception commonly found in the literature. In fact we will now consider an example which will show that the separation of the positions is not a condition to have entanglement. The *only* condition is the nonseparability of the states. In the following example there will be a violation of Bell inequalities that cannot be considered as a manifestation of nonlocality.

Let us consider a photon with its polarization at 45 with respect to some reference axis. In the occupation number notation, this time with respect to the horizontally and vertically polarized modes, this photon is described by the state (4.17). If the two modes have polarizations for example of 45 and 135, the state is separable, otherwise it is entangled. By using a beam splitter it is possible to spatially separate the two modes and go back to the preceding procedure. As the beam splitter is part of the measuring apparatus, we can affirm that we measured entanglement in one spatial position. The beam splitter does generate nonlocality (it delocalizes the photon), but it is not correct to say that it generates entanglement. It is already present in the photon. We can interpret the beam splitter as a device which changes the tensor product structure i.e. it converts the modes that are being measured in other modes which exhibit entanglement.

4.3 Flavor states as entangled states: static and dynamic entanglement

Let us now see how entanglement is associated to neutrino mixing and oscillations in the simpler case of Quantum Mechanics (see [24] for a recent review). In the following we consider the case of two flavors, leaving the treatment of multiflavor (i.e. multipartite) case to later sections. In our case case, the Pontecorvo–Maki–Nakagawa–Sakata (PMNS) matrix reduces to the 2×2 rotation Pontecorvo matrix $\mathbf{U}(\theta)$,

$$\mathbf{U}(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \quad (4.19)$$

which as we saw in chapter 1 connects the neutrino states with definite flavor with those with definite masses:

$$|\underline{\nu}^{(f)}\rangle = \mathbf{U}(\theta) |\underline{\nu}^{(m)}\rangle \quad (4.20)$$

where $|\underline{\nu}^{(f)}\rangle = (|\nu_e\rangle, |\nu_\mu\rangle)^T$ and $|\underline{\nu}^{(m)}\rangle = (|\nu_1\rangle, |\nu_2\rangle)^T$.

Both $\{|\nu_\alpha\rangle\}$ and $\{|\nu_i\rangle\}$ are orthonormal, i.e. $\langle \nu_\alpha | \nu_\beta \rangle = \delta_{\alpha,\beta}$ and $\langle \nu_i | \nu_j \rangle = \delta_{i,j}$.

We now establish the following correspondence with two-qubit states (a tensor product symbol \otimes is understood):

$$|\nu_1\rangle \equiv |1\rangle_1|0\rangle_2 \equiv |10\rangle, \quad |\nu_2\rangle \equiv |0\rangle_1|1\rangle_2 \equiv |01\rangle, \quad (4.21)$$

where $|\rangle_i$ denotes states in the Hilbert space for neutrinos with mass m_i . Thus, the occupation number allows to interpret the flavor states as constituted by entangled superpositions of the mass eigenstates. Quantum entanglement as usual emerges as a direct consequence of the superposition principle. It is important to remark that the Fock space associated with the neutrino mass eigenstates is physically well defined. In fact, at least in principle, the mass eigenstates can be produced or detected in experiments performing extremely precise kinematical measurements [90]. In this framework, as discussed in Ref. [25], the quantum mechanical state (4.45) is entangled in the field modes, although being a single-particle state.

In the dynamical regime, we saw in chapter 1 as flavor mixing (and neutrino mass differences) generates the phenomenon of neutrino oscillations. The mass eigenstates $|\nu_j\rangle$ have definite masses m_j and definite energies ω_j . Their propagation can be described by plane wave solutions of the form $|\nu_j(t)\rangle = e^{-i\omega_j t}|\nu_j\rangle$. The time evolution of the flavor neutrino states Eq.(4.20) is given by:

$$|\underline{\nu}^{(f)}(t)\rangle = \tilde{\mathbf{U}}(t)|\underline{\nu}^{(f)}\rangle, \quad \tilde{\mathbf{U}}(t) \equiv \mathbf{U}(\theta) \mathbf{U}_0(t) \mathbf{U}(\theta)^{-1}, \quad (4.22)$$

where $|\underline{\nu}^{(f)}\rangle$ are the flavor states at $t = 0$, $\mathbf{U}_0(t) = \text{diag}(e^{-i\omega_1 t}, e^{-i\omega_2 t})$, and $\tilde{\mathbf{U}}(t = 0) = \mathbb{I}$. Remember that at time t , the probability associated with the transition $\nu_\alpha \rightarrow \nu_\beta$ is

$$P_{\nu_\alpha \rightarrow \nu_\beta}(t) = |\langle \nu_\beta | \nu_\alpha(t) \rangle|^2 = |\tilde{\mathbf{U}}_{\alpha\beta}(t)|^2, \quad (4.23)$$

where $\alpha, \beta = e, \mu$. Let us rewrite the explicit form for the transition probabilities in the two flavor case:

$$P_{\nu_e \rightarrow \nu_e}(t) = 1 - \sin^2(2\theta) \sin^2\left(\frac{\omega_2 - \omega_1}{2}t\right), \quad (4.24)$$

$$P_{\nu_e \rightarrow \nu_\mu}(t) = \sin^2(2\theta) \sin^2\left(\frac{\omega_2 - \omega_1}{2}t\right). \quad (4.25)$$

Flavor neutrino states are well defined in the context of Quantum Field Theory (QFT), where they are obtained as eigenstates of the flavor neutrino charges [39, 41]. We discussed how in the relativistic limit, the exact QFT flavor states reduce to the usual Pontecorvo flavor states Eq.(4.20): flavor modes are thus legitimate and physically well-defined individual entities and mode entanglement can be defined and studied in analogy with the static case of Ref.[25]. We can thus establish the following correspondence with two-qubit states:

$$|\nu_e\rangle \equiv |1\rangle_e|0\rangle_\mu, \quad |\nu_\mu\rangle \equiv |0\rangle_e|1\rangle_\mu. \quad (4.26)$$

States $|0\rangle_\alpha$ and $|1\rangle_\alpha$ correspond, respectively, to the absence and the presence of a neutrino in mode α . Entanglement is thus established among flavor modes, in a single-particle setting. Eq. (4.80) can then be recast as

$$|\nu_\alpha(t)\rangle = \tilde{\mathbf{U}}_{\alpha e}(t)|1\rangle_e|0\rangle_\mu + \tilde{\mathbf{U}}_{\alpha\mu}(t)|0\rangle_e|1\rangle_\mu, \quad (4.27)$$

with the normalization condition $\sum_{\beta} |\tilde{U}_{\alpha\beta}(t)|^2 = 1$ ($\alpha, \beta = e, \mu$). The time-evolved states $|\underline{\nu}^{(f)}(t)\rangle$ are entangled superpositions of the two flavor eigenstates with time-dependent coefficients. Thus, flavor oscillations can be related to bipartite (flavor) entanglement of single-particle states [23].

To summarize, the flavor neutrino state at a given time, say $|\nu_e(t)\rangle$ for definiteness, can be regarded as an entangled state either in terms of the mass eigenstates or in terms of the flavor eigenstates (at a fixed time). In the first instance, which was studied in detail for the multipartite case in [25], we have a static entanglement, in the sense that the result of the entanglement measures on the state $|\nu_e(t)\rangle$ do not depend on time. In the second case, considered for the general three flavor case in [23], the entanglement varies with time as it is related to the oscillations of flavor(s). Let us now discuss both cases in the two flavor case by using the standard entropic measures of bipartite entanglement. The extension to more than two flavors of both kinds of entanglement will be the subject of later sections, as will be the use of the dynamical symmetry approach to entanglement to the dynamic entanglement of two flavors in both QM and QFT. The extension of the latter approach to multiflavor cases is the subject of ongoing work.

4.3.1 Static entanglement

Let us study the entanglement properties of the Pontecorvo states using the standard entropic measures of bipartite entanglement. We will use both von Neumann and linear entropy, the latter giving results which are particular cases of the ones obtained in the multipartite case. Recall the explicit expression of these states:

$$|\nu_e\rangle = \cos\theta |\nu_1\rangle + \sin\theta |\nu_2\rangle \quad (4.28)$$

$$|\nu_{\mu}\rangle = -\sin\theta |\nu_1\rangle + \cos\theta |\nu_2\rangle, \quad (4.29)$$

With the identification (4.21) they can be written as

$$|\nu_e\rangle = \cos\theta |1\rangle_1 |0\rangle_2 + \sin\theta |0\rangle_1 |1\rangle_2 \quad (4.30)$$

$$|\nu_{\mu}\rangle = -\sin\theta |1\rangle_1 |0\rangle_2 + \cos\theta |0\rangle_1 |1\rangle_2. \quad (4.31)$$

The density matrices associated to these states are of course:

$$\rho^{(e)} \equiv |\nu_e\rangle\langle\nu_e| \quad ; \quad \rho^{(\mu)} \equiv |\nu_{\mu}\rangle\langle\nu_{\mu}|. \quad (4.32)$$

The reduced density matrices obtained by tracing out ν_2 are:

$$\rho_1^{(e)} = \cos^2\theta |1\rangle_1\langle 1| + \sin^2\theta |0\rangle_1\langle 0| \quad (4.33)$$

$$\rho_1^{(\mu)} = \sin^2\theta |1\rangle_1\langle 1| + \cos^2\theta |0\rangle_1\langle 0|, \quad (4.34)$$

while the ones obtained by tracing out ν_1 are:

$$\rho_2^{(e)} = \cos^2\theta |0\rangle_2\langle 0| + \sin^2\theta |1\rangle_2\langle 1| \quad (4.35)$$

$$\rho_2^{(\mu)} = \sin^2\theta |0\rangle_2\langle 0| + \cos^2\theta |1\rangle_2\langle 1| \quad (4.36)$$

where we used the usual definition $\rho_1^e \equiv Tr_2(\rho^e) = \sum_j 2^j \langle j | (|\nu_e\rangle\langle\nu_e|) | j \rangle_2$. The von Neumann entropies relative to these states are given by the usual definition $S(\rho) = -Tr(\rho \log_2 \rho)$ and are:

$$S(\rho_1^{(e)}) = S(\rho_1^{(\mu)}) = -\cos^2 \theta \log_2 \cos^2 \theta - \sin^2 \theta \log_2 \sin^2 \theta \quad (4.37)$$

In the maximal mixing case $\theta = \frac{\pi}{4}$, the two Pontecorvo states are maximally entangled, and they coincide with the $|\Psi^+\rangle$ and $|\Psi^-\rangle$ states in the Bell basis:

$$|\Phi^\pm\rangle = \frac{1}{\sqrt{2}} (|0\rangle_A \otimes |0\rangle_B \pm |1\rangle_A \otimes |1\rangle_B) \quad (4.38)$$

$$|\Psi^\pm\rangle = \frac{1}{\sqrt{2}} (|0\rangle_A \otimes |1\rangle_B \pm |1\rangle_A \otimes |0\rangle_B). \quad (4.39)$$

The linear entropies associated to the electron neutrino state are equal:

$$S_L^{(1;2)}(\rho_e) = 2 \left(1 - Tr_1[(\rho_1^{(e)})^2] \right) = \sin^2(2\theta), \quad (4.40)$$

$$S_L^{(2;1)}(\rho_e) = 2 \left(1 - Tr_2[(\rho_2^{(e)})^2] \right) = \sin^2(2\theta), \quad (4.41)$$

and similar results hold for the muon neutrino state. The above results are particular cases of the more general ones obtained for the three flavor neutrino states in Ref.[25] and exposed in the following sections, where it was found that such states can be classified as generalized W states.

Eqs.(4.40)-(4.41) express the fact that flavor neutrino states at any time can be regarded as entangled superpositions of the mass qubits $|\nu_i\rangle$, where the entanglement is a function of the mixing angle only.

4.3.2 Dynamic entanglement

Let us now turn to the dynamic entanglement arising in connection with flavor oscillations [23]. To this aim, we rewrite the electron neutrino state $|\nu_e(t)\rangle$ as

$$|\nu_e(t)\rangle = \tilde{\mathbf{U}}_{ee}(t) |\nu_e\rangle + \tilde{\mathbf{U}}_{e\mu}(t) |\nu_\mu\rangle, \quad (4.42)$$

where $|\nu_e\rangle, |\nu_\mu\rangle$ are the flavor neutrino states at time $t = 0$ and are now taken as the relevant qubits (see Eq.(4.26)). By proceeding in a similar way as we did for the static case, we arrive at the following expression for the linear entropies associated to the above state:

$$\begin{aligned} S_L^{(\mu;e)}(\rho_e) = S_L^{(e;\mu)}(\rho_e) &= 4|\tilde{\mathbf{U}}_{ee}(t)|^2 |\tilde{\mathbf{U}}_{e\mu}(t)|^2 \\ &= 4|\tilde{\mathbf{U}}_{ee}(t)|^2 (1 - |\tilde{\mathbf{U}}_{ee}(t)|^2) \end{aligned} \quad (4.43)$$

Eq.(4.43) establishes that the linear entropy of the reduced state is equal to the product of the two-flavor transition probabilities given in Eqs.(4.81)-(4.25). It is remarkable that simple expressions similar to those of Eq. (4.43) hold also for the three flavor case [23].

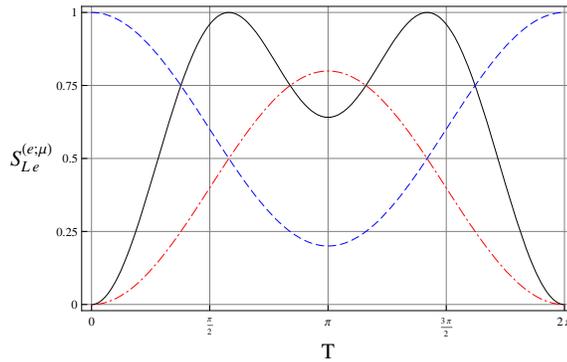


Figure 4.1: (Color online) Linear entropy $S_L^{(e;\mu)}(\rho_e)$ (full) as a function of the scaled time $T = \frac{2Et}{\Delta m_{12}^2}$. The mixing angle θ is fixed at the experimental value $\sin^2 \theta = 0.314$. The transition probabilities $P_{\nu_e \rightarrow \nu_e}$ (dashed) and $P_{\nu_e \rightarrow \nu_\mu}$ (dot-dashed) are reported as well for comparison.

Note also that, for any reduced state ρ of a two-level system one has that $S_L = 2[1 - \text{Tr}(\rho^2)] = 4\text{Det}\rho = 4\lambda_1(1 - \lambda_1)$, where λ_1 is one of the two non-negative eigenvalues of ρ , and the relation $\lambda_1 + \lambda_2 = 1$ has been exploited. Comparing with Eq. (4.43), one sees that the transition probabilities coincide with the eigenvalues of the reduced state density matrix.

In Fig. 4.1 we show the behavior of $S_L^{(e;\mu)}(\rho_e)$ as a function of the scaled, dimensionless time $T = \frac{2Et}{\Delta m_{12}^2}$. In the same figure, we also report the behavior of the transition probabilities $P_{\nu_e \rightarrow \nu_e}$ and $P_{\nu_e \rightarrow \nu_\mu}$. The plots have a clear physical interpretation. At time $T = 0$, the entanglement is zero, the global state of the system is factorized, and the two flavors are not mixed. For $T > 0$, flavors start to oscillate and the entanglement is maximal at largest mixing: $P_{\nu_e \rightarrow \nu_e} = P_{\nu_e \rightarrow \nu_\mu} = 0.5$, and minimum at $T = \pi$.

4.4 Flavor states as generalized W states

Now we turn to the study of static entanglement in the multiflavor case. We will consider generalized W states of the form:

$$|W^{(N)}(\alpha_1, \alpha_2, \dots, \alpha_N)\rangle = \sum_{k=1}^N \alpha_k |\delta_{1,k}, \delta_{2,k}, \dots, \delta_{N,k}\rangle \equiv \sum_{k=1}^N \alpha_k |\nu_k^{(N)}\rangle, \quad \sum_{k=1}^N |\alpha_k|^2 = 1, \quad (4.44)$$

in particular in the cases $N = 3, 4$. Although the $N = 3$ case is the physically relevant one, the $N = 4$ case is worth studying because of some additional subtleties.

4.4.1 Generalized $W^{(3)}$ states from the CKM matrix

In this subsection we discuss the $N = 3$ case. For the coefficients $\{\alpha_k\}$ we shall use the well-known Cabibbo–Kobayashi–Maskawa (CKM) parametrization [53]:

$$|\underline{\nu}_f\rangle = U(\tilde{\theta}, \delta) |\underline{\nu}_m\rangle \quad (4.45)$$

$$U(\tilde{\theta}, \delta) = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}, \quad (4.46)$$

where $|\underline{\nu}_f\rangle = (|\nu_e\rangle, |\nu_\mu\rangle, |\nu_\tau\rangle)^T$ are the definite flavor states and $|\underline{\nu}_m\rangle = (|\nu_1\rangle, |\nu_2\rangle, |\nu_3\rangle)^T$ are the definite mass states.

In Eqs. (4.45) and (4.46) we used the following short notation: $(\tilde{\theta}, \delta) \equiv (\theta_{12}, \theta_{13}, \theta_{23}; \delta)$, $c_{ij} \equiv \cos \theta_{ij}$ and $s_{ij} \equiv \sin \theta_{ij}$. The mixing matrix is parametrized by three mixing angles θ_{13} , θ_{23} and a phase δ , in total four parameters. It is possible to show that maximal mixing is attained for [95]:

$$\theta_{12}^{max} = \frac{\pi}{4}; \quad \theta_{23}^{max} = \frac{\pi}{4}; \quad \theta_{13}^{max} = \arccos \sqrt{\frac{2}{3}}; \quad \delta^{max} = \frac{\pi}{2}. \quad (4.47)$$

For these values of the parameters the *CKM* matrix elements all have the same value $\frac{1}{\sqrt{3}}$.

We define the generalized *W* states of three qubits through application of the matrix defined below, which is obtained from the *CKM* matrix where the third column has been multiplied by $e^{i\delta}$:

$$|\underline{W}^{(3)}(\tilde{\theta}; \delta)\rangle \equiv U^{(3f)}(\tilde{\theta}, \delta) |\underline{\nu}^{(3)}\rangle \quad (4.48)$$

$$U^{(3f)}(\tilde{\theta}, \delta) = U(\tilde{\theta}, \delta) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\delta} \end{pmatrix} \quad (4.49)$$

$$= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13}e^{i\delta} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13}e^{i\delta} \end{pmatrix},$$

where $|\underline{W}^{(3)}(\tilde{\theta}; \delta)\rangle = (|W_e^{(3)}(\tilde{\theta}, \delta)\rangle, |W_\mu^{(3)}(\tilde{\theta}, \delta)\rangle, |W_\tau^{(3)}(\tilde{\theta}, \delta)\rangle)^T e |\underline{\nu}^{(3)}\rangle = (|\nu_1^{(3)}\rangle, |\nu_2^{(3)}\rangle, |\nu_3^{(3)}\rangle)^T$. The two matrices (4.45) and (4.49) produce states whose entanglement properties are identical. The reason for using the parametrization (4.49) is that when the mixing angles assume their maximal mixing values, the first state coincides with the usual $|W^{(3)}\rangle$ state independently on the phase, and this property will turn out to be useful.

In the maximal mixing case the matrix $U^{(3f)}$ is:

$$U_{max}^{(3f)} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ iy & iy^2 & i \\ iy^2 & iy & i \end{pmatrix}. \quad (4.50)$$

where $y = \exp(2i\pi/3)$. In this case the states $|W_2^{(3)}(\tilde{\theta}, \delta)\rangle, |W_3^{(3)}(\tilde{\theta}, \delta)\rangle$ contain the same amount of entanglement as $|W^{(3)}\rangle$:

$$E_{vN}^{(A_2; B_1)}(|\underline{W}^{(3)}(\tilde{\theta}^{max}; \delta^{max})\rangle) = \langle E_{vN}^{(2;1)}(|\underline{W}^{(3)}(\tilde{\theta}^{max}; \delta^{max})\rangle) \rangle = E_{21}^{(3)}, \quad (4.51)$$

where $E_{21}^{(3)}$ is defined in (4.13). In general the three flavor states $|W_1^{(3)}(\tilde{\theta}, \delta)\rangle, |W_2^{(3)}(\tilde{\theta}, \delta)\rangle$ and $|W_3^{(3)}(\tilde{\theta}, \delta)\rangle$ defined in (4.48) represent generalized $|W\rangle$ states.

In the next section we shall analyze the entanglement properties of the states $|W_\alpha^{(3)}(\tilde{\theta}, \tilde{\delta})\rangle$ and their behavior as the mixing parameters vary. Before doing this, let us construct the analogues of the states $|W_\alpha^{(3)}(\tilde{\theta}, \tilde{\delta})\rangle$ in the four flavor case.

4.4.2 Generalized $W^{(4)}$ states

In the $N = 4$ case the associated mixing matrix will depend on 9 independent parameters, of which 6 mixing angles and 3 phases: $(\tilde{\theta}; \tilde{\delta}) = (\theta_{12}, \theta_{13}, \theta_{14}, \theta_{23}, \theta_{24}, \theta_{34}; \delta_{14}, \delta_{23}, \delta_{34})$. We can build this matrix as the product of elementary matrices:

$$U^{(4f)}(\tilde{\theta}; \tilde{\delta}) = U_{34}(\theta_{34}, \delta_{34})U_{24}(\theta_{24})U_{23}(\theta_{23}, \delta_{23})U_{14}(\theta_{14}, \delta_{14})U_{13}(\theta_{13})U_{12}(\theta_{12})U_\delta(\delta_{14}), \quad (4.52)$$

where

$$\begin{aligned} U_\delta(\delta_{14}) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\delta_{14}} \end{pmatrix}; & U_{12} &= \begin{pmatrix} \cos \theta_{12} & \sin \theta_{12} & 0 & 0 \\ -\sin \theta_{12} & \cos \theta_{12} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \\ U_{13} &= \begin{pmatrix} \cos \theta_{13} & 0 & \sin \theta_{13} & 0 \\ 0 & 1 & 0 & 0 \\ -\sin \theta_{13} & 0 & \cos \theta_{13} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ U_{14} &= \begin{pmatrix} \cos \theta_{14} & 0 & 0 & e^{-i\delta_{14}} \sin \theta_{14} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -e^{i\delta_{14}} \sin \theta_{14} & 0 & 0 & \cos \theta_{14} \end{pmatrix}; \\ U_{23} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta_{23} & e^{-i\delta_{23}} \sin \theta_{23} & 0 \\ 0 & -e^{i\delta_{23}} \sin \theta_{23} & \cos \theta_{23} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & (4.53) \\ U_{24} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta_{24} & 0 & \sin \theta_{24} \\ 0 & 0 & 1 & 0 \\ 0 & -\sin \theta_{24} & 0 & \cos \theta_{24} \end{pmatrix}; \\ U_{34} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \theta_{34} & e^{-i\delta_{34}} \sin \theta_{34} \\ 0 & 0 & -e^{i\delta_{34}} \sin \theta_{34} & \cos \theta_{34} \end{pmatrix}. \end{aligned}$$

As before we define the generalized W states as follows:

$$|\underline{W}^{(4)}(\tilde{\theta}; \tilde{\delta})\rangle \equiv U^{(4f)}(\tilde{\theta}; \tilde{\delta}) |\underline{\nu}^{(4)}\rangle. \quad (4.54)$$

The maximal mixing case is attained when the parameters assume the values

$$\theta_{12}^{max} = \theta_{34}^{max} = \frac{\pi}{4}; \quad \theta_{14}^{max} = \theta_{23}^{max} = \frac{\pi}{6}; \quad \theta_{13}^{max} = \arccos \sqrt{\frac{2}{3}}; \quad (4.55)$$

$$\theta_{24}^{max} = \arcsin \sqrt{\frac{1}{3}}; \delta_{14}^{max} = \phi; \quad \delta_{23}^{max} = \pi - \phi; \quad \delta_{34}^{max} = \phi. \quad (4.56)$$

and in this case all the elements of the matrix (4.52) assume their maximal value 1/2. In correspondence of the values (4.55) and (4.56), $U_{max}^{(4f)}(\phi)$ takes the simple form

$$U_{max}^{(4f)}(\phi) = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -e^{i\phi} & e^{i\phi} \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -e^{i\phi} & e^{i\phi} \end{pmatrix}. \quad (4.57)$$

All the states $|\underline{W}^{(4)}(\tilde{\theta}^{max}; \tilde{\delta}^{max})\rangle$ contain the same amount of entanglement of the usual $|W^{(4)}\rangle$ state:

$$E_{vN}^{(A_3:B_1)}(|\underline{W}^{(4)}(\tilde{\theta}^{max}; \tilde{\delta}^{max})\rangle) = \langle E_{vN}^{(3:1)}(|\underline{W}^{(4)}(\tilde{\theta}^{max}; \tilde{\delta}^{max})\rangle) \rangle = E_{31}^{(4)}. \quad (4.58)$$

$$E_{vN}^{(A_2:B_2)}(|\underline{W}^{(4)}(\tilde{\theta}^{max}; \tilde{\delta}^{max})\rangle) = \langle E_{vN}^{(2:2)}(|\underline{W}^{(4)}(\tilde{\theta}^{max}; \tilde{\delta}^{max})\rangle) \rangle = E_{22}^{(4)}, \quad (4.59)$$

where the bipartition (A_2, B_2) and (A_3, B_1) has been considered. As in the three qubit case, the state $|W_1^{(4)}(\tilde{\theta}; \tilde{\delta})\rangle$ reduces to the usual $|W^{(4)}\rangle$ when the mixing angles assume their maximal mixing values (4.55), independently of the phases δ_{ij} .

4.5 The role of phases in the $W^{(N)}$ state entanglement

Let us now study the correlation properties of the W -like states defined in (4.48) and (4.54). These properties are completely determined by the mixing angles θ_{ij} and by the phases δ_{ij} .

We observed that our definition of generalized $|W^{(N)}\rangle$ states is characterized by the fact that the usual $|W^{(N)}\rangle$ state can be obtained by the first row of the mixing matrix by setting the mixing angles to their maximal mixing values, i.e. $|W^{(N)}\rangle = |W_1^{(N)}(\tilde{\theta}^{max}; \tilde{\delta})\rangle$. In other words, in general an $N - 1$ -dimensional subspace exists which is orthogonal to the state $|W^{(N)}\rangle$ and is generated by the vectors $\left\{ |W_2^{(N)}(\tilde{\theta}^{max}; \tilde{\delta})\rangle, \dots, |W_N^{(N)}(\tilde{\theta}^{max}; \tilde{\delta})\rangle \right\}$. We will study the entanglement properties of this subclass of generalized $|W^{(N)}\rangle$ states, which are parametrized just by the phases of the mixing matrix. We will not treat the general case of varying mixing angles as well.

4.5.1 States $W_k^{(3)}(\theta_{ij}^{max}; \delta)$

Let us study how the entanglement of the states defined through the matrix (4.48) depends on the phase δ . Let us set the mixing angles to their maximal mixing values θ_i^{max} given by

(4.47). We thus obtain the three orthogonal generalized W states $|W_q^{(3)}(\delta)\rangle \equiv |W_q^{(3)}(\tilde{\theta}^{max}; \delta)\rangle$ ($q = 1, 2, 3$), of which the first state coincides with the usual W state. With this choice of the parameters, the matrix $U^{(3f)}$ assumes the form:

$$U^{(3f)}(\delta) = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ -\frac{1}{2}(\sqrt{3} + e^{i\delta}) & \frac{1}{2}(\sqrt{3} - e^{i\delta}) & e^{i\delta} \\ \frac{1}{2}(\sqrt{3} - e^{i\delta}) & -\frac{1}{2}(\sqrt{3} + e^{i\delta}) & e^{i\delta} \end{pmatrix}. \quad (4.60)$$

Now let us compute the quantities $E_{vN}^{(A_2:B_1)}$ and $\langle E_{vN}^{(2:1)} \rangle$ defined by (4.2) and (4.3). We obtain:

$$E_{vN1}^{(1,2:3)} = E_{vN1}^{(1,3:2)} = E_{vN1}^{(2,3:1)} = E_{vN2}^{(1,2:3)} = E_{vN3}^{(1,2:3)} = \log_2 3 - \frac{2}{3}, \quad (4.61)$$

$$\begin{aligned} E_{vN2}^{(1,3:2)} &= E_{vN3}^{(2,3:1)} = -\left(\frac{1}{3} - \frac{\cos \delta}{2\sqrt{3}}\right) \log_2 \left[\frac{1}{3} - \frac{\cos \delta}{2\sqrt{3}}\right] \\ &\quad - \left(\frac{2}{3} + \frac{\cos \delta}{2\sqrt{3}}\right) \log_2 \left[\frac{2}{3} + \frac{\cos \delta}{2\sqrt{3}}\right], \end{aligned} \quad (4.62)$$

$$\begin{aligned} E_{vN2}^{(2,3:1)} &= E_{vN3}^{(1,3:2)} = -\left(\frac{2}{3} - \frac{\cos \delta}{2\sqrt{3}}\right) \log_2 \left[\frac{2}{3} - \frac{\cos \delta}{2\sqrt{3}}\right] \\ &\quad - \left(\frac{1}{3} + \frac{\cos \delta}{2\sqrt{3}}\right) \log_2 \left[\frac{1}{3} + \frac{\cos \delta}{2\sqrt{3}}\right], \end{aligned} \quad (4.63)$$

wher the notation $(i, j : k)$ represents the explicit composition of the bipartitions $A_2 = \{S_i, S_j\}$ and $B_1 = \{S_k\}$, with $i, j, k = 1, 2, 3$ and $i \neq j \neq k$. We introduced the synthetic notation $E_{vNq}^{(i,j:k)} \equiv E_{vN}^{(i,j:k)}(|W_q^{(3)}(\delta)\rangle)$.

Notice that the correlation properties of the states $|W_q^{(3)}(\delta)\rangle$, with $q = 2, 3$ depend on δ . Consider for example the state $|W_2^{(3)}(\delta)\rangle$. In Fig. 4.2, the plots I and II represent respectively the behavior of $E_{vN2}^{(i,j:k)}$ and $\langle E_{vN2}^{(2:1)} \rangle$ as functions of δ in the range $[-\pi, \pi]$

While $E_{vN2}^{(1,2:3)}$ (dotted line) assumes the constant value $E_{21}^{(3)}$ (the same of the state $|W^{(3)}\rangle$), the quantities $E_{vN2}^{(1,3:2)}$ (dashed line) and $E_{vN2}^{(2,3:1)}$ (mixed line) vary with δ , attaining the absolute minimum 1 in $\delta_1 = \pm \arccos\left(-\frac{1}{\sqrt{3}}\right) \pm 2p\pi$ and $\delta_2 = \pm \arccos\left(\frac{1}{\sqrt{3}}\right) \pm 2p\pi$ (with p an integer), respectively. This means that the state $|W_2^{(3)}(\delta_i)\rangle$, with $i = 1, 2$, contains in a given bipartition maximal entanglement which coincides with the amount of entanglement contained in the state $|GHZ^{(3)}\rangle$, which is bigger than the amount of entanglement contained in the W state. Moreover, for a given range of δ , one of the two quantities stays higher than $E_{21}^{(3)}$. On the other hand Fig. 4.2 II, shows that the average von Neumann entropy $\langle E_{vN2}^{(2:1)} \rangle$ is always lower than the reference value $E_{21}^{(3)}$, assuming this maximal value in $\delta = \frac{\pi}{2} \pm p\pi$.

We thus saw that the phase δ can be used to concentrate entanglement in a particular bipartition, thus getting a ‘‘squeezing’’ of entanglement, at the expenses of the average von Neumann entropy.

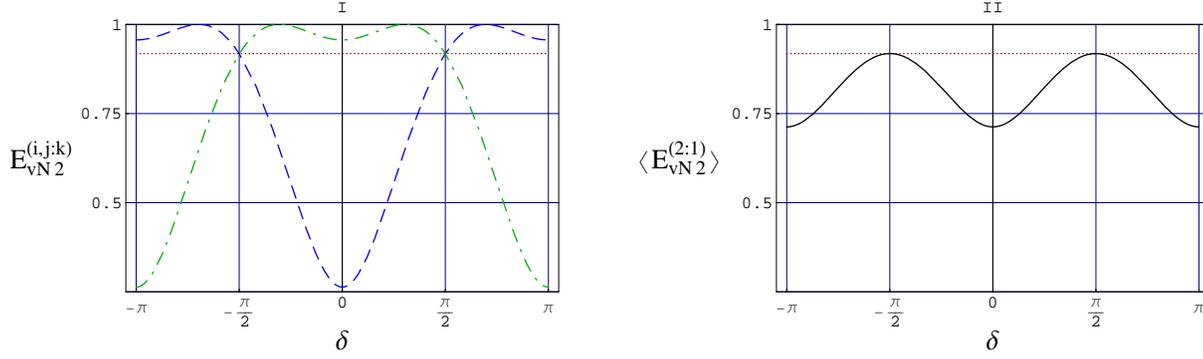


Figure 4.2: I-II Plots of $E_{vN2}^{(i,j;k)}$ e $\langle E_{vN2}^{(2:1)} \rangle$ as functions of δ . I shows the values of $E_{vN2}^{(i,j;k)}$ corresponding to the following values of i, j, k : (a) $i = 1, j = 2, e k = 3$ (dotted line); (b) $i = 1, j = 3, e k = 2$ (dashed line); (c) $i = 2, j = 3, e k = 1$ (mixed line). $E_{vN2}^{(1,2;3)}$ is constant and assumes the reference value $E_{21}^{(3)} = 0.918296$. In II, the average entropy $\langle E_{vN2}^{(2:1)} \rangle$ assumes its maximal value $E_{21}^{(3)}$ (dotted line) in $\delta = \frac{\pi}{2} \pm p\pi$, with p integer.

4.5.2 States $W_k^{(4)}(\theta_{ij}^{max}; \delta_{lm})$

The class of W -like states for $N = 4$, given by (4.54), gives a bigger arena to explore, since the number of degrees of freedom grows. We proceed in the same way as in sect. 4.5.1, that is, we fix the mixing angles to their maximal values given by θ_{ij}^{max} , leaving the phases δ_{ij} as free parameters. The matrix $U^{(4f)}(\tilde{\theta}; \tilde{\delta})$ assumes the form:

$$U^{(4f)}(\tilde{\delta}) = \frac{1}{2} \quad (4.64)$$

$$\times \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 - \frac{z_{14}}{3} - \frac{z_{23}^*}{3} & 1 - \frac{z_{14}}{3} - \frac{z_{23}^*}{3} & -\frac{z_{14}}{3} + \frac{2z_{23}^*}{3} & z_{14} \\ -\frac{1}{2} + \frac{z_{23}}{2} - \frac{z_{14}z_{34}^*}{3} + \frac{z_{23}z_{34}^*}{6} + \frac{z_{34}^*}{2} & -\frac{1}{2} - \frac{z_{23}}{2} - \frac{z_{14}z_{34}^*}{3} + \frac{z_{23}z_{34}^*}{6} - \frac{z_{34}^*}{2} & 1 - \frac{z_{14}z_{34}^*}{3} - \frac{z_{23}z_{34}^*}{3} & z_{34}^*z_{14} \\ \frac{1}{2} - \frac{z_{14}}{3} + \frac{z_{23}^*}{6} - \frac{z_{23}z_{34}}{2} + \frac{z_{34}}{2} & -\frac{1}{2} - \frac{z_{14}}{3} + \frac{z_{23}^*}{6} + \frac{z_{23}z_{34}}{2} + \frac{z_{34}}{2} & -\frac{z_{14}}{3} - \frac{z_{23}^*}{3} - z_{34} & z_{14} \end{pmatrix}$$

where $z_{ij} \equiv e^{i\delta_{ij}}$. The analytic expressions of the entanglement measures of the states $|W_q^{(4)}(\tilde{\delta})\rangle \equiv |W_q^{(4)}(\tilde{\theta}^{max}; \tilde{\delta})\rangle$ ($q = 1, \dots, 4$) are given in appendix A. As anticipated, the state $|W_1^{(4)}(\tilde{\delta})\rangle$ coincides with the usual $|W^{(4)}\rangle$.

Let us analyze the dependence of the phases δ_{14} and δ_{23} of the entanglement measures corresponding to the state $|W_2^{(4)}(\tilde{\delta})\rangle$. In Fig. 4.3 the plots I–III respectively show $E_{vN2}^{(1,2;3,4)}$, $E_{vN2}^{(1,3;2,4)}$ and $E_{vN2}^{(1,4;2,3)}$ as functions of δ_{14} and δ_{23} ; plot IV shows the behavior of the averaged entropy $\langle E_{vN2}^{(2:2)} \rangle$. Entanglement assumes the maximal value 1 at the values given by (4.56), that is $\delta_{14} + \delta_{23} = \pm p\pi$, with p an odd integer. Moreover, while $E_{vN2}^{(1,2;3,4)}$ displays an oscillating behavior along the direction parallel to the vector $(\delta_{14}, \delta_{23}) = (1, 1)$, the quantities $E_{vN2}^{(1,3;2,4)}$, $E_{vN2}^{(1,4;2,3)}$, and $\langle E_{vN2}^{(2:2)} \rangle$ display a periodic array of holes.

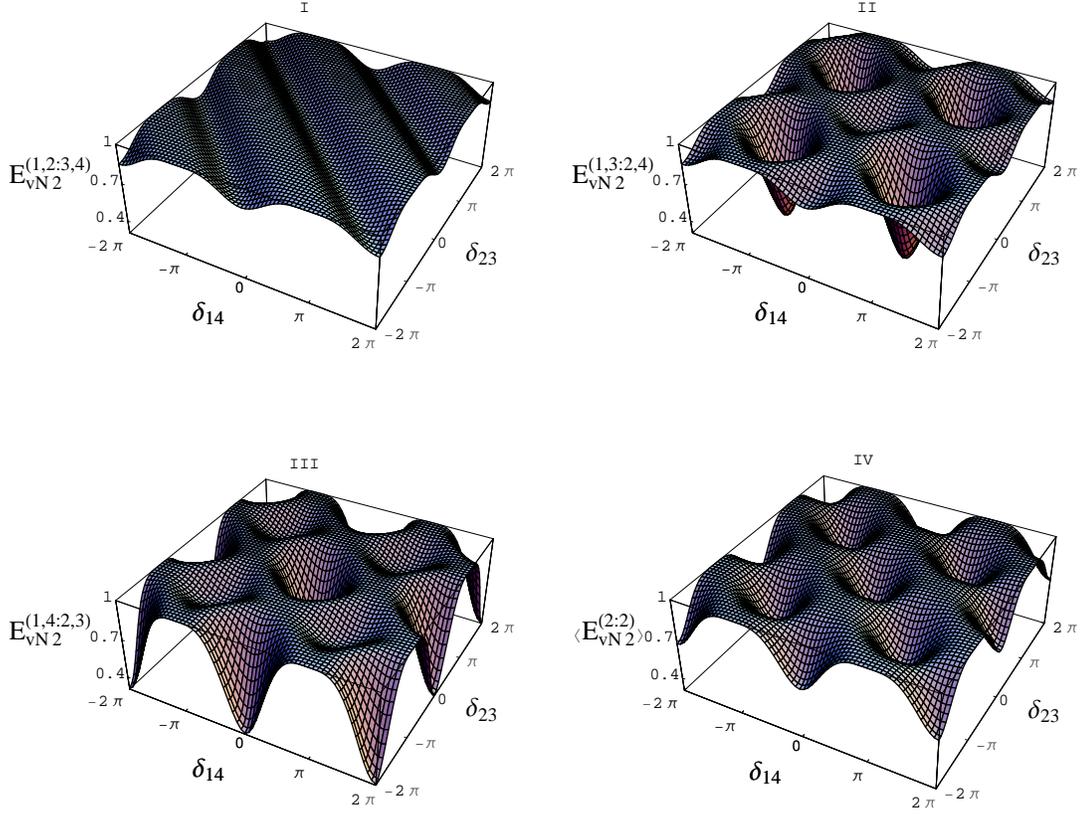


Figure 4.3: I-IV Plots of $E_{vN 2}^{(i,j;k,l)}$ and $\langle E_{vN 2}^{(2;2)} \rangle$ as functions of the phases δ_{14} and δ_{23} .

Let us now consider the entropies corresponding to the unbalanced bipartitions $E_{vN 2}^{(i;j,k,l)}$, shown in Fig. 4.4. As in the three qubit case, entanglement (which stays in the range $[E_{31}^{(4)}, 1]$) can be concentrated in the bipartitions $(1 : 2, 3, 4)$ and $(2 : 1, 3, 4)$, to the expense of the averaged entropy $\langle E_{vN 2}^{(1;3)} \rangle$, which takes the maximal value $E_{31}^{(4)}$ in $\delta_{14} + \delta_{23} = \pm p\pi$.

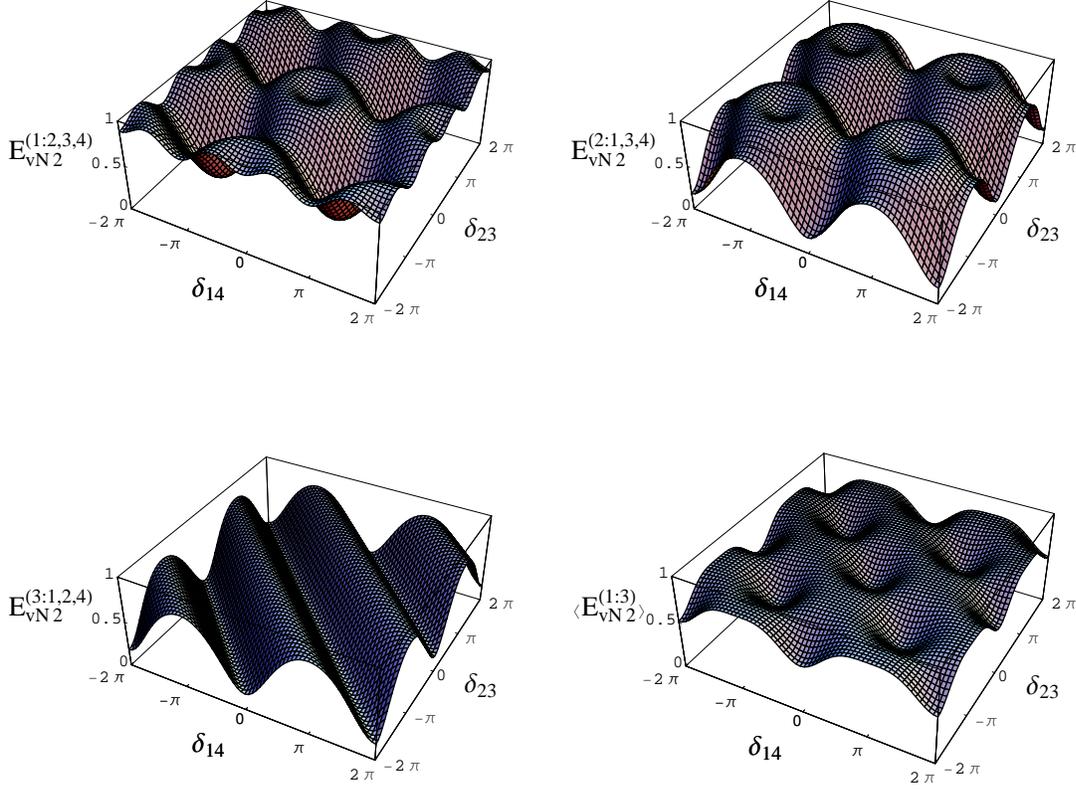


Figure 4.4: I-IV Plots of the entropies $E_{vN2}^{(i:j,k,l)}$ and of their averages $\langle E_{vN2}^{(1:3)} \rangle$ as functions of the phases δ_{14} and δ_{23} . The plot of $E_{vN2}^{(4:1,2,3)}$ was omitted, as this quantity assumes the constant value $E_{31}^{(4)}$.

This effect is best seen in Fig.4.5 where we display sections of the surfaces of Fig. 4.4 which belong to the plane $\delta_{14} = \delta_{23}$. In the range $[-\pi, \pi]$, both $E_{vN2}^{(1:2,3,4)}$ (dotted line) and $E_{vN2}^{(2:1,3,4)}$ (dashed line) can be higher than the value $E_{31}^{(4)}$, and they take their maximal value 1 in the points $\delta_a = \pm \arccos \left[\frac{3}{2}(\sqrt{2}-1) \right]$ and $\delta_b = \pm \arccos \left[-\frac{3}{2}(\sqrt{2}-1) \right]$ respectively. As in the three qubit case, the average entropy, which displays an oscillating behavior, stays under the value $E_{31}^{(4)}$, which is attained in $\delta = \pm \frac{\pi}{2}$.

The states $|W_3^{(4)}(\tilde{\delta})\rangle$ and $|W_4^{(4)}(\tilde{\delta})\rangle$ depends nontrivially on all the phases δ_{ij} , so they show a richer entanglement structure. In both cases we find effects similar to the ones just studied in the case of $|W_2^{(4)}(\tilde{\delta})\rangle$.

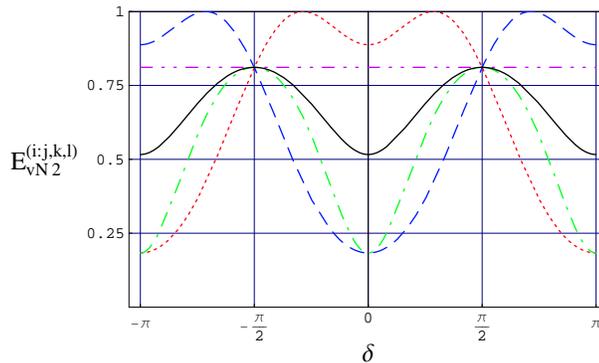


Figure 4.5: Plots of the entropies $E_{vN 2}^{(1:2,3,4)}$ (dotted line), $E_{vN 2}^{(2:1,3,4)}$ (dashed line), $E_{vN 2}^{(3:1,2,4)}$ (mixed line with one dot), $E_{vN 2}^{(4:1,2,3)}$ (mixed line with two dots), and their averages $\langle E_{vN 2}^{(1:3)} \rangle$ (continuous line) as functions of the phase $\delta \equiv \delta_{14} = \delta_{23}$.

4.6 $W^{(N)}$ state entanglement with random phases

In the preceding section we have showed how the presence of the phases in generalized W states can lead to an increase of the entanglement of a given bipartition in correspondence of some value of the phases.

Let us now study the effect of a statistical distribution of phases on the entanglement of such states. This is useful for two reasons. It gives a “global” picture of the generalized W states as a function of the states, and it allows to estimate the robustness of the entanglement increase against variations of the control parameters which are the phases. We will consider the cases of a uniform distribution and of a Gaussian distribution.

In Fig. 4.6 we show the histograms of the state $|W_2^{(3)}(\delta)\rangle$. In Fig. 4.7 and 4.8 we show the histograms of the state $|W_4^{(4)}(\tilde{\delta})\rangle$ respectively for the balanced and unbalanced bipartition. In Fig. 4.9, we consider the entropy of the state $|W_4^{(4)}(\tilde{\delta})\rangle$ (for unbalanced bipartitions) in the situation in which the distributions of the phases $\delta_{14}, \delta_{23}, \delta_{34}$ are Gaussian.

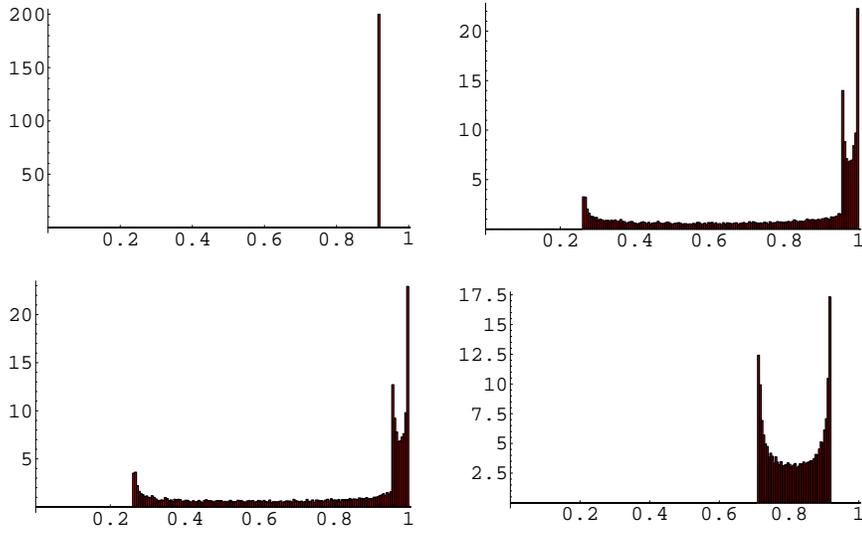


Figure 4.6: I-IV Histograms of $E_{vN2}^{(i,j:k)}$ and $\langle E_{vN2}^{(2:1)} \rangle$ for δ randomly distributed $[0, \pi]$.

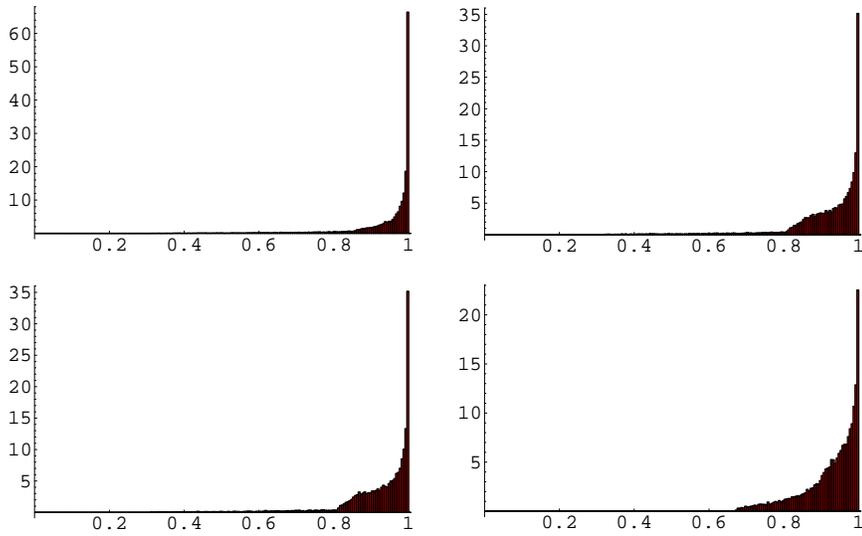


Figure 4.7: I-IV Histograms of $E_{vN4}^{(i,j:k,l)}$ and $\langle E_{vN4}^{(2:2)} \rangle$ for randomly distributed phases in the interval $[0, \pi]$.

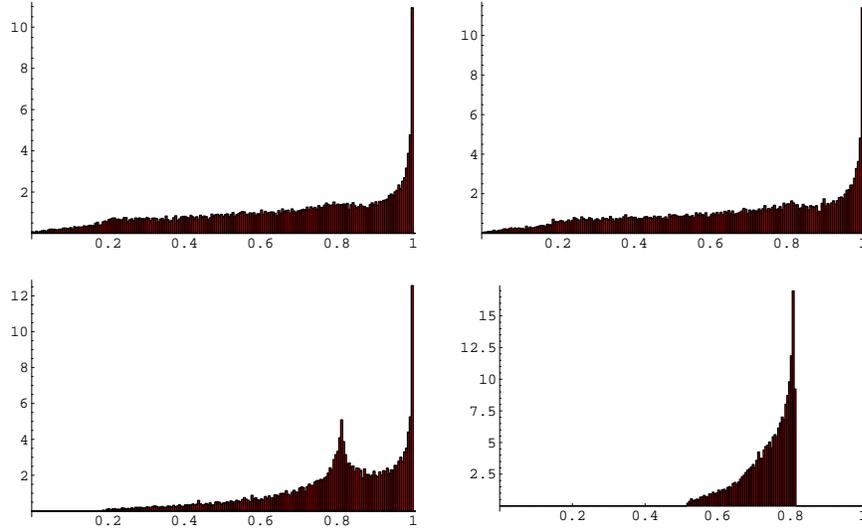


Figure 4.8: I-IV Histograms of $E_{vN4}^{(i;j,k,l)} e \langle E_{vN4}^{(1;3)} \rangle$ for randomly distributed phases in the interval $[0, \pi]$.

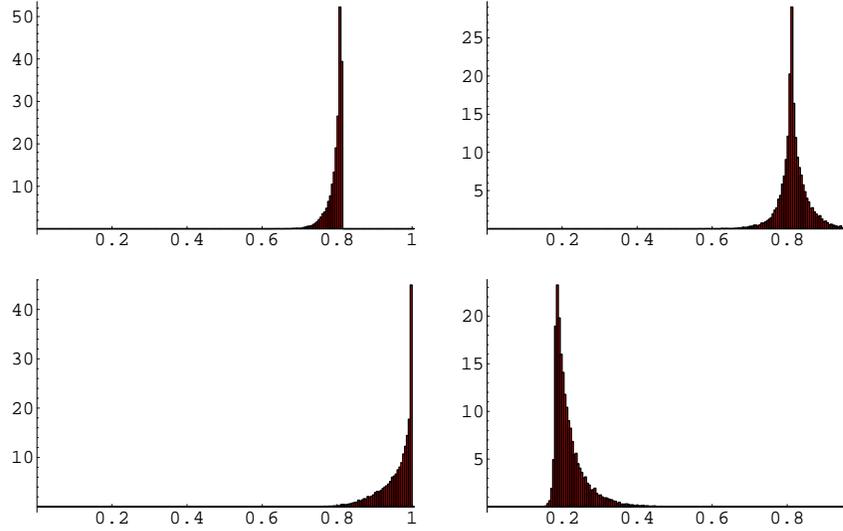


Figure 4.9: I-IV Histograms of $E_{vN4}^{(i;j,k,l)}$ and $\langle E_{vN4}^{(1;3)} \rangle$ for a Gaussian distribution of phases $\delta_{14}, \delta_{23}, \delta_{34}$ with mean values $\bar{\delta}_{ij}$ and variances σ_{ij} . I: $\langle E_{vN4}^{(1;3)} \rangle$ for $\bar{\delta}_{ij} = \frac{\pi}{2}$ and $\sigma_{ij} = 0.3$. II: $E_{vN4}^{(3;1,2,4)}$ for $\bar{\delta}_{ij} = \frac{\pi}{2}$ and $\sigma_{ij} = 0.3$. III: $E_{vN4}^{(3;1,2,4)}$ for $\bar{\delta}_{ij} = \arccos \left[\frac{3}{2}(\sqrt{2} - 1) \right]$ and $\sigma_{ij} = 0.3$. IV: $E_{vN4}^{(1;2,3,4)}$ for $\bar{\delta}_{ij} = 0$ and $\sigma_{ij} = 0.2$.

4.7 Quantification of entanglement in quark and neutrino mixing

In this section we are going to quantify the amount of entanglement that is present in quark and neutrino mixing which is described by the flavor states defined by Eq.(4.45). In the case of neutrinos, for the parameters we will take the experimental values as given in [121, 118, 107, 66]. In the case of quarks the mixing angles of the CKM matrix are given by [118]:

$$\theta_{12}^{CKM} = 13.0^\circ \pm 0.1^\circ, \quad \theta_{13}^{CKM} = 0.2^\circ \pm 0.1^\circ, \quad \theta_{23}^{CKM} = 2.4^\circ \pm 0.1^\circ. \quad (4.65)$$

A CP violation measure has given the following value for the phase [121]

$$\delta^{CKM} = 1.05 \pm 0.24. \quad (4.66)$$

In table 4.1 we list the von Neumann entropies $E_{vN\alpha}^{(i,j;k)}$, with $\alpha = d', s', b'$ and $i, j, k = d, s, b$, and $\langle E_{vN\alpha}^{(2:1)} \rangle$ corresponding to the states (4.45), with the mixing angles and the phase fixed to 4.65) and (4.66) respectively, without taking into account the experimental errors.

| α | $E_{vN\alpha}^{(d,s;b)}$ | $E_{vN\alpha}^{(d,b;s)}$ | $E_{vN\alpha}^{(s,b;d)}$ | $\langle E_{vN\alpha}^{(2:1)} \rangle$ |
|----------|--------------------------|--------------------------|--------------------------|--|
| d' | 0.0002 | 0.2889 | 0.2890 | 0.1927 |
| s' | 0.0185 | 0.2960 | 0.2887 | 0.2011 |
| b' | 0.0186 | 0.0180 | 0.0010 | 0.0126 |

Table 4.1: Von Neumann entropies $E_{vN\alpha}^{(i,j;k)}$ and $\langle E_{vN\alpha}^{(2:1)} \rangle$ ($\alpha = d', s', b'$) for the quark mixing states.

Entanglement stays low. It concentrates on the bipartitions $(d, b; s)$ and $(s, b; d)$ of the states $|d'\rangle$ and $|s'\rangle$, while it is very small for the state $|b'\rangle$.

In the case of neutrinos, recent estimates for the PMNS matrix parameters are expressed by [66]:

$$\sin^2 \theta_{12}^{MNSP} = 0.314(1 \begin{smallmatrix} +0.18 \\ -0.15 \end{smallmatrix}), \quad \sin^2 \theta_{13}^{MNSP} = (0.8 \begin{smallmatrix} +2.3 \\ -0.8 \end{smallmatrix}) \times 10^{-2}, \quad (4.67)$$

$$\sin^2 \theta_{23}^{MNSP} = 0.45(1 \begin{smallmatrix} +0.35 \\ -0.20 \end{smallmatrix}). \quad (4.68)$$

The lepton mixing phase δ^{MNSP} is still undetermined, so it could in principle assume any value in the interval $[0, 2\pi)$. In table 4.2, using (4.68) (without considering experimental errors) and for arbitrary δ^{MNSP} , we give the entropies corresponding to the neutrino flavor states. The presence of intervals is due to the possibility of choosing the phase.

By comparing 4.1 and 4.2, we observe that in neutrino mixing there is more entanglement and a more homogeneous distribution with respect to the quark case. On the other hand, in the neutrino case there are big uncertainties. Moreover, the value of the mixing angle θ_{13}^{MNSP}

| α | $E_{vN\alpha}^{(1,2;3)}$ | $E_{vN\alpha}^{(1,3;2)}$ | $E_{vN\alpha}^{(2,3;1)}$ | $\langle E_{vN\alpha}^{(2;1)} \rangle$ |
|----------|--------------------------|--------------------------|--------------------------|--|
| e | 0.0672 | 0.8948 | 0.9038 | 0.5995 |
| μ | 0.9916 | 0.9220 – 0.9813 | 0.5679 – 0.7536 | 0.8469 – 0.8891 |
| τ | 0.9939 | 0.8397 – 0.9352 | 0.4784 – 0.6922 | 0.8025 – 0.8419 |

Table 4.2: Von Neumann entropies $E_{vN\alpha}^{(i,j;k)}$ and $\langle E_{vN\alpha}^{(2;1)} \rangle$ ($\alpha = e, \mu, \tau$) for the states associated to neutrino mixing.

is crucial, because the entropies depend on the phase only if this angle does not vanish. It is thus interesting to study the behavior of entanglement by taking into account the experimental uncertainties in the mixing angles. To do this we assume that the angles θ_{ij}^{MNSP} assume Gaussianly distributed random values, and as mean values we take the experimentally measured values. For example, in Fig.4.10 we plot $E_{vN\mu}^{(i,j;k)}$ and $\langle E_{vN\mu}^{(2;1)} \rangle$ as functions of the free parameter $\delta^{MNSP} \equiv \delta$.

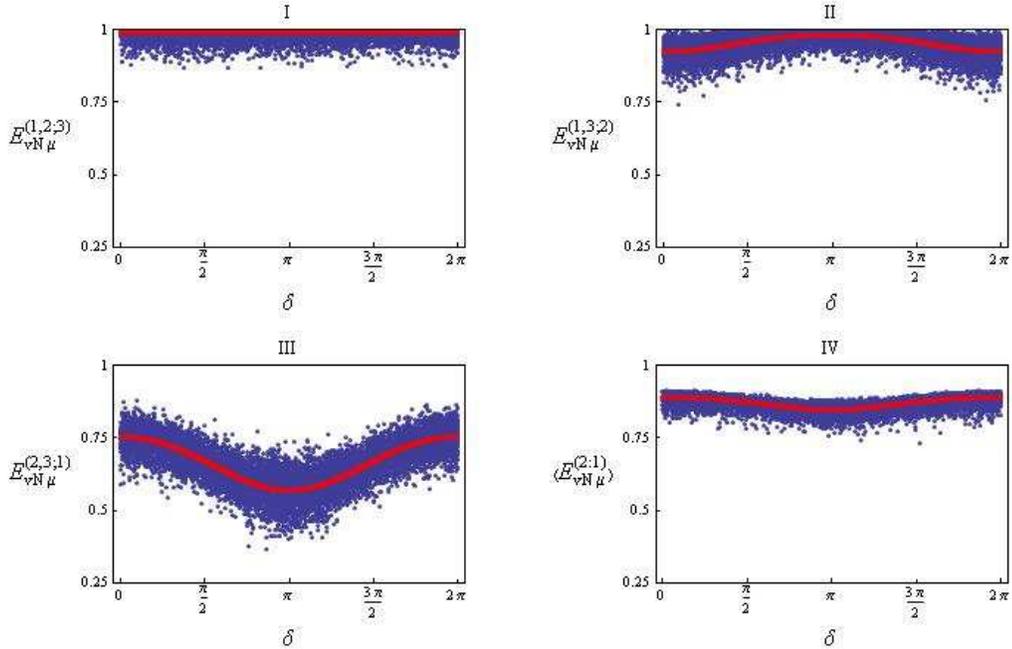


Figure 4.10: I-IV Plots of $E_{vN\mu}^{(i,j;k)}$ and $\langle E_{vN\mu}^{(2;1)} \rangle$ as functions of δ . θ_{ij}^{MNSP} are random variables distributed according to Gaussian distributions centered around the mean values $\bar{\theta}_{ij}^{MNSP}$, given by the experimental values (4.68), with standard deviation $\sigma_{ij} = \frac{\delta\theta_{ij}^{MNSP}}{3}$. The uncertainties $\delta\theta_{ij}^{MNSP}$ are fixed to the maximum between the left and right interval given in Eq. (4.68). Continuous thick lines represent the entropies with $\theta_{ij}^{MNSP} = \bar{\theta}_{ij}^{MNSP}$ and zero uncertainty.

We notice that the entanglement corresponding to the bipartitions (1, 2; 3) and (1, 3; 2) stays high, (I and II); on the other hand, the bipartition (2, 3; 1) has lower entanglement (III), so that

the average global entanglement decreases (IV). We conclude that in the μ system the parts 2 and 3 are more strongly correlated than the couples 1, 2 and 1, 3. A similar analysis can be performed for the systems e and τ .

4.8 Decoherence in neutrino oscillations

The above results have been obtained in the plane wave approximation. In this approximation, these results hold for any given time. However, as we have discussed in chapter 1, a more realistic description of the phenomenon can be achieved using the wave packet approach outlined there. As we have discussed, the fact that the three massive neutrinos have different masses leads to different propagation speeds of the packets, so the coherent interference effects which lead to the oscillations decrease in time. This will lead in turn to the disappearance of the oscillation phenomenon and to the vanishing of multipartite entanglement. In this section we are going to study this situation.

Let us consider for simplicity only one spatial dimension. A neutrino with definite flavor, propagating along the x direction, is described by the state:

$$|\nu_\alpha(x, t)\rangle = \sum_j U_{\alpha,j} \psi_j(x, t) |\nu_j\rangle, \quad (4.69)$$

where the $U_{\alpha,j}$ is the corresponding element of the mixing matrix, $|\nu_j\rangle$ is the mass eigenstate with mass m_j , and $\psi_j(x, t)$ is its wave function. Assuming that the momentum of the massive neutrino $|\nu_j\rangle$ has a Gaussian distribution $\psi_j(p)$, the wave function is given by:

$$\psi_j(x, t) = \frac{1}{\sqrt{2\pi}} \int dp \psi_j(p) e^{ipx - iE_j(p)t}, \quad (4.70)$$

$$\psi_j(p) = \frac{1}{(2\pi\sigma_p^2)^{1/4}} e^{-(1/(4\sigma_p^2))(p-p_j)^2}, \quad (4.71)$$

where p_j is the average momentum, σ_p is the momentum uncertainty, and $E_j(p) = \sqrt{p^2 + m_j^2}$. The density matrix associated with the pure state Eq. (4.69) writes:

$$\rho_\alpha(x, t) = |\nu_\alpha(x, t)\rangle \langle \nu_\alpha(x, t)|. \quad (4.72)$$

If the inequality $\sigma_p \ll E_j^2(p_j)/m_j$ holds, the energy $E_j(p)$ can be approximated by $E_j(p) \simeq E_j + v_j(p - p_j)$, with $E_j \equiv \sqrt{p_j^2 + m_j^2}$, and $v_j \equiv \frac{\partial E_j(p)}{\partial p} \Big|_{p=p_j} = \frac{p_j}{E_j}$ is the group velocity of the wave packet of the massive neutrino $|\nu_j\rangle$. In this case, the integration over p in Eq. (4.70) is Gaussian and can be easily performed, yielding the following expression for $\rho_\alpha(x, t)$

$$\rho_\alpha(x, t) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \sum_{j,k} U_{\alpha j} U_{\alpha k}^* e^{-i(E_j - E_k)t + i(p_j - p_k)x - \frac{1}{4\sigma_x^2}[(x - v_j t)^2 + (x - v_k t)^2]} |\nu_j\rangle \langle \nu_k|, \quad (4.73)$$

where $\sigma_x = (2\sigma_p)^{-1}$. In the instance of extremely relativistic neutrinos, the following approximations are usually assumed

$$E_j \simeq E + \xi \frac{m_j^2}{2E}, \quad p_j \simeq E - (1 - \xi) \frac{m_j^2}{2E}, \quad v_j \simeq 1 - \frac{m_j^2}{2E^2} \quad (4.74)$$

where E is the neutrino energy in the limit of zero mass, and ξ is a dimensionless constant depending on the characteristic of the production process [72, 73]. The density matrix (4.73) provides a space-time description of neutrino dynamics. However, in realistic situations, it is convenient to consider the corresponding stationary process, which is associated with the time-independent density matrix $\rho_\alpha(x)$ obtained by the time average of $\rho_\alpha(x, t)$ [73]. By taking into account Eq. (4.74), and by computing a Gaussian integration over the time, the density matrix becomes [73]

$$\rho_\alpha(x) = \sum_{j,k} U_{\alpha j} U_{\alpha k}^* \exp \left[-i \frac{\Delta m_{jk}^2 x}{2E} - \left(\frac{\Delta m_{jk}^2 x}{4\sqrt{2}E^2\sigma_x} \right)^2 - \left(\xi \frac{\Delta m_{jk}^2}{4\sqrt{2}E\sigma_p} \right)^2 \right] |\nu_j\rangle \langle \nu_k|, \quad (4.75)$$

with $\Delta m_{jk}^2 = m_j^2 - m_k^2$. The density matrix (4.75) can be used to study, in the wave packet approach, the phenomenon of neutrino oscillations for stationary neutrino beams [72, 73, 71]. Here, we intend to analyze the coherence of the quantum superposition of the neutrino mass eigenstates, by looking at the spatial behavior of the multipartite entanglement of the state (4.75). By establishing the identification $|\nu_i\rangle = |\delta_{i,1}\rangle_1 |\delta_{i,2}\rangle_2 |\delta_{i,3}\rangle_3 \equiv |\delta_{i,1}\delta_{i,2}\delta_{i,3}\rangle$ ($i = 1, 2, 3$), we can easily construct from Eq. (4.75) the matrix with elements $\langle lmn | \rho_\alpha(x) | ijk \rangle$, where $i, j, k, l, m, n = 0, 1$. Let us notice that the density matrix $\rho_\alpha(x)$ describes a mixed state, whose non-diagonal elements are suppressed by a Gaussian function of x . An appropriate quantifier of multipartite entanglement for the state $\rho_\alpha(x)$ is based on the set of logarithmic negativities defined above. We analytically compute the quantities $E_{\mathcal{N}_\alpha}^{(i,j;k)}$, for $i, j, k = 1, 2, 3$ and $i \neq j \neq k$, and the average logarithmic negativity $\langle E_{\mathcal{N}_\alpha}^{(2:1)} \rangle$, for the neutrino states with flavor $\alpha = e, \mu, \tau$. We assume for the mixing angles θ_{ij}^{MNSP} the experimental values (4.68). The squared mass differences are fixed at the experimental values reported in Ref. [66]:

$$\begin{aligned} \Delta m_{21}^2 &= \delta m^2, & \Delta m_{31}^2 &= \Delta m^2 + \frac{\delta m^2}{2}, & \Delta m_{32}^2 &= \Delta m^2 - \frac{\delta m^2}{2}, \\ \delta m^2 &= 7.92 \times 10^{-5} eV^2, & \Delta m^2 &= 2.6 \times 10^{-3} eV^2. \end{aligned} \quad (4.76)$$

The parameters E and σ_p in Eq. (4.75) are fixed at the values $E = 10 GeV$ and $\sigma_p = 1 GeV$. Moreover, although depending on the particular production process [69], the parameter ξ is put to zero for simplicity. In Fig. 4.11, we plot the logarithmic negativities for the electronic neutrino, i.e. $E_{\mathcal{N}_e}^{(i,j;k)}$ as function of the distance x . The bipartitions (1, 3; 2) and (2, 3; 1), see panel I, exhibit a high entanglement content (> 0.93) that keeps almost constant for $x \lesssim 10^8 m$; finally, it goes to zero for $x \approx 3 \times 10^9 m$. The bipartition (1, 2; 3) exhibits a low entanglement (< 0.24), that goes to zero for $x \approx 9 \times 10^7 m$. Furthermore, let us remark that the logarithmic negativities $E_{\mathcal{N}_e}^{(i,j;k)}$ and $\langle E_{\mathcal{N}_e}^{(2:1)} \rangle$ for the electronic neutrino are independent of the CP-violating phase δ .

In the muonic and tauonic instances, the independence from the CP-violating phase δ holds no more. Therefore, first we choose to study the quantum correlations of these states for $\delta = 0$; then we consider separately the influence of a non-zero δ . In Fig. 4.12, we plot the logarithmic negativities for the muonic and tauonic neutrinos as functions of the distance x with $\delta = 0$. We see that the spatial behavior of multipartite entanglement for muonic and tauonic neutrinos are similar. The logarithmic negativities $E_{\mathcal{N}_\mu}^{(1,2;3)}$ and $E_{\mathcal{N}_\tau}^{(1,2;3)}$ are initially close to 1, and they go to

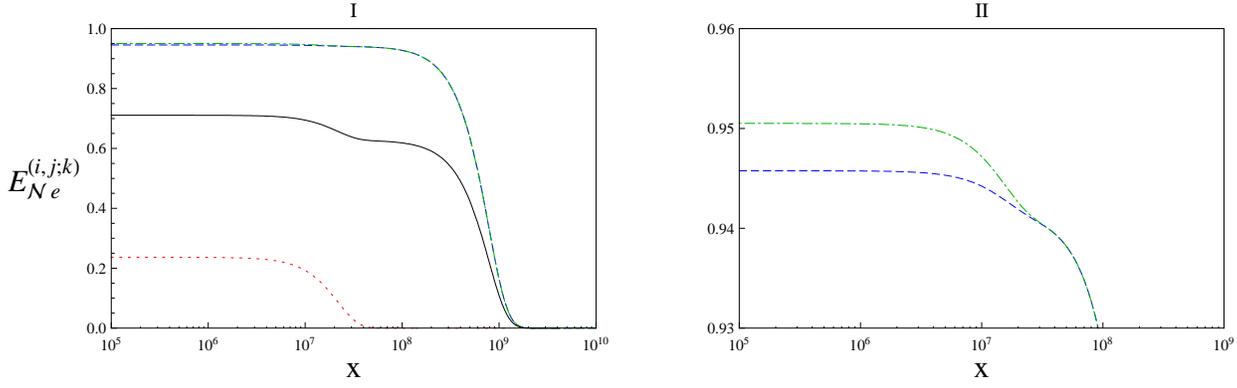


Figure 4.11: (Color online) The logarithmic negativities $E_{\mathcal{N}_e}^{(i,j;k)}$ for all possible bipartitions and the average logarithmic negativity $\langle E_{\mathcal{N}_e}^{(2;1)} \rangle$ as functions of the distance x . The quantities $E_{\mathcal{N}_e}^{(1,3;2)}$ (dashed line) and $E_{\mathcal{N}_e}^{(2,3;1)}$ (dot-dashed line), see panel I, show a high amount of entanglement content in the corresponding bipartitions, and seem to be superimposed. In panel II we plot a zoom of $E_{\mathcal{N}_e}^{(1,3;2)}$ and $E_{\mathcal{N}_e}^{(2,3;1)}$ to observe the differences in their behaviors: the two curves are initially separated, and then they superimpose each other. The bipartition $(1, 2; 3)$, associated with the quantity $E_{\mathcal{N}_e}^{(1,2;3)}$ (dotted line), exhibits the lowest amount of entanglement. The full line corresponds to the average logarithmic negativity $\langle E_{\mathcal{N}_e}^{(2;1)} \rangle$. The mixing angles θ_{ij}^{MNSP} and the squared mass differences Δm_{ij}^2 are fixed at the experimental values (4.68) and (4.76), respectively. We assume the values $E = 10 \text{ GeV}$, $\sigma_p = 1 \text{ GeV}$, and $\xi = 0$ for the remaining parameters in Eq. (4.75). All the plotted quantities are independent of the CP-violating phase δ , that can be assumed arbitrary. The x axis is in logarithmic scale, and the dimensions are meters.

zero for $x \approx 10^8 m$. On the other side, $E_{\mathcal{N}_\mu}^{(1,3;2)}$, $E_{\mathcal{N}_\mu}^{(2,3;1)}$, $E_{\mathcal{N}_\tau}^{(1,3;2)}$, and $E_{\mathcal{N}_\tau}^{(2,3;1)}$ exhibit alternating regimes with slowly decreasing slope and with rapidly decreasing slope; moreover, all vanish for $x \approx 3 \times 10^9 m$.

The average logarithmic negativity $\langle E_{\mathcal{N}_\alpha}^{(2;1)} \rangle$ can be used to define a *decoherence length* L_{decoh} as

$$L_{decoh} : \langle E_{\mathcal{N}_\alpha}^{(2;1)} \rangle(L_{decoh}) = 0. \quad (4.77)$$

From Figs. 4.11, 4.12, for assigned experimental parameters, we see that the common decoherence length for the neutrinos of flavor $\alpha = e, \mu, \tau$ can be estimated at a value of $L_{decoh} \approx 3 \times 10^6 Km$.

Finally, we consider the influence of a non-vanishing phase δ in determining the spatial behavior of multipartite entanglement of stationary neutrino beams. To this aim, in Fig. 4.13 we plot the logarithmic negativities for the muonic neutrino $E_{\mathcal{N}_\mu}^{(1,3;2)}$ and $E_{\mathcal{N}_\mu}^{(2,3;1)}$, with δ fixed at the values $\delta = 0, \frac{\pi}{2}, \pi$. The behavior of $E_{\mathcal{N}_\mu}^{(1,2;3)}$ is not reported as it is independent of δ . We observe that the CP-violating phase δ does not lead to a change of the decoherence length L_{decoh} . However, we see that it may lead a lowering or an increasing of the amount of entanglement in a given bipartition, in agreement with the results obtained for the instance of static neutrinos.

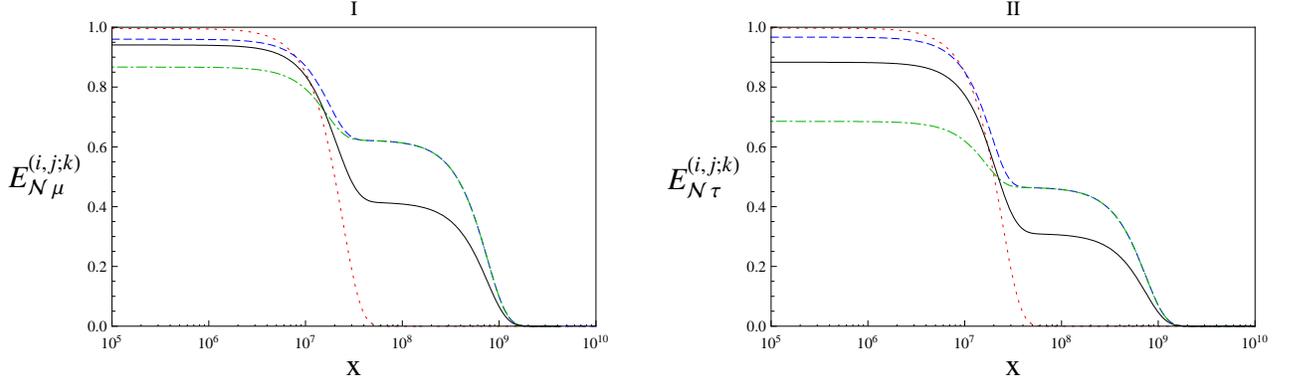


Figure 4.12: (Color online) The logarithmic negativities $E_{\mathcal{N}\alpha}^{(i,j;k)}$ for all possible bipartitions and the average logarithmic negativity $\langle E_{\mathcal{N}\alpha}^{(2:1)} \rangle$, with $\alpha = \mu, \tau$, as functions of the distance x . In panel I we plot the negativities for the muonic neutrino. The bipartition $(1, 2; 3)$, associated with the quantity $E_{\mathcal{N}\mu}^{(1,2;3)}$ (dotted line), shows the highest initial amount of entanglement, that goes to zero for a lower of x with respect to the other bipartitions. $E_{\mathcal{N}\mu}^{(1,3;2)}$ (dashed line) and $E_{\mathcal{N}\mu}^{(2,3;1)}$ (dot-dashed line) show peculiar behaviors, that consist in alternating slowly decreasing and rapidly decreasing slopes. The average logarithmic negativity $\langle E_{\mathcal{N}\mu}^{(2:1)} \rangle$ (full line) summarizes the behavior of the global entanglement. In panel II we plot the negativities for the tauonic neutrino; The behaviors of the negativities for the tauonic instance are similar to the negativities for the muonic instance. The curves associated to a given bipartition are plotted with the same plotstyle. The mixing angles θ_{ij}^{MNSP} and the squared mass differences Δm_{ij}^2 are fixed at the experimental values (4.68) and (4.76), respectively. We assume the values $E = 10 \text{ GeV}$, $\sigma_p = 1 \text{ GeV}$, and $\xi = 0$ for the remaining parameters in Eq. (4.75). The CP-violating phase δ is put to zero. The x axis is in logarithmic scale, and the dimensions are meters.

Similar results can be obtained for the tauonic instance.

4.9 Dynamic entanglement in the three flavor case

Let us write again the flavor states (4.45):

$$|\underline{\nu}_f\rangle = U(\tilde{\theta}, \delta) |\underline{\nu}_m\rangle \quad (4.78)$$

$$U(\tilde{\theta}, \delta) = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}, \quad (4.79)$$

Their time evolution is given by:

$$|\underline{\nu}^{(f)}(t)\rangle = \mathbf{U}(\tilde{\theta}, \delta) \mathbf{U}_0(t) \mathbf{U}(\tilde{\theta}, \delta)^{-1} |\underline{\nu}^{(f)}\rangle \equiv \tilde{\mathbf{U}}(t) |\underline{\nu}^{(f)}\rangle, \quad (4.80)$$

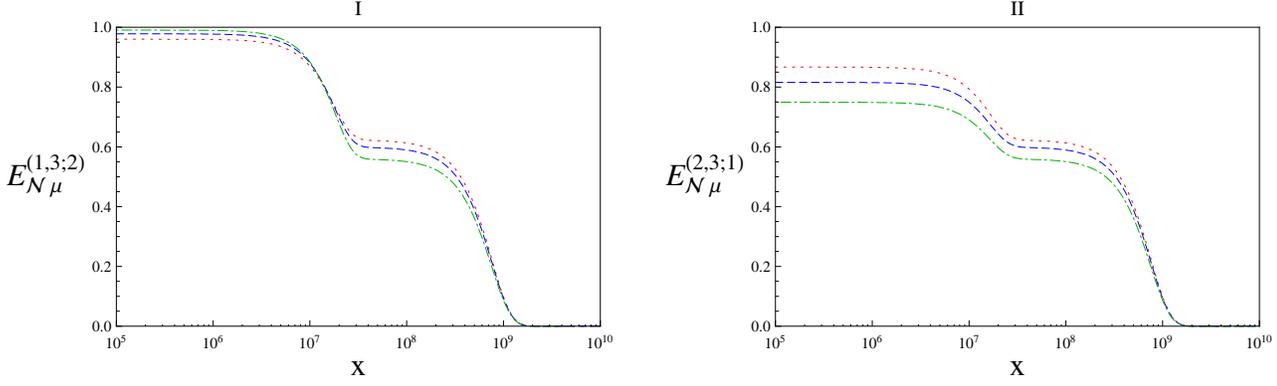


Figure 4.13: (Color online) The logarithmic negativities $E_{N\mu}^{(1,3;2)}$ (panel I) and $E_{N\mu}^{(2,3;1)}$ (panel II) as functions of the distance x for different choices of the CP-violating phase δ : (a) $\delta = 0$ (dotted line); (b) $\delta = \frac{\pi}{2}$ (dashed line); (c) $\delta = \pi$ (dot-dashed line). $E_{N\mu}^{(1,2;3)}$ is independent of δ . The mixing angles θ_{ij}^{MNSP} , the squared mass differences Δm_{ij}^2 , the parameters E , σ_p , and ξ are fixed as in Figs. 4.11 and 4.12. The x axis is in logarithmic scale, and the dimensions are meters.

where $|\underline{\nu}^{(f)}\rangle$ are the flavor states at $t = 0$, $\mathbf{U}_0(t) = \text{diag}(e^{-iE_1t}, e^{-iE_2t}, e^{-iE_3t})$, and $\tilde{\mathbf{U}}(t) = \mathbf{U}(\tilde{\theta}, \delta) \mathbf{U}_0(t) \mathbf{U}(\tilde{\theta}, \delta)^{-1}$, con $\tilde{\mathbf{U}}(t = 0) = \mathbb{I}$. The transition probability relative to $\nu_\alpha \rightarrow \nu_\beta$ is given by:

$$P_{\nu_\alpha \rightarrow \nu_\beta}(t) = |\langle \nu_\beta | \nu_\alpha(t) \rangle|^2 = |\tilde{\mathbf{U}}_{\alpha\beta}(t)|^2. \quad (4.81)$$

In the notation $\tilde{\mathbf{U}}_{\alpha\beta}(t)$ the index $\alpha = e, \mu, \tau$ refers to the time evolution of the initial state with flavor α , while $\beta = e, \mu, \tau$ denotes the flavor at time t . Eq. (4.80) can be rewritten in the W -like form:

$$\begin{aligned} |\nu_\alpha(t)\rangle &= \tilde{\mathbf{U}}_{\alpha e}(t) |1\rangle_e |0\rangle_\mu |0\rangle_\tau + \tilde{\mathbf{U}}_{\alpha \mu}(t) |0\rangle_e |1\rangle_\mu |0\rangle_\tau \\ &+ \tilde{\mathbf{U}}_{\alpha \tau}(t) |0\rangle_e |0\rangle_\mu |1\rangle_\tau, \quad \alpha = e, \mu, \tau, \end{aligned} \quad (4.82)$$

where the normalization condition $\sum_\beta |\tilde{\mathbf{U}}_{\alpha\beta}(t)|^2 = 1$ ($\alpha = e, \mu, \tau$) is automatically satisfied. This means that the time evolution $|\underline{\nu}^{(f)}(t)\rangle$ can be seen as an entangled superposition of the flavor eigenstates with time dependent coefficients.

By tracing for example on the τ mode we obtain:

$$S_{L\alpha}^{(e,\mu;\tau)} = 4|\tilde{\mathbf{U}}_{\alpha\tau}(t)|^2 (|\tilde{\mathbf{U}}_{\alpha e}(t)|^2 + |\tilde{\mathbf{U}}_{\alpha\mu}(t)|^2) = 4|\tilde{\mathbf{U}}_{\alpha\tau}(t)|^2 (1 - |\tilde{\mathbf{U}}_{\alpha\tau}(t)|^2). \quad (4.83)$$

The linear entropies corresponding to the other bipartitions are easily obtained by permuting e, μ, τ . Then the average linear entropy for the state (4.82) is:

$$\langle S_{L\alpha}^{(2;1)} \rangle = \frac{8}{3} (|\tilde{\mathbf{U}}_{\alpha e}(t)|^2 |\tilde{\mathbf{U}}_{\alpha\mu}(t)|^2 + |\tilde{\mathbf{U}}_{\alpha e}(t)|^2 |\tilde{\mathbf{U}}_{\alpha\tau}(t)|^2 + |\tilde{\mathbf{U}}_{\alpha\mu}(t)|^2 |\tilde{\mathbf{U}}_{\alpha\tau}(t)|^2). \quad (4.84)$$

It is easy to generalize the relations (4.83) and (4.84) for the general N flavor case.

4.10 Dynamic symmetry approach to quantum entanglement

In this section we briefly describe a characterization of quantum entanglement which is particularly appealing since it has a very clear physical interpretation and it seems to be amenable of some generalization which could be used to study some quantum field theoretical situation. This approach was pioneered by A. Klyachko [93, 94]. According to this approach, quantum entanglement measures the magnitude of quantum fluctuations of certain basic observables, thus matching the intuitive picture that entangled states are the ones that exhibit the most non-classical behavior. As has been shown by Klyachko, this measure of entanglement is applicable to pure states of arbitrary systems and reduces to other known measures such as the concurrence [155] in the cases in which these are applicable as well.

The starting point is the realization by Wick, Wightman and Wigner [150] that put in question the idealized von Neumann approach to Quantum Mechanics, according to which any Hermitian operator represents a measurable quantity. They introduced the so-called superselection rules which generally express "restrictions on the nature and scope of possible measurements" (of which an example is the already mentioned Bargmann superselection rule). Later Robert Hermann [83] argued that the principles of Quantum Mechanics require that measurable observables should form a Lie algebra \mathcal{L} of (skew-)Hermitian operators acting on the Hilbert space H of the quantum system. This Lie algebra is known as the Lie algebra of observables and its exponentiation $G = \exp(i\mathcal{L})$ is known as the dynamical symmetry group of the quantum system. The algebra of observables is determined by the available measurements, for example in the case of a multipartite system with spatially separated components only local measurements are possible. Phenomena like entanglement are precisely due to the fact that there are physical systems whose dynamical group does not act transitively on the Hilbert space of quantum states.

Let us consider a quantum system with Hilbert space H and Lie algebra of observables \mathcal{L} . Given the observable $X \in \mathcal{L}$ and the quantum state $|\psi\rangle \in H$, the quantum uncertainty of X in the state $|\psi\rangle$ is given by the variance:

$$V(X, \psi) = \langle \psi | X^2 | \psi \rangle - \langle \psi | X | \psi \rangle^2. \quad (4.85)$$

It is possible to choose a basis in the Lie algebra L which is orthonormal with respect to its Cartan-Killing form $(X, Y)_K$ and define the total variance of the quantum state $|\psi\rangle$ by

$$\mathbb{V}(\psi) = \sum_{\alpha} (\langle \psi | X_{\alpha}^2 | \psi \rangle - \langle \psi | X_{\alpha} | \psi \rangle^2) \quad (4.86)$$

which can be shown to be basis-independent. This quantity measures the overall level of the quantum fluctuations of the system in the state $|\psi\rangle$. In the first sum of (4.86) we recognize the Casimir operator $C = \sum_{\alpha} X_{\alpha}^2$ of the Lie algebra, which as well known acts as a scalar $C_{\mathcal{H}}$ in every irreducible representation of the algebra on the vector space \mathcal{H} . We thus get:

$$\mathbb{V}(\psi) = C_{\mathcal{H}} - \sum_{\alpha} \langle \psi | X_{\alpha} | \psi \rangle^2 \quad (4.87)$$

We can write the second sum in a more useful way. Let us consider the quantity

$$X_\psi = \sum_{\alpha} \langle \psi | X_{\alpha} | \psi \rangle X_{\alpha} \quad (4.88)$$

which can be understood as the centre of the quantum fluctuations of the system in the state ψ . For example to a spin where this operator is a suitably scaled projection of the spin onto the mean spin direction in the state $|\psi\rangle$. This operator has the property

$$\langle \psi | X | \psi \rangle = (X, X_\psi)_K, \quad \forall X \in \mathcal{L}. \quad (4.89)$$

which, being the Cartan–Killing form nondegenerate (for semisimple Lie algebras) uniquely determines it, thus showing its basis independence.

From (4.87) it is possible to infer the bound

$$\mathbb{V}(\psi) \leq C_{\mathcal{H}} \quad (4.90)$$

which is saturated if and only if

$$\langle \psi | X | \psi \rangle = 0 \quad \forall X \in \mathcal{L} \quad (4.91)$$

Of particular importance for our purposes is the case of multipartite systems. These are described by a tensor product Hilbert space $\mathcal{H} = \bigotimes_A \mathcal{H}_A$. If we assume that we have full access to the local degrees of freedom the relevant Lie algebra is $\bigoplus_A su(H_A)$. In this case it is possible to show [94] that the total variance takes the form

$$\mathbb{V}(\psi) = \sum_A [\dim H_A - \text{Tr}_{H_A}(\rho_A^2)]. \quad (4.92)$$

where ρ_A are the reduced density matrices associated to the various subsystems. For such systems (4.91) implies that all one party reduced states are completely disordered. In other words, there exists a basis such that the reduced state is given by a diagonal matrix ρ_A corresponding to a uniform probability distribution. This is a well known characterization of entangled states. Thus (4.91) is usually referred to as *entanglement equation* and the corresponding states, which saturate the bound (4.90), are called *completely entangled states*.

Completely entangled states are those for which the total variance is maximal i.e. they are characterized by the fact that quantum fluctuations come to their extreme. On the opposite side there are states which have the minimal total level of quantum fluctuations. These are the coherent states [124]. In the case of multipartite systems these states are factorizable i.e. unentangled.

It has been noticed [93, 94] that in the case of a bipartite system the square of the concurrence [155] coincides with the total variance normalized to the interval $[0, 1]$:

$$C^2(\psi) = \frac{\mathbb{V}(\psi) - \mathbb{V}_{coh}}{\mathbb{V}_{ent} - \mathbb{V}_{coh}} \quad (4.93)$$

where \mathbb{V}_{ent} and \mathbb{V}_{coh} are the total variances in completely entangled and coherent states respectively. Besides clarifying the physical meaning of concurrence, this suggests the following natural measure of entanglement of pure states:

$$\mu(\psi) = \sqrt{\frac{\mathbb{V}(\psi) - \mathbb{V}_{coh}}{\mathbb{V}_{ent} - \mathbb{V}_{coh}}} \quad (4.94)$$

which is valid for pure states of an arbitrary multipartite system.

It is important to remark that the measure of entanglement defined in (4.94) makes sense only for systems whose Hilbert space is finite dimensional. In the applications we will describe in the following section this requirement is met both in the quantum mechanical and in the field theoretical case since we restrict to single particle flavor neutrino states.

4.11 Applications to neutrino mixing in QM and QFT

In this section we will apply the dynamical approach to entanglement to the case of neutrino mixing and oscillations (in the static and dynamic case), both in the case of QM and in the case of QFT. Instead of the measure (4.94) we will use the total variance itself. In the quantum mechanical case this will give a result proportional to the linear entropy, while the results obtained in the latter case are the direct generalization of the quantum mechanical ones.

4.11.1 Quantum Mechanics

In order to apply the formalism outlined in the preceding section to the case of neutrino mixing and oscillations we introduce the (fermionic) annihilation operator α_i for a neutrino with mass m_i , with anticommutators $\{\alpha_i, \alpha_j\} = \delta_{ij}$. We then define neutrino states with definite masses as:

$$|\nu_i\rangle \equiv \alpha_i^\dagger |0\rangle_m, \quad i = 1, 2 \quad (4.95)$$

where $|0\rangle_m \equiv |0\rangle_1 \otimes |0\rangle_2$ is the vacuum for the mass eigenstates.

Next we define the flavor annihilation operators by means of the following mixing relations:

$$\alpha_e(t) = \cos \theta \alpha_1(t) + \sin \theta \alpha_2(t) \quad (4.96)$$

$$\alpha_\mu(t) = -\sin \theta \alpha_2(t) + \cos \theta \alpha_1(t) \quad (4.97)$$

where $\alpha_i(t) = e^{i\omega_i t} \alpha_i$, with $i = 1, 2$.

The flavor states are given by:

$$|\nu_\sigma(t)\rangle \equiv \alpha_\sigma^\dagger(t) |0\rangle_m, \quad \sigma = e, \mu. \quad (4.98)$$

We use in the following the notation $|\nu_\sigma\rangle \equiv |\nu_\sigma(t=0)\rangle$.

The Hamiltonian for the system is given by

$$\begin{aligned} H &= \omega_{ee}\alpha_e^\dagger(t)\alpha_e(t) + \omega_{\mu\mu}\alpha_\mu^\dagger(t)\alpha_\mu(t) + \omega_{e\mu}(\alpha_e^\dagger(t)\alpha_\mu(t) + \alpha_\mu^\dagger(t)\alpha_e(t)) \\ &= \omega_1\alpha_1^\dagger\alpha_1 + \omega_2\alpha_2^\dagger\alpha_2 \end{aligned} \quad (4.99)$$

where we used the relations $\omega_{ee} = \omega_1 \cos^2 \theta + \omega_2 \sin^2 \theta$, $\omega_{\mu\mu} = \omega_1 \sin^2 \theta + \omega_2 \cos^2 \theta$, $\omega_{e\mu} = (\omega_2 - \omega_1) \sin \theta \cos \theta$.

The relevant Lie algebra to use for characterizing static entanglement is the $u(1)_1 \oplus u(1)_2$ algebra generated by the two number operators, since all the other generators have vanishing variance on single particle states. This choice is dictated by the fact that the relevant measurements we can perform here are essentially particle countings. Alternatively we could consider the double Weyl–Heisenberg algebra generated by $\{I_e, \alpha_e, \alpha_e^\dagger, N_e; I_\mu, \alpha_\mu, \alpha_\mu^\dagger, N_\mu\}$. The two choices are equivalent since the variances of all the other generators vanish on the single particle states. The static entanglement of the electron neutrino state $|\nu_e(t)\rangle$ defined in Eq.(4.98), is characterized, in the present formalism, by the variances associated with the numbers N_i , relative to the mass qubits:

$$\begin{aligned} \Delta N_i(\nu_e) &\equiv \langle \nu_e(t) | N_i^2 | \nu_e(t) \rangle - \langle \nu_e(t) | N_i | \nu_e(t) \rangle^2 \\ &= \frac{1}{4} \sin^2(2\theta), \quad i = 1, 2. \end{aligned} \quad (4.100)$$

This result differs by a factor 4 from that obtained by means of the linear entropy, Eqs.(4.40)-(4.41). The total variance is

$$\mathbb{V}(|\nu_e\rangle) = \Delta N_1(\nu_e) + \Delta N_2(\nu_e) = \frac{1}{4} \left(S_L^{(1;2)}(\rho_e) + S_L^{(2;1)}(\rho_e) \right). \quad (4.101)$$

In order to discuss the dynamical entanglement of the state $|\nu_e(t)\rangle$, we need to introduce flavor oscillations, which can be seen either in terms of overlaps of states at different times:

$$P_{\nu_e \rightarrow \nu_e}(t) = |\langle \nu_e | \nu_e(t) \rangle|^2 \quad (4.102)$$

$$P_{\nu_e \rightarrow \nu_\mu}(t) = |\langle \nu_\mu | \nu_e(t) \rangle|^2 \quad (4.103)$$

with $P_{\nu_e \rightarrow \nu_e}(t) + P_{\nu_e \rightarrow \nu_\mu}(t) = 1$, or equivalently in terms of expectation values of number operators at time t :

$$P_{\nu_e \rightarrow \nu_e}(t) = \langle \nu_e | N_e(t) | \nu_e \rangle \quad (4.104)$$

$$P_{\nu_e \rightarrow \nu_\mu}(t) = \langle \nu_e | N_\mu(t) | \nu_e \rangle \quad (4.105)$$

$$N_\sigma(t) = \alpha_\sigma^\dagger(t)\alpha_\sigma(t) \quad \sigma = e, \mu \quad (4.106)$$

The explicit expressions of the transition probabilities are reminded in Eqs.(4.24),(4.25).

In analogy with the static entanglement case, dynamic entanglement can be characterized by using the algebra $u(1)_e \oplus u(1)_\mu$ generated by the two number operators, or equivalently the double Weyl–Heisenberg algebra relative to the flavor ladder operators, whose basis is $\{I_e, \alpha_e, \alpha_e^\dagger, N_e; I_\mu, \alpha_\mu, \alpha_\mu^\dagger, N_\mu\}$, since the variances of all the other generators vanish. Flavor entanglement is given by the variances of the above flavor numbers.

In the electron neutrino case we find

$$\Delta N_e(\nu_e)(t) \equiv \langle \nu_e(t) | N_e^2(t) | \nu_e(t) \rangle - \langle \nu_e(t) | N_e(t) | \nu_e(t) \rangle^2 \quad (4.107)$$

$$= P_{\nu_e \rightarrow \nu_e}(t) (1 - P_{\nu_e \rightarrow \nu_e}(t)) = P_{\nu_e \rightarrow \nu_e}(t) P_{\nu_e \rightarrow \nu_\mu}(t) \quad (4.108)$$

with the same result for $\Delta N_\mu(\nu_e)(t)$. The above result coincides (again up to a factor 4) with the one obtained in Eq.(4.43) by means of the linear entropy.

The total variance is given by:

$$\begin{aligned} \mathbb{V}(|\nu_e(t)\rangle) &= \Delta N_e(\nu_e)(t) + \Delta N_\mu(\nu_e)(t) = 2P_{\nu_e \rightarrow \nu_e}(t) P_{\nu_e \rightarrow \nu_\mu}(t) \\ &= \frac{1}{4} \left(S_L^{(\mu;e)}(\rho_e) + S_L^{(e;\mu)}(\rho_e) \right) \end{aligned} \quad (4.109)$$

Analogous results are easily obtained for the state $|\nu_\mu(t)\rangle$.

4.11.2 Quantum Field Theory

Following what done in the QM case, we now calculate the entanglement associated to an electron neutrino state at time t , by means of the variances of the above discussed charge operators.

Let us start with the $U(1)$ Noether charges Q_{ν_i} , which are expected to characterize the amount of static entanglement present in the states Eq.(2.123). In this case the relevant Lie algebra is $u(1)_1 \oplus u(1)_2$. We obtain:

$$\begin{aligned} \Delta Q_{\nu_i}(\nu_e) &= \langle \nu_{\mathbf{k},e}^r | Q_{\nu_i}^2 | \nu_{\mathbf{k},e}^r \rangle - \langle \nu_{\mathbf{k},e}^r | Q_{\nu_i} | \nu_{\mathbf{k},e}^r \rangle^2 \\ &= \frac{1}{4} \sin^2(2\theta), \quad i = 1, 2. \end{aligned} \quad (4.110)$$

in perfect agreement with the quantum mechanical result Eq.(4.100). The total variance is

$$\mathbb{V}(|\nu_e\rangle) = \Delta Q_{\nu_1}(\nu_e) + \Delta Q_{\nu_2}(\nu_e) = \frac{1}{2} \sin^2(2\theta). \quad (4.111)$$

Next we consider dynamic entanglement, which is described by the variances of the flavor charges defined in section 2.1. We have:

$$\begin{aligned} \Delta Q_{\nu_e}(\nu_e)(t) &= \langle \nu_{\mathbf{k},e}^r | Q_{\nu_e}^2(t) | \nu_{\mathbf{k},e}^r \rangle - \langle \nu_{\mathbf{k},e}^r | Q_{\nu_e}(t) | \nu_{\mathbf{k},e}^r \rangle^2 \\ &= \langle \nu_{\mathbf{k},e}^r | \left[\sum_s \int d^3\mathbf{p} \left(\alpha_{\mathbf{p},e}^{s\dagger}(t) \alpha_{\mathbf{p},e}^s(t) - \beta_{-\mathbf{p},e}^{s\dagger}(t) \beta_{-\mathbf{p},e}^s(t) \right) \right]^2 | \nu_{\mathbf{k},e}^r \rangle - [Q_{\nu_e \rightarrow \nu_e}^{\mathbf{k}}(t)]^2 \\ &= \langle \nu_{\mathbf{k},e}^r | \alpha_{\mathbf{k},e}^{r\dagger}(t) \alpha_{\mathbf{k},e}^r(t) | \nu_{\mathbf{k},e}^r \rangle + \langle \nu_{\mathbf{k},e}^r | \beta_{-\mathbf{k},e}^{r\dagger}(t) \beta_{-\mathbf{k},e}^r(t) | \nu_{\mathbf{k},e}^r \rangle \\ &\quad - 2 \langle \nu_{\mathbf{k},e}^r | \alpha_{\mathbf{k},e}^{r\dagger}(t) \alpha_{\mathbf{k},e}^r(t) \beta_{-\mathbf{k},e}^{r\dagger}(t) \beta_{-\mathbf{k},e}^r(t) | \nu_{\mathbf{k},e}^r \rangle - [Q_{\nu_e \rightarrow \nu_e}^{\mathbf{k}}(t)]^2. \end{aligned} \quad (4.112)$$

We now consider the third term in Eq.(4.112). We have:

$$\langle \nu_{\mathbf{k},e}^r | \alpha_{\mathbf{k},e}^{r\dagger}(t) \alpha_{\mathbf{k},e}^r(t) \beta_{-\mathbf{k},e}^{r\dagger}(t) \beta_{-\mathbf{k},e}^r(t) | \nu_{\mathbf{k},e}^r \rangle = {}_{e,\mu} \langle 0 | \alpha_{\mathbf{k},e}^{r\dagger}(t) \alpha_{\mathbf{k},e}^r(t) \beta_{-\mathbf{k},e}^{r\dagger}(t) \beta_{-\mathbf{k},e}^r(t) | 0 \rangle_{e,\mu}$$

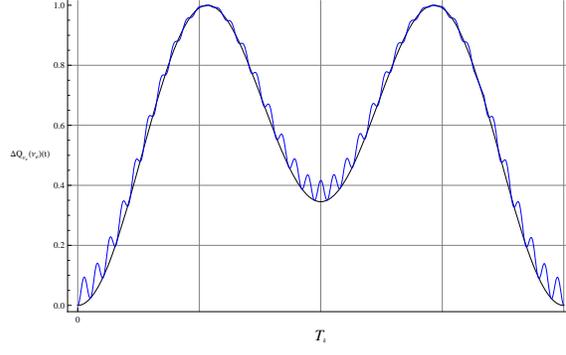


Figure 4.14: QM vs. QFT flavor entanglement for $|\nu_e(t)\rangle$ as a function of the scaled time $T = \frac{2Et}{\Delta m_{12}^2}$ with θ fixed at the value $\sin^2 \theta = 0.314$.

$$\begin{aligned}
& + \left| \left\{ \alpha_{\mathbf{k},e}^r(t), \alpha_{\mathbf{k},e}^{r\dagger}(0) \right\} \right|_{e,\mu}^2 \langle 0 | \beta_{-\mathbf{k},e}^{r\dagger}(t) \beta_{-\mathbf{k},e}^r(t) | 0 \rangle_{e,\mu} \\
& - \left| \left\{ \alpha_{\mathbf{k},e}^r(0), \beta_{-\mathbf{k},e}^r(t) \right\} \right|_{e,\mu}^2 \langle 0 | \alpha_{\mathbf{k},e}^{r\dagger}(t) \alpha_{\mathbf{k},e}^r(t) | 0 \rangle_{e,\mu} \\
& - \left\{ \alpha_{\mathbf{k},e}^{r\dagger}(0), \beta_{-\mathbf{k},e}^{r\dagger}(t) \right\} \left\{ \alpha_{\mathbf{k},e}^r(0), \alpha_{\mathbf{k},e}^{r\dagger}(t) \right\} \langle 0 | \alpha_{\mathbf{k},e}^r(t) \beta_{-\mathbf{k},e}^r(t) | 0 \rangle_{e,\mu} \\
& + \left\{ \alpha_{\mathbf{k},e}^r(0), \beta_{-\mathbf{k},e}^r(t) \right\} \left\{ \alpha_{\mathbf{k},e}^r(t), \alpha_{\mathbf{k},e}^{r\dagger}(0) \right\} \langle 0 | \alpha_{\mathbf{k},e}^{r\dagger}(t) \beta_{-\mathbf{k},e}^{r\dagger}(t) | 0 \rangle_{e,\mu} \quad (4.113)
\end{aligned}$$

Explicit calculation of the above quantity shows that:

$$\langle \nu_{\mathbf{k},e}^r | \alpha_{\mathbf{k},e}^{r\dagger}(t) \alpha_{\mathbf{k},e}^r(t) \beta_{-\mathbf{k},e}^{r\dagger}(t) \beta_{-\mathbf{k},e}^r(t) | \nu_{\mathbf{k},e}^r \rangle = \langle \nu_{\mathbf{k},e}^r | \beta_{-\mathbf{k},e}^{r\dagger}(t) \beta_{-\mathbf{k},e}^r(t) | \nu_{\mathbf{k},e}^r \rangle \quad (4.114)$$

so that we have

$$\Delta Q_{\nu_e}(\nu_e)(t) = \mathcal{Q}_{\nu_e \rightarrow \nu_e}^{\mathbf{k}}(t) \mathcal{Q}_{\nu_e \rightarrow \nu_\mu}^{\mathbf{k}}(t) \quad (4.115)$$

which formally resembles the quantum mechanical result Eq.(4.107). The differences are now due to the presence of the flavor condensate, which affects the oscillation formulae (see Eqs.(2.118),(2.119)). In Fig. 4.14, flavor entanglement formula is plotted in the QFT case against the corresponding QM case. The correction due to the nontrivial vacuum condensate is evident.

As already emphasized, a serious drawback of the dynamic symmetry approach is its limitation to finite dimensional Hilbert spaces, which of course prevents its use in more general situations in which there can be creation or annihilation of particles. Nevertheless, the previous study provides a simple, exactly solvable example of a possible extension of entanglement to a genuine quantum field theoretical situation. Apart from the differences with QM due to the flavor vacuum contributions, the QFT result is interesting from a more conceptual point of view. Indeed, these results show that both the static and the dynamical entanglement arise in connection with unitarily inequivalent representations: in the case of the static entanglement, the flavor Hilbert space at time t to which the entangled state $|\nu_\sigma(t)\rangle$ belongs, is unitarily inequivalent to the Hilbert space for the qubit states $|\nu_i\rangle$ [39]; on the other hand, in the case of

dynamical entanglement, where the qubits are taken to be the flavor states at time $t = 0$, the inequivalence is among the flavor Hilbert space at different times [19]. In the first case, the relevant orthogonality relation is $\lim_{V \rightarrow \infty} \langle 0|0(t) \rangle_f = 0$, in the second $\lim_{V \rightarrow \infty} \langle 0(t')|0(t) \rangle_f = 0$, with $t \neq t'$.

Since the inequivalent representations are associated with a non-trivial condensate vacuum structure, the above conjecture suggests that, in the context of QFT, many interpretational issues connected with entanglement could be revisited in this new light. A similar point of view has been expressed by other authors, for example in [84].

The extension of the above results to the three flavor case is under investigation [26]. In the quantum mechanical case the relevant algebra is $u(1)_1 \oplus u(1)_2 \oplus u(1)_3$ (resp. $u(1)_e \oplus u(1)_\mu \oplus u(1)_\tau$) in the static (resp. dynamic) case. The relevant charges for the QFT case are discussed in [20, 21].

Chapter 5

Particle mixing by external fields, flavor states and Lorentz invariance

The subject of neutrino mixing and oscillations traditionally has tight links with Lorentz, Poincaré and CPT violations (a very partial list of references is [29, 120, 99, 97, 92, 59, 85, 62, 58]), and to the broader emerging subject of Quantum Gravity Phenomenology [6]. Neutrinos have often been proposed as viable probes to such subtle and small effects that are expected to occur because of unknown and uncharted physics beyond the walls of our established knowledge of Nature. In particular it is expected that quantum gravity induced decoherence should affect neutrino oscillations [5, 3, 98]. Such effects have also been connected to the nontrivial flavor vacuum structure discussed in chapter 2 [111].

Because of their nature, mixed particles are at odds with Poincaré invariance, because of the fact that they do not have a definite mass and so they cannot be classified using the standard wisdom descending from Wigner [151].

Let us state this more clearly, using the formalism we have been developing so far, and focusing for definiteness on the case of neutrinos. The flavor states we have characterized and studied, while being eigenstates of the momentum, are not eigenstates of the Hamiltonian, so it is not possible to define as usual the mass of flavored particles as the eigenvalue of the zero momentum Hamiltonian. This has led to a number of speculations with the purpose of reconciling particle mixing with Poincaré invariance, for example by using some nonlinear realization of the Lorentz group [36, 37] (along the lines of [105, 106]), or to embed it in an appropriate symmetry structure¹. In this chapter we shall show that this problem can be at least partially overcome by considering mixing not as an intrinsic property of the particles, but as the result of the interaction of some otherwise ordinary particles with an external field. This is a somewhat more conservative approach than the ones mentioned above since it does not involve some fundamental violation of relativistic symmetry but rather an effective violation due to the non scalar nature of the background field, which selects a preferred direction, thus being analogue of some other model present in the literature (for a review of these and other proposals see e.g. [109]). The background field necessary to implement mixing turns out to

¹Some unsuccessful attempt to embed mixed particles in some extended symmetry structure was carried out in [30]

be a Lie algebra valued vector field, which can be interpreted, at least in the maximal mixing limit, as the gauge field corresponding to a symmetry group which horizontally connects the various generations of particles. Thus our approach, besides addressing the above mentioned problems, unveils some hidden approximate non abelian gauge structure in particle mixing, which adds to other similar structures previously discovered, related to geometric phases [22]. Keeping this discussion in mind, in the following we will often use such gauge theoretic terms like connection and field strength.

This background field acts like a medium with nontrivial optical properties with respect to neutrinos. In particular in the two neutrino case, it acts on the two flavor case much like a birefringent medium acts on the two polarization of photons. The extension of this analogy to the three flavor case is of course not so obvious, since it would involve such exotic concepts as a “trirefringent” medium which would act on some sort of massive light. Nevertheless, the emerging physical picture is very appealing.

One line of thought suggested by these consideration, which would be very interesting to investigate, is the possibility of writing down mixing as a kind of Aharonov–Bohm [4] interaction. If this is the case, the vector field would just be a fictitious field much like the Chern–Simons field used to embed fractional statistics in a local field theory context (see e.g. [101, 152]).

Another possibility would be to interpret the vector field as a real field which, besides suggesting the above mentioned optical analogy, could be interpreted as a dark matter component. This would open the way to cosmological applications and to spacetime dependent oscillation length, susceptible of observational tests [28].

Both of these line of research are still in their infancy and will not be addressed at all in this work, apart from some comments on the optical analogy.

In the next sections, we will develop the background field approach to mixing in the case of two neutrinos and see how this allows to address the above mentioned problems. We will see that the ambiguity in the field theoretical formalism for mixing described in section 2.5 will be fixed in this new framework.

The extension to the cases of bosons and more than two flavors presents some technical difficulties and is at present object of study. We will limit ourselves to a sketch of the partial results we got in the boson case in the last section.

5.1 Neutrino mixing induced by an external vector field

We begin with the Lagrangian density describing two mixed neutrino fields:

$$\mathcal{L} = \bar{\nu}_e (i \not{\partial} - m_e) \nu_e + \bar{\nu}_\mu (i \not{\partial} - m_\mu) \nu_\mu - m_{e\mu} (\bar{\nu}_e \nu_\mu + \bar{\nu}_\mu \nu_e). \quad (5.1)$$

This can be consistently viewed as the interaction of the flavor neutrino fields with a constant external Lie algebra valued vector field. The most direct way of seeing this goes through the Euler–Lagrange equations corresponding to the Lagrangian (5.1), namely:

$$i\partial_0 \nu_e = (-i\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} + \beta m_e) \nu_e + \beta m_{e\mu} \nu_\mu \quad (5.2)$$

$$i\partial_0 \nu_\mu = (-i\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} + \beta m_\mu) \nu_\mu + \beta m_{e\mu} \nu_e, \quad (5.3)$$

where α_i , $i = 1, 2, 3$ and β are the usual Dirac matrices in a given representation. For definiteness let us choose the following representation:

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad (5.4)$$

where σ_i are the Pauli matrices and $\mathbb{1}$ is the 2×2 identity matrix. The Euler–Lagrange equations can be compactly written as follows:

$$iD_0\nu_f = (-i\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} + \beta M_d)\nu_f, \quad (5.5)$$

where $\nu_f = (\nu_e, \nu_\mu)^T$ is the flavor doublet and $M_d = \text{diag}(m_e, m_\mu)$ is a diagonal mass matrix. We have defined the (non-abelian) covariant derivative:

$$D_0 := \partial_0 + i m_{e\mu} \beta \sigma_1, \quad (5.6)$$

where $m_{e\mu} = \frac{1}{2} \tan 2\theta \delta m$, and $\delta m := m_\mu - m_e$.

We thus see that flavor mixing can be seen as an interaction of the flavor fields with an $su(2)$ -valued constant vector field having the following structure:

$$A_\mu := \frac{1}{2} A_\mu^a \sigma_a = n_\mu \delta m \frac{\sigma_1}{2} \in su(2), \quad n^\mu := (1, 0, 0, 0)^T, \quad (5.7)$$

that is, having only the temporal component in spacetime and only the first component in $su(2)$ space. In terms of this connection, the covariant derivative can be written in the form:

$$D_\mu = \partial_\mu + i g \beta A_\mu, \quad (5.8)$$

where we have defined $g := \tan 2\theta$ as the coupling constant for the mixing interaction. Note that in the case of maximal mixing ($\theta = \pi/4$), the coupling constant grows to infinity while δm goes to zero. We further note that, since the connection (5.7) is a constant, with just one non-zero component in group space, its field strength vanishes identically:

$$F_{\mu\nu}^a = \epsilon^{abc} A_\mu^b A_\nu^c = 0, \quad (5.9)$$

with $a, b, c = 1, 2, 3$. The fact that, despite $F_{\mu\nu}$ vanishes identically, the gauge field has physical effects, leads to an analogy with the Aharonov–Bohm effect [4], as we already mentioned in the introduction.

Finally, the equations of motion for the mixed fields can be cast in a manifestly covariant form:

$$(i\gamma^\mu D_\mu - M_d)\nu_f = 0, \quad (5.10)$$

and the Lagrangian density (5.1) has the form of the one describing a doublet of Dirac fields in interaction with an external Yang-Mills field:

$$\mathcal{L} = \bar{\nu}_f (i\gamma^\mu D_\mu - M_d)\nu_f. \quad (5.11)$$

The energy momentum tensor associated with the flavor neutrino fields in interaction with the external vector field can be computed by means of the standard procedure (described for example in [100]). One finds:

$$\tilde{T}_{\rho\sigma} = \bar{\nu}_f i\gamma_\rho D_\sigma \nu_f - \eta_{\rho\sigma} \bar{\nu}_f (i\gamma^\lambda D_\lambda - M_d) \nu_f. \quad (5.12)$$

This expression is to be compared with the one of the canonical energy momentum tensor

$$\begin{aligned} T_{\rho\sigma} &= \bar{\nu}_e i\gamma_\rho \partial_\sigma \nu_e - \eta_{\rho\sigma} \bar{\nu}_e (i\gamma^\lambda \partial_\lambda - m_e) \nu_e + \bar{\nu}_\mu i\gamma_\rho \partial_\sigma \nu_\mu + \\ &\quad - \eta_{\rho\sigma} \bar{\nu}_\mu (i\gamma^\lambda \partial_\lambda - m_\mu) \nu_\mu + \eta_{\rho\sigma} m_{e\mu} (\bar{\nu}_e \nu_\mu + \bar{\nu}_\mu \nu_e) \\ &= \bar{\nu}_1 i\gamma_\rho \partial_\sigma \nu_1 - \eta_{\rho\sigma} \bar{\nu}_1 (i\gamma^\lambda \partial_\lambda - m_1) \nu_1 + \bar{\nu}_2 i\gamma_\rho \partial_\sigma \nu_2 - \eta_{\rho\sigma} \bar{\nu}_2 (i\gamma^\lambda \partial_\lambda - m_2) \nu_2, \end{aligned} \quad (5.13)$$

where $\eta_{\rho\sigma} = \text{diag}(+1, -1, -1, -1)$ is the Minkowskian metric tensor. We see that the difference between the two is just the presence of the interaction terms in the 00 component, i.e. $T_{00} - \tilde{T}_{00} = m_{e\mu} (\bar{\nu}_e \nu_\mu + \bar{\nu}_\mu \nu_e)$, while we have $T_{0i} = \tilde{T}_{0i}$, $T_{ij} = \tilde{T}_{ij}$.

The tensor $\tilde{T}_{\mu\nu}$ is not conserved on-shell. In particular we have:

$$\partial^\rho \tilde{T}_{\rho i} = 0; \quad \partial^\rho \tilde{T}_{\rho 0} \neq 0. \quad (5.14)$$

Note that without the β matrix appearing in the covariant derivative (5.6) we would have found: $\partial^\mu \tilde{T}_{\mu\nu} = g F_{\mu\nu\alpha} j_\alpha^\mu = 0$, i.e. the energy-momentum tensor would have been conserved. In the present case $[\gamma_\mu, D_0] \neq 0$, in consequence of the presence of the β matrix in D_0 .

We also note that the matter current j_a^μ has only one component in group space:

$$j_1^\rho = \bar{\nu}_f \gamma^\rho \frac{\sigma_1}{2} \nu_f = \frac{1}{2} (\bar{\nu}_e \gamma^\rho \nu_\mu + \bar{\nu}_\mu \gamma^\rho \nu_e) = J_{f,1}^\rho, \quad (5.15)$$

where $J_{f,1}^\rho$ is the Noether current associated to the Lagrangian density (5.11) under the $SU(2)$ transformation [34]:

$$\nu'_f = e^{im_{e\mu}\lambda_1 \frac{\sigma_1}{2}} \nu_f. \quad (5.16)$$

Following the usual procedure, we now define a 4-momentum operator \tilde{P}^μ by taking the 0i and 00 components of $\tilde{T}^{\mu\nu}$ and integrating them over 3-space. We obtain a conserved 3-momentum operator:

$$\begin{aligned} \tilde{P}^i &= \int d^3\mathbf{x} \tilde{T}^{0i} = i \int d^3\mathbf{x} \nu_f^\dagger \partial^i \nu_f \\ &= i \int d^3\mathbf{x} \nu_e^\dagger \partial^i \nu_e + i \int d^3\mathbf{x} \nu_\mu^\dagger \partial^i \nu_\mu \\ &\equiv \tilde{P}_e^i(x_0) + \tilde{P}_\mu^i(x_0), \quad i = 1, 2, 3 \end{aligned} \quad (5.17)$$

and a non conserved Hamiltonian operator:

$$\begin{aligned} \tilde{P}^0(x_0) \equiv \tilde{H}(x_0) &= \int d^3\mathbf{x} \tilde{T}^{00} = \int d^3\mathbf{x} \bar{\nu}_f (i\gamma_0 D_0 - i\gamma^\mu D_\mu + M_d) \nu_f \\ &= \int d^3\mathbf{x} \nu_e^\dagger (-i\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} + \beta m_e) \nu_e + \int d^3\mathbf{x} \nu_\mu^\dagger (-i\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} + \beta m_\mu) \nu_\mu \\ &\equiv \tilde{H}_e(x_0) + \tilde{H}_\mu(x_0). \end{aligned} \quad (5.18)$$

We see that both the Hamiltonian and the momentum operators split in a natural way in a contribution involving only the electron neutrino field and in another where only the muon neutrino field appears. In such a way, we have a natural definition of a Hamiltonian and momentum operators for each flavor field.

We remark that the tilde Hamiltonian is *not* the generator of time translations. This role competes to the complete Hamiltonian $H = \int d^3\mathbf{x} T^{00}$, which includes the interaction term.

5.2 Recovery of the Poincaré algebra

Till now our considerations have been purely classical. Now we want to pass to the quantum theory. Our purpose is to construct flavor neutrino states which are simultaneous eigenstates of the 4-momentum operators above constructed and of the flavor charges. Of course this is a highly nontrivial request. We will see that such states can indeed be constructed, but this involves a nontrivial redefinition of the flavor vacuum which will also erase any reference to the ν_1 and ν_2 fields.

As we have seen in section 2.5, the flavor neutrino field operator expansion $((\sigma, j) = (e, 1)(\mu, 2))$

$$\nu_\sigma(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \sum_r \left[u_{\mathbf{k},j}^r(x_0) \alpha_{\mathbf{k},\sigma}^r(x_0) + v_{-\mathbf{k},j}^r(x_0) \beta_{-\mathbf{k},\sigma}^{r\dagger}(x_0) \right] e^{i\mathbf{k}\cdot\mathbf{x}}, \quad (5.19)$$

relies on a special choice of the bases of spinors, namely those referring to the free field masses m_1, m_2 . It is always possible to perform a Bogoliubov transformation in order to expand the field operators in a different basis of spinors, referring to an arbitrarily chosen couple of mass parameters [67].

In the context of the above reformulation of mixing, it seems natural to expand the flavor fields in the bases corresponding to the couple of masses (m_e, m_μ) . We will discover that precisely those values are singled out by the requirement that the flavor states be eigenstates of the Hamiltonian operator.

The new spinors are defined as the solutions of the equations:

$$(-\alpha \cdot \mathbf{k} + m_\sigma \beta) u_{\mathbf{k},\sigma}^r = \omega_{\mathbf{k},\sigma} u_{\mathbf{k},\sigma}^r \quad (5.20)$$

$$(-\alpha \cdot \mathbf{k} + m_\sigma \beta) v_{-\mathbf{k},\sigma}^r = -\omega_{\mathbf{k},\sigma} v_{-\mathbf{k},\sigma}^r, \quad (5.21)$$

where $\omega_{\mathbf{k},\sigma} = \sqrt{\mathbf{k}^2 + m_\sigma^2}$. These are the momentum space version of the *free* Dirac equation with mass m_σ .

The flavor field operators are then expanded as follows:

$$\nu_\sigma(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \sum_r \left[u_{\mathbf{k},\sigma}^r(x_0) \tilde{\alpha}_{\mathbf{k},\sigma}^r(x_0) + v_{-\mathbf{k},\sigma}^r(x_0) \tilde{\beta}_{-\mathbf{k},\sigma}^{r\dagger}(x_0) \right] e^{i\mathbf{k}\cdot\mathbf{x}}, \quad \sigma = e, \mu, \quad (5.22)$$

with $u_{\mathbf{k},\sigma}^r(x_0) = u_{\mathbf{k},\sigma}^r e^{-i\omega_{\mathbf{k},\sigma} x_0}$, $v_{-\mathbf{k},\sigma}^r(x_0) = v_{-\mathbf{k},\sigma}^r e^{i\omega_{\mathbf{k},\sigma} x_0}$. Here and in the following the tilde operators are those corresponding to the specific couple of mass parameters (m_e, m_μ) . With these definitions all the calculations at a fixed instant of time x_0 can be performed in exactly

the same way they are done in the free field case. The explicit time dependence of the creation and destruction operators, which is of course due to the interaction with the external field and is not present in the free field case, does not create problems as the states which are acted upon by the operators are evaluated at the same time as the operators themselves and the commutators are all considered at equal times.

In terms of the tilde flavor ladder operators, the Hamiltonian and momentum operators Eqs.(5.17),(5.18) read:

$$\tilde{\mathbf{P}}_\sigma(x_0) = \sum_r \int d^3\mathbf{k} \mathbf{k} \left(\tilde{\alpha}_{\mathbf{k},\sigma}^{r\dagger}(x_0) \tilde{\alpha}_{\mathbf{k},\sigma}^r(x_0) + \tilde{\beta}_{\mathbf{k},\sigma}^{r\dagger}(x_0) \tilde{\beta}_{\mathbf{k},\sigma}^r(x_0) \right), \quad (5.23)$$

$$\tilde{H}_\sigma(x_0) = \sum_r \int d^3\mathbf{k} \omega_{\mathbf{k},\sigma} \left(\tilde{\alpha}_{\mathbf{k},\sigma}^{r\dagger}(x_0) \tilde{\alpha}_{\mathbf{k},\sigma}^r(x_0) - \tilde{\beta}_{\mathbf{k},\sigma}^r(x_0) \tilde{\beta}_{\mathbf{k},\sigma}^{r\dagger}(x_0) \right). \quad (5.24)$$

The new flavor states are defined by the action of the tilde creation operator on the tilde flavor vacuum:

$$|\tilde{\nu}_{\mathbf{k},\sigma}^r(x_0)\rangle = \tilde{\alpha}_{\mathbf{k},\sigma}^{r\dagger}(x_0) |\tilde{0}(x_0)\rangle_{e\mu}. \quad (5.25)$$

We easily find the result that these single particle states are eigenstates of both the Hamiltonian and the momentum operator:

$$\begin{pmatrix} \tilde{H}_\sigma(x_0) \\ \tilde{\mathbf{P}}_\sigma(x_0) \end{pmatrix} |\tilde{\nu}_{\mathbf{k},\sigma}^r(x_0)\rangle = \begin{pmatrix} \omega_{\mathbf{k},\sigma} \\ \mathbf{k} \end{pmatrix} |\tilde{\nu}_{\mathbf{k},\sigma}^r(x_0)\rangle, \quad (5.26)$$

making explicit the 4–vector structure.

It can be also verified that the flavor charges commute with the tilde Hamiltonian operator: $[\tilde{Q}_\sigma(x_0), \tilde{H}(x_0)] = 0$, as a consequence of:

$$[\tilde{Q}_\sigma(x_0), \tilde{H}_{\sigma'}(x_0)] = 0, \quad \sigma, \sigma' = e, \mu. \quad (5.27)$$

This is of course a consequence of the fact that the flavor nonconservation is entirely due to the interaction term, which is absent in \tilde{H} . This fact ensures the existence of a common set of eigenstates of these operators. Indeed the flavor states (5.25) are straightforwardly seen to be also eigenstates of the flavor charges:

$$\tilde{Q}_\sigma(x_0) |\tilde{\nu}_{\mathbf{k},\sigma}^r(x_0)\rangle = |\tilde{\nu}_{\mathbf{k},\sigma}^r(x_0)\rangle, \quad (5.28)$$

thus confirming that these are precisely the states we were looking for.

Let us now make some observations on the algebra of the generators descending from the energy-momentum tensor (5.12). All the generators are defined in the usual way. Besides the translation generators defined by Eqs.(5.17) and (5.18), we have the Lorentz generators, defined as:

$$\tilde{M}^{\lambda\rho}(x_0) = \int d^3\mathbf{x} \left(\tilde{T}^{0\rho} x^\lambda - \tilde{T}^{0\lambda} x^\rho \right) + \frac{1}{2} \int d^3\mathbf{x} \nu_f^\dagger \sigma^{\lambda\rho} \nu_f = \tilde{M}_e^{\lambda\rho}(x_0) + \tilde{M}_\mu^{\lambda\rho}(x_0), \quad (5.29)$$

where $\sigma^{\mu\nu} = -\frac{i}{2}[\gamma^\mu, \gamma^\nu]$. The algebra of (equal-time) commutators of these generators will be just the direct sum of two Poincaré algebras (we omit the specification of the instant of time):

$$\begin{aligned} [\tilde{P}_\sigma^\mu, \tilde{P}_{\sigma'}^\nu] &= 0 \quad ; \quad [\tilde{M}_\sigma^{\mu\nu}, \tilde{P}_{\sigma'}^\lambda] = i\delta_{\sigma\sigma'} \left(\eta^{\mu\lambda} \tilde{P}_\sigma^\nu - \eta^{\nu\lambda} \tilde{P}_\sigma^\mu \right); \\ [\tilde{M}_\sigma^{\mu\nu}, \tilde{M}_{\sigma'}^{\lambda\rho}] &= i\delta_{\sigma\sigma'} \left(\eta^{\mu\lambda} \tilde{M}_\sigma^{\nu\rho} - \eta^{\nu\lambda} \tilde{M}_\sigma^{\mu\rho} - \eta^{\mu\rho} \tilde{M}_\sigma^{\nu\lambda} + \eta^{\nu\rho} \tilde{M}_\sigma^{\mu\lambda} \right), \quad \sigma, \sigma' = e, \mu \end{aligned} \quad (5.30)$$

Note that the above construction and the consequent Poincaré invariance, holds *at a given time* x_0 . Thus, for each different time, we have a different Poincaré structure. This is strongly reminiscent of a theory with a local Poincaré structure, that is, a theory of gravity. Speculations about the possible gravitational origin of mixing will not be pursued in this work.

5.3 Phenomenological consequences

The above analysis leads us to the view that the flavor fields ν_e and ν_μ should be regarded as fundamental. This fact has some interesting consequences at phenomenological level. Indeed, if we consider a charged current process in which for example an electron neutrino is created, we see that the hypothesis that mixing is due to interaction with an external field, implies that what is created in the vertex is really $|\nu_e\rangle$, rather than $|\nu_1\rangle$ or $|\nu_2\rangle$. As remarked above, such an interpretation is made possible because we can regard, at any given time, flavor fields as on shell fields, associated with masses m_e and m_μ .

We consider the case of a beta decay process, say for definiteness tritium decay, which allows for a direct investigation of neutrino mass. In the following we compare the various possible outcomes of this experiment predicted by the different theoretical possibilities for the nature of mixed neutrinos. As we shall see, the scenario described above presents significative phenomenological differences with respect to the standard theory.

Let us then consider the decay:

$$A \rightarrow B + e^- + \bar{\nu}_e,$$

where A and B are two nuclei (e.g. ${}^3\text{H}$ and ${}^3\text{He}$).

The electron spectrum is proportional to phase volume factor $E p E_e p_e$:

$$\frac{dN}{dK} = C E p (Q - K) \sqrt{(Q - K)^2 - m_\nu^2} \quad (5.31)$$

where $E = m + K$ and $p = \sqrt{E^2 - m^2}$ are electron's energy and momentum. The endpoint of β decay is the maximal kinetic energy K_{max} the electron can take (constrained by the available energy $Q = E_A - E_B - m \approx m_A - m_B - m$). In the case of tritium decay, $Q = 18.6$ KeV. Q is shared between the (unmeasured) neutrino energy and the (measured) electron kinetic energy K . It is clear that if the neutrino were massless, then $m_\nu = 0$ and $K_{max} = Q$. On the other hand, if the neutrino were a mass eigenstate with $m_\nu = m_1$, then $K_{max} = Q - m_1$.

We now consider the various possibilities which can arise in the presence of mixing. If, following the common wisdom, neutrinos with masses m_1 and m_2 are considered as fundamental,

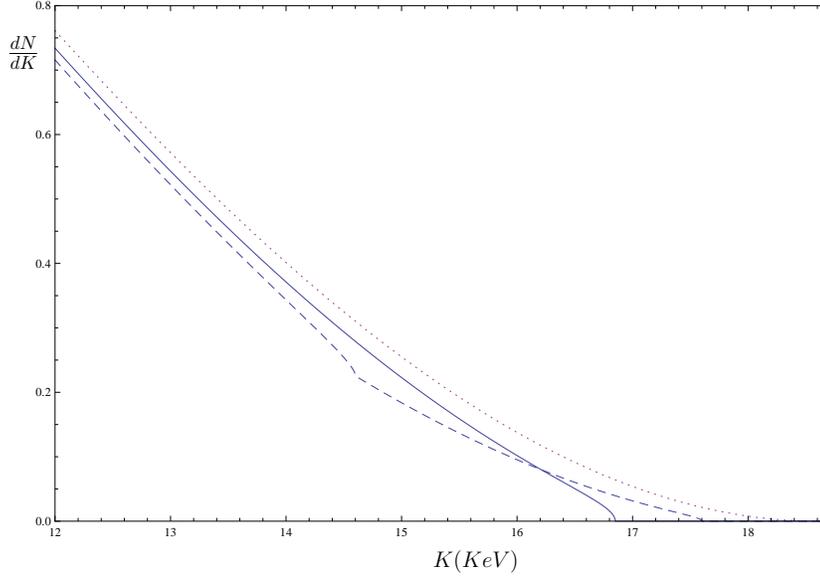


Figure 5.1: The tail of the tritium β spectrum for: - a massless neutrino (dotted line); - fundamental flavor states (continuous line); - superposed prediction for 2 mass states (short-dashed line): notice the inflexion in the spectrum where the most massive state switches off. We used $m_e = 1.75$ KeV, $m_1 = 1$ KeV, $m_2 = 4$ KeV, $\theta = \pi/6$.

the β spectrum is:

$$\frac{dN}{dK} = CEp E_e \sum_j |U_{ej}|^2 \sqrt{E_e^2 - m_j^2} \Theta(E_e - m_j), \quad (5.32)$$

where $E_e = Q - K$ and $U_{ej} = (\cos \theta, \sin \theta)$ and $\Theta(E_e - m_j)$ is the Heaviside step function. The end point is at $K = Q - m_1$ and the spectrum has an inflexion at $K \simeq Q - m_2$.

If on the other hand we take flavor neutrinos as fundamental according to the above scheme, we have that $m_\nu = m_e$ and $K_{max} = Q - m_e$ and the spectrum is proportional to the phase volume factor $EpE_e p_e$:

$$\frac{dN}{dK} = CEp (Q - K) \sqrt{(Q - K)^2 - m_e^2} \Theta(E_e - m_e). \quad (5.33)$$

The above discussed possibilities are plotted in Fig.(5.1), together with the spectrum for a massless neutrino, for comparison. We note that the next generation tritium beta decay experiments will allow a sub-eV sensitivity for the electron neutrino mass [123], thus hopefully allowing to unveil the true nature of mixed neutrinos.

Finally we point out that also in the neutrino detection process, it would be possible to discriminate among the various scenarios above considered. In such a case, our scheme would imply that in each detection vertex, either an electron neutrino or a muon neutrino would take part to the process. Again, this is in contrast with the standard view, which assumes that either ν_1 or ν_2 are entering in the elementary processes.

5.4 Discussion

In the framework outlined in this chapter flavor neutrino fields are taken to be on-shell fields with definite masses m_e and m_μ , which are different from those of the mass eigenstates of the standard approach, m_1 and m_2 . Flavor oscillations then arise as a consequence of the interaction with an external field, which acts as a sort of refractive medium which can be called *neutrino aether*.

It would be interesting to explore the properties of such a medium and possible optical analogs of this situation. A very interesting example in this respect has been given recently in Ref.[146]. Another interesting analogy can be drawn with recent studies in which, for the case of photons, the vacuum has been thought to act as a refractive medium in consequence of quantum gravity fluctuations [63, 79].

The gauge structure associated to flavor mixing has the very interesting property of arising across different fermion generations, thus having a different (“horizontal”) nature with respect to the gauge structure of the Standard Model. The idea of horizontal (family) symmetries has been already invoked long ago in the literature to explain mass hierarchy and mixing patterns of quarks and leptons [11, 91], and has been connected also to such problems as dark matter and strong CP violation [13, 12].

A natural question that comes in concerns the origin of the external vector field which causes the mixing. This could also have some connection to quantum gravity models that sometimes are invoked to explain the origin of mixing [110, 2].

Another outcome of our analysis is that we could recover, at least locally in time, a Poincaré structure for the flavor states. This is possible since we could define a Hamiltonian operator that commutes with the flavor charges, thus allowing for simultaneous eigenstates. In this scheme where the fields ν_e and ν_μ are taken to be fundamental, one avoids any reference to the fields ν_1 and ν_2 . As pointed out, this leads to phenomenological consequences that can be possibly tested in experiments on beta decay.

A final consideration concerns the interpretation of the Hamiltonian operator \tilde{H} which, as already remarked, does not take into account the interaction energy, i.e. the energy associated with mixing. We can thus view \tilde{H} as the sum of the kinetic energies of the flavor neutrinos, or equivalently as the energy which can be extracted from flavored neutrinos by scattering processes, the mixing energy being “frozen” (there’s no way to turn off the mixing!). This suggests the interpretation of such a quantity as a “free” energy $F \equiv \tilde{H}$, so that we can write:

$$H - F = TS. \tag{5.34}$$

This quantity defines an entropy associated with flavor mixing. It is natural to identify the “temperature” T with the coupling constant $g = \tan 2\theta$, thus leading to:

$$S = \int d^3\mathbf{x} \bar{\nu}_f A_0 \nu_f = \frac{1}{2} \delta m \int d^3\mathbf{x} (\bar{\nu}_e \nu_\mu + \bar{\nu}_\mu \nu_e). \tag{5.35}$$

The appearance of an entropy should not be surprising, since each of the two flavor neutrinos can be considered as an open system which presents some kind of (cyclic) dissipation. This situation can be handled by use of well known methods of Thermo Field Dynamics [136] developed for the study of quantum dissipative systems [52].

The explicit expression for the expectation values of the entropy on the flavor neutrino states is quite complicated, and thus not very illuminating. An attempt at an interpretation of it is given in Appendix B in the much simpler context of Quantum Mechanics. There it is shown that at a given time, the difference of the expectation values of the muon and electron free energies is less than the total initial energy of the flavor neutrino state. The missing part is proportional to the expectation value of the entropy.

The scenario emerged in this chapter, and in particular the last considerations, is consistent with an interpretation of the gauge field as a reservoir, first put forward in [51].

Of course, in view of the time dependent entanglement entropy associated with neutrino mixing and oscillations discussed in the previous chapter, it is an interesting question the one of the connection of the latter to the entropy discussed in this chapter.

5.5 Mixing of boson fields

In this section we outline preliminary work done to get a procedure to formulate the mixing of scalar fields in an analogous way [32]. The Lagrangian density describing the mixing of two scalar fields is:

$$\mathcal{L}(x) = \partial_\mu \Phi_f^\dagger(x) \partial^\mu \Phi_f(x) - \Phi_f^\dagger(x) M_b \Phi_f(x) = -\Phi_f^\dagger(x) (\square + M_b) \Phi_f(x), \quad (5.36)$$

where in the second equality we have integrated by parts, $\Phi_f^T = (\phi_A, \phi_B)$ and

$$M_b = \begin{pmatrix} m_A^2 & m_{AB}^2 \\ m_{AB}^2 & m_B^2 \end{pmatrix}. \quad (5.37)$$

In complete analogy to the fermion case, this Lagrangian can be diagonalized [18] by writing:

$$\Phi_f(x) = U \Phi_m(x) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \Phi_m(x), \quad (5.38)$$

where $\Phi_m^T = (\phi_1, \phi_2)$, so to have:

$$\mathcal{L}(x) = \partial_\mu \Phi_m^\dagger(x) \partial^\mu \Phi_m(x) - \Phi_m^\dagger(x) M_d^2 \Phi_m(x), \quad (5.39)$$

with $M_d^2 = \text{diag}(m_1^2, m_2^2)$ and

$$m_A^2 = m_1^2 \cos^2 \theta + m_2^2 \sin^2 \theta \quad (5.40)$$

$$m_B^2 = m_1^2 \sin^2 \theta + m_2^2 \cos^2 \theta \quad (5.41)$$

$$m_{AB}^2 = (m_2^2 - m_1^2) \sin \theta \cos \theta. \quad (5.42)$$

In order to identify the gauge structure hidden in boson mixing, let us consider the following identity:

$$M^2 = (UM_d U^T)^2 = UM_d U^T U M_d U^T = UM_d^2 U^T = M_b. \quad (5.43)$$

where the orthogonality of U has been used. The matrices M and M_d are the same as the ones defined above in the fermion case.

We also have the relation:

$$(i \not{\partial} + M_d)(i \not{\partial} - M_d) = -(\square + M_d^2) \quad (5.44)$$

which, multiplied on the left by U and on the right by U^T , gives:

$$(i \not{\partial} + M)(i \not{\partial} - M) = -(\square + M_b). \quad (5.45)$$

consistently with Eq. (5.43).

Now, by remembering the expression of the covariant derivative:

$$\mathcal{D} = \not{\partial} + im_{e\mu}\sigma_1 \quad (5.46)$$

and using $\widetilde{M}_d = \text{diag}(m_e, m_\mu)$, we can write the left-hand side of Eq. (5.45) as follows:

$$\begin{aligned} (i \not{\partial} + M)(i \not{\partial} - M) &= (i \not{\mathcal{D}}^\dagger + \widetilde{M}_d)(i \not{\mathcal{D}}^\dagger - \widetilde{M}_d) \\ &= -\not{\mathcal{D}}^\dagger \not{\mathcal{D}} - i \not{\mathcal{D}}^\dagger \widetilde{M}_d + i \widetilde{M}_d \not{\mathcal{D}} - \widetilde{M}_d^2 \end{aligned} \quad (5.47)$$

that is, apart from the minus sign:

$$\not{\mathcal{D}}^\dagger \not{\mathcal{D}} + m_{e\mu}(m_e + m_\mu)\sigma_1 \equiv \widetilde{\Delta}. \quad (5.48)$$

Now we use the fact that

$$\not{\mathcal{D}}^\dagger \not{\mathcal{D}} = (\not{\partial} - im_{e\mu}\sigma_1)(\not{\partial} + im_{e\mu}\sigma_1) = \square + m_{e\mu}^2 I \quad (5.49)$$

to write:

$$\widetilde{\Delta} = \square + m_{e\mu}^2 I + C\sigma_1 \equiv \square + A \quad (5.50)$$

where we have defined $C = m_\mu(m_e + m_\mu)$ and $A = m_{e\mu}^2 I + C\sigma_1$. Our purpose is to identify the covariant derivative ∇_μ such that

$$\square + A = \nabla_\mu^\dagger \nabla^\mu \quad (5.51)$$

i.e. the appropriate vector field to implement boson mixing. By putting:

$$\nabla_\mu = \begin{pmatrix} \partial_0 + ig\mathcal{A}_0 \\ \partial_i \end{pmatrix} \quad (5.52)$$

we get $\nabla_\mu^\dagger \nabla^\mu = \square + g^2 \mathcal{A}_0^\dagger \mathcal{A}_0$, so that the new vector field must fulfill the condition $g^2 \mathcal{A}_0^\dagger \mathcal{A}_0 = A$. Further study along this line is in progress.

Appendix A

Entropic measures for the states

$$W_q^{(4)}(\delta_{ij})$$

Here we give the analytic expressions of the eigenvalues corresponding to the reduced density matrices of the states $|W_q^{(4)}(\tilde{\delta})\rangle$ ($q = 1, \dots, 4$). Let $\underline{\lambda}_q^{(i;j:k,l)}$ and $\underline{\lambda}_q^{(i,j:k,l)}$ the eigenvalue vectors associates respectively to the reduced density matrices $Tr_{j,k,l}[|W_q^{(4)}(\tilde{\delta})\rangle\langle W_q^{(4)}(\tilde{\delta})|]$ e $Tr_{k,l}[|W_q^{(4)}(\tilde{\delta})\rangle\langle W_q^{(4)}(\tilde{\delta})|]$. We have:

$$\begin{aligned} \underline{\lambda}_1^{(1:2,3,4)} &= \underline{\lambda}_1^{(2:1,3,4)} = \underline{\lambda}_1^{(3:1,2,4)} = \underline{\lambda}_1^{(4:1,2,3)} = \underline{\lambda}_2^{(4:1,2,3)} = \underline{\lambda}_3^{(4:1,2,3)} = \underline{\lambda}_4^{(4:1,2,3)} \\ &= \frac{1}{4}\{3, 1\}, \end{aligned}$$

$$\begin{aligned} \underline{\lambda}_2^{(1:2,3,4)} &= \frac{1}{36}\left\{25 - 6 \cos \delta_{14} - 6 \cos \delta_{23} - 2 \cos(\delta_{14} + \delta_{23}), \right. \\ &\quad \left. 11 + 6 \cos \delta_{14} + 6 \cos \delta_{23} + 2 \cos(\delta_{14} + \delta_{23})\right\}, \end{aligned}$$

$$\begin{aligned} \underline{\lambda}_2^{(2:1,3,4)} &= \frac{1}{36}\left\{11 - 6 \cos \delta_{14} - 6 \cos \delta_{23} + 2 \cos(\delta_{14} + \delta_{23}), \right. \\ &\quad \left. 25 + 6 \cos \delta_{14} + 6 \cos \delta_{23} - 2 \cos(\delta_{14} + \delta_{23})\right\}, \end{aligned}$$

$$\underline{\lambda}_2^{(3:1,2,4)} = \frac{1}{36}\left\{5 - 4 \cos(\delta_{14} + \delta_{23}), 31 + 4 \cos(\delta_{14} + \delta_{23})\right\},$$

$$\begin{aligned} \underline{\lambda}_3^{(1:2,3,4)} &= \frac{1}{72}\left\{16 - 6 \cos \delta_{14} - 6 \cos \delta_{23} - 2 \cos(\delta_{14} + \delta_{23}) + 6 \cos(\delta_{14} - \delta_{34}), \right. \\ &\quad -6 \cos(\delta_{14} - \delta_{23} - \delta_{34}) - 9 \cos \delta_{34} + 6 \cos(\delta_{23} + \delta_{34}) + 3 \cos(2\delta_{23} + \delta_{34}), \\ &\quad 56 + 6 \cos \delta_{14} + 6 \cos \delta_{23} + 2 \cos(\delta_{14} + \delta_{23}) - 6 \cos(\delta_{14} - \delta_{34}) \\ &\quad \left. + 6 \cos(\delta_{14} - \delta_{23} - \delta_{34}) + 9 \cos \delta_{34} - 6 \cos(\delta_{23} + \delta_{34}) - 3 \cos(2\delta_{23} + \delta_{34})\right\}, \end{aligned}$$

$$\underline{\lambda}_3^{(2:1,3,4)} = \frac{1}{72}\left\{56 - 6 \cos \delta_{14} - 6 \cos \delta_{23} + 2 \cos(\delta_{14} + \delta_{23}) - 6 \cos(\delta_{14} - \delta_{34})\right\}$$

$$\begin{aligned}
& -6 \cos(\delta_{14} - \delta_{23} - \delta_{34}) - 9 \cos \delta_{34} - 6 \cos(\delta_{23} + \delta_{34}) + 3 \cos(2\delta_{23} + \delta_{34}), \\
& 16 + 6 \cos \delta_{14} + 6 \cos \delta_{23} - 2 \cos(\delta_{14} + \delta_{23}) + 6 \cos(\delta_{14} - \delta_{34}) \\
& + 6 \cos(\delta_{14} - \delta_{23} - \delta_{34}) + 9 \cos \delta_{34} + 6 \cos(\delta_{23} + \delta_{34}) - 3 \cos(2\delta_{23} + \delta_{34}) \},
\end{aligned}$$

$$\begin{aligned}
\underline{\lambda}_3^{(3:1,2,4)} = & \frac{1}{36} \left\{ 11 + 2 \cos(\delta_{14} + \delta_{23}) - 6 \cos(\delta_{14} - \delta_{34}) - 6 \cos(\delta_{23} + \delta_{34}), \right. \\
& \left. 25 - 2 \cos(\delta_{14} + \delta_{23}) + 6 \cos(\delta_{14} - \delta_{34}) + 6 \cos(\delta_{23} + \delta_{34}) \right\},
\end{aligned}$$

$$\begin{aligned}
\underline{\lambda}_4^{(1:2,3,4)} = & \frac{1}{72} \left\{ 56 + 6 \cos \delta_{14} + 6 \cos \delta_{23} + 2 \cos(\delta_{14} + \delta_{23}) + 6 \cos(\delta_{14} - \delta_{34}) \right. \\
& - 6 \cos(\delta_{14} - \delta_{23} - \delta_{34}) - 9 \cos \delta_{34} - 6 \cos(\delta_{23} + \delta_{34}) + 3 \cos(2\delta_{23} + \delta_{34}), \\
& 16 - 6 \cos \delta_{14} - 6 \cos \delta_{23} - 2 \cos(\delta_{14} + \delta_{23}) - 6 \cos(\delta_{14} - \delta_{34}) \\
& \left. + 6 \cos(\delta_{14} - \delta_{23} - \delta_{34}) + 9 \cos \delta_{34} - 6 \cos(\delta_{23} + \delta_{34}) - 3 \cos(2\delta_{23} + \delta_{34}) \right\},
\end{aligned}$$

$$\begin{aligned}
\underline{\lambda}_4^{(2:1,3,4)} = & \frac{1}{72} \left\{ 16 + 6 \cos \delta_{14} + 6 \cos \delta_{23} - 2 \cos(\delta_{14} + \delta_{23}) - 6 \cos(\delta_{14} - \delta_{34}) \right. \\
& - 6 \cos(\delta_{14} - \delta_{23} - \delta_{34}) - 9 \cos \delta_{34} - 6 \cos(\delta_{23} + \delta_{34}) + 3 \cos(2\delta_{23} + \delta_{34}), \\
& 56 - 6 \cos \delta_{14} - 6 \cos \delta_{23} + 2 \cos(\delta_{14} + \delta_{23}) + 6 \cos(\delta_{14} - \delta_{34}) \\
& \left. + 6 \cos(\delta_{14} - \delta_{23} - \delta_{34}) + 9 \cos \delta_{34} + 6 \cos(\delta_{23} + \delta_{34}) - 3 \cos(2\delta_{23} + \delta_{34}) \right\},
\end{aligned}$$

$$\begin{aligned}
\underline{\lambda}_4^{(3:1,2,4)} = & \frac{1}{36} \left\{ 25 - 2 \cos(\delta_{14} + \delta_{23}) - 6 \cos(\delta_{14} - \delta_{34}) - 6 \cos(\delta_{23} + \delta_{34}), \right. \\
& \left. 11 + 2 \cos(\delta_{14} + \delta_{23}) + 6 \cos(\delta_{14} - \delta_{34}) + 6 \cos(\delta_{23} + \delta_{34}) \right\},
\end{aligned}$$

$$\underline{\lambda}_1^{(1,2:3,4)} = \underline{\lambda}_1^{(1,3:2,4)} = \underline{\lambda}_1^{(1,4:2,3)} = \frac{1}{2} \{0, 0, 1, 1\},$$

$$\underline{\lambda}_2^{(1,2:3,4)} = \frac{1}{18} \left\{ 0, 0, 7 - 2 \cos(\delta_{14} + \delta_{23}), 11 + 2 \cos(\delta_{14} + \delta_{23}) \right\},$$

$$\begin{aligned}
\underline{\lambda}_2^{(1,3:2,4)} = & \frac{1}{18} \left\{ 0, 0, 10 - 3 \cos \delta_{14} - 3 \cos \delta_{23} + \cos(\delta_{14} + \delta_{23}), \right. \\
& \left. 8 + 3 \cos \delta_{14} + 3 \cos \delta_{23} - \cos(\delta_{14} + \delta_{23}) \right\},
\end{aligned}$$

$$\begin{aligned}
\underline{\lambda}_2^{(1,4:2,3)} = & \frac{1}{18} \left\{ 0, 0, 8 - 3 \cos \delta_{14} - 3 \cos \delta_{23} - \cos(\delta_{14} + \delta_{23}), \right. \\
& \left. 10 + 3 \cos \delta_{14} + 3 \cos \delta_{23} + \cos(\delta_{14} + \delta_{23}) \right\},
\end{aligned}$$

$$\begin{aligned}
\underline{\lambda}_3^{(1,2:3,4)} = & \frac{1}{18} \left\{ 0, 0, 10 + \cos(\delta_{14} + \delta_{23}) - 3 \cos(\delta_{14} - \delta_{34}) - 3 \cos(\delta_{23} + \delta_{34}), \right. \\
& \left. 8 - \cos(\delta_{14} + \delta_{23}) + 3 \cos(\delta_{14} - \delta_{34}) + 3 \cos(\delta_{23} + \delta_{34}) \right\},
\end{aligned}$$

$$\begin{aligned}
\lambda_3^{(1,3:2,4)} &= \frac{1}{72} \left\{ 0, 0, 38 - 6 \cos \delta_{14} - 6 \cos \delta_{23} + 2 \cos(\delta_{14} + \delta_{23}) + 6 \cos(\delta_{14} - \delta_{34}) \right. \\
&\quad - 6 \cos(\delta_{14} - \delta_{23} - \delta_{34}) - 9 \cos \delta_{34} - 6 \cos(\delta_{23} + \delta_{34}) + 3 \cos(2\delta_{23} + \delta_{34}), \\
&\quad 34 + 6 \cos \delta_{14} + 6 \cos \delta_{23} - 2 \cos(\delta_{14} + \delta_{23}) + 6 \cos(\delta_{14} - \delta_{34}) \\
&\quad \left. + 6 \cos(\delta_{14} - \delta_{23} - \delta_{34}) + 9 \cos \delta_{34} + 6 \cos(\delta_{23} + \delta_{34}) - 3 \cos(2\delta_{23} + \delta_{34}) \right\}, \\
\lambda_3^{(1,4:2,3)} &= \frac{1}{72} \left\{ 0, 0, 34 - 6 \cos \delta_{14} - 6 \cos \delta_{23} - 2 \cos(\delta_{14} + \delta_{23}) + 6 \cos(\delta_{14} - \delta_{34}) \right. \\
&\quad - 6 \cos(\delta_{14} - \delta_{23} - \delta_{34}) - 9 \cos \delta_{34} + 6 \cos(\delta_{23} + \delta_{34}) + 3 \cos(2\delta_{23} + \delta_{34}), \\
&\quad 38 + 6 \cos \delta_{14} + 6 \cos \delta_{23} + 2 \cos(\delta_{14} + \delta_{23}) - 6 \cos(\delta_{14} - \delta_{34}) \\
&\quad \left. + 6 \cos(\delta_{14} - \delta_{23} - \delta_{34}) + 9 \cos \delta_{34} - 6 \cos(\delta_{23} + \delta_{34}) - 3 \cos(2\delta_{23} + \delta_{34}) \right\}, \\
\lambda_4^{(1,2:3,4)} &= \frac{1}{18} \left\{ 0, 0, 8 - \cos(\delta_{14} + \delta_{23}) - 3 \cos(\delta_{14} - \delta_{34}) - 3 \cos(\delta_{23} + \delta_{34}), \right. \\
&\quad \left. 10 + \cos(\delta_{14} + \delta_{23}) + 3 \cos(\delta_{14} - \delta_{34}) + 3 \cos(\delta_{23} + \delta_{34}) \right\}, \\
\lambda_4^{(1,3:2,4)} &= \frac{1}{72} \left\{ 0, 0, 34 + 6 \cos \delta_{14} + 6 \cos \delta_{23} - 2 \cos(\delta_{14} + \delta_{23}) - 6 \cos(\delta_{14} - \delta_{34}) \right. \\
&\quad - 6 \cos(\delta_{14} - \delta_{23} - \delta_{34}) - 9 \cos \delta_{34} - 6 \cos(\delta_{23} + \delta_{34}) + 3 \cos(2\delta_{23} + \delta_{34}), \\
&\quad 38 - 6 \cos \delta_{14} - 6 \cos \delta_{23} + 2 \cos(\delta_{14} + \delta_{23}) + 6 \cos(\delta_{14} - \delta_{34}) \\
&\quad \left. + 6 \cos(\delta_{14} - \delta_{23} - \delta_{34}) + 9 \cos \delta_{34} + 6 \cos(\delta_{23} + \delta_{34}) - 3 \cos(2\delta_{23} + \delta_{34}) \right\}, \\
\lambda_4^{(1,4:2,3)} &= \frac{1}{72} \left\{ 0, 0, 38 + 6 \cos \delta_{14} + 6 \cos \delta_{23} + 2 \cos(\delta_{14} + \delta_{23}) + 6 \cos(\delta_{14} - \delta_{34}) \right. \\
&\quad - 6 \cos(\delta_{14} - \delta_{23} - \delta_{34}) - 9 \cos \delta_{34} + 6 \cos(\delta_{23} + \delta_{34}) + 3 \cos(2\delta_{23} + \delta_{34}), \\
&\quad 34 - 6 \cos \delta_{14} - 6 \cos \delta_{23} - 2 \cos(\delta_{14} + \delta_{23}) - 6 \cos(\delta_{14} - \delta_{34}) \\
&\quad \left. + 6 \cos(\delta_{14} - \delta_{23} - \delta_{34}) + 9 \cos \delta_{34} - 6 \cos(\delta_{23} + \delta_{34}) - 3 \cos(2\delta_{23} + \delta_{34}) \right\}.
\end{aligned}$$

The von Neumann entropies are given by:

$$E_{vN q}^{(\cdot)} = - \sum_n \lambda_q^{(\cdot)}(n) \log_2 \lambda_q^{(\cdot)}(n). \quad (\text{A.1})$$

Appendix B

Mixing in external field in Quantum Mechanics

In this appendix we develop a similar analysis to the one given in chapter 5 to the case of mixing in Quantum Mechanics that is, of a single mode. This is useful for the interpretation of the results, which in this case have a much simpler form.

In a QM context, the flavor (fermionic) annihilation operators are defined by the relations:

$$\alpha_e(t) = \cos \theta \alpha_1(t) + \sin \theta \alpha_2(t) \quad (\text{B.1})$$

$$\alpha_\mu(t) = -\sin \theta \alpha_1(t) + \cos \theta \alpha_2(t), \quad (\text{B.2})$$

where $\alpha_i(t) = e^{i\omega_i t} \alpha_i$, $i = 1, 2$. The flavor states are given by:

$$|\nu_\sigma(t)\rangle = \alpha_\sigma^\dagger(t) |0\rangle_m, \quad \sigma = e, \mu, \quad (\text{B.3})$$

where $|0\rangle_m = |0\rangle_1 \otimes |0\rangle_2$ is the vacuum for the mass eigenstates. We use the notation $|\nu_\sigma\rangle = |\nu_\sigma(t=0)\rangle$. The Hamiltonian of the system is:

$$\begin{aligned} H &= \omega_e \alpha_e^\dagger(t) \alpha_e(t) + \omega_\mu \alpha_\mu^\dagger(t) \alpha_\mu(t) + \omega_{e\mu} [\alpha_e^\dagger(t) \alpha_\mu(t) + \alpha_\mu^\dagger(t) \alpha_e(t)] \\ &= \omega_1 \alpha_1^\dagger \alpha_1 + \omega_2 \alpha_2^\dagger \alpha_2, \end{aligned} \quad (\text{B.4})$$

where $\omega_e = \omega_1 \cos^2 \theta + \omega_2 \sin^2 \theta$, $\omega_\mu = \omega_1 \sin^2 \theta + \omega_2 \cos^2 \theta$, $\omega_{e\mu} = (\omega_2 - \omega_1) \sin \theta \cos \theta$.

In analogy with the QFT case we define the covariant derivative:

$$D_t = \frac{d}{dt} + igA = \frac{d}{dt} + i\omega_{e\mu} \sigma_1, \quad (\text{B.5})$$

where we have $\omega_{e\mu} = \frac{1}{2} \tan 2\theta \delta\omega$, $\delta\omega = \omega_\mu - \omega_e$, and we have defined $A := \delta\omega \frac{\sigma_1}{2}$. Using this covariant derivative the equations of motion read:

$$D_t \alpha_f = -i\omega_d \alpha_f, \quad (\text{B.6})$$

where $\alpha_f = (\alpha_e, \alpha_\mu)^T$ and $\omega_d = \text{diag}(\omega_e, \omega_\mu)$. The Hamiltonian can then be written in the form:

$$H = \alpha_f^\dagger \omega_d \alpha_f + g \alpha_f^\dagger A \alpha_f. \quad (\text{B.7})$$

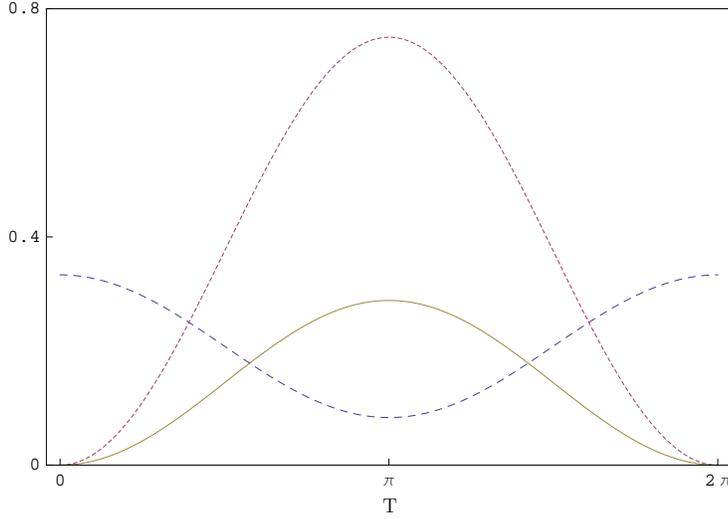


Figure B.1: Plot of expectation values on $|\nu_e(0)\rangle$ of $F_e(t)$ (long-dashed line), $F_\mu(t)$ (short-dashed line) and $2TS_e(t)$ (solid line). We used rescaled dimensionless time $T = (\omega_2 - \omega_1)t$ and $\theta = \pi/6$. The scale on the vertical axis is normalized to ω_μ .

The diagonal part of the above expression can be readily split into separate contributions for each flavor

$$\tilde{H}(t) = \alpha_f^\dagger \omega_d \alpha_f = \omega_e \alpha_e^\dagger(t) \alpha_e(t) + \omega_\mu \alpha_\mu^\dagger(t) \alpha_\mu(t) = \tilde{H}_e(t) + \tilde{H}_\mu(t). \quad (\text{B.8})$$

Note that expectation values of the flavor number operators on the single particle flavor neutrino states at time zero give the oscillation probabilities:

$$\langle \nu_e(0) | N_e(t) | \nu_e(0) \rangle = P_{\nu_e \rightarrow \nu_e}(t) = 1 - \sin^2 2\theta \sin^2 \left(\frac{\omega_2 - \omega_1}{2} t \right); \quad (\text{B.9})$$

$$\langle \nu_e(0) | N_\mu(t) | \nu_e(0) \rangle = P_{\nu_e \rightarrow \nu_\mu}(t) = \sin^2 2\theta \sin^2 \left(\frac{\omega_2 - \omega_1}{2} t \right). \quad (\text{B.10})$$

Thus we have:

$$\langle \nu_e(0) | \tilde{H}_e(t) | \nu_e(0) \rangle = \omega_e P_{\nu_e \rightarrow \nu_e}(t); \quad (\text{B.11})$$

$$\langle \nu_e(0) | \tilde{H}_\mu(t) | \nu_e(0) \rangle = \omega_\mu P_{\nu_e \rightarrow \nu_\mu}(t). \quad (\text{B.12})$$

In analogy with the field theoretical case, we regard these “free” Hamiltonians as free energies, and we write:

$$H = \sum_{\sigma=e,\mu} (F_\sigma(t) + TS_\sigma(t)), \quad (\text{B.13})$$

where we make the identifications $g \equiv T$ and:

$$S_\sigma(t) = \frac{1}{4} \delta\omega [\alpha_e^\dagger(t) \alpha_\mu(t) + \alpha_\mu^\dagger(t) \alpha_e(t)]. \quad (\text{B.14})$$

We have:

$$\langle \nu_e(0) | S_e(t) | \nu_e(0) \rangle = \langle \nu_e(0) | S_\mu(t) | \nu_e(0) \rangle = -\frac{1}{4} \delta\omega \sin 4\theta \sin^2 \left[\frac{1}{2}(\omega_2 - \omega_1)t \right]. \quad (\text{B.15})$$

All the expectation values obtained are summarized in the following table, from which we immediately see how the energetic balance is recovered. The situation for an electron neutrino state is represented in Fig. B.1 for sample values of the parameters.

| | H | F_e | F_μ | $TS_e = TS_\mu$ |
|----------------------|--------------|----------------------|----------------------|---------------------------------|
| $ \nu_e(0)\rangle$ | ω_e | $\omega_e(1 - P(t))$ | $\omega_\mu P(t)$ | $\frac{1}{2}\delta\omega P(t)$ |
| $ \nu_\mu(0)\rangle$ | ω_μ | $\omega_\mu P(t)$ | $\omega_e(1 - P(t))$ | $-\frac{1}{2}\delta\omega P(t)$ |

Table B.1: Energetic balance for flavor neutrino states. $P(t)$ denotes the transition probability $P_{\nu_e \rightarrow \nu_\mu}(t)$.

Note finally that the integral of the entropy expectation value over an oscillation cycle, is only dependent on the mixing angle:

$$\int_0^\tau \langle \nu_e(0) | S_e(t) | \nu_e(0) \rangle dt = \pi \cos^2 2\theta \sin 2\theta. \quad (\text{B.16})$$

where the period $\tau = \frac{2\pi}{\omega_2 - \omega_1}$. It would be interesting to compare this result with the geometric invariants discussed in Ref.[33].

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