Università degli studi di Salerno

DIPARTIMENTO DI FISICA "E.R. CAIANIELLO"

DOCTOR OF PHILOSOPHY IN PHYSICS

XIV CICLO, II SERIE

Non-Markovian Quantum Dynamics

Gianpaolo Torre

SUPERVISOR

Prof. Fabrizio Illuminati

COORDINATOR

Prof. Canio Noce

2014/2015

Contents

Introduction

Quantum Mechanics, together with General Relativity, is one of the best achievement of the human mind in the last Century. From its formulation it has passed an uncountable number of experimental tests and it has been of fundamental importance in the comprehension of the laws of Nature.

Quantum systems can necessarily be viewed as open systems. As in classical physics, any realistic description of a system must take into account the coupling to an environment that strongly influences the system itself. Since perfect isolation of a quantum system is not feasible and the complete description of the degrees of freedom of the environment is not possible, it is required a description that accounts for these aspects. Furthermore, not all the degrees of freedom are of interest in order to effectively describe the system. Hence a probabilistic approach to a quantum evolution is most appropriate: the idea is to consider only the degrees of freedom that are useful, thus reducing to a small set the number of variables needed to describe the evolution of the system.

Another reason forces to consider open quantum systems. Any information about the system can be obtained only through a measurement process that requires a coupling of the system with the measuring apparatus. At variance with the classical case, in the quantum instance the measuring apparatus influences in a non trivial way the systems. Therefore, the notion of open system is implicitly induced in quantum mechanics by the measurement process.

In last years the remarkable progress of quantum technologies opened up new perspective in the investigation of the dynamics of open systems; in this sense, particularly relevant are the techniques which allow to control the degrees of freedom of the environment that influence the system of interest. Till now the attention was focused on the methods to reduce the detrimental effect on the quantum properties of the system-environment interaction; namely, on the methods allowing to make the system the more isolated as possible. On the other hand, it has been experimentally checked that in situations requiring effective quantum transfer a system-environment coupling can become a

resource: an example is provided by efficient quantum-information processing , in which the present thesis is specifically framed. The question is that a set of approximations usually exploited in describing quantum evolution (collectively known as Markovian approximation) are too strong to carefully manage some quantum phenomena. Therefore, in recent years a complete review of the definition of Markovian process is in progress, due to our ever deeper understanding of the multiple aspects of a quantum evolution.

Historically, the fundamental idea of Markovian dynamics was connected to the integro-differential form of the equation describing the evolution which implies that the state of the system at a given time t depends on all the previous instants of time; this has been traditionally interpreted as a "memory" effect, i.e. as the fact that, during the evolution, the system retains memory of its past history. Finding the solution of the problem in this fully general form is highly difficult; so this difficulty has led to describe the evolution through an ordinary differential equation, the master equation in Lindblad form, that neglects the memory aspects of the dynamics. In the framework of this approximation the evolution is known as Markovian evolution. The remarkable accuracy of this equation in describing various systems, above all in quantum optics, has decreed its enormous success.

On the other hand, studying of new systems which are not well described by the Lindblad master equation, as superconducting circuits, photonic band gap materials [1, 2] and light harvesting processes in photosynthesis [3], has led to a renewed interest for non-Markovian dynamics and to the development of various mathematical techniques allowing to describe the evolution in full generality. The scientific community has realized that it is possible to describe dynamics incorporating memory effects even with a master equation local in time; furthermore, the master equation in Lindblad form is not the only equation that can describe memoryless evolutions. Consequently the following problems arise: what do we mean when we say that the dynamics is non-Markovian, and how can we characterize non-Makovianity? This question is known as the *characterization problem*. Different approaches have been proposed, which have led in some cases to inequivalent conclusions. A logically subsequent problem is the possibility to quantify the degree of non-Markovianity of a channel: in other words, how can we quantify the deviation from a Markovian evolution of a given process? This is known as the *quantification problem*. Finally, non-Markovian dynamics are really useful for quantum technologies?

The aim of this thesis is twofold. On the one hand, the characterization and quantification of non-Markovian content for continuous-variable quantum systems; on the other hand, its possible usefulness as a resource in the framework of Quantum Information. The attention is focused mainly on the

class of Gaussian states and Gaussian channels; this choice is motivated by their experimental relevance, and by the advantage to pass from the infinite-dimensional Hilbert space to a finite-dimensional Hilbert space because, in this case, we can exploit the finite-dimensional matrix analysis. However, we consider also some non-Gaussian resources, which are anyway needful for implementing universal quantum computation, an potentially more powerful for all the quantum protocols.

The thesis is organized as follows.

The first Chapter presents a short introduction to the concept of isolated system in Quantum Mechanics and the related concept of decoherence.

Chapter 2 contains a short review of the main formalisms describing the dynamics of open systems, together with the concepts of Markovian dynamics. In Chapter 3 we describe the concepts of non-Markovian dynamics, together with the principal approaches to characterize and quantify its properties.

In Chapter 4 we briefly review the phase-space formulation of Quantum Mechanics. Indeed, this is the best formalism in managing continuous variable systems, and in particular for the description of Gaussian states and Gaussian channels. We also describe the states and channels we use in this thesis in order to highlight the new ideas that will be proposed, and the actual approaches to non-Markovianity for Gaussian channels.

Chapter 5 and Chapter 6 contain the original part of this thesis. In Chapter 5 [4] we introduce a new measure of non-Markovianity for Gaussian channels; the measure is based on the definition of non-Markovianity as violation of the divisibility condition. We provide also a comparison with the equivalent measure (the Rivas-Huelga-Plenio measure) introduced for finite-dimensional systems, and we present some considerations about the possible range of applicability. In the same chapter we finally discuss some simple examples, i.e. the Damping master equation and the Quantum Brownian Motion in the secular approximation, and we apply the new measure to this cases. In Chapter 6 we describe the generalization of the realistic Braunstein-Kimble Continuous Variable Quantum information protocol to the case of non-Markovian evolution [5], namely in the case in which the mode sent to Bob travels through a non-Markovian channel. In fact, we exploit the performance of the teleportation fidelity to quantify the detrimental effects of the environment, with the aim of contrast these effects by maximizing the fidelity with respect to the available resources and parameters. As relevant result, we show that a crucial role in highly improving the performance is played by the phases. This is anyway true if one considers Gaussian resources; furthermore, we study the interplay between the non-Markovianity and another possible key feature of quantum states, the non-Gaussianity, and we show that the combined effect of these two key features, non-Markovian channels

and non-Gaussian resources, allows to obtain a further sensible improvement of the teleportation fidelity.

In Chapter 7 we briefly discuss some recent proposals to measure the non-Markovianity of a quantum evolution.

In the conclusions we sum up and comment the results, and we propose some possible future developments.

With the aim of avoid to introduce inside the chapters cumbersome details, we added also some technical appendices.

Finally, in concluding this introduction we stress that the improvement, shown in the thesis, which is obtained by using non-Markovian channels was not at all obvious "a priori", as not all non-Markovian evolutions influence the state dynamics.

CHAPTER 1

Introduction to decoherence

1.1 The concept of isolated system

To study the laws of Nature, i.e. to grasp the physical law underlying the particular aspect of a phenomenon under scrutiny, it is necessary to devise the experiment in a "controlled environment", i.e. in such a way to minimize every form of disturbance on the desired aspect of the system of interest. For example, suppose an experimenter is interested in measuring the effect of gravity on free falling bodies. As done by Galileo [6] a possibility is to consider a ball rolling down an inclined plane. To arrive at the "exact" laws of motion it is then necessary to minimize all the sources of noise, primarily the friction.

This method of investigation has proven extremely successful and fruitful in the history of physics, leading to the idea that it is always possible to shield our system of interest from unwanted environmental disturbances in a sufficient manner such that to highlight the interesting properties of the system. Consequently, even when the advent of Quantum Mechanics led to a change of perspective in the way of looking at nature, the notion of isolated systems remains accepted without being subjected to a thorough review. On the other hand, in the early days of Quantum Mechanics the tremendous success in explaining some experimental facts, as the discrete spectrum of the hydrogen atom, was based on the model of completely isolated atom. Furthermore the laws of Quantum Mechanics seem to apply only to the microscopic word, in which the systems may be considered isolated to a good approximation.

Notwithstanding these spectacular results, it soon became clear that some consequences of the theory were non-intuitive and "spooky", that is the intrinsic probabilistic nature of the theory (i.e. its non-epistemic character)

and the macro-objectification problem. We will briefly discuss this two problems and how they forced us to completely change our way of thinking about the concept of physical systems (we refer to the literature [7, 8] for further discussion).

1.2 The Einstein-Podolsky-Rosen paradox and the non-local aspect of Quantum Mechanics

Despite the remarkable experimental results of Quantum Mechanics, the theory was not universally accepted by the physics community, due to its unsatisfactory intrinsic probabilistic nature. Indeed the Copenhagen interpretation [8], according to which physical systems, before being measured, do not have definite properties, imposed a new viewpoint about the fundamental nature of reality, namely it required to give up the existence of an objective physical reality in favor of the privileged point of view by the experimenter.

The debate on the epistemic/non-epistemic nature of Quantum Mechanics dates to a seminal paper appeared in 1935 by Einstein, Podolsky and Rosen [9]. Their goal was to reconcile the experimental results with the view of the world arising from classical physics, showing that Nature is deterministic and that its apparent intrinsic probabilistic nature is an artifact of the theory, due to our ignorance of some parameters; namely, Quantum Mechanics is not a *complete theory.*

Following the authors line of reasoning, to show the incompleteness of Quantum Mechanics it is necessary to introduce the following definitions:

- *Element of reality*: if, without disturbing the system, it is possible to predict with certainty the value of a physical quantity, then there is an element of reality corresponding to this quantity (*the hypothesis of realism*).

- *Completeness*: a theory is complete if every element of reality has a counterpart in the theory.

Furthermore special relativity imposes a constraint on an admissible physical theory:

- *Locality*: Any non-local action on a given system is forbidden.

The above assumptions defines the concept of *local realism.* We now show that, under this hypothesis, Quantum mechanics *is not a complete theory*[1]

---

[1]For simplicity, here we present the Bohm version of the EPR argument.

Let us consider the singlet state of two spin 1/2 particles:

$$|\psi_0\rangle = \frac{|\uparrow\rangle\,|\downarrow\rangle - |\downarrow\rangle\,|\uparrow\rangle}{\sqrt{2}}, \tag{1.1}$$

where $|\uparrow\rangle$ and $|\downarrow\rangle$ represent, respectively, a particle with spin up and spin down. Suppose now that the particles are separated and sent to two measurement apparatuses. The two experimenters Alice and Bob, standing the possibility of only local action, can take any measurement on the respective particle; in particular they can measure the spin of their own particle along a generic direction. Let us suppose that Alice measures the z component of the spin of its particle; from Eq. (1.1) follows that we can infer the spin z component of the second particle with certainty without the need to take a measurement. Thus, according to the previous definition, the z component of the second particle is an element of reality.

Now, since the singlet state Eq. (1.1) is invariant under rotation, we can repeat the procedure referring to another direction (as x or y) and obtain that the respective component of the spin of the second particle is an element of reality. However the spin components on different axes are incompatible observable in Quantum Mechanics, they does not commute; consequently the theory cannot be complete, in the sense of EPR argument, since it does not allow a prediction for all elements of reality.

We do not enter in the long debate about the various ways to resolve the EPR paradox, such as the construction, in the sense of EPR paper, of complete theories that contain variables corresponding to all the "elements of reality" (the so called *hidden variables theories*), and whose Quantum Mechanics represents a kind of statistical approximation [10]. It is sufficient to say that, thanks to the theoretical formulation, due to Bell [11, 12, 13], of a way to distinguish between any local hidden variable theory and Quantum Mechanics[2], it was possible to experimentally test the validity of the EPR argument: the results was in agreement with Quantum Mechanics.

We are thus forced to reconsider the concept of locality. The correct way to describe a physical system is through a non-local theory; indeed, though the interaction remains local, the states obtained by this interaction *can be non-local*: the measurement of an observable for one of the subsystems fixes instantaneously[3] the value of the correlated observable of the other subsystem, independently of the distance between the subsystems. The two

---

[2]The reader interested to the Bell's theorem and the other Bell type inequalities can refer to [10].

[3]It can be proven that these results are not in contrast with special relativity. For a proof see [10].

subsystems, loosely speaking, are *entangled*: it no longer make sense to speak of two individual systems, as the physical properties are at least partially encapsulated in the non-local quantum correlations that have been created by the interaction. Citing Schrödinger [14]:

> "When two systems, of which we know the states by their respective representatives, enter into temporary physical interaction due to known forces between them, and when after a time of mutual influence the systems separate again, then they can no longer be described in the same way as before, viz. by endowing each of them with a representative of its own. I would not call that one but rather the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought. By the interaction the two representatives [the quantum states] have become entangled."

Now the following question arises: is the entanglement a property of quantum systems in accordance with the established idealization of isolated systems? Indeed in the classical realm this concept is closely connected with the property of locality.

As an example we consider a small grain of pollen immersed in a liquid. The particle is subjected to a random motion due to the collision with the molecules of the fluid (known as Brownian motion [15, 16]), which act as a source of noise. Nevertheless, during the motion the system is always well identifiable, i.e. the grain of pollen maintains all its physical properties, so we can always identify the physical system, separating it ideally from the surrounding particles (the environment). But in the quantum realm *the same concept of system loses meaning*: the local interaction lead to entanglement between the two subsystems, changing the nature of the objects itself and altering them in a fundamental way.

1.3 The macro-objectification problem and the Schrödinger's cat paradox

The other largely debated foundation problem of Quantum Mechanics is the macro-objectification problem: as a consequence of the linearity of the Schrödinger equation, every macroscopic system in interaction with an entangled system gets entangled as well.

Consider the following measurement scheme discussed for the first time by Von-Neumann [17]. A quantum system $\mathcal{S}$ is in interaction with a measurement apparatus $\mathcal{A}$. Suppose that the apparatus has some indicator whose

$$|\phi>= \alpha|\text{decayed} > |c_{\text{dead}} > +\beta|\text{not decayed} > |c_{\text{alive}} >$$

$$P(\text{alive}) = |\beta|^2$$

$$P(\text{dead}) = |\alpha|^2$$

Figure 1.1: Schematic illustration of the Schrödinger's cat paradox. *Left*: The laws of Quantum Mechanics allow for a superposition of the states corresponding respectively to the two distinct situations, one of which contain an alive cat, the other one a dead cat. *Right*: When the observer opens the box the superposition "collapses" onto either one of the two components. However, this begs the question of the state of the cat before the box was opened.

value is i, represented by the state $|a_i\rangle$, when the system is in the state $|s_i\rangle$. If the instrument is in the initial default state $|a_{\text{init}}\rangle$, the effect of the measurement will be to shift the state of the apparatus in the state corresponding to the i position:

$$|s_i\rangle |a_{\text{init}}\rangle \longrightarrow |s_i\rangle |a_i\rangle . \tag{1.2}$$

We note that this measurement scheme is *ideal*, as we have tacitly assumed that the interaction does not change the state of the system.

Now we face the key point. Consider the case in which the system is in the following superposition of states:

$$|\psi\rangle = \sum_i c_i |s_i\rangle ; \tag{1.3}$$

from the linearity of Schrödinger equation the system state evolves according to:

$$\left(\sum_i c_i |s_1\rangle \right) |a_{\text{init}}\rangle \longrightarrow \sum_i c_i |s_i\rangle |a_i\rangle . \tag{1.4}$$

The last equation shows that the state after the interaction is in general an entangled state, i.e. it is no longer possible to distinguish between the system and the measurement apparatus.

In describing the procedure we do not refer to the "size" of the system (i.e. its microscopic/macroscopic nature), so the argument is still valid when

the measurement apparatus, now in the broader sense of a system in inter-action with the given system, is macroscopic.

A very illuminating example is the Schrödinger's cat paradox [18] Fig. (1.1). Imagine a cat confined to a box. In that box there is an unstable atom that triggers an hammer to break a vial of poison, so killing the cat. The micro-scopic system $\mathcal{S}$ is the decaying atom, that in general lives in a superposition of decayed/non-decayed states:

$$|\psi\rangle = \alpha |\text{decayed}\rangle + \beta |\text{not decayed}\rangle, \qquad (1.5)$$

where α and β are time-independent coefficients, fulfilling the constraint $|\alpha|^2 + |\beta|^2 = 1$. The macroscopic apparatus $\mathcal{A}$ is represented by the cat state[4], in the sense that its vitality is an "indicator" of the state of the atom. If $|c_\text{init}\rangle$ is the initial state of the cat, the measurement scheme returns:

$$|\text{not decayed}\rangle |c_\text{init}\rangle \longrightarrow |\text{not decayed}\rangle |c_\text{alive}\rangle, \qquad (1.6)$$

$$|\text{decayed}\rangle |c_\text{init}\rangle \longrightarrow |\text{decayed}\rangle |c_\text{dead}\rangle. \qquad (1.7)$$

From Eq. (1.5), Eq (1.6) and Eq (1.7) the final atom-cat state will be:

$$|\psi\rangle |c_\text{init}\rangle \longrightarrow \alpha |\text{decayed}\rangle |c_\text{dead}\rangle + \beta |\text{not decayed}\rangle |c_\text{alive}\rangle, \qquad (1.8)$$

namely, it is not possible to assign a individual quantum state to the cat.

So is seems that *we should observe quantum properties even at the macro-scopic level*. Why is not this the case? Indeed, when an external observer opens the box, he does not observe the superposition but only the situation represented by the states $|c_\text{alive}\rangle$ and $|c_\text{dead}\rangle$ respectively with probability $|\beta|^2$ and $|\alpha|^2$.

A first answer can be to postulate the existence of intrinsically classical mea-surement apparatuses not subject to the laws of quantum mechanics, i.e. to split the word into a microscopic world that follows Quantum Mechanics, and a macroscopic one, described by the Classical Mechanics (substantially the Copenhagen interpretation of quantum mechanics). This working hy-pothesis, indeed useful for practical calculation, is not satisfactory from a conceptual point of view. First of all it is not clear what we have to consider as "macroscopic": how many particles a body must have to be considered a macroscopic object? Furthermore this situation does not account for the macroscopic systems that exhibit quantum properties, such as superconduc-tors.

---

[4]Strictly speaking the system is represented by the hammer, the vial and the cat. Due to the von-Neumann chain, for simplicity, we consider this state uniquely as the cat state.

Indeed, instead of introducing the ad hoc hypothesis to distinguish between microscopic/macroscopic objects, it is reasonable that the measurement procedure requires some process breaking the system/apparatus entanglement. In the next paragraph we discuss a possible solution to this problem.

1.4 Introduction to decoherence and the appearance of the classical world

In Sec. (1.2) we have introduced the most distinguishing characteristic of Quantum Mechanics, namely the property of quantum systems to lose their individuality and become entangled; furthermore in Sec. (1.3) we have shown how the linearity of Quantum Mechanics allows to "transmit" this quantum property between systems.

We now discuss qualitatively, in the particular case of the Collisional Model, how the appearance of the classical world, a phenomenon known as *decoherence*, can be explained as a consequence of these distinctive aspects of Quantum Mechanics [7].

Consider as a system a body subject to incident light (i.e. photons), Fig. (1.2). In the classical situation the motion of the body is only negligibly influenced, as the amount of momentum transferred by the photons is very small; furthermore the incident light is distributed isotropically, i.e. the average of the momentum is zero.

In the Quantum realm the situation is completely different. Now the interaction between the body and the incident photons causes the formation of quantum correlation between the objects which "carry away" coherence from the body, diminishing in this way its degree of "quantumness"; i.e. the body is subject to decoherence. This also does not depend on the momentum transferred: the body can also remain unperturbed classically. Hence the decoherence can be thought of as a continuous monitoring process of the system by the environment; in the next Chapter we will see effectively how the environment performs non-demolition measurements on the system.

Hence, next to the classical process of dissipation (namely the loss of energy from the system) the body is subject to a pure Quantum decoherence effect.

The following considerations arise immediately. In principle, if we are sufficiently able to isolate the system of interest, we can avoid the decoherence effect due to the environmental interaction. However in practice it is impossible to shield the system completely: even in extreme conditions such as for example deep-vacuum the density of air molecules is sufficient to cause

Classical setting   Quantum setting
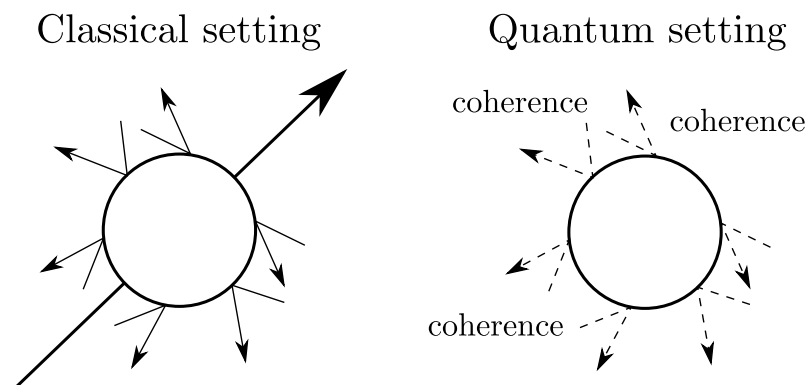
Figure 1.2: The influence of the environment in the Classical and Quantum Settings. *Left*: In the classical case, even if the system-environment interaction is strong, the motion of the system is influenced only in a negligible way. *Right*: The effect of the interaction causes the formation of an entangled object–photon state; the coherence becomes de-localized from the object, making quantum effects unobservable at the level of the system.

the decoherence; furthermore, the inscapable influence of some environments, as microwave background radiation, is already sufficient to cause a fast decoherence. Hence, in the vast majority of cases, *a realistic description of a quantum system requires necessarily to take into account the influence of the environment to the system evolution.*
Another important point are the characteristic time-scales that rule the dynamics. Between two interacting systems, an energy exchange will take place up reaching a thermal equilibrium. The characteristic time necessary to reach the equilibrium, known as *relaxation time*, depends on the system-environment interaction: the weaker the interaction, the greater the time required to reach equilibrium. Furthermore, depending on the particular type of interaction, it may be that the dissipation is negligible. However, even if the interaction does not cause dissipation, *it still always cause decoherence*. It can hence be possible to have decoherence without dissipation, but not vice versa. If both are present, theoretical and experimental studies have shown that the characteristic time-scale of the decoherence is many orders of magnitude shorter than the relaxation time-scale. Zurek [19] gave the first estimation of the decoherence time-scale for a coherent superposition of two different objects with mass m and spatial separation Δx:

$$\frac{\tau_R}{\tau_D} \sim \left(\frac{\Delta x}{\lambda_{dB}}\right)^2, \tag{1.9}$$

where λ_{dB} is the de Broglie wavelength of the object:

$$\lambda_{dB} = \frac{\hbar}{\sqrt{2mk_BT}},\qquad(1.10)$$

T is the temperature of the object and k_B is the Boltzmann constant. For a macroscopic object Eq. (1.9) shows that in the vast majority of cases[5] the decoherence time-scale is completely negligible. For example, for a mass $m = 1g$ at room temperature $T = 300K$ the de Broglie wavelength is $\lambda_{dB} \approx 10^{-23}m$; consequently, for a macroscopic separation for $\Delta x = 1cm$ Eq. (1.9) returns $\tau_R/\tau_D \sim 10^{40}$, that is the decoherence time-scale is orders of magnitude shorter than the relaxation time-scale.

Hence the entanglement is ubiquitous especially in the macroscopic domain. Furthermore, except in particularly controlled environments, *the decoherence is virtually instantaneous*. This aspect shows that it is very difficult to manipulate and exploit entanglement and other quantum properties of the systems in such a controlled way to be used as a technology resource. Historically, this perspective paved the way to a deeper understanding of Quantum Mechanics and of the open quantum systems dynamics.

1.5 Quantum Information science

The first scientist that highlights the possibility to use Quantum-mechanical systems to simulate physical systems was Richard Feynman [20]; in a seminal paper of 1982 he argued that "nature isn't classical, dammint, and if you want to make a simulation of nature you'd better make it quantum mechanical". In the same year Bienoff [21] shows that every Quantum-mechanical system can model Turing machine, proving that a quantum computer is powerful at least as the classical one. In the eighties, a series of paper [22, 23] showed the power of Quantum Mechanics in processing and transmitting information; furthermore, it was shown that a Universal Quantum Computer can do things that the Universal Turing machine cannot do, as generate genuinely random numbers or perform some parallel calculations in a single register.

Later, a series of discoveries [24, 25, 26, 27] aroused great expectations from the scientific community. First of all, the definition of a model of Computation in the mid-ninenties through the discovery of the sets of quantum gates necessary for quantum computing. Furthermore, in 1984 H. Bennett

---

[5]It is important to note that, in particular controlled environment, even a macroscopic object must be treated as quantum; an example is the Weber bar, a macroscopic aluminium cylinders, used for the detection of gravitational waves, maintained at cryogenic temperature and with $\Delta x = 10^{-17}cm$.

and Gilles Brassard [28] introduced a cryptographic method, the "Quantum Key Distribution", that permits to encrypt messages in such a way that an eavesdropper cannot intercept messages undetected, regardless of computational resources; and in the same year David Deutsch and Richard Jozsa [29] discovered the first quantum algorithm that determines whether a function f is constant over all inputs (i.e., either equal to 1 for all x or equal to 0 for all x), or balanced (namely, equal to 1 for half of the values of x and equal to 0 for the other half), and that runs faster than its classical counterparts. So, they showed the great technological possibility of the new field of Quantum Information.

As a consequence of these enormous successes it soon became clear that the development of Quantum technologies needed, on the one hand, a better understanding of the "quantumness" of the systems that can be exploited to obtain an advantage with respect to the classical physics; on the other hand, an efficient protection of these quantum properties from errors caused by channel noise, and in particular from decoherence. As discussed by Di Vincenzo in a famous paper [30], to obtain a physical implementation of quantum computation it is essential to have long decoherence times, sufficient to perform Quantum operation.

Consequently, a great effort of the scientific community was directed to contrast the decoherence through the exploitation of new techniques, such as the Quantum error correction [31] and the decoherence-free subspaces[6] [32]. In the last years a deeper understanding of the open system dynamics has led to a change of paradigm: the system-environment interaction *can be a resource* to be exploited in quantum technologies, as we shall discuss later.

Moreover, a complete description of the dynamics shows that the process of decoherence, through the information flow between the system and the environment, *is not a one way process*: during the evolution, the quantum properties can be in part recovered *through a back flow of information from the environment to the system*. Due to this aspect, theoretical and experimental analysis have shown that, in some cases, the effect of the system-environment interaction *can assist* quantum processes, as in some biological systems [33, 34, 35]. Furthermore, the possibility to experimentally control the system-environment interaction [36, 37] can pave the way to *an environment assisted* Quantum Information science, namely the system-environment interaction *can be a resource* for the Quantum Information technologies.

Great effort from the physics community has been spent in the last years to characterize this property, i.e. the *non-Markovianity*, of the dynamics.

---

[6]We do not discuss these strategies and we remand to the literature for further investigation.

In the next Chapters we will see how this aspect is related to the memory effect of the dynamics, i.e. the fact that the system retains memory of its past history during the evolution, and how at present it is characterized and quantified.

CHAPTER 2

Open systems dynamics

Before discussing in detail the concept of non-Markovianity of a quantum evolution and its implication in Quantum Information science, it is useful to review some basic principles and tools used for the description of open systems dynamics. The scientific community devoted great efforts to the study of non-Markovianity mainly in the case of finite dimensional systems, that we discuss in this section.

2.1 Dynamics of Open Systems

Following the principles of Quantum Mechanics, an isolated quantum system is described at a given reference time $t = t_0$ by a vector $|\psi(t_0)\rangle \in \mathcal{H}$ in an Hilbert space, and its evolution in time is determined by the Schrödinger equation [38]:

$$\imath\hbar\frac{\mathrm{d}}{\mathrm{dt}}|\psi(t)\rangle = H|\psi(t_0)\rangle.$$

The evolution generated by this equation is unitary:

$$|\psi(t)\rangle = U(t, t_0)|\psi(t_0)\rangle, \qquad U^\dagger(t, t_0) = U^{-1}(t, t_0);$$

in particular, when the Hamiltonian which generates the dynamics is time-independent, the unitary evolution takes the form $U(t, t_0) = \exp(-\imath H(t - t_0)/\hbar)$.

In more general situations we lack a complete control on the preparation of the initial state of the system S, and we can associate to the system only a mixture of possible initial states $(\{p_i\}, \{|\psi_i\rangle\}), i = 1, \ldots, n$, where p_i is the probability of the system being in the $|\psi_i\rangle$ state. In this instance a more general framework is needed: the system must be described by a density

operator $\rho_S : \mathcal{H} \to \mathcal{H}$, whose spectral decomposition is given by:

$$\rho_S = \sum_i p_i \left| \psi_i \right\rangle \left\langle \psi_i \right|, \qquad \sum_i \mathrm{p_i} = 1, \qquad \mathrm{p_i} \geq 0. \tag{2.1}$$

The knowledge of the density operator is sufficient to obtain all information about a physical system. In fact, the expectation values of a generic operator, in particular of an observable, is given by the relation:

$$\langle A \rangle = \mathrm{Tr}\left[A\rho \right]. \tag{2.2}$$

The properties of the set of probabilies $\{p_i\}$ imply that ρ must be positive and trace-preserving:

$$\rho_S \geq 0, \qquad \mathrm{tr}\left[\rho_S\right] = 1;$$

furthermore, when one probability is one, i.e. when $p_j = 1$ for some index j, the state of the system must be of necessity the vector $|\psi_j\rangle$, and the density formalism is reduced to the Schrödinger vectorial description. We denote with $\mathcal{D}(\rho)$ the set of the density operators.

The dynamic equation describing the time evolution of the density operator is the von Neumann equation:

$$\frac{\mathrm{d}}{\mathrm{d}t} \rho\left(t\right) = -\frac{\imath}{\hbar} \left[H\left(t\right), \rho \right], \tag{2.3}$$

and even in this case the evolution is unitary:

$$\rho\left(t\right) = U^\dagger\left(t, t_0\right) \rho\left(t_0\right) U\left(t, t_0\right).$$

Let us suppose now that the system S under scrutiny is not isolated but is in interaction with a second system E, the *environment* (Fig. (2.1)). The most general system-environment model can be described by the following Hamiltonian structure:

$$\mathcal{H} = \mathcal{H}_S + \mathcal{H}_E + \mathcal{H}_I,$$

where $\mathcal{H}_S$ is the free Hamiltonian of the system, $\mathcal{H}_S$ is the free Hamiltonian of the environment, and $\mathcal{H}_I$ is the interaction term. The goal is to describe the evolution of the system taking into account the mutual influence between the system and the environment. Indeed, as discussed in the previous Chapter, in a realistic situation a quantum system is never isolated; the complete description of all the details of the composite system may be too complicated or uninteresting, or it may happen that the modes of E are not accessible. However, we are really interested in effectively describing the evolution of the system S and, from Eq. (2.2), this kind of information is

clearly encoded in the density operator ρ_S; therefore, the problem is how to obtain the detectable physical content of its evolution.

Various mathematical formulations have been introduced to describe open quantum systems, and in the subsequent sections we will discuss the most important techniques. Before to further proceed, we establish that, unless otherwise stated, from now on we will choose the initial system-environment state as a product state:

$$\rho(t_0) = \rho_S(t_0) \otimes \rho_E(t_0). \tag{2.4}$$

2.1.1 Dynamical map

In the previous Section we have shown that, in the case of an isolated system, there exists an unitary operator such that the density operator at time t is given by $\rho(t) = U^\dagger \rho(0) U$. For not isolated systems we want to define, by analogy, a superoperator $\Phi(t, t_0)$, known as *dynamical map*, such that:

$$\rho_S(t) = \Phi(t, t_0) \rho_S(t_0). \tag{2.5}$$

By construction this superoperator must to map density operators into density operators and it must preserve superpositions, i.e. it must satisfy:

$$\begin{aligned}
&\Phi(t, t_0) \rho = \tilde{\rho} \in \mathcal{D}(\mathcal{H}), \\
&\Phi(t, t_0) [\lambda \rho_1 + (1 - \lambda) \rho_2] = \lambda \Phi(t, t_0) \rho_1 + (1 - \lambda) \Phi(t, t_0) \rho_2;
\end{aligned} \tag{2.6}$$

for every $t \geq 0$, $0 \leq \lambda \leq 1$ and $\rho, \rho_1, \rho_2 \in \mathcal{D}(\mathcal{H})$. Furthermore it must as well preserve the trace of the density operator:

$$\mathrm{Tr}\left[\Phi(t, t_0) \rho_S(t_0)\right] = 1. \tag{2.7}$$

The first condition in Eq. (2.6) is called the positivity condition: a map is positive if it sends positive states into positive states; however, we require a stronger property, the *complete positivity*. Consider the situation in which we extend the map in an Hilbert space $\tilde{\mathcal{H}}$ of arbitrary dimensions by introducing the extended map as the direct product of the original map with a unit matrix of arbitrary dimension: $\Phi(t, t_0) \rightarrow \Phi(t, t_0) \otimes \mathbb{1}$. The requirement of complete positivity means that even the combined operation is a positive map. This is a reasonable condition from a physical point of view, because the combined map may be viewed as a map that operates locally on the first of two spatially separated systems, without thus influencing the second system.
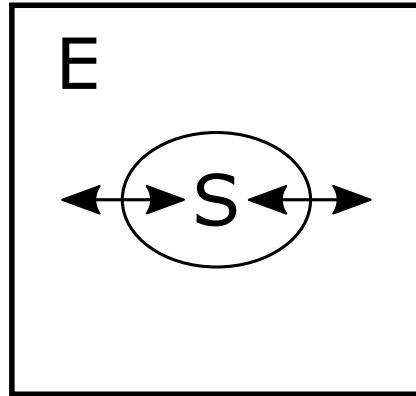
Figure 2.1: Schematic picture of an open quantum system.

$$\rho\left(0\right) \xrightarrow{\text{Unitary evolution}} \rho\left(t\right)$$

$$\downarrow \text{Tr}_{\text{E}} \qquad\qquad\qquad \downarrow \text{Tr}_{\text{E}}$$

$$\rho_S\left(0\right) \xrightarrow{\text{dynamical map}} \rho_S\left(t\right)$$

Figure 2.2: Commutative diagram showing the construction of the dynamical map.

Intuitively, since we know how to describe the isolated system made up by the system and the environment, we can start from Eq. (2.1)

$$\rho(t) = U_{SE}^{\dagger}(t, t_0)\, \rho(t_0)\, U_{SE}(t, t_0), \qquad (2.8)$$

and, by applying the partial trace rule, we can neglect all the degrees of freedom which are not interesting for us; so, we obtain the density operator which provides an effective description of the system of our interest (see Fig. (2.2)):

$$\rho_S(t) = \mathrm{Tr}_E\left[U_{SE}^{\dagger}(t, t_0)\, \rho(t_0)\, U_{SE}(t, t_0)\right] = \Phi(t, t_0)\, \rho_S(t_0). \qquad (2.9)$$

Thus, a completely positive evolution can be seen as the reduced dynamics of some unitary evolution which acts on an extended state of the form $\rho_S \otimes \rho_E$, where ρ_E remains unchanged independently of ρ_S.

A more explicit form of the dynamical map can be obtained [38, 7]. Consider the diagonal decomposition of the initial state of the environment $\rho_E(t_0) = \sum_i p_i |E_i\rangle\langle E_i|$, where the vectors $|E_i\rangle$ form an orthonormal basis in the Hilbert space of the environment and $\sum_i p_i = 1$. If one considers the hypothesis that the Hamiltonian which describes the system-plus-environment evolution is time-independent, using Eq. (2.9) and the form of the initial state Eq. (2.4) it is straightforward to obtain for the density ρ_S the expression:

$$\rho_S(t) = \sum_{ij} p_i \langle E_j|U(t)|E_i\rangle\, \rho_S(t_0)\, \langle E_i|U^{\dagger}(t)|E_j\rangle\,.$$

By introducing the *Kraus operators* $W_{ij} := \sqrt{p_i}\,\langle E_j|U(t)|E_i\rangle$, the last equation can be recast in the form:

$$\rho_S(t) = \sum_{ij} W_{ij}\rho_S(t_0)W_{ij}^{\dagger}. \qquad (2.10)$$

It is customary to rewrite the Kraus operators with only one index in the form $W_k := \sqrt{p_i}\,\langle E_{j_k}|U(t)|E_{i_k}\rangle$; consequently the density matrix of the system can be expressed as

$$\rho_S(t) = \sum_k W_k\rho_S(t_0)W_k^{\dagger}. \qquad (2.11)$$

Furthermore, since the global evolution is unitary, the Kraus operators satisfy the completeness constraint:

$$\sum_k W_k W_k^{\dagger} = I_S. \qquad (2.12)$$

Eq. (2.11) and Eq. (2.12) show that the Kraus operators are the generators of the dynamical map defined in Eq. (2.9); they describe the effect of the environment as a sequence of (generally non-unitary) transformations of the density matrix of the system generated by the operators W_k.

We now want to motivate, in a more formal way, the description of the decoherence process in terms of environmental monitoring.

We denote by M a projective measurement on the environment, whose projectors are $P_\alpha = |\alpha\rangle\langle\alpha|$, with $P_\alpha^\dagger = P_\alpha = P_\alpha^2$, and where the set $\{\alpha\}$ is made up by the eigenvalues of M. Let us suppose that the measurement process returns the outcome α; from the postulates of Quantum Mechanics, the resulting density matrix will be:

$$\rho_S^{(\alpha)}(t) = \frac{\text{Tr}_E\left[(I \otimes P_\alpha)\rho(t)(I \otimes P_\alpha)\right]}{\Pr(\alpha|\rho_s(t))}, \tag{2.13}$$

where $\Pr(\alpha|\rho_s(t)) = \text{Tr}(P_\alpha \rho_E(t))$ is the probability of obtaining the outcome α. By exploiting the diagonal decomposition of the initial state of the environment, Eq. (2.8) and Eq. (2.4), the last equation becomes:

$$\rho_S^{(\alpha)}(t) = \sum_k \frac{M_{\alpha,k}\rho_S(t)M_{\alpha,k}^\dagger}{\Pr(\alpha|\rho_s(t))}, \tag{2.14}$$

where we introduced the measurement operator:

$$M_{\alpha,k} := \sqrt{p_k}\,\langle\alpha|U(t)|E_k\rangle, \qquad \sum_{\alpha,k} M_{\alpha,k}M_{\alpha,k}^\dagger = I_S. \tag{2.15}$$

As we are not interested in the result of the measurement, the system will be described by a density operator obtained through a weighted sum over all the possible states $\rho_S^{(\alpha)}(t)$, with weights given by respective probabilities:

$$\rho_S(t) = \sum_\alpha \Pr(\alpha|\rho_s(t))\rho_S^{(\alpha)}(t) = \sum_{\alpha,k} M_{\alpha,k}\rho_S(t)M_{\alpha,k}^\dagger. \tag{2.16}$$

This expression is formally analogous to Eq. (2.10); furthermore, in the decoherence process we do not read out the environment, i.e. we do not inquire about the result of the indirect measurement. Hence the decoherence process can be viewed as a monitoring of the system carried out by the environment.

2.1.2 Master equation

In the previous section we have illustrated how, starting from an unitary evolution, the open system dynamics can be described by Eq. (2.9). This

approach inevitably requires the knowledge of the dynamics of the global system $\{\mathcal{S}, \mathcal{E}\}$ before we can obtain, through the trace operation, the reduced description for the system $\mathcal{S}$. This task is not practically achievable for systems which are reasonably more complex. On the other hand, as we have already noticed in the previous paragraph, the dynamics of the environment or of the global system is not important. The really important aspect is the influence of the environment on the system of interest.

We will now introduce another way to describe the decoherence dynamics in terms of so-called *master equations*. Such equations directly yield the time evolution of the reduced density matrix $\rho_S(t)$, because they allow us to calculate $\rho_S(t)$ directly from Eq. (2.9):

$$\rho_S(t) = \Phi\left(t, t_0\right) \rho_S\left(t_0\right).$$

Various methods have been developed to obtain a valid master equation, e.g. influence functional methods [39], projectors operators [40, 41], time convolutionless techniques [42, 43, 44]. As the system evolution may be quite complicated, the equation is, in general, an integro-differential equation. A quite general example is the Nakajima-Zwanzing equation [42, 43, 44], whose only significant assumption is the initial factorizing condition. Since we are only interested in the system dynamics, we can define a superoperator $\mathcal{P}$ that projects on the relevant part of the dynamics, in the sense that, having the total density matrix $\rho(t)$, it allows to reconstruct the density operator ρ_S of the system under scrutiny: $\mathcal{P}\rho(t) = \mathrm{Tr}_E[\rho(t)] \otimes \rho_E = \rho_S \otimes \rho_E$. It can be proven that the master equation describing the evolution of the projected state is:

$$\frac{\partial}{\partial t}\mathcal{P}\rho = \int_{t_0}^{t} \mathrm{d}s \, \mathcal{K}\left(t, s\right) \mathcal{P}\rho\left(s\right),$$

where $\mathcal{K}(t, s)$, called the convolution kernel, depends on the total Hamiltonan and on the state of the environment. As this is an integro-differential equation, the solution at time t depends on the history of the state evolution.

This dependence of the state of the system from the whole dynamical evolution can be intuitively interpreted saying that the system "remember" its past history. At the end of the previous Chapter, we have briefly pointed how this memory effect, the non-Markovianity of the evolution, is connected with a back-flow of information from the environment to the system, how this is necessary to describe some physical systems, and how this property can be useful for the Quantum Information science. Despite its importance, the difficulties to face this problem in full generality has led historically to neglect this memory effect by exploiting a series of approximations. The resulting evolution is called *Markovian*.

In the next Section we will describe this situation both from an historical perspective and from the modern point of view. For our aim, its importance lies in the fact that the various characterizations of the more general non-Markovian evolution will be defined as the violation of the properties of the Markovian one.

2.2 Markovian dynamics

The mathematical difficulties deriving by the solution in full generality of the problem of open system dynamics have led to consider a series of approximations that, roughly speaking, consist in completely neglecting any memory effect (Markovian approximation). The system is now described by a simple ODE (see next Section), and the relatively simple structure of the master equation obtained in this approximation, added to the greatly accurate way in the description of systems in quantum optics, has decreed its tremendous success. Nowadays the definition of Markovian dynamic is under discussion: indeed, if we want to assume the memoryless dynamics as the primary definition, the usual approach with the master equation in the Lindblad form, Eq. (2.19), is not the only possible.

We will briefly review the various approaches. For a deeper analysis we refer to the literature [45, 46].

2.2.1 Markovianity through semigroup evolution

Historically the absence of memory effects is associated with the semigroup property of the dynamical map [38]:

$$\Phi\left(t_1, 0\right)\Phi\left(t_2, 0\right) = \Phi\left(t_1 + t_2, 0\right). \tag{2.17}$$

Under proper mathematical conditions, there exists a time-independent generator $\mathcal{L}$ such that:

$$\Phi\left(t, t_0\right) = e^{\mathcal{L}(t-t_0)}.$$

It is possible to construct the most general form of the generator by starting only from this condition. It is however possible to obtain a second, more physical, derivation that starts from the von-Neumann equation (2.3). Even if we do not present in detail this construction, it is interesting to discuss the various approximations needed to obtain the form of the generator, because this allows us to have a reference with which the non-Markovian property of the master equations that we present in Chapter 4 can be compared.

Generally, it is possible to identify three time scales of interest that describe the evolution. The first is the characteristic time scale of the system

τ_S that, for an harmonic oscillator, is the inverse of the frequency, $\tau_S \simeq 1/\omega_0$; another one is the correlation time scale of the environment τ_E, that represents the characteristic time of the decay correlation function of the reservoir; and, finally, it can be also defined the relaxation time scale τ_R, that represents the characteristic time along which the system varies appreciably. The usually assumed approximations, listed below, introduce some hypotheses on these characteristic time scales.

- *Born approximation.* As we showed in Sec. (2.1) the most general quantum evolution takes into account a system-environment initial state of generic form. We assume instead that the system-environment initial state is of the factorized form $\rho(t_0) = \rho_S(t_0) \otimes \rho_E(t_0)$. This feature is not a specific characteristic of Markovian evolution: all the non-Markovian evolutions we will consider satisfy this property. Furthermore we assume that the coupling of the system with the environment is weak. Under these hypotheses the system dynamics does not affect the environment so that, for every instant of time, $\rho(t) \simeq \rho_S(t) \otimes \rho_E$, where ρ_E is the stationary state of the environment.

- *Markov approximation.* Even after the decoupling ($\rho(t) = \rho_S(t) \otimes \rho_E$) the state of the system at a given time depends by all previous instants of time. Furthermore, in general the dynamics depends on the initial state of the system. The Markov approximation imposes a condition on the time scales of the system-environment dynamics, given by $\tau_E \ll \tau_R$. In this way the dynamics becomes "local in time", i.e. the state of the system at time t depends only by this instant of time.

- *Secular approximation.* Even if, by assuming the previous two approximations, the dynamics becomes local in time, the dynamics is not yet a semigroup dynamics. The secular approximation consists in assuming the condition $\tau_S \ll \tau_R$ that allows to neglect the rapidly oscillating terms appearing in the master equation.

Under these hypotheses it is possible to show that the dynamics is a semigroup dynamics and that the generator has the form [47, 48, 49]:

$$\mathcal{L}\rho = -\imath \left[H\rho \right] + \sum_k \gamma_k \left[V_k \rho V_k^\dagger - \frac{1}{2} \left\{ V_k^\dagger V_k, \rho \right\} \right], \qquad (2.18)$$

where $\{\gamma_k \geq 0\}$ are the decay rates and V_k are the Lindblad operators.

The corresponding master equation:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \mathcal{L}\rho, \qquad (2.19)$$

with $\mathcal{L}$ given by Eq. (2.18), is called Markovian. The absence of memory effects is associated with the form local in time of the master equation Eq. (2.19). This equation will be useful as a reference dynamics in the study of non-Markovianity.

2.2.2 Markovanity through divisibility condition

Due to the semigroup property of the dynamical map, Eq. (2.17), the master equation Eq. (2.19) describes a memoryless evolution. Nevertheless, it is possible to obtain an evolution without memory even when the condition (2.17) is not satisfied.
Consider the following master equation local in time:

$$\frac{d}{dt}\rho_S(t) = \mathcal{K}(t)\rho(t), \tag{2.20}$$

where $\mathcal{K}(t)$ is the time-dependent generator. With the help of the chronological time-ordering operator T it is possible to define the following family of Completely Positive and Trace Preserving (CPTP) maps:

$$\Phi(t_2, t_1) = T \exp\left[\int_{t_1}^{t_2} dt' \mathcal{K}(t')\right].$$

From this definition it is easy to show that this dynamical map satisfies the following property:

$$\Phi(t_3, t_1) = \Phi(t_3, t_2)\,\Phi(t_2, t_1), \qquad \forall\, t_3 \geq t_2 \geq t_1. \tag{2.21}$$

Eq. (2.21) expresses the *divisibility property*; indeed the possibility to subdivide the dynamics in two separated evolutions is a clear signature of a memoryless dynamics.
It is important to remark that the condition (2.21) is more general than the semigroup property, Eq. (2.17), to which it reduces when $\Phi(t_2, t_1) = \Phi(t_2 - t_1)$.

The definition expressed by Eq. (2.21) can be justified in terms of the analogue classical concept. Consider a classical stochastic process, i.e. a family of random variables $\{X(t),\ t \in I \subset \mathbb{R}\}$ that take values on a finite set denoted by $\mathcal{X}$. It is by definition a *Markovian process* [50, 51] if and only if the probability that the random variable X takes the value x_n at time t_n depends only on the value x_{n-1} that it assumes at the previous instant of time t_{n-1}

$$\mathbb{P}(x_n, t_n | x_{n-1}, t_{n-1}, \ldots, x_0.t_0) = \mathbb{P}(x_n, t_n | x_{n-1}, t_{n-1}), \qquad \forall t_n \geq \ldots \geq t_0; \tag{2.22}$$

this condition represents exactly the concept of memoryless evolution. A direct consequence of this property is the Chapman-Kolmogorov equation, obtained directly from the definition of conditional probability: for every $t_3 > t_2 > t_1$:

$$\mathbb{P}\left(x_3, t_3 | x_1, t_1\right) = \sum_{x_2 \in \mathcal{X}} \mathbb{P}(x_3, t_3 | x_2, t_2) \mathbb{P}\left(x_2, t_2 | x_1, t_1\right). \qquad (2.23)$$

Consider now the stochastic evolution of one-point probabilities $\mathbb{P}(x, t)$. The matrix which connects the probability at different instants of time t_0 and t_1

$$\mathbb{P}\left(x_1, t_1\right) = \sum_{x_0 \in \mathcal{X}} T\left(x_1, t_1 | x_0, t_0\right) \mathbb{P}(x_0, t_0)$$

must satisfy the following conditions:

$$\sum_{x_1 \in \mathcal{X}} T\left(x_1, t_1 | x_0, t_0\right) = 1,$$
$$T\left(x_1, t_1 | x_0, t_0\right) \geq 0, \quad x_1, x_0 \in \mathcal{X}, \qquad (2.24)$$

as directly follows from the relations $\sum_{x_1 \in \mathcal{X}} \mathbb{P}(x_1, t_1) = 1$ and $\mathbb{P}(x_1, t_1) \geq 0$. Matrices satisfying these properties are called *stochastic matrices*.

If t_0 is the initial instant of time of the stochastic process, then $\mathbb{P}(x_2, t_2) = \sum_{x_0 \in \mathcal{X}} \mathbb{P}(x_2, t_2 | x_0, t_0) \mathbb{P}(x_0, t_0)$; as a consequence we have $T(x_2, t_2 | x_0, t_0) = \mathbb{P}(x_2, t_2 | x_0, t_0)$. Furthermore, for a Markovian process even the intermediate evolution is defined as $T(x_2, t_2 | x_1, t_1) = \mathbb{P}(x_2, t_2 | x_1, t_1)$, as follows from the definition (2.22); finally, using Eq. (2.23) it is straightforward to obtain the following further condition:

$$T\left(x_3, t_3 | x_1, t_1\right) = \sum_{x_2 \in \mathcal{X}} T\left(x_3, t_3 | x_2, t_2\right) T\left(x_2, t_2 | x_1, t_1\right). \qquad (2.25)$$

A process which satisfies Eq. (2.24) and Eq. (2.25) is called *divisible*. Thus *Markovianity condition and divisibility condition are equivalent*.

We now return to the Quantum case. The problem of defining a Quantum Markovian process as in Eq. (2.22) is more subtle: at variance with the classical process, sample a Quantum system means make a measurement that disturbs the state of the system and affects the subsequent outcomes. Consequently, the quantity $\mathbb{P}\left(x_n, t_n | x_{n-1}, t_{n-1}, \ldots, x_0.t_0\right)$ depends also on the measurement process.

However, we can focus only on the one-time probabilities $\mathbb{P}(x, t)$ that do not depend on the measurement process; as a consequence, even in the Quantum case the Markovian character of the dynamics is equivalent to the

concept of divisibility, and a very nice property is that divisibility may be defined in the quantum case without referring to measurement processes.

Let us now consider the spectral decomposition (2.1) of the density matrix at a reference time t_0:

$$\rho\left(t_0\right) = \sum_x p\left(x, t_0\right) \left|\Psi\left(x\right)\right\rangle \left\langle\Psi\left(x\right)\right|,$$

where the set of eigenvalues $p(x, t_0)$ forms a classical probability distribution. If the evolution preserves the spectral decomposition:

$$\rho(t_0) \rightarrow \rho\left(t\right) = \sum_x p\left(x, t\right) \left|\Psi\left(x\right)\right\rangle \left\langle\Psi\left(x\right)\right|, \qquad (2.26)$$

we can consider this process as a classical stochastic process for the x variable; in particular we consider this process divisible if and only if $p(x,t)$ satisfies the divisibility condition, namely if the transition matrices defined by

$$p\left(x_1, t_1\right) = \sum_{x_0 \in \mathcal{X}} T\left(x_1, t_1 | x_0, t_0\right) p\left(x_0, t_0\right) \qquad (2.27)$$

satisfy Eq. (2.24) and Eq. (2.25). As illustrated in Sec (2.1) the quantum evolution can be described by a linear dynamical map $\rho(t_1) = \Phi(t_1, t_0)\rho(t_0)$ that preserves the positivity of the state and the trace; furthermore, applying the map to a density operator, from Eq. (2.27) we obtain:

$$\Phi\left(t_1, t_0\right) \rho\left(t_0\right) = \sum_{x_1, x_0 \in \mathcal{X}} T\left(x_1, t_1 | x_0, t_0\right) p\left(x_0, t_0\right) \left|\psi\left(x_1\right)\right\rangle \left\langle\psi\left(x_1\right)\right|.$$

From the conditions Eq. (2.24) and Eq. (2.25) we finally obtain the composition law (2.21).

We are now able to formulate the definition of a positive divisible (P-divisible) process: *a quantum evolution is P-divisible if, $\forall t_1, t_2$, $\Phi(t_2, t_1)$ is a positive map and satisfies Eq. (2.21).* On the other hand, as described in Sec (2.2.2), the P-divisibility is not sufficient to preserve the positivity of the density matrices if the selected system is in contact with an ancilla system; consequently we add a more strong property, the Complete Positivity (CP) condition, so arriving at the following definition of quantum Markovianity: *a quantum evolution is Markovian* [52] *if it is described by a trace-preserving complete positive map satisfying the divisibility condition Eq. (2.21).*

Finally, we want to characterize the divisibility property in terms of master equation. For quantum Markovian processes for which the limit

$\mathcal{K}(t) := \lim_{\epsilon \to o^+} (\Phi(t + \epsilon, t) - \mathbb{1})/\epsilon$ is well defined, it is possible to characterize the Markovian evolution through the form of the generator (Gorini-Kossakowski-Sudarshan-Lindblad theorem): the operator $\mathcal{K}(t)$ is a generator of a Markovian dynamic if and only if it can be written in the form:

$$\mathcal{K}(t)\rho = -\imath \left[H(t)\rho(t) \right] + \tag{2.28}$$
$$\sum_k \gamma_k(t) \left[V_k(t)\rho(t) V_k^\dagger(t) - \frac{1}{2} \left\{ V_k^\dagger(t) V_k(t), \rho(t) \right\} \right],$$

where $H(t)$ is the Hamiltonian operator and $\gamma(t) \geq 0$ for every k and t. This is the most general form of the generator appearing in Eq. (2.20) [53, 54].

From now on *we assume the divisibility property as definition of Markovian evolution.*

2.2.3 Markovianity through the contractive property

In the previous paragraph we have shown how the concept of memoryless dynamics is connected with the divisibility property. In addition to the definition, a divisible process possesses another remarkable feature that will be useful in the characterization of the dynamics [55, 56]. Furthermore this property allows to connect the abstract concept of divisibility with the more intuitive concept of the information flow between the system and the environment.

As before, we start by discussing the case of a classical stochastic process. The following theorem allows to characterize divisible processes: *a stochastic process is divisible if and only if, for every vector* $v(x), x \in \mathcal{X}$ *and* $\forall\, t_1, t_2$:

$$\left\| \sum_{x_1 \in \mathcal{X}} T(x_2, t_2 | x_1, t_1) v(x_1) \right\|_1 \leq \|v(x_2)\|_1, \qquad t_1 \leq t_2, \tag{2.29}$$

where $\|v(x)\|_1 = \sum_x |v(x)|$ *is the* L^1*-norm.*

Now let us suppose that we want to discriminate among the distributions between $p_1(x)$ and $p_2(x)$ that the random variable X can follow. It is possible to show that the minimum (averaged) probability of wrong answer is given by:

$$\mathbb{P}_{\text{fail}} = \frac{1 - \|w(x)\|_1}{2},$$

where $w(x) = q p_1(x) - (1 - q) p_2(x)$, and q is the probability that X is distributed according to $p_1(x)$. From the theorem it follows that $\mathbb{P}_{\min}(\text{fail})$, for a Markovian process, increases monotonically: the best chance to distinguish

between the two probabilities is in the first stage of the dynamics. The system losses information continuously and "does not remember" its previous state. This behaviour hence expresses in a more intuitive way the concept of memoryless evolution.

For a quantum mechanical process the situation is similar. Suppose that we want to distinguish between two states $\rho_1, \rho_2 \in \mathcal{H}$ in which the system can be with probability q and $1 - q$, respectively. We want to determine, by performing a measurement, which density matrix describes the state of the system. It is possible to show that, by choosing appropriately the measurement, the minimum total error is:

$$\mathbb{P}_{\min}\left(\text{fail}\right) = \frac{1 - \|\Delta\|_1}{2}, \tag{2.30}$$

where $\|\Delta\|_1 = \text{Tr}\sqrt{\Delta^\dagger \Delta}$ is the trace norm and $\Delta = q\rho_1 - (1 - q)\rho_2$ is the Helstrom matrix [57].

Now it is possible to show [47, 58] that a trace-preserving linear map is positive if and only if

$$\|\Phi\left(\Delta\right)\|_1 \leq \|\Delta\|_1 \tag{2.31}$$

for any Hermitian operator Δ acting on $\mathcal{H}$. On the other hand, as shown in Sec. (2.1.1), the positivity condition of a dynamical map is too weak to represent a physical evolution, so we need to impose the more stringent requirement of complete positivity. It is possible to obtain the following fundamental result: *A quantum evolution $\Phi(t_2, t_1), t_2 \geq t_1$ is Markovian if and only if:*

$$\|\left(\Phi\left(t_2, t_1\right) \otimes \mathbb{1}\right)(\tilde{\Delta})\|_1 \leq \|\tilde{\Delta}\|_1, \qquad \forall\, t_2 \geq t_1, \tag{2.32}$$

for every Hermitian operator $\tilde{\Delta}$ acting on $\mathcal{H} \otimes \mathcal{H}$. This relation is the equivalent of Eq. (2.29) for a quantum dynamical map.

Therefore, as follows from this theorem and from Eq. (2.30), and in parallel with the classical case, the signature of a Markovian evolution is the monotonic increase of the $\mathbb{P}_{\min}(\text{fail})$ quantity: the information about the state of the system is degrading during the evolution, namely we must make the measurement as soon as possible. The same result applies even if we make a measurement including an ancillary system, as follows from the imposed condition of complete positivity of the map.

Finally we note that, in the scientific literature, a particular case of this condition has been proposed by Breuer, Laine, Piilo (BLP definition) [55, 59, 60] as a definition of Markovianity. The contractive property, Eq. (2.32),

clearly implies Eq. (2.31); choosing $\Delta = \frac{1}{2}(\rho_1 - \rho_2)$, we obtain:

$$\| \Phi\left(t_2, t_0\right)\left(\Delta\right) \|_1 \leq \| \Phi\left(t_1, t_0\right)\left(\Delta\right) \|_1, \Leftrightarrow \tag{2.33}$$
$$\| \rho_1\left(t_2\right) - \rho_2\left(t_2\right) \|_1 \leq \| \rho_1\left(t_1\right) - \rho_2\left(t_1\right) \|_1, \quad \forall\, t_2 \geq t_1.$$

Thus, if the evolution is Markovian the distance between two generic states is monotonically decreasing. However the reverse implication does not hold: since we have relaxed the CP hypothesis, there are dynamics which are compatible with Eq. (2.33), but not with Eq. (2.32). We will discuss in more detail this definition in Sec. (3.2.1).

CHAPTER 3

Measures of non-Markovianity

In this Chapter we present the common approaches to detect and quantify the non-Markovian character of a quantum evolution. Indeed, as we have already characterized a Markovian dynamics, the usual approach to define a non-Markovian evolution is based on a "negative" definition: a non-Markovian dynamics is a dynamics that violates the condition of Markovianity. Consequently it depends on the specific definition of Markovianity that is chosen. Indeed if we assume the semigroup property as the fundamental property, as it was historically been done, we consider as non-Markovian even some dynamics that are memoryless, and hence Markovian, in the sense of sections (2.2.2), and (2.2.3). On the other hand, there are also some evolutions which are apparently non-Markovian, but which anyway can be re-expressed in the local form of Eq. (2.28) with positive decay rates for every instant of time.

In Sec. (2.2.2) we have shown how the intuitive concept of memoryless evolution is connected with the divisibility property. Consequently, we assume the violation of divisibility as the definition of non-Markovian evolution. In addition to this quantity we can resort even to the contractive property in the form of trace distance between states (the BLP definition) illustrated in Sec. (2.2.3) and in Fig. (3.1.1). Nevertheless there are non-Markovian evolutions which are not detected by this approach; in this case it is only a sufficient condition. As a consequence we can define two types of quantities: *genuine measure of non-Markovianity that are based on the divisibility condition* and *witnesses that are based on the contractive property.* These last quantities even if providing only sufficient conditions, have a more clear physical meaning and are more simply experimentally implemented.

In the remaining part of the chapter we discuss the principal techniques to detect and quantify non-Markovianity.
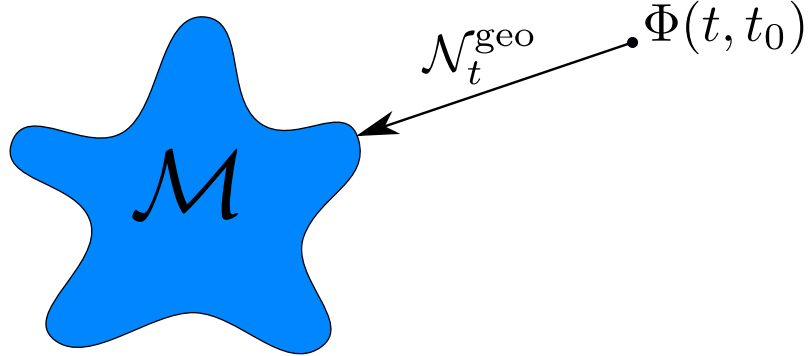
Figure 3.1: Schematic representation of the geometric measure of non-Markovianity. The quantity $\mathcal{N}_t^{\text{geo}}\left[\Phi\left(t,t_0\right)\right]$ measures, at each instant of time t, the distance between the given map $\Phi\left(t,t_0\right)$ and the non-convex set $\mathcal{M}$ of Markovian maps. For a given interval of time the measure is the maximum of $\mathcal{N}_t^{\text{geo}}\left[\Phi\left(t,t_0\right)\right]$.

3.1 Measures of Non-Markovianity

In this section we describe the more relevant measures of non-Markovianity introduced in literature. It is important to note that, following an approach that is usual in Quantum Information Science, it is possible to define through a geometric approach an intuitive measure (i.e. a necessary and sufficient condition for the non-Markovianity of a given map) that it is not based on a particular property of the map (see Fig. (3.1)). At a given instant of time t we define the *punctual non-Markovianity* through the distance between the given channel and the set of Markovian channels:

$$\mathcal{N}_t^{\text{geo}}\left[\Phi\left(t,t_0\right)\right] := \min_{\Phi^M \in \mathcal{M}} \mathcal{D}\left(\Phi\left(t,t_0\right), \Phi^M\left(t,t_0\right)\right), \tag{3.1}$$

where $\mathcal{M}$ is the set of Markovian channels and $\mathcal{D}$ is an appropriate normalized measure of distance. As a consequence of Eq. (3.1), for a given interval of time I the geometric measure of non-Markovianity can be defined as the maximum value of the punctual non-Markovianity in that interval:

$$\mathcal{D}_{\text{GNM}}^I := \max_{t \in I} \mathcal{N}_t^{\text{geo}}\left[\Phi\left(t,t_0\right)\right]. \tag{3.2}$$

As the value of the distance in Eq. (3.1) is confined between 0 and 1, and since it is 0 if and only if the channel is Markovian, the definition provided by Eq. (3.2) is a well defined measure of non-Markovianity.

Despite the conceptually clear meaning of this measure, it presents an important drawback: it is mostly very hard to compute in practice because of the involved optimization process. It has been shown that the set of

Markovian maps is not convex [61], and so the problem becomes intractable when the dimension of the system grows. Furthermore, from a computational point of view, deciding if a given channel is Markovian is a very hard problem in the sense of the complexity theory [62].

A better approach then would be to directly assume as starting point the violation of the divisibility property that characterizes a Markovian evolution.

3.1.1 Measure based on the Helstrom matrix

In section (2.2.3) the contractive property of a Markovian evolution allows us to define in mathematical terms the intuitive concept of flow of information. There is in fact a continuous flow of information from the system to the environment; as a consequence, the minimum-error probability $\mathbb{P}_{\min}(\text{fail})$, used to distinguish between two initial states, is a monotonically increasing function of the time, implying that the states become less and less distinguishable as time passes.

Consider now the situation in which, for some instant of time t and for $\epsilon > 0$, we have $\|\Delta(t + \epsilon),\|_1 > \|\Delta(t)\|_1$, where Δ is the Helstrom matrix defined in Sec. (2.2.3); i.e., we have that the probability to distinguish whether the system was in state ρ_1 or in state ρ_2 is higher at $t + \epsilon$ than it was at the time t. Roughly speaking, this can be interpreted as a backflow of information from the environment to the system. Furthermore, we have also to consider the possible presence of an ancillary state A; consequently, from Eq. (2.32) we have $\|\tilde{\Delta}(t+\epsilon),\|_1 > \|\tilde{\Delta}(t)\|_1$, where $\tilde{\Delta} = q\rho_{1\mathcal{A}} - (1-q)\rho_{2\mathcal{A}}$ is the enlarged Helstrom matrix. Therefore, the quantity

$$\tilde{\sigma}(\tilde{\Delta}, t) := \frac{\mathrm{d}\|\tilde{\Delta}(t)\|_1}{\mathrm{d}t} = \lim_{\epsilon \to 0^+} \frac{\|\tilde{\Delta}(t + \epsilon)\|_1 - \|\tilde{\Delta}(t)\|_1}{\epsilon} \tag{3.3}$$

encodes this increment of information: $\tilde{\sigma}(\tilde{\Delta}, t)$ is positive if and only if the evolution is non-Markovian.

However, the quantity (3.3) is not a characteristic of the channel, as it depends on the chosen initial states. In order to obtain a state-independent definition it is hence necessary to maximize over the initial Helstrom matrix (i.e. over $\rho_{1\mathcal{A}}$, $\rho_{2\mathcal{A}}$ and the bias q) and, finally, it is necessary to add every increment of information. We thus get:

$$\mathcal{N}_H^I := \max_{\tilde{\Delta}} \int_{t \in I, \tilde{\sigma} > 0} \mathrm{d}t \, \tilde{\sigma}(\tilde{\Delta}, t). \tag{3.4}$$

Due to the "if and only if condition" expressed by Eq. (2.32), the quantity Eq. (3.4) is a well defined measure of non-Markovianity: it is 0 only for
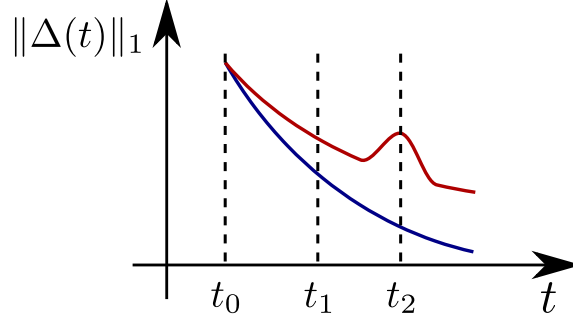
Figure 3.2: In the case of Markovian dynamics the evolution of $\|\Delta(t)\|_1$ decreases monotonically from its initial value t_0 (blue line). When the dynamics is non-Markovian there exist revivals at some instants of time (red line).

Markovian channels. This measure so defined, as for the geometric approach, has the main drawback of the optimization process that makes the measure rather impractical.

3.1.2 Rivas-Huelga-Plenio measure

The approach proposed by Rivas, Huelga, Plenio (RHP) [63] is based directly on the divisibility property of the dynamical map.

Consider a generic evolution from a reference time t_0 to another instant of time t_2. The divisibility condition, Eq. (2.21), expresses the possibility, for every $t_0 < t_1 < t_2$, to split the evolution in terms of other two valid dynamical maps which are CPTP (see Fig. (3.3)). Indeed, as the evolution is memoryless, the description of the dynamics is independent of the initial instant of time: starting from t_1 is equivalent to start from t_0.

At variance, if the evolution has memory, we cannot choose arbitrarily the starting point, but we must start necessarily from t_0. Stated in another way, the intermediate dynamical map $\Phi(t_1, t_2)$ that describe the dynamics from t_1 to t_2 is not a valid dynamical map.

Hence we can construct a criterion to verify the non-Markovianity of a given evolution checking the properties of the intermediate map.

Under the hypothesis that the dynamical map $\Phi(t_1, t_0)$ is invertible, starting from Eq. (2.21) we can express this evolution in terms of a backward evolution from t_1 to t_0 and a forward evolution from t_0 to t_2 (see Fig. (3.4)):

$$\Phi(t_2, t_1) = \Phi(t_2, t_0)\,\Phi^{-1}(t_1, t_0), \qquad t_2 \geq t_1 \geq t_0. \tag{3.5}$$

For a non-Markovian evolution, Eq. (3.5) *is not* a CPTP map. In order to justify this statement we start once again from the classical case.
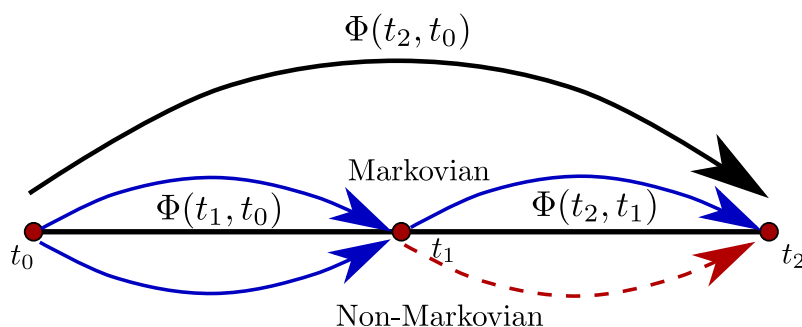
Figure 3.3: Schematic representation of the divisibility property. In the case of Markovian (divisible) evolution the dynamics can be split in another two valid dynamical maps (blue arrow). If the dynamics is non-Markovian the intermediate map does not exists (red dashed arrow).
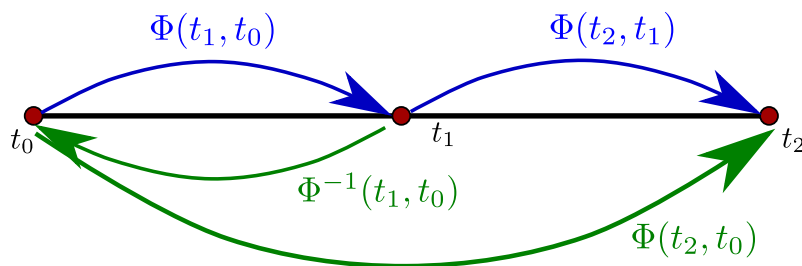
Figure 3.4: Schematic representation of the construction of the intermediate map.

In Sec. (2.2.2) we have shown that for a Markovian process the intermediate evolution is defined as $T(x_2, t_2|x_1, t_1) = \mathbb{P}(x_2, t_2|x_1, t_1)$. In general this relation is not valid for $t_1 > t_0$: $\mathbb{P}(x_2, t_2|x_1, t_1)$ is not well defined, as depends on the previous instant of time: $\mathbb{P}(x_2, t_2|x_1, t_1; x_0, t_0) \neq \mathbb{P}(x_2, t_2|x_1, t_1; x'_0, t_0)$ for $x_0 \neq x'_0$. Nevertheless, if $\mathbb{P}(x_1, t_1|x_0, t_0)$ is invertible, as the evolution from t_1 to t_2 must be the composition of a backward evolution from t_1 to t_0 and of a forward evolution from t_1 to t_2, we have:

$$
\begin{aligned}
T(x_2, t_2|x_1, t_1) &= \sum_{x_o \in \mathcal{X}} T(x_2, t_2|x_0, t_0) \, T(x_0, t_0|x_1, t_1) = \\
&\sum_{x_o \in \mathcal{X}} \mathbb{P}(x_2, t_2|x_0, t_0) \, \mathbb{P}^{-1}(x_0, t_0|x_1, t_1).
\end{aligned}
\tag{3.6}
$$

although it is divisible, the intermediate map, Eq. (3.6), is not positive semidefinite, i.e. does not satisfy the second condition in Eq. (2.24), thus highlighting the non-Markovian character of the evolution. Eq. (3.5) is nothing more than the quantum equivalent of the Eq. (3.6).

Hence, for a non-Markovian dynamics there must be some t_1 such that

$\Phi(t_2, t_1)$ is not CP. To quantify the degree of non-Markovianity of a dynamical map we can then measure how the intermediate dynamics depart from the complete positivity condition.

We can resort to the Choi-Jamolkowski (CJ) isomorphism [64, 65]. Starting from the map $\Phi(t_2, t_1)$, we can construct the CJ matrix:

$$[\Phi(t_2, t_1) \otimes (\mathbb{1})](|\Psi\rangle \langle\Psi|), \tag{3.7}$$

where $|\Psi\rangle = \frac{1}{\sqrt{d}} \sum_{n=0}^{d-1} |n\rangle \langle n|$ is the maximally entangled state. Then we can check the CP condition of the map through the Choi theorem: the map $\Phi(t_2, t_1)$ is CP if and only if the corresponding matrix, Eq. (3.7), is positive semidefinite.

A good way to quantify the CP violation is through the trace norm: in fact, since the map is trace preserving, we have

$$\| [\Phi(t_2, t_1) \otimes \mathbb{1}](|\Psi\rangle \langle\Psi|) \|_1 = \begin{cases} 1 \text{ if } \Phi(t_2, t_1) \text{ is CP} \\ > 1 \text{ otherwise} \end{cases},$$

and consequently the function [63]

$$g(t) := \lim_{\epsilon \to 0^+} \frac{\| [\Phi(t_2, t_1) \otimes \mathbb{1}()](|\Psi\rangle \langle\Psi|) \|_1 - 1}{\epsilon} \tag{3.8}$$

is greater than zero if and only if the evolution is non-Markovian. Finally, the total amount of non-Markovianity in an interval $t \in I$ will be given by:

$$\mathcal{N}_{RHP}^I := \int_I g(t) \mathrm{d}t.$$

It is important to note that, for some t_1, the map $\Phi(t_1, t_0)$ may be not invertible, but the singularity can be removed; a possible way is to compute the inverse of $\mathbb{1}\eta + \Phi(t_1, t_0)$ (which always exists), taking at the end of computation the limit $\eta \to 0^+$.

This measure has the great advantage of not requiring optimization procedures, though it is less intuitive if compared with some other approaches to non-Markovianity as the BLP witness (see Sec. (3.2.1)).

The divisibility condition will be our starting point to define a measure of non-Markovianity for the class of Gaussian channels.

3.1.3   Hall-Cresser-Lee-Andersson measure

Finally we want to present another measure that it is based directly on the decay rates appearing in the generator, Eq. (2.28), of the master equation (2.20) [66]. This is not a serious drawback, as it is possible to show that

for finite dimensional systems it is always possible to pass from a dynamical map description of the dynamics to a quantum master equation and vice versa, even if the problem is computationally hard [62].

For a finite N-dimensional system we start from the the Lindblad generator:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(t)\rho = -\imath\left[H(t)\rho(t)\right] +$$

$$+ \sum_{k,l=1}^{N} \gamma_{kl}(t)\left[V_k(t)\rho(t)W_l^{\dagger}(t) - \frac{1}{2}\left\{W_l^{\dagger}(t)V_k(t), \rho(t)\right\}\right]. \qquad (3.9)$$

From sec. (2.2.2) we know that for differential evolutions the Markovian/non-Markovian nature of the process depends on the sign of the decay rates coefficients. But the master equation Eq. (3.9) may be written in many ways; as a consequence it seems that it is not possible to base a definition of non-Markovianity directly on the form of the generator, unless a unique form of the equation exists.

Hall, Cresser, Li and Andersson [66] proved that such a canonical form exists. The first step is to rewrite the generator of the master equation in a basis $\{G_j\}_{j=0}^{N^2-1}$ which is orthonormal with respect to the Hilbert-Schmidt product $\mathrm{Tr}(G_m^{\dagger}G_n) = \delta_{mn}$. As the obtained matrix of the decay rates is Hermitian, it can be diagonalized via some unitary operation; furthermore, as the eigenvalues are independent of the chosen basis, the obtained form is unique up to degeneracy. The final resulting canonical form of the generator is Eq. (2.28), where the decay coefficients are now the canonical decay rates.

Through this result and Eq. (2.28) of Markovian dynamics, it is then possible to quantify the non-Markovianity of a given evolution directly by the negativity of the decay rates.
The functions:

$$f_j(t) := \max\left\{-\gamma_j(t), 0\right\}, \quad j = 1, \ldots, N^2 - 1$$

are greater than zero if and only if the coefficients are negative, i.e. if the channel is non-Markovian. Consequently the quantity:

$$\mathcal{N}_{\gamma}^{I} := \int_{I} f(t)\,\mathrm{d}t,$$

where $f(t) = \sum_{j=1}^{N^2-1} f_j(t)$, is a good measure of non Markovianity. It is possible to show that this measure is equivalent to the RHP measure: $f(t) = (N/2)g(t)$, where $g(t)$ is defined in Eq. (3.8). This is not surprising, as both

are necessary and sufficient conditions and, for every dynamical map, it is possible to construct the corresponding generator.

We discuss briefly this measure in Chapter 4 in connection with the infinite dimensional case and its difference with the finite-dimensional case.

3.2 Non-Markovian witnesses

In this Section we describe some of the different ways proposed in literature to detect non-Markovianity via witnesses. A witness is a quantity that is zero when the dynamics is Markovian; however it provides only a sufficient condition because it can be zero even when the dynamics is non-Markovian. We do not discuss all the proposed witnesses[1], but we limit ourselves to present the fundamental idea and the most used witnesses. In general they can be classified according to their monotonic behaviour under completely positive maps or under local completely positive maps.

3.2.1 Witnesses monotonic under completely positive maps

These witnesses are based on the behaviour of certain quantities under completely positive maps. A quantity $F(\rho_1, \rho_2)$ is contractive under CP map if and only if

$$F\left(\rho_1(t), \rho_2(t)\right) \leq F(\rho_1, \rho_2),$$

where $\rho_i(t) = \Phi(t, t_0)\rho, i = 1, 2$. As a consequence the quantity

$$\sigma\left(\rho_1, \rho_2, t\right) := \frac{\mathrm{d}F\left(\rho_1\left(t\right), \rho_2\left(t\right)\right)}{\mathrm{d}t}$$

is certainly greater than zero when the contractive property is violated, i.e. when the dynamics is non-Markovian. The two examples we present in this section are based on this idea.

- *Breuer-Laine-Piilo (BLP) witness* [55, 59]. This witness is based on the contractive property of the trace distance under CPTP maps. In fact, in Sec. (2.2.3) we have described how the non-Matkovianity is linked to the intuitive idea of information flow from the environment to the system (Eq. (2.33)): an increasing of the distance between any two initial states is a signature of non-Markovian evolution.

---

[1]The interesting reader can be found a more in depth discussion in [45] and reference therein.

As a consequence, the quantity

$$\mathcal{N}_{\text{BLP}} := \max_{\rho_1(t_0),\rho_2(t_0)} \int_{\sigma>0} \mathrm{d}t \, \sigma\left(\rho_1(t),\rho_2(t),t\right),$$

where $F(\rho_1(t),\rho_2(t)) = D_1(\rho_1(t),\rho_2(t)) = (1/2)\|\rho_1(t) - \rho_2(t)\|_1$ is the trace distance between $\rho_1(t)$ and $\rho_2(t)$, is a non-Markovianity quantifier because, if it is greater than zero, the evolution is non-Markovian. The maximization over all the possible initial pairs $\{\rho_1(t_0),\rho_2(t_0)\}$ is required in order to obtain a state-independent quantity, as it can happen that there are two particular initial states for which the distance is a monotonic decreasing function.

- *Quantum relative entropies.* Another quantity that is monotone under CPTP map is the relative entropy between two quantum states [67], defined as

$$S\left(\rho_1\|\rho_2\right) := \mathrm{Tr}\left(\rho_1 \log \rho_1\right) - \mathrm{Tr}\left(\rho_1 \log \rho_2\right).$$

Despite the fact that it is not symmetric, $S(\rho_1\|\rho_2) \neq S(\rho_2\|\rho_1)$, nor satisfies the triangle inequality, it fulfills the properties $S(\rho_1\|\rho_2) \geq 0$, and $S(\rho_1\|\rho_2) = 0$ if and only if $\rho_1 = \rho_2$. Consequently, it is a good quantity to define a non-Markovian witness. The same considerations as in the previous case can be applied.

This approach has the undoubted advantage of a clear physical interpretation in terms of the backflow of information from the environment to the system; furthermore it is more easily accessible from the experiment. For example, the BLP measure is used in the experiments to test non-Markovianity; therefore it is extensively used in the scientific literature. Despite these remarkable aspects, this method suffer of a serious drawback: the optimization over the initial states that it is often impracticable.

3.2.2 Witnesses monotonic under local completely positive maps

All these witnesses are correlation measures between the system S under study and an ancillary system A, which do not increase under local operations of the form $\Phi \otimes \mathbb{1}_A$. We present two examples:

- *Entanglement.* In the first chapter we have briefly discussed the property that quantum systems have to lose their individuality and become entangled. Furthermore an alternative characterization of the

entanglement is possible from the point of view of the resource theory: the entanglement are those correlations among quantum systems that cannot be generated by local operations and classical communication (LOCC) [68, 69]. Every entanglement measure[2] must fulfill some conditions: in particular, it cannot increase under (LOCC). Consequently, as local operations of the form $\Phi \otimes \mathbb{1}$, where Φ is a CP map, are particular cases of LOCC, then the entanglement measure cannot increase when the system evolves according to the dynamical map $\Phi \otimes \mathbb{1}$.

Hence, in order to characterize the evolution it is possible to study the evolution of the system S coupled with an ancilla A: $\rho_{SA}(t) = [\Phi(t, t_0) \otimes \mathbb{1}](\rho_{SA}(t_0))$. We consider the case in which, the $S + A$ system is initially in the maximally entangled state $\rho_{SA}(0) = |\Phi\rangle \langle\Phi|$, with $|\Phi\rangle = (1/\sqrt{d}) \sum_{n=0}^{d-1} |n\rangle |n\rangle$.

The quantity:

$$\mathcal{I}^{(E)} := \Delta E + \int_{t_0}^{t_1} \left| \frac{\mathrm{d}E[\rho_{SA}(t)]}{\mathrm{d}t} \right| \mathrm{d}t,$$

where $\Delta E := E[\rho_{SA}(t_1)] - E[\rho_{SA}(t_0)]]$, is different from zero only if $\Phi(t, t_0)$ is non-Markovian in the interval (t_0, t_1) [63].

- *Quantum mutual information.* The quantum mutual information between two given systems is a measure of the total amount of correlation (classical and quantum).
  It is defined as:

$$I(\rho_{SA}) = S(\rho_S) + S(\rho_A) - S(\rho_{SA}),$$

where $S(\rho) = -\mathrm{Tr}(\rho \log \rho)$ is the von Neumann entropy, ρ_{SA} is the system-plus-ancilla state, and $\rho_{S,A} = \mathrm{Tr}_{A,S}(\rho_{SA})$.

Expressing this definition in terms of the relative entropy as $I(\rho_{SA}) = S(\rho_{SA} \| \rho_S \otimes \rho_A)$, it is possible to show that:

$$I[(\Phi \otimes \mathbb{1}) \rho_{SA}] \leq S(\rho_{SA} \| \rho_S \otimes \rho_A) = I(\rho_{SA}).$$

Hence, the quantum mutual information is monotonic under local CPTP maps, and thus it can be used to study non Markovianity [70].

The main drawback of these approaches is that they provide only sufficient conditions: there are non-Markovian evolution compatible with a monotonic decreasing of these quantities.

---

[2]The same argument applies to the more general case of entanglement monotone, i.e. entanglement quantifiers that do not coincide with the entropy of entanglement for pure states.

CHAPTER 4

Quantum Mechanics in phase space: an overview

Before facing the central problem, i.e. the characterization and quantification of non-Markovianity in Continuous Variable (CV) quantum systems, it is necessary to "setting the stage". In this chapter we present a brief review of the phase space formulation of Quantum Mechanics [71, 72], aiming at introducing the necessary concepts and tools for subsequent chapters. Particular attention is dedicated to the case of Gaussian State and Gaussian Channels [73, 74, 75, 76] because of the central role they have in the current theoretical and experimental scenario, and because of their extensive use throughout the manuscript. However, also some especially interesting non-Gaussian states are introduced.

4.1 Bosonic systems and symplectic transformations

Let us consider the system of N quantized radiation modes of the electromagnetic field represented by N bosonic modes. These modes live in the Hilbert space $\mathcal{H}^{\otimes N} = \otimes_{i=1}^{N} \mathcal{H}_i$ that is the tensor product of the Hilbert spaces associated to each of the modes; to the N modes are associated the bosonic field operators $\{a_k, a_k^\dagger\}, k = 1, ..., N$ satisfying the commutation relations $[a_k, a_l^\dagger] = \delta_{kl}$.

The system is usually described even in terms of another pair of field operators, the quadrature field operators $\{q_k, p_k\}_{k=1}^{N}$, defined as:

$$q_k = \frac{1}{\sqrt{2}}(a_k + a_k^\dagger), \qquad p_k = \frac{1}{\sqrt{2}\imath}(a_k - a_k^\dagger);$$

these operators satisfy the canonical commutation relations[1] (CCR) $[q_k, p_l] = \imath \delta_{kl}$ which are easily derivable from the commutator relations of the bosonic modes.

A convenient way to deal with the calculations is to introduce the vectorial operator $\mathbf{R} = (q_1, p_1, \cdots, q_N, p_N)^{\intercal}$. With this notation the CCR are rewritten as:

$$[R_k, R_l] = \imath \Omega_{kl}, \qquad \mathbf{\Omega} = \bigoplus_{k=1}^{N} \boldsymbol{\omega}, \qquad \boldsymbol{\omega} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where $\mathbf{\Omega}$ is the symplectic form.

Now we may ask: what are the transformations of the canonical variables that preserve the CCR? Stated more formally, let us consider a transformation $R'_k = f_k(R_1, \ldots, R_N)$, $k = 1, \ldots, N$; what are the conditions on the transformation ensuring that $[R'_k, R'_l] = \imath \Omega_{kl}$? We focus the attention on affine transformations which assume the form:

$$\mathbf{R}' = \mathbf{FR} + \mathbf{d}, \tag{4.1}$$

where $\mathbf{F}$ is a $2N \times 2N$ matrix that represents a rotation in phase space, and $\mathbf{d}$ is a displacement of the quadrature operator in phase space. By using the commutation relations it can be proven that:

$$[R'_k, R'_l] = \imath \Omega_{kl} \quad \Longleftrightarrow \quad \mathbf{F \Omega F}^{\dagger} = \mathbf{\Omega}. \tag{4.2}$$

A transformation that satisfies Eq. (4.2) is called *symplectic* and the matrix $\mathbf{F}$ is called *symplectic matrix*. By exploiting the expression of the symplectic form and the condition expressed by Eq. (4.2), it can be easily proved that the $\mathbf{F}$ matrices form a group, the *symplectic group* $\mathrm{Sp}(2N, \mathbb{R})$. Together with the displacements they form the *inhomogeneous symplectic group* $\mathrm{Isp}(2N, \mathbb{R})$.

In what follows we deal exclusively with quantum operations in Hilbert space that correspond to symplectic affine transformation in phase space. For a more detailed discussion see Ref. [77, 78, 79] for the case of the real symplectic group in quantum mechanics, and Ref. [80] for the single mode case.

4.2 Phase-space representation of Continuous Variable quantum states

As discussed in Sec. (2.1), all the information about the physical system of interest are contained in its quantum state represented by the positive,

---

[1]From now on, for ease of notation, we set $\hbar = 1$, unless otherwise stated.

trace-one density operator $\rho : \mathcal{H}^{\otimes N} \to \mathcal{H}^{\otimes N}$; we denote the set of the density operators with $\mathcal{D}(\mathcal{H}^{\otimes N})$. When a density operator satisfies the additional property $\rho^2 = \rho$ (i.e. it is a projector), the state is termed *pure* and ρ is represented in terms of the corresponding vector state $|\phi\rangle \in \mathcal{H}^{\otimes N}$ as $\rho = |\phi\rangle \langle\phi|$.

An equivalent description of quantum systems can be obtained in terms of quasi-probability distribution in phase space, as we now move to describe.

First we introduce the displacement or Weyl operator:

$$D\left(\boldsymbol{\alpha}\right) = \bigotimes_{k=1}^{N} D\left(\alpha_k\right), \qquad D\left(\alpha_k\right) = e^{\alpha_k a_k^\dagger - \alpha_k^* a_k}, \tag{4.3}$$

where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N)^\intercal$ and $\alpha_k \in \mathbb{C}, k = 1, \ldots, N$. Every operator O can be expressed in terms of Eq. (4.3) as:

$$O = \int_{\mathbb{C}^N} \frac{\mathrm{d}\boldsymbol{\alpha}^N}{\pi^N} \operatorname{Tr}\left[O D\left(\boldsymbol{\alpha}\right)\right] D\left(\boldsymbol{\alpha}\right)^\dagger, \tag{4.4}$$

i.e. the operators Eq. (4.3) form a *complete set.* In the particular case $O = \rho$ the last equation establishes a one-to-one correspondence between the density operator and the *characteristic function,* given by:

$$\chi\left(\boldsymbol{\alpha}\right) = \chi\left[\rho\right]\left(\boldsymbol{\alpha}\right) = \operatorname{Tr}\left[\rho D\left(\boldsymbol{\alpha}\right)\right]. \tag{4.5}$$

Equivalently, the equation establishes a one-to-one correspondence between the density operator and the *Wigner quasi-probability distribution* that is defined as the Fourier transform of the characteristic function:

$$W\left(\boldsymbol{\xi}\right) = \int_{\mathbb{C}^N} \frac{\mathrm{d}\boldsymbol{\alpha}^{2N}}{\pi^N} \chi\left(\boldsymbol{\alpha}\right) e^{\boldsymbol{\alpha}^\dagger \boldsymbol{\xi} + \boldsymbol{\xi}^\dagger \boldsymbol{\alpha}}.$$

Henceforth we will use exclusively the characteristic function, that will be useful in Chapter 6 when we will describe the continuous-variable quantum teleportation protocol.

The phase space representation of the density operator allows to obtain in a straightforward way all the statistical moments $< (a_k^\dagger)^p a_l^q >_S$ by exploiting the formula:

$$< (a_k^\dagger)^p a_l^q >_S = (-1)^q \left. \frac{\partial^{p+q}}{\partial \alpha_k^p \partial \alpha_l^{q*}} \chi(\boldsymbol{\alpha}) \right|_{\boldsymbol{\alpha}=0}, \tag{4.6}$$

where $< (a_k^\dagger)^p a_l^q >_S$ are the symmetrically ordered moments of mode operators.

From Eq. (4.6) in particular it is possible to obtain the first moment, called the displacement vector or the mean value:

$$\bar{\mathbf{R}} := \langle \mathbf{R} \rangle = \mathrm{Tr}\left[\rho \mathbf{R}\right], \tag{4.7}$$

and the second moment, called the covariance matrix $\boldsymbol{\sigma}$:

$$\sigma_{kl} := [\boldsymbol{\sigma}]_{kl} = \frac{1}{2} \langle \{\Delta R_k, \Delta R_l\} \rangle, \tag{4.8}$$

where $\Delta R_l = R_l - \langle R_l \rangle$ and $\{,\}$ is the anticommutator. The covariance matrix, Eq. (4.8), is a $2N \times 2N$ real and symmetric matrix by construction. Using the definition of Eq. (4.8) and the commutation relations Eq. (4.2), it can be immediately shown that uncertainty relations impose on the covariance matrix the constraint:

$$\boldsymbol{\sigma} + \frac{\imath}{2} \boldsymbol{\Omega} \geq 0, \tag{4.9}$$

that imply the positivity condition $\boldsymbol{\sigma} > 0$.

The first and second moments play a central role in the description of quantum systems in phase space; in particular, they are sufficient to characterize completely the state of a system if this state is described by a Gaussian characteristic function (Gaussian states). As we will see in the next section, the vacuum state of quantum electrodynamics is itself a Gaussian state; furthermore all states which are commonly produced in the laboratories are Gaussian, since the current technology allows to easily implement quantum evolutions described by Hamiltonians at most quadratic in the field operators, that have the property to preserve in time the Gaussian character of an initial Gaussian state.

4.3   Gaussian states

The relevant class of Gaussian states plays a predominant role in Quantum Information science. A state ρ is called Gaussian if its characteristic function has a Gaussian form:

$$\chi(\boldsymbol{\alpha}) = \exp\left(-\frac{1}{2} \boldsymbol{\alpha}^{\mathsf{T}} \boldsymbol{\sigma} \boldsymbol{\alpha} + \imath \boldsymbol{\alpha}^{\mathsf{T}} \bar{\mathbf{R}}\right).$$

The particular shape of the characteristic function allows to characterize the state of the system by the knowledge only of the first and of the second moments (Eqs. (4.7) and Eq. (4.8)).

It is possible to set a general theory for the N-modes Gaussian states. The most general form for these states is:

$$\rho = U \rho^{\text{th}} U^\dagger \qquad \rho^{\text{th}} = \bigotimes_{k=1}^{N} \rho_k^{\text{th}}, \tag{4.10}$$

where ρ_k^{th} is the one mode thermal state[2] and U is a Gaussian unitary (i.e. reversible) transformation $U^{-1} = U^\dagger$ that sends Gaussian states into Gaussian states.

Now the following question arises: what is the most general form of a Gaussian unitary transformation? Let us consider the following general form of a bilinear Hamiltonian:

$$H = \mathbf{a}^\dagger \mathbf{G}^{(1)} + \mathbf{a}^\dagger \mathbf{G}^{(2)} \mathbf{a} + \mathbf{a}^\dagger \mathbf{G}^{(3)} \mathbf{a}^\dagger + \text{h.c.}, \tag{4.11}$$

where $\mathbf{a} := (a_1, \ldots, a_N)^\intercal$, $\mathbf{a}^\dagger := (a_1^\dagger, \ldots, a_N^\dagger)^\intercal$, $\mathbf{G}^{(1)} \in \mathbb{C}^N$, $\mathbf{G}^{(2)}$ and $\mathbf{G}^{(3)}$ are complex $N \times N$ matrices. This Hamiltonian generates unitary transformations $U = \exp(-\imath H)$ that, according to the rule $\rho \to U \rho U^\dagger$, preserves the Gaussian property, i.e. transforms Gaussian states into Gaussian states. In terms of the quadrature operators this unitary transformation corresponds to an affine map of the form given in Eq. (4.1) with the constraining condition expressed by Eq. (4.2). Clearly, also the eigenvalues of the quadrature operators transform according to the same affine transformation[3], i.e. $(\mathbf{F}, \mathbf{d}) : \mathbf{R} \to \mathbf{F}\mathbf{R} + \mathbf{d}$. Consequently *each Gaussian unitary transformation is a transformation generated by a bilinear Hamiltonian whose general form is described in Eq. (4.11). In phase space a Gaussian unitary transformation corresponds to an affine symplectic transformation $(\mathbf{F}, \mathbf{d})$. In particular it is possible to decompose every U as $U(\mathbf{F}, \mathbf{d}) = D(\mathbf{d})U(\mathbf{F})$, where $U(\mathbf{F})$ is the canonical unitary transformation corresponding to the linear map $\mathbf{R} \to \mathbf{F}\mathbf{R}$, and $D(\mathbf{d}) = e^{\imath \mathbf{R}^\intercal \mathbf{\Omega} \mathbf{d}}$ is the Weyl operator of Eq. (4.3) and corresponds to the phase space translation $\mathbf{R} \to \mathbf{R} + \mathbf{d}$.*

Hence, each Gaussian state is obtained starting from a thermal state and applying the appropriate Gaussian unitary transformation. Moreover, by applying the corresponding affine map of Eq. (4.1) to the definitions in Eqs. (4.7) and (4.8), it follows that the action of $U(\mathbf{F}, \mathbf{d})$ on the first and second moments is given by:

$$\bar{\mathbf{R}} \to \mathbf{F}\bar{\mathbf{R}} + \mathbf{d}, \qquad \boldsymbol{\sigma} \to \mathbf{F}\boldsymbol{\sigma}\mathbf{F}^\intercal. \tag{4.12}$$

---

[2]By definition a thermal state is a bosonic state that maximize the von Neumann entropy $S := -\text{Tr}(\rho \log(\rho))$ at fixed energy.

[3]For ease of notation we indicate in the same way the vectorial operator $\mathbf{R}$ and the corresponding vector formed by the eigenvalues, because the difference will be made clear by the context.

Since each Gaussian state is completely characterized from the knowledge of these moments, *a convenient way to obtain a generic Gaussian state is to start from the first and second moments of the thermal state, and to apply the affine map that corresponds to the unitary transformation.*

For our purposes it is sufficient to consider only the cases $N = 1$ and $N = 2$, i. e. one-mode and two-mode Gaussian states. Furthermore, we introduce only the Gaussian states which play a major role in the CV Quantum Information theory and which will be used in the manuscript. For a more exhaustive discussion, we refer to the excellent reviews [71, 73].

4.3.1 One-mode Gaussian states

In the case of only one mode ($N = 1$) the Hamiltonian of Eq. (4.11) reduces to $H = G^{(1)}a^\dagger + G^{(2)}a^\dagger a + G^{(3)}a^\dagger a^\dagger + \text{h.c.}$, and the decomposition (4.10) reads:

$$\rho = D\left(\alpha\right) S\left(\zeta\right) \rho_{\text{th}}\left(\bar{n}\right) S\left(\zeta\right)^\dagger D\left(\alpha\right)^\dagger,$$

where $\rho_{\text{th}}(\bar{n})$ is the thermal state with $\bar{n}$ (≥ 0) mean number of photons in the bosonic mode, $D(\alpha)$ is the one-dimensional Weyl operator of Eq. (4.3) generated by the linear terms of the Hamiltonian ($H = G^{(1)}a^\dagger + \text{h.c.}$) and $S\left(\zeta\right) = \exp(\frac{\xi}{2}(a^\dagger)^2 - \frac{\xi^*}{2}a^2)$ is the one-mode squeezing operator generated by the Hamiltonian terms $H = G^{(3)}a^\dagger a^\dagger + \text{h.c.}$.

We now introduce the more relevant Gaussian states which will be exploited in the thesis.

Vacuum state

The simplest and most important Gaussian state is the state with zero photons ($\bar{n} = 0$), obtained from Eq. (4.3.1) putting $\theta = \zeta = \alpha = 0$, i.e., the vacuum state $\rho = |0\rangle \langle 0|$. The vacuum state minimizes the Heisenberg uncertainty principle, and is also defined as the eigenstate of the annihilation operator with zero eigenvalue ($a |0\rangle = 0$); its covariance matrix is proportional to the identity matrix, i.e. $\boldsymbol{\sigma} = \frac{1}{2}\mathbf{I}$.

Coherent state

Coherent states are defined as the eigenstates of the annihilation operator: $a |\alpha\rangle = \alpha |\alpha\rangle$, $\alpha \in \mathbb{C}$. They are generated by displacing the vacuum state: $|\alpha\rangle = D(\alpha) |0\rangle$. Their covariance matrix coincides with that of the vacuum state, and the vector of the mean values is $\bar{\mathbf{R}} = \mathbf{d}_\alpha$, where $\mathbf{d}_\alpha = (q, p)^\intercal$, and q and p are, respectively, the real part and complex part of α. Also the coherent states minimize the Heisenberg uncertainty principle, with the two quadratures constrained to be equal.

Squeezed state

Finally, the squeezed states are obtained by applying the one-mode squeezing operator to the vacuum state: $|\zeta\rangle = S(\zeta)|0\rangle$, $\zeta \in \mathbb{C}$. The corresponding covariance matrix (for $\mathrm{Im}(\zeta) = 0$) is given by:

$$\boldsymbol{\sigma}(r) = \frac{1}{2}\begin{pmatrix} e^{-2r} & 0 \\ 0 & e^{-2r} \end{pmatrix}.$$

These states define a further class of states which minimize the Heisenberg uncertainy principle but, at variance with the coherent states, now the variance of one quadrature is "squeezed" below the quantum shot noise, i.e. below the corresponding value of the vacuum state, while the variance of the other quadrature is "anti-squeezed" and placed above this value.

4.3.2   Two-mode Gaussian states

The most general form of two mode Gaussian states is given by Eq. (4.10) for $N = 2$. The general form of the covariance matrix is expressed in block form by the equation

$$\boldsymbol{\sigma} = \begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C}^{\intercal} & \mathbf{B} \end{pmatrix},$$

where $\mathbf{A} = \mathbf{A}^{\intercal}$, $\mathbf{B} = \mathbf{B}^{\intercal}$ and $\mathbf{C}$ are 2×2 real matrices.

An important class of two-mode Gaussian states has a covariance matrix of the form:

$$\boldsymbol{\sigma} = \begin{pmatrix} a\mathbf{I} & \mathbf{C} \\ \mathbf{C} & b\mathbf{I} \end{pmatrix},$$

where $\mathbf{C} = \mathrm{diag}(c_1, c_2)$ and $a, b, c_1, c_2 \in \mathbb{R}$. In this case the physical constraints required from the Heisenberg principle, Eq. (4.9), read $\boldsymbol{\sigma} > 0$, $\det \boldsymbol{\sigma} \geq \frac{1}{2}$ and $\Delta \leq \frac{1}{2} + 2\det \boldsymbol{\sigma}$, with $\Delta = \det \mathbf{A} + \det \mathbf{B} + 2\det \mathbf{C}$.

Among the two-mode Gaussian states we consider two subclasses of states.

Two-mode squeezed thermal state

The first subclass we consider is the two-mode squeezed thermal state:

$$\rho = S_2(\zeta)\, \rho_\nu\, S_2^{\dagger}(\zeta),$$

where $\rho_\nu = \rho_1^{th} \otimes \rho_2^{th}$ is the two-mode thermal state and where $S_2(\zeta) = \exp\left(\zeta a_1^{\dagger} a_2^{\dagger} - \zeta^* a_1 a_2\right)$ is the two-mode squeezing operator generated by the

Hamiltonian contribution of the form $H = G^{(3)} a_1^\dagger a_2^\dagger + h.c.$. The corresponding covariance matrix is:

$$\boldsymbol{\sigma} = \frac{1}{2} \begin{pmatrix} A\mathbf{I}_2 & C\mathbf{R} \\ C\mathbf{R} & B\mathbf{I}_2 \end{pmatrix}, \tag{4.13}$$

with:

$$\begin{aligned} A &= \cosh(2r) + 2\bar{n}_1 \cosh^2(r) + 2\bar{n}_2 \sinh^2(r), \\ B &= \cosh(2r) + 2\bar{n}_1 \sinh^2(r) + 2\bar{n}_2 \cosh^2(r), \\ C &= (1 + \bar{n}_1 + \bar{n}_2) \sinh(2r), \end{aligned} \tag{4.14}$$

$$\mathbf{R} = \begin{pmatrix} \cos(\phi) & \sin(\phi) \\ \sin(\phi) & -\cos(\phi) \end{pmatrix}, \tag{4.15}$$

where $\mathbf{I}_2$ is the 2×2 identity matrix and where $\bar{n}_1$ and $\bar{n}_2$ are the number of thermal photons respectively in the first and in the second mode.

Twin Beam states

The second subclass is made up by the so called Twin Beam States (TWBS) which are obtained from the two-mode squeezed thermal state when $\bar{n}_1 = \bar{n}_2 = 0$, leading to:

$$A = B = \cosh(2r), \qquad C = \sinh(2r).$$

As TWBS are entangled, we will consider these states when we generalize the continuous variable quantum teleportation protocol in presence of a noisy environment.

In the next section we will introduce also some classes of non-Gaussian states. The Gaussian states described in the present section and the non-Gaussian states introduced in the next section will be exploited in Chapter 6 as teleportation resources in the framework of realistic quantum teleportation protocol.

4.4 Non-Gaussian states

Although the class of Gaussian states is of high relevance, exploitation of non-Gaussian resources is not avoidable in order to realize universal quantum computation [81]; furthermore, since various non-classical properties are minimized by Gaussian states [82], non-Gaussian resources can provide a potentially relevant improvement of performance in all the quantum information protocols [83, 84, 85, 86], and more specifically when an interplay

between the non-Gaussian character of the states and the non-Markovianity of the channel is considered. It is also to be remarked that experimental implementations of non-Gaussian states have been already realized or proposed [87, 88, 89]. Therefore, in this section we introduce two classes of entangled resources: the two-mode squeezed-Bell (SB) states $|\psi\rangle_{SB}$ [90], and the two-mode squeezed-cat (SC) states $|\psi\rangle_{SC}$ [91, 92], whose expressions are given, respectively, by

$$|\psi\rangle_{SB} = S_{12}(\zeta)\{\cos\delta|0,0\rangle + e^{i\theta}\sin\delta|1,1\rangle\}, \qquad (4.16)$$

$$|\psi\rangle_{SC} = \mathcal{N}_{SC}S_{12}(\zeta)\{\cos\delta|0,0\rangle + e^{i\theta}\sin\delta|\gamma,\gamma\rangle\}, \qquad (4.17)$$

where $S_{12}(\zeta)$ is the two-mode squeezing operator, $\zeta = re^{i\phi}$, $|m,n\rangle \equiv |m\rangle_1 \otimes |n\rangle_2$ is the two-mode Fock state, $|\gamma,\gamma\rangle \equiv |\gamma\rangle_1 \otimes |\gamma\rangle_2$ is a symmetric two-mode coherent state with complex amplitude $\gamma = |\gamma|e^{i\varphi}$, and $\mathcal{N}_{SC}$ is the normalization factor. The presence of tunable parameters, as δ, θ, will be very important in Chapter 6 in order to optimize the performance of the continuous variable quantum teleportation protocol. In SB states the parameters δ, θ rule the non-Gaussian character of the states, allowing a transition from the Gaussian Twin Beam (TB) to the fully non-Gaussian squeezed number states, passing through intermediate states which include photon-added (PA) squeezed states and photon-subtracted (PS) squeezed states.

4.5 Gaussian channels

As described in Sec. (2.1), every quantum operation is represented by a linear map, Eq. (2.5), that has to be completely positive, Eqs. (2.6), and trace-decreasing; moreover, an operation $\Phi : \rho \to \Phi(\rho)$ is a quantum channel if the map is trace-preserving, i.e. if $\text{Tr}[\Phi(\rho)] = 1$ (Eq. (2.7)).

On the same line of reasoning adopted for the classification of Gaussian states we may ask: what is the general form of a channel that preserves the Gaussian property of the input state?

Indeed, for a generic N-mode bosonic channel, a useful way to represent the linear transformation $\Phi : \rho \to \Phi(\rho)$ is through Eq. (2.2):

$$\Phi = \text{Tr}_{\text{E}}\left[U\left(\rho \otimes |\Phi\rangle\langle\Phi|_E\right)U^\dagger\right], \qquad (4.18)$$

where ρ is a N-mode bosonic state and $|\Phi\rangle_E$ is the ancillary N-mode state associated with the environment; it can be shown [93] that Eq. (4.18) is unique up to partial isometries, so one can always choose $|\Phi\rangle_E \equiv |\mathbf{0}\rangle_E$, the multimode vacuum state.

In the particular case of Gaussian bosonic channels [94, 95, 96, 97] the unitary

U is Gaussian and the environment is composed by $N_E \leq 2N$ modes [95, 96]. It is possible to show that the action of a N-mode Gaussian channel on a generic Gaussian state ρ in terms of first and second moments is expressed by:

$$\bar{\mathbf{R}} \to \mathbf{X}\bar{\mathbf{R}} + \mathbf{d}, \qquad \boldsymbol{\sigma} \to \mathbf{X}\boldsymbol{\sigma}\mathbf{X}^\intercal + \mathbf{Y}, \tag{4.19}$$

where $\mathbf{d} \in \mathbb{R}^{2N}$ is a displacement vector and $\mathbf{X}$ and $\mathbf{Y} = \mathbf{Y}^\intercal$ are real $2N \times 2N$ matrices. Clearly, not all the $(\mathbf{X}, \mathbf{Y})$ matrices describe allowable physical transformations: indeed the state after the evolution must be a physical state, i.e. the covariance matrix of the input state after the evolution must satisfy the uncertainty relation Eq. (4.9). It can be shown that the matrix that defines the channel must satisfy the complete positivity condition [98]:

$$\mathbf{Y} + \frac{\imath}{2}\boldsymbol{\Omega} - \frac{\imath}{2}\mathbf{X}\boldsymbol{\Omega}\mathbf{X}^\intercal \geq 0, \tag{4.20}$$

where $\boldsymbol{\Omega}$ is the symplectic form.

The matrices $(\mathbf{X}, \mathbf{Y})$ have a clear physical significance [97]: the first represents an amplification or attenuation and rotation in phase space, the second can be regarded as a quantum or classical noise term. From Eq. (4.20) it follows that $\mathbf{Y} \geq 0$; furthermore, if $\mathbf{Y} = 0$ the same condition reduces to $\mathbf{X}\boldsymbol{\Omega}\mathbf{X}^\intercal = \boldsymbol{\Omega}$, implying that the matrix $\mathbf{X}$ is an element of the symplectic group $\mathrm{Sp}(2N, \mathbb{R})$. Thus the subset of Gaussian channels with $\mathbf{Y} = 0$ corresponds to the unitary (reversible) transformations already discussed in Sec (4.3).

Clearly the concatenation of two Gaussian channels is another Gaussian channel, as follow directly from the definition; furthermore for $(\mathbf{X}, \mathbf{Y}) = (\mathbb{1}, 0)$ the covariance matrix does not change. Consequently the set of Gaussian channels *forms a semigroup*. It can be shown that the semigroup product is given by:

$$(\mathbf{X}_1, \mathbf{Y}_1) \cdot (\mathbf{X}_2, \mathbf{Y}_2) = \left(\mathbf{X}_1\mathbf{X}_2, \mathbf{Y}_1 + \mathbf{X}_1\mathbf{Y}_2\mathbf{X}_1^\intercal\right). \tag{4.21}$$

The characterization of Gaussian channels and of their semigroup structure is a rich and active field of research; in particular, at moment a complete characterization of infinitesimal divisible quantum channels is still lacking. We remand to Ref. [97] for a deeper discussion.

4.5.1 Gaussian master equation

As illustrated in Sec. (2.1.2) another way to describe the evolution of a quantum system is through the master equation approach. In the case of Gaussian channels this description is less general: not every Gaussian channel admits

a generator, i.e. not all Gaussian channels can be described by a master equation [97]. On the other hand, when a master equation exists, from its solution it is straightforward to obtain the form of the corresponding $(\mathbf{X}, \mathbf{Y})$ matrices. Indeed it is possible to convert the master equation into a differential equation in phase space for the characteristic function representing the state of the system. Every master equation (2.20):

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = \mathcal{K}(t)\rho(t),$$

where $\mathcal{K}$ is a superoperator quadratic in terms of the ladder operators, becomes a linear partial differential equation in the canonical variables for the characteristic function.

For Gaussian channels the solution of the corresponding PDE is given by:

$$\chi\left(\mathbf{R}\right) \to \chi\left(\mathbf{X(t)R}\right) e^{-\frac{1}{2}\mathbf{R}^{\intercal}\mathbf{Y}(t)\mathbf{R}},$$

as it can be also derived from the corresponding evolution, in the Heisenberg picture, of the Weyl operator, Eq. (4.3), expressed in terms of Cartesian coordinates:

$$\mathcal{D}\left(\mathbf{R}\right) \to \mathcal{D}\left(\mathbf{X(t)R}\right) e^{-\frac{1}{2}\mathbf{R}^{\intercal}\mathbf{Y}(t)\mathbf{R}}.$$

4.6 Examples of Gaussian quantum channels

In this section we describe the channels we consider hereafter: in detail we focus on two paradigmatic master equations, the Damping master equation and the Quantum Brownian Motion.

4.6.1 Damping master equation

The Damping master equation [99]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho\left(t\right) = \alpha\frac{\gamma\left(t\right)}{2}\left[2a\rho a^{\dagger} - \left\{a^{\dagger}a\rho\right\}\right], \qquad (4.22)$$

where a, $a^{\dagger}$ are the ladder operators, α is the coupling constant and $\gamma(t)$ is the time-dependent decay rate, is a phenomenological generalization of the corresponding Markovian equation, Eq. (2.19), allowing the decay rate to be time-dependent. It is the simplest arena to test criteria of non-Markovianity because in this case we only have a single field mode and one time-dependent

decay parameter. From Eq. (4.22) the $(\mathbf{X}, \mathbf{Y})$ matrices defining the channel are found to be:

$$\mathbf{X}(t,0) = e^{-\frac{\Gamma(t)}{2}} \mathbb{1} \, , \tag{4.23}$$

$$\mathbf{Y}(t,0) = \left[1 - e^{-\Gamma(t)}\right] \frac{\mathbb{1}}{2} \, , \tag{4.24}$$

where $\Gamma(t) = 2\alpha \int_0^t \gamma(s)ds$.

4.6.2 Quantum Brownian Motion

The model consists of an harmonic oscillator in interaction with an environment made up of independent harmonic oscillators [100, 101, 102]. The Hamiltonian of the total system is:

$$\mathcal{H} = \mathcal{H}_S + \mathcal{H}_E + \mathcal{H}_I + \mathcal{H}_R,$$

where $\mathcal{H}_S$ and $\mathcal{H}_E$ are the free Hamiltonians respectively of the system and the environment:

$$\begin{aligned}
\mathcal{H}_S &= \frac{p^2}{2M} + \frac{1}{2}M\omega_0^2 q^2, \\
\mathcal{H}_E &= \sum_n \left(\frac{p_n^2}{2m_n} + \frac{1}{2}m_n\omega_n^2 q_n^2 \right),
\end{aligned} \tag{4.25}$$

the term $\mathcal{H}_I$ represents the system-environment coupling, and the term $\mathcal{H}_R$ represents a normalization factor that allows to cancel the non-physical change of the free energies of the mode of the system $(\omega_0 \to \omega_0 + \delta\omega_0)$.

Various models of interaction can be considered: for our purposes we consider the following coupling forms:

$$\mathcal{H}_I^{(RWA)} = \alpha \sum_n k_n \left(ab_n^\dagger + a^\dagger b_n \right), \tag{4.26}$$

$$\mathcal{H}_I = -\alpha q \sum_n k_n q_n, \tag{4.27}$$

where a, $a^\dagger$ and b_n, $b_n^\dagger$ are the ladder operators, respectively, of the oscillator representing the system and of the k-th bath mode, k_n denote the interaction strengths and α is a coupling constant. We don't describe in detail the two models corresponding to the different coupling forms; we make use of the first interaction (the rotating wave coupling that contains only the terms conserving the number of excitation of the total system) in the discussion of

the measure of non-Markovianity defined in the next Chapter, and we will observe the effect of the second, more complete interaction (the position-position coupling) on the fidelity of teleportation in Chapter 6.

The equation that describes the harmonic particle dynamics in the case of position-position coupling, under the assumption of factorized initial condition and the assumption of the environment in a thermal state at temperature T is the Hu-Paz-Zhang equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_S = \frac{1}{\imath\hbar}\left[\mathcal{H}_S, \rho_S\right] + \imath r\left(t\right)\left[q^2, \rho_S\right] - \imath\gamma\left(t\right)\left[q, \{p, \rho_S\}\right]$$
$$- \Delta\left(t\right)\left[q, \left[q, \rho_S\right]\right] + \Pi\left(t\right)\left[q, \left[p, \rho_S\right]\right]. \tag{4.28}$$

The first term is the unitary evolution, while the remaining terms represent the system-bath interaction. The coefficient $r(t)$ is called the normalization term as it represents a change in the free-oscillator frequency. The coefficient $\gamma(t)$ is the damping coefficient. Finally, the coefficients $\Delta(t)$ and $\Pi(t)$ are, respectively, the normal and the anomalous diffusion coefficients. The form of the coefficients depends strictly on the properties of the environment through the spectral density $J(\omega)$, that encapsulate the physical properties of the environment, namely is a measure of the coupling strength between the system and the environment. For weak coupling ($\alpha \ll 1$), at the second order in α the coefficients read:

$$\Delta\left(t\right) = \alpha^2 \int_0^t \mathrm{d}s \int_0^\infty \mathrm{d}\omega J\left(\omega\right)\left[2N\left(\omega\right) + 1\right]\cos\left(\omega s\right)\cos\left(\omega_0 s\right),$$

$$\Pi\left(t\right) = \alpha^2 \int_0^t \mathrm{d}s \int_0^\infty \mathrm{d}\omega J\left(\omega\right)\left[2N\left(\omega\right) + 1\right]\cos\left(\omega s\right)\sin\left(\omega_0 s\right),$$

$$\gamma\left(t\right) = \alpha^2 \int_0^t \mathrm{d}s \int_0^\infty \mathrm{d}\omega J\left(\omega\right)\sin\left(\omega s\right)\sin\left(\omega_0 s\right), \tag{4.29}$$

$$r\left(t\right) = \alpha^2 \int_0^t \mathrm{d}s \int_0^\infty \mathrm{d}\omega J\left(\omega\right)\sin\left(\omega s\right)\cos\left(\omega_0 s\right),$$

where $N(\omega) = (\exp\{\hbar\omega/k_B T\} - 1)^{-1}$ is the mean number of photons at frequency ω. In terms of characteristic function the solution of Eq. (4.28) is [101]:

$$\chi\left(\mathbf{R}, t\right) = \chi\left(e^{-\frac{\Gamma(s)}{2}}\mathbf{F}^{-1}\left(t\right)\mathbf{R}, 0\right)e^{-\mathbf{R}^\intercal \bar{\mathbf{W}}(t)\mathbf{R}}, \tag{4.30}$$

where:

$$\Gamma(t) = 2\int_0^t \gamma(s)\,ds \tag{4.31}$$

and:

$$\bar{\mathbf{W}}\left(t\right) = e^{-\Gamma(t)} \left[\mathbf{F}^{-1}\left(t\right)\right]^{\mathsf{T}} \mathbf{W}\left(t\right) \mathbf{F}^{-1}\left(t\right), \tag{4.32}$$

$$\mathbf{W}\left(t\right) = \int_0^t e^{\Gamma(s)} \mathbf{F}^{\mathsf{T}}\left(s\right) \begin{pmatrix} \Delta\left(s\right) & -\Pi\left(s\right)/2 \\ -\Pi\left(s\right)/2 & 0 \end{pmatrix} \mathbf{F}\left(s\right). \tag{4.33}$$

The general form of the $\mathbf{F}(t)$ matrix depends on the coefficient $r(t)$ and is quite complicated; at the second order in α it is possible to show that it assumes the $r(t)$-independent form:

$$\mathbf{F}\left(t\right) = \begin{pmatrix} \cos\left(\omega_0 t\right) & \sin\left(\omega_0 t\right) \\ -\sin\left(\omega_0 t\right) & \cos\left(\omega_0 t\right) \end{pmatrix}.$$

From the solution, Eq. (4.30), and from the definition of Gaussian characteristic function, Eq. (4.5), it is straightforward to show that the evolution preserves the Gaussian property of the initial state. The first and second moments evolve as:

$$\begin{aligned} \bar{\mathbf{R}}\left(t\right) &= e^{-\frac{\Gamma(t)}{2}} \mathbf{F}\left(t\right) \bar{\mathbf{R}}\left(0\right), \\ \boldsymbol{\sigma}\left(t\right) &= e^{-\Gamma(t)} \mathbf{F}\left(t\right) \boldsymbol{\sigma}\left(0\right) \mathbf{F}^{\mathsf{T}}\left(t\right) + 2\bar{\mathbf{W}}\left(t\right), \end{aligned} \tag{4.34}$$

and we use this last relations in Chapter 6 to study the behaviour in time of the teleportation fidelity.

It is important to note that the long-time behaviour of the dynamics is independent of the particular form of the spectral density: for $t \gg \tau_E$, where τ_E is the reservoir correlation time, it is possible to show that the dynamics reduces to the Markovian equation Eq. (2.19).

Finally, when we introduce the non-Markovian measure for Gaussian channels in the next Chapter, we refer to an approximate version of the Eq. (4.28), that is called the secular approximation: it consists in neglecting the fastly oscillating terms when $\tau_S \ll \tau_R$, i.e. when the free system dynamics time-scale is less than the relaxation time-scale. In this case the master equation Eq. (4.28), in the interaction picture, reduces to:

$$\begin{aligned} \frac{d\rho\left(t\right)}{dt} =& \frac{\Delta\left(t\right) + \gamma\left(t\right)}{2} \left[2a\rho a^\dagger - \left\{a^\dagger a, \rho\right\}\right] + \\ & \frac{\Delta\left(t\right) - \gamma\left(t\right)}{2} \left[2a^\dagger \rho a - \left\{aa^\dagger, \rho\right\}\right]. \end{aligned} \tag{4.35}$$

For a generic Gaussian state the solution of Eq. (4.35) reads:

$$\begin{aligned} \bar{\mathbf{R}}\left(t\right) &= e^{-\Gamma(t)/2} \bar{\mathbf{R}}\left(0\right), \\ \boldsymbol{\sigma}\left(t\right) &= e^{-\Gamma(t)} \boldsymbol{\sigma}\left(0\right) + 2\Delta_\Gamma\left(t\right) \mathbf{I}, \end{aligned} \tag{4.36}$$

where $\Delta_\Gamma(t) = e^{-\Gamma(t)} \int_0^t \mathrm{d}s e^{\Gamma(s)} \Delta(s)$.

It is possible to show [103] that the Markovian/non-Markovian behaviour of Eq. 4.28 and its approximated version Eq. 4.35 depends on the reservoir parameters, in particular on the ratio $x = \omega_c/\omega_0$ between the cut-off frequency of the bath and the free frequency of the harmonic oscillator, in particular for $x \ll 1$ the evolution is non-Markovian, while it is Markovian in the regime $x \gg 1$. This property allows us to test the measure of non-Markovianity that will be introduced in the next Chapter.

4.7 Witness of non-Markovianity for Continuous Variable systems

The approaches to characterize and quantify non-Markovianity of a quantum evolution in the case of infinite dimensional systems follow the same line of reasoning presented in the finite dimensional case. Nevertheless the optimization procedure is more cumbersome: the number of parameters which characterize a state in the infinite dimensional Hilbert space is infinite. As a consequence all the quantities of interest are typically defined only for Gaussian states and channels, for which a description in terms of finite dimensional vector analysis exists.

In this section we present the two most common approach to non-Markovianity in the Gaussian realm.

4.7.1 Witness of non-Markovianity based on the fidelity

Following the line of reasoning presented in Sec. (3.2.1), in Ref. [99] the authors define a witness of non-Markovianity, for the particular case of one mode channels, based on the contractive property of appropriate distance measures under CPTP maps. In particular they consider the Bures distance:

$$\mathcal{D}_B(\rho_1, \rho_2) = \sqrt{2 - 2\sqrt{F(\rho_1, \rho_2)}},$$

where $F(\rho_1, \rho_2) = \mathrm{Tr}\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}}$ is the fidelity, as this last quantity is well know for the set of Gaussian states. Furthermore, as the fidelity and the distance are monotonic each other, it is convenient to base the measure of non-Markovianity directly on the fidelity. We have:

$$\mathcal{N}_{\mathcal{F}}^G = \max_{\mathbf{P}} \left[-\int_{\dot{\mathcal{F}} < 0} \frac{d}{dt} \mathcal{F}(\mathbf{P}, t) \, dt \right],$$

where $\mathbf{P} = \{\bar{n}_1, \bar{n}_2, r_1, r_2, \phi_1, \phi_2 \, \alpha_1, \alpha_2\}$ are the parameters that define the two one-mode Gaussian input states.

This approach suffers of some drawbacks. As we have described in the last Chapter, it is only a sufficient condition. Furthermore the optimization is only over the restricted set of Gaussian states. Finally, although it is generalizable to the N-mode case, it is useless, due to the rapid increasing of the parameters to be optimized.

4.7.2 Witness of non-Markovianity based on the Interferometric Power

This quantity is based on the monotonic behaviour of the Gaussian interferometric power (GPI) under local completely positive maps [104, 105], i.e. it follows the approach discussed in Sec. (3.2.2).

The GIP allows estimation of a parameter embedded in a unitary dynamics applied to one subsystem only, in the worse-case scenario, through a procedure known as a black-box interferometry.
Consider a two-mode Gaussian state ρ_{AB} that constitutes the probe for an interferometer (Fig. (4.1 a)); the mode B is sent in a black box in which the unitary transformation $U_B^\phi = e^{\imath \phi H_B}$ occurs, with ϕ an unknown phase, while the spectrum of the generator H_B is known. In the particular case of transformations which preserve the Gaussian character of the state, U_B^ϕ can be written as $U_B^\phi = V_B^\dagger W_B V_B$, where $W_B = e^{-\imath \phi b^\dagger b}$ is the shift operator and V_b is a unitary Gaussian transformation. Given the state at the output of the black box, ρ_{AB}^{ϕ, V_B}, a measurement can be performed on the output state to estimate the value ϕ_{est} of the parameter ϕ.

This estimation, in the case of N repetition, having N identical copies of the initial state, is given by the Cramèr-Rao bound:

$$N\Delta\phi^2 \geq \frac{1}{\mathcal{F}(\rho_{AB}^{\phi, V_B})}, \tag{4.37}$$
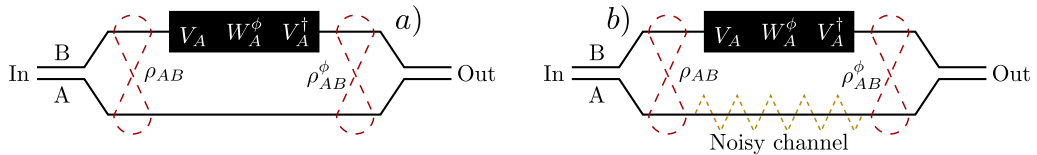

Figure 4.1: Schematic representation of the black box optical interferometry setup. a) the evolution of A is unitary. b) the mode A is sent through a quantum channel.

where $\Delta\phi^2 = <(\phi_{est}-\phi)^2>$ is the variance and $\mathcal{F}(\rho_{AB}^{\phi,V_B})$ is the Quantum Fisher Information (QFI)[4]. The QFI quantifies, for any given black-box generator, the precision of estimation of the parameter ϕ; therefore to obtain, for a given resource ρ_{AB}, the worse-case scenario, it is necessary to minimize the (QFI) over the set of local generators. The GIP is then defined as:

$$\mathcal{Q}_A^G(\rho_{AB}) = \frac{1}{4}\inf_{V_B}\mathcal{F}(\rho_{AB}^{\phi,V_B}). \qquad (4.38)$$

It is possible to show that Eq. (4.38) satisfies the following properties: it is zero if and only if the resource ρ_{AB} is a product state, it is invariant under local unitaries transformation and, especially for what concern us, it is monotonically non increasing under local CPTP operation on subsystem A.

Exploiting this last property it is possible to check the non-Markovianity of a quantum evolution. Indeed suppose now that the mode A, that does not enter in the black-box, undergoes open evolution (Fig. (4.1 b)). Due to the contractive property, for a Markovian evolution the derivative:

$$\mathcal{D}(t) \doteq \frac{d}{dt}\mathcal{Q}^G(\rho_{AB}(t))$$

is a strictly non-positive quantity for all $t \geq 0$. As a consequence, if this quantity is positive the evolution must be non-Markovian. We thus can define the following witness with respect the given input resource as:

$$\mathcal{N}_{\mathcal{Q}}^{\boldsymbol{\sigma}_{AB}} = \int_{\mathcal{D}(t)>0}\mathcal{D}(t)dt,$$

where $\boldsymbol{\sigma}_{AB}$ is the covariance matrix of the two- mode input state. Finally we can obtain a state-independent definition through an optimization procedure over the set of possible initial states:

$$\mathcal{N}_{\mathcal{Q}} = \max_{\boldsymbol{\sigma}_{AB}(t_0)}\mathcal{N}_{\mathcal{Q}}^{\boldsymbol{\sigma}_{AB}}.$$

The main advantage of this approach is that it can be very useful to achieve an experimental characterization of the non-Markovianity of a dynamical map. However the optimization procedure remains difficult to achieve in complete generality, and numerical techniques are necessary.

---

[4]Under the smoothness hypothesis the QFI it is defined as:

$$\mathcal{F} = -2\lim_{\epsilon\to0}\frac{\partial^2}{\partial\epsilon^2}F(\rho_{AB}^{\phi},\rho_{AB}^{\phi+\epsilon}),$$

where F is the fidelity.

CHAPTER 5

Measure of non-Markovianity for Gaussian channels

The problem of characterization and quantification of non-Markovianity has been mostly devoted, till now, to the case of finite-dimensional systems, while less work has been done for the infinite-dimensional case.

In this chapter, starting from the definition provided in Eq. (4.20), we introduce a proper measure of non-Markovianity for Gaussian channels based directly on the divisibility property of the map [4]. We discuss the range of applicability of this definition, and the differences with respect to the finite-dimensional case. Finally, we apply the new measure to some of the non-Markovian channels which has been previously introduced.

5.1 Non-Markovianity for Continuous Variable systems

In Chapter 3 we illustrated various approaches recently introduced in order to characterize and quantify quantum non-Markovianity. We have then discussed proper measures of non-Markovianity for finite-dimensional systems, and we have introduced various witnesses, i.e. quantities whose non monotonic behaviour in the presence of memory effects allows to identity and quantify the non-Markovianity of the evolution. These quantities provide only sufficient, but not necessary, conditions: in fact, there are non-Markovian channels which are not identified by these quantities. Furthermore, the most drawback is perhaps the not avoidable optimization procedure over states that must be performed in order to obtain a state-independent quantity.

The importance to characterize and quantify non-Markovianity in the infinite-dimensional systems is due to the relevant role that these systems

(as for instance Gaussian states and Gaussian channels) play in quantum optics, quantum information, and quantum technologies. If one considers the most general bosonic channels the problem at moment is clearly irresolvable, because the need for an optimization over the states in an infinite-dimensional space makes pointless the definition of witness. Furthermore, a measure of non-Markovianity which is based on its definition requires a method to characterize the complete positivity of the map, a condition that currently lacks. If only Gaussian channels are considered, however, the situation becomes less problematic since, as illustrated in Chapter 4, the characterization of Gaussian states and Gaussian maps requires simply the use of the finite-dimensional matrix algebra. But, if we aim to construct a proper measure of non-Markovianity which is based on its definition, a direct generalization of the Rivas measure is impossible due to the present lacks of the Choi-Jamołkovsky isomorphism for a generic Gaussian map. On the other hand, for Gaussian channels a condition of complete positivity does exist (Eq. (4.20)), and this will be our starting point to define a proper measure of non-Markovianity .

5.2   Measure of Non-Markovianity

We consider a quantum evolution from the instant t_0 to the instant t_2, described in full generality by some family of trace-preserving linear maps $\{\Phi(t_2, t_1), t_2 \geq t_1 \geq t_0\}$, and we recall that this evolution is said to be divisible or Markovian if, for each pair of times t_2 and t_1, it is fulfilled the property

$$\Phi(t_2, t_0) = \Phi(t_2, t_1)\Phi(t_1, t_0), \qquad t_2 \geq t_1 \geq t_0 ,$$

and the intermediate map $\Phi(t_2, t_1)$ is completely positive (CP). Our strategy is to obtain the characterization of divisible map $\Phi(t_2, t_1)$ for a generic Gaussian channel, and then use this characterization in order to introduce a definition of non-Markovianity based on the breaking of the Markovian property. Since, as shown in Sec. (4.5), it is possible to characterize the map through the set $(\mathbf{X}, \mathbf{Y})$ of $2N \times 2N$ matrices, we want to describe the intermediate evolution in terms of these matrices. To this aim, let us introduce an auxiliary vectorial notation [106]. Given a generic Gaussian input state, its time evolution in a Gaussian channel is given (see Sec. (4.5)) by the following transformation on the initial covariance matrix $\boldsymbol{\sigma}(0)$ that defines the state (Eq. (4.19)):

$$\boldsymbol{\sigma}(t) = \mathbf{X}(t)\,\boldsymbol{\sigma}(0)\,\mathbf{X}^{\mathsf{T}}(t) + \mathbf{Y}(t) .$$

Now it is expedient to reorder the component of σ according to a lexico-graphical ordering, as follows: $\vec{\boldsymbol{\sigma}}_k(t) \doteq \boldsymbol{\sigma}_{ij(t)}$, where $k = N(i-1) + j$ and $i, j = 1, \ldots, 2N$. In the more familiar Dirac notation we can write $\langle k|\vec{\boldsymbol{\sigma}}(t)\rangle \equiv \langle ij|\vec{\boldsymbol{\sigma}}(t)\rangle \equiv \langle i|\boldsymbol{\sigma}(t)|j\rangle$. Furthermore, we add to $\vec{\boldsymbol{\sigma}}$ an auxiliary vector entry of value 1, obtaining the vector $(\vec{\boldsymbol{\sigma}}, 1)^\mathsf{T} = (\sigma_{11}, \ldots, \sigma_{nn}, 1)^\mathsf{T}$ Consequently we obtain the following representation: $[\mathbf{X}(t)\boldsymbol{\sigma}(0)\mathbf{X}^\mathsf{T}(t)]_{ij} = [(\mathbf{X}(t)\otimes \mathbf{X}(t))\vec{\boldsymbol{\sigma}}(0)]_k \equiv [\boldsymbol{\Phi}(t)\vec{\boldsymbol{\sigma}}(0)]_k$, where $\boldsymbol{\Phi}(t) = \mathbf{X}(t) \otimes \mathbf{X}(t)$; in Dirac notation:

$$\langle i|\mathbf{X}(t)\boldsymbol{\sigma}\mathbf{X}^\mathsf{T}(t)|j\rangle = \sum_{n,m}\langle i|\mathbf{X}(t)|n\rangle\langle n|\boldsymbol{\sigma}(0)|m\rangle\langle m|\mathbf{X}^\mathsf{T}(t)|j\rangle =$$

$$= \sum_{n,m}\langle ij|\mathbf{X}(t)\otimes\mathbf{X}(t)|nm\rangle\langle nm|\vec{\boldsymbol{\sigma}}(0)\rangle = \langle ij|\mathbf{X}(t)\otimes\mathbf{X}(t)|\vec{\sigma}(0)\rangle,$$

where $\sum_n |n\rangle\langle n| = \mathbb{1}$ is the identity resolution. One can now express Eq. (4.19) in terms of a vector by matrix multiplication:

$$\left(\begin{array}{c} \vec{\boldsymbol{\sigma}}(t) \\ 1 \end{array}\right) = \left(\begin{array}{cc} \boldsymbol{\Phi}(t) & \vec{\mathbf{Y}}(t) \\ \vec{\mathbf{0}}^\mathsf{T} & 1 \end{array}\right)\left(\begin{array}{c} \vec{\boldsymbol{\sigma}}(0) \\ 1 \end{array}\right), \tag{5.1}$$

where $\vec{0} = (0, \ldots, 0)^\mathsf{T}$ is the $2N$-dimensional null vector and $\vec{\mathbf{Y}}(t))$ is the vectorial form of the matrix $\mathbf{Y}(t)$. Vectorization is an isomorphism, thus reversible: de-vectorizing Eq. (5.1) yields exactly the standard representation.

Consider now the semigroup composition law of Gaussian channels, Eq. (4.21):

$$(\mathbf{X}_1, \mathbf{Y}_1) \cdot (\mathbf{X}_2, \mathbf{Y}_2) = (\mathbf{X}_1\mathbf{X}_2, \mathbf{X}_1\mathbf{Y}_2\mathbf{X}_1^\mathsf{T} + \mathbf{Y}_1). \tag{5.2}$$

In vectorial notation this expression assumes the following matrix multiplication form:

$$\left(\begin{array}{cc} \boldsymbol{\Phi}_2 & \vec{\mathbf{Y}}_2 \\ \vec{0}^\mathsf{T} & 1 \end{array}\right)\left(\begin{array}{cc} \boldsymbol{\Phi}_1 & \vec{\mathbf{Y}}_1 \\ \vec{0}^\mathsf{T} & 1 \end{array}\right) = \left(\begin{array}{cc} \boldsymbol{\Phi}_2\boldsymbol{\Phi}_1 & \boldsymbol{\Phi}_2\vec{\mathbf{Y}}_1 + \vec{\mathbf{Y}}_2 \\ \vec{0}^\mathsf{T} & 1 \end{array}\right). \tag{5.3}$$

Setting for ease of notation $t_0 = 0$, $t_1 = t$ and $t_2 = t + \epsilon$ for generic instants of time t and ϵ, continuity of time implies that the dynamics can be split in two steps as $[0, t + \epsilon] = [0, t] \cup [t, t + \epsilon]$, and one can obtain the vectorial expression for the intermediate map in the interval $[t, t + \epsilon]$:

$$\left(\begin{array}{cc} \boldsymbol{\Phi}(t + \epsilon, t) & \vec{\mathbf{Y}}(t + \epsilon, t) \\ \vec{0} & 1 \end{array}\right) = \left(\begin{array}{cc} \boldsymbol{\Phi}(t + \epsilon, 0) & \vec{\mathbf{Y}}(t + \epsilon, 0) \\ \vec{0} & 1 \end{array}\right)\left(\begin{array}{cc} \boldsymbol{\Phi}(t, 0) & \vec{\mathbf{Y}}(t, 0) \\ \vec{0} & 1 \end{array}\right)^{-1}, \tag{5.4}$$

or more explicitly

$$\left(\begin{array}{cc} \boldsymbol{\Phi}(t + \epsilon, t) & \vec{\mathbf{Y}}(t + \epsilon, t) \\ \vec{0} & 1 \end{array}\right) =$$

$$\left(\begin{array}{cc} \boldsymbol{\Phi}(t + \epsilon, 0)\boldsymbol{\Phi}^{-1}(t, 0) & -\boldsymbol{\Phi}(t + \epsilon, 0)\boldsymbol{\Phi}^{-1}(t, 0)\vec{\mathbf{Y}}(t, 0) + \vec{\mathbf{Y}}(t + \epsilon, 0) \\ \vec{0} & 1 \end{array}\right). \tag{5.5}$$

Here we must face the question of the invertibility of the matrix $\mathbf{X}(t,0)$. The possibility to invert the $\mathbf{\Phi}(t,0)$ matrix is connected to the possibility to invert the $\mathbf{X}(t,0)$ matrix. Examples of Gaussian channels characterized by a non invertible $\mathbf{X}$ matrix, based on the classification of one-mode Gaussian channels, can be found in Ref. [107]. Up to Gaussian unitary equivalence, channels for which the matrix $\mathbf{X}$ is non invertible include the completely depolarising channel, which projects every input state on a thermal state, and channels which transform the canonical quadrature Q and P as: $P \rightarrow p$, $Q \rightarrow Q + q$. However, following the same reasoning used for the RHP measure in Sec. (3.1.2), non invertible cases do not impose any restriction because one can always introduce the matrix $\mathbb{1}\eta + \mathbf{X}(t,0)$, determine its inverse, and evaluate the limit $\eta \rightarrow 0$ that is always non-singular [45, 108, 106].

De-vectorizing the intermediate Gaussian map, we obtain its complete expression in terms of the $\mathbf{X}$ and $\mathbf{Y}$ matrices:

$$\mathbf{X}(t+\epsilon,t){=}\mathbf{X}(t+\epsilon,0)\mathbf{X}^{-1}(t,0)\,,$$

$$\tag{5.6}$$

$$\mathbf{Y}(t+\epsilon,t){=}\mathbf{Y}(t+\epsilon,0){-}\mathbf{X}(t+\epsilon,t)\mathbf{Y}(t,0)\mathbf{X}^{\mathsf{T}}(t+\epsilon,t)\,.$$

The condition of divisibility, Eq. (2.21), is equivalent to the complete positivity of the intermediate map, expressed in terms of the matrices defining the channel by Eqs. (5.6), that, for Gaussian channels, gives Eq. (4.20):

$$\mathbf{Y}\left(t\right) - \frac{\imath}{2}\mathbf{\Omega} + \frac{\imath}{2}\mathbf{X}\left(t\right)\mathbf{\Omega}\mathbf{X}^{\mathsf{T}}\left(t\right) \geq 0.$$

Therefore, from Eqs. (4.20), (2.21) and (5.6), the condition of non-Markovianity at any given time t reads:

$$\mathbf{Y}\left(t+\epsilon,t\right) - \frac{\imath}{2}\mathbf{\Omega} + \frac{\imath}{2}\mathbf{X}\left(t+\epsilon,t\right)\mathbf{\Omega}\mathbf{X}^{\mathsf{T}}\left(t+\epsilon,t\right) < 0\,. \tag{5.7}$$

Since Eq. (4.20) is a necessary and sufficient condition for the complete positivity of Gaussian channels, it follows that Eq. (5.7) is a necessary and sufficient criterion for the non-Markovianity of Gaussian channels.

Eq. (5.7) allows to introduce a proper measure of non-Markovianity for Gaussian channels by quantifying the extent by which the intermediate dynamics fails to be CP. This corresponds clearly to the quantification of the negative part of the spectrum of the matrix that appears in the l.h.s. of Eq. (5.7). Denoting the set of eigenvalues by $\{\nu_k(t+\epsilon,t)\}_{k=1,\dots,2N}$, the following functions

$$f_k\left(t\right) = \frac{1}{2}\lim_{\epsilon\to 0^+}\left[|\nu_k(t+\epsilon,t)| - \nu_k(t+\epsilon,t)\right] \tag{5.8}$$

quantify the negative contribution at time t given by the kth eigenvalue. Therefore, the amount of non-Markovianity quantified by the negative part of the spectrum at a given time t is expressed by the *punctual nonMarkovianity*:

$$F\left(t\right) \equiv \sum_{k=1}^{2N} f_k\left(t\right) \ .$$ (5.9)

Since $F(t) > 0$ if and only if the evolution is non-Markovian, and $F(t) = 0$ otherwise, the total amount of non-Markovianity in a generic time interval I is

$$\mathcal{N}^I \equiv \int_I F\left(t\right) \ dt.$$ (5.10)

It is worthy to be remarked that, when the dynamics is described by means of a master equation, the expressions in the phase space formalism of the matrices $(\mathbf{X}, \mathbf{Y})$ which define the channel, as illustrated in Sec. (4.5.1), are obtained directly from the expression of the characteristic function of the evolved Gaussian state.

The measure we have introduced is based directly on the definition of Markovian evolution and it checks the CP condition of the intermediate map. So this measure seems the Gaussian infinite-dimensional equivalent of the RHP measure described in Sec. (3.1.2). However, as discussed in Sec. (3.1.3), Hall, Cresser, Li, and Andersson [66] recently showed that in the finite-dimensional case for which (at variance with the infinite-dimensional case) all processes always admit a generator, the necessary and sufficient criterion for non-Markovianity based on divisibility is equivalent to the criterion based on the negativity of the decoherence rates appearing in the canonical form of the master equation. Therefore, one is tempting to conjecture that this equivalence holds also in the infinite-dimensional case for channels that admit a generator, but actually we are not able to provide a proof of this conjecture.

In the next section we apply the above measure to some of the Gaussian channels defined in Chapter 4, and we compare the results with known facts about the non-Markovianity of these maps.

5.3   Two simple examples

In this section, we discuss two paradigmatic cases that admit a representation in terms of master equations: the Damping master equation and the Quantum Brownian motion.

5.3.1 Non-Markovianity of the Damping master equation

As discussed in Sec. (4.6.1), the Damping master equation (4.22) is the simplest example we can begin with as it describes the damping process for a single field mode with a single decay rate:

$$\frac{d\rho\left(t\right)}{dt} = \alpha\,\gamma\left(t\right)\left[a\rho a^{\dagger} - \frac{1}{2}\left\{a^{\dagger}a, \rho\right\}\right]\,,$$

where $\alpha \ll 1$ is the coupling constant and $\gamma(t)$ is the damping rate. The evolution of a generic Gaussian state in this Gaussian channel is described by the corresponding evolution of the $\mathbf{X}$ and $\mathbf{Y}$ matrices (Eqs. (4.23) and (4.24)):

$$\mathbf{X}\left(t, 0\right) = e^{-\frac{\Gamma(t)}{2}}\mathbb{1}\,,$$

$$\mathbf{Y}\left(t, 0\right) = \left[1 - e^{-\Gamma(t)}\right]\frac{\mathbb{1}}{2}\,,$$

where $\Gamma(t) = 2\alpha\int_0^t \gamma(s)ds$. These equations allow to obtain, by exploiting Eqs. (5.6), the matrix that appears in the l.h.s. of the CP condition, Ineq. (5.7). It is straightforward to verify that the eigenvalues of this matrix are negative if $\exp(-\Gamma(t + \epsilon, t)) < 1$, where $\Gamma(t + \epsilon, t) = \Gamma(t + \epsilon, 0) - \Gamma(t, 0)$. Moreover, to first order in ϵ we have $\Gamma(t + \epsilon, t) \approx 2\gamma(t)\epsilon$. Consequently, the evolution is non-Markovian if and only if $\gamma(t) < 0$. This result corresponds exactly to the violation of differential Markovian evolution given in Sec. (2.2.2), and concerning the generator of the master equation. Using Eqs. (5.8) and (5.9), the corresponding measure reads:

$$\mathcal{N}^I = -\alpha\int_{I'}\gamma\left(t\right)dt\,, \tag{5.11}$$

where I' are the sub-intervals of I in which $\gamma(t) < 0$.

5.3.2 Non-Markovianity of the Quantum Brownian motion under the secular approximation

We next consider the Quantum Brownian Motion in the weak coupling limit and under the secular approximation [103] presented in Sec. (4.6.2). It is described in the interaction picture by the Lindblad-type master equa-

tion (4.35):

$$\frac{d\rho\left(t\right)}{dt} = \frac{\Delta\left(t\right) + \gamma\left(t\right)}{2}\left[2a\rho a^\dagger - \left\{a^\dagger a, \rho\right\}\right] +$$
$$+ \frac{\Delta\left(t\right) - \gamma\left(t\right)}{2}\left[2a^\dagger\rho a - \left\{aa^\dagger, \rho\right\}\right] ,$$

where the coefficients $\gamma(t)$ and $\Delta(t)$ are the damping coefficient and the diffusion coefficient, respectively. The general solution allows to obtain the evolution of the displacement and covariance matrices for any input Gaussian state (Eq. (4.36)). The corresponding $\mathbf{X}$ and $\mathbf{Y}$ matrices read:

$$\mathbf{X}\left(t, 0\right) = e^{-\frac{\Gamma(t)}{2}}\mathbf{R}\left(t\right) , \qquad (5.12)$$

$$\mathbf{Y}\left(t, 0\right) = e^{-\Gamma(t)}\tilde{\Delta}\left(t\right)\mathbb{1} , \qquad (5.13)$$

where $\Gamma(t) = 2\int_0^t \gamma(s)ds$, $\tilde{\Delta}(t) = \int_0^t e^{\Gamma(s)}\Delta(s)ds$, $\mathbf{R}(t)$ is the matrix describing a rotation by the angle $\omega_0 t$, and ω_0 is the system's characteristic frequency. These expressions and Eqs. (5.6) determine the eigenvalues of the matrix in the l.h.s. of Ineq. (5.7):

$$\nu_1(t + \epsilon, t) = \frac{1}{2}\left[e^{-\Gamma(t+\epsilon, t)} + 2\tilde{\Delta}\left(t + \epsilon, t\right)e^{-\Gamma(t+\epsilon, 0)} - 1\right],$$

$$\nu_2(t + \epsilon, t) = \frac{1}{2}\left[1 - e^{-\Gamma(t+\epsilon, t)} + 2\tilde{\Delta}\left(t + \epsilon, t\right)e^{-\Gamma(t+\epsilon, 0)}\right], \qquad (5.14)$$

where $\Gamma(t + \epsilon, t) = \Gamma(t + \epsilon, 0) - \Gamma(t, 0)$ and $\tilde{\Delta}(t + \epsilon, t) = \tilde{\Delta}(t + \epsilon, 0) - \tilde{\Delta}(t, 0)$. To first order in ϵ, we have: $e^{-\Gamma(t+\epsilon, t)} \approx 1 - 2\gamma(t)\epsilon$ and $\tilde{\Delta}(t + \epsilon, t) \approx e^{\Gamma(t, 0)}\Delta(t)\epsilon$. Then, condition Eq. (5.7) on the eigenvalues, i.e. the violation of the divisibility condition, implies $\Delta(t) < |\gamma(t)|$. This is again equivalent to negativity of the decoherence rates $[\Delta(t) + \gamma(t)]/2$ and $[\Delta(t) - \gamma(t)]/2$ appearing in Eq. (5.12). Finally, exploiting Eqs. (5.14) and (5.8) we obtain the following expression for the punctual measure of non-Markovianity:

$$F\left(t\right) = \frac{1}{2}\left[\left|\Delta\left(t\right) - \gamma\left(t\right)\right| + \left|\Delta\left(t\right) + \gamma\left(t\right)\right|\right] - \Delta\left(t\right) . \qquad (5.15)$$

As discussed in Sec. (4.6.2), in order to investigate explicitly the behaviour of non-Markovianity in the Quantum Brownian Motion we need to specify the spectral density to obtain explicit expressions of the damping and diffusion coefficients $\gamma(t)$ and $\Delta(t)$. By considering the rather typical case of an Ohmic bath with an exponential cut-off ω_c, the parameters which control the dynamics are the temperature T and the ratio, $x = \omega_c/\omega_0$, between the cut-off frequency of the bath and the characteristic frequency of the system.
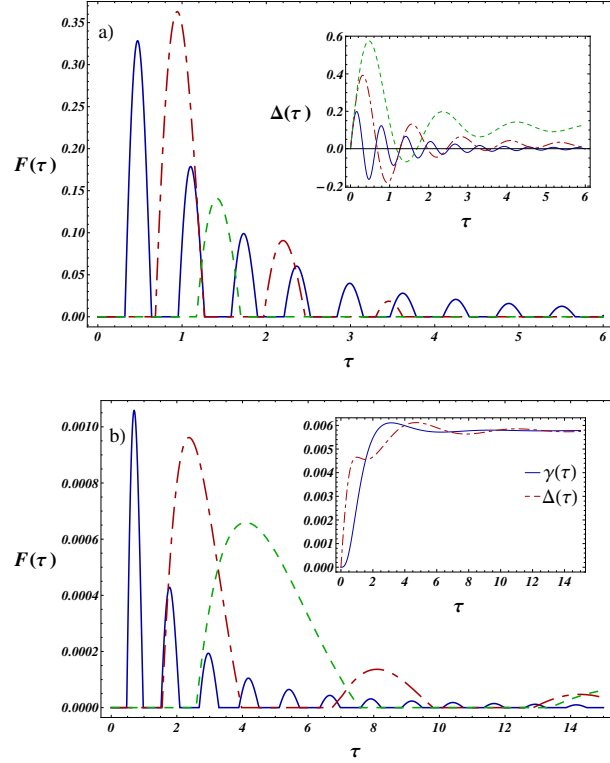
Figure 5.1: Punctual non-Markovianity $F(\tau)$, Eq. (5.9). a): High-temperature limit for $x = 0.1$ (blue full line), $x = 0.2$ (red dot-dashed line), and $x = 0.3$ (green dashed line). Inset: $\Delta(\tau)$ for $x = 0.1$ (blue full line), $x = 0.2$ (red dot-dashed line) and $x = 0.3$ (green dashed line). b): Low-temperature limit for $x = 0.2$ (blue full line), $x = 1.0$ (red dot-dashed line) and $x = 2.0$ (green dashed line). Inset: $\Delta(\tau)$ (red dot-dashed line) and $\gamma(\tau)$ (blue line) for $x = 0.1$.

As discussed in Sec. (4.6.2), it is expected that in the regime $x \ll 1$ the dynamics should be non-Markovian, while Markovianity should be recovered for $x \gg 1$ [103]. It is also convenient to express the evolution in terms of the dimensionless reduced time $\tau = \omega_c t$. Furthermore, explicit analytic expressions of the diffusion coefficient $\Delta(\tau)$ can be quite straightforwardly obtained in the high- and low-temperature regimes. The explicit expressions are reported in Appendix (C). By considering first, both in the high-temperature and in the low-temperature regimes [109], the asymptotic values of the damping and diffusion coefficients in the large-time limit $\tau \to \infty$, it is straightforward to verify that the asymptotic punctual non-Markovianity $F(\infty) = 0$: at large times Markovianity is always recovered, independently of the values of the parameters that govern the dynamics.

Considering now generic times, in Fig. (5.1a) we report the behaviour of

the punctual non-Markovianity F, Eq. (5.15), as a function of the reduced time τ at fixed values of the parameter $x = \omega_c/\omega_0$ in the high-temperature limit. In this regime $\Delta(\tau) \gg \gamma(\tau)$ [103], and the non-Markovianity of the dynamics depends essentially only on the diffusion coefficient: $F(\tau) \simeq |\Delta(\tau)| - \Delta(\tau)$. Hence, the time interval for which the evolution is non-Markovian ($F(\tau) > 0$) corresponds to the negativity of the decoherence rate, $\Delta(\tau) < 0$. Non-Markovianity is strong in the regime $x \ll 1$, corresponding to the characteristic time of the bath much larger than the characteristic time of the system. When x increases, the negative part of the oscillations and $F(\tau)$ quickly vanish, and one recovers the Markovian regime.

In the low-temperature regime, see Fig. (5.1*b*), the diffusion and damping coefficients are comparable, and the non-Markovianity $F(\tau)$ is given by the full expression, Eq. (5.15). In this situation, a non-Markovian regime is observed also for $\Delta(\tau) > 0$, provided $\Delta(\tau) < \gamma(\tau)$, and even if the characteristic times of the bath start to be comparable or smaller than the characteristic times of the system, $x \gtrsim 1$. In these examples, the criterion based on the **X** and **Y** matrices defining a Gaussian channel turns out to correspond to the negativity of the decoherence rates. Indeed, even this is not a proof, it is anyway an indication pointing in this direction. Furthermore, it should be stressed that the criterion is much more general and applies to any Gaussian evolution, including those that do not admit a generator and hence cannot be described in terms of master equations. Finally, the criterion always allows, at least in principle, the experimental verification of the Markovianity of the evolution, in particular when it is expressed through the master equation coefficients that can be experimentally reconstructed [110]. Finally, we stress what is probably the main advantage of this measure: it does not require optimization over the set of input states since it is based directly on the characteristic matrices that define intrinsically the dynamical map.

CHAPTER 6

Non-Markovian Continuous Variable Quantum Teleportation

In previous Chapters we addressed the problem of characterization and quantification of the non-Markovian character of a quantum channel. However, a primarily important aspect is the usefulness of non-Markovian channels, in particular as a tool in Quantum Information theory. In fact, the role of the non-Markovianity as a prominent tool to contrast the decoherence effects generated by the interaction of a quantum system with the environment and, consequently, as a promising strategy to improve the efficiency of quantum technologies, gave a decisive boost to his study. Here we focus our attention on the Realistic Continuous-Variable (CV) Quantum Teleportation protocol, historically one of the paradigmatic arenas to test new quantum ideas. In particular, the teleportation fidelity, which is the figure of merit for evaluating the success of the protocol, can provide a quantitative estimate of the advantage of exploiting non-Markovian channels; furthermore its simple expression in terms of the characteristic functions of the input state and of resources allows us to manage in a simple way even non Gaussian resources.

6.1 The realistic Braunstein-Kimble Quantum Teleportation protocol

In this section we describe the realistic Braunstein-Kimble (BK) Quantum Teleportation protocol, by extending the scheme presented in Ref. [85] to the non-Markovian case [5]. Even following the same procedure, we present the protocol from a different perspective. In fact, a Markovian channel that describes the uncontrollable noise affecting the protocol can only have a detrimental effect on the positive realization of the teleportation. At variance, a
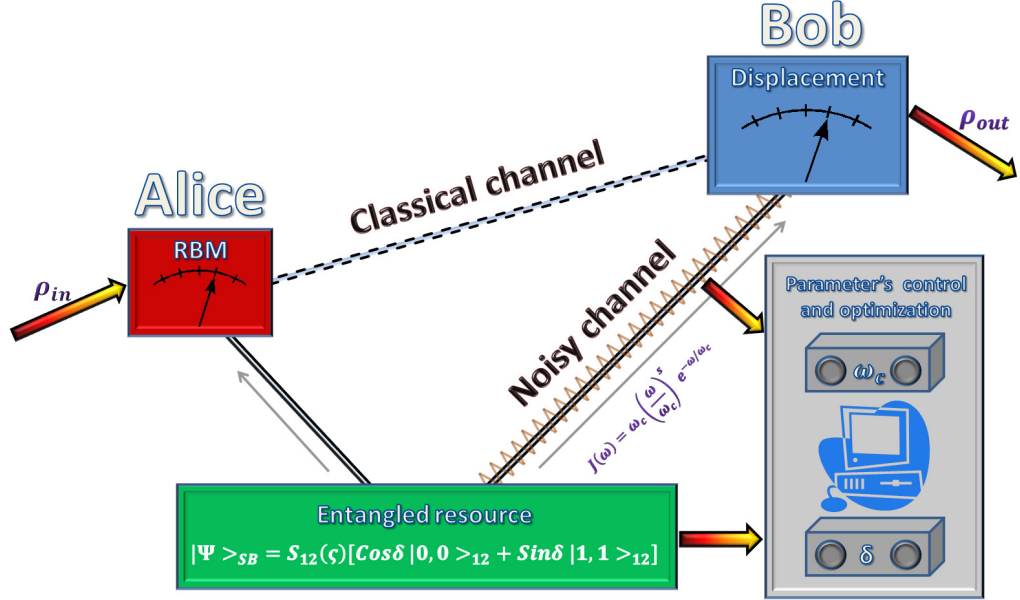
Figure 6.1: Realistic Braunstein-Kimble continuous variable quantum teleportation protocol.

non Markovian channel is a *structured environment*, because it is possible to experimentally control the parameters characterizing the dynamics; *this allows to extend also to the parameters associated to the channel the optimization procedures exploited in* Refs. [86, 83, 85] *on the parameters which characterize the quantum resources.* Fig. (6.1) illustrates the protocol. In the standard procedure the first user (Alice) wants to teleport a single-mode input state ρ_{in} to a second user (Bob). They share a classical communication channel, characterized by the gain g, and an entangled two-mode quantum state ρ_{12} (the resource); in particular, the mode 1 is available to Alice, while mode 2 is sent to Bob through a non-Markovian quantum channel[1]. The success of the procedure is measured by the teleportation fidelity that, for a pure input state, is expressed in terms of the characteristic functions of the input and output state as [111]:

$$\mathcal{F} = \frac{1}{\pi} \int \mathrm{d}^2\alpha \chi_{\text{in}}\left(\alpha\right) \chi_{\text{out}}\left(-\alpha\right). \qquad (6.1)$$

The fidelity is equal to 1 if and only if the input and output states coincide, and is equal to 0 if and only if the two states can be distinguished with

---

[1]A more general situation would be to consider the case in which even the Alice's mode is affected by noise. However the results do not change significantly.

certainty by performing a quantum measurement.

Now we proceed to compute the characteristic function of the output state, and in the following we illustrate the transformations on the characteristic function which correspond to the various steps of the protocol.

We assume that the input state is a pure state: this is not a serious limitation because one can always map the case of a non ideal teleportation protocol with noisy (mixed) inputs and resources to an equivalent protocol with pure inputs and resources, but with a correspondingly larger amount of noise affecting the protocol. As a first step Alice mixes the single-mode input state with the mode 1 of the resource (her available mode) through a 50-50 beam splitter. Before the operation the characteristic function of the initial three-mode field is given by $\rho_0 = \rho_{\text{in}} \otimes \rho_{\text{res}}$ and the global characteristic function, from the definition Eq. (4.5), is $\chi_0 (\alpha_{\text{in}}; \alpha_1; \alpha_2) = \chi_{\text{in}} (\alpha_{\text{in}}) \chi_{res} (\alpha_1; \alpha_2)$. Furthermore, it is convenient to express all the quantities in terms of quadrature operators (x_i, p_1), $i = \text{in}, 1, 2$:

$$\chi_0 (x_{\text{in}}, p_{\text{in}}; x_1, p_1; x_2, p_2) = \chi_{\text{in}} (x_{\text{in}}, p_{\text{in}}) \chi_{\text{res}} (x_1, p_1; x_2, p_2). \qquad (6.2)$$

Applying the beam splitter transformation, Eq. (A.5), with $t = r = \frac{1}{\sqrt{2}}$ Alice obtains:

$$\begin{aligned}
\chi_0' (x_{\text{in}}', p_{\text{in}}'; x_1', p_1'; x_2, p_2) = & \chi_{\text{in}} \left(\frac{1}{\sqrt{2}} (x_{\text{in}}' + x_1'), \frac{1}{\sqrt{2}} (p_{\text{in}}' + p_1') \right) \times \\
& \chi_{\text{res}} \left(\frac{1}{\sqrt{2}} (x_{\text{in}}' - x_1'), \frac{1}{\sqrt{2}} (p_{\text{in}}' - p_1'); x_2, p_2 \right).
\end{aligned}$$
$$(6.3)$$

Alice then faces the next step, that is a (non-ideal) Bell measurement (a homodyne detection) on the modes in and 1. In order to describe a non-ideal measurement, we need to consider the inefficiencies of the photo-detectors. As depicted in Fig. (6.2), a realistic detector can be modelled by placing a fictitious beam-splitter, i.e. a partly transmitting mirror, in front of an ideal detector. The modes in$'$ and 1$'$ at the output of the first beam splitter are sent into the ports of the two remaining beam splitters, characterized by the same transmissivity η. The remaining input ports of the beam splitters are then fed with two vacuum states $|0\rangle_{3'}$, $|0\rangle_{4'}$, whose characteristic function is of the form:

$$\chi_k (x_k', p_k') = \exp \left[-\frac{1}{2} (x_k'^2 + p_k'^2) \right], \qquad (6.4)$$

with $k = 3, 4$. With this procedure, starting from the characteristic function Eq. (6.3) and exploiting again the beam splitter transformation Eq. (A.5),

we obtain:

$$\chi_{\mathrm{BS}}\left(x_{\mathrm{in}}'', p_{\mathrm{in}}''; x_1'', p_1''; x_2, p_2, x_3'', p_3'', x_4'', p_4''\right) =$$
$$= \chi_{\mathrm{in}}\left(\frac{1}{\sqrt{2}}\left[Tx_{\mathrm{in}}'' + Rx_3'' + Tx_1'' + Rx_4''\right], \frac{1}{\sqrt{2}}\left[Tp_{\mathrm{in}}'' + Rp_3'' + Tp_1'' + Rp_4''\right]\right)\times$$
$$\times \chi_{\mathrm{res}}\left(\frac{1}{\sqrt{2}}\left[Tx_{\mathrm{in}}'' + Rx_3'' - Tx_1'' - Rx_4''\right], \frac{1}{\sqrt{2}}\left[Tp_{\mathrm{in}}'' + Rp_3'' - Tp_1'' - Rp_4''; x_2, p_2\right]\right)\times$$
$$\times \chi_3\big(Tx_3'' - Rx_{\mathrm{in}}'', Tp_3'' - Rp_{\mathrm{in}}''\big)\, \chi_4\left(Tx_4'' - Rx_1'', Tp_4'' - Rp_1''\right).$$
(6.5)

Alice applies the two homodyne measurements, by measuring the first quadrature on the mode 1 and the second quadrature on the mode in, and obtains as a result $\tilde{x}$ and $\tilde{p}$, respectively. After the measurements, the remaining mode 2 is left in a mixed state. In Appendix B it is shown that the characteristic function after the measurements becomes:

$$\chi_{Bm}(x_2, p_2) = \frac{\mathcal{P}^{-1}(\tilde{p}, \tilde{x})}{(2\pi)^2}\int dx_{in}'' dp_1''\, e^{ix_{in}''\tilde{p} - i\tilde{x}p_1''}\chi_{BS}(x_{in}'', 0; 0, p_1''; x_2, p_2; 0, 0; 0, 0),$$

with:

$$\mathcal{P}(\tilde{p}, \tilde{x}) = \mathrm{Tr}[|\tilde{p}\rangle_{in''\ in''}\langle\tilde{p}| \otimes |\tilde{x}\rangle_{1''\ 1''}\langle\tilde{x}|\ \rho_{BS}] =$$
$$= \frac{1}{(2\pi)^2}\int dx_{in}'' dp_1''\ \exp\{ix_{in}''\tilde{p} - i\tilde{x}p_1''\}\chi_{BS}(x_{in}'', 0; 0, p_1''; 0, 0; 0, 0; 0, 0).$$

The mode 2 of the resource is sent to Bob in a non-Markovian noisy channel. Clearly the dynamics depends on the particular channel that has been selected. We choose the Quantum Brownian Motion, Eq. (4.28); consequently, in the characteristic function description the evolution of the characteristic function χ_{Bm} is given by the Eq. (4.28). After receiving the mode 2, Bob finally performs on it a displacement $\lambda = g(\tilde{x} + \tilde{p})$ depending by the results communicated by Alice through the classical channel.

After the entire process, the characteristic function of the output state reads:

$$\chi_{out}(x_2, p_2) = \chi_{in}(g\mathcal{T}x_2, g\mathcal{T}p_2)\times$$
$$\times \chi_{res}\Big(g\mathcal{T}x_2, -g\mathcal{T}p_2; e^{-\frac{\Gamma(t)}{2}}[x_2\cos\omega_0 t - p_2\sin\omega_0 t], e^{-\frac{\Gamma(t)}{2}}[x_2\sin\omega_0 t + p_2\cos\omega_0 t]\Big)\times$$
$$\times \exp\left[\left(\bar{W}_{11} + \frac{g^2\mathcal{R}^2}{2}\right)x_2^2 + \left(\bar{W}_{22} + \frac{g^2\mathcal{R}^2}{2}\right)p_2^2 + 2\bar{W}_{12}x_2 p_2\right].$$
(6.6)

The teleportation fidelity is then finally obtained by using Eqs. (6.1), (6.6), and by choosing the input state, namely the characteristic function $\chi_{\mathrm{in}}(x_{\mathrm{in}}, p_{\mathrm{in}})$.

Once given the general description of the non ideal protocol in terms of the characteristic functions, we must now chose the states used as a resource. We will
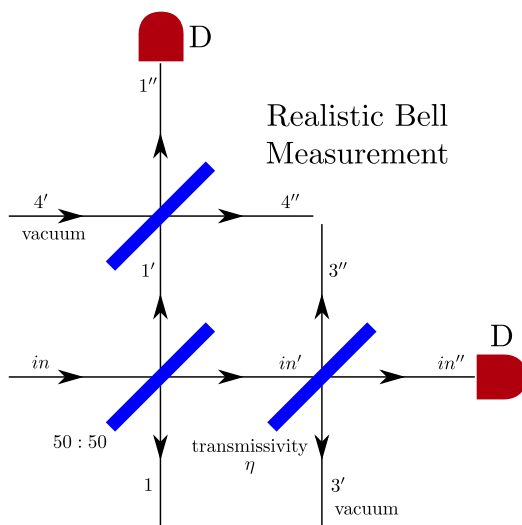
Figure 6.2: Scheme of a realistic Bell measurement. The model takes into account that the detectors D performing the homodyne measurement are not ideal (their efficiency is not 1). In the scheme, such inefficiency is simulated by the introduction of two fictitious beam splitters, BS_2 and BS_3, with equal transmissivity η.

refer to two classes of entangled resources already discussed in Sec. (4.4): the two-mode squeezed-Bell (SB) states $|\psi\rangle_{SB}$, and the two-mode squeezed-cat (SC) states $|\psi\rangle_{SC}$. In fact, although our main interest are the non-Markovian effects, the use of non Gaussian resources allows us, on the one hand a more complete investigation targeted to a possible engineering of effective experimental implementations of the protocol, and on the other hand a better insight on the relation between the quantum property of a state and the corresponding behaviour of the fidelity in a non-Markovian environment.

By summarizing, we want to observe the interplay between the non-Markovian property of the channel and its relation with the quantum properties of the resources, as this allows to improve the teleportation fidelity with respect to the Markovian case [85]. From an operational point of view, identify the values of the parameters associated to these aspects is equivalent to assuming the control on the characteristics of the experimental apparatus, including the characteristics of the entangled resources and the length and memory of the noisy channel. The tunable experimental parameters are: *for all the resources* (included the Gaussian TB), the squeezing phase ϕ; *for the SB resources*, the parameters δ, θ (the angle in cosine and sine, and the relative phase within the superposition, respectively); *for the SC resources*, the parameters δ, θ, $|\gamma|$, φ (the angle in cosine and sine, the relative phase, the modulus and the phase of the coherent parameter within the superposition); *for the channel*, the parameter x of non-Markovianity (or, equivalently, the cut-off frequency ω_c) and the dimensionless time $\tau = \omega_c t$. In the next

section we present an analysis of the performance of the teleportation protocol based on a procedure of optimization of these controllable parameters.

6.2 Results

As resources for the CV Quantum Teleportation protocol we use the TB state described in Sec. (4.3.2) (as Gaussian reference state) and the non-Gaussian states described in Sec. (4.4), namely the Squeezed Cat (SC) state, the (optimized) Squeezed Bell (SB) state and the Photon-Subtracted (PS) state. Furthermore we choose as input state the coherent state $\rho_{in} = |\beta\rangle_{in\ in}\langle\beta|$ with complex amplitude β. Finally, in order to make the fidelity β-independent, we fix the value of the gain as $g = 1/\mathcal{T}$, and we choose $\mathcal{R}^2 = 0.05$.

Under the above assumptions the analytical expressions of the teleportation fidelities associated with the entangled resources depend explicitly on the remaining tunable parameters: $\mathcal{F}_{SB} = \mathcal{F}_{SB}(r, \phi, \delta, \theta)$, $\mathcal{F}_{SC} = \mathcal{F}_{SC}(r, \phi, \delta, \theta, |\gamma|, \varphi)$, $\mathcal{F}_{PS} = \mathcal{F}_{PS}(r, \phi)$, $\mathcal{F}_{TB} = \mathcal{F}_{TB}(r, \phi)$; the last two fidelities can be obtained as particular cases of the fidelity for the SB resources. The numerical maximization of the teleportation fidelity are carried out over a subset, or over all the tunable parameters ϕ, δ, θ, $|\gamma|$, φ, at fixed (finite) values of the squeezing r, and at fixed values of x, τ. The analytic expressions of the above fidelities, being very cumbersome, are not reported. Nevertheless, in order to better clarify the role in the optimization procedure of the various parameters, and in particular of the phases, it is sufficient to look at the exact expression of the fidelity in the case of Gaussian TB resources, that is reported in the Appendix D.

6.2.1 Partially optimized fidelities

At first, as in the articles where Markovian channels were considered, we limit the optimization procedure to the parameter δ for the squeezed Bell resources $|\psi_{SB}\rangle$, and to the parameters δ and $|\gamma|$ for the squeezed cat states $|\psi_{SC}\rangle$, by choosing for the phases ϕ, θ, φ the fixed values $\phi = \pi$ and $\theta = \varphi = 0$. Indeed, in the case of propagation along Markovian channels these values of the phases are sufficient to guarantee always the best performance (see Appendix D). Therefore, the partially optimized fidelities of teleportation are defined as:

$$\mathcal{F}_{p-opt}^{(SB)} = \max_{\delta} \left.\mathcal{F}_{SB}(r, \delta)\right|_{\phi=\pi, \theta=0}, \tag{6.7}$$

$$\mathcal{F}_{p-opt}^{(SC)} = \max_{\delta, |\gamma|} \left.\mathcal{F}_{SC}(r, \delta, |\gamma|)\right|_{\phi=\pi, \theta=\varphi=0}. \tag{6.8}$$

As previously remarked, the fidelities associated with PS resources and with Gaussian TB resources can be easily obtained from $\mathcal{F}_{SB}(r, \delta)$ by suitably choosing the value of δ. Fig. (6.3) contains an array of plots which display the partially optimized fidelities of teleportation $\mathcal{F}_{p-opt}$ as functions of the dimensionless time τ,

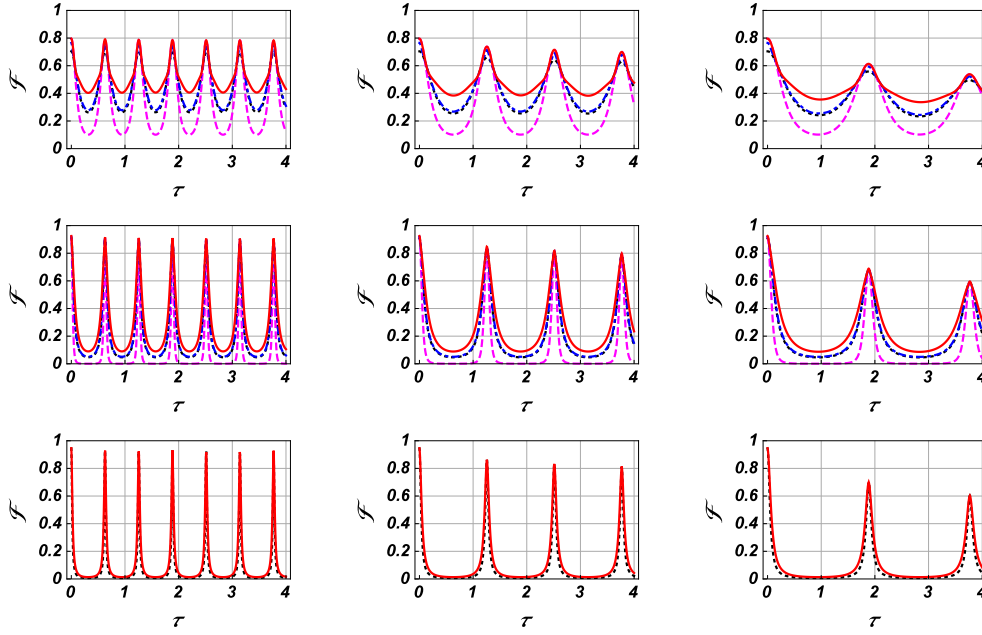Figure 6.3: Array of plots of the partially optimized fidelities of teleportation $\mathcal{F}_{p-opt}$, as functions of the dimensionless time τ, for different values of the parameter $x = 0.1$, 0.2, 0.3 (plots ordered from left to right for increasing x) and of the squeezing $r = 0.5$, 1.5, 2.5 (plots ordered from top to bottom for increasing r). The curves correspond to the fidelities of teleportation of single-mode input coherent states $|\beta\rangle$ obtained by using SB (full line), SC (dashed line), PS (dotted line), and TB (long-dashed line), as entangled resources. In the last row ($r = 2.5$), the fidelities associated with SC and PS states are omitted.

for different choices of the squeezing parameter r and of the parameter of non-Markovianity x. Inside each row, the value of the squeezing parameter is kept fixed, while the parameter x takes, from left to right, the values $x = 0.1$, 0.2, 0.3; so the environment ranges from a strongly non-Markovian regime ($x = 0.1$), to a behaviour characterized by the presence of both Markovian and non-Markovian regimes ($x = 0.3$). Instead, if one moves from the first to the third row, the squeezing parameter increases from a low value ($r = 0.5$) to the maximum value that in principle is at present attainable ($r = 2.5$). The oscillating behaviour exhibited by each fidelity is a clear signature of the non-Markovian regime; however, the curves associated with the value $x = 0.3$, i. e. with a lower content of non-Markovianity, clearly exhibit a global decreasing trend for increasing values of τ, due to the emergence of the Markovian regime. The first row of plots is associated with a low value of the squeezing parameter ($r = 0.5$); the optimized SB resources provide the best performance with respect to the other resources. In particular, the fidelities satisfy the hierarchy $\mathcal{F}_{p-opt}^{(SB)} \geq \mathcal{F}_{p-opt}^{(SC)} \geq \mathcal{F}_{p-opt}^{(TB)}$, that do not include PS states. In

fact, the PS resources exhibit a performance comparable with the SB resources around the peaks, but the worst performance elsewhere. Such a behaviour of the PS resources is due to the lack of an optimizable parameter ruling the efficiency of the state. Passing to the second ($r = 1.5$) and the third ($r = 2.5$) rows of plots, we observe the same features as in the first row, but we see also that the curves around the peaks (relative maxima) are tighter and the relative minima are lower. For high values of the squeezing parameter, the weight of the non-Gaussianity (at least for the here considered non-Gaussian resources) becomes less significant, and the curves become quite indistinguishable around the peaks.

6.2.2 Fully optimized fidelities

Now we show the unexpected role that phases can play in the non-Markovian channels. In fact, we include in the optimization procedure also the phases which previously were kept fixed, i. e. the squeezing phase ϕ and the phases of the non-Gaussian resources. We thus consider the fully optimized fidelities of teleportation, defined as:

$$\mathcal{F}_{f-opt}^{(SB)} = \max_{\phi,\delta,\theta} \mathcal{F}_{SB}(r,\phi,\delta,\theta)\,, \tag{6.9}$$

$$\mathcal{F}_{f-opt}^{(SC)} = \max_{\phi,\delta,\theta,|\gamma|,\varphi} \mathcal{F}_{SC}(r,\phi,\delta,\theta,|\gamma|,\varphi)\,, \tag{6.10}$$

$$\mathcal{F}_{f-opt}^{(PS)} = \max_{\phi} \mathcal{F}_{PS}(r,\phi)\,, \tag{6.11}$$

$$\mathcal{F}_{f-opt}^{TB)} = \max_{\phi} \mathcal{F}_{TB}(r,\phi)\,. \tag{6.12}$$

As shown in the Appendix D, in the instance of TB resources the optimal phase ϕ is given by the τ-dependent relation $\phi = \pi + \dfrac{\tau}{x}$. We have numerically checked that, although non-trivially connected with the polynomial structure of the fidelity corresponding to non-Gaussian resources, the optimal phase is again given by the same relation. Therefore, the optimization on the phase ϕ in Eqs. (6.9), (6.10), (6.11), (6.12) is equivalent to the substitution $\phi = \pi + \dfrac{\tau}{x}$ (Eq. (D.6)). Fig.(6.4) contains an array of plots which display the fully optimized fidelities of teleportation $\mathcal{F}_{f-opt}$ as functions of the dimensionless time τ, for different choices of the squeezing parameter r and of the parameter of non-Markovianity x. We observe that the present situation is quite different from the case of partial optimization: the further optimization carried out on the phases leads to the curves of Fig. (6.4), which can be obtained as a sort of interpolation of the absolute maxima of the corresponding curves of Fig. (6.3). Therefore, at variance with the Markovian channels, non-Markovian channels are very sensitive to the phases included in the resources, and the exploitation of these phases allows a complete control of the optimized fidelity at any instant of time τ, by suppressing the oscillations typical of the non-Markovian regime. This is true both for Gaussian and for non-Gaussian
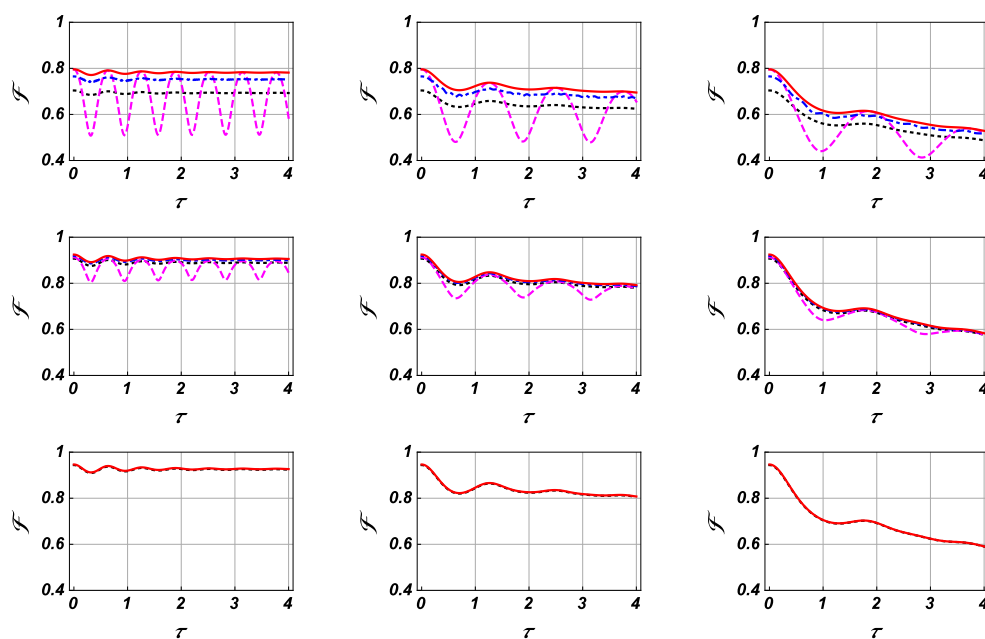
Figure 6.4: Array of plots of the fully optimized fidelities of teleportation $\mathcal{F}_{f-opt}$, as functions of the dimensionless time τ, for different values of the parameter $x = 0.1, 0.2, 0.3$ (plots ordered from left to right for increasing x) and of the squeezing $r = 0.5, 1.5, 2.5$ (plots ordered from top to bottom for increasing r). The curves correspond to the fidelities of teleportation of single-mode input coherent states $|\beta\rangle$ obtained by using SB (full line), SC (dashed line), PS (dotted line), and TB (long-dashed line), as entangled resources. In the last row ($r = 2.5$), the fidelities associated with SC and PS states are omitted.

resources. The behaviour of the fidelity in each row, at fixed squeezing and for increasing values of the parameter x, is similar to that described from the corresponding row in the case of partial optimization: the performance deteriorates if the degree of non-Markovianity decreases (moving from left to right inside a single row). Increasing the value of the squeezing (moving from the first to the last row) increases the performance; concerning the comparison between Gaussian and non-Gaussian resources, we see that optimized SB non-Gaussian resources retain an advantage for not too high values of the squeezing parameter (till about $r = 1$), but that their performance becomes practically indistinguishable from that of the Gaussian TB if $r \geq 1.5$.

Summarizing, the dependence of the teleportation fidelity from the phases is very different, and much more complex, in the non-Markovian regime if compared with the ideal case or with the Markovian limit. In these last two cases, the maximum value of the fidelity, for each value of the other parameters, is always obtained for a unique fixed value, while in the non-Markovian channels the complex intertwinement among time and phases makes the optimization much less trivial.

The crucial point is the sensitivity of the non-Markovian channels to the values assumed by the free phases of the entangled resources, an aspect that is not present in Markovian channels. Therefore, in principle, in the presence of a high degree of non-Markovianity one can obtain at intermediate times highest and almost constant values of the fidelities. Obviously, efficiency grows for increasing values of the squeezing parameter and decreases if the degree of non-Markovianity decreases.

Concerning the comparison between Gaussian and non-Gaussian resources, we verified that non-Gaussian resources, and in particular optimized SB states, guarantee a sensibly better efficiency with respect to the Gaussian TB if the value of the squeezing parameter is not overly large, while Gaussian and non-Gaussian resources become practically equivalent for high squeezing.

From a practical point of view, it is clear that a continuous tuning, at each instant, of the optimizable parameter is not accessible, but the experimentalist can compute "a priori" the length of the fiber (and thus the travelling time) and the values of the free parameters, in such a way to determine a high value of the fidelity at the receiver's location.

CHAPTER 7

Some recent proposals

Till now we have identified the non-Markovianity of a quantum evolution with the violation of the complete positivity of the intermediate map. Some recent works [112, 113, 114] try to generalize the definition, following different directions. In this Chapter we give a brief review of some of the latest proposals.

7.1 Degree of non-Markovianity

The assumption of the divisibility property as the definition of a Markovian evolution is in some sense a coarse definition; a more fine one can be obtained using the concept of k-divisible map. A family of map $\{\Phi(t_0, t_2), t_2 \geq t_0\}$ is k-divisible if and only if its extension $\Phi(t_0, t_2) \otimes \mathbb{1}_k$ is positive, while $\Phi(t_0, t_2) \otimes \mathbb{1}_{k+1}$ is not. In particular, if the dimension of the system is d, when the map is k-divisible, with $k \geq d$, we return to the complete positivity condition; at the other side, if the intermediate map is 0-divisible, the process is not even positive.

Therefore, considering the k-divisible property of the intermediate map $\Phi(t_1, t_2)$, the authors in [112] define the *degree k of non-Markvovianity*. They start from the generalization to k-divisible maps of the Eq. (2.31) and Eq. (2.32) proving that, for a k-divisible process:

$$\|[\Phi(t_0, t) \otimes \mathbb{1}_k]\Delta\|_1 \leq \|\Delta\|_1, \quad t \geq t_0,$$

for every Helstrom matrix Δ with an ancillary space of k dimension. As a consequence they define the k-degree of non-Markovianity quantifying the departure from the k-divisibility condition, in a way similar to Eq. (3.4):

$$\mathcal{D}_k^I = \sup_\Delta \frac{N_k^+(\Delta, I)}{|N_k^-(\Delta, I)|},$$

where $N_k^\pm(\Delta, I) := \int_{t \in I, \sigma \lessgtr 0} dt \sigma(\Delta, t)$ and $\sigma(\Delta, t) := \frac{d}{dt}\|[\Phi(t_0, t) \otimes \mathbb{1}_k]\Delta\|_1$.

It can be show that $N_k^+(\Delta, I) \leq |N_k^-(\Delta, I)|$ and hence $0 \leq \mathcal{D}_k^I \leq 1$. Furthermore, as the dimension of the optimization space increases with k increasing, it follows that:

$$0 \leq \mathcal{D}_k^I \leq \ldots \leq \mathcal{D}_d^I \leq 1,$$

where $\mathcal{D}_d^I$ is the degree of non-Markovianity corresponding to the complete positivity violation. This hierarchy allows to define a maximally non-Markovian dynamics: it is the dynamics for which $\mathcal{D}_1^I = 1$ and, consequently, $\mathcal{D}_2^I = \ldots = \mathcal{D}_d^I = 1$.

7.2 Generalized trace distance measure

In Sec. (3.1.2) and Sec. (3.2) we have described two different approaches to non-Markovianity, namely the divisibility and the contractive properties of the dynamical map. Furthermore we have introduced the BLP definition based on the distinguishably between states and we have discussed its intuitive physical meaning in terms of flow of information between the system and the environment. Despite its clear interpretation, as this quantity is not based on the divisibility of the dynamical map, it is only a sufficient condition. Now we may ask is there is a way to reconcile these two different aspects of the dynamics.

Indeed the difference is due to the fact that the BLP definition is based on Eq. (2.31) and not on the stronger condition Eq. (2.32); as a consequence there are non-completely positive evolutions that are not identified by this approach. On the other hand in the previous paragraph we have discussed the possibility to resort to a more general definition of non-Markovianity, through the concept of k-divisible map. It is then possible to generalize the contractive approach so as to be equivalent to the violation of the P divisibility of the dynamical map [113]. With this generalization the until now assumption of divisibility as the violation of complete positivity condition turns out to be only a sufficient condition.

We start from the property Eq. (2.31); by this condition, for a dynamical map $\Phi(t_0, t)$, it follows that:

$$\|p_1 \Phi(t_0, t)\rho_1(t_0) - p_2 \Phi(t_0, t)\rho_2(t_0)\|_1 \leq \|p_1 \rho_1(t_0) - p_2 \rho_2(t_0)\|, \qquad (7.1)$$

where $\Delta = p_1 \rho_1(t_0) - p_2 \rho_2(t_0)$ is the Helstrom matrix introduced in Sec. (3.1.1) and where $\{p_i | p_{1,2} \geq 0, p_1 + p_2 = 1\}$ is an arbitrary probability distribution. As a consequence we now assume the following definition for a Markovian process: *a quantum process is Markovian if and only if* $\|p_1 \Phi(t_0, t)\rho_1(t_0) - p_2 \Phi(t_0, t)\rho_2(t_0)\|_1$ *is a monotonically decreasing function of the time* $t \geq 0$ *for all sets* $\{p_i, \rho_i\}$. Furthermore, we define a P-divisible evolution *a dynamical process in which the intermediate map* $\Phi(t_1, t_2)$ *is positive for all* $t \geq s \geq 0$. It can be shown that, under the hypothesis that the dynamical map is bijective, Φ *is Markovian if and only if it is P-divisible.*

As a consequence of this result it is possible to define a measure of non-Markovianity as :

$$\mathcal{N}_{GTD} \equiv \max_{\{p_i, \rho_i\}} \int_{\sigma > 0} dt \, \sigma(t, p_i, \rho_i), \qquad (7.2)$$

where $\sigma(t, p_i, \rho_i) = \frac{d}{dt} \| p_1 \Phi(t_0, t) \rho_1(t_0) - p_2 \Phi(t_0, t) \rho_2(t_0) \|_1$ and "GTD" means "'Generalized trace distance". By construction the process is non-Markovian if and only if $\mathcal{N}_{GTD} > 0$

We have reconciled in this way the concept of memoryless evolution, expressed by the divisibility of the map, with the concept of flow of information, expressed by the monotonic decreasing of the possibility to distinguish between states.

We finally note that this definition of Markovian process can be connected to the corresponding definition of classical Markovian stochastic processes [113].

7.3 Generalization of information distinguishability measure

In Sec. (7.2) we success to connect the flow of information between the system and the environment with the divisibility property, provided to extend the latter in terms of P-divisibility. Now we present another approach [114] that allows to obtain the same equivalence, provided to track the evolution of an arbitrary ensemble of quantum states with arbitrary *a priori* probabilities.

We consider a discrete evolution $t_0 \leq t_1 \leq \ldots \leq t_n$ and an initial system-environment state of the form Eq. (2.4). Furthermore, for ease of notation, we indicate the dynamical map Eq. (2.9) as: $\Phi(t_0, t_i) \rho(t_0) = \mathrm{Tr}_E[U_{SE}^\dagger(t, t_i) \rho_S(t_0) \otimes \rho_E U_{SE}(t, t_i)] = \Phi^i(\rho_s(t_0))$.

Suppose that the initial state of the system is modelled by the ensemble $\mathcal{E} = \{p(x), \rho_S^x\}$, namely the system state is ρ_S^x with probability $p(x)$. The information on the initial state therefore depends on the distinguishability of the states in the ensemble. A measure of this information is given by the *guessing probability*:

$$P_{guess}(\mathcal{E}) = \max \sum_x p(x) \mathrm{Tr}[P_S^x \rho_S^x],$$

where the maximization is carried on all over the POVM[1] P_S^x defined on the system Hilbert space.

We consider the case in which the initial state evolves according to the discrete time mapping $\{\Phi^i\}_{i \geq 0}$. The evolution will be *information decreasing* by definition if, for any ensemble $\mathcal{E} = \{p(x), \rho_S^x\}$:

$$P_{guess}(\mathcal{E}_i) \geq P_{guess}(\mathcal{E}_{i+1})$$

---

[1]A *positive operator valued measure* (POVM) is a family $\{P^x\}_{x \in \mathcal{X}}$ of positive semidefinite operators defined on a Hilbert space $\mathcal{H}$ such that $\sum_x P^x = \mathbb{1}$.

for all $i > 0$, where $\mathcal{E}_i = \{p(x), \Phi^i(\rho_S^x)\}$.

Furthermore we can consider the case in which the discrete time mapping is given by a sequence of linear, complete positive, trace preserving map. In this case the evolution is said to be *completely information decreasing* if:

$$P_{guess}(\tilde{\mathcal{E}}_i) \geq P_{guess}(\tilde{\mathcal{E}}_{i+1}),$$

fo every ensemble $\tilde{\mathcal{E}}_i$ where now $\tilde{\mathcal{E}}_i = \{p(x)|[\mathbb{1}_{S'} \otimes \Phi_S^i)]\rho_{S'S}^x\}$ is the ensemble obtained extending the map with an arbitrary Hilbert space $\mathcal{H}_{S'}$.

It can be shown that *a given discrete dynamical map is divisible, in the sense of Sec. (2.2.2), if and only if it is completely information decreasing.* As a consequence a given evolution is non-Markovian if and only if there exists an auxiliary Hilbert space $\mathcal{H}_{S'}$, an ensemble $\tilde{\mathcal{E}}$ and a time t_k such that:

$$P_{guess}(\tilde{\mathcal{E}}_k) > P_{guess}(\tilde{\mathcal{E}}_{k-1}),$$

namely there is a time step in which the distinguishability increases with respect to the initial ensemble.

Conclusions and outlook

In this thesis we have discussed the concept of quantum non-Markovianity, particularly regarding its characterization and quantification. Indeed for finite dimensional systems a lot of work has yet been done. For the infinite dimensional case the situation is more subtle: we have shown that the major part of the approaches requires some optimization, that in the infinite dimensional Hilbert space is infeasible. As a consequence the possibility to resort to the divisibility property is particularly advantageous in the case of Gaussian channels: the violation of divisibility condition is expressed by a simple matrix relation involving the matrices that define the channels.

Another great advantage of this procedure is the easy generalization of the measure to the more interesting case of many modes. Indeed, if we consider the channel $\Phi_{\mathbf{X},\mathbf{Y}}$ given by the tensor product of the channels $\Phi_{\mathbf{X}_i,\mathbf{Y}_i}$: $\Phi_{\mathbf{X},\mathbf{Y}} = \bigotimes_i \Phi_{\mathbf{X}_i,\mathbf{Y}_i}$, since the tensor products structure in Hilbert spaces correspond to direct sums in phase space, we have $\Phi_{\mathbf{X},\mathbf{Y}} = \Phi_{\bigoplus \mathbf{X}_i, \bigoplus \mathbf{Y_i}}$.

Regarding the usefulness of non-Markovianity in Quantum Information science, namely its potential applications, some work has yet been done, such as the possibility to prepare steady state entanglement [115], to enhance the resolution in quantum metrology [116] and help in the realization of some tasks [117, 118, 119]. In this direction we have generalized the Continuous Variable Quantum Teleportation protocol when the mode of the resource sent to Bob undergoes an open non-Markovian evolution. We have show how the non-Markovian property of the bath leads to an optimized phase that results time-dependent. As a consequence the possibility to choose in an appropriate manner the squeezed resource allows to improve significantly, at a given time, the teleportation fidelity.

Notwithstanding these significant results achieved by the scientific community, there are many unresolved issues.

First of all the relation between the various measures of non-Markovianity: it is not clear if the various measures induce the same ordering, and the unavoidable optimization procedure makes difficult to achieve some explicit results, except in very simple cases. Furthermore a more in depth study of the behaviour of interesting properties, such as quantum correlations, is necessary. Indeed some results are known, but their relation with the defined measures of non-Markovianity is in general unknown. Some recent results show [120], for particular choice of a

Gaussian channel and input states, that the revival of quantum correlations is not strictly connected with the divisibility of the map: the map is always not divisible, but the revival is present only when there is a backflow of information. Indeed this is due to the non-equivalence between the divisibility and the contractive property and, as we have discussed in Sec. (7.2) and in Sec. (7.3), some recent results try to recover this equivalence.

Regarding the potential applications of non-Markovianity there are two major directions to follow. First of all the conceptual problem to link non-Markovianity with other phenomena. Some work has already been done regarding, for example, criticality and phase transitions [121, 122, 123, 124], Loschmidt echo [125, 121, 122] and symmetry breaking [126]. The second important point is to exploit the non-Markovianity to eventually enhance other tasks in quantum information and quantum computation.

Probably the most important objective is to formulate *a resource theory for non-Markovianity*. Indeed this is of fundamental interest, as allows us to understand if there are some tasks that cannot be done without non-Markovianity and how to implement them.

Finally we want to stress another important point that we do not have treated in this thesis. When the initial system-environment state is entangled, and hence it is not of the form Eq. (2.4), the description of the system evolution can be very different from the approach presented: in particular the map can not be completely positive. Without entering in this long debate we want to stress that this problem requires some completely new approaches. In particular in [127] the authors define a measure of non-Markovianity without violating the complete positivity condition.

The understanding of the non-Markovianity property of a quantum evolution is yet at its infancy. Apart the great expectations for quantum technologies, a deep understanding of the phenomenon is indeed necessary to acquire a better understanding of the quantum world.

Appendices

APPENDIX A

Beam splitter in phase space

In order to discuss the beam splitter in quantum optics, first of all it is necessary to discuss the classical case, as the quantum case is obtained directly through modes quantization.

A beam splitter, represented in Fig. (A.1), consists in a dielectric medium, typically a thin plate, that mixes two input modes of the electromagnetic field. If the dielectric medium is linear the modes are coupled linearly: if $a_{i'}$ is the amplitude of the i'-th mode on one side of the beam splitter, the amplitude a_i of the i-th mode on the other side is obtained through the relation:

$$a_{i'} = \sum_i T_{i'i} a_i,$$

where the elements $T_{i'i}$ of the linear transformation are determined by the properties of the dielectric medium and by the boundary conditions. Explicitly:

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} t & r \\ r & t \end{pmatrix} \begin{pmatrix} a_{1'} \\ a_{2'} \end{pmatrix}, \tag{A.1}$$

where t and r denotes respectively the transmission and reflection coefficients. Moreover, for a lossless beam splitter the energy conservation imposes the constraint:

$$|a_1^2| + |a_2^2| = |a_{1'}^2| + |a_{2'}^2|; \tag{A.2}$$

through the matrix in Eqs. (A.1) and (A.2) it is straightforward to obtain the relations:

$$\begin{aligned} |t^2| + |r^2| &= 1, \\ tr^* + t^*r &= 1 = 0; \end{aligned} \tag{A.3}$$

the first equation states that there is no absorption in the beam splitter, the second simply correlates the modes.

The quantum case is obtained directly by replacing the amplitudes with the corresponding operators: $(a_i \to \hat{a}_i, a_{i'} \to \hat{a}_{i'})$, while the transformation matrix remains the same.
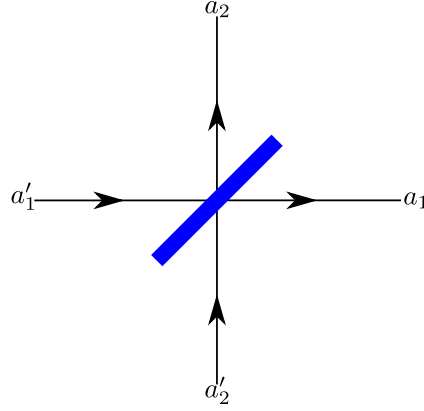
Figure A.1: Schematic representation of a beam splitter.

We now want to obtain the effect of the beam splitter transformation on a generic input state ρ_{in}. As we use the phase-space formulation of Quantum Mechanics, we are interested in the corresponding transformation on the characteristic function of the input state. We start by expressing the density operator ρ_{in} in terms of the characteristic function, Eq. (4.4), for the two modes case:

$$\rho_{in} = \frac{1}{\pi^2} \int_{\mathbb{C}^2} \mathrm{d}^2\alpha_{1'}\mathrm{d}^2\alpha_{2'} \chi_{\mathrm{in}}\left(\alpha_{1'}, \alpha_{2'}\right) \mathcal{D}_{1'}^\dagger\left(\alpha_{1'}\right) \mathcal{D}_{2'}^\dagger\left(\alpha_{2'}\right); \qquad (A.4)$$

the beam splitter transformation $\rho_{in} \rightarrow U_{BS}\rho_{in}U_{BS}^\dagger$ thus corresponds to the transformation $U_{BS}\mathcal{D}_{1'}^\dagger\left(\alpha_{1'}\right)\mathcal{D}_{2'}^\dagger\left(\alpha_{2'}\right)U_{BS}^\dagger$. As U_{SB} is a unitary operator, expanding $\mathcal{D}_{1'}^\dagger\left(\alpha_{1'}\right)$ and $\mathcal{D}_{2'}^\dagger\left(\alpha_{2'}\right)U_{BS}^\dagger$ in series and using the Baker-Campbell-Hausdorf formula we obtain:

$$\mathcal{D}_{1'}^\dagger\left(\alpha_{1'}\right) \rightarrow \mathcal{D}_1^\dagger\left(t\alpha_{1'}\right)\mathcal{D}_2^\dagger\left(r\alpha_{2'}\right), \qquad \mathcal{D}_{2'}^\dagger\left(\alpha_{2'}\right) \rightarrow \mathcal{D}_1^\dagger\left(r\alpha_{1'}\right)\mathcal{D}_2^\dagger\left(t\alpha_{2'}\right);$$

substituting these expressions in Eq. (A.4) we have:

$$\rho_{out} = \frac{1}{\pi^2} \int_{\mathbb{C}^2} \mathrm{d}^2\alpha_{1'}\mathrm{d}^2\alpha_{2'} \chi_{\mathrm{in}}\left(\alpha_{1'}, \alpha_{2'}\right) \mathcal{D}_1^\dagger\left(\alpha_{1'}t + \alpha_{2'}r\right) \mathcal{D}_2^\dagger\left(\alpha_{1'}r + \alpha_{2'}t\right).$$

Finally, considering the transformation rules $\xi_1 = \alpha_{1'}t + \alpha_{2'}r$, $\xi_2 = \alpha_{1'}r + \alpha_{2'}t$ and their hermitian conjugates, and nothing, looking at Eqs. (A.3,) that the Jacobian of the transformation is equal to one, we obtain:

$$\rho_{out} = \frac{1}{\pi^2} \int_{\mathbb{C}^2} \mathrm{d}^2\xi_1\mathrm{d}^2\xi_2 \chi_{\mathrm{in}}\left(t^*\xi_1 + r^*\xi_2, r^*\xi_1 + t^*\xi_2\right) \mathcal{D}_1^\dagger\left(\xi_1\right) \mathcal{D}_2^\dagger\left(\xi_2\right).$$

Hence applying the beam splitter transformation we obtain the following relation between the characteristic functions of the input and output state:

$$\chi_{\mathrm{out}}\left(\xi_1, \xi_2\right) = \chi_{\mathrm{in}}\left(t^*\xi_1 + r^*\xi_2, r^*\xi_1 + t^*\xi_2\right).$$

Finally, as it is always possible to take t real and choose an arbitrary phase for r, the last relation can be put in the equivalent, final form:

$$\chi_{\text{out}}\left(\xi_1,\xi_2\right) = \chi_{\text{in}}\left(t\xi_1 + r\xi_2, r\xi_1 + t\xi_2\right). \tag{A.5}$$

APPENDIX B

Homodyne measurement in the characteristic function description

In this appendix we describe the homodyne measurement in the characteristic function formalism. In particular, we calculate the expression of the density operator and of its characteristic function for the reduced state that is obtained applying the homodyne measurements on appropriately selected modes. For this Appendix we refer to [85].

Let us consider the three-mode density operator ρ_{123} whose characteristic function is given by Eq. (6.2). In terms of Weyl expansion the density operator can be written explicitly through Eq. (4.4) as:

$$\rho_{123} = \frac{1}{\pi^3} \int d^2\alpha_1 d^2\alpha_2 d^2\alpha_3 \chi_{123}(\alpha_1, \alpha_2, \alpha_3) D_1(-\alpha_1) D_2(-\alpha_2) D_3(-\alpha_3).$$

Homodyne measurements reduce the three-mode state to a single-mode one. Suppose that $p_1 = \tilde{p}$ and $x_2 = \tilde{x}$ are the results of the measurements of the quadratures p_1 and x_2. If $|\tilde{p}\rangle_1$ and $|\tilde{x}\rangle_2$ are the projectors on the quadrature eigenvalues, the state after the measurements will be:

$$\tilde{\rho}_3 = \mathcal{P}^{-1}(\tilde{p}, \tilde{x}) \mathrm{Tr}_{12} \left[|\tilde{p}\rangle \langle\tilde{p}| \otimes |\tilde{x}\rangle \langle\tilde{x}| \rho_{123} \right],$$

where $\mathcal{P}(\tilde{p}, \tilde{x}) = \mathrm{Tr}_3[\tilde{\rho}_3]$ is the distribution function of the outcomes $\tilde{p}$ and $\tilde{x}$. Using the relations:

$$\langle\tilde{p}|D_1(-\alpha_1)|\tilde{p}\rangle = e^{-\frac{i}{2}x_1 p_1 + i x_1 \tilde{p}} \delta(p_1),$$
$$\langle\tilde{x}|D_2(-\alpha_2)|\tilde{x}\rangle = e^{-\frac{i}{2}x_2 p_2 + i p_2 \tilde{x}} \delta(x_2),$$

it is possible to rewrite $\tilde{\rho}_3$ in the following form:

$$\tilde{\rho}_3 = \frac{\mathcal{P}^{-1}(\tilde{p}, \tilde{x})}{(2\pi)^3} \int dx_1 dp_2 dx_3 dp_3 \, e^{i x_1 \tilde{p} - i \tilde{x} p_2} \times$$
$$\chi_{123}(x_1, 0; 0, p_2; x_3, p_3) D_3 \left(-\frac{x_3 + ip_3}{\sqrt{2}} \right);$$

99

furthermore, remembering the expression of the density operator in the characteristic function description, given by:

$$\tilde{\rho}_3 = \frac{1}{2\pi} \int dx_3 dp_3 \; \chi_3(x_3, p_3) D_3 \left(-\frac{x_3 + ip_3}{\sqrt{2}} \right) ,$$

and comparing the last two equations we obtain:

$$\chi_3(x_3, p_3) = \frac{\mathcal{P}^{-1}(\tilde{p}, \tilde{x})}{(2\pi)^2} \int dx_1 dp_2 \; e^{ix_1\tilde{p} - i\tilde{x}p_2} \chi_{123}(x_1, 0; 0, p_2; x_3, p_3) .$$

Finally, from this last relation and the definition $\mathcal{P}(\tilde{p}, \tilde{x}) = \text{Tr}_3[\tilde{\rho}_3]$ we obtain:

$$\mathcal{P}(\tilde{p}, \tilde{x}) = \frac{1}{(2\pi)^2} \int dx_1 dp_2 \; e^{ix_1\tilde{p} - i\tilde{x}p_2} \chi_{123}(x_1, 0; 0, p_2; 0, 0) . \qquad \text{(B.1)}$$

APPENDIX C

Coefficients of Quantum Brownian motion master equation

In this Appendix we report the explicit expression of the coefficients of the master equation Eq. (4.28) and, consequently, of its secular approximation (4.35) for the ohmic spectral density of the form:

$$J(\omega) = \omega e^{-\omega/\omega_c}.$$

The explicit expressions are obtained through Eqs. (4.29). The function $\text{Ci}(x)$ and $\text{Si}(x)$ are respectively the cosine integral and the sine integral [128]. The coefficients are expressed in terms of the ratio $x = \omega_c/\omega_0$ and the dimensionless time $\tau = \omega_c t$ (see Sec. (4.6.2)).

The dissipation coefficient is temperature-independent:

$$
\begin{aligned}
\gamma(\tau) = {} & \frac{\alpha^2 \omega 0}{2\left(\tau^2 + 1\right)} \times \\
& \times \left(-2x \sin\left(\frac{\tau}{x}\right) + \left(\tau^2 + 1\right)\left(-i \cosh\left(\frac{1}{x}\right)\left(\text{Ci}\left(\frac{-i+\tau}{x}\right) - \right.\right.\right. \\
& \left. -\text{Ci}\left(\frac{i+\tau}{x}\right) + \log\left(\frac{i}{x}\right) - \log\left(-\frac{i}{x}\right)\right) - \\
& \left.\left.\left. - \sinh\left(\frac{1}{x}\right)\left(\text{Si}\left(\frac{-i+\tau}{x}\right) + \text{Si}\left(\frac{i+\tau}{x}\right)\right)\right)\right).
\end{aligned}
\tag{C.1}
$$

High temperature limit

In the high-temperature limit the coefficients, Eq. (4.29), can be computed using the approximation $2N(\omega) + 1 \simeq \frac{2K_B T}{\hbar\omega}$:

$$\Delta(\tau) = \frac{1}{2}\alpha^2 \frac{k_B T}{\hbar}\left(\cosh\left(\frac{1}{x}\right)\left(-i\text{Ci}\left(\frac{-i+\tau}{x}\right) + i\text{Ci}\left(\frac{i+\tau}{x}\right) + \pi\right) - \right.$$
$$\left. - \sinh\left(\frac{1}{x}\right)\left(\text{Si}\left(\frac{-i+\tau}{x}\right) + \text{Si}\left(\frac{i+\tau}{x}\right)\right)\right), \qquad (\text{C.2})$$

$$\Pi(\tau) = \frac{1}{2}\alpha^2 \frac{k_B T}{\hbar}\left(\sinh\left(\frac{1}{x}\right)\left(\text{Ci}\left(\frac{-i+\tau}{x}\right) + \text{Ci}\left(\frac{i+\tau}{x}\right) - \text{Ci}\left(\frac{i}{x}\right) - \right.\right.$$
$$\left.- \text{Ci}\left(-\frac{i}{x}\right)\right) + \cosh\left(\frac{1}{x}\right)\left(2\text{Shi}\left(\frac{1}{x}\right) - \right.$$
$$\left.\left.- i\left(\text{Si}\left(\frac{-i+\tau}{x}\right) - \text{Si}\left(\frac{i+\tau}{x}\right)\right)\right)\right). \qquad (\text{C.3})$$

Zero temperature limit

In the zero temperature limit the coefficients Eq. (4.29) can be calculated using the approximation $2N(\omega) + 1 \simeq 1$.

$$\Delta(\tau) = \frac{\alpha^2 \omega_0}{2(\tau^2 + 1)}\left(2\tau x \cos\left(\frac{\tau}{x}\right) + (\tau^2 + 1)\left(i\sinh\left(\frac{1}{x}\right)\left(\text{Ci}\left(\frac{-i+\tau}{x}\right)\right.\right.\right.$$
$$\left.- \text{Ci}\left(\frac{i+\tau}{x}\right) + \log\left(\frac{i}{x}\right) - \log\left(-\frac{i}{x}\right)\right) + $$
$$\left.\left.+ \cosh\left(\frac{1}{x}\right)\left(\text{Si}\left(\frac{-i+\tau}{x}\right) + \text{Si}\left(\frac{i+\tau}{x}\right)\right)\right)\right) \qquad (\text{C.4})$$

$$\Pi(\tau) = \frac{\alpha^2 \omega_0}{2(\tau^2 + 1)}\left(2\tau x \sin\left(\frac{\tau}{x}\right) + (\tau^2 + 1)\left(\cosh\left(\frac{1}{x}\right)\left(-\text{Ci}\left(\frac{-i+\tau}{x}\right) - \right.\right.\right.$$
$$\left.- \text{Ci}\left(\frac{i+\tau}{x}\right) + \text{Ci}\left(\frac{i}{x}\right) + \text{Ci}\left(-\frac{i}{x}\right)\right) + \sinh\left(\frac{1}{x}\right)\left(-2\text{Shi}\left(\frac{1}{x}\right) + \right.$$
$$\left.\left.\left.+ i\left(\text{Si}\left(\frac{-i+\tau}{x}\right) - \text{Si}\left(\frac{i+\tau}{x}\right)\right)\right)\right)\right) \qquad (\text{C.5})$$

APPENDIX D

Non-Markovian teleportation fidelity in the case of Gaussian TWBS resources

We now give the explicit expression of the teleportation fidelity when the resource is a Twin Beam state.

From Eq. (6.1) we know how to compute the fidelity in terms of characteristic functions of the input and output states; furthermore, as both the input coherence state and the entangled resource are Gaussian, at every transformation on the characteristic function corresponds a transformation on the corresponding covariance matrix. Consequently we exploit this property to perform the calculation.

The starting point are the covariance matrix of the unknown input coherence state ($\boldsymbol{\sigma}_{in} = \frac{1}{2}\mathbf{I}$) and the covariance matrix of the TWBS resource, Eq. (4.13) and Eqs. (4.14). From these expressions, through Eq. (6.6), the covariance matrix of the output state is given by $\boldsymbol{\sigma}_{out} = \boldsymbol{\sigma}_{in} + \tilde{\boldsymbol{\sigma}}_{res} + \boldsymbol{\sigma}_{exp}$, with:

$$\boldsymbol{\sigma}_{exp} = -\begin{pmatrix} g^2\mathcal{R}^2 + 2\bar{W}_{11} & 2\bar{W}_{12} \\ 2\bar{W}_{12} & g^2\mathcal{R}^2 + 2\bar{W}_{22} \end{pmatrix}, \tag{D.1}$$

where $\bar{W}_{11}$, $\bar{W}_{12}$ and $\bar{W}_{22}$ are the coefficients (Eq. (4.32)) expressing the state evolution described by the Quantum Brownian Motion master equation (4.28), g is the gain of the classical communication channel and $\mathcal{R}$ is the reflectivity of the beam splitters. Furthermore:

$$\tilde{\boldsymbol{\sigma}}_{res} = \begin{pmatrix} -\frac{1}{2}e^{-\Gamma(t)}\left(\left(e^{\Gamma(t)}g^2\mathcal{T}^2 + 1\right)\cosh(2r) + 2e^{\frac{\Gamma(t)}{2}}g\mathcal{T}\cos(\phi - t\omega_0)\sinh(2r)\right) \\ 0 \\ \\ 0 \\ -\frac{1}{2}e^{-\Gamma(t)}\left(\left(e^{\Gamma(t)}g^2\mathcal{T}^2 + 1\right)\cosh(2r) + 2e^{\frac{\Gamma(t)}{2}}g\mathcal{T}\cos(\phi - t\omega_0)\sinh(2r)\right) \end{pmatrix}, \tag{D.2}$$

where $\mathcal{T}$ is the transmissivity of the beam splitters and $\Gamma(t)$ is the master equation coefficient Eq. (4.31). Finally, the covariance matrix of the Gaussian state appearing in the Eq. (6.1) is $\boldsymbol{\sigma}_{fin} = \boldsymbol{\sigma}_{in} + \boldsymbol{\sigma}_{out}$.

It can be shown [129] that, when the resource is a Gaussian state, the teleportation fidelity, Eq. (6.1), has a simple expression in terms of the covariance matrix as:

$$\mathcal{F}_{TWBS} = \frac{1}{\sqrt{\det\boldsymbol{\sigma}}};$$ (D.3)

consequently, through $\boldsymbol{\sigma}_{fin}$, this last expression and Eq. (D.3), we obtain:

$$F(r,\phi) = \frac{1}{\sqrt{-4\bar{W}_{12}^2 + \dfrac{1}{4}e^{2\Gamma(\tau)}\Lambda_{11}\Lambda_{22}}},$$ (D.4)

with:

$$\Lambda_{ii} = \cosh(2r) + e^{\Gamma(\tau)}\left(2 + 4\bar{W}_{ii}\right.$$ (D.5)

$$+2\frac{\mathcal{R}^2}{\mathcal{T}^2} + \cosh(2r)\Big) + 2e^{\frac{\Gamma(\tau)}{2}}\cos\left(\phi - \frac{\tau}{x}\right)\sinh(2r) \quad \Lambda_{ii} = 1, 2.$$

Let us analyse the behaviour of $\mathcal{F}_{TWBS}$ Eq. (D.3) as a function of the squeezing phase ϕ, with the other variables fixed. The dependence of the above fidelity on ϕ appears only in the Λ_{ii} terms (the argument of the cosine is $\phi - \frac{\tau}{x}$); it is easy to show that the minimization of these terms (and of their product) and, consequently, the maximization of the fidelity, are obtained by letting ϕ to satisfy the simple relation:

$$\phi = \pi + \frac{\tau}{x}$$ (D.6)

At variance with the Markovian case, with which the relation Eq. (D.6) is in agreement in the limit $x \gg 1$ ($\phi = \pi$), the optimal phase is time-dependent.

Bibliography

[1] E. Yablonovitch, "Inhibited spontaneous emission in solid-state physics and electronics," *Phys. Rev. Lett.*, vol. 58, pp. 2059–2062, May 1987.

[2] P. Lambropoulos, G. M. Nikolopoulos, T. R. Nielsen, and S. Bay, "Fundamental quantum optics in structured reservoirs," *Reports on Progress in Physics*, vol. 63, no. 4, p. 455, 2000.

[3] M. Sarovar, A. Ishizaki, G. R. Fleming, and K. B. Whaley, "Quantum entanglement in photosynthetic light-harvesting complexes," *Nat Phys*, vol. 6, pp. 462–467, June 2010.

[4] G. Torre, W. Roga, and F. Illuminati, "Non-markovianity of gaussian channels," *Phys. Rev. Lett.*, vol. 115, p. 070401, Aug 2015.

[5] G. Torre, F. Dell'Anno, S. D. Siena, and F. Illuminati, "In preparation,"

[6] G. Galilei, *Discorsi e dimostrazioni matematiche intorno a due nuove scienze attinenti la meccanica e i movimenti locali.* Elzeviri, 1638.

[7] M. A. Schlosshauer, *Decoherence and the Quantum-To-Classical Transition.* Springer, 2007.

[8] G. Auletta, *Foundations and Interpretation of Quantum Mechanics.* World Scientific, 2000.

[9] A. Einstein, B. Podolsky, and N. Rosen, "Can quantum-mechanical description of physical reality be considered complete?," *Phys. Rev.*, vol. 47, pp. 777–780, May 1935.

[10] M. Genovese, "Research on hidden variable theories: A review of recent progresses," *Physics Reports*, vol. 413, no. 6, pp. 319 – 396, 2005.

[11] J. S. Bell, "On the einstein podolsky rosen paradox," *Physics*, vol. 1, no. 2, pp. 195–290, 1964.

[12] J. S. Bell, "On the problem of hidden variables in quantum mechanics," *Rev. Mod. Phys.*, vol. 38, pp. 447–452, Jul 1966.

[13] J. S. Bell, M. Bell, K. Gottfried, and M. Veltman, eds., *John S. Bell on the Foundations of Quantum Mechanics*. World Scientific, 2001.

[14] E. Schrödinger, "Discussion of probability relations between separated systems," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 31, pp. 555–563, 10 1935.

[15] R. Brown, "A brief account of microscopical observations made in the months of june, july and august, 1827, on the particles contained in the pollen of plants; and on the general existence of active molecules in organic and inorganic bodies," *Phil. Mag.*, vol. 4, p. 161–173, 1828.

[16] A. Einstein, "Über die von der molekularkinetischen theorie der wärme geforderte bewegung von in ruhenden flüssigkeiten suspendierten teilchen," *Annalen der Physik*, vol. 322, no. 8, pp. 549–560, 1905.

[17] J. Von Neumann, *Mathematische grundlagen der quantenmechanik*. J. Springer, 1932.

[18] E. Schrödinger, "Die gegenwärtige situation in der quantenmechanik i-iii," *Die Naturwissenschaften*, vol. 35, pp. 807–812; 823–828; 844–849, 1935.

[19] W. H. Zurek, "Decoherence and the transition from quantum to classical," *Physics Today*, vol. 44, no. 10, pp. 36–44, 1991.

[20] R. P. Feynman, "Simulating physics with computers," *International Journal of Theoretical Physics*, vol. 21, no. 6, pp. 467–488, 1982.

[21] P. Benioff, "Quantum mechanical models of turing machines that dissipate no energy," *Phys. Rev. Lett.*, vol. 48, pp. 1581–1585, Jun 1982.

[22] D. Deutsch, "Quantum theory, the church-turing principle and the universal quantum computer," *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, vol. 400, no. 1818, pp. 97–117, 1985.

[23] D. Deutsch, "Quantum computational networks," *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, vol. 425, no. 1868, pp. 73–90, 1989.

[24] A. Barenco, C. H. Bennett, R. Cleve, D. P. DiVincenzo, N. Margolus, P. Shor, T. Sleator, J. A. Smolin, and H. Weinfurter, "Elementary gates for quantum computation," *Phys. Rev. A*, vol. 52, pp. 3457–3467, Nov 1995.

[25] D. P. DiVincenzo, "Two-bit gates are universal for quantum computation," *Phys. Rev. A*, vol. 51, pp. 1015–1022, Feb 1995.

[26] A. Barenco, D. Deutsch, A. Ekert, and R. Jozsa, "Conditional quantum dynamics and logic gates," *Phys. Rev. Lett.*, vol. 74, pp. 4083–4086, May 1995.

[27] S. Lloyd, "Almost any quantum logic gate is universal," *Phys. Rev. Lett.*, vol. 75, pp. 346–349, Jul 1995.

[28] C. H. Bennett and G. Brassard, "Quantum cryptography: Public key distribution and coin tossing," *Proceedings of IEEE International Conference on Computers, Systems and Signal Processing*, pp. 175–179, 1984.

[29] D. Deutsch and R. Jozsa, "Rapid solution of problems by quantum computation," *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, vol. 439, no. 1907, pp. 553–558, 1992.

[30] D. P. Divincenzo, "The physical implementation of quantum computation," *Fortschritte der Physik*, vol. 48, pp. 771–783, 2000.

[31] S. J. Devitt, W. J. Munro, and K. Nemoto, "Quantum error correction for beginners," *Reports on Progress in Physics*, vol. 76, no. 7, p. 076001, 2013.

[32] D. A. Lidar, *Review of Decoherence-Free Subspaces, Noiseless Subsystems, and Dynamical Decoupling*, pp. 295–354. John Wiley & Sons, Inc., 2014.

[33] M. Thorwart, J. Eckel, J. Reina, P. Nalbach, and S. Weiss, "Enhanced quantum entanglement in the non-markovian dynamics of biomolecular excitons," *Chemical Physics Letters*, vol. 478, no. 4–6, pp. 234 – 237, 2009.

[34] A. W. Chin, J. Prior, R. Rosenbach, F. Caycedo-Soler, S. F. Huelga, and M. B. Plenio, "The role of non-equilibrium vibrational structures in electronic coherence and recoherence in pigment-protein complexes," *Nat Phys*, vol. 9, pp. 113–118, Feb. 2013.

[35] H.-B. Chen, N. Lambert, Y.-C. Cheng, Y.-N. Chen, and F. Nori, "Using non-markovian measures to evaluate quantum master equations for photosynthesis," *Scientific Reports*, vol. 5, pp. 12753–, Aug. 2015.

[36] C. J. Myatt, B. E. King, Q. A. Turchette, C. A. Sackett, D. Kielpinski, W. M. Itano, C. Monroe, and D. J. Wineland, "Decoherence of quantum superpositions through coupling to engineered reservoirs," *Nature*, vol. 403, pp. 269–273, Jan. 2000.

[37] M. J. Biercuk, H. Uys, A. P. VanDevender, N. Shiga, W. M. Itano, and J. J. Bollinger, "Optimized dynamical decoupling in a model quantum memory," *Nature*, vol. 458, pp. 996–1000, Apr. 2009.

[38] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems*. Oxford University Press, 2002.

[39] J. R. P. Feynman, F. L. Vernon, "The theory of a general quantum system interacting with a linear dissipative system," *Ann. Phys. (N.Y.)*, vol. 24, pp. 118–173, 1963.

[40] S. Nakajima, "On quantum theory of transport phenomena: Steady diffusion," *Progress of Theoretical Physics*, vol. 20, no. 6, pp. 948–959, 1958.

[41] R. Zwanzig, "Ensemble method in the theory of irreversibility," *The Journal of Chemical Physics*, vol. 33, no. 5, pp. 1338–1341, 1960.

[42] F. Shibata, Y. Takahashi, and N. Hashitsume, "A generalized stochastic liouville equation. non-markovian versus memoryless master equations," *Journal of Statistical Physics*, vol. 17, no. 4, pp. 171–187, 1977.

[43] S. Chaturvedi and F. Shibata, "Time-convolutionless projection operator formalism for elimination of fast variables. applications to brownian motion," *Zeitschrift für Physik B Condensed Matter*, vol. 35, no. 3, pp. 297–308, 1979.

[44] F. Shibata and T. Arimitsu, "Expansion formulas in nonequilibrium statistical mechanics," *Journal of the Physical Society of Japan*, vol. 49, no. 3, pp. 891–897, 1980.

[45] Ángel Rivas, S. F. Huelga, and M. B. Plenio, "Quantum non-markovianity: characterization, quantification and detection," *Reports on Progress in Physics*, vol. 77, no. 9, p. 094001, 2014.

[46] H.-P. Breuer, E.-M. Laine, J. Piilo, and B. Vacchini, "Non-Markovian dynamics in open quantum systems," *arXiv:1505.01385*, May 2015.

[47] A. Kossakowski, "On quantum statistical mechanics of non-hamiltonian systems," *Reports on Mathematical Physics*, vol. 3, no. 4, pp. 247 – 274, 1972.

[48] V. Gorini, A. Kossakowski, and E. C. G. Sudarshan, "Completely positive dynamical semigroup of n-level systems," *Journal of Mathematical Physics*, vol. 17, no. 5, pp. 821–825, 1976.

[49] G. Lindblad, "On the generators of quantum dynamical semigroups," *Communications in Mathematical Physics*, vol. 48, no. 2, pp. 119–130, 1976.

[50] G. P. Basharin, A. N. Langville, and V. A. Naumov, "The life and work of a.a. markov," *Linear Algebra and its Applications*, vol. 386, pp. 3 – 26, 2004. Special Issue on the Conference on the Numerical Solution of Markov Chains 2003.

[51] R. Durrett, *Probability: Theory and Examples*. Cambridge University Press, 2010.

[52] M. M. Wolf and J. I. Cirac, "Dividing quantum channels," *Communications in Mathematical Physics*, vol. 279, no. 1, pp. 147–168, 2008.

[53] A. Rivas and S. F. Huelga, *Open Quantum Systems: An Introduction.* Springer, 2012.

[54] D. Chruściński and A. Kossakowski, "Markovianity criteria for quantum evolution," *Journal of Physics B: Atomic, Molecular and Optical Physics*, vol. 45, no. 15, p. 154002, 2012.

[55] H.-P. Breuer, E.-M. Laine, and J. Piilo, "Measure for the degree of non-markovian behavior of quantum processes in open systems," *Phys. Rev. Lett.*, vol. 103, p. 210401, Nov 2009.

[56] D. Chruściński, A. Kossakowski, and A. Rivas, "Measures of non-markovianity: Divisibility versus backflow of information," *Phys. Rev. A*, vol. 83, p. 052128, May 2011.

[57] C. W. Helstrom, *Quantum Detection and Estimation Theory.* Academic Press, 1976.

[58] M. B. Ruskai, "Beyond strong subadditivity? improved bounds on the contraction of generalized relative entropy," *Reviews in Mathematical Physics*, vol. 06, no. 05a, pp. 1147–1161, 1994.

[59] E.-M. Laine, J. Piilo, and H.-P. Breuer, "Measure for the non-markovianity of quantum processes," *Phys. Rev. A*, vol. 81, p. 062115, Jun 2010.

[60] H.-P. Breuer, "Foundations and measures of quantum non-markovianity," *Journal of Physics B: Atomic, Molecular and Optical Physics*, vol. 45, no. 15, p. 154001, 2012.

[61] M. M. Wolf, J. Eisert, T. S. Cubitt, and J. I. Cirac, "Assessing non-markovian quantum dynamics," *Phys. Rev. Lett.*, vol. 101, p. 150402, Oct 2008.

[62] T. S. Cubitt, J. Eisert, and M. M. Wolf, "The complexity of relating quantum channels to master equations," *Communications in Mathematical Physics*, vol. 310, no. 2, pp. 383–418, 2012.

[63] A. Rivas, S. F. Huelga, and M. B. Plenio, "Entanglement and non-markovianity of quantum evolutions," *Phys. Rev. Lett.*, vol. 105, p. 050403, Jul 2010.

[64] M.-D. Choi, "Completely positive linear maps on complex matrices," *Linear Algebra and its Applications*, vol. 10, no. 3, pp. 285 – 290, 1975.

[65] A. Jamiołkowski, "Linear transformations which preserve trace and positive semidefiniteness of operators," *Reports on Mathematical Physics*, vol. 3, no. 4, pp. 275 – 278, 1972.

[66] M. J. W. Hall, J. D. Cresser, L. Li, and E. Andersson, "Canonical form of master equations and characterization of non-markovianity," *Phys. Rev. A*, vol. 89, p. 042120, Apr 2014.

[67] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*. Cambridge University Press, 2001.

[68] M. B. Plenio, M. B.nio and S. Virmani, "An introduction to entanglement measures," *Quantum Info. Comput.*, vol. 7, pp. 1–51, Jan. 2007.

[69] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, "Quantum entanglement," *Rev. Mod. Phys.*, vol. 81, pp. 865–942, Jun 2009.

[70] S. Luo, S. Fu, and H. Song, "Quantifying non-markovianity via correlations," *Phys. Rev. A*, vol. 86, p. 044101, Oct 2012.

[71] S. O. A. Ferraro and M. G. A. Paris, *Gaussian States in Quantum Information*. BIBLIOPOLIS, 2005.

[72] S. Barnett and P. Radmore, *Methods in Theoretical Quantum Optics*. Calendon Press, 2002.

[73] C. Weedbrook, S. Pirandola, R. García-Patrón, N. J. Cerf, T. C. Ralph, J. H. Shapiro, and S. Lloyd, "Gaussian quantum information," *Rev. Mod. Phys.*, vol. 84, pp. 621–669, May 2012.

[74] X.-B. Wang, T. Hiroshima, A. Tomita, and M. Hayashi, "Quantum information with gaussian states," *Physics Reports*, vol. 448, no. 1–4, pp. 1 – 111, 2007.

[75] G. Adesso, S. Ragy, and A. R. Lee, "Continuous variable quantum information: Gaussian states and beyond," *Open Systems & Information Dynamics*, vol. 21, no. 01n02, p. 1440001, 2014.

[76] G. Adesso and F. Illuminati, "Entanglement in continuous-variable systems: recent advances and current perspectives," *Journal of Physics A: Mathematical and Theoretical*, vol. 40, no. 28, p. 7821, 2007.

[77] R. G. Littlejohn, "The semiclassical evolution of wave packets," *Physics Reports*, vol. 138, no. 4–5, pp. 193 – 291, 1986.

[78] R. Simon, E. C. G. Sudarshan, and N. Mukunda, "Gaussian pure states in quantum mechanics and the symplectic group," *Phys. Rev. A*, vol. 37, pp. 3028–3038, Apr 1988.

[79] Arvind, B. Dutta, N. Mukunda, and R. Simon, "The real symplectic groups in quantum mechanics and optics," *Pramana*, vol. 45, no. 6, pp. 471–497, 1995.

[80] A. Wünsche, "The quantum-mechanical inhomogeneous symplectic group," *Journal of Optics B: Quantum and Semiclassical Optics*, vol. 4, no. 1, p. 1, 2002.

[81] N. C. Menicucci, P. van Loock, M. Gu, C. Weedbrook, T. C. Ralph, and M. A. Nielsen, "Universal quantum computation with continuous-variable cluster states," *Phys. Rev. Lett.*, vol. 97, p. 110501, Sep 2006.

[82] M. M. Wolf, G. Giedke, and J. I. Cirac, "Extremality of gaussian quantum states," *Phys. Rev. Lett.*, vol. 96, p. 080502, Mar 2006.

[83] F. Dell'Anno, S. De Siena, L. Albano, and F. Illuminati, "Continuous-variable quantum teleportation with non-gaussian resources," *Phys. Rev. A*, vol. 76, p. 022301, Aug 2007.

[84] F. Dell'Anno, S. De Siena, L. Albano Farias, and F. Illuminati, "Continuous variable quantum teleportation with sculptured and noisy non-gaussian resources," *The European Physical Journal Special Topics*, vol. 160, no. 1, pp. 115–126, 2008.

[85] F. Dell'Anno, S. De Siena, and F. Illuminati, "Realistic continuous-variable quantum teleportation with non-gaussian resources," *Phys. Rev. A*, vol. 81, p. 012333, Jan 2010.

[86] F. Dell'Anno, S. De Siena, G. Adesso, and F. Illuminati, "Teleportation of squeezing: Optimization using non-gaussian resources," *Phys. Rev. A*, vol. 82, p. 062329, Dec 2010.

[87] G. Björk and Y. Yamamoto, "Generation of nonclassical photon states using correlated photon pairs and linear feedforward," *Phys. Rev. A*, vol. 37, pp. 4229–4239, Jun 1988.

[88] Z. Zhang and H. Fan, "Properties of states generated by excitations on a squeezed vacuum state," *Physics Letters A*, vol. 165, no. 1, pp. 14 – 18, 1992.

[89] M. Dakna, T. Anhut, T. Opatrný, L. Knöll, and D.-G. Welsch, "Generating schrödinger-cat-like states by means of conditional measurements on a beam splitter," *Phys. Rev. A*, vol. 55, pp. 3184–3194, Apr 1997.

[90] F. Dell'Anno, D. Buono, G. Nocerino, A. Porzio, S. Solimeno, S. De Siena, and F. Illuminati, "Tunable non-gaussian resources for continuous-variable quantum technologies," *Phys. Rev. A*, vol. 88, p. 043818, Oct 2013.

[91] F. Dell'Anno, S. D. Siena, and F. Illuminati, "Multiphoton quantum optics and quantum state engineering," *Physics Reports*, vol. 428, no. 2–3, pp. 53 – 168, 2006.

[92] A. Ourjoumtsev, H. Jeong, R. Tualle-Brouri, and P. Grangier, "Generation of optical /'schrodinger cats/' from photon number states," *Nature*, vol. 448, pp. 784–786, Aug. 2007.

[93] V. Paulsen, *Completely Bounded Maps and Operator Algebras*. Cambridge University Press, 2003. Cambridge Books Online.

[94] J. Eisert and M. M. Wolf, "Gaussian quantum channels," in *Quantum Information with Continuous Variables of Atoms and Light*, pp. 23–42–, Published by Imperial College Press and distributed by World Scientific Publishing Co., Feb. 2007.

[95] F. Caruso, J. Eisert, V. Giovannetti, and A. S. Holevo, "Optimal unitary dilation for bosonic gaussian channels," *Phys. Rev. A*, vol. 84, p. 022306, Aug 2011.

[96] F. Caruso, J. Eisert, V. Giovannetti, and A. S. Holevo, "Multi-mode bosonic gaussian channels," *New Journal of Physics*, vol. 10, no. 8, p. 083030, 2008.

[97] M. M. W. Teiko Heinosaari, Alexander S. Holevo, "The semigroup structure of gaussian channels," *Quantum Inf. Comp.*, vol. 10, pp. 0619–0635, 2010.

[98] G. Lindblad, "Cloning the quantum oscillator," *Journal of Physics A: Mathematical and General*, vol. 33, no. 28, p. 5059, 2000.

[99] R. Vasile, S. Maniscalco, M. G. A. Paris, H.-P. Breuer, and J. Piilo, "Quantifying non-markovianity of continuous-variable gaussian dynamical maps," *Phys. Rev. A*, vol. 84, p. 052118, Nov 2011.

[100] B. L. Hu, J. P. Paz, and Y. Zhang, "Quantum brownian motion in a general environment: Exact master equation with nonlocal dissipation and colored noise," *Phys. Rev. D*, vol. 45, pp. 2843–2861, Apr 1992.

[101] F. Intravaia, S. Maniscalco, and A. Messina, "Density-matrix operatorial solution of the non-markovian master equation for quantum brownian motion," *Phys. Rev. A*, vol. 67, p. 042108, Apr 2003.

[102] F. Intravaia, S. Maniscalco, and A. Messina, "Comparison between the rotating wave and feynman-vernon system-reservoir couplings in the non-markovian regime," *The European Physical Journal B - Condensed Matter and Complex Systems*, vol. 32, no. 1, pp. 97–107, 2003.

[103] S. Maniscalco, J. Piilo, F. Intravaia, F. Petruccione, and A. Messina, "Lindblad- and non-lindblad-type dynamics of a quantum brownian particle," *Phys. Rev. A*, vol. 70, p. 032113, Sep 2004.

[104] G. Adesso, "Gaussian interferometric power," *Phys. Rev. A*, vol. 90, p. 022321, Aug 2014.

[105] L. A. M. Souza, H. S. Dhar, M. N. Bera, P. Liuzzo-Scorpo, and G. Adesso, "Gaussian interferometric power as a measure of continuous-variable non-markovianity," *Phys. Rev. A*, vol. 92, p. 052122, Nov 2015.

[106] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*. Cambridge University Press, 1991. Cambridge Books Online.

[107] A. S. Holevo, "One-mode quantum gaussian channels: Structure and quantum capacity," *Problems of Information Transmission*, vol. 43, no. 1, pp. 1–11, 2007.

[108] R. Bhatia, *Matrix Analysis*. Springer, 1997.

[109] R. Vasile, S. Olivares, M. G. A. Paris, and S. Maniscalco, "Continuous-variable-entanglement dynamics in structured reservoirs," *Phys. Rev. A*, vol. 80, p. 062324, Dec 2009.

[110] B. Bellomo, A. De Pasquale, G. Gualdi, and U. Marzolino, "Reconstruction of time-dependent coefficients: A check of approximation schemes for non-markovian convolutionless dissipative generators," *Phys. Rev. A*, vol. 82, p. 062104, Dec 2010.

[111] P. Marian and T. A. Marian, "Continuous-variable teleportation in the characteristic-function description," *Phys. Rev. A*, vol. 74, p. 042306, Oct 2006.

[112] D. Chruściński and S. Maniscalco, "Degree of non-markovianity of quantum evolution," *Phys. Rev. Lett.*, vol. 112, p. 120404, Mar 2014.

[113] S. Wißmann, H.-P. Breuer, and B. Vacchini, "Generalized trace-distance measure connecting quantum and classical non-markovianity," *Phys. Rev. A*, vol. 92, p. 042108, Oct 2015.

[114] F. Buscemi and N. Datta, "Equivalence between divisibility and monotonic decrease of information in classical and quantum stochastic processes," *Phys. Rev. A*, vol. 93, p. 012101, Jan 2016.

[115] S. F. Huelga, A. Rivas, and M. B. Plenio, "Non-markovianity-assisted steady state entanglement," *Phys. Rev. Lett.*, vol. 108, p. 160402, Apr 2012.

[116] A. W. Chin, S. F. Huelga, and M. B. Plenio, "Quantum metrology in non-markovian environments," *Phys. Rev. Lett.*, vol. 109, p. 233601, Dec 2012.

[117] B. Bylicka, D. Chruściński, and S. Maniscalco, "Non-markovianity and reservoir memory of quantum channels: a quantum information theory perspective," *Scientific Reports*, vol. 4, pp. 5720–, July 2014.

[118] E.-M. Laine, H.-P. Breuer, and J. Piilo, "Nonlocal memory effects allow perfect teleportation with mixed states," *Scientific Reports*, vol. 4, pp. 4620–, Apr. 2014.

[119] G.-Y. Xiang, Z.-B. Hou, C.-F. Li, G.-C. Guo, H.-P. Breuer, E.-M. Laine, and J. Piilo, "Entanglement distribution in optical fibers assisted by nonlocal memory effects," *EPL (Europhysics Letters)*, vol. 107, no. 5, p. 54006, 2014.

[120] J. Trapani and M. G. A. Paris, "Non-divisibility vs backflow of information in understanding revivals of quantum correlations for continuous-variable systems interacting with fluctuating environments," *ArXiv: 1512.03600T*, Dec. 2015.

[121] P. Haikka and S. Maniscalco, "Non-markovian quantum probes," *Open Systems & Information Dynamics*, vol. 21, no. 01n02, p. 1440005, 2014.

[122] A. Sindona, J. Goold, N. Lo Gullo, S. Lorenzo, and F. Plastina, "Orthogonality catastrophe and decoherence in a trapped-fermion environment," *Phys. Rev. Lett.*, vol. 111, p. 165303, Oct 2013.

[123] M. Borrelli, P. Haikka, G. De Chiara, and S. Maniscalco, "Non-markovian qubit dynamics induced by coulomb crystals," *Phys. Rev. A*, vol. 88, p. 010101, Jul 2013.

[124] M. Borrelli and S. Maniscalco, "Effect of temperature on non-markovian dynamics in coulomb crystals," *International Journal of Quantum Information*, vol. 12, no. 02, p. 1461006, 2014.

[125] P. Haikka, J. Goold, S. McEndoo, F. Plastina, and S. Maniscalco, "Non-markovianity, loschmidt echo, and criticality: A unified picture," *Phys. Rev. A*, vol. 85, p. 060101, Jun 2012.

[126] N. Chancellor, C. Petri, and S. Haas, "Non-markovian equilibration controlled by symmetry breaking," *Phys. Rev. B*, vol. 87, p. 184302, May 2013.

[127] F. A. Pollock, C. Rodríguez-Rosario, T. Frauenheim, M. Paternostro, and K. Modi, "Complete framework for efficient characterisation of non-Markovian processes," *ArXiv: 1512.00589P*, Dec. 2015.

[128] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions: with Formulas, Graphs, and Mathematical Tables.* Dover Publications, 1983.

[129] J. Fiurášek, "Improving the fidelity of continuous-variable teleportation via local operations," *Phys. Rev. A*, vol. 66, p. 012304, Jul 2002.

Acknowledgements