Università degli Studi di Salerno
Dipartimento di Fisica "E. R. Caianiello"
INFN, Sezione di Napoli
Gruppo collegato di Salerno

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Theoretical and phenomenological implications of fundamental principles in Physics

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Author
Luciano Petruzziello

Ph.D. School Coordinator
Ch.mo Prof. Carmine Attanasio

Advisor
Ch.mo Prof. Gaetano Lambiase

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To the undying faith my parents placed in me and
to the vivid love from my grandparents I carry within.

Abstract

The relentless quest for unraveling the true mechanisms that govern the Nature we observe and are part of have led physicists to rely on the adoption of fundamental principles. These assumptions have acted as a guiding light for the development of both theory and experiments since the moment in which Physics was born. Therefore, in order to further broaden the knowledge of the Universe surrounding us, it is natural to still have faith in the implications that a rational (though most of the times unexplainable) insight entails.

The aim of this thesis is to provide a self-contained analysis centered around some of the most important physical principles we currently have at our disposal: general covariance, equivalence principle and Heisenberg uncertainty relations. However, the attention is not exclusively focused on the relevant consequences of the aforementioned concepts, as we also insist on the possibility of going beyond them, thus allowing for the existence of a novel phenomenology which can only be unfolded by means of new physics. In this direction, we prove that intriguing perspectives for future investigations can be achieved in several ways. In particular, we show that:

- the requirement of general covariance fulfillment unambiguously leads to a theoretical check of the Unruh effect and to peculiar properties associated with the mixed nature of neutrinos in connection with the Unruh radiation;

- equivalence principle violation is a viable outcome both in the quantum realm and at finite temperature, thus showing that it might not be always valid at all regimes and regardless of the interaction of the studied system;

- Heisenberg uncertainty relations are not exact in the presence of a gravitational field, which induces modifications that become relevant at the Planck scale and that might in principle be revealed also at current energies.

Furthermore, we also remark that tests involving the Casimir effect are particularly sensitive and hence useful in the above frameworks, in that the measurable quantities related to it acquire a contribution that accounts for any violation/generalization of the aforementioned principles.

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Contents

Conventions and abbreviations

Units and metric notation

Throughout the thesis, we set

$$\hbar = c = k_B = 1\,, \tag{1}$$

unless explicitly stated otherwise. We work in $1+3$-dimensions, and we adopt the mostly negative signature for the metric

$$(+\,-\,-\,-)\,. \tag{2}$$

Riemann and Ricci curvature tensors are expressed as follows:

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma} \tag{3}$$

$$R_{\mu\nu} = R^{\rho}{}_{\sigma\rho\nu} \tag{4}$$

where the Christoffel symbols are defined in terms of the metric tensor $g_{\mu\nu}$ assuming metric compatibility and vanishing torsion

$$\Gamma^{\rho}_{\nu\sigma} = \frac{1}{2} g^{\rho\lambda} \left(\partial_\sigma g_{\lambda\nu} + \partial_\nu g_{\lambda\sigma} - \partial_\lambda g_{\nu\sigma}\right)\,. \tag{5}$$

Moreover, the usual Minkowski metric is denoted as $\eta_{\mu\nu}$ to distinguish it from the non-flat case.

Special characters and abbreviations

The following special characters and abbreviations are employed:

Character	Meaning
*	complex conjugate

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$\dagger$ or h.c.	Hermitian conjugate
$^{-}$	Dirac adjoint
$\frac{\partial}{\partial x^\mu}$ or ∂_μ	partial derivative
∇_μ	covariant derivative
Re (Im)	real (imaginary) part
tr	trace
ln	natural logarithm
Γ	Gamma function
$K_\alpha(x)$	K-Bessel function of order α and argument x
$[A, B]$	$AB - BA$
$\{A, B\}$	$AB + BA$
$\equiv$	defined to be equal to
$\propto$	proportional to
$::$	normal ordering
QM	Quantum Mechanics
SR	Special Relativity
GR	General Relativity
QFT	Quantum Field Theory
QFTCS	Quantum Field Theory in Curved Spacetime
SM	Standard Model
SME	Standard Model Extension
QG	Quantum Gravity
GC	General Covariance
LI	Lorentz Invariance
LLI	Local Lorentz Invariance
LIV	Lorentz Invariance Violation
EP	Equivalence Principle
HUP	Heisenberg Uncertainty Principle

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PPN	Parametrized post-Newtonian
GUP	Generalized Uncertainty Principle
CG	Corpuscular Gravity
IR	Infrared
UV	Ultraviolet

We use greek letters for 4-dimensional spacetime indices ranging from 0 to 3, whereas latin letters are reserved for 3-dimensional spatial indices running from 1 to 3. Moreover, when vierbein formalism is employed, latin letters with a hat denote locally flat indices belonging to the local tangent bundle of the Riemannian manifold.

Introduction

In the yesteryear of Physics, the adoption of fundamental principles has turned out to be an unprecedented information carrier. The application of such guiding notions represents an important step towards the understanding of the mechanisms with which Nature explains itself through any physical manifestation. Although intimately related to the phenomenological realm from which they stem, once elevated to the status of postulates, physical principles act as a lighthouse for the development of a consistent theoretical apparatus. History is crawling with illustrious examples in support of the aforementioned statement.

A paradigmatic episode in this direction can be recognized in the birth of classical mechanics itself. Indeed, in 1630 Galileo Galilei published one of his most significant work, titled *"Dialogo sopra i due massimi sistemi del mondo"*. In this book, relevant physical concepts are exhibited and explained by means of clever gedanken experiments, the most famous of which is related to the principle of relativity. Galilei imagined an observer to be locked within the innermost cabin of a huge vessel that floats on an ideal calm sea. In similar conditions, he realized that it is impossible to distinguish whether the vessel is sailing in absence of waves and with uniform speed or it is at rest at the harbor:

> *fate muover la nave con quanta si voglia velocità; ché (pur che il moto sia uniforme e non fluttuante in qua e in là) voi non riconoscerete una minima mutazione in tutti li nominati effetti, né da alcuno di quelli potrete comprender se la nave cammina o pure sta ferma* [...]

Several years later, the principle of relativity and all concepts contained in the works of Galilei and other philosophers and mathematicians (Copernicus, Descartes, etc.) inspired the mind and researches of Sir Isaac Newton, who laid the foundations of classical mechanics with his groundbreaking book *"PhilosophiæNaturalis Principia Mathematica"*. The game-changing impact of this opus on the scientific community of that time can be summarized in the words of the French mathematical physicist Alexis Clairaut:

> *The famous book of* Mathematical Principles of Natural Philosophy *marked the epoch of a great revolution in physics. The method followed by its illustrious author Sir Newton* [...] *spread the light of mathematics on a science which up to then had remained in the darkness of conjectures and hypotheses.*

Moreover, the efforts of Newton are deemed to be the first great unification, in that he was able to formulate a single description for phenomena occurring both at the "human" and at the astronomical scale (Kepler's laws of planetary motion). The recurrent idea of unification is at the basis of the modern prospect of having a theory of everything, which encodes and perfectly depicts all aspects of the Universe, from the quantum realm up to the cosmological domain.

In such an extended journey across radically different energy scales and phenomenology, it is reasonable to rely on some fundamental principles that can guide wit to overcome problems and difficulties throughout the uphill path. This is exactly what happened in the development of the two theories that have become the cornerstone of modern physics in the early twentieth century: quantum mechanics and special relativity.

The harbingers of the first theory were introduced in the attempt of finding a motivation for the flawed spectral radiance of black bodies as a function of wavelength. Classical calculations performed by Lord Rayleigh and Sir James Jeans showed that the radiation emitted by a blackbody increases as the frequency grows, thus resulting in an inconsistency between theoretical prediction and experiments. Such a problem is nowadays addressed as ultraviolet catastrophe (coined by Paul Ehrenfest in 1911), and it was brilliantly solved by Max Planck [1] by assuming the existence of energy quanta. Borrowing Poincaré's words, we can briefly say that:

> *Planck* [...] *devised his quanta theory, according to which the exchange of energy between the matter and the ether–or rather between ordinary matter and the small resonators whose vibrations furnish the light of incandescent matter–can take place only intermittently. A resonator can not gain energy or lose it in a continuous manner. It can not gain a fraction of a quantum; it must acquire a whole quantum or none at all.*

This working hypothesis shed light on a series of rather obscure phenomena which can only be explained by taking Planck's postulate for granted. For instance, the first application of the principle of quantized energy is the model associated to the well-known photoelectric effect devised by Albert Einstein [2], which earned him the Nobel Prize in 1921.

On the other hand, a similar course of events led to the formulation of special relativity. Before its advent, it was believed that light waves propagate into a luminiferous aether, and several methods were conceived with the aim of detecting it by using the relative motion of matter through the aether. Among the proposals, the most renowned experiment is attributable to Michelson and Morley [3]. However, according to the acquired data, the speed of light seemed to be a constant quantity, and this was in conflict with the existence of a medium in which light waves spread. A fine quote to express the inadequacy of the aether is attributable to Hendrik Lorentz:

> *The impressions received by the two observers A_0 and A would be alike in all respects. It would be impossible to decide which of them moves or stands still with respect to the ether, and there would be no reason for preferring the times and lengths measured by the one to those determined by the other, nor for saying that either of them is in possession of the "true" times or the "true" lengths.*

Such a crucial outcome enshrined the failure of the luminiferous aether conjecture, which was superseded by the principles of special relativity, that made their appearance in literature for the first time in 1905 [4].

The aforementioned examples are only the most emblematic ones that convey the relevant role covered by the adoption of principles in Physics. From Newton to Einstein, from classical mechanics to QFT and GR, fundamental postulates have been established for the edification of solid and well-grounded theories. The worth of a new theory is measured with its capability of recovering successful predictions of old models in a suitable limit and at the same time going further for the description of still inexplicable phenomena. In both cases, reasonable physical principles may potentially open a window for the refinement of actual theoretical models and for the foundation of new ones. In this perspective, we want to stress that the above-mentioned view is recently impinging on quantum gravity domain. Indeed, it is common knowledge that GR exhibits severe incompatibilities when it comes to connect its regime of validity with the realm of QM. The hope is to formulate a consistent QG theory, so that it would be feasible to investigate the behavior of any interaction including gravity even when quantum effects are not negligible. In current literature, many sharp proposals can be found (string theory, loop quantum gravity, asymptotic safety in QG, causal set theory, etc.), but the sensation is that there is still a great amount of conceptual issues and drawbacks to deal with. In spite of this, all the candidates possess a discrete number of underlying similarities, namely supposed properties that should become manifest at the QG scale, which is of the order of the Planck scale $m_p \sim 10^{19}$ GeV. By virtue of this fact, the scientific community advocates the possibility of providing QG with a set of principles that any would-be model should satisfy [5]. Such a requirement restricts the list of possible candidates, thus narrowing the circle of potential theories that rigorously describe quantum and gravitational effects altogether.

The aim of this thesis consists in the analysis of selected physical principles of major importance in the contemporary scientific landscape. However, this study is not only focused on their theoretical aspects and/or their mathematical formulation, but it also comprises a broader scenario. Indeed, in the next Chapters we will discuss phenomenological implications and important predictions associated to the investigated postulates. Moreover, we explore the possibility of coping with eventual violations/generalizations of some of them, together with a discussion revolving around immediate consequences and potential experimental windows in which detect tiny deviations from the standard framework. In view of the aforesaid concepts, the essay is organized as follows:

– *Chapter 1* deals with the introduction and the description of the selected fundamental principles. In particular, we present the main topics with a high degree of accuracy and we discuss about each one of them separately. The treatment is intended to be self-consistent, but not complete; for further details, the reader is invited to consult the quoted references.

– *Chapter 2* is devoted to the investigation centered around the Casimir effect. After a brief overview of the phenomenon, we then shift the focus on its application in different contexts. The purpose of this action lies in the opportunity to evaluate the tiny deviation from the usual measurable results that is directly related to the violation of the principles presented in the first Chapter. In so doing, we also emphasize the sensitivity of the Casimir experiment to new physics phenomenology.

– *Chapter 3* addresses fundamental claims related to the possibility of identifying the decay of an accelerated proton as a "theoretical check" of the Unruh effect. The existence of a similar phenomenon is due to the requirement of general covariance in QFT computations; therefore, the fulfillment of a fundamental principle necessarily entails the occurrence of a physical manifestation not yet spotted. The intriguing line of research developed after this idea is riddled with controversies when neutrino mixing is taken into account. However, by still relying on general covariance, we prove that a plausible solution to the dispute does exist.

– *Chapter 4* tackles one of the cornerstones of GR, namely the equivalence principle. Indeed, we point out different contexts in which classical EP appears to be violated. Among other things, we encompass the case in which a charged point particle object is freely falling within a region with non-vanishing temperature. Moreover, we explore the intertwining between neutrinos and EP violation, which is already an existing topic in literature. This analysis involves some remarks on the non-relativistic behavior of neutrinos as well as flavor transitions occurring throughout a free propagation on a curved background.

– *Chapter 5* investigates the consequences of generalizing the usual Heisenberg uncertainty relations to the case in which gravitational effects cannot be neglected. Such a scenario sinks its roots into quantum gravity; several QG models predict the existence of a minimum length at Planck scale, which should be envisaged by suitably modifying HUP. By resorting to the so-called generalized uncertainty principle, it is intriguing to check how physical quantities associated to a given quantum effect vary in response to the changes made to the standard HUP. In addition, such a procedure may also be employed to compare predictions of different QG candidates, thus getting closer to a satisfactory framework in which it is possible to handle both gravity and quantum mechanics.

– *Chapter 6* contains conclusions and future perspective associated to possible implementations of the arguments treated in the current work.

Chapter 1

A first glance at the selected principles in Physics

In this Chapter, we introduce the physical principles under examination. A meticulous concern is devoted to the presentation of the chosen topics so as to render them self-consistent. Moreover, the analysis not only concentrates on the illustration of the treated arguments, but also endeavors to go beyond a mere overview of them. Indeed, for all the investigated principles, we mention some considerations about eventual generalizations and/or violations, both from a theoretical and an experimental perspective. Such a scenario should be neither underestimated nor ignored, since it conceals the possibility of encountering unexpected signatures of new physics. Therefore, the thorough study and the continuous questioning of the available physical postulates may be regarded as a valuable probe for predictions and phenomenological implications that cannot be deduced from the current theoretical models. A more detailed analysis of such setting will be tackled in the next Chapters.

On the other hand, an accurate research on this topic is also a bearer of cues related to the contingency of a "dismissal" of one or more adopted principles. As a matter of fact, it is not always indispensable to come up with novel proposals which are added to the already existing ones; in some cases, it suffices to give up on some postulates to attain a workable model. Along this line, remarkable examples in the list of QG candidates are given by the departure from the perturbative renormalizability of GR (firstly explored in Refs. [6]) which led to the concept of asymptotic safety [7] and the requirement of non-locality that fostered the advent of infinite derivative gravity models [8].

In the next Sections, we analyze the following physical arguments with the ensuing order: general covariance, equivalence principle and Heisenberg uncertainty relations. Such a sequence does not suggest the relevance of one topic over the others, but it has been chosen this way for the sake of convenience. All the aforementioned principles have a huge impact on the modern setting of theoretical physics, and all of them deserve the same prominent role they factually occupy.

1.1 General covariance

The principle of general covariance is one of the essential building blocks that led Einstein to the implementation of general relativity [9]. In its most famous formulation (cf. Ref. [10]), GC states that *all physical laws retain the same form under any arbitrary differentiable coordinate transformation* (*diffeomorphisms*). On the other hand, it can also be found within a different shape, as it is demonstrated by the expression reported in the book by Wald [11]:

> *The principle of general covariance* [...] *states that the metric of space is the only quantity pertaining to space that can appear in the laws of physics.*

This concept has represented a constant guide for Einstein throughout the development of GR, even though he himself was on the verge of discarding it for a short period of time. A similar occurrence is due to the fact that, although straightforward in its statement, the magnitude of GC advent is not completely evident. Indeed, as claimed in Ref. [10], the revolutionary implications withheld by GC *fathered half a century of confusion*. A well-known case in this direction is due to Kretschmann [12], who recognized no physical motivations behind the adoption of general covariance, which in his opinion can be introduced ad hoc in any theory.

Moreover, the requirement of having the mathematical apparatus imposed by GC has been object of a plethora of discussions also in more recent years. For instance, in Ref. [13] it is argued that GC needs to be revisited and reformulated in a proper way so as to clarify and to better expound the issues raised against it in the last century. A more radical point of view is contemplated by the authors of Ref. [14], in which GC is addressed as a "dogma" that should be reconsidered. According to their reasoning, in all the works appeared in literature after the emergence of GR, physicists have always preferred a given coordinate system instead of other ones with the purpose of both defining quantities of physical interest and simplifying calculations. Such a tendency suggests the possibility that GC can either be overcome in favor of a different principle or be regarded as a totally negligible requisite. In this sense, we believe the authors of Ref. [14] are tackling the matter in an improper way. The message of their paper is the same expressed in the remarkable works [15] in which Weinberg introduced the *unitary gauge*. In a nutshell, in particle physics and in the context of spontaneous symmetry breaking, the aforesaid choice of gauge fixing allows for the true degrees of freedom to become manifest in the Lagrangian, thus ruling Goldstone bosons out of the analysis. In the framework of GR, we have the "classical" counterpart of the above phenomenon, since GC implies that the set of chosen coordinates are nothing but *gauge functions*.

As a clarifying introduction to the concepts treated up to now, it is worth concisely recalling the steps taken by Einstein for the development of GR. In doing so, we mainly follow the path traced in Refs. [16, 17].

1.1.1 The advent of general covariance

In 1912, Einstein moved to Zurich and started his collaboration with Marcel Grossmann, who introduced him to the mathematical developments in the field of absolute differential calculus accomplished by Ricci and Levi-Civita. In these years, the seminal papers aiming at the emergence of a general theory of relativity were published. In similar circumstances, the guiding light of GC was brighter than ever, since several works contain hints in the direction of a set of equations for the gravitational field of the kind

$$G_{\mu\nu}\left(g, \partial g, \partial^2 g\right) = k\, T_{\mu\nu}\,, \tag{1.1}$$

in which $T_{\mu\nu}$ is the stress-energy tensor of the source of gravity, whereas $G_{\mu\nu}$ is a function of the metric tensor and its field derivatives only[1].

Although the physical intuition was flawless, Einstein believed that $G_{\mu\nu} = R_{\mu\nu}$. However, such a choice would not return the experimentally successful Newtonian limit, and this occurrence was seen as the first signal of a premature reappraisal of GC role. Moreover, the requirement that the same metric solution of (1.1) is still a solution after a change of coordinates, namely

$$g'_{\mu\nu}(y) = \frac{\partial x^\alpha}{\partial y^\mu}\frac{\partial x^\beta}{\partial y^\nu}\, g_{\alpha\beta}(x)\,, \tag{1.2}$$

led Einstein to further question the requirement of a generally covariant theory [18]. The idea underlying the previous hypothesis can be summarized in the *hole argument*[2].

[1] As well-known, later on the above quantity was discovered to be the Einstein tensor we are familiar with.

[2] For a pedagogical explanation involving the Schwarzschild solution explicitly, see Ref. [19].

The hole argument

Suppose to consider a portion of spacetime where there is no matter (i.e. $T_{\mu\nu} = 0$), which will be labeled as "hole". The solution of the field equations (1.1) provides us with a metric tensor $g_{\mu\nu}(x)$ that defines the gravitational field in a given coordinate system. For the sake of transparency, we can focus the attention on two points belonging to the hole, A and B, and suppose that the former is located in a flat region whereas the latter is not (see Fig. 1.1, left part).

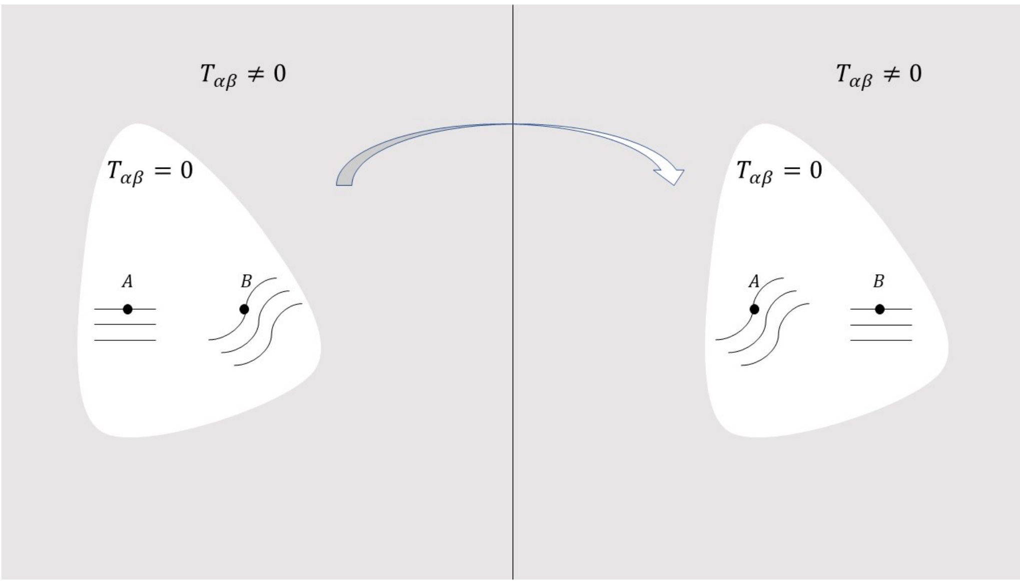

Figure 1.1: In this picture, the grey portion is where the stress-energy tensor is non-vanishing, whilst no matter is present in the white part. The straight lines specify a flat region of spacetime while the wiggly ones denote the presence of curvature.

Let us now perform a change of coordinate system $x^\mu \to y^\mu(x)$, which means that $g_{\mu\nu}(x) \to g'_{\mu\nu}(y)$ with the transformation law exhibited in (1.2). In performing such step, we demand $y^\mu(x)$ to be made in such a way that $x^\mu = y^\mu$ outside the hole while smoothly changing inside of it. In particular, we want to exchange the positions of the points A and B introduced before. Then, we define a *new* metric $g'_{\mu\nu}(x)$, which is the starting metric g written in the new coordinate system y^μ, but expressed by employing the old coordinates x^μ instead of y^μ. By so doing, we now have two *distinct* gravitational fields expressed in the *same* coordinate system. At this point, GC tells us that $g'_{\mu\nu}(x)$ is still a solution for (1.1), but as such it produces a radically different interpretation. Indeed, because of the choice made for the set of y^μ, we know for sure that inside the hole there are two solutions of the same field equations that behave differently. To ensure this, we note that, according to the setting given by $g'_{\mu\nu}(x)$, the flat region is now occupied by the point B whilst A is placed in the curved one (see Fig. 1.1, right part).

The above outcome conveys that (1.1) is not capable of describing physics at the spacetime points A and B, thus lacking determinism. Actually, there are two conclusions one can come up with after having analyzed the hole argument:

- general covariance is not a necessary principle for the theory;
- points belonging to the spacetime manifold have no physical meaning.

Point-coincidence argument

The key to solve the dichotomy lies in the second option: the spacetime manifold has no per se physical interpretation. In the complete formulation of GR, regarding this point Einstein wrote [9]:

> *That this requirement of general covariance, which takes away from space and time the last remnant of physical objectivity, is a natural one, will be seen from the following reflexion. All our space-time verifications invariably amount to a determination of spacetime coincidences.*

Such a statement finally settles the misunderstanding revolving around the crucial role played by GC in the construction of general relativity. It is often addressed to as the spacetime coincidence argument [17], but also as point-coincidence argument [16, 20]. The concept behind these names is simple but at the same time astonishing, and in order to effectively illustrate it we refer to the configuration already used for the hole argument.

As shown before, we have realized that the points A and B on the spacetime manifold do not cover a relevant role from a physical perspective; equivalently, we can say that they are *not* associated to any observable quantity. Suppose now to introduce two point-like test particles[3] inside the hole whose world lines intersect in the point B in the reference frame where the metric tensor is $g_{\mu\nu}(x)$ (see Fig. 1.2, left part). Such intersection may represent some sort of interaction that takes place in B, and therefore it is an event which can be detected. Then, if we perform the same description according to the metric tensor $g'_{\mu\nu}(x)$, we note that the interaction between particles does not occur in B anymore, but rather in A (see Fig. 1.2, right part)! In light of this, it is now meaningful to ask whether the gravitational field is vanishing or not *in the spacetime point where the test particles interact.* The answer is identical both for $g_{\mu\nu}(x)$ and $g'_{\mu\nu}(x)$, which implies that determinism is

[3]By test particle we mean a physical entity whose presence does not alter the environment in which its motion takes place.

left untouched. In order to achieve this result, we have had to require *background independence*, which can be loosely explained by saying that the spacetime structure has a relevance only when dynamical entities (physical fields) are present [17].

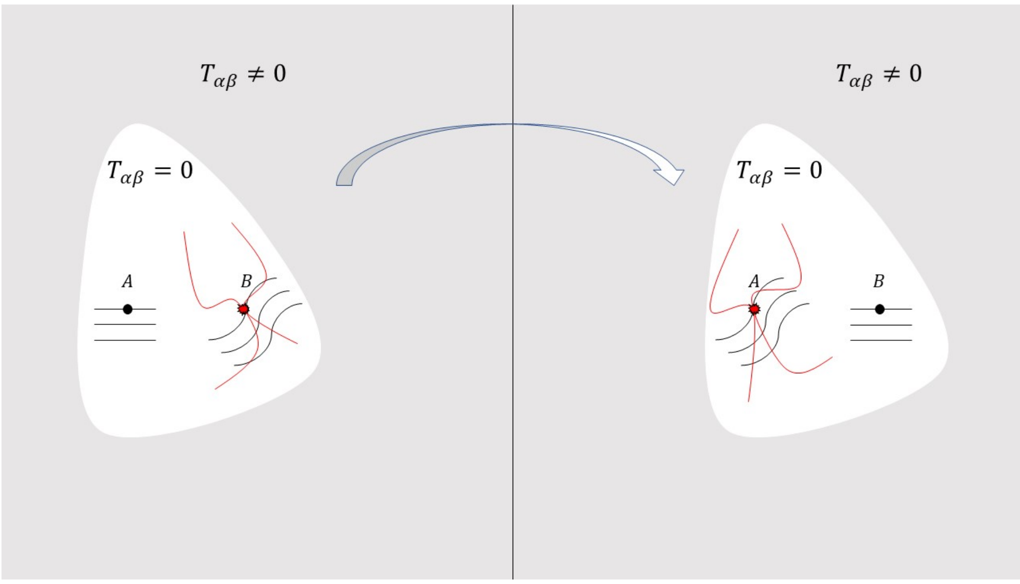

Figure 1.2: The difference between this figure and Fig. 1.1 consists in the world lines of point-like test particles appearing here and sketched in red. Scribbles denote interactions between particles.

We then conclude that both $g_{\mu\nu}(x)$ and $g'_{\mu\nu}(x)$ describe the same field, as it should correctly be. Another rephrasing to express the aforesaid concept conveys that the localization on the manifold is simply a *gauge*, thus being not relevant at all. A diffeomorphism acting on a field simply changes its position on the spacetime manifold (i.e. the redefinition of the metric tensor in the previous example), but such a freedom is harmless, since the physical properties and events whose description should remain invariant for any observer (i.e. the interaction of the test particles in Fig. 1.2) are "dragged" along. Therefore, in Rovelli's words [17]:

> *A state of the universe does not correspond to a configuration of fields on M* (spacetime manifold). *It corresponds to an equivalence class of field configurations under active diffeomorphisms.*

With the aforementioned considerations, we stop the overview of general covariance; for further details, we remand the reader to Refs. [16, 17, 20] and references therein. However, before moving on with the next principle, we must introduce another concept that is strictly related to GC, namely Lorentz invariance, which will also be taken into account in the next Chapters.

1.1.2 Lorentz invariance

In order to verify the relation between GC and LI, it is opportune to resort to another "representation" of general relativity, which is based on the notion of *vierbein* fields (also known as *tetrads*) rather than the metric tensor. Such a description immediately stems from another principle Einstein relied upon for the development of GR, namely the equivalence principle, but its meaning will be thoroughly tackled in the next Section. For our purpose, all we need to know here is that, by virtue of EP, the effects of a gravitational field are always *locally* removable. By this definition, we learn that it is possible to eliminate the impact of gravity in a small neighborhood I_x of a given point x belonging to the spacetime manifold M. Such a procedure entails that in I_x we are allowed to approximate the region of M with its projection on the *flat* tangent space in x, denoted as $T_M(x)$ (see Fig. 1.3).

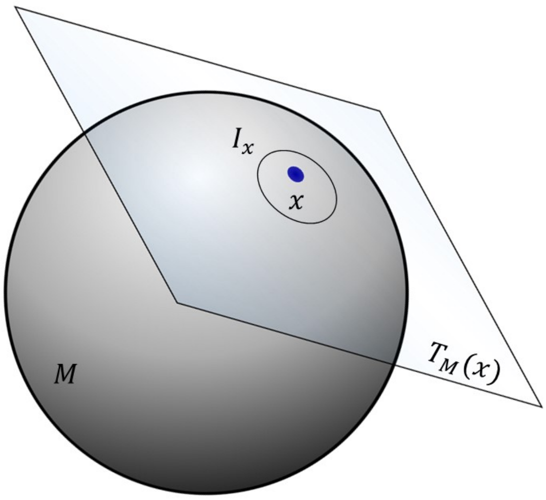

Figure 1.3: This figure depicts a streamlined view of a generic spacetime manifold M and its tangent space $T_M(x)$ in x. An arbitrary neighborhood of x is indicated as I_x.

To achieve such a goal, we define a set of four covariant vectors called vierbein, $e^{\hat{a}}_{\mu}(x)$, $a = 0, \ldots, 3$, which are an orthonormal basis for $T_M(x)$. Therefore, they are orthonormal with respect to the metric of the tangent space [10, 11, 21], that is $\eta^{\hat{a}\hat{b}}$

$$g^{\mu\nu}\, e^{\hat{a}}_{\mu}\, e^{\hat{b}}_{\nu} = \eta^{\hat{a}\hat{b}}\,. \tag{1.3}$$

From the above expression, one can derive also the dual basis $e^{\mu}_{\hat{a}}$, which clearly satisfies the conditions

$$g_{\mu\nu}\, e^{\mu}_{\hat{a}}\, e^{\nu}_{\hat{b}} = \eta_{\hat{a}\hat{b}}\,, \qquad e^{\hat{a}}_{\mu}\, e^{\mu}_{\hat{b}} = \delta^{\hat{a}}_{\hat{b}}\,. \tag{1.4}$$

With the aid of vierbein, the metric tensor $g_{\mu\nu}$ is locally determined by the fields $e^{\hat{a}}_{\mu}$ up to an arbitrariness due to local Lorentz transformations. Indeed, if Λ is a generic Lorentz group transformation[4], which thus leaves η invariant, then $\tilde{e}^{\hat{a}}_{\mu} = \Lambda^{\hat{a}}_{\hat{b}} e^{\hat{b}}_{\mu}$ and hence

$$\tilde{g}_{\mu\nu} = e^{\hat{c}}_{\mu}\, \Lambda^{\hat{a}}_{\hat{c}}\, \eta_{\hat{a}\hat{b}}\, \Lambda^{\hat{b}}_{\hat{d}}\, e^{\hat{d}}_{\nu} = e^{\hat{c}}_{\mu}\, e^{\hat{d}}_{\nu}\, \eta_{\hat{c}\hat{d}} = g_{\mu\nu}\,. \tag{1.5}$$

The description according to either g or e is completely equivalent, but with the employment of vierbein we are able to project every tensorial quantity from the manifold to the tangent space and vice-versa. In fact, given a generic (m, n) tensor on M, say $T^{\mu_1 \ldots \mu_m}_{\nu_1 \ldots \nu_n}$, which under an arbitrary diffeomorphism transforms as

$$\tilde{T}^{\mu'_1 \ldots \mu'_m}_{\nu'_1 \ldots \nu'_n}(y) = \frac{\partial y^{\mu'_1}}{\partial x^{\mu_1}} \cdots \frac{\partial y^{\mu'_m}}{\partial x^{\mu_m}} \frac{\partial x^{\nu_1}}{\partial y^{\nu'_1}} \cdots \frac{\partial x^{\nu_n}}{\partial y^{\nu'_n}}\, T^{\mu_1 \ldots \mu_m}_{\nu_1 \ldots \nu_n}(x)\,, \tag{1.6}$$

and we let tetrads act on it as

$$T^{\hat{a}_1 \ldots \hat{a}_m}_{\hat{b}_1 \ldots \hat{b}_n} = e^{\hat{a}_1}_{\mu_1} \ldots e^{\hat{a}_m}_{\mu_m}\, e^{\nu_1}_{\hat{b}_1} \ldots e^{\nu_n}_{\hat{b}_n}\, T^{\mu_1 \ldots \mu_m}_{\nu_1 \ldots \nu_n}\,, \tag{1.7}$$

we are left with an object which transforms as a (m, n) tensor under the local Lorentz group

$$\tilde{T}^{\hat{a}'_1 \ldots \hat{a}'_m}_{\hat{b}'_1 \ldots \hat{b}'_n} = \Lambda^{\hat{a}'_1}_{\hat{a}_1} \ldots \Lambda^{\hat{a}'_m}_{\hat{a}_m}\, \Lambda^{\hat{b}_1}_{\hat{b}'_1} \ldots \Lambda^{\hat{b}_n}_{\hat{b}'_n}\, T^{\hat{a}_1 \ldots \hat{a}_m}_{\hat{b}_1 \ldots \hat{b}_n}\,, \tag{1.8}$$

but as a scalar under diffeomorphisms.

For what concerns vierbein, instead, they transform as vectors both under diffeomorphisms and under local Lorentz transformations

$$\tilde{e}^{\hat{a}}_{\mu}(y) = \Lambda^{\hat{a}}_{\hat{b}}\, \frac{\partial x^{\nu}}{\partial y^{\mu}}\, e^{\hat{b}}_{\nu}(x)\,. \tag{1.9}$$

The stiff thread that connects GC and LI is bonded to the work made by e, which links with a one-to-one correspondence geometric elements from the curved spacetime manifold to the flat tangent space. Indeed, the principle of general covariance on M finds its "counterpart" in the local Lorentz invariance on the flat tangent space. Evidently, should M be flat everywhere, we talk of global Lorentz invariance, thus recovering special relativity. In this perspective, LI and LLI are to be understood

[4]Since the transformation is local, in the following Λ must be regarded as $\Lambda(x)$, but for notational simplicity we omit the spacetime dependence.

as one of the multiple facets with which GC manifests itself in general relativity.

Lorentz violation

Although beautifully embedded in the GR framework, in recent years LI (but equivalently also LLI) has been challenged both theoretically and experimentally (for a complete survey, see Ref. [22]). As a matter of fact, several QG theories predict the existence of Lorentz invariance violation at all energy scales, even though for current feasible laboratory tests such breaking is expected to be extremely small. In this direction, the most famous model that accounts for LIV is represented by the "Standard Model Extension", arising from investigations in the context of covariant string field theory [23]. In few words, SME[5] is built by obtaining scalars from the contraction of SM and gravitational field with opportune coefficients that induce Lorentz (and CPT) violation [25, 26]. As expected, such coefficients turn out to be heavily suppressed, and thus assumed extremely tiny if analyzed at current scales. However, many focused experiments have been performed with the purpose of establishing constraints on their values and to gather precious information on them [27].

The academic appeal stirred up by SME opened the doors for novel generalizations of LI so as to include the energy scale at which QG effects are believed to be relevant, namely the Planck length $\ell_p \simeq 10^{-35}\, m$. For instance, an interesting model that extends the results of SR in order to render ℓ_p invariant as well as the speed of light c is the so-called DSR, which stands for "Doubly special relativity" [28]. After the first works on this issue, other papers have approached the same idea but with other Planck units, as the Planck mass $m_p \simeq 10^{-8}\, Kg$ [29] and the Planck energy $E_p \simeq 10^{9}\, J$ [30]. Although the theoretical apparatus suffers from several problems that still need to find a definite answer [31], the phenomenological implications of DSR prove to be extremely helpful in constraining Lorentz violation [22]. This holds true since DSR does not contain any preferred reference frame, as it instead occurs for other LIV models. In this connection, it is worth emphasizing that all our study is intended to be performed at zero temperature, $T = 0$; if we had $T \neq 0$, then LIV would automatically become manifest. A similar consideration is easily supported by recalling that the presence of a thermal bath naturally induces spatial anisotropies, and the emergence of a preferred reference frame (i.e. the one in which the thermal bath is at rest) immediately breaks LI. An interesting derivation of LIV at finite temperature in the context of QFT can be found in Ref. [32].

A question that spontaneously arises at this point is: what about general co-

[5]For a pedagogical introduction on this topic, see Ref. [24].

variance? At the end of Sec. 1.1.2 we have said that LLI is an aspect that can be directly related to GC, which means that an hypothetical LIV could in principle interfere with general covariance implications. However, this is not the case, since GC can still be preserved in a Lorentz-violating theory [22]. On the other hand, as suggested in Ref. [22], should Lorentz-violating fields not be treated as dynamical entities, LIV would be at odds with lack of prior geometry [25, 26, 33]. Additionally, LIV also implies a violation of the equivalence principle, in that it induces a mass-dependent acceleration that contradicts one of EP formulations. In order to better elucidate the meaning of the last statement, we now turn the attention on another fundamental postulate which is counted within GR building blocks.

1.2 Equivalence principle

As already anticipated at some point in the previous Section, we have given a definition for the equivalence principle. For the sake of completeness, we recall the formulation stated before: "*the effects of a gravitational field are always locally removable*" [10, 11, 34]. Subsequently, we have partially clarified what is meant for the word "locally", which is a crucial aspect that should not be underestimated. Here, we expand the discussion centered around the above concept and try to carry out an exhaustive survey on EP, which was addressed by Einstein as "the happiest thought of my life".

To begin with this topic, it must be said that the message of the equivalence principle was disclosed long before the advent of GR. Indeed, it was Galilei who first observed that the acceleration of a test body attributable to gravity is mass-independent. Later on, also Kepler (in his work "*Astronomia Nova*") and Newton (with his "*Principia*") dealt with several implications related to the implementation of EP. However, only in 1907 such principle was actually spread and explicitly tackled by Einstein; in its first form, EP can be summarized by the words of Einstein (1907):

> *we* [...] *assume the complete physical equivalence of a gravitational field and a corresponding acceleration of the reference system.*

To put it differently, there is no way an observer closed in a room with no windows and openings can discriminate between the presence of a gravitational field and a uniformly accelerated motion in absence of gravity. This holds true since it is always possible to simulate the effects of gravity by introducing an *ad hoc* magnitude for the acceleration. In particular, if with ϕ we denote the gravitational potential, we immediately guess that

$$\mathbf{a} = -\boldsymbol{\nabla}\phi\,, \tag{1.10}$$

is the exact acceleration a system should possess so as to mimic the presence of gravity (see Fig. 1.4). Note that, in writing down (1.10), we have tacitly assumed the *equality* between the *inertial* mass m_i and the *gravitational* mass m_g. Such an enforcement is another convenient formulation to regard the equivalence principle.

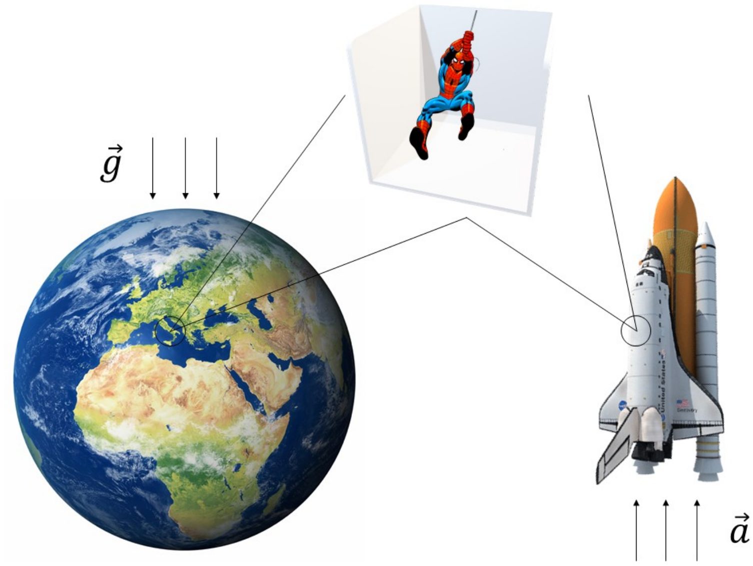

Figure 1.4: Spiderman swinging in a closed room cannot distinguish whether he is in a region of spacetime where the gravitational field produces an acceleration equal to $\mathbf{g}$ or in a rocket with an unchanging rate of acceleration $\mathbf{a}$ if $|\mathbf{g}| = |\mathbf{a}|$.

As a further remark related to the above arguments, we underline that, should the previous observer freely fall together with the room and all pieces of furniture, he/she would experience no effects that can be associated to a gravitational field. Such an occurrence is traceable to the *universality* of the gravitational interaction, which renders it radically different from the other fundamental interactions. Additionally, the universality of free-fall implies that, in these conditions, there is no such thing as acceleration, and hence it is as if no forces are detectable. In this sense, a freely-falling reference frame is comparable to a *local* inertial frame, which thus legitimates the notion of LLI introduced in the previous Section. Along this line, another powerful proposition of the equivalence principle claims that *in any local inertial (Lorentz) frame, all the laws of physics must reduce to the description predicted by special relativity* [10, 35]. The strength of the last rephrasing of EP lies in the opportunity to suitably extend all the known physical laws of SR according to the *minimal coupling principle* [34]. In practice, by virtue of EP, we know how to properly generalize a SR framework to the case in which gravitational effects are no longer negligible[6]. A remarkable result that stems from this correspondence can be

[6]For instance, the "comma-goes-to-semicolon" rule [10] is a consequence of that.

recognized with QFT in curved spacetime, which essentially resorts to the aforesaid procedure to account for the presence of gravity in quantum field theoretical processes. Such a conceptually simple extension of QFT is of crucial importance, since it is thought that the analysis of quantum phenomena on curved background represents a semi-classical limit of a complete quantum gravity theory, thereby binding all QG predictions to be consistent with QFTCS under certain circumstances.

At this stage, there is still one point that needs to be clarified, namely the relevance of the adjective "local" within our statements concerning EP. As a matter of fact, the equality between an inertial and a freely-falling system has only a local significance. No sooner do we deal with a wider portion of spacetime than all implications of EP fail to apply. This conclusion is attributable to the fact that gravity *is* curvature of spacetime. In order to show it, let us recall that a freely-propagating test body in presence of a gravitational field moves along a given *geodesic*, which (geometrically speaking) is the curve that minimizes the distance between two points on a manifold. Obviously, in absence of gravity, geodesics are mere straight lines. However, when the spacetime manifold is curved due to the presence of a gravitational source, geodesics follow the background curvature, thus bending themselves in return [10, 34]. Therefore, it should be clear by now that the locality of the equivalence principle is related to the fact that only in a small portion of spacetime geodesics can be approximated by the straight lines tangent to the manifold. Such an idea is easily conveyed by means of a gedanken experiment[7]: suppose to have two objects "floating" (i.e. no forces are present) in an elevator whose relative distance at a given time t is $d(t)$. To achieve the desired outcome, we have to require that all relative gravitational interactions are totally negligible. If the system is globally inertial, at a later time $t' > t$ an observer inside the elevator would measure the same distance $d(t') = d(t)$ between the objects (see Fig. 1.5, right part). On the other hand, if the system is only locally inertial and the initial condition are identical to the previous setting, after a sufficient time $t' > t$ an observer comoving with the freely-falling objects and elevator would measure $d(t') < d(t)$ (see Fig. 1.5, left part). Indeed, if the considered region of spacetime is sufficiently large, the gap among geodesics followed by the test bodies starts narrowing, thus allowing the objects to come closer throughout the free-fall.

If we want to highlight this aspect even more, we can rely on the existence of the so-called Riemann normal coordinates, which can be used to cast the metric tensor in a form that resembles Minkowski metric. Without entering the geometric details

[7] For a more rigorous derivation of this concept, see Appendix A.

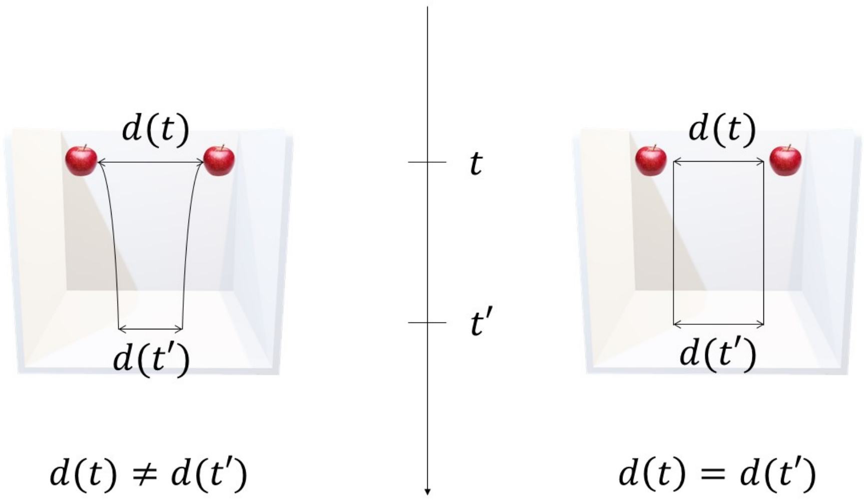

Figure 1.5: The relative distance between two apples is not the same if they are freely falling under the influence of an external gravitational field. Note that the height of the apples with respect to the floor is constant for an observer inside the elevator. Therefore, the lines that go down should be regarded as the temporal evolution of the system, and not as an actual fall.

(see Ref. [10]), one can prove that in such coordinate system $g_{\mu\nu}$ is denoted by [36]

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{3}\,R_{\mu\alpha\nu\beta}\,x^\alpha\,x^\beta + \mathcal{O}\left(x^3\right)\,. \tag{1.11}$$

From the above expression, one can comprehend that locality is a necessary requisite to treat any freely-falling reference frame as an inertial frame. The second term of the r.h.s. is the error committed in performing such an approximation, and as long as spacetime is curved the Riemann tensor is always non-vanishing.

1.2.1 Different formulations

As we have already seen so far, there is a plethora of ways with which to refer to EP: although several rephrasing are completely equivalent, some of them are more general than others. In what follows, we try to succinctly give a panorama of all the currently affordable formulations. To this aim, we essentially follow the scheme adopted in Refs. [37, 38], in such a manner to avoid the confusion persisting in literature about the different statements to indicate EP.

Newton Equivalence Principle (NEP): This principle has already been tackled in the comment after (1.10). As a matter of fact, this axiom affirms that, *in the Newtonian limit,* $m_i = m_g$. An equivalent expression to say it consists in recalling once again the universality of gravity, which operates indiscriminately on any form of matter and energy.

Weak Equivalence Principle (WEP): *In a gravitational field, the motion of test particles with negligible self-gravity does not depend on their physical features.* In order to quantify self-gravity, we can duly introduce the dimensionless parameter

$$\lambda = \frac{Gm}{rc^2}\,, \tag{1.12}$$

with G being the Newton constant, c the speed of light, m the mass of the test body and r its linear size. Equation (1.12) is the ratio between the gravitational and the rest energy provided that NEP holds. Therefore, we can assert that as long as $\lambda \ll 1$ self-gravity can be safely neglected.

Roughly speaking, this notion can be easily visualized by imagining that, if two test particles have the same initial conditions, they travel along the same geodesic, regardless of their properties (i.e. mass, charge, etc.).

Gravitational Weak Equivalence Principle (GWEP): *In a gravitational field and in vacuum, the motion of test particles does not depend on their physical features.* Differently from the previous statement, we have relaxed the condition on self-gravity, which implies that GWEP $\to$ WEP as $\lambda \to 0$. However, such modification entails a further requirement, namely the presence of vacuum. This is crucial, otherwise gravitational field of test bodies interacts with the physical environment in which they propagate. In so doing, due to the action-reaction principle, they would feel a net force that would jeopardize the universality of the gravitational interaction, since the aforementioned force would be a function of the particles' properties.

Einstein Equivalence Principle (EEP): *The presence of a gravitational field does not affect fundamental non-gravitational physical tests locally and in any point of spacetime.* At this point, it is worth stressing the relevance of some expressions contained in the previous sentence. By "fundamental physical tests" we refer to experiments that probe the validity of equations describing the behavior of single particles, thereby undermining the ones which can be deduced by them. For instance, the rules that govern the motion of composite systems are not contemplated by EEP, since for such systems local gravitational effects may be detected via experimental tests. The term "locally" should be clear by now and it is manifestly related to LLI. Finally, with the line "in any point of spacetime" we intend that there are no privileged points on the spacetime manifold. Consequently, it is unimportant to know *where* and *when* in the universe the experiment is conducted, since the outcome shall not depend on it. A similar idea is often reported as *local position invariance* [35].

In light of this, we observe that EEP is the simultaneous requirement of weak equivalence principle, local Lorentz invariance and local position invariance [35, 37].

Strong Equivalence Principle (SEP): *The presence of a gravitational field does not affect all fundamental physical tests (including gravitational physics) locally and in any point of spacetime.* Insofar, from its definition it should be clear that SEP is the simultaneous requirement of gravitational weak equivalence principle, local Lorentz invariance and local position invariance. Hence, from the perspective of such principle, it is possible to perform even gravitational local experiments in presence of an external gravitational field, with the results not being invalidated by that.

For a more detailed reading on this topic and on the interplay between the above non-equivalent EPs, see Refs. [35, 37, 38].

1.2.2 Parametrized post-Newtonian formalism

As predictable, the equivalence principle is the protagonist of a vast variety of laboratory tests. For its easiest form, namely $m_i = m_g$, many experiments have been developed throughout the years. It is sufficient to mention that Newton himself was the first one to indirectly verify the assumption of equal inertial and gravitational mass by evaluating the period of a pendulum with a test body attached to its free end. Later on, the tests were refined by Loránd Eötvös through the adoption of a torsion balance, which sharply increased the accuracy of the data acquirement. After his pioneering works, several upgrades have been performed to the experimental setting, thus resulting in a net improvement of the sensitivity of the instruments.

However, for the purpose of the current essay, the relevant theoretical tool to be discussed is represented by the post-Newtonian formalism [35, 37, 39]. Such an approximation is applicable to a system of slowly-moving particles[8] which are tied together by gravitational interactions. For this reason, it is typically confused with the weak-field limit, which instead is a completely different analysis. In order to tackle the above topic, we can assume the typical values of mass, mutual distance among two bodies and velocity of the particles composing the studied system to be

[8]Here, by particles we can generally denote also extended objects.

m, r and v, respectively. In classical mechanics, it is well-known that, if the kinetic and the gravitational energy are of the same magnitude, then

$$v^2 \sim \frac{Gm}{r}\,. \tag{1.13}$$

To go beyond Newtonian mechanics, we must require higher powers in the small parameter v^2 or equivalently Gm/r. In this sense, by post-Newtonian approximation we basically mean an expansion of the metric tensor around the parameter v^2, which is dimensionless[9]. The details of calculations involving such expansion are beautifully covered in Ref. [39], and do not play an important aspect to approach here.

On the other hand, what we need to emphasize is that the aforesaid procedure may be applied to verify eventual predictions of extended theories of gravity by means of simple tests which do not require experiments in the relativistic and in the strong gravity regime. Indeed, there exists a classification of gravitational theories based on the introduction of ad hoc coefficients that are model-dependent. A similar differentiation is encountered within the so-called parametrized post-Newtonian formalism, which juxtaposes the metric potentials arising from the post-Newtonian expansion with distinct parameters, each of them having a precise significance [35, 37]. According to the PPN method, there are currently ten available parameters used to catalogue a given gravitational model [35]; their meaning and values for GR are displayed in Table 1.1.

Table 1.1: The ten PPN parameters

Parameter	Meaning	Value in GR
γ	Space curvature produced by unit rest mass	1
β	Nonlinearity effects for gravity	1
ξ	Preferred-location effects	0
α_1		0
α_2	Preferred-frame effects	0
α_3		0
α_3		0
ζ_1		0
ζ_2	Violation of total momentum conservation	0
ζ_3		0
ζ_4		0

In the following Chapters, we will be mainly concerned with the parameters γ

[9] Recall that we have set $c = 1$, which implies that the expansion parameter is actually v^2/c^2.

and β, also known as Eddington-Robertson-Schiff parameters. They are the only non-vanishing parameters both in GR and in scalar-tensor theories of gravity.

In particular, γ and β will be employed to properly quantify the violation of SEP. In this direction, it is conventional to introduce the so-called Nordtvedt parameter η [40, 41], defined as

$$\eta = 4\,(\beta - 1) - (\gamma - 1)\ . \tag{1.14}$$

Strong equivalence principle is violated as long as $\eta \neq 0$ [41]. In reporting the expression (1.14), we have tacitly required the absence of anisotropies and preferred-frame effects [35, 37, 39], which should have been described by other PPN parameters that will be set to zero in all the upcoming considerations (see Ref. [41] for more pieces of information). Furthermore, we will assume that nonlinear effects are essentially described by the contributions coming from GR, thus yielding $\beta = 1$ (see Table 1.1), which in turn entails

$$\eta = 1 - \gamma\,. \tag{1.15}$$

With this final remark, we have concluded the preparatory dissertation on EP by studying all of its possible aspects and rephrasing. Further comments and observations on such a fundamental principle of theoretical physics are retained for later. We now turn the attention on the last principle left to take into account.

1.3 Heisenberg uncertainty principle

Let us now move the focus from GR aspects to a completely different scenario involving features of quantum mechanics. As well-known, one of the cornerstones of QM is represented by Heisenberg uncertainty relations, which were heuristically deduced for the first time in 1927 [42]. In a nutshell, the message hidden in this principle is plainly summarized in few words [42]:

> *The more precisely the position is determined, the less precisely the momentum is known in this instant, and vice-versa.*

The impact of such a simple observation is ground-breaking; it is impossible to *exactly learn at the same time* both the position and the momentum of a particle. In some sense, such an awareness totally dismantles determinism as long as the analysis is carried out in the quantum regime. A similar "epiphany" finds its inspiration from several considerations regarding the *wave-particle duality.* The aforementioned concept was firstly introduced by Louis de Broglie, who claimed that all matter possesses a wave-like behavior, which can be manifest up to some opportune scale.

The core assumption for this strong statement consists in relating the momentum p of an arbitrary object with the corresponding wavelength λ of the associated wave as

$$\lambda = \frac{h}{p}\,, \tag{1.16}$$

where h is the Planck constant.

After few years, de Broglie's hypothesis received separate experimental confirmations [43, 44] which exhibited *electron diffraction*, thus proving the wave-like nature of an elementary particle. The laboratory tests that coped with this kind of phenomenon resorted to the double-slit experiment performed by Thomas Young in 1801. In order to grasp its connection with HUP, it is worth briefly addressing this argument before moving on.

1.3.1 Double-slit experiment

Suppose to have an apparatus that generates electron beams having nearly the same energy and diffused in a sufficiently large solid angle. Not distant from the source, let us place a wall made up of some material that absorbs all the incoming electrons, and having two narrow slits[10] which allow for the passage of the elementary particles. Finally, locate a screen that is capable of revealing particles just behind the electron absorber (see Fig. 1.6).

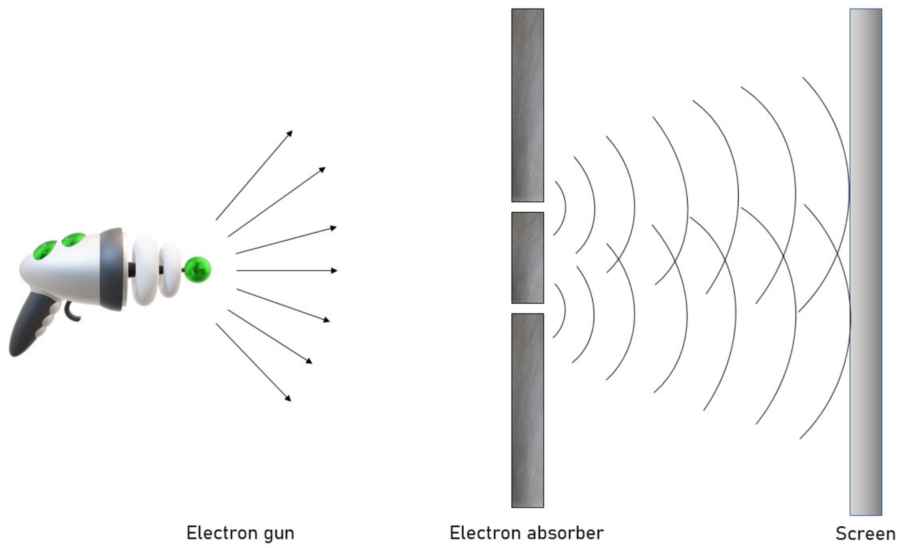

Figure 1.6: In this figure, the experimental setup for the double-slit experiment is sketched.

At the beginning, only few particles are visible on the screen, and their distri-

[10] Such that only few electrons at a time are permitted to go through.

bution appears to be completely random. However, as the screen starts enriching with more and more electrons, the typical fringes of wave interference make their appearance on the detecting surface. Particles tend to fill the sector where the interference is constructive, while leaving the area of destructive interference empty (see Fig. 1.7).

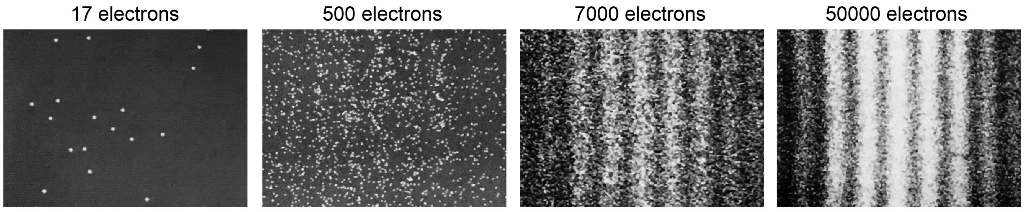

Figure 1.7: As the number of electrons captured by the screen increases, the interference pattern becomes manifest. This picture is taken from Ref. [45].

Therefore, we observe that the final pattern is not just the superposition of the two patterns we would have obtained by considering the passage of electrons through each single slit; in a similar situation, no interference fringes should have become manifest on the screen. In fact, particles crossing one slit should in principle follow their own path without being affected by particles traversing the other slit [46]. Such is the "classical" explanation one may expect before looking at the outcome of the experiment, but this is not the case. The occurrence of a wave-like behavior acknowledges the impossibility of talking about the notion of a *trajectory* for particles. We cannot tell whether an electron has passed through one slit or another, because we simply cannot regard the electron as a classical object with a well-defined trajectory.

However, it would be feasible to perform a laboratory test analogous to the one presented before with the addition of a device that is able to tell which particle crosses which slit (for more details, see for example Ref. [47]). Within these conditions, the outcome of the experiment would *not* return the interference fringes on the screen, but rather a distribution which complies with the aforesaid classical superposition! What happens here is that the detection of the electrons before they reach the screen has introduced an interaction between the particles and the device, thus altering the properties of the analyzed entities. Consequently, this phenomenon (and many others in this direction) leads us to the ensuing conclusion: *any measurement of a physical system incontrovertibly perturbs its state.* Until the first tests were performed at a microscopic scale, no one could have ever been aware of that, since in the classical realm perturbations induced by observations are always negligible. On the other hand, when it comes to the quantum regime, this aspect cannot be overlooked anymore, and the experience currently treated is a straightforward indication of such concept.

In view of the above remarks, the link between HUP and the fundamental tests tackling electron diffraction via double-slit experiments should now be apparent. Both the failure in leaving a physical system unperturbed after a measurement and the absence of a proper notion of a well-defined trajectory entail the requirement of recognizing some fundamental restriction imposed by Nature itself. A similar unbreakable bound can be identified with the Heisenberg uncertainty principle.

So far, we still have not exhibited a proper formulation to give mathematical significance to HUP. Heisenberg himself was not capable of including in his article [42] a rigorous expression that can formally accommodate the content of the principle he developed. In what follows, we will fill this gap by relying on the operational definition of Howard Percy Robertson [48], who generalized the earlier works by Earle Hesse Kennard [49] and Erwin Schrödinger [50].

1.3.2 Mathematical formulation

Let us consider a generic Hermitian operator $\mathcal{O}$; we define the associated standard deviation as

$$\sigma_{\mathcal{O}} = \sqrt{\langle \mathcal{O}^2 \rangle - \langle \mathcal{O} \rangle^2}\,, \tag{1.17}$$

with $\langle \dots \rangle$ denoting the expectation value of the operator on a generic state $|\Psi\rangle$ belonging to a given Hilbert space. We can then take two such operators A and B and examine the product of their respective variance, that is

$$\sigma_A^2\, \sigma_B^2 = \left(\langle A^2 \rangle - \langle A \rangle^2\right) \left(\langle B^2 \rangle - \langle B \rangle^2\right) . \tag{1.18}$$

By invoking Chauchy-Schwarz inequality, one can verify that

$$\sigma_A^2\, \sigma_B^2 \geq \Big| \langle \Psi |\, (A - \langle A \rangle)\, (B - \langle B \rangle)\, |\Psi\rangle \Big|^2 . \tag{1.19}$$

At this point, simple mathematical manipulations allow us to deduce the following inequality, also known as Schrödinger uncertainty relation [50]:

$$\sigma_A\, \sigma_B \geq \sqrt{\left(\frac{1}{2}\langle\{A, B\}\rangle - \langle A \rangle \langle B \rangle\right)^2 + \left(\frac{1}{2\,i}\langle[A, B]\rangle\right)^2}\,. \tag{1.20}$$

Starting from (1.20), it is a straightforward task to check that also Robertson uncertainty relation

$$\sigma_A\, \sigma_B \geq \sqrt{\left(\frac{1}{2\,i}\langle[A, B]\rangle\right)^2}\,, \tag{1.21}$$

holds. However, it must be said that, for the above inequalities to be reliable, there are some conditions the domain of the operators A and B must satisfy, but we are not going to discuss them here. For more information on these technical issues, we remand the reader to Ref. [51].

In order to recover Kennard result [49], we have to assume our generic operators to be the momentum and position operators, and thus make the identification $A = x$ and $B = p_x$. With such a choice, by recalling that the canonical commutation relation for the above quantities is $[x, p_x] = i\hbar$, we obtain the well-known inequality that encompasses HUP

$$\sigma_x\,\sigma_{p_x} \geq \frac{\hbar}{2}\,. \tag{1.22}$$

Equation (1.22) implies the existence of a region $\sigma_x\,\sigma_{p_x}$ of size $\hbar$ in the phase space in which physical predictions cannot be tested. On the other hand, if taken separately, there is no limit to precise measurements of either position or momentum; to put it differently, arbitrarily short distances may in principle be detected via arbitrarily high energy probes, and vice–versa. Despite this, we would like to stress one more time that a similar characteristic does not involve the accuracy of current tests and technology, but it rather expresses an intrinsic property of any quantum system.

Generalized uncertainty principle

The distinctive features and information contained in (1.22) are drastically modified if gravity is taken into account. Indeed, several QG models support the presence of a minimum length at the Planck scale [52], thus ending up in a limited resolution of spacetime, which would no longer appear smooth beyond the threshold ℓ_p due to quantum fluctuations [53]. In light of these findings, it is natural to conclude that HUP is not suitable for the description of a physical system at the QG scale.

In this regard, many studies (for instance, Refs. [54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82] and references therein) have converged on the idea that the HUP should be properly modified to fit in a consistent QG picture, so as to account for the existence of a fundamental minimum length. In this sense, one of the most adopted generalizations of the uncertainty principle (GUP) is given by

$$\sigma_x\,\sigma_{p_x} \geq \frac{\hbar}{2} \pm 2|\beta|\hbar\frac{\sigma_{p_x}^2}{m_p^2\,c^2}\,, \tag{1.23}$$

where the sign $\pm$ refers to positive/negative values of the dimensionless *deformation parameter* β, which is assumed to be of order unity in some QG models, and in particular in string theory [58, 60].

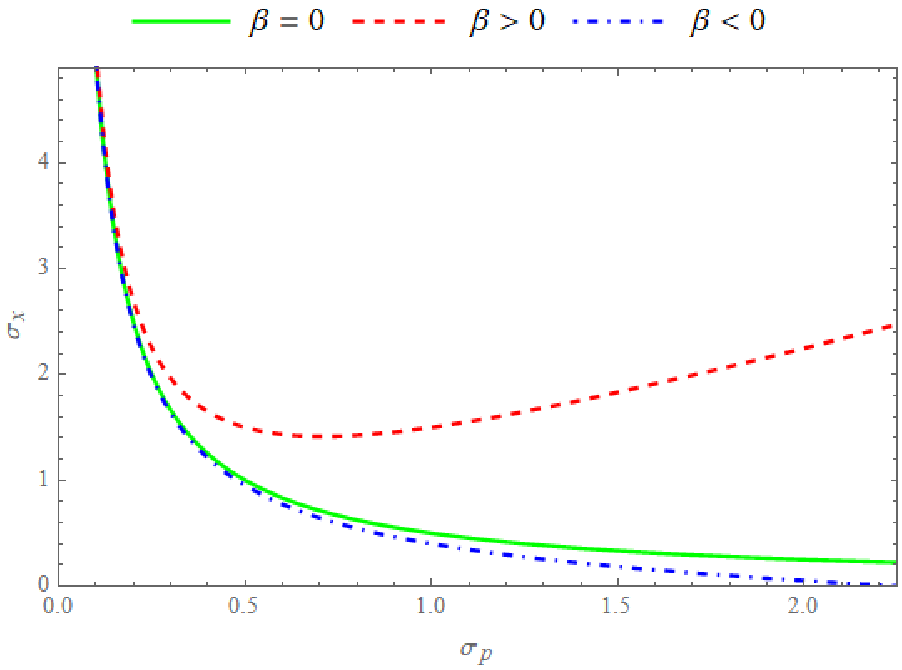

Figure 1.8: In this figure, the qualitative difference between HUP and GUP is displayed when inequalities (1.22) and (1.23) are saturated. For simplicity, we have set $\hbar = c = G = 1$.

Although seminal papers on this topic always assumed β to be a positive number, several claims in literature [78, 79, 80, 82, 83] affirm that it might come with a minus sign in (1.23). If such an outcome is taken to be valid, then there exists a maximal value for σ_{p_x} around the Planck scale for which $\sigma_x\, \sigma_{p_x} \geq 0$, thus implying a "classical" behavior of physical phenomena for energetic regimes close to m_p. The scenario depicted so far is not entirely surprising; as a matter of fact, the intertwining between classical and quantum physics has already been discussed by Gerardus 't Hooft [84] by introducing the so-called "cellular automaton interpretation" of QM.

Before ending the current Chapter, we remark that, starting from (1.23), one can go back to the commutator between the momentum and position operators, thus obtaining the *deformed* canonical commutation relation

$$[x, p] = i\hbar \left(1 \pm |\beta| \frac{p^2}{m_p^2\, c^2}\right), \tag{1.24}$$

which is exact for mirror-symmetric states (i.e. $\langle p \rangle = 0$). By means of (1.24), it is possible to study in detail the quantum mechanical structure underlying GUP [85].

As a final interesting observation, we would like to stress that the existence of modifications to (1.22) can be found in the context of quantum mechanics without invoking the presence of gravity. In this perspective, it is worth mentioning the so-called Ozawa uncertainty relations [86], and in particular their extension to the framework of noncommutative QM [87], which possess several contact points with QG models.

Chapter 2

Casimir effect as a probe for new physics

In the previous Chapter, the dissertation was exclusively centered around the introduction of the physical principles the current essay deals with. In passing, we have also mentioned several hypotheses and scenarios that may jeopardize the validity of the aforesaid principles. However, only accurate laboratory tests and observations have the last word on such issues, since the worth of a brand-new theoretical model is enshrined by its predictive power. Therefore, in order to render our discussion quantitatively meaningful, it is convenient to identify an efficient probe that is able to detect tiny deviations from standard outcomes so as to describe small signatures of new physics. In this direction, the Casimir effect [88] proves to be a valuable tool for the cause. This physical phenomenon is extremely important, because it can be regarded as the first-ever manifestation of the zero-point energy.

The Casimir effect occurs whenever a quantum field is confined in a small region of space. The confinement gives rise to a net attractive force between the binding objects, whose intensity depends not only on the geometry of the volume in which the field is bound, but also on its nature (i.e. scalar, fermion, etc.) and on the spacetime in which the experiment takes place (for a complete overview, see Ref. [89]). Indeed, a rich literature can be found in connection with the Casimir effect in flat spacetime and with different geometrical settings (i.e. Refs. [90] and references therein) as well as for the cases in which the background is curved by the presence of gravity (i.e. Refs. [91, 92, 93, 94] and references there contained). In light of the aforementioned findings, it is not hard to guess that the properties of an external gravitational field are closely intertwined with the outcome of an experiment involving the Casimir apparatus. Therefore, according to the theoretical model with which gravity is described, we expect the observable quantities to change their usual behavior due to the existence of tiny corrections that can in principle be detected.

In the next Sections, we exhibit how the Casimir effect can be employed to test the range of validity and the generalization of the physical principles presented in the previous Chapter. In particular, we will show:

- the interplay between the Casimir effect and the gravitational sector of SME by imposing an upper bound for one of the free parameters of the model;
- how to properly quantify the violation of EP for several quadratic theories of gravity;
- the modification of the standard outcomes related to the measurable quantities of the Casimir effect when HUP is superseded by GUP.

Before thoroughly tackling the above points, it is worth briefly recalling the key aspects of the Casimir effect with a proper introduction to the subject. To this aim, we first review the essence of the studied phenomenon by following Ref. [95], where the authors investigate a confined two-dimensional massless scalar field on a flat spacetime.

2.1 A quick glimpse at the Casimir effect

Let us then assume to have the aforesaid scalar field $\psi(x,t)$ which satisfies the boundary conditions

$$\psi(0,t) = \psi(D,t) = 0\,, \tag{2.1}$$

where D defines the distance between the two binding objects.

By virtue of the Klein-Gordon equation $\left(\partial_t^2 - \partial_x^2\right)\psi = 0$, the complete expansion that accounts for (2.1) is [95]

$$\psi(x,t) = \sqrt{\frac{1}{D}}\sum_{n=1}^{\infty}\left(a_n\, e^{-i\omega_n t} + a_n^\dagger\, e^{i\omega_n t}\right)\frac{\sin\left(\omega_n\, x\right)}{\sqrt{\omega_n}}\,, \quad \omega_n = \frac{|n|\,\pi}{D}\,. \tag{2.2}$$

Consequently, the zero-point energy per unit length between the plates can be written as

$$\varepsilon_0 = \frac{1}{D}\langle 0|H|0\rangle = \frac{\pi}{2\,D^2}\sum_{n=1}^{\infty} n\,, \tag{2.3}$$

where H is the Hamiltonian operator of the free scalar field.

Equation (2.3) clearly yields an infinite quantity which needs to be renormalized. For this specific situation, the easiest method to get rid of the divergence is the

Riemann zeta function regularization, which relies on the use of the same-named special function defined as

$$\zeta(x) = \sum_{n=1}^{\infty} n^{-x} \,. \tag{2.4}$$

At this point, one recognizes the divergent part of (2.3) with the zeta function $\zeta(-1)$, which converges at $-1/12$ [96], thus giving

$$\varepsilon_0 = -\frac{\pi}{24\, D^2} \,. \tag{2.5}$$

However, the energy per unit length does not represent a measurable quantity, which means that there is still some effort to be done. As a matter of fact, starting from ε_0, one can compute the strength between the binding objects, which is simply

$$F = -\frac{d}{dD}\left(D\, \varepsilon_0\right) = -\frac{\pi}{24\, D^2} \,. \tag{2.6}$$

The minus sign reminds us that there is a faint attraction between the confining object which scales as the inverse of the square of their relative distance. It is worth emphasizing that such behavior strictly depends on the dimensionality of the spacetime; for instance, should the same analysis be carried out in four dimensions, the resulting attractive strength per unit area would be

$$f = -\frac{\pi^2}{240\, D^4} \,. \tag{2.7}$$

We are now ready to delve into the core of the current Chapter. As already anticipated, we will start with the connection between the Casimir effect and the gravitational sector of SME.

2.2 Casimir effect in Post-Newtonian gravity with Lorentz-violation

The most general Lagrangian density for the SME gravitational sector of Ref. [26] contains both a Lorentz-invariant and a Lorentz-violating term. The background is represented by a Riemann-Cartan spacetime, but for our purposes we take the limit of vanishing torsion, in such a way that the Lorentz-invariant part is the usual Einstein-Hilbert contribution. The effective action in which we consider only the leading-order Lorentz-violating terms is thus given by

$$S = S_{EH} + S_{LV} + S_m, \tag{2.8}$$

where

$$S_{EH} = \frac{1}{2\kappa}\int d^4x\,\sqrt{-g}\,\mathcal{R}, \tag{2.9}$$

is the aforementioned Einstein-Hilbert action, with $\kappa = 8\pi G$, S_m the matter action and S_{LV} the Lorentz-violating term [97]

$$S_{LV} = \frac{1}{2\kappa}\int d^4x\,\sqrt{-g}\left(-u\,\mathcal{R} + s^{\mu\nu}\,R^T_{\mu\nu} + t^{\rho\lambda\mu\nu}\,C_{\rho\lambda\mu\nu}\right). \tag{2.10}$$

Here, $\mathcal{R}$ is the Ricci scalar, $R^T_{\mu\nu}$ the trace-free Ricci tensor, $C_{\rho\lambda\mu\nu}$ the Weyl conformal tensor and all other terms contain the information of Lorentz violation. Of course, they must depend on spacetime position and have to be treated as dynamical fields, in order to be compatible with lack of prior geometry, a typical feature of GR[1].

Since the fields u, $s^{\mu\nu}$ and $t^{\rho\lambda\mu\nu}$ in (2.10) are the ones responsible for Lorentz violation, they acquire a vacuum expectation value, so that it is possible to write fluctuations around them as

$$u = \bar{u} + \tilde{u}\,, \qquad s^{\mu\nu} = \bar{s}^{\mu\nu} + \tilde{s}^{\mu\nu}, \qquad t^{\rho\lambda\mu\nu} = \bar{t}^{\rho\lambda\mu\nu} + \tilde{t}^{\rho\lambda\mu\nu}. \tag{2.11}$$

Furthermore, we require that each first element of the r.h.s. of (2.11) is constant in asymptotically inertial Cartesian coordinates [97]. However, the fundamental assumption is that, when dealing with Lorentz violation, one always takes into account only the vacuum expectation values of (2.11), completely neglecting fluctuations. This ansatz is reasonable, because we expect to have extremely small deviations from Lorentz symmetry realized in nature.

Without entering the details of calculation[2], it is possible to derive the most general linearized metric tensor for a point-like source of gravity, whose non-null components are given by

$$g_{00} = 1 - \frac{GM}{r}\left(2 + 3\,\bar{s}^{00}\right)\,, \quad g_{ij} = \left[-1 - \frac{GM}{r}\left(2 - \bar{s}^{00}\right)\right]\delta_{ij}\,. \tag{2.12}$$

[1]See Ref. [22] for a detailed explanation of this concept.

[2]See Appendix B for a complete treatment.

2.2.1 Dynamics of a massless scalar field

Let us now consider a conventional massless scalar field $\psi(\mathbf{x}, t)$ in curved background (i.e. we consider the SME parameters are only into gravity sector). In general, the $\bar{s}^{\mu\nu}$ parameters can be moved from the gravity sector into the scalar sector using a coordinate choice [98]. The choice does not change the physics, so although the calculation looks different it must give the same result.

In our analysis, the Klein-Gordon equation reads [99]

$$(\Box + \zeta\, \mathcal{R})\, \psi(\mathbf{x}, t) = 0, \tag{2.13}$$

where $\Box$ is the d'Alembert operator in curved space and ζ is the coupling parameter between geometry and matter.

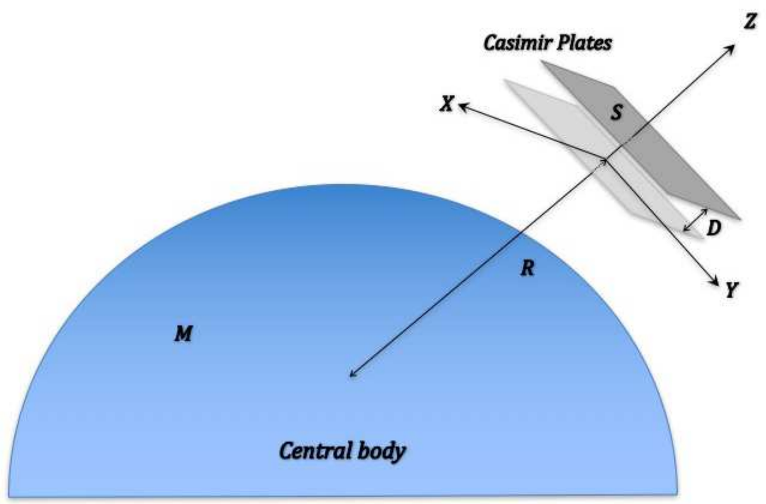

Figure 2.1: The Casimir-like system in a gravitational field is represented above. Here, D denotes the distance between the plates, S their surface and R the distance from the source of gravity of mass M, with $D < \sqrt{S} \ll R$.

As it can be seen in Fig. (2.1), the configuration is simple: the plates are set in such a way that the one nearer to the source of gravity is distant R from it, and hence we can choose Cartesian coordinates so that $r = R + z$, where the variable z is free to vary in the interval $[0,\ D]$, if we denote with D the separation between the plates. Clearly, the relation $D \ll R$ holds.

A further simplification comes from the fact that the only Cartesian coordinate explicitly present in the quantities appearing in (2.13) is z. In fact, denoting $\phi =$

$-GM/R$ and recalling that $z/R \ll 1$, the metric tensor becomes

$$g_{00} \simeq 1 - \phi \left(1 - \frac{z}{R}\right) \left(2 + 3\,\bar{s}^{00}\right), \quad g_{ij} \simeq \left[-1 - \phi \left(1 - \frac{z}{R}\right) \left(2 - \bar{s}^{00}\right)\right] \delta_{ij}\,, \tag{2.14}$$

with the scalar curvature that assumes the form[3] $\mathcal{R} \equiv \mathcal{R}_1 + z\,\mathcal{R}_2$.

At this point, it is clear that the interest is focused on the variation of the field along the radial direction (namely, along the z-axis). Because there is no explicit dependence on other coordinates, one can think of a solution of the form $\psi\left(\mathbf{x}, t\right) = N\, e^{[i(\omega\, t - \mathbf{k}_\perp \cdot \mathbf{x}_\perp)]} \varphi\left(z\right)$, where $\mathbf{k}_\perp = (k_x, k_y)$, $\mathbf{x}_\perp = (x, y)$ and N is the normalization factor.

The field equation can thus be rewritten as

$$\partial_z^2\, \varphi + C_1\, \partial_z\, \varphi + C_2\, \varphi = 0\,, \tag{2.15}$$

where

$$C_1 = -2\,\frac{\phi}{R}\,\bar{s}^{00}, \qquad C_2 = a + b\,z\,, \tag{2.16}$$

with

$$a = \omega^2 \left[1 - 2\,\phi\left(\bar{s}^{00} + 2\right)\right] + \zeta\,\frac{\phi}{R^2}\left(4 - 10\,\bar{s}^{00}\right) - \mid \mathbf{k}_\perp \mid^2, \quad b = 4\,\frac{\phi}{R}\left(\omega^2 - 3\,\frac{\zeta}{R^2}\right). \tag{2.17}$$

The solution of this differential equation is a linear combination of Airy functions of the first and of the second kind, with argument $x\left(z\right) = \left[\left(C_1^2 - 4\,a\right)/4 - b\,z\right]\left(-b\right)^{-2/3}$. In the considered approximation, the solution can be written as

$$\varphi\left(z\right) = k_1\, \mathrm{Ai}\left(\frac{-a - b\,z}{\left(-b\right)^{\frac{2}{3}}}\right) + k_2\, \mathrm{Bi}\left(\frac{-a - b\,z}{\left(-b\right)^{\frac{2}{3}}}\right). \tag{2.18}$$

Airy functions can be expressed in terms of Bessel functions [100]. Due to the form of a and b, it is clear that the argument of the Bessel functions $\eta(z) \equiv \left[a + bz\right]\left(-b\right)^{-2/3} \gg 1$, and hence their asymptotic behavior yields

$$\varphi\left(z\right) \simeq \sqrt{\frac{3}{\pi\sqrt{\eta(z)}}}\,\sin\left[\frac{2}{3}\eta^{\frac{3}{2}}(z) + \tau\right]. \tag{2.19}$$

If we impose the Dirichlet boundary conditions on the plates for the field $\varphi(z)$, that is $\varphi(0) = \varphi(D) = 0$, we get the relation $2/3\left[\eta^{3/2}(0) - \eta^{3/2}(L)\right] = n\,\pi$, where n is an

[3] The presence of Lorentz-violating terms allows for a non-vanishing scalar curvature. However, details of its form will not be necessary in the next steps, since it contributes to the mean vacuum energy density only at higher orders.

integer. From these boundary conditions, we find the energy spectrum

$$\omega_n^2 = \left[1 - 2\,\phi\,(\bar{s}^{00} + 2) + 4\,\frac{\phi}{R}\,D\right]\left[\mathbf{k}_\perp^2 + \left(\frac{n\pi}{D}\right)^2\right] + \frac{\zeta\,\phi}{R}\left[10\,\bar{s}^{00} - 4 + \frac{6\,D}{R}\right]. \quad (2.20)$$

Finally, using the scalar product defined for quantum fields in curved spacetimes [99], one derives the normalization constant

$$N_n^2 = \frac{a}{3\,S\;b^{\frac{1}{3}}\omega_n\,n\left[1 - \phi\left(1 + \frac{3}{2}\bar{s}^{00}\right)\right]}\,, \quad (2.21)$$

with S being the surface of the plates.

2.2.2 Mean vacuum energy density and pressure

In order to calculate the mean vacuum energy density ε between the plates, we use the general relation [99]

$$\varepsilon = \frac{1}{V_p}\sum_n \int d^2\,\mathbf{k}_\perp \int dx\,dy\,dz\,\sqrt{-g_\Sigma}\,\left(g_{00}\right)^{-1} T_{00}\,, \quad (2.22)$$

where $T_{00} \equiv T_{00}\big(\psi_n, \psi_n^*\big)$ is a component of the energy-momentum tensor

$$T_{\mu\nu} = \partial_\mu\,\psi\,\partial_\nu\,\psi - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\,\psi\,\partial_\beta\,\psi\,,$$

and

$$V_p = \int dx\,dy\,dz\,\sqrt{-g_\Sigma}$$

is the proper volume; g_Σ is the determinant of the induced metric on a spacelike Cauchy hypersurface Σ. Using the Schwinger proper-time representation and zeta function regularization, we find the mean vacuum energy density

$$\varepsilon = \varepsilon_0 + \varepsilon_{GR} + \varepsilon_{LV}\,, \quad (2.23)$$

with

$$\varepsilon_0 = -\frac{\pi^2}{1440\,D_p^4}\,, \quad \varepsilon_{GR} = -\frac{\phi\,D_p}{R}\,\varepsilon_0\,, \quad \varepsilon_{LV} = -6\,\phi\,\bar{s}^{00}\,\varepsilon_0\,, \quad (2.24)$$

where ε_0 is the standard term of Casimir effect, ε_{GR} is the contribution due to GR and ε_{LV} is the Lorentz-violating term, with $D_p = \int dz\,\sqrt{-g_{33}}$ being the proper length of the cavity. Note that we have neglected higher-order contributions in our analysis.

Equation (2.23) gives us the expression of Casimir vacuum energy density at the second order $\mathcal{O}(R^{-2})$ in the framework of SME. We note that the part related to GR does not have contributions at the first order in $\mathcal{O}(R^{-1})$, but only at higher orders, such as $\mathcal{O}(R^{-2})$. The Lorentz-violating sector, instead, exhibits a first order factor in $\mathcal{O}(R^{-1})$ connected to $\bar{s}^{00}$.

To obtain a plausible bound on $\bar{s}^{00}$, we make the assumption $|\varepsilon_{LV}| \lesssim |\varepsilon_{GR}|$. This agrees with several considerations and results expressed in Refs. [23, 26, 97] and ensures the fact that Lorentz-violating manifestations are small, as widely employed in Lorentz violation phenomenology [97]. However, the reasonableness of the constraint we derive cannot be directly tested, since ε is still an unmeasurable quantity. This is why we need to compare the heuristic constraint with a physical one, which can only be calculated using the pressure.

Apart from the previous comment, considering the case of the Earth and requiring $D_p \sim 10^{-7}\,$m (a typical choice for the proper length in standard literature) for the plausible assumption exhibited above, we get [93]

$$\bar{s}^{00} \lesssim \frac{D_p}{6\,R_\oplus} \lesssim 10^{-14}, \tag{2.25}$$

where $R_\oplus \sim 6.4 \times 10^6\,$m.

It must be pointed out that recent developments in nanotechnology can further strengthen the above bound by one or two orders of magnitude. In fact, in the near future, the value of D_p could reach scales even smaller than nanometers (as already contemplated, for example, in Ref. [101]), thus transforming (2.25) into a more stringent constraint, $\bar{s}^{00} \lesssim 10^{-15}$.

Let us now turn the attention to the pressure. The attractive force observed between the cavity plates is obtained by the relation $F = -\partial\mathcal{E}/\partial D_p$, where $\mathcal{E} = \varepsilon\, V_P$ is the Casimir vacuum energy. Then, the pressure is simply given by $P = F/S_p$, where $S_p = \int dx\, dy\, \sqrt{g_{11}\, g_{22}}$ is the proper area. Hence, we get

$$P = P_0 + P_{GR} + P_{LV}\,, \tag{2.26}$$

with

$$P_0 = -\frac{\pi^2}{480 D_p^4}\,, \quad P_{GR} = -\frac{2}{3}\frac{\phi\, D_p}{R}\, P_0\,, \quad P_{LV} = -6\,\phi\,\bar{s}^{00}\, P_0\,, \tag{2.27}$$

where P_0 is the pressure in the flat case, while P_{GR} is the pressure in GR and P_{LV} is the contribution connected to Lorentz-violation.

Note that the result (2.25) is achievable also if we require the inequality $|P_{LV}| \lesssim |P_{GR}|$ to hold. Indeed, we obtain exactly the same order of magnitude for the upper bound of $\bar{s}^{00}$, even when the heuristic approach carried out for the mean vacuum energy density applies to the pressure. However, the above consideration on the elusive manifestation of Lorentz violation acquires a substantial meaning for the case of P, since such a quantity is measurable. In this perspective, the expression of P_{GR} and P_{LV} for the contribution to the pressure of GR and SME, respectively, further strengthen our plausible constraint on the Lorentz-violating factor.

We now want to test the compatibility of SME with the experimental data to check how much a concrete bound differs from the heuristic one obtained in (2.25). This can be achieved by using the pressure as a measurable physical quantity. In fact, imposing the constraint $|P_{LV}| \lesssim \delta P$, where δP is the experimental error, we obtain the following relation:

$$\bar{s}^{00} \lesssim \frac{\delta P}{P_0}\frac{1}{6\phi} = \frac{1}{3}\frac{\delta P}{P_0}\frac{R}{R_S}\,, \tag{2.28}$$

where R_S is Schwarzschild radius.

The total absolute experimental error of the measured Casimir pressure [102] is 0.2% ($\delta P/P_0 \simeq 0.002$). Typical values of the ratio R/R_S in the Solar System are included between $10^7 \div 10^{10}$. In particular, for the Earth we have 7.2×10^8, which means that the term on the r.h.s. of (2.28) is of order 10^6. The comparison of such a result with (2.25) clearly shows that we still cannot use[4] the Casimir experiment to measure the pressure in order to significantly constrain the parameter $\bar{s}^{00}$. To do this, we need to enhance the experimental sensitivity on Earth by at least six order of magnitude, in such a way that $\delta P/P \lesssim 10^{-9}$.

2.3 Casimir effect in quadratic theories of gravity

In what follows, we will see that considerations analogous to the ones carried out in Sec. 2.2.1 can be used in the context of quadratic theories of gravity[5] to check the accordance with EP by means of the PPN formalism introduced in Sec. 1.2.2. For this purpose, we observe that a generic line element in proximity of a point-like gravitational source for a given extended model of gravity can be written as

$$ds^2 = [1 + 2\Phi(r)]\, dt^2 - [1 - 2\Psi(r)]\, d\mathbf{r} \cdot d\mathbf{r}\,, \tag{2.29}$$

where Φ and Ψ are metric potentials which can in principle be different. By relying on the same setting, assumptions and terminology contained in Sec. 2.2.1, in the

[4]Unless we believe the heuristic bound to be true and thus physically consistent.

[5]See Appendix C for further details.

linearized approximation it is possible to prove that [92, 94]

$$g_{00}(r) \simeq 1 + 2\,\Phi_0 + 2\,\Phi_1\,z\,, \quad g_{ij}(r) \simeq -1 + 2\,\Psi_0 + 2\,\Psi_1\,z\,, \tag{2.30}$$

with

$$\Phi_0 = \Phi(R), \quad \Phi_1 = \frac{d\Phi(r)}{dr}\bigg|_{r=R}, \quad \Psi_0 = \Psi(R), \quad \Psi_1 = \frac{d\Psi(r)}{dr}\bigg|_{r=R}. \tag{2.31}$$

If we retrace the path of Sec. 2.2.1 towards the evaluation of the mean vacuum energy density due to the presence of a massless scalar field bounded between the plates, we reach the following expression for ε [94]:

$$\varepsilon = -\Big[1 + 3\big(\Phi_0 - \Psi_0\big) - \big(2\Psi_1 - \Phi_1\big)D_p\Big]\frac{\pi^2}{1440\,D_p^4}\,. \tag{2.32}$$

In the above expression, we clearly identify the first term of the r.h.s. with the usual contribution to the Casimir mean vacuum energy density, whereas the remaining part is the correction attributable to the effects of gravity. From (2.32), we can compute the Casimir pressure as seen in Sec. 2.2.2, which naturally inherits the same structure of ε; namely, we have

$$P = P_0 \,+\, P_G\,, \tag{2.33}$$

with

$$P_0 = -\frac{\pi^2}{480\,D_p^4}\,, \quad P_G = \Big[3\big(\Phi_0 - \Psi_0\big) - \frac{2}{3}\,(2\Psi_1 - \Phi_1)\,D_p\Big]P_0\,. \tag{2.34}$$

Henceforth, we restrict the attention to the factor P_G which is the relevant one for our goal. Indeed, by recalling the contents of Sec. 1.2.2, we observe that the parameter γ is defined as

$$\gamma(r) = \frac{\Psi(r)}{\Phi(r)}\,, \tag{2.35}$$

which in terms of the Nordtvedt parameter η in (1.15) turns out to be

$$\eta(r) = 1 - \gamma(r) = \frac{\Phi(r) - \Psi(r)}{\Phi(r)}\,. \tag{2.36}$$

Therefore, if we focus only on the gravitational leading-order correction to the flat case, we notice that P_G can be cast in the form

$$P_G = 3\,\eta_0\,\Phi_0\,P_0\,, \tag{2.37}$$

with $\eta_0 \equiv \eta(R)$. In light of the arguments discussed in the previous Chapter, from (2.37) it is straightforward to deduce that a given gravitational model violates SEP if it predicts the existence of a non-vanishing lowest-order contribution to the Casimir pressure. Furthermore, should this term be non-vanishing, we know for sure that it is by no means related to GR, for which SEP holds and $\Phi = \Psi$.

2.3.1 Applications

For the sake of clarity, we will apply the aforementioned reasoning to several noteworthy quadratic theories of gravity. Although exemplified by a set of plausible assumptions in Sec. 1.2.2, the analysis we will approach is capable of shedding light on some fundamental properties of the aforesaid theories. However, we will overlook most of the features and theoretical implications of such models; the interested reader can find all information in the quoted references.

$f(\mathcal{R})$ gravity

The gravitational models belonging to the $f(\mathcal{R})$ gravity can be deduced from the action [103]

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g}\, f(\mathcal{R})\,, \tag{2.38}$$

with f being a generic function of the Ricci scalar. The most immediate generalization of the Einstein-Hilbert action (2.9) is selected by choosing $f(\mathcal{R}) = \mathcal{R} + \alpha\mathcal{R}^2$. Under these circumstances, the metric potentials appearing in (2.29) become[6]

$$\Phi(r) = -\frac{Gm}{r}\left(1 + \frac{1}{3}e^{-m_0 r}\right), \quad \Psi(r) = -\frac{Gm}{r}\left(1 - \frac{1}{3}e^{-m_0 r}\right), \tag{2.39}$$

where $m_0 = 1/\sqrt{3\,\alpha}$ is the mass of the spin-0 massive degree of freedom coming from the Ricci scalar squared contribution.

The Eddington-Robertson-Schiff parameter γ for this model turns out to be

$$\gamma = \frac{1 - \frac{1}{3}e^{-m_0 r}}{1 + \frac{1}{3}e^{-m_0 r}}\,. \tag{2.40}$$

Therefore, the leading-order expression of P_G for the current theory is

$$P_G = 2\, e^{-m_0\, R}\, P_0\,, \tag{2.41}$$

which undoubtedly signals the presence of SEP violation.

[6]For a thorough derivation of metric tensors of different quadratic theories, see Appendix C.

Fourth order gravity

The action for Stelle's fourth order gravity [104] is given by

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\,\left[\mathcal{R} + \frac{1}{2}\left(\alpha\,\mathcal{R}^2 + \beta\,R_{\mu\nu}R^{\mu\nu}\right)\right]. \tag{2.42}$$

The metric potentials now read

$$\Phi(r) = -\frac{Gm}{r}\left(1 + \frac{1}{3}e^{-m_0 r} - \frac{4}{3}e^{-m_2 r}\right), \quad \Psi(r) = -\frac{Gm}{r}\left(1 - \frac{1}{3}e^{-m_0 r} - \frac{2}{3}e^{-m_2 r}\right), \tag{2.43}$$

where $m_0 = 2/\sqrt{12\,\alpha + \beta}$ and $m_2 = \sqrt{2/(-\beta)}$ correspond to the masses of the spin-0 and of the spin-2 massive mode, respectively. In order to avoid tachyonic solutions, we need to require $\beta < 0$. In addition to that, the spin-2 mode is a ghost-like degree of freedom. Such an outcome is not surprising, since it is known that, for any local higher derivative theory of gravity, ghost-like degrees of freedom always appear [105].

The factor γ for Stelle's fourth-order gravity is given by

$$\gamma = \frac{1 - \frac{1}{3}e^{-m_0 r} - \frac{2}{3}e^{-m_2 r}}{1 + \frac{1}{3}e^{-m_0 r} - \frac{4}{3}e^{-m_2 r}}. \tag{2.44}$$

As for the previous case, the limit of large masses $m_0, m_2 \to \infty$ returns GR. The SEP-violating correction to the Casimir pressure P_G given by this model is

$$P_G = 2\left(e^{-m_0 R} - e^{-m_2 R}\right)P_0\,. \tag{2.45}$$

Sixth order gravity

Let us now deal with a sixth-order gravity model, which is an example of super-renormalizable theory [106, 107]. The starting action is

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\,\left[\mathcal{R} + \frac{1}{2}\left(\alpha\,\mathcal{R}\,\Box\,\mathcal{R} + \beta\,R_{\mu\nu}\,\Box\,R^{\mu\nu}\right)\right]. \tag{2.46}$$

It is possible to show that the two metric potentials Φ and Ψ assume the following expressions:

$$
\begin{aligned}
\Phi(r) &= -\frac{Gm}{r}\left(1+\frac{1}{3}e^{-m_0 r}\cos(m_0 r)-\frac{4}{3}e^{-m_2 r}\cos(m_2 r)\right),\\
\Psi(r) &= -\frac{Gm}{r}\left(1-\frac{1}{3}e^{-m_0 r}\cos(m_0 r)-\frac{2}{3}e^{-m_2 r}\cos(m_2 r)\right),
\end{aligned}
\tag{2.47}
$$

where the masses of the spin-0 and spin-2 degrees of freedom are now given by $m_0 = 2^{-1/2}(-3\,\alpha-\beta)^{-1/4}$ and $m_2 = (2\,\beta)^{-1/4}$, respectively. Note that, in this case, tachyonic solutions are avoided for $-3\,\alpha-\beta > 0$, which can be satisfied by the requirement $\alpha < 0$ and $-3\,\alpha > \beta$, with $\beta > 0$. The current higher derivative theory of gravity has no real ghost-modes around the Minkowski background, but a pair of complex conjugate poles with equal real and imaginary parts [107], and corresponds to the so-called Lee-Wick gravity [108]. It is worthwhile noting that in this model the unitarity condition is not violated, since the optical theorem still holds [109].

The parameter γ related to SEP violation now reads

$$
\gamma = \frac{1-\frac{1}{3}e^{-m_0 r}\cos(m_0 r)-\frac{2}{3}e^{-m_2 r}\cos(m_2 r)}{1+\frac{1}{3}e^{-m_0 r}\cos(m_0 r)-\frac{4}{3}e^{-m_2 r}\cos(m_2 r)},
$$

which leads to the following correction to the pressure:

$$
P_G = 2\left[e^{-m_0 R}\cos(m_0 R)-e^{-m_2 R}\cos(m_2 R)\right]P_0\,. \tag{2.48}
$$

Ghost-free infinite derivative gravity

We now consider an example of ghost-free non-local theory of gravity (for a complete literature, see Refs. [8, 110] and references therein). In particular, we analyze the model that comes from the action

$$
S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\left[\mathcal{R}+\frac{1}{2}\left(\mathcal{R}\,\frac{1-e^{\Box/M_s^2}}{2\Box}\,\mathcal{R}-R_{\mu\nu}\,\frac{1-e^{\Box/M_s^2}}{\Box}\,R^{\mu\nu}\right)\right], \tag{2.49}
$$

where M_s is the scale at which the non-locality of the gravitational interaction should become manifest. Note that, for the special ghost-free choice in (2.49), no extra degrees of freedom other than the massless transverse spin-2 graviton propagate around the Minkowski background.

Because of the aforesaid peculiar choice, the metric potentials coincide

$$\Phi(r) = \Psi(r) = -\frac{Gm}{r}\,\mathrm{Erf}\left(\frac{M_s r}{2}\right), \tag{2.50}$$

where

$$\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt\,, \tag{2.51}$$

is the error function [100]. In light of this, it is immediate to conclude that $\gamma = 1$ (just like GR), thus negating the opportunity to evaluate any leading-order correction to P_0. Actually, one can estimate the next-to-leading-order correction, but this would go against the assumptions of Sec. 1.2.2 upon which the whole treatment is based. Moreover, in a similar approximation, both GR and gravitational non-local corrections would arise, hence rendering the latter irrelevant with respect to the former[7].

The example of non-local theories is useful to convey the idea that the current study needs to be seriously improved in order to get a sharper distinction between GR predictions and signatures of new physics arising from extended models of gravity.

Non-local gravity

Differently from the previous theory that could be seen as a UV-completion of GR, the last model we describe is rather an infrared extension of it, achievable by means of non-analytic functions of $\Box$. These theories are inspired by quantum corrections to the effective action of quantum gravity [111]. The particular case we deal with is deducible from the action

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\left(\mathcal{R} + \frac{1}{2}\mathcal{R}\,\frac{\alpha}{\Box}\,\mathcal{R}\right). \tag{2.52}$$

The two metric potentials are IR modifications of the Newtonian one

$$\Phi(r) = -\frac{Gm}{r}\left(\frac{4\alpha-1}{3\alpha-1}\right), \quad \Psi(r) = -\frac{Gm}{r}\left(\frac{2\alpha-1}{3\alpha-1}\right). \tag{2.53}$$

Since we expect α to be small, we can deduce that the Eddington-Robertson-Schiff parameter for this model is represented by

$$\gamma = \frac{2\alpha-1}{4\alpha-1} \simeq 1 + 2\alpha\,. \tag{2.54}$$

[7]If we reasonably assume that GR contributes to the Casimir pressure the most.

Consequently, the SEP-violating contribution to the pressure is given by

$$P_G = 6\,\alpha\,P_0\,. \tag{2.55}$$

With this final result, we have depicted an exhaustive portrait of how the Casimir effect can be employed to detect signals of EP violations, which in turn are open windows towards novel physical insights. In the next Section, we will see that similar achievements can also be obtained by merging the Casimir experiment within the framework of GUP [77].

2.4 Heuristic derivation of the Casimir effect from GUP

In order to depart from the rigorous computation of the important quantities into play and to shape an intuitive scheme to explain how the Casimir effect occurs, here we adopt a heuristic derivation for the aforementioned phenomenon by relying on the seminal paper by Giné [112]. In his article, the author starts from HUP and recovers the usual behavior of the Casimir energy per unit area (2.7) by virtue of several observations regarding the production and annihilation of photons from the vacuum. However, the reasoning there contained is rather unclear to some extent; for this reason, we refine the considerations of Ref. [112] according to Ref. [77], in which the same investigation is then extended to the case of GUP. For the sake of clarity, we first review the case with HUP and after that we turn the attention to the more general scenario.

2.4.1 Casimir effect from Heisenberg uncertainty principle

In Ref. [112], the Casimir effect is derived from the idea that the contribution to the vacuum energy at a point A of a plate is affected by the presence of the other boundary. Specifically, the author considers virtual photons produced by vacuum fluctuations somewhere in the space and arriving at A. In order to compute the total Casimir energy ΔE, one has to take into account all the points on the surface S of the plate. Therefore, from HUP

$$\Delta x \Delta E \simeq \frac{1}{2}\,, \tag{2.56}$$

($p = E$ for photons), the total contribution to the energy fluctuation ΔE is given by those photons in a volume $S\Delta x$ around the plate, where Δx is the position uncertainty of the single particle. Note that, if we had only one plate, Δx would be infinite, since photons may be created in any point of the space. However, this is no longer true in the presence of both the boundaries. In that case, indeed, virtual particles originating from behind the second plate cannot reach A. Thus, the additional plate acts as a sort of shield.

The above situation can be depicted as follows: consider a sphere of radius R centered at the point A and enclosing both the plates. In the single-plate configuration, the effective volume $S\Delta x$ corresponding to the entire space can be thought of as the total volume of the sphere $V_T = 4/3\pi R^3$, with $R \to \infty$. Clearly, such a volume will be reduced by including the second plate: as a result, we can write $S\Delta x = V_T - V_C$, where V_C is the volume shielded by the second plate (see Fig. 2.2).

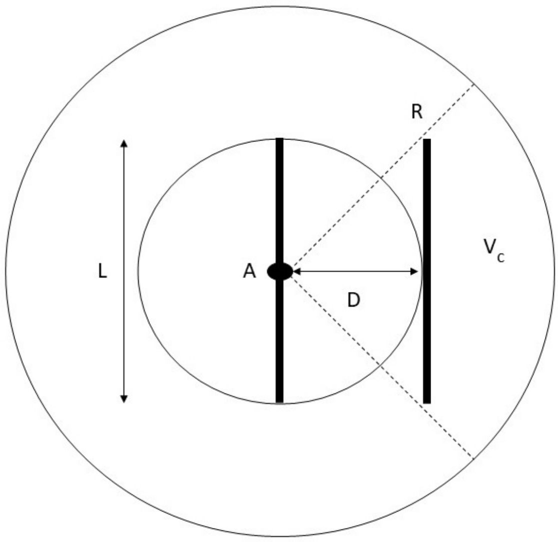

Figure 2.2: A section of the analyzed system. In particular, we want to put the emphasis on the spatial displacement of the shielded volume V_C, contained within the surface of the sphere and the right plate.

In the case of infinite boundaries, or even better when $L/(2D) \to \infty$, one can show that $V_C = 2/3\pi R^3$, yielding

$$S\Delta x \simeq \frac{2}{3}\pi R^3\,, \tag{2.57}$$

(see Ref. [112] for more details).

In the above treatment, no length scale has been considered, hence the volume $S\Delta x$ diverges as the radius R increases. To cure such a pathological behavior, in Ref. [112] the author introduces a cutoff r_e representing the effective distance beyond which photons have a negligible probability to reach the plate. In this way, Eq. (2.57) can be rewritten as

$$S\Delta x \simeq \frac{2}{3}\pi r_e^3 \,, \tag{2.58}$$

which has indeed a finite value. Combining this relation with the HUP, we then obtain

$$|\Delta E(r_e)| = \frac{3}{4}\frac{S}{\pi r_e^3} \,, \tag{2.59}$$

which implies

$$r_e \simeq D \,, \tag{2.60}$$

from comparison with the exact expression

$$\Delta E(d) = -\frac{\pi^2}{720}\frac{S}{D^3} \,. \tag{2.61}$$

for the Casimir energy (strictly speaking, we have $r_e = \sqrt[3]{540}D/\pi \simeq 2.6\, D$).

Although the above derivation is straightforward and very intuitive, the discussion on the physical origin of the length cutoff r_e appears to be rather obscure in some points. Therefore, in order to clarify the meaning of (2.60), let us focus on the computation of the Casimir effect in a simplified one-dimensional system: similar reasoning can be promptly extended to three dimensions.

From the Heisenberg uncertainty relation, it is well-known that large energy fluctuations live for very short time and, thus, hard virtual photons of energy ΔE can only travel short distances of order $1/\Delta E$. As a consequence, the further these particles are created from a plate, the more negligible their contribution to the energy around that plate will be. Let us apply these considerations to the apparatus in Fig. 2.3.

It is easy to see that virtual photons popping out in the strip of width D on the right side of the right plate do not contribute to the Casimir effect, since their pressure is balanced by those photons originating between the plates. By contrast, photons coming from a distance greater than D in the right region do not experience any compensation, because their symmetric "partners" on the left side are screened by the first plate. The overall result is a net force acting on the right plate from right to left. Of course, this argument can be symmetrically applied to the left boundary and provides a qualitative explanation for the origin of the attractive Casimir force.

Now, consider a point at a distance $x_0 > D$ from one of the plates, as in Fig. 2.3. Virtual photons originate from quantum fluctuations in a small region around that

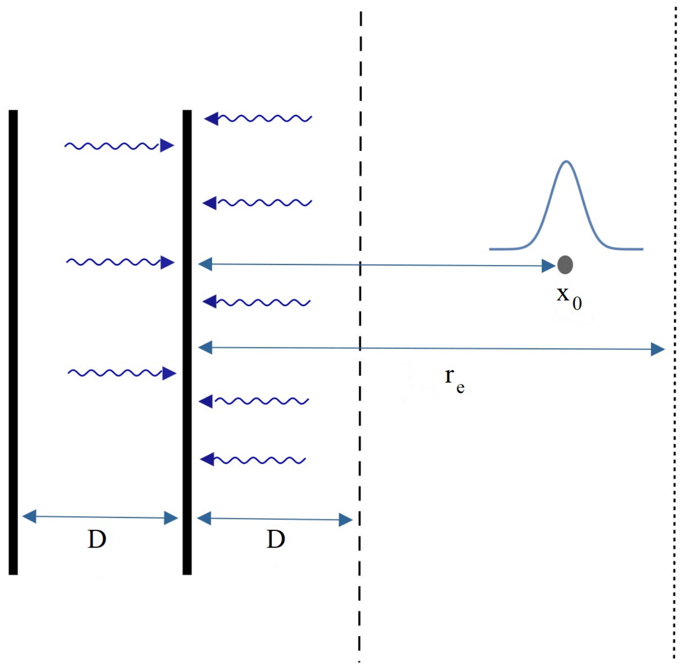

Figure 2.3: Setup for the heuristic derivation of the Casimir effect: two infinite parallel plates (bold lines) at distance D. The effective radius beyond which the creation of virtual photons does not give a significant contribution to the Casimir energy is denoted by r_e (see text for a more comprehensive explanation).

point. Such a region, however, cannot be smaller than the Compton length of the electron, $\lambda_C = 1/m_e$, otherwise the energy amplitude of the fluctuation would exceed the threshold $E \simeq m_e$ for the production of electron-positron pairs. Besides, photons produced at x_0 can impact on the plate (and therefore contribute to the Casimir force) only if their energy E is such that $0 < E < E_0$, where $E_0 = 1/x_0$. Particles of higher energy $E > E_0$, indeed, would recombine before reaching the plate, since the distance they travel is $x = 1/E < 1/E_0 = x_0$.

We can now assume that photons coming from x_0 originate from fluctuations of energy E with a probability given by a Boltzmann-like factor $f(E) = e^{-E/m_e}$. Thus, the total linear energy density (i.e. the energy per unit length) arriving on the plate will be

$$|\Delta\epsilon(E_0)| = \int_0^{E_0} \frac{dE}{\lambda_C}\frac{E}{m_e}\, f(E) = \int_0^{E_0} dE\, E\, e^{-\frac{E}{m_e}}\,, \tag{2.62}$$

where, since we are dealing with the electromagnetic field, we have introduced the natural threshold of the electron mass/energy m_e. In terms of the distance x_0, the above integral becomes

$$|\Delta\epsilon(x_0)| = \int_{x_0}^{\infty} \frac{dx}{x^3}\, e^{-\frac{1}{m_e x}}\,. \tag{2.63}$$

Finally, in order to get the contribution to the Casimir energy from all the photons which impact on the plate, we integrate over all the points x_0 such that $D < x_0 < \infty$, obtaining

$$|\Delta E(D)| = \int_D^\infty dx_0 \, \Delta\epsilon(x_0) \,. \tag{2.64}$$

The integrals in (2.63) and (2.64) can be easily evaluated by observing that, for x large enough, the Boltzmann factor $e^{-1/(m_e x)}$ becomes approximately of order unity. This yields

$$|\Delta\epsilon(x_0)| \simeq \frac{1}{2\,x_0^2} \,, \tag{2.65}$$

and hence

$$|\Delta E(D)| \simeq \frac{1}{2\,D} \,, \tag{2.66}$$

which is in good agreement with the QFT prediction (see Sec. 2.1).

The physical relevance of the above discussion becomes clearer if we observe that probability distributions like those in (2.62) or (2.63) allow us to naturally interpret the effective radius r_e in (2.58) as the distance from the plate below which the vast majority of photons contribute to the large part of the Casimir energy. More rigorously, we can define $r_e > D$ as the distance within which photons carrying the fraction γ $(0 < \gamma < 1)$ of the total Casimir energy are created. In other terms, we can write

$$\frac{1}{2}\int_D^{r_e} \frac{dx}{x^2} = \gamma \Delta E(D) \,, \tag{2.67}$$

from which

$$r_e = \frac{D}{1-\gamma} \,. \tag{2.68}$$

Thus, setting $r_e \simeq 2.6\,D$ amounts to consider a fraction $\gamma \simeq 0.62$ of the total energy responsible for the Casimir effect.

The above picture is quite rough, since it relies on the adoption of a Boltzmann-like distribution for the energy of quantum vacuum fluctuations. As a result, it underestimates the fraction of photons produced within the distance r_e from the plate. Considerable improvements can be achieved by employing more realistic functions $f(E)$ in (2.62). For further details on this research topic, see Refs. [113] and therein.

2.4.2 Casimir effect from generalized uncertainty principle

Let us now extend the above arguments to the context of the GUP. As in the previous Subsection, we shall focus for simplicity on the one-dimensional case, since the analysis in three dimensions proceeds in a very similar fashion.

Let us start from the modified uncertainty relation (1.23), here recast in the form

$$\Delta x \Delta E \simeq \frac{1}{2}\left[1 + \beta\left(\frac{\Delta E}{E_p}\right)^2\right]. \tag{2.69}$$

If we neglect those photons coming from distance greater than the effective radius r_e, it is natural to assume the uncertainty position Δx of the single photon to be of the order of r_e and thus of D, according to (2.60). Then, by replacing $\Delta x \simeq D$ into (2.69), the contribution to the Casimir energy at a given point reads

$$|\Delta E(D)| \simeq \frac{E_p^2 D}{\beta}\left[1 - \sqrt{1 - \beta\left(\frac{1}{E_p D}\right)^2}\,\right]. \tag{2.70}$$

After expanding to the leading order in β, we obtain

$$|\Delta E(D)| \simeq \frac{1}{2D}\left[1 + \frac{\beta}{4}\left(\frac{1}{E_p D}\right)^2\right], \tag{2.71}$$

which indeed agrees with the usual result in the limit $\beta \to 0$, up to a coefficient. More precisely, the exact formula for the Casimir energy can be recovered by assuming $\Delta x = \alpha D$ ($\alpha \sim \mathcal{O}(1)$), and then setting α in such a way that the standard outcome and (2.71) match up for vanishing β.

The above considerations can now be generalized to three dimensions by taking into account the contribution to the zero-point energy at any point of the plates of surface area S. In so doing, straightforward calculations lead to [77]

$$|\Delta E(D)| \simeq \frac{S}{2\,D^3}\left[1 + \frac{\beta}{4}\left(\frac{1}{E_p D}\right)^2\right]. \tag{2.72}$$

It must be pointed out that, in spite of our minimal setting, the behavior of the field theoretical GUP correction which can be found in literature [114] is recovered up to an overall numerical factor.

Concluding remarks

In this Chapter, we have seen how to employ the Casimir effect as a valuable probe to test the implications of GC, EP and HUP not only at a theoretical level, but also experimentally. Indeed, in all the analyzed cases, we have emphasized how physical quantities associated with measurable observables acquire an extra term that signals the existence of a violation/generalization of the studied fundamental principles. In what follows, we briefly review the obtained results.

- In Sec. 2.2, in the context of SME and working in the weak-field approximation, we have studied the dynamics of a massless scalar field confined between two nearby parallel plates in a static spacetime background generated by a point-like source. In order to obtain a reasonable constraint on Lorentz-violating terms, we have derived the corrections to the flat spacetime Casimir vacuum energy density (2.23) in the framework of SME. We have found that, both in the energy density and in the pressure, GR gives us only contributions at the second order $\mathcal{O}(R^{-2})$, while Lorentz-violating corrections occur at first order $\mathcal{O}(R^{-1})$. After that, we have evaluated the pressure (2.27) to observe how it changes from the usual expression in flat spacetime, but in the presence of gravity and with SME coefficients [93].

  By requiring $|\varepsilon_{LV}| \lesssim |\varepsilon_{GR}|$ but also $|P_{LV}| \lesssim |P_{GR}|$, we have then been able to find a significant bound on the SME coefficient $\bar{s}^{00}$. Such an assumption is related to the fact that manifestations of Lorentz violation in nature are expected to be extremely evanescent. If the above inequality did not hold true, it would have been possible to detect traces of Lorentz-violating terms in the tests proposed in Ref. [97] and in other experiments involving the intertwining between SME and gravity, but this is not the case.

- By resorting to the same setting introduced above and to PPN formalism, in Sec. 2.3 we have seen how the Casimir pressure gains an additional term due to the employment of quadratic models of gravity (2.37); it has then been shown that such contribution arises whenever SEP violation occurs. Therefore, SEP is directly linked to a phenomenological manifestation that is in principle observable even with current laboratory tests. However, due to the smallness of the aforementioned corrections, the only feasible perspective with the available experimental sensitivity consists in putting a somewhat stringent bound on the free parameters of the extended theories under examination [94].

  These achievements have been attainable despite our exemplifying approximations concerning the weak-field regime and the negligence of non-linear effects which could have been described by means of several PPN parameters other than γ. For such a reason, not all of the discussed quadratic models have provided us with a clear outcome (i.e. infinite derivative gravity), which suggests that a more rigorous treatment on the topic can lead to further intriguing developments.

- Finally, in Sec. 2.4 we have evaluated the Casimir energy when HUP is replaced by GUP, thus investigating how the standard result changes in response to the generalization of the uncertainty relations underlying QM. Differently from the previous cases, we have made use of a heuristic approach to derive the main outcome [77], in order to disclose a pictorial representation that clarifies the role of the zero-point energy in the analyzed framework. The same considerations have been carried out both for the case of HUP and GUP.

  Unfortunately, as for the above scenarios, direct observations of GUP effects on the Casimir force are extremely challenging. However, current experiments [115] might enable us to fix an upper bound on the parameter β. For the sake of completeness, it must be said that the value of such constraints is still far from the ones that can be put by means of a thorough theoretical reasoning. A more accurate esteem that conveys the aforesaid concept will be tackled in Chapter 5.

Chapter 3

General covariance implications: the case of the inverse β-decay

In the first Chapter and in particular in Sec. 1.1, we have discussed about the outstanding achievement represented by the establishment of general covariance. Furthermore, we have investigated all the issues Einstein had to face to embed such principle into the theory of general relativity. However, this is not the end of the story, since general covariance fulfillment encloses the possibility to theoretically prove debated features related to QFTCS. In order to clarify the previous statement, it is opportune to chronicle a series of crucial results recently appeared in literature.

First and foremost, we start from a beautiful work by Muller [116] which explores the decay properties of particles that are constantly accelerating due to the presence of an external source (i.e. an electric field for charged particles). In his simplified analysis, the author transparently shows that the decay rate of several physical processes acquires a dependence on the acceleration the particle is subject to. Specifically, the last treated example is extremely illustrative, since it exhibits that also a supposedly stable particle such as the proton may decay via a channel that is typically addressed as *inverse β-decay*, for which $p \to n + e^- + \bar{\nu}_e$, with n being the neutron, e^- the electron and $\bar{\nu}_e$ the electron antineutrino. Finally, the author claims that such processes may share a profound connection with the Unruh effect [117], which in a nutshell states that an accelerated observer experiences a thermal radiation when moving through the inertial vacuum[1].

In view of the aforesaid brilliant intuition, several remarkable papers [118] have demonstrated Muller's hypothesis to hold true. Indeed, by means of a thorough study centered around the inverse β-decay in two dimensions with massless neutrinos and by enforcing general covariance, the authors of Refs. [118] have explicitly proven

[1] More details on this topic can be found in Chapter 5

the *absolute need* of the Unruh effect for the internal consistency of QFTCS. Such a relevant result not only confers a crucial role to the Unruh radiation (whose existence is still questioned, as for instance in Refs. [119]), but it also opens new perspectives towards the "theoretical check" of formal aspects of modern physics. For instance, the reasoning carried out here for accelerated protons can be employed to explain some quantum field theoretical features related to bremsstrahlung [120].

In what follows, we review the formalism upon which the whole analysis of the inverse β-decay is based. To this aim, we follow Ref. [121], which extends the treatment of Refs. [118] to four dimensions and provides neutrino field with a non-vanishing mass. Subsequently, we extend the whole argument to the case with neutrino mixing, which is useful to derive a preliminary result on the properties of particles belonging to the Unruh radiation. Note that such an apparently harmless generalization is the main source of disagreement between different approaches present in literature. As a matter of fact, we would like to stress that the study of the accelerated proton decay with neutrino mixing has been tackled in a couple of papers [122] in which the authors encounter several theoretical difficulties. Later on, such complications have been cured with distinct methods in Refs. [123] and [124]. However, in the next Sections we will only look at the approach contemplated in Refs. [123], with a brief mention to the other one. For a detailed comment on the inconsistencies of the latter approach, see Ref. [125].

3.1 A formal investigation of the inverse β-decay

In this Section, we discuss the decay of accelerated protons both in the laboratory and comoving frame. By virtue of general covariance, we expect the mean proper lifetime of the proton (or its inverse, namely the decay rate) to be equal in all reference frames. However, we will observe that a similar requirement entails profound implications at a theoretical level which are closely related to the Unruh effect, as already anticipated.

Throughout the whole analysis, neutron $|n\rangle$ and proton $|p\rangle$ are considered as excited and unexcited states of the nucleon, respectively. Moreover, we assume that they are energetic enough to have a well-defined trajectory. As a consequence, the current-current interaction of Fermi theory can be treated with a classical hadronic current $\hat{J}^{\mu}_{\ell}\hat{J}_{h,\mu} \to \hat{J}^{\mu}_{\ell}\hat{J}^{(cl)}_{h,\mu}$, where

$$\hat{J}^{(cl)}_{h,\mu} \;=\; \hat{q}(\tau)u_{\mu}\delta(x)\delta(y)\delta(u-a^{-1})\,. \tag{3.1}$$

Here $u = a^{-1} =$ const is the spatial Rindler coordinate describing the world line of the uniformly accelerated nucleon with proper acceleration a, and $\tau = v/a$ is its proper time, with v being the Rindler time coordinate. The nucleon four-velocity u^μ is given by

$$u^\mu \;=\; (a, 0, 0, 0), \qquad u^\mu \;=\; \left(\sqrt{a^2t^2+1}, 0, 0, at\right), \tag{3.2}$$

in Rindler and Minkowski coordinates, respectively[2]. According to Refs. [99, 118], the Hermitian monopole $\hat{q}(\tau)$ is defined as

$$\hat{q}(\tau) \;\equiv\; e^{i\hat{H}\tau}\hat{q}_0 e^{-i\hat{H}\tau}, \tag{3.3}$$

where $\hat{H}$ is the nucleon Hamiltonian and $\hat{q}_0$ is related to the Fermi constant G_F by

$$G_F \;\equiv\; \langle p|\hat{q}_0|n\rangle. \tag{3.4}$$

Next, the minimal coupling of the electron $\hat{\Psi}_e$ and neutrino $\hat{\Psi}_{\nu_e}$ fields to the nucleon current $\hat{J}^{(cl)}_{h,\mu}$ can be expressed through the Fermi action

$$\hat{S}_I \;=\; \int d^4x\sqrt{-g}\,\hat{J}^{(cl)}_{h,\mu}\left(\hat{\bar{\Psi}}_{\nu_e}\gamma^\mu\hat{\Psi}_e \,+\, \hat{\bar{\Psi}}_e\gamma^\mu\hat{\Psi}_{\nu_e}\right), \tag{3.5}$$

where $g \equiv \det(g_{\mu\nu})$ and γ^μ are the gamma matrices in Dirac representation (see, e.g., Ref. [126]).

3.1.1 Inertial frame

Let us firstly analyze the decay process in the inertial frame. In this case, the proton is accelerated by an external field and converts into a neutron by emitting a positron and a neutrino, according to (see Fig. 3.1 below)

$$p \,\to\, n \,+\, e^+ \,+\, \nu_e\,. \tag{3.6}$$

In order to calculate the transition rate, we quantize fermionic fields in the usual way [118, 126]

$$\hat{\Psi}(t,\mathbf{x}) \;=\; \sum_{\sigma=\pm}\int d^3k\left[\hat{b}_{\mathbf{k}\sigma}\psi^{(+\omega)}_{\mathbf{k}\sigma}(t,\mathbf{x}) \,+\, \hat{d}^\dagger_{\mathbf{k}\sigma}\psi^{(-\omega)}_{-\mathbf{k}-\sigma}(t,\mathbf{x})\right], \tag{3.7}$$

[2]We assume that the proton is accelerated along the z-direction. Hence, the Rindler coordinates (v, x, y, u) are related with the Minkowski coordinates (t, x, y, z) by: $t = u\sinh v$, $z = u\cosh v$, with x and y left unchanged.

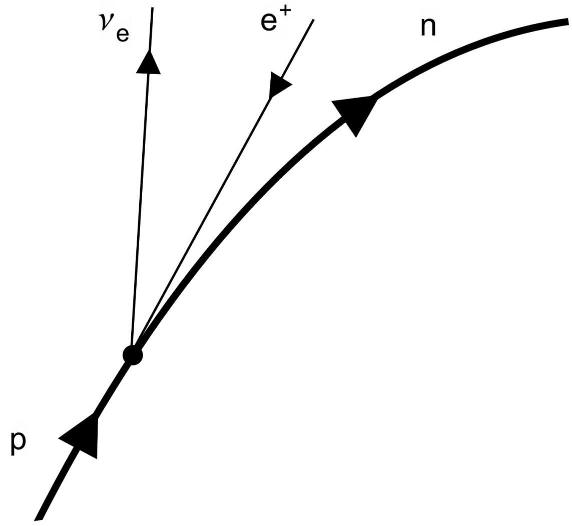

Figure 3.1: A pictorial representation of the decay process in the laboratory frame.

where $\mathbf{x} \equiv (x, y, z)$. Here we have denoted by $\hat{b}_{\mathbf{k}\sigma}$ ($\hat{d}_{\mathbf{k}\sigma}$) the canonical annihilation operators of fermions (antifermions) with momentum $\mathbf{k} \equiv (k^x, k^y, k^z)$, polarization $\sigma = \pm$ and frequency $\omega = \sqrt{\mathbf{k}^2 + m^2} > 0$, m being the mass of the field. The modes $\psi^{(\pm\omega)}_{\mathbf{k}\sigma}$ are positive and negative energy solutions of the Dirac equation in Minkowski spacetime

$$\big(i\gamma^\mu\partial_\mu - m\big)\psi^{(\pm\omega)}_{\mathbf{k}\sigma}(t, \mathbf{x}) = 0. \tag{3.8}$$

In the adopted representation of γ matrices, they take the form [118]

$$\psi^{(\pm\omega)}_{\mathbf{k}\sigma}(t, \mathbf{x}) = \frac{e^{i(\mp\omega t + \mathbf{k}\cdot\mathbf{x})}}{2^2\pi^{\frac{3}{2}}}\, u^{(\pm\omega)}_{\sigma}(\mathbf{k}), \tag{3.9}$$

where

$$u^{(\pm\omega)}_{+}(\mathbf{k}) = \frac{1}{\sqrt{\omega(\omega \pm m)}}\begin{pmatrix} m \pm \omega \\ 0 \\ k^z \\ k^x + ik^y \end{pmatrix}, \qquad u^{(\pm\omega)}_{-}(\mathbf{k}) = \frac{1}{\sqrt{\omega(\omega \pm m)}}\begin{pmatrix} 0 \\ m \pm \omega \\ k^x - ik^y \\ -k^z \end{pmatrix}. \tag{3.10}$$

It is easy to show that the modes (3.9) are orthonormal with respect to the inner product

$$\Big\langle \psi^{(\pm\omega)}_{\mathbf{k}\sigma}, \psi^{(\pm\omega')}_{\mathbf{k}'\sigma'} \Big\rangle = \int_\Sigma d\Sigma_\mu\, \overline{\psi}^{(\pm\omega)}_{\mathbf{k}\sigma}\gamma^\mu\psi^{(\pm\omega')}_{\mathbf{k}'\sigma'} = \delta_{\sigma\sigma'}\delta^3(\mathbf{k} - \mathbf{k}')\delta_{\pm\omega\pm\omega'}, \tag{3.11}$$

where $\overline{\psi} = \psi^\dagger\gamma^0$, $d\Sigma_\mu = n_\mu d\Sigma$, with n_μ being a unit vector orthogonal to the arbitrary spacelike hypersurface Σ and pointing to the future.

Next, by using the definition (3.5) of the Fermi action and expanding leptonic fields according to (3.7), we obtain the following expression for the tree-level transition amplitude:

$$\mathcal{A}_{in}^{p\to n} \equiv \langle n| \otimes \langle e^+_{k_e\sigma_e}, \nu_{k_\nu\sigma_\nu}|\hat{S}_I|0\rangle \otimes |p\rangle = \frac{G_F}{2^4\pi^3}\,\mathcal{I}_{\sigma_\nu\sigma_e}(\omega_\nu,\omega_e), \tag{3.12}$$

where

$$\mathcal{I}_{\sigma_\nu\sigma_e}(\omega_\nu,\omega_e) = \int_{-\infty}^{+\infty} d\tau\, e^{i\left[\Delta m\tau + a^{-1}(\omega_\nu+\omega_e)\sinh a\tau - a^{-1}(k_\nu^z + k_e^z)\cosh a\tau\right]}\, u_\mu \left[\bar{u}^{(+\omega_\nu)}_{\sigma_\nu}\gamma^\mu u^{(-\omega_e)}_{-\sigma_e}\right]. \tag{3.13}$$

Here Δm is the difference between the nucleon masses. By defining the differential transition rate as

$$\begin{aligned}\frac{d^6\mathcal{P}_{in}^{p\to n}}{d^3k_\nu\, d^3k_e} &\equiv \sum_{\sigma_\nu,\sigma_e} |\mathcal{A}_{in}^{p\to n}|^2 \\ &= \frac{G_F^2}{2^8\pi^6}\int_{-\infty}^{+\infty} d\tau_1 d\tau_2\, u_\mu u_\nu \sum_{\sigma_\nu,\sigma_e}\left[\bar{u}^{(+\omega_\nu)}_{\sigma_\nu}\gamma^\mu u^{(-\omega_e)}_{-\sigma_e}\right]\left[\bar{u}^{(+\omega_\nu)}_{\sigma_\nu}\gamma^\nu u^{(-\omega_e)}_{-\sigma_e}\right]^* \\ &\times e^{i\left[\Delta m(\tau_1-\tau_2) + a^{-1}(\omega_\nu+\omega_e)(\sinh a\tau_1 - \sinh a\tau_2) - a^{-1}(k_\nu^z+k_e^z)(\cosh a\tau_1 - \cosh a\tau_2)\right]}\,,\end{aligned} \tag{3.14}$$

the total transition rate is simply given by

$$\Gamma_{in}^{p\to n} = \mathcal{P}_{in}^{p\to n}/T\,, \tag{3.15}$$

where $T = \int_{-\infty}^{+\infty} ds$ is the nucleon proper time. The above integrals can be solved by introducing the new variables

$$\tau_1 = s + \xi/2, \qquad \tau_2 = s - \xi/2\,, \tag{3.16}$$

and using the spin sum

$$\begin{aligned}& u_\mu u_\nu \sum_{\sigma_\nu,\sigma_e}\left[\bar{u}^{(+\omega_\nu)}_{\sigma_\nu}\gamma^\mu u^{(-\omega_e)}_{-\sigma_e}\right]\left[\bar{u}^{(+\omega_\nu)}_{\sigma_\nu}\gamma^\nu u^{(-\omega_e)}_{-\sigma_e}\right]^* \\ &= \frac{2^2}{\omega_\nu\omega_e}\Big[\left(\omega_\nu\omega_e + k_\nu^z k_e^z\right)\cosh 2as - \left(\omega_\nu k_e^z + \omega_e k_\nu^z\right)\sinh 2as \\ &+ \left(k_\nu^x k_e^x + k_\nu^y k_e^y - m_\nu m_e\right)\cosh a\xi\Big].\end{aligned} \tag{3.17}$$

By explicit calculation, we obtain

$$\Gamma^{p\to n}_{in} = \frac{G_F^2}{a\,\pi^6 e^{\pi\Delta m/a}} \int d^3k_\nu d^3k_e \Big[K_{2i\Delta m/a}\left(\frac{2(\omega_\nu + \omega_e)}{a}\right) + \frac{m_\nu m_e}{\omega_\nu \omega_e} \mathrm{Re}\left\{ K_{2i\Delta m/a+2}\left(\frac{2(\omega_\nu + \omega_e)}{a}\right)\right\}\Big]. \tag{3.18}$$

The analytic evaluation of the integral (3.18) can be found in Ref. [121].

3.1.2 Comoving frame

We now analyze the same decay process in the proton comoving frame. As well-known, the natural manifold to describe phenomena for uniformly accelerated observers is the Rindler wedge, i.e., the Minkowski spacetime region defined by $z > |t|$. Within such a manifold, fermionic fields are expanded in terms of the positive and negative frequency solutions of the Dirac equation with respect to the boost Killing vector $\partial/\partial v$ [121]

$$\hat{\Psi}(v,\mathbf{x}) = \sum_{\sigma=\pm} \int_0^{+\infty} d\omega\, d^2k \left[\hat{b}_{\mathbf{w}\sigma} \psi^{(+\omega)}_{\mathbf{w}\sigma}(v,\mathbf{x}) + \hat{d}^\dagger_{\mathbf{w}\sigma} \psi^{(-\omega)}_{\mathbf{w}-\sigma}(v,\mathbf{x}) \right], \tag{3.19}$$

where now $\mathbf{x} \equiv (x, y, u)$ and $\mathbf{w} \equiv (\omega, k^x, k^y)$. We recall that the Rindler frequency ω may assume arbitrary positive real values. In particular, unlike the inertial case, there are massive Rindler particles with zero frequency.

The modes $\psi^{(\pm\omega)}_{\mathbf{k}\sigma}$ in (3.19) are positive and negative energy solutions of the Dirac equation in Rindler spacetime

$$(i\gamma^\mu_R \widetilde{\nabla}_\mu - m)\psi^{(\omega)}_{\mathbf{w}\sigma}(v,\mathbf{x}) = 0, \tag{3.20}$$

where

$$\gamma^\mu_R \equiv e^\mu_{\hat{a}}\gamma^{\hat{a}}, \qquad e^\mu_{\hat{0}} = u^{-1}\delta^\mu_0, \qquad e^\mu_{\hat{i}} = \delta^\mu_i, \qquad \tilde{\nabla}_\mu \equiv \partial_\mu + \frac{1}{8}\left[\gamma^{\hat{a}}, \gamma^{\hat{b}}\right] e_{\hat{a}}{}^\lambda \nabla_\mu e_{\hat{b}\lambda}. \tag{3.21}$$

By virtue of these relations and using the Rindler coordinates, Eq. (3.20) becomes[3]

$$i\frac{\partial\psi^{(\omega)}_{\mathbf{w}\sigma}(v,\mathbf{x})}{\partial v} = \left(\gamma^0 m u - \frac{i\alpha^3}{2} - iu\alpha^i\partial_i\right)\psi^{(\omega)}_{\mathbf{w}\sigma}(v,\mathbf{x}), \qquad \alpha^i = \gamma^0\gamma^i, \;\; i = 1,2,3 \tag{3.22}$$

[3]For notational simplicity, the Dirac matrices with curved indexes will be labeled with a subscript R. Therefore, the ones without it are to be intended as the flat ones.

whose solutions can be written in the form [121]

$$\psi^{(\omega)}_{\mathbf{w}\sigma}(v,\mathbf{x}) \;=\; \frac{e^{i(-\omega v/a\,+\,k_\alpha x^\alpha)}}{(2\pi)^{\frac{3}{2}}}u^{(\omega)}_\sigma(u,\mathbf{w}), \qquad \alpha=1,2, \tag{3.23}$$

with

$$u^{(\omega)}_+(u,\mathbf{w}) = N\begin{pmatrix} ilK_{i\omega/a-1/2}(ul)\,+\,mK_{i\omega/a+1/2}(ul) \\ -(k^x\,+\,ik^y)K_{i\omega/a+1/2}(ul) \\ ilK_{i\omega/a-1/2}(ul)\,-\,mK_{i\omega/a\,+\,1/2}(ul) \\ -(k^x+ik^y)K_{i\omega/a+1/2}(ul)\end{pmatrix},$$

$$u^{(\omega)}_-(u,\mathbf{w}) = N\begin{pmatrix} (k^x\,-\,ik^y)K_{i\omega+1/2}(ul) \\ ilK_{i\omega/a-1/2}(ul)\,+\,mK_{i\omega/a+1/2}(ul) \\ -(k^x\,-\,ik^y)K_{i\omega+1/2}(ul) \\ -ilK_{i\omega/a-1/2}(ul)\,+\,mK_{i\omega/a+1/2}(ul)\end{pmatrix}. \tag{3.24}$$

Here we have denoted by $K_{i\omega/a+1/2}(ul)$ the modified Bessel function of the second kind with complex order, $N = \sqrt{a\cosh(\pi\omega/a)/\pi l}$ and $l = \sqrt{m^2+(k^x)^2+(k^y)^2}$. Again, one can verify that the modes in (3.23) are normalized with respect to the inner product (3.11) expressed in Rindler coordinates.

As it will be shown, in the comoving frame the proton decay is represented as the combination of the following three processes in terms of the Rindler particles [118] (see Fig. 3.2 below):

$$(i)\quad p^+ + e^- \,\to\, n+\nu_e, \qquad (ii)\quad p^+ + \overline{\nu}_e \,\to\, n+e^+, \qquad (iii)\quad p^+ + e^- + \overline{\nu}_e \,\to\, n. \tag{3.25}$$

These processes are characterized by the conversion of protons in neutrons due to the absorption of e^- and $\bar{\nu}_e$, and emission of e^+ and $\bar{\nu}_e$ from and to the Unruh thermal bath [117]. Since the strategy for calculating the transition amplitude is the same for each of these processes, by way of illustration we shall focus on the first.

By exploiting the Rindler expansion (3.19) for the electron and neutrino fields, it can be shown that

$$\mathcal{A}^{p\to n}_{(i)} \;\equiv\; \langle n|\otimes\langle \nu_{\omega_\nu\sigma_\nu}|\hat{S}_I|e^-_{\omega_{e^-}\sigma_{e^-}}\rangle\otimes|p\rangle \;=\; \frac{G_F}{(2\pi)^2}\,\mathcal{I}_{\sigma_\nu\sigma_e}(\omega_\nu,\omega_e), \tag{3.26}$$

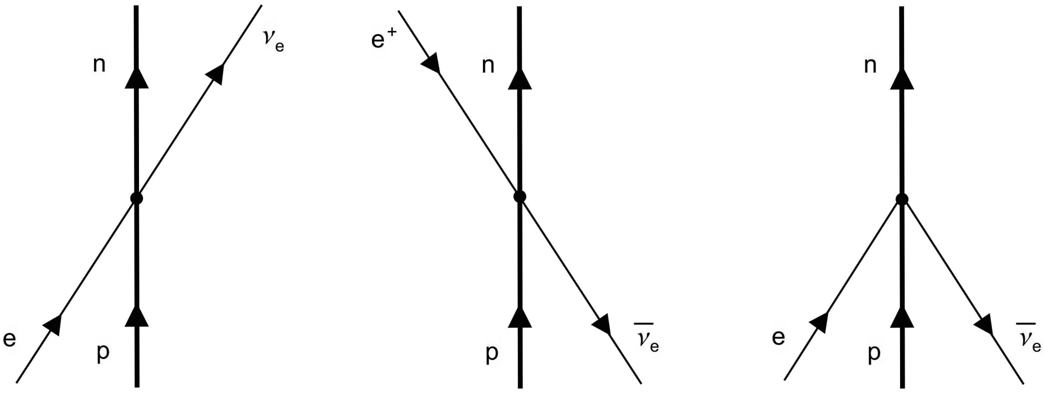

Figure 3.2: A pictorial representation of the decay processes in the comoving frame.

where $\hat{S}_I$ is given by (3.5) with γ^μ replaced by the Rindler gamma matrices γ_R^μ defined in (3.21) and

$$\mathcal{J}_{\sigma_\nu\sigma_e}(\omega_\nu,\omega_e) \;=\; \delta\big(\omega_e-\omega_\nu-\Delta m\big)\,\bar{u}^{(\omega_\nu)}_{\sigma_\nu}\gamma^0 u^{(\omega_e)}_{\sigma_e}. \tag{3.27}$$

Now, bearing in mind that the probability for the proton to absorb (emit) a particle of frequency ω from (to) the thermal bath is $n_F(\omega) = 1/(e^{2\pi\omega/a}+1)$ $\big(1-n_F(\omega)\big)$ [118], the differential transition rate per unit time for the process (i) can be readily evaluated, thus leading to

$$\begin{aligned}\frac{1}{T}\frac{d^6\mathcal{P}^{p\to n}_{(i)}}{d\omega_\nu d\omega_e d^2k_\nu d^2k_e} \;&\equiv\; \frac{1}{T}\sum_{\sigma_\nu,\sigma_e}\big|\mathcal{A}^{p\to n}_{(i)}\big|^2 n_F(\omega_e)\big[1\,-\,n_F(\omega_\nu)\big] \\ &=\; \frac{G_F^2}{2^7\pi^5}\,\frac{\sum_{\sigma_\nu,\sigma_e}\big|\bar{u}^{(\omega_\nu)}_{\sigma_\nu}\gamma^0u^{(\omega_e)}_{\sigma_e}\big|^2\,\delta\left(\omega_e-\omega_\nu-\Delta m\right)}{e^{\pi\Delta m/a}\cosh(\pi\omega_\nu/a)\cosh(\pi\omega_e/a)}\,,\end{aligned} \tag{3.28}$$

where $T = 2\pi\delta(0)$ is the total proper time of the proton. In order to finalize the evaluation of the transition rate, we observe that

$$\begin{aligned}\sum_{\sigma_\nu,\sigma_e}\big|\bar{u}^{(\omega_\nu)}_{\sigma_\nu}\gamma^0u^{(\omega_e)}_{\sigma_e}\big|^2 \;&=\; \frac{2^4}{(a\,\pi)^2}\cosh(\pi\omega_\nu/a)\cosh(\pi\omega_e/a) \\ &\times\;\Bigg[l_\nu l_e\Big|K_{i\omega_\nu/a+1/2}\left(\frac{l_\nu}{a}\right)K_{i\omega_e/a+1/2}\left(\frac{l_e}{a}\right)\Big|^2+\big(k^x_\nu k^x_e \\ &+\;k^y_\nu k^y_e\,+\,m_\nu m_e\big)\mathrm{Re}\left\{K^2_{i\omega_\nu/a-1/2}\left(\frac{l_\nu}{a}\right)K^2_{i\omega_e/a+1/2}\left(\frac{l_e}{a}\right)\right\}\Bigg].\end{aligned} \tag{3.29}$$

Using this equation, the differential transition rate for the process (i) takes the form

$$\begin{aligned}
\frac{1}{T}\frac{d^6\mathcal{P}^{p\to n}_{(i)}}{d\omega_\nu d\omega_e d^2k_\nu d^2k_e} &\equiv \frac{G_F^2}{2^3\, a^2\, \pi^7\, e^{\pi\Delta m/a}}\delta\left(\omega_e-\omega_\nu-\Delta m\right) \\
&\times \left[l_\nu l_e\left|K_{i\omega_\nu/a+1/2}\left(\frac{l_\nu}{a}\right)K_{i\omega_e/a+1/2}\left(\frac{l_e}{a}\right)\right|^2 + m_\nu m_e\right. \\
&\times \left.\mathrm{Re}\left\{K^2_{i\omega_\nu/a-1/2}\left(\frac{l_\nu}{a}\right)K^2_{i\omega_e/a+1/2}\left(\frac{l_e}{a}\right)\right\}\right]. \qquad (3.30)
\end{aligned}$$

Next, by performing similar calculation for the processes (ii) and (iii) and adding up the three contributions, we end up with the following integral expression for the total decay rate in the comoving frame:

$$\Gamma^{p\to n}_{acc} \equiv \Gamma^{p\to n}_{(i)}+\Gamma^{p\to n}_{(ii)}+\Gamma^{p\to n}_{(iii)} = \frac{2G_F^2}{a^2\pi^7\, e^{\pi\Delta m/a}}\int_{-\infty}^{+\infty}d\omega\,\mathcal{R}(\omega), \qquad (3.31)$$

where

$$\begin{aligned}
\mathcal{R}(\omega) &= \int d^2k_\nu d^2k_e\, l_\nu\, l_e\left|K_{i(\omega-\Delta m)/a+1/2}\left(\frac{l_\nu}{a}\right)\right|^2\left|K_{i\omega/a+1/2}\left(\frac{l_e}{a}\right)\right|^2 \qquad (3.32) \\
&+ m_\nu m_e \mathrm{Re}\left\{\int d^2k_\nu d^2k_e\, K^2_{i(\omega-\Delta m)/a-1/2}\left(\frac{l_\nu}{a}\right)K^2_{i\omega/a+1/2}\left(\frac{l_e}{a}\right)\right\}.
\end{aligned}$$

The analytic resolution of the integral (3.31) is performed in Ref. [121]. Comparing this result to the one in the inertial frame (Eq. (3.18)), it is possible to show that the resulting expressions for the decay rates perfectly agree with each other, thus corroborating the necessity of the Unruh effect for the consistency of QFTCS. As a matter of fact, such an achievement is made possible by requiring general covariance to hold. Indeed, the same decay rate has been evaluated by using two different sets of coordinates (namely, Minkowski and Rindler ones), but since the choice of a coordinate system is nothing but a gauge fixing (see Sec. 1.1) both calculations must converge at the same result. The sole method to allow this is to unambiguously demand the Unruh radiation to be a real physical occurrence. Therefore, we have proven the necessity of the existence of the Unruh effect without invoking any experimental argument.

However, this is not the end of the story, since general covariance fulfillment is still capable of unraveling novel features. In this perspective, it is intriguing to extend the discussion of the current Section to the case in which neutrino field exhibits a mixed nature, as a plethora of recent tests confirms.

3.2 The inverse β-decay with neutrino mixing

So far, in the evaluation of the transition amplitude, we have treated the electron neutrino as a particle with definite mass m_ν. However, it is well-known that neutrinos exhibit flavor mixing[4]: in a simplified two-flavor model, by denoting with θ the mixing angle, the transformations relating the flavor eigenstates $|\nu_\ell\rangle$ ($\ell = e, \mu$) and mass eigenstates $|\nu_i\rangle$ ($i = 1, 2$) are determined by the well-known Pontecorvo unitary mixing matrix[5] [127]

$$\begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} |\nu_1\rangle \\ |\nu_2\rangle \end{pmatrix}. \tag{3.33}$$

Along the line of Refs. [122], the question thus arises whether such a transformation is consistent with the framework of Sec. 3.1. Clearly, since general covariance must hold, we are already aware of the fact that the results of the analysis in the laboratory and in the comoving frame must necessarily be consistent. The search for a similar compatibility is the main task of the current Section. To this aim, we essentially follow the approach of Refs. [123]

3.2.1 Inertial frame

Let us then implement the Pontecorvo rotation (3.33) on both the neutrino fields and states appearing in (3.12). Note that in Refs. [122] this step is missing in the inertial frame calculation since $\hat{\Psi}_{\nu_e}$ is treated as a free-field even when taking into account flavor mixing, and indeed the same result as in the case of unmixed fields is obtained. We explicitly show that the decay rate exhibits a dependence on θ in the inertial frame, a feature which is not present in the analysis of Refs. [122].

By assuming equal momenta and polarizations for the two neutrino mass eigenstates, the transition amplitude (3.12) now becomes

$$\mathcal{A}_{in}^{p\to n} = \frac{G_F}{2^4\pi^3}\Big[\cos^2\theta\, \mathcal{I}_{\sigma_\nu\sigma_e}(\omega_{\nu_1}, \omega_e) + \sin^2\theta\, \mathcal{I}_{\sigma_\nu\sigma_e}(\omega_{\nu_2}, \omega_e)\Big], \tag{3.34}$$

where $\mathcal{I}_{\sigma_\nu\sigma_e}(\omega_{\nu_j}, \omega_e)$, $j = 1, 2$, is defined as in (3.13) for each of the two mass eigenstates, and we have rotated the electron neutrino field according to

$$\hat{\Psi}_{\nu_e}(t, \mathbf{x}) = \cos\theta\, \hat{\Psi}_{\nu_1}(t, \mathbf{x}) + \sin\theta\, \hat{\Psi}_{\nu_2}(t, \mathbf{x}). \tag{3.35}$$

[4]For more pieces of information, see Appendix D.

[5]Note that the number of neutrino generations does not affect the results of our analysis.

Using (3.14), the differential transition rate takes the form

$$\frac{d^6\mathcal{P}_{in}^{p\to n}}{d^3k_\nu d^3k_e} = \sum_{\sigma_\nu,\sigma_e}\frac{{G_F}^2}{2^8\pi^6}\Big\{\cos^4\theta\left|\mathcal{I}_{\sigma_\nu\sigma_e}(\omega_{\nu_1},\omega_e)\right|^2 + \sin^4\theta\left|\mathcal{I}_{\sigma_\nu\sigma_e}(\omega_{\nu_2},\omega_e)\right|^2$$
$$+ \cos^2\theta\sin^2\theta\Big[\mathcal{I}_{\sigma_\nu\sigma_e}(\omega_{\nu_1},\omega_e)\,\mathcal{I}^*_{\sigma_\nu\sigma_e}(\omega_{\nu_2},\omega_e) + \text{c.c.}\Big]\Big\}. \tag{3.36}$$

The total decay rate $\Gamma_{in}^{p\to n}$ is obtained after inserting this equation into the definition (3.15)

$$\Gamma_{in}^{p\to n} = \cos^4\theta\,\Gamma_1^{p\to n} + \sin^4\theta\,\Gamma_2^{p\to n} + \cos^2\theta\sin^2\theta\,\Gamma_{12}^{p\to n}, \tag{3.37}$$

where we have introduced the shorthand notation

$$\Gamma_j^{p\to n} \equiv \frac{1}{T}\sum_{\sigma_\nu,\sigma_e}\frac{{G_F}^2}{2^8\pi^6}\int d^3k_\nu d^3k_e\left|\mathcal{I}_{\sigma_\nu\sigma_e}(\omega_{\nu_j},\omega_e)\right|^2, \qquad j=1,2, \tag{3.38}$$

and

$$\Gamma_{12}^{p\to n} \equiv \frac{1}{T}\sum_{\sigma_\nu,\sigma_e}\frac{{G_F}^2}{2^8\pi^6}\int d^3k_\nu d^3k_e\Big[\mathcal{I}_{\sigma_\nu\sigma_e}(\omega_{\nu_1},\omega_e)\,\mathcal{I}^*_{\sigma_\nu\sigma_e}(\omega_{\nu_2},\omega_e) + \text{c.c.}\Big]. \tag{3.39}$$

We observe that, for $\theta\to 0$, the obtained result correctly reduces to (3.18), as it should be in absence of mixing. Unfortunately, due to technical difficulties in the evaluation of the integral (3.39), at this stage we are not able to give the exact expression of the inertial decay rate (3.37). A preliminary result, however, can be obtained in the limit of small neutrino mass difference $\delta m/m_{\nu_1} \equiv (m_{\nu_2}-m_{\nu_1})/m_{\nu_1} \ll 1$. In this case, indeed, we can expand $\Gamma_{12}^{p\to n}$ according to

$$\Gamma_{12}^{p\to n} = 2\Gamma_1^{p\to n} + \frac{\delta m}{m_{\nu_1}}\,\Gamma^{(1)} + \mathcal{O}\left(\frac{\delta m^2}{m_{\nu_1}^2}\right), \tag{3.40}$$

where $\Gamma_1^{p\to n}$ is defined as in (3.38) and we have denoted by $\Gamma^{(1)}$ the first-order term of the Taylor expansion. The explicit expression of $\Gamma^{(1)}$ is rather awkward to exhibit. Nevertheless, for $m_{\nu_1}\to 0$, it can be substantially simplified, thus giving

$$\frac{\Gamma^{(1)}}{m_{\nu_1}} = \frac{1}{T}\frac{{G_F}^2\,m_e}{2^7\pi^6}\int\frac{d^3k_\nu}{|k_\nu|}\frac{d^3k_e}{\omega_e}\int_{-\infty}^{+\infty}ds\,d\xi\cosh a\xi$$
$$\times\left[e^{i\left\{\Delta m\,\xi+\frac{2\sinh a\xi/2}{a}\left[(|k_\nu|+\omega_e)\cosh as-(k_\nu^z+k_e^z)\sinh as\right]\right\}} + \text{c.c.}\right], \tag{3.41}$$

where s and ξ are defined in (3.16). Note that such a "limit" has a purely mathematical sense. It is the most suitable scenario to have an easy-to-handle esteem for the studied quantity.

By performing a boost along the z-direction

$$k'^x_\ell \;=\; k^x_\ell, \qquad k'^y_\ell \;=\; k^y_\ell, \qquad k'^z_\ell \;=\; -\omega_\ell \sinh as \,+\, k^z_\ell \cosh as, \qquad \ell \;=\; \nu_1, e, \tag{3.42}$$

equation (3.41) can be cast in the form

$$\frac{\Gamma^{(1)}}{m_{\nu_1}} \;=\; \lim_{\varepsilon\to 0} \frac{2\, G_F^2\, m_e}{a\, \pi^6\, e^{\pi\Delta m/a}} \int \frac{d^3k_\nu}{\omega_\varepsilon} \frac{d^3k_e}{\omega_e} \mathrm{Re}\left\{K_{2i\Delta m/a+2}\left(\frac{2(\omega_\varepsilon \,+\, \omega_e)}{a}\right)\right\}, \tag{3.43}$$

where $\omega_\varepsilon \;=\; \sqrt{\mathbf{k}_\nu^2 \,+\, \varepsilon^2}$, with ε acting as a regulator. In order to perform k-integration, we use the following representation of the modified Bessel function:

$$K_\mu(z) \;=\; \frac{1}{2}\int_{C_1} \frac{ds}{2\pi i}\Gamma(-s)\Gamma(-s-\mu)\left(\frac{z}{2}\right)^{2s+\mu}, \tag{3.44}$$

where Γ is the Euler's Gamma function. C_1 is the path in the complex plane including all the poles of $\Gamma(-s)$ and $\Gamma(-s\,-\,\mu)$, chosen in such a way that the integration with respect to the momentum variables does not diverge [121].

Using spherical coordinates, Eq. (3.43) becomes

$$\begin{aligned}\frac{\Gamma^{(1)}}{m_{\nu_1}} \;&=\; \lim_{\varepsilon\to 0} \frac{2^3\, G_F^2\, m_e}{a\, \pi^4\, e^{\pi\Delta m/a}} \int_0^{+\infty} dk_\nu dk_e \frac{k_\nu^2\, k_e^2}{\omega_\varepsilon\, \omega_e} \int_{C_s} \frac{ds}{2\pi i} \left(\frac{\omega_\varepsilon \,+\, \omega_e}{a}\right)^{2s} \left[\Gamma\left(-s + \frac{i\Delta m}{a} + 1\right)\right. \\ &\times\; \left.\Gamma\left(-s - \frac{i\Delta m}{a} - 1\right) + \Gamma\left(-s + \frac{i\Delta m}{a} - 1\right)\Gamma\left(-s - \frac{i\Delta m}{a} + 1\right)\right].\end{aligned} \tag{3.45}$$

Let us observe at this point that [121]

$$\left(\frac{\omega_\varepsilon \,+\, \omega_e}{a}\right)^{2s} \;=\; \int_{C_2} \frac{dt}{2\pi i} \frac{\Gamma(-t)\Gamma(t-2s)}{\Gamma(-2s)} \left(\frac{\omega_\varepsilon}{a}\right)^{-t+2s} \left(\frac{\omega_e}{a}\right)^{t}, \tag{3.46}$$

where C_2 is the contour in the complex plane separating the poles of $\Gamma(-t)$ from the ones of $\Gamma(t-2s)$. Exploiting this relation and properly redefining the integration variables, we finally obtain

$$\begin{aligned}
\frac{\Gamma^{(1)}}{m_{\nu_1}} &= \lim_{\varepsilon\to 0}\frac{G_F^2\, m_e\, a^3}{\pi^3\, e^{\pi\Delta m/a}}\int_{C_s}\frac{ds}{2\pi i}\int_{C_t}\frac{dt}{2\pi i}\left(\frac{\varepsilon}{a}\right)^{2s+2}\left(\frac{m_e}{a}\right)^{2t+2} \\
&\times \frac{\Gamma(-2s)\Gamma(-2t)\Gamma(-t\,-\,1)\Gamma(-s\,-\,1)}{\Gamma(-s\,+\,\frac{1}{2})\Gamma(-t\,+\,\frac{1}{2})\Gamma(-2s\,-\,2t)} \\
&\times \left[\Gamma\left(-s\,-\,t\,+\,1\,+\,i\frac{\Delta m}{a}\right)\Gamma\left(-s-t-1-i\frac{\Delta m}{a}\right)\right. \\
&+ \left.\Gamma\left(-s-t+1-i\frac{\Delta m}{a}\right)\Gamma\left(-s-t-1+i\frac{\Delta m}{a}\right)\right]. \qquad (3.47)
\end{aligned}$$

where the contour $C_{s(t)}$ includes all poles of gamma functions in s (t) complex plane.

From (3.40) and (3.47), we thus infer that the off-diagonal term $\Gamma_{12}^{p\to n}$ is non-vanishing, thereby leading to a structure of the inertial decay rate (3.37) that is different from the corresponding one in Refs. [122].

3.2.2 Comoving frame

Let us now extend the above discussion to the proton comoving frame. As done in the inertial case, we require the asymptotic neutrino states to be flavor eigenstates (the choice of mass eigenstates would inevitably lead to a contradiction). Note that the same assumption is contemplated also in Refs. [122]. In spite of this, those authors exclude such an alternative on the basis of the KMS condition, claiming that the accelerated neutrino vacuum must be a thermal state of neutrinos with definite masses rather than definite flavors. Actually, this argument does not apply, at least within the first-order approximation we are dealing with (see (3.40)). Indeed, as shown in Refs. [128], non-thermal corrections to the Unruh spectrum for flavor (mixed) neutrinos only appear at orders higher than $\mathcal{O}\left(\delta m/m\right)$.

Relying on these considerations, let us evaluate the decay rate in the comoving frame. A straightforward calculation leads to the following expression for the transition amplitude (3.26):

$$\mathcal{A}_{(i)}^{p\to n} = \frac{G_F}{(2\pi)^2}\left[\cos^2\theta\,\mathcal{J}^{(1)}_{\sigma_\nu\sigma_e}(\omega_\nu,\omega_e) + \sin^2\theta\,\mathcal{J}^{(2)}_{\sigma_\nu\sigma_e}(\omega_\nu,\omega_e)\right], \qquad (3.48)$$

where $\mathcal{J}^{(j)}_{\sigma_\nu\sigma_e}(\omega_\nu,\omega_e)$, $j\,=\,1,2$, is defined as in (3.27) for each of the two neutrino mass eigenstates. The differential transition rate per unit time thus reads

$$
\begin{aligned}
\frac{1}{T}\frac{d^6\mathcal{P}^{p\to n}_{(i)}}{d\omega_\nu d\omega_e d^2k_\nu d^2k_e} &= \frac{1}{T}\frac{G_F^2}{2^6\pi^4}\frac{1}{e^{\pi\Delta m/a}\cosh(\pi\omega_\nu/a)\cosh(\pi\omega_e/a)} \\
&\times \sum_{\sigma_\nu,\sigma_e}\Big\{\cos^4\theta\left|\mathcal{J}^{(1)}_{\sigma_\nu\sigma_e}(\omega_\nu,\omega_e)\right|^2 + \sin^4\theta\left|\mathcal{J}^{(2)}_{\sigma_\nu\sigma_e}(\omega_\nu,\omega_e)\right|^2 \\
&+ \cos^2\theta\sin^2\theta\left[\mathcal{J}^{(1)}_{\sigma_\nu\sigma_e}(\omega_\nu,\omega_e)\,\mathcal{J}^{(2)*}_{\sigma_\nu\sigma_e}(\omega_\nu,\omega_e) + \text{c.c.}\right]\Big\}.
\end{aligned}
\tag{3.49}
$$

The spin sum for the process (i) is given by

$$
\begin{aligned}
&\frac{1}{T}\sum_{\sigma_\nu,\sigma_e}\left[\mathcal{J}^{(1)}_{\sigma_\nu\sigma_e}(\omega_\nu,\omega_e)\,\mathcal{J}^{(2)*}_{\sigma_\nu\sigma_e}(\omega_\nu,\omega_e) + \text{c.c.}\right] = \frac{2^3\,\delta(\omega_e-\omega_\nu-\Delta m)}{a^2\,\pi^3\,\sqrt{l_{\nu_1}l_{\nu_2}}} \\
&\times\cosh\left(\pi\omega/a\right)\cosh\left(\pi\omega_e/a\right)\Bigg[l_e\left(\kappa_\nu^2 + m_{\nu_1}m_{\nu_2} + l_{\nu_1}l_{\nu_2}\right)\left|K_{i\omega_e/a+1/2}\left(\frac{l_e}{a}\right)\right|^2 \\
&\times\mathrm{Re}\left\{K_{i\omega_\nu/a+1/2}\left(\frac{l_{\nu_1}}{a}\right)K_{i\omega_\nu/a-1/2}\left(\frac{l_{\nu_2}}{a}\right)\right\} + \Big[\left(k_\nu^x k_e^x + k_\nu^y k_e^y\right)\left(l_{\nu_1}+l_{\nu_2}\right) \\
&+m_e\left(l_{\nu_1}m_{\nu_2}+l_{\nu_2}m_{\nu_1}\right)\Big]\mathrm{Re}\left\{K^2_{i\omega_e/a+1/2}\left(\frac{l_e}{a}\right)K_{i\omega_\nu/a+1/2}\left(\frac{l_{\nu_1}}{a}\right)\right. \\
&\left.\times K_{i\omega_\nu/a+1/2}\left(\frac{l_{\nu_2}}{a}\right)\right\}\Bigg],
\end{aligned}
\tag{3.50}
$$

where $\kappa_\nu \equiv (k_\nu^x, k_\nu^y)$.

Next, by performing similar calculations for the other two processes and adding up the three contributions, we finally obtain the total transition rate in the comoving frame

$$
\Gamma^{p\to n}_{acc} = \cos^4\theta\,\widetilde{\Gamma}^{p\to n}_1 + \sin^4\theta\,\widetilde{\Gamma}^{p\to n}_2 + \cos^2\theta\sin^2\theta\,\widetilde{\Gamma}^{p\to n}_{12}, \tag{3.51}
$$

where $\widetilde{\Gamma}^{p\to n}_j$, $j = 1, 2$, is defined as

$$
\widetilde{\Gamma}^{p\to n}_j \equiv \frac{2\,G_F^2}{a^2\,\pi^7\,e^{\pi\Delta m/a}}\int_{-\infty}^{+\infty}d\omega\,R_j(\omega), \qquad j = 1, 2, \tag{3.52}
$$

with $\mathcal{R}_j(\omega)$ being defined as in (3.32) for each of the two neutrino mass eigenstates, and

$$\begin{aligned}\widetilde{\Gamma}_{12}^{p\to n} &= \frac{2\,G_F^2}{a^2\,\pi^7\,e^{\pi\Delta m/a}}\int_{-\infty}^{+\infty}\frac{d\omega}{\sqrt{l_{\nu_1}l_{\nu_2}}}\left\{\int d^2k_e d^2k_\nu\, l_e\left|K_{i\omega/a+1/2}\left(\frac{l_e}{a}\right)\right|^2\right.\\ &\times\ \left(\kappa_\nu^2+m_{\nu_1}m_{\nu_2}+l_{\nu_1}l_{\nu_2}\right)\mathrm{Re}\left\{K_{i(\omega-\Delta m)/a+1/2}\left(\frac{l_{\nu_1}}{a}\right)\right.\\ &\times\ \left.K_{i(\omega-\Delta m)/a-1/2}\left(\frac{l_{\nu_2}}{a}\right)\right\}+m_e\int d^2k_e d^2k_\nu\left(l_{\nu_1}m_{\nu_2}+l_{\nu_2}m_{\nu_1}\right)\\ &\times\ \mathrm{Re}\left\{K^2_{i\omega/a+1/2}\left(\frac{l_e}{a}\right)K_{i(\omega-\Delta m)/a-1/2}\left(\frac{l_{\nu_1}}{a}\right)\right.\\ &\times\ \left.\left.K_{i(\omega-\Delta m)/a-1/2}\left(\frac{l_{\nu_2}}{a}\right)\right\}\right\}. \end{aligned}\tag{3.53}$$

It is now possible to verify that

$$\Gamma_j^{p\to n} = \widetilde{\Gamma}_j^{p\to n} \qquad j = 1,2. \tag{3.54}$$

By comparing (3.37) and (3.51) and using the above equality, we thus realize that inertial and comoving calculations would match, provided that the integrals (3.39) and (3.53) coincide. As in the inertial case, however, the treatment of the $\widetilde{\Gamma}_{12}^{p\to n}$ is absolutely nontrivial. A clue to a preliminary solution can be found by expanding $\widetilde{\Gamma}_{12}^{p\to n}$ in the limit of small neutrino mass difference, as in Sec. 3.2.1

$$\widetilde{\Gamma}_{12}^{p\to n} = 2\widetilde{\Gamma}_1^{p\to n} + \frac{\delta m}{m_{\nu_1}}\,\widetilde{\Gamma}^{(1)} + \mathcal{O}\left(\frac{\delta m^2}{m_{\nu_1}^2}\right), \tag{3.55}$$

where $\widetilde{\Gamma}_1^{p\to n}$ is defined in (3.52) and we have denoted by $\widetilde{\Gamma}^{(1)}$ the first-order term of the expansion. For $m_{\nu_1}\to 0$, it is possible to show that

$$\frac{\widetilde{\Gamma}^{(1)}}{m_{\nu_1}} = \lim_{\varepsilon\to 0}\frac{2^2\,G_F^2\,m_e}{a^2\,\pi^7\,e^{\pi\Delta m/a}}\int_{-\infty}^{+\infty}d\omega\,\mathrm{Re}\left\{\int d^2k_\nu d^2k_e\,K^2_{i(\omega-\Delta m)/a-1/2}\left(\frac{l_\varepsilon}{a}\right)K^2_{i\omega/a+1/2}\left(\frac{l_e}{a}\right)\right\}, \tag{3.56}$$

where $l_\varepsilon = \sqrt{(k_\nu^x)^2+(k_\nu^y)^2+\varepsilon^2}$, with ε acting as a regulator.

Equation (3.56) can be now further manipulated by introducing the following relation involving Meijer G-function (see, e.g., Ref. [100]):

$$
\begin{aligned}
x^\sigma K_\nu(x) K_\mu(x) &= \frac{\sqrt{\pi}}{2} \\
&\times G^{40}_{24}\left(x^2 \middle| \begin{matrix} \frac{1}{2}\sigma,\ \frac{1}{2}\sigma + \frac{1}{2} \\ \frac{1}{2}(\nu+\mu+\sigma), \frac{1}{2}(\nu-\mu+\sigma), \frac{1}{2}(-\nu+\mu+\sigma), \frac{1}{2}(-\nu-\mu+\sigma) \end{matrix}\right).
\end{aligned}
\tag{3.57}
$$

A somewhat laborious calculation then leads to

$$
\begin{aligned}
\frac{\widetilde{\Gamma}^{(1)}}{m_{\nu_1}} &= \lim_{\varepsilon\to 0} \frac{2\, G_F^2\, m_e}{a^2\, \pi^4\, e^{\pi\Delta m/a}} \int_{-\infty}^{+\infty} d\omega \int_{C_s} \frac{ds}{2\pi i} \int_{C_t} \frac{dt}{2\pi i} \int_0^{+\infty} dk_\nu dk_e\, k_\nu\, l_\varepsilon^{2s}\, k_e\, l_e^{2t} \\
&\times \Bigg[\frac{\Gamma(-s)\,\Gamma(-t)\,\Gamma\left(\frac{i\omega}{a} + \frac{1}{2} - t\right)\Gamma\left(-\frac{i\omega}{a} - \frac{1}{2} - t\right)}{\Gamma\left(-s+\frac{1}{2}\right)\Gamma\left(-t+\frac{1}{2}\right)} \\
&\times \Gamma\left(\frac{i(\omega-\Delta m)}{a} - \frac{1}{2} - s\right)\Gamma\left(-\frac{i(\omega-\Delta m)}{a} + \frac{1}{2} - s\right) \\
&+ \frac{\Gamma(-s)\,\Gamma(-t)\,\Gamma\left(\frac{i\omega}{a} - \frac{1}{2} - t\right)\Gamma\left(-\frac{i\omega}{a} + \frac{1}{2} - t\right)}{\Gamma\left(-s+\frac{1}{2}\right)\Gamma\left(-t+\frac{1}{2}\right)} \\
&\times \Gamma\left(\frac{i(\omega-\Delta m)}{a} + \frac{1}{2} - s\right)\Gamma\left(-\frac{i(\omega-\Delta m)}{a} - \frac{1}{2} - s\right)\Bigg].
\end{aligned}
\tag{3.58}
$$

In order to perform the integration with respect to ω, let us use the first Barnes lemma, according to which [100]

$$
\int_{-i\infty}^{+i\infty} d\omega\, \Gamma(a+\omega)\Gamma(b+\omega)\Gamma(c-\omega)\Gamma(d-\omega) = 2\pi i\, \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}.
\tag{3.59}
$$

Inserting this relation into (3.58), it follows that

$$
\begin{aligned}
\frac{\widetilde{\Gamma}^{(1)}}{m_{\nu_1}} &= \lim_{\varepsilon\to 0} \frac{G_F^2\, m_e\, a^3}{\pi^3\, e^{\pi\Delta m/a}} \int_{C_s} \frac{ds}{2\pi i} \int_{C_t} \frac{dt}{2\pi i} \left(\frac{\varepsilon}{a}\right)^{2s+2} \left(\frac{m_e}{a}\right)^{2t+2} \\
&\times \frac{\Gamma(-2s)\Gamma(-2t)\Gamma(-t-1)\Gamma(-s-1)}{\Gamma(-s+\frac{1}{2})\Gamma(-t+\frac{1}{2})\Gamma(-2s-2t)} \\
&\times \left[\Gamma\left(-s-t+1+i\frac{\Delta m}{a}\right)\Gamma\left(-s-t-1-i\frac{\Delta m}{a}\right)\right. \\
&+ \left.\Gamma\left(-s-t+1-i\frac{\Delta m}{a}\right)\Gamma\left(-s-t-1+i\frac{\Delta m}{a}\right)\right].
\end{aligned}
\tag{3.60}
$$

which is exactly the same expression obtained in the inertial frame (3.47).

So far, we have seen the motivations for which it is mandatory to require the existence of the Unruh effect by relying on the generally covariant formulation of QFTCS. Moreover, we have shown how such an accomplishment does not depend on the mixed nature of the emitted (for the inertial observer) or absorbed (for the comoving observer) neutrino. However, there are still further considerations to be performed that are directly related to the occurrence of neutrino flavor transitions [123]. In the following Section, we will see that investigations in this direction shed light on a crucial aspect of neutrinos belonging to the Unruh radiation.

3.3 Inverse β-decay with oscillating neutrinos

In Sec. 3.2.2, we have demonstrated general covariance to imply Unruh effect even when neutrino is regarded as a mixed field. Furthermore, due to their intrinsic nature, we know that flavor transitions take place throughout the propagation of the particle [127]. Therefore, it is licit to assume that the emitted neutrino can change its flavor after its production via inverse β-decay. Indeed, it must be emphasized that in the above calculations of Sec 3.2.1 an infinite proper time interval is considered, which allows for the electron neutrino produced in the proton decay to oscillate. Thus, we should take into account not only the process contemplated in (3.6), but also the following one:

$$p \to n + e^+ + \nu_\mu \,. \tag{3.61}$$

The above relation must be intended in the sense of Fig. 3.3: although it is true that the lepton charge must necessarily be conserved in the vertex (at tree-level), as soon as the outgoing neutrino is produced, flavor oscillations will inevitably occur.

The transition amplitude for such process is non-vanishing

$$\begin{aligned} \mathcal{A}^{(\nu_\mu)} &= \langle n| \otimes \langle e^+, \nu_\mu|\hat{S}_I|0\rangle \otimes |p\rangle \\ &= -\frac{G_F}{2^4\pi^3}\cos\theta\sin\theta\Big[\mathcal{I}_{\sigma_\nu\sigma_e}(\omega_{\nu_1},\omega_e) - \mathcal{I}_{\sigma_\nu\sigma_e}(\omega_{\nu_2},\omega_e)\Big]. \end{aligned} \tag{3.62}$$

In terms of Γ, the quantity $\mathcal{A}^{(\nu_\mu)}$ of (3.62) associated to the process (3.61) leads to the following transition rate:

$$\Gamma^{(\nu_\mu)}_{in} = \cos^2\theta\sin^2\theta\,(\Gamma_1 + \Gamma_2 - \Gamma_{12}) \ , \tag{3.63}$$

where all the contributions in the r.h.s. of the above expression have already been introduced in (3.38) and (3.39). We notice that the above transition rate is proportional to $\sin^2 2\theta$, thus showing that it is originated by interference.

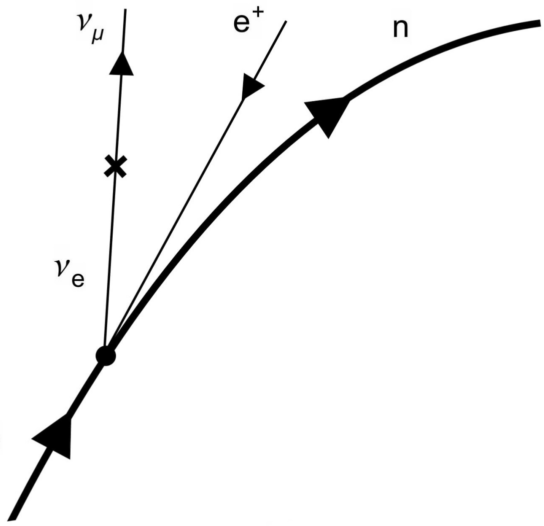

Figure 3.3: Decay process in the inertial frame that accounts for flavor transitions.

Finally, observe that

$$\Gamma_{in} \equiv \Gamma_{in}^{(\nu_e)} + \Gamma_{in}^{(\nu_\mu)} = \cos^2\theta\,\Gamma_1 + \sin^2\theta\,\Gamma_2\,. \tag{3.64}$$

As it should be plain by now, due to the general covariance of QFTCS an outcome similar to the one expressed in (3.64) must be manifest also in the comoving frame. In Sec. 3.2.2 it was shown that, by taking into account the three interactions in (3.25) of the proton with the particles of the thermal bath, the quantity $\Gamma_{acc}^{(\nu_e)}$ matches the corresponding decay rate $\Gamma_{in}^{(\nu_e)}$ evaluated in the inertial frame. In particular, in Sec. 3.2.2 it has been possible to exhibit that $\Gamma_i = \widetilde{\Gamma}_i$ for $i = 1, 2$, whereas Γ_{12} and $\widetilde{\Gamma}_{12}$ are equal to each other only up to a first-order expansion in the parameter $\delta m \equiv m_{\nu_2} - m_{\nu_1}$.

On the other hand, we have seen above that an additional contribution to the proton decay rate has to be considered, which in the inertial frame is represented by the process in Fig. 3.3. Guided by the principle of general covariance, we now seek the corresponding processes in the comoving frame which should lead to the same result. To this aim, we consider the three following contributions as potential candidates for the non-inertial counterpart of the decay (3.61):

$$(i)\ p^+ + e^- \to n + \nu_\mu, \quad (ii)\ p^+ + \overline{\nu}_\mu \to n + e^+, \quad (iii)\ p^+ + e^- + \overline{\nu}_\mu \to n, \tag{3.65}$$

which are depicted in Fig. 3.4. Note that, whilst the first process (3.65) is of the same type of (3.61) since it entails an oscillation of the emitted (electron) neutrino, the other processes (3.65) are essentially due to the oscillation of an (muon) antineutrino that is already present in the Unruh thermal bath.

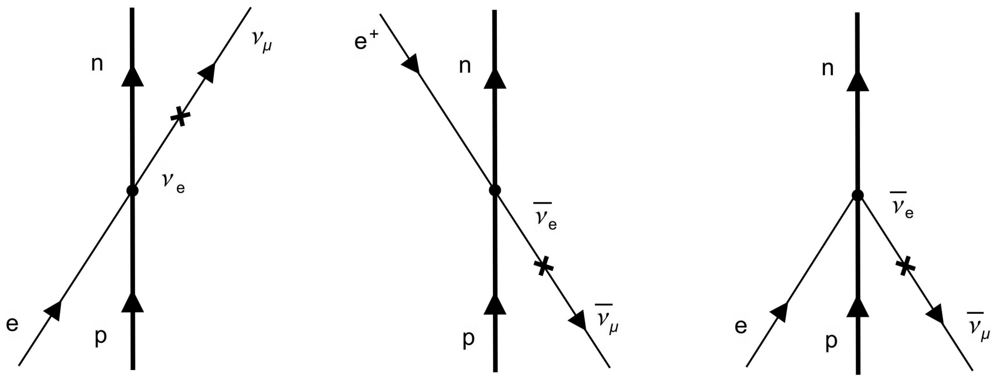

Figure 3.4: Decay processes in the accelerated frame. Oscillations of neutrinos in the Unruh thermal bath are considered in the last two diagrams.

In order to legitimate the validity of our assumption, we need to perform the same calculations that lead to the decay rate (3.51). The outcome of this procedure turns out to be

$$\Gamma_{acc}^{(\nu_\mu)} = \cos^2\theta\,\sin^2\theta\left(\widetilde{\Gamma}_1 + \widetilde{\Gamma}_2 - \widetilde{\Gamma}_{12}\right)\,. \tag{3.66}$$

By virtue of the aforesaid observations contained in detail in Sec. 3.2 which allow us to state that $\Gamma_{in}^{(\nu_e)} = \Gamma_{acc}^{(\nu_e)}$, it is possible to infer that such an equivalence holds also between the decay rates of (3.63) and (3.66).

Moreover, if we compute the total comoving decay rate which includes neutrino oscillations, we deduce that

$$\Gamma_{acc} = \Gamma_{acc}^{(\nu_e)} + \Gamma_{acc}^{(\nu_\mu)} = \cos^2\theta\,\widetilde{\Gamma}_1 + \sin^2\theta\,\widetilde{\Gamma}_2\,. \tag{3.67}$$

By comparing this with the total inertial decay rate (3.64), we find that

$$\Gamma_{in} = \Gamma_{acc}\,, \tag{3.68}$$

which means that such a result does not depend on the quantities Γ_{12} and $\widetilde{\Gamma}_{12}$, whose treatment would require additional computational effort [123], as we have already shown before.

Remarkably, Eq. (3.68) not only involves a generalization of the analysis of the accelerated proton decay to the case in which the produced neutrino oscillates, but it also unambiguously corroborates our guess of selecting the processes (3.65) as the counterpart for the decay (3.6) in the inertial frame. Hence, the requirement of the principle of general covariance clearly results in the necessity of having an Unruh thermal bath containing flavor neutrinos which do oscillate.

This represents another achievement which has been obtained by resorting to the underlying general covariance of QFTCS. Although at first it may appear predictable, it is by no means evident that particles should retain their properties when they are taken to be the constituents of the Unruh radiation. Therefore, the current description is the first proof along such an intriguing path, which can potentially lead to considerable theoretical attainments. As a matter of fact, the literature based on the ideas discussed in this Chapter has had a remarkable impact on the scientific community.

A final interesting observation that can be deduced from our analysis is related to the identities (3.64) and (3.67) that are true for the inertial and the comoving frame, respectively. For this purpose, we recall that the decay rates appearing in the aforementioned equations have been computed by employing neutrino flavor states as asymptotic states. However, we note that similar relations also hold for the quantities $\Gamma^{(\nu_1)}$ and $\Gamma^{(\nu_2)}$ calculated in Ref. [124] using neutrino mass eigenstates as fundamental objects. We then have

$$\Gamma^{(\nu_1)} + \Gamma^{(\nu_2)} = \Gamma^{(\nu_e)} + \Gamma^{(\nu_\mu)}\,, \tag{3.69}$$

where $\Gamma^{(\nu_1)}$ and $\Gamma^{(\nu_2)}$ are inclusive of the elements of Pontecorvo matrix. The above equality has to be regarded both in the inertial and the comoving frames. Such an equation constitutes a consistency check for the correctness of the calculations in Refs. [123] and [124]. The physical meaning of (3.69) can be understood by considering the charges for mixed neutrino fields as derived from Noether's theorem [129]. Indeed, by denoting with

$$Q_i = \int d^3x\, \Psi^\dagger_{\nu_i}(x)\Psi_{\nu_i}(x)\,, \quad i = 1, 2, \tag{3.70}$$

the conserved charges for the neutrino fields with definite masses and with

$$Q_\alpha(t) = \int d^3x\, \Psi^\dagger_{\nu_\alpha}(x)\Psi_{\nu_\alpha}(x)\,, \quad \alpha = e, \mu, \tag{3.71}$$

the (time-dependent) flavor charges, one can see that

$$Q = \sum_i Q_i = \sum_\alpha Q_\alpha(t)\,, \tag{3.72}$$

where Q represents the total charge [129]. The above relation can be interpreted as the conservation of the total lepton number. On the one hand, this can be viewed as the sum of two separately conserved family lepton numbers, when no mixing is present; on the other hand, the same conserved number is obtained by the sum of non-conserved flavor charges, which are associated to oscillations.

Concluding remarks

In the present Chapter, we have discussed the decay of uniformly accelerated protons. Following the line of reasoning of Refs. [118, 121], we have reviewed the calculation of the total decay rate both in the laboratory and comoving frame in Sec. 3.1, highlighting the incompatibility between the two results when taking into account neutrino flavor mixing [122]. Such an inconsistency would not be striking if the underlying theory were not generally covariant, but this is not the case, since the fundamental ingredients for analyzing the process, namely the SM and QFT in curved space-time, are by construction generally covariant. On the other hand, the authors of Ref. [122] argue their result claiming that mixed neutrinos are not representations of the Lorentz group with a well-defined invariant p^2, and that the mathematical origin of the disagreement arises from the noncommutativity of weak and energy-momentum currents. Furthermore, they propose the experimental investigation as the only way to resolve such a controversial issue. Even assuming there are no flaws in this reasoning, we believe the last statement to be basically incorrect: an experiment, indeed, should not be used as a tool for checking the internal consistency of theory against a theoretical paradox.

Led by these considerations, we have thus revised calculations of Ref. [122] modifying some of the key assumptions of that work. In particular, in Sec. 3.2 we have required the asymptotic neutrino states to be flavor rather than mass eigenstates. Within this framework, by comparing the obtained expressions for the two decay rates, it has been shown that they would coincide [123], provided that the off-diagonal terms (3.39) and (3.53) are equal to each other. In order to check whether this is the case, we have performed the reasonable approximation of small neutrino mass difference, pushing our analysis up to the first order in $\delta m/m_\nu$. However, due to computational difficulties, the further assumption of vanishing neutrino mass $m_{\nu_1} \to 0$ has proved to be necessary for getting information about these terms. In such a regime, we have found that (3.47) and (3.60) are perfectly in agreement, thus removing the aforementioned ambiguity at a purely theoretical level.

Subsequently, in Sec. 3.3 we have seen that the above technical difficulties can be solved by extending the study of the inverse β-decay to the case in which neutrino oscillations are taken into account. Once again, on the basis of the requirement of general covariance of QFT, we have shown that the Unruh radiation "seen" by the accelerated proton must necessarily be made up of oscillating neutrinos [123]. This is a novel feature which had been surprisingly neglected in the previous literature on this topic but that should emerge in a very natural way.

As a final remark, we stress that in the current work we have made use of the simplest framework of neutrino mixing among two generations. The extension to three flavors is in principle straightforward and represents one of the future directions of our investigation. We envisage that the presence of CP violation may introduce interesting additional features which would enrich the non-trivial structure of the Unruh radiation.

Chapter 4

An insight on equivalence principle violation

In light of the arguments contained in Sec. 1.2, we are aware that the equivalence principle has played a crucial role for the development of GR. Together with general covariance, EP has led Einstein to the realization of one of the most elegant and experimentally successful theoretical apparatuses. Nevertheless, GR is still not capable of describing all observed phenomena and acquired data related to gravity, thus leaving an opening for novel physical models. Inevitably, most of such gravitational theories try to overcome the aforesaid shortcomings by relaxing one or more hypotheses on which GR is firmly grounded. If this is the case for EP, a series of examples in this direction can be found in Sec. 2.3, where we have seen that several quadratic models of gravity allow for the emergence of a measurable quantity related to SEP violation in the context of the Casimir effect.

Apart from the previous scenario, there is also another remark that must be done, and it concerns the interplay between GR and QM. Indeed, it is by no means obvious that such principles as EP which hold at macroscopic scales should be regarded as fundamental also in the quantum domain. For instance, a noteworthy elaboration that tackles a similar topic is Ref. [130], where the EEP is questioned for quantum systems. In this perspective, we stress that many theoretical and experimental works have identified neutrinos with a potential probe to test the validity of EP, in particular WEP [131]. Therefore, it appears that EP formulation should be properly modified to fit in a quantum mechanical framework.

Furthermore, in connection with WEP, if the analysis of a given physical system is performed at finite temperature T, we have also to account for the "thermal" contribution to the inertial and gravitational masses separately. In fact, in principle it is impossible to state whether the two types of mass would receive the same amount of energy coming from the interaction with the thermal bath or not. Actually, a thorough QFT investigation can prove that for $T \neq 0$ WEP is explicitly violated; this has been demonstrated for the first time in Refs. [132] by studying an electron in equilibrium with a photon heat bath.

Since we have already treated the case of SEP violation in connection with quadratic theories of gravity, in the current Chapter we will be mainly interested in presenting the ways in which WEP violation is viable. For this purpose, we essentially deepen two aspects that have already been mentioned in the above paragraphs. In particular:

- inspired by Ref. [130], we will resort to a novel derivation of the non-relativistic limit for flavor neutrinos showing that it naturally leads to WEP violation;
- we will follow the reasoning of Refs. [132] by relying on a simplified approach that can be found in Ref. [133]. We then apply the same method to the case of Brans-Dicke theory [134].

4.1 Non-relativistic neutrinos and WEP violation

The subject of non-relativistic neutrinos is per se an intriguing one. For instance, neutrinos that constitute the so-called cosmic neutrino background (CNB), also known as relic neutrinos, may open new scenarios in our understanding of the early universe [135]; in fact, it is estimated that the CNB decoupled from matter few seconds after the Big Bang [136]. In this sense, the CNB contains more information on the primordial characteristics of the universe than the photon-based cosmic microwave background radiation. Since the temperature of the CNB is estimated [136] to be $T \simeq 2K$, it is reasonable to think of relic neutrinos as non-relativistic particles.

Bearing these concepts in mind, we start our analysis from the non-relativistic limit of the Dirac equation for flavor neutrinos. A similar choice finds its justification in Sec. 3.3, in which we have seen an interesting example that shows why flavor basis better fits the description of neutrino phenomenology. Along this line, there is a vast literature that supports the employment of flavor basis as the fundamental description for neutrinos (see for instance Refs. [129, 137]), but this issue will not be tackled in the current essay.

4.1.1 Non-relativistic neutrinos without external field

Let us consider the Dirac equation associated with flavor neutrinos ν_e and ν_μ. In the simplest case of a two-flavor model and no external field, it reads

$$(i\gamma^\alpha\partial_\alpha \,-\, \mathbb{M})\,\Psi \,=\, 0\,. \tag{4.1}$$

Here, γ^α is implicitly meant to be the 8×8 matrix $\mathbb{I}_{2\times 2}\otimes\gamma^\alpha$ and $\mathbb{M}$ is the 8×8 (non-diagonal) mass matrix, which in the 4×4 block formalism reads

$$\mathbb{M} \,=\, \begin{pmatrix} m_e & m_{e\mu} \\ m_{e\mu} & m_\mu \end{pmatrix}. \tag{4.2}$$

The wave-function Ψ contains the bispinors related both to ν_e and ν_μ

$$\Psi = \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix}. \tag{4.3}$$

If we explicitly write the two Dirac equations, we get

$$(i\gamma^\alpha\partial_\alpha - m_e)\,\psi_e \;=\; m_{e\mu}\psi_\mu\,, \tag{4.4}$$

$$(i\gamma^\alpha\partial_\alpha - m_\mu)\,\psi_\mu \;=\; m_{e\mu}\psi_e\,. \tag{4.5}$$

Unless stated otherwise, we will focus only on (4.4), since the ensuing results for the muon neutrino are easily obtained by exchanging the subscripts $e\leftrightarrow\mu$. In addition, with foresight of a non-relativistic treatment of (4.1) we will employ the standard Dirac representation of γ matrices. Consequently, the positive-energy wave functions satisfy algebraic equations

$$\begin{aligned} (i\partial_0 - m_e)\,\varphi_e \,+\, i\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\chi_e \;&=\; m_{e\mu}\varphi_\mu\,, \\ -i\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\varphi_e \,-\, (i\partial_0 + m_e)\,\chi_e \;&=\; m_{e\mu}\chi_\mu\,. \end{aligned} \tag{4.6}$$

Here, $\varphi_{e,\mu}$ and $\chi_{e,\mu}$ denote the "large" (upper) and "small" (lower) spin components of respective bispinors. At this point, we can perform the non-relativistic limit, by assuming that the dominant contribution to the energy comes from the rest mass. Hence, in (4.6) we can assume the kinetic energy to be much smaller than the rest mass. One can thus pull out from the bispinor the fast oscillating factor $e^{-im_\sigma t}$ (for the positive energy solutions), so that

$$\psi_\sigma \;=\; e^{-im_\sigma t}\widetilde{\psi}_\sigma\,, \qquad \sigma \;=\; e,\mu\,, \tag{4.7}$$

with the field $\widetilde{\psi}_\sigma$ oscillating much slower than $e^{-im_\sigma t}$ in time. Then, one drops the term $\partial_0\widetilde{\psi}_\sigma$ as small compared to $-2im_\sigma\widetilde{\psi}_\sigma$ (more specifically, one assumes that $|\partial_0\widetilde{\psi}_\sigma| \ll |2m_\sigma\widetilde{\psi}_\sigma|$). In light of this, Eqs. (4.6) reduce to

$$i\partial_0\widetilde{\varphi}_e \;+\; i\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\widetilde{\chi}_e \;=\; m_{e\mu}e^{i(m_e-m_\mu)t}\widetilde{\varphi}_\mu\,, \tag{4.8}$$

$$-i\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\widetilde{\varphi}_e \;-\; 2m_e\widetilde{\chi}_e \;=\; m_{e\mu}e^{i(m_e-m_\mu)t}\widetilde{\chi}_\mu\,. \tag{4.9}$$

Analogous relations hold for ν_μ. In what follows, for notational simplicity we remove the tilde from the components of Dirac bispinors, but it should not be forgotten that henceforth we deal with non-relativistic quantities.

Note that, in the usual non-mixing case, the small spin component is much smaller than the large one. In presence of mixing, however, the small component χ_μ can be in the non-relativistic limit of the same order as φ_e provided $m_{e\mu}$ is sufficiently small (namely, when $m_{e\mu} \approx |\boldsymbol{\sigma}\cdot\boldsymbol{p}| = |\boldsymbol{p}|$). This should be contrasted with the ultra-relativistic limit, where for small $m_{e\mu}$ the component φ_e can substantially dominate over χ_μ. We shall see shortly that these observations have interesting and non-trivial implications.

Let us now plug χ_e in the expression for φ_e. We get

$$i\partial_0\,\varphi_e \;=\; -\frac{\nabla^2}{2m_e}\,\varphi_e + e^{i(m_e-m_\mu)t}\left[m_{e\mu}\varphi_\mu + \frac{i\,m_{e\mu}}{2m_e}\,(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\;\chi_\mu\right]. \tag{4.10}$$

As expected, the first term on the r.h.s. of (4.10) represents the kinetic part, whereas the information about mixing is imprinted in two remaining terms.

One can push the above analysis beyond (4.10) by employing the ensuing non-relativistic relation for χ_μ stemming from (4.5). Indeed, by using the fact that

$$\chi_\mu \;=\; -\frac{i\,\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}}{2m_\mu}\,\varphi_\mu \;-\; e^{i(m_\mu-m_e)t}\,\frac{m_{e\mu}}{2m_\mu}\,\chi_e\,,$$

and inserting it into (4.10), we obtain

$$i\partial_0\,\varphi_e = -\frac{\nabla^2}{2m_e}\varphi_e + e^{i(m_e-m_\mu)t}\left[m_{e\mu}\varphi_\mu + \frac{m_{e\mu}}{2m_e}\frac{\nabla^2}{2m_\mu}\varphi_\mu\right] - \frac{i\,m_{e\mu}^2}{4m_em_\mu}\,(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\,\chi_e\,. \tag{4.11}$$

It is clear that we can continue this iteration procedure indefinitely. If the corresponding infinite sum converges, we can get rid of the small spin components in both ψ_e and ψ_μ and obtain two coupled field equations for φ_e and φ_μ only — as it could be expect from the non-relativistic limit, where only (equal parity) large bispinor components (Pauli spinors) appear.

The aforesaid iterative procedure brings (4.11) to the form

$$i\partial_0\,\varphi_e \;=\; -\,A(\mathbb{M})\frac{\nabla^2}{2m_e}\,\varphi_e\;+e^{i(m_e-m_\mu)t}\,B(\mathbb{M})\,\varphi_\mu\,, \tag{4.12}$$

where

$$A(\mathbb{M}) \;=\; \sum_{n=0}^{\infty}\left(\frac{m_{e\mu}^2}{4m_e m_\mu}\right)^n, \tag{4.13}$$

and

$$B(\mathbb{M}) \;=\; m_{e\mu}\;+\;\frac{m_{e\mu}}{2m_e}\,A(\mathbb{M})\,\frac{\nabla^2}{2m_\mu}\,. \tag{4.14}$$

Since for two flavors the relations between m_e, m_μ, $m_{e\mu}$ and the mass parameters m_1 and m_2 are known to be [138]

$$\begin{aligned} m_e &= m_1\cos^2\theta\;+\;m_2\sin^2\theta\,,\\ m_\mu &= m_1\sin^2\theta\;+\;m_2\cos^2\theta\,,\\ m_{e\mu} &= (m_2-m_1)\sin\theta\cos\theta\,, \end{aligned} \tag{4.15}$$

one might easily check that $m_{e\mu}^2 < m_e m_\mu$. For future convenience, let us denote the expansion parameter $m_{e\mu}^2/(4m_e m_\mu)$ as ω. Because $\omega<1$, the geometric series $A(\mathbb{M})$ converges and it sums up to

$$A(\mathbb{M}) \;=\; \frac{1}{1-\omega}\,. \tag{4.16}$$

With this, we obtain the equation for the Pauli spinors (large bispinor components) in the Schrödinger form

$$i\partial_0\varphi_e = -\left(\frac{1}{1-\omega}\right)\frac{\nabla^2}{2m_e}\,\varphi_e\;+\;m_{e\mu}\,e^{i(m_e-m_\mu)t}\left\{1+\frac{\nabla^2}{4m_e m_\mu\,(1-\omega)}\right\}\varphi_\mu\,. \tag{4.17}$$

Equation (4.17) is the sought non-relativistic limit of the Dirac equation for an electron neutrino. As already stressed, when we exchange $e\leftrightarrow\mu$ we obtain the corresponding equation for φ_μ.

By looking at the formula (4.17), we can immediately draw two important conclusions. First, in order to have the standard kinetic contribution in (4.17), the would-be inertial mass m_e should be modified. In fact, we should require that the inertial mass is $m_e^{\text{eff}} = m_e\,(1-\omega)$. A similar redefinition must be performed also for

m_μ. The existence of $m_e^{\text{eff}} \neq m_e$ might be at first surprising, since it is not clear why mixing should affect the inertial masses related to flavor states. In this connection, it is worth noting that the presence of the correction term $A(\mathbb{M})$ is due to the fact that Dirac equation (4.6) simultaneously deals with large and small bispinor components (φ_e and χ_μ), that in the case of mixing are comparably important. In fact, to reach (4.17) one has to work interchangeably with small and large components because these are interlocked at all energy scales. Should the same analysis were performed with the Klein–Gordon equation for mixed fields (i.e., the ones describing mixed composite particles with spin 0, such as K^0, D^0 or B^0 mesons [139]), an analogous redefinition of the inertial mass would be found.

Second, the part related to φ_μ characterizes the oscillation phenomenon. Note that the factor inside $\{\ldots\}$ in (4.17) is the same as for the φ_μ apart for the time-dependent phase factor. When $\{\ldots\}$ were zero (i.e. when $m_{e\mu} = 0$), these two equations would just describe two uncoupled equations for free electron and muon neutrinos, with masses $m_e = m_1$ and $m_\mu = m_2$, respectively. However, there is coupling between the two flavor neutrinos by the amplitude $\{\ldots\}$, thus implying that there may be "leakage" from one flavor to the other. This is nothing but the "flip-flop" amplitude of a two-state system [47]. Note also that its modulus is manifestly invariant under the exchange of flavors $e \leftrightarrow \mu$ which reflects *detailed balance* of the oscillation phenomena.

4.1.2 Non-relativistic neutrinos in gravitational field

Let us now focus the attention on what happens if we switch a gravitational potential on. It is not a priori evident that the effective inertial masses m_e^{eff} and m_μ^{eff} will also couple to the gravitational potential. To explore this point, we will restrict our attention to a metric in the post-Newtonian approximation that goes up to the order $\mathcal{O}\left(c^{-2}\right)$. Moreover, without loss of generality, we will consider the isotropic reference frame, so that for the gravitational potential we consider $\phi\left(\vec{x}\right) \equiv \phi\left(|\vec{x}|\right)$. The ensuing line element reads [130]

$$ds^2 = \left(1 + 2\,\phi\,\right) dt^2 \; - \left(1 - 2\,\phi\right) \left(dx^2 + dy^2 + dz^2\right) \, . \tag{4.18}$$

In order to couple gravity with the Dirac equation (4.1), we use the conventional spin connection formalism. In particular, we should substitute Feynman's Dirac operator $\partial\!\!\!/$ with $\gamma^\mu \tilde{\nabla}_\mu$, where $\gamma^\mu = e^\mu_{\hat{a}} \gamma^{\hat{a}}$ and $\tilde{\nabla}_\mu = \partial_\mu + \Gamma_\mu$, as seen in Sec. 3.1.2, see (3.21). Γ_μ is the Fock–Kondratenko connection

$$\Gamma_\mu \;=\; -\frac{i}{4}\,\sigma^{\hat{a}\hat{b}}\,\omega_{\mu\hat{a}\hat{b}} \;=\; \frac{1}{8}\left[\gamma^{\hat{a}},\gamma^{\hat{b}}\right] e^{\lambda}_{\hat{a}}\nabla_\mu e_{\hat{b}\lambda}\,. \tag{4.19}$$

Here, $\sigma^{\hat{a}\hat{b}} = i/2\left[\gamma^{\hat{a}},\gamma^{\hat{b}}\right]$ are generators of the bi-spinorial representation of Lorentz group, $\omega_{\mu\hat{a}\hat{b}} = e^{\lambda}_{\hat{a}}\nabla_\mu e_{\hat{b}\lambda}$ are the spin connection components, $\gamma^{\hat{a}}$ represent the gamma matrices in flat spacetime, ∇_μ is the usual covariant derivative and $e^{\mu}_{\hat{a}}$ is the vierbein field.

Because in our case both $g_{\mu\nu}$ and $\eta_{\hat{a}\hat{b}}$ are diagonal, it is simple to evaluate the non-vanishing components of the vierbein fields. By using the relation

$$g^{\mu\nu} \;=\; e^{\mu}_{\hat{a}}\,e^{\nu}_{\hat{b}}\,\eta^{\hat{a}\hat{b}}\,,$$

which is the same one introduced in (1.4), we obtain

$$e^{0}_{\hat{0}} \;=\; 1-\phi\,, \qquad e^{x}_{\hat{x}} \;=\; e^{y}_{\hat{y}} \;=\; e^{z}_{\hat{z}} \;=\; 1+\phi\,, \tag{4.20}$$

and the ensuing Fock–Kondratenko connection

$$\Gamma_\mu \;=\; \frac{1}{8}\left[\gamma^{\hat{a}},\gamma^{\hat{b}}\right] e^{\lambda}_{\hat{a}}\big(\eta_{\mu\lambda}\partial_\rho\phi - \eta_{\mu\rho}\partial_\lambda\phi\big)e^{\rho}_{\hat{b}}\,. \tag{4.21}$$

Let us discuss the modifications of (4.17) that are induced by the presence of a weak gravitational field. Using the fact that (4.1) is now replaced by

$$\Big(i\gamma^\alpha\tilde{\nabla}_\alpha - M\Big)\,\Psi \;=\; 0\,, \tag{4.22}$$

we obtain the equations for electron neutrino in the form

$$\begin{aligned}(i\partial_0 - m_e - i\phi\,\partial_0)\,\varphi_e + i(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\,\chi_e \;&=\; m_{e\mu}\varphi_\mu\,,\\ -i(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\,\varphi_e - (i\partial_0 + m_e - i\phi\,\partial_0)\,\chi_e \;&=\; m_{e\mu}\chi_\mu\,.\end{aligned} \tag{4.23}$$

The assumption at the basis of (4.23) is that we consider only a weak gravitation field, and hence the gravitational potential is slowly varying (as on the Earth surface). In particular, we consider that $\partial_i\phi \approx 0$, $\forall i$, and so ϕ enters in (4.23) only via vierbeins in γ_α matrices. Despite this, the subsequent conclusions would still remain qualitatively correct.

At this point, we can take the non-relativistic limit in (4.23). This yields

$$
\begin{aligned}
i\partial_0\varphi_e &= m_e\phi\,\varphi_e \,+\, e^{i(m_e-m_\mu)t}\, m_{e\mu}\varphi_\mu \,-\, i\left(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\right)\chi_e\,, \\
\chi_e &= -\frac{i\,\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}}{2m_e}\varphi_e \,-\, e^{i(m_e-m_\mu)t}\,\frac{m_{e\mu}}{2m_e}\chi_\mu\,.
\end{aligned}
\tag{4.24}
$$

By following the same procedure which we already adopted in the previous Section, one arrives at the non-relativistic Dirac equation in the presence of a weak gravitational field in the form

$$
i\partial_0\varphi_e = \left(-\frac{\nabla^2}{2m_e^{\mathrm{eff}}} + m_e\,\phi\right)\varphi_e + e^{i(m_e-m_\mu)t}\left[\frac{m_{e\mu}}{2m_e}\left(2m_e + \frac{\nabla^2}{2m_\mu^{\mathrm{eff}}}\right)\right]\varphi_\mu\,. \tag{4.25}
$$

As expected, for the electron neutrino we recover the sum of the kinetic and the potential contribution, but also the same "flip-flop" amplitude as in (4.17) (at least in the lowest non-trivial order considered here). However, note that whilst the inertial mass undergoes the same redefinition as in the free-field case (4.17), the gravitational mass remains m_e. This might be seen as a straightforward example of the violation of WEP for flavor neutrinos, since $m^{\mathrm{eff}} = m_{\mathrm{i}} \neq m_{\mathrm{g}}$.

Let us finally stress that, should we have performed an analogous treatment in the mass basis, we would not have found any distinction between inertial and gravitational masses. This holds true because electron and muon state vectors are in such a case completely decoupled and the absence of off-diagonal mass terms leads to $m_j^{\mathrm{eff}} = m_{j\mathrm{i}} = m_{j\mathrm{g}}$ with $j = 1, 2$. Such an unexpected occurrence is the starting point for a deeper investigation concerning the nature of neutrinos, but for the present goal it can be neglected.

To summarize, in the non-relativistic regime we have seen how everything works as if mixing can be reinterpreted as a sort of "interaction" which does not contribute with the same weight to the inertial and the gravitational mass. It can be proven that the same happenstance is encountered when dealing with a quantum system at finite temperature, in conjunction with what has been done in Refs. [132]. This subject will be the central topic of the next Section [134].

4.2 WEP violation at finite temperature

As already anticipated at the beginning of the Chapter, two approaches can be used to prove that WEP is violated at finite temperature: the first one involves a thorough QFT analysis, whereas the second one is based on a modification of the geodesic equation which accounts for the fact that $T \neq 0$. The former will be briefly sketched; for further details, the interested reader can consult Refs. [132]. On the other hand, the latter will be better developed, since it will be applied also to the case of Brans-Dicke model.

4.2.1 WEP violation via quantum field theory

There exists an extremely elegant way to treat EP violation in a QFT and GR framework [132] (for modified gravity, see for example Refs. [140] and Ref. [72] for the generalized uncertainty principle). The system we want to study consists of an electron with mass m_0 (the renormalized mass of the particle when the temperature is zero) in thermal equilibrium with a photon heat bath. The aim of the analysis is the evaluation of electron's gravitational and inertial mass in the low-temperature limit (namely, $T \ll m_0$). We remark that the presence of a non-zero temperature is fundamental, since calculations clearly show that $m_g = m_i$ for $T = 0$.

The gravitational and inertial masses are derived by adopting a Foldy–Wouthuysen transformation [141] on the Dirac equation. Such a procedure gives the opportunity to study the non-relativistic limit of particles with spin-1/2 (i.e. electrons). In other words, it is possible to derive a Schrödinger equation in which the expression for the mass is easily recognizable.

In order to find a proper expression for m_i, one can imagine to switch an electric field on, so that the Dirac equation which includes the electromagnetic interaction turns out to be [132]

$$\left(\not{p} - m_0 - \frac{\alpha}{4\pi^2}\not{I}\right)\psi = e\Upsilon_\mu A^\mu \psi. \tag{4.26}$$

In (4.26), α is the fine-structure constant, as usual $\not{p} = \gamma^\mu p_\mu$, with γ^μ being the Dirac matrices, A^μ is the electromagnetic four-potential, namely $A^\mu = (\phi, \mathbf{A})$, where ϕ is the scalar potential and $\mathbf{A}$ the vector potential and the quantity I_μ is defined as

$$I_\mu = 2\int d^3k \frac{n_B(k)}{k_0}\frac{k_\mu}{\omega_p k_0 - \mathbf{p}\cdot\mathbf{k}}, \tag{4.27}$$

with $k_\mu = (k_0, \mathbf{k})$ and where ω_p and $\mathbf{p}$ are connected by

$$\omega_p = \sqrt{m_0^2 + |\mathbf{p}|^2}. \tag{4.28}$$

In (4.27), $n_B(k)$ represents the Bose-Einstein distribution

$$n_B(k) = \frac{1}{e^{\beta k} - 1}, \tag{4.29}$$

where $\beta = 1/k_B T$, with k_B being the Boltzmann constant. Finally, Υ_μ is

$$\Upsilon_\mu = \gamma_\mu \left(1 - \frac{\alpha}{4\pi^2}\frac{I_0}{E}\right) + \frac{\alpha}{4\pi^2} I_\mu . \tag{4.30}$$

At this point, a Foldy–Wouthuysen transformation converts (4.26) into a Schrödinger equation, which reads

$$i\frac{\partial \psi_s}{\partial t} = \left[m_0 + \frac{\alpha\pi T^2}{3m_0} + \frac{|\mathbf{p}|^2}{2\left(m_0 + \frac{\alpha\pi T^2}{3m_0}\right)} + e\phi + \frac{\mathbf{p}\cdot\mathbf{A} + \mathbf{A}\cdot\mathbf{p}}{2\left(m_0 + \frac{\alpha\pi T^2}{3m_0}\right)} + \dots \right]\psi_s, \tag{4.31}$$

from which one extracts the inertial mass

$$m_i = m_0 + \frac{\alpha\pi T^2}{3m_0}. \tag{4.32}$$

We immediately notice that the difference between the inertial mass of an electron at finite temperature and m_0 is due exclusively to the thermal radiative correction of (4.32).

An analogous reasoning can be performed also for the gravitational mass, which can be derived in the same way, but starting from a different Dirac equation that takes into account the gravitational interaction. In a similar circumstance, one can write [132]

$$\left(\not{p} - m_0 - \frac{\alpha}{4\pi^2}\not{I}\right)\psi = \frac{1}{2} h_{\mu\nu}\tau^{\mu\nu}\psi, \tag{4.33}$$

where a weak gravitational field is considered (the fluctuation with respect to the background metric are defined by $h_{\mu\nu}$) and with $\tau^{\mu\nu}$ being the renormalized stress-energy tensor. In the previous expression, following Refs. [132], it is assumed $h_{\mu\nu} = 2\,\phi_g\,\mathrm{diag}\,(1,1,1,1)$, where ϕ_g is a gravitational potential.

Once again, a Foldy–Wouthuysen transformation yields another Schrödinger equation

$$i\frac{\partial \psi_s}{\partial t} = \left[m_0 + \frac{\alpha\pi T^2}{3m_0} + \frac{|\mathbf{p}|^2}{2\left(m_0 + \frac{\alpha\pi T^2}{3m_0}\right)} + \left(m_0 - \frac{\alpha\pi T^2}{3m_0}\right)\phi_g \right]\psi_s, \tag{4.34}$$

from which the identification of the gravitational mass is an easy task

$$m_g = \left(m_0 - \frac{\alpha\pi T^2}{3m_0}\right). \tag{4.35}$$

Such an outcome implies that there is no difference at all between m_g and m_i at zero temperature, because they both equal the renormalized mass. This means that only radiative corrections render the violation of the equivalence principle feasible.

At this point, it is straightforward to check that (4.32) and (4.35) entail

$$\frac{m_g}{m_i} = 1 - \frac{2\alpha\pi T^2}{3m_0^2}, \tag{4.36}$$

in the first-order approximation in T^2, legitimated by the choice of evaluating the low-temperature limit of the analyzed quantum system. The last expression is a direct consequence of the fact that Lorentz invariance of the finite-temperature vacuum is broken, which means that it is possible to define an absolute motion through the vacuum (i.e. the one at rest with the heat bath).

Equation (4.36) is the core of our argumentation that will be developed below. The central result is the violation of EP, achievable by means of QFT at finite temperature. Regarding this point, the question arises whether T can be inserted into GR with the aim to reproduce the same outcome bypassing radiative correction computations. If this proposal is viable, it should be possible to develop similar calculations for several physical frames describing different spacetimes.

4.2.2 WEP violation via modified geodesic equation

The opportunity to check whether (4.36) can be derived involving exclusively GR properties is the goal of this Section. The derivation exhibited below closely follows the original one contained in Ref. [133].

Let us then study the aforementioned procedure to reach (4.36) once again, but from a different path. The starting point is the analysis of a charged test particle of renormalized mass at zero temperature m_0 in thermal equilibrium with a photon heat bath in the low-temperature limit $T \ll m_0$. Hence, the dispersion relation is modified by an additional term [132]

$$E = \sqrt{m_0^2 + |\mathbf{p}|^2 + \frac{2}{3}\alpha\pi T^2}, \tag{4.37}$$

which can be easily identified with the first-order correction in T^2 that descends from the finite-temperature analysis.

Now, let us introduce the stress-energy tensor $T^{\mu\nu}$ related to the test particle, whose world line can be contained in a narrow "world tube" in which $T^{\mu\nu}$ is non-vanishing. The conservation equation for the stress-energy tensor can be integrated over a three-dimensional hypersurface Σ and defined as

$$\int_{\Sigma} d^3x' \sqrt{-g}\, T^{\mu\nu}(x') = \frac{p^\mu p^\nu}{E}, \tag{4.38}$$

where p^μ is the four-momentum and $E = p^0$ the energy given by

$$E = \int_{\Sigma} d^3x' \sqrt{-g}\, T^{00}(x'). \tag{4.39}$$

These equations hold in the limit for the world tube radius going to zero [142].

A more accurate study [132] gives the term that should be viewed as the source of gravity at finite temperature and in weak-field approximation, which in the rest frame of the heat bath turns out to be

$$\Xi^{\mu\nu} = T^{\mu\nu} - \frac{2}{3}\alpha\pi \frac{T^2}{E^2}\delta_0^\mu \delta_0^\nu T^{00}, \tag{4.40}$$

where $\Xi^{\mu\nu}$ contains not only the information on the Einstein tensor $G^{\mu\nu}$, but also thermal corrections to it.

Equation (4.40) is explicitly derived after the choice of the privileged reference frame at rest with the heat bath, and this fact produces a Lorentz invariance violation of the finite-temperature vacuum. In fact, in the flat tangent space, one cannot consider a Minkowski vacuum anymore, since it is substituted by a thermal bath. For this reason, Lorentz group is no longer the symmetry group of the local tangent space to the Riemannian manifold, even though general covariance still holds there (recall the arguments of Sec. 1.1). The last consideration allows us to proceed with the awareness that the current situation is slightly different from the usual GR scheme.

However, since the analyzed case deals with weak-field approximation and quadratic thermal corrections in low-temperature limit, the generalization of (4.40) to curved spacetime can be

$$\Xi^{\mu\nu} = T^{\mu\nu} - \frac{2}{3}\alpha\pi \frac{T^2}{E^2} e_{\hat{0}}^\mu e_{\hat{0}}^\nu T^{\hat{0}\hat{0}}, \tag{4.41}$$

where $e_{\hat{0}}^\mu$ denotes the vierbein field.

Another fundamental assumption has be made to proceed further: effects of temperature on geometry have not to be taken into account [133]. If the last assertion holds, it is possible to write the Einstein field equations as

$$G^{\mu\nu} = \Xi^{\mu\nu}, \tag{4.42}$$

otherwise several contributions would arise from a relativistic investigation on temperature (because also T would have an influence on spacetime structure), but for our purposes they can be safely neglected.

If now we employ the Bianchi identity (namely, $\nabla_\nu G^{\mu\nu}=0$), it is straightforward to check that

$$\nabla_\nu T^{\mu\nu} = \nabla_\nu \left(\frac{2}{3}\alpha\pi\frac{T^2}{E^2}e^\mu_{\hat{0}}e^\nu_{\hat{0}}T^{\hat{0}\hat{0}}\right), \tag{4.43}$$

which can be rewritten as

$$\begin{aligned}\partial_\nu\left(\sqrt{-g}\,T^{\mu\nu}\right)+\Gamma^\mu_{\nu\alpha}\sqrt{-g}\,T^{\alpha\nu} &= \partial_\nu\left(\sqrt{-g}\,\frac{2}{3}\alpha\pi\frac{T^2}{E^2}e^\mu_{\hat{0}}e^\nu_{\hat{0}}T^{\hat{0}\hat{0}}\right)\\ &+ \frac{2}{3}\alpha\pi\Gamma^\mu_{\nu\alpha}\sqrt{-g}\,\frac{T^2}{E^2}e^\mu_{\hat{0}}e^\nu_{\hat{0}}T^{\hat{0}\hat{0}}.\end{aligned} \tag{4.44}$$

By denoting $\dot{x}^\mu \equiv dx^\mu/ds$, it can be shown [133] that (4.44) is equal to

$$\ddot{x}^\mu+\Gamma^\mu_{\alpha\nu}\dot{x}^\alpha\dot{x}^\nu = \frac{d}{ds}\left(\frac{2}{3}\alpha\pi\frac{T^2}{mE}e^\mu_{\hat{0}}\right)+\frac{2}{3}\alpha\pi\frac{T^2}{m^2}\Gamma^\mu_{\alpha\nu}e^\alpha_{\hat{0}}e^\nu_{\hat{0}}, \tag{4.45}$$

which can be cast into another form by using the fact that

$$E = m\dot{x}^{\hat{0}} = m\dot{x}^\rho e^{\hat{0}}_\rho. \tag{4.46}$$

This substitution finally gives

$$\ddot{x}^\mu+\Gamma^\mu_{\alpha\nu}\dot{x}^\alpha\dot{x}^\nu = \frac{2}{3}\alpha\pi T^2\left[\frac{\dot{x}^\nu\partial_\nu e^\mu_{\hat{0}}}{mE}-\frac{e^\mu_{\hat{0}}\left(\ddot{x}^\nu e^{\hat{0}}_\nu+\dot{x}^\nu\dot{x}^\beta\partial_\beta e^{\hat{0}}_\nu\right)}{E^2}+\frac{\Gamma^\mu_{\alpha\nu}e^\alpha_{\hat{0}}e^\nu_{\hat{0}}}{m^2}\right]. \tag{4.47}$$

Equation (4.47) represents a generalization of the geodesic equation to the case in which the temperature is non-vanishing.

Application to Schwarzschild solution

We are ready to analyze (4.47) in the context of the Schwarzschild solution. To this aim, we can write the metric tensor as[1]

$$g_{\mu\nu} = \mathrm{diag}\left(e^\nu,-e^\lambda,-r^2,-r^2\sin^2\theta\right), \qquad e^\nu=e^{-\lambda}=1-2\phi=1-\frac{2M}{r}. \tag{4.48}$$

[1] In the following, not only we use natural units, but we also set $G=1$.

Moreover, let us recall that $\partial_t\phi = 0$ and let us assume that only radial motion is considered ($\dot\vartheta = \dot\varphi = 0$). The vierbeins for the metric (4.48) are

$$e^0_{\hat 0} = e^{-\frac{\nu}{2}}; \quad e^1_{\hat 1} = e^{-\frac{\lambda}{2}}. \tag{4.49}$$

In addition, we report the expression of the non-vanishing Christoffel symbols

$$\Gamma^0_{00} = 0; \quad \Gamma^0_{01} = \frac{\nu'}{2}; \quad \Gamma^0_{11} = 0; \quad \Gamma^1_{00} = \frac{\nu'}{2}e^{2\nu}; \quad \Gamma^1_{01} = 0; \quad \Gamma^1_{11} = -\frac{\nu'}{2}, \tag{4.50}$$

where $\nu = \ln(1-2\phi)$ and $\nu' = d\nu/dr$. The geodesic equation for $\mu = 0$ is

$$\ddot t + \nu'\dot r\dot t = -\frac{2}{3}\alpha\pi T^2\left[\frac{\dot r\nu'}{2mE} + \frac{\ddot t + \frac{\dot r\dot t\nu'}{2}}{E^2}e^{\frac{\nu}{2}}\right]e^{-\frac{\nu}{2}}, \tag{4.51}$$

but if one recalls that $E = m\dot x^{\hat 0} = m\dot x^\alpha e^{\hat 0}_\alpha = m\,\dot t\, e^{\nu/2}$, Eq. (4.51) can be once again manipulated to obtain

$$\ddot t + \nu'\dot r\dot t = -\frac{2\alpha\pi T^2}{3E^2}\left(\ddot t + \nu'\dot r\dot t\right), \tag{4.52}$$

and since $\dot\nu = \nu'\dot r$, the final relation for the temporal part will be

$$\left(1 + \frac{2\alpha\pi T^2}{3E^2}\right)\left(\ddot t + \dot\nu\dot t\right) = 0. \tag{4.53}$$

The radial contribution can be computed involving (4.47) for $\mu = 1$

$$\ddot r + \frac{\nu'}{2}\left(\dot t^2 e^{2\nu} - \dot r^2\right) = \frac{2\alpha\pi T^2}{3m^2}\frac{e^\nu\nu'}{2}, \tag{4.54}$$

which can be reformulated in a different fashion

$$\ddot r + \frac{\nu'}{2}\left(\dot t^2 e^{\nu-\lambda} - \dot r^2 - \frac{2\alpha\pi T^2}{3m^2}e^{-\lambda}\right) = 0. \tag{4.55}$$

Equations (4.53) and (4.55) constitute a coupled system of differential equations, which, in general, can be quite difficult to solve. In this case, however, simple calculations lead to a handy relation between $\dot t^2$ and $\dot r^2$ which can be adopted to find the desired outcome.

In fact, Eq. (4.55) can be cast in the form

$$2\ddot r - \dot r^2\nu' + \dot t^2\nu' e^{2\nu} - \frac{2\alpha\pi T^2}{3m^2}\nu' e^\nu = 0. \tag{4.56}$$

The previous expression can be written as

$$e^{\nu}\frac{d}{dr}\left(e^{\lambda}\dot{r}^2 - e^{\nu}\dot{t}^2 - \frac{2\alpha\pi T^2}{3m^2}\nu\right) = 0, \tag{4.57}$$

which implies

$$e^{\lambda}\dot{r}^2 - e^{\nu}\dot{t}^2 - \frac{2\alpha\pi T^2}{3m^2}\nu = \text{const.} \tag{4.58}$$

The constant can be determined from the condition of normalization on four-velocity in the limit $\phi \to 0$. Such a requirement is possible due to the hypothesis made above, namely the independence of the geometric structure on temperature. Hence, normalization of $\dot{x}^{\mu}$ implies

$$\dot{x}^{\mu}\dot{x}_{\mu} = g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = 1, \tag{4.59}$$

or explicitly

$$e^{\lambda}\dot{r}^2 - e^{\nu}\dot{t}^2 = -1, \tag{4.60}$$

because angles are fixed quantities.

In the limit of vanishing gravitational field (namely, $\nu, \lambda \to 0$ as $r \to \infty$), Eq. (4.60) reduces to

$$\dot{r}^2_{\infty} - \dot{t}^2_{\infty} = -1. \tag{4.61}$$

Such an expression clearly fits also (4.58), and thus we have

$$e^{\lambda}\dot{r}^2 - e^{\nu}\dot{t}^2 - \frac{2\alpha\pi T^2}{3m^2}\nu = -1. \tag{4.62}$$

At this point, let us invoke the weak-field approximation. Within this regime and by virtue of (4.62), it is immediate to find that (4.55) gets modified as

$$\ddot{r} = -\frac{M}{r^2}\left(1 - \frac{2\alpha\pi T^2}{3m^2}\right), \tag{4.63}$$

and if one considers the first-order approximation in T^2 just like in QFT considerations, the outcome is

$$\frac{m_g}{m_i} = 1 - \frac{2\alpha\pi T^2}{3m_0^2},$$

which is exactly equal to (4.36).

With the last result, the equivalence between the QFT approach and the one formulated in Ref. [133] has been clarified, even though they differ in some aspects. However, both of the two investigations base their development on a finite-temperature analysis, that enables the emergence of the radiative correction in the ratio m_g/m_i.

At this point, it is reasonable to guess that the aforementioned apparatus can be promptly extended to any scenario involving extended theories of gravity. However, since quadratic theories have already been analyzed in Chapter 2, here we will deal with Brans-Dicke model [143], which is the most famous example of scalar-tensor theories [144].

Application to Brans-Dicke model

The information we need is the expression of the metric tensor. If a static and isotropic solution is sought, it is possible to find an expression for the line element [143]

$$ds^2 = e^v dt^2 - e^u \left[dr^2 + r^2 \left(d\vartheta^2 + \sin^2\vartheta d\Phi^2\right)\right], \tag{4.64}$$

where

$$e^v = e^{2\alpha_0} \left(\frac{1-\frac{B}{r}}{1+\frac{B}{r}}\right)^{\frac{2}{\lambda}}, \qquad e^u = e^{2\beta_0} \left(1+\frac{B}{r}\right)^4 \left(\frac{1-\frac{B}{r}}{1+\frac{B}{r}}\right)^{\frac{2(\lambda-C-1)}{\lambda}}, \tag{4.65}$$

with α_0, β_0, B, C and λ being constants that can be connected to the free parameter of the theory[2] ω. Since it is a scalar-tensor theory, a solution for φ must also be found; in the considered case, the outcome turns out to be

$$\varphi = \varphi_0 \left(\frac{1-\frac{B}{r}}{1+\frac{B}{r}}\right)^{-\frac{C}{\lambda}}, \tag{4.66}$$

where φ_0 is another constant.

The desired physical quantities are the Christoffel symbols and the tetrads, which once again are easy to calculate since $g_{\mu\nu}$ is diagonal

$$e^0_{\hat{0}} = e^{-\frac{v}{2}}; \quad e^1_{\hat{1}} = e^{-\frac{u}{2}}, \tag{4.67}$$

$$\Gamma^0_{00} = 0; \quad \Gamma^0_{01} = \frac{v'}{2}; \quad \Gamma^0_{11} = 0; \quad \Gamma^1_{00} = \frac{v'}{2} e^{v-u}; \quad \Gamma^1_{01} = 0; \quad \Gamma^1_{11} = -\frac{u'}{2}. \tag{4.68}$$

[2]See Appendix E for further details.

Explicit formulas for u and v are

$$v = 2\alpha_0 + \frac{2}{\lambda}\ln\left(\frac{1-\frac{B}{r}}{1+\frac{B}{r}}\right), \quad u = 2\beta_0 + 4\ln\left(1+\frac{B}{r}\right) + \frac{2}{\lambda}\left(\lambda - C - 1\right)\ln\left(\frac{1-\frac{B}{r}}{1+\frac{B}{r}}\right), \tag{4.69}$$

where it is clear that the information related to ω is hidden in the constants, whereas the mass of the gravitational source is contained in the parameter B.

After this digression, it is possible to evaluate the temporal differential equation

$$\ddot{t} + \dot{r}\dot{t}v' = \frac{2}{3}\alpha\pi T^2\left[-\frac{\dot{r}v'e^{-\frac{v}{2}}}{2mE} + \frac{\ddot{t}+\dot{r}\dot{t}\frac{v'}{2}}{E^2}\right]. \tag{4.70}$$

Being $E = m\dot{t}\,e^{v/2}$, the previous relation can be reformulated as

$$\left[1 + \frac{2\alpha\pi T^2}{3E^2}\right]\left(\ddot{t} + \dot{v}\dot{t}\right) = 0, \tag{4.71}$$

which is formally equal to the expression obtained for Schwarzschild, but in this case v has a different meaning.

The radial equation is similar to (4.55), namely

$$\ddot{r} + \dot{r}^2\frac{u'}{2} + \dot{t}^2\frac{v'e^{v-u}}{2} = \frac{2\alpha\pi T^2}{3m^2}\frac{v'e^{-u}}{2}, \tag{4.72}$$

or equivalently

$$\ddot{r} + \frac{v'}{2}\left[\dot{t}^2 e^{v-u} + \dot{r}^2\frac{u'}{v'} - \frac{2\alpha\pi T^2}{3m^2}e^{-u}\right] = 0. \tag{4.73}$$

If $u' = -v'$ one exactly obtains the above results of the Schwarzschild solution.

However, this is not the final expression for the Brans-Dicke case. In fact, Eq. (4.73) can be further simplified adopting the same method that leads to (4.62) in the previous section. Here, the situation is similar, and thus

$$\dot{r}^2 e^u - \dot{t}^2 e^v - \frac{2\alpha\pi T^2}{3m^2}v = -1. \tag{4.74}$$

Neglecting higher-order terms with respect to φ would exclude interesting contributions to the ratio m_g/m_i. Hence, another way to simplify (4.73) must be found. In order to do that, $\dot{t}$ and $\dot{r}$ are the quantities that should be explicitly expressed, since their evolution has not been determined yet. However, this turns out to be easy by virtue of (4.71) and (4.74). Indeed, the first one tells that

$$\frac{\ddot{t}}{\dot{t}} = -\frac{d}{ds}v, \tag{4.75}$$

but it is evident that $\ddot{t}/\dot{t} = d/ds\left[\ln \dot{t}\right]$, and for this reason the differential equation can be immediately solved

$$\dot{t} = e^{-v}. \tag{4.76}$$

Thanks to (4.76), it is possible to find an expression also for $\dot{r}^2$

$$\dot{r}^2 = \left(e^{-v} + \frac{2\alpha\pi T^2}{3m^2}v - 1\right)e^{-u}, \tag{4.77}$$

and with these two expressions, Eq. (4.73) can be rewritten as

$$\ddot{r} = -\frac{v'}{2}\left[e^{-v}\left(1+\frac{u'}{v'}\right) - \frac{u'}{v'} - \frac{2\alpha\pi T^2}{3m^2}\left(1-\frac{u'}{v'}v\right)\right]e^{-u}. \tag{4.78}$$

Nevertheless, the above-mentioned quantities are not so easy to handle within this framework, since their expression is rather convoluted. However, a further step can link u' with v' starting from (4.69). One then has

$$v' = -\frac{4B}{\lambda}\left(\frac{1}{B^2-r^2}\right), \qquad u' = -\frac{4B}{\lambda}\left(\frac{\frac{\lambda B}{r} - C - 1}{B^2-r^2}\right). \tag{4.79}$$

With these definitions, the Brans-Dicke constants appear in (4.78). Such a statement is non-trivial, because the shift between gravitational and inertial mass will depend on ω.

Apart from the previous prediction, it can be easily observed that

$$\frac{u'}{v'} = \frac{\lambda B}{r} - C - 1, \tag{4.80}$$

and as a consequence

$$\ddot{r} = -\frac{v'}{2}\left\{1 + \left(e^{-v}-1\right)\left(\frac{\lambda B}{r} - C\right) - \frac{2\alpha\pi T^2}{3m^2}\left[1 + v - \left(\frac{\lambda B}{r} - C\right)v\right]\right\}e^{-u}. \tag{4.81}$$

By looking at (4.81), one can observe that there is not only the radiative correction to the ratio m_g/m_i, but also another contribution which exclusively depends on ω and that correctly vanishes in the limit $\omega \to \infty$, that is when GR is recovered. Moreover, the evaluation of the second quantity of (4.81) represents an important opportunity to put a lower bound to the parameter of the Brans-Dicke theory. In fact, if one imposes that $|(m_g - m_i)/m_i| < 10^{-14}$ [145] and studies the factor $(e^{-v} - 1)(\lambda B/r - C)$ in the first-order approximation in ϕ_g casting radiative corrections aside momentarily, it is possible to easily constrain ω. In order to perform that, we need the quantities of the Brans-Dicke theory to be written in terms of the free parameter of the model and in the weak-field limit [146]

$$\alpha_0 = \beta_0 = 0; \;\; C = -\frac{1}{2+\omega}; \;\; B = \frac{GM\lambda}{2}; \;\; \lambda = \sqrt{\frac{2\omega+3}{2\omega+4}}. \tag{4.82}$$

It is easy to put B in terms of ϕ_g

$$B = \frac{\lambda r \phi_g}{2}, \tag{4.83}$$

and to expand the function e^{-v}

$$e^{-v} = \left(\frac{1 - \frac{\lambda\phi_g}{2}}{1 + \frac{\lambda\phi_g}{2}}\right)^{-\frac{2}{\lambda}} \sim 1 + 2\phi_g. \tag{4.84}$$

Since we neglect higher-order terms of ϕ_g, the examined factor is simply

$$\frac{2\phi_g}{2+\omega}, \tag{4.85}$$

and thus from $2\phi_g/(2+\omega) < 10^{-14}$, we get

$$\omega > \frac{2GM}{r} \cdot 10^{14}, \tag{4.86}$$

which is the final expression for the lower bound of the Brans-Dicke parameter in the weak-field approximation.

A similar result is easy to achieve only if weak-field approximation is performed, otherwise the complete dependence of constants λ and C with respect to ω would have been more difficult to handle. For instance, let us consider the gravitational field of the Earth by recalling that $M_\oplus = 5.97 \cdot 10^{24}\ Kg; \quad R_\oplus = 6.37 \cdot 10^6\ m$. It is immediate to achieve

$$\omega > 1.40 \cdot 10^5, \tag{4.87}$$

that is similar to a bound recently obtained by experiments [147], which gives $\omega > 3 \cdot 10^5$. For the sake of completeness, it is useful to look at a table that contains a prediction of the most reliable bounds for ω [148].

Detector	System	Specification	Expected bound on ω
aLIGO	$(1.4+5)\,M_\odot$	100 Mpc	~ 100
ET	$(1.4+5)\,M_\odot$	100 Mpc	$\sim 10^5$
ET	$(1.4+2)\,M_\odot$	100 Mpc	$\sim 10^4$
eLISA	$(1.4+400)\,M_\odot$	SNR=10	$\sim 10^4$
LISA	$(1.4+400)\,M_\odot$	SNR=10	$\sim 10^5$
DECIGO	$(1.4+10)\,M_\odot$	SNR=10	$\sim 10^6$
Cassini	Solar System		$\sim 10^4$

Table 4.1: This table includes expected outcomes of experimental observations, in addition to a known bound deduced by the probe Cassini through the analysis of the Solar System.

Therefore, the case of the Brans-Dicke model is particularly interesting, since it exhibits a temperature-independent contribution that violates WEP. In view of such a result, it is straightforward to deduce that the same line of reasoning can be carried out for a generic extended theory of gravity in order to check the extent of EP violation, both the one with a thermal nature and the non-thermal one.

Concluding remarks

The equivalence principle has been analyzed from various standpoints, and in all cases we have encountered a violation of EP in its easiest formulation, namely WEP. We have investigated the crucial reasons behind the disagreement of the inertial and gravitational mass of totally different systems, and for the sake of clarity it is opportune to briefly summarize the work performed up to this point.

- In Sec. 4.1, we have analyzed the non-relativistic limit of the Dirac equation for mixed neutrinos both in the absence and presence of an external gravitational field. In its absence, we have shown that the small components of the flavor bispinor wave functions inevitably induce a redefinition of the inertial mass. This rather unexpected behavior is a consequence of the fact that, when mixing is present, in the Dirac equation one simultaneously deals with large and small bispinor components that are comparably important in the non-relativistic approximation.

  Furthermore, when an external gravitational field is considered in the weak-field limit, we have observed that the gravitational mass does not undergo the same redefinition as the inertial mass, and hence a violation of WEP arises.

Accordingly, a non-relativistic regime appears to be a suitable playground for testing the violation of the equivalence principle in neutrino physics. Among other things, this may become of relevant interest for the case of relic neutrinos in the CNB, which may be experimentally detectable in the not-so-distant future [149].

- In Sec. 4.2, we have analyzed EP violation triggered by the presence of a non-vanishing temperature. We have briefly summarized the procedure that firstly led to such an outcome by adopting QFT techniques. After that, we have shown how the same result can be achieved without relying on loop computations, but rather focusing the attention on the modification of the geodesic equation for $T \neq 0$. With this simplified treatment, it is possible to study not only the space-time described by the Schwarzschild solution in GR, but also other physical environments, even the ones arising from extended theories of gravity. In this perspective, the Brans-Dicke theory has been examined, and we have highlighted the possibility to put a constraint on the free parameter of this model by resorting to the current data related to EP.

  In particular, we want to recall the result (4.87), which directly depends on the ratio of the gravitational and inertial mass, as it could be seen in (4.86), where the factor 10^{14} is an immediate consequence of $|(m_g - m_i)/m_i| < 10^{-14}$. If experiments were able to reach an even higher precision, i.e. 10^{-17} [150], one would have $\omega > 2\,\phi_g \cdot 10^{17}$, instead of (4.86). As a consequence, the lower bound on ω for the Earth would be $\omega > 1.40 \cdot 10^{8}$, which far exceeds the expected outcomes exposed in Table 4.1.

Chapter 5

Novel perspectives from generalized uncertainty principle

In Sec. 1.3, we have explored the most important features related to HUP. Furthermore, we have seen what happens when the presence of gravity cannot be regarded as negligible, which hence leads to a modification of the quantum uncertainty relation that is typically addressed as GUP. As we have already mentioned, a similar generalization has a significant impact on all aspects of quantum mechanics [85]. However, as Sec. 2.4.2 attests, it is licit to expect profound differences with respect to the usual scenario also for what concerns quantum field theory. In this sense, GUP may be viewed as a valuable probe to verify a first and non-trivial interplay between gravitational and quantum effects. For this reason, after the seminal work of Refs. [58, 59, 60, 61] based on gedanken experiments from string theory, GUP acquired a primary role in QG investigations.

The purpose of the current Chapter is to investigate the implications of GUP in the context of well-established theoretical results, such as the Hawking [151] and the Unruh [117] effect. As a matter of fact, both these two physical manifestations are genuinely quantum, which thus naturally entails a close correlation between their derivation and the uncertainty relations. Therefore, it is immediate to foresee the existence of corrections to the standard formulations of the aforementioned phenomena that are attributable to the implementation of GUP. In the following, the first Section is entirely dedicated to this analysis.

In addition to the above prospect, we will also compare several results coming from GUP and from other theoretical models that try to formalize QG features. In particular, we will verify the similarities between the predictions of GUP with the ones arising from corpuscular gravity picture for black holes (i.e. see Refs. [152, 153, 154, 155] and references therein) and with the findings belonging to the model of maximal acceleration (see Refs. [156, 157]). In both frameworks, the focus will be dedicated to the deformation parameter β introduced in Sec. 1.3, which will play a crucial part for the establishment of a solid consistency between these theoretical models. An intriguing aspect that will be pointed out is centered around the sign of β; indeed, the two different approaches imply the same magnitude for the deformation parameter, but opposite signs.

Before we proceed, it is opportune to recall that for the sake of conciseness we will adopt natural units together with $k_B = 1$. Therefore, Eqs. (1.22) and (1.23) shall be rewritten as

$$\delta x\, \delta p \geq \frac{1}{2}\,, \tag{5.1}$$

and

$$\delta x\, \delta p \geq \frac{1}{2} \pm 2|\beta| \frac{\delta p^2}{m_p^2}\,, \tag{5.2}$$

respectively.

5.1 GUP corrections to Hawking and Unruh effect

In this Section, we will see how the formalism underlying the Hawking and the Unruh effect is modified by the presence of GUP instead of HUP. In so doing, in light of Refs. [63, 64, 69, 75, 76, 83] we resort to a heuristic treatment which is useful to better figure out the essence of the analyzed physical manifestations. Furthermore, we will perform these considerations both for HUP and GUP.

5.1.1 Hawking effect

In a nutshell, the Hawking effect [151] is the physical phenomenon according to which a black hole "evaporates" by emitting blackbody radiation. The source of a similar occurrence is purely quantum. As a matter of fact, there is no classical calculation that is capable of explaining the Hawking radiation; only a full-fledged QFT computation can shed light on the peculiar features of this effect. However, there are several heuristic arguments which can be carried out that help the visualization of such phenomenon and at the same time recover the standard behavior of the radiation's temperature as a function of the black hole mass. In what follows, we will assume that the black hole radiates only photons, but our working hypothesis does not harm the generality of the outcome.

In light of the previous remarks, in order to tackle the analysis of the Hawking effect, we can consider for simplicity a spherically symmetric black hole with mass M and Schwarzschild radius $r_s = 2MG$. Let us observe that, just outside the event horizon, the position uncertainty of photons emitted by the black hole is of the order of its Schwarzschild radius, i.e. $\delta x \simeq \mu r_s$, where the constant μ is of order of unity and will be fixed below. From (5.1), the corresponding momentum uncertainty is given by

$$\delta p \simeq \frac{1}{4\mu M G}, \tag{5.3}$$

which also represents the characteristic energy of the emitted photons, since $\delta p \simeq p = E$. According to the equipartition theorem, this can be now identified with the temperature T of the ensemble of photons,

$$E = T \simeq \frac{1}{4\mu M G} = \frac{m_p^2}{4\mu M}\,, \tag{5.4}$$

which agrees with the Hawking temperature,

$$T_{\mathrm{H}} = \frac{1}{8\pi M G} \equiv \frac{m_p^2}{8\pi M}\,, \tag{5.5}$$

provided that $\mu = 2\pi$.

Therefore, on the basis of the HUP and thermodynamic consistency, we have recovered the standard Hawking formula (5.5) for the temperature of the radiation emitted by the black hole.

Now, it is well known that black holes with temperature greater than the background temperature (about 2.7 K for the present universe) shrink over time by radiating energy in the form of photons and other ordinary particles. In certain conditions [158], however, it is reasonable to assume that the evaporation is dominated by photon emission. In this case, we can exploit the Stefan–Boltzmann law to estimate the radiated power P as

$$P = A_s \varepsilon \sigma T^4 \simeq A_s \sigma T^4, \tag{5.6}$$

where $A_s = 4\pi r_s^2$ is the black hole sphere surface area at Schwarzschild radius r_s, $\sigma = \pi^2/60\hbar^3$ is the Stefan–Boltzmann constant, and we have assumed for simplicity the black hole to be a perfect blackbody, i.e. $\varepsilon \simeq 1$.

Using (5.5) and (5.6), the black hole energy loss can be easily evaluated as a function of time, yielding

$$\frac{dM}{dt} = -P \simeq -\frac{1}{60(16)^2\pi\sqrt{G}}\frac{m_p^3}{M^2} = -\frac{1}{60(16)^2\pi}\frac{m_p^4}{M^2}. \tag{5.7}$$

Therefore, the evaporation process leads black holes to vanish entirely with both the temperature (5.4) and emission rate (5.7) blowing up as the mass decreases.

The above results have been derived starting from HUP (5.1). Let us now perform similar calculations by resorting to the GUP in (5.2), so as to realize to what extent GUP affects the black hole thermodynamics. In this case, solving (5.2) with respect to the momentum uncertainty δp and setting again δx of the order of the Schwarzschild radius, we obtain the following expression for the modified Hawking temperature

$$T_{\rm GUP} = \pm\frac{M}{4\pi|\beta|}\left(1 \pm \sqrt{1 \mp |\beta|\frac{m_p^2}{M^2}}\right). \tag{5.8}$$

In the semiclassical limit $\sqrt{|\beta|}m_p/M \ll 1$, this agrees with the standard Hawking result in (5.4), provided that the negative sign in front of the square root is chosen, whereas the positive sign has no physical meaning. Similarly, the emission rate in (5.7) is modified as

$$\left(\frac{dM}{dt}\right)_{\rm GUP} \simeq \frac{-1}{60\,(16)\,\pi}\,\frac{M^6}{|\beta|^4\,m_p^4}\left(1-\sqrt{1 \mp |\beta|\,\frac{m_p^2}{M^2}}\right)^4. \tag{5.9}$$

In what follows, the implications of (5.8) and (5.9) will be discussed separately for the cases of $\beta > 0$ and $\beta < 0$.

GUP with $\beta > 0$

Let us start by analyzing the most common setting of GUP with positive deformation parameter. In this case, from (5.8) it is easy to see that the GUP naturally introduces a minimum size allowed for black holes: for $M < \sqrt{\beta}\,m_p$ (i.e. $r_s < 2\sqrt{\beta}\,\ell_p$), indeed, the temperature would become complex, in contrast with predictions of ordinary black hole thermodynamics. Remarkably, one can verify that the evaporation process should stop at $M \sim \sqrt{\beta}\,m_p$ by observing that the specific heat of the black hole dM/dT is negative under GUP modification and it vanishes as $M \to \sqrt{\beta}\,m_p$ [159], thus leading to an *inert* remnant with finite temperature and size. On the other hand, different models predict a change of sign in the specific heat as the mass is wiped out; in that case, black hole remnants may undergo a phase transition which renders them unstable. Notice also that the idea of black hole remnants dates back to Aharonov–Casher–Nussinov, who first addressed the issue in the context of the black hole unitarity puzzle [160].

Similarly, concerning the modified emission rate (5.9), we find that it is finite at the endpoint of black hole evaporation $M \sim \sqrt{\beta}\,m_p$, whereas the corresponding HUP result (5.7) diverges at the endpoint when $M = 0$.

GUP with $\beta < 0$

Although the GUP with $\beta > 0$ cures the undesired infinite final temperature predicted by Hawking's formula (5.5), it would create several complications, such as the entropy/information problem [161], or the removal of the Chandrasekhar limit [80]. The latter prediction, in particular, would allow white dwarfs to be arbitrarily large, a result that is at odds with astrophysical observations. An elegant way to overcome these ambiguities was proposed in Ref. [80], where it was shown that both the infinities in black hole and white dwarf physics can be avoided by choosing a negative deformation parameter in (5.2). A similar scenario had previously been encountered in Ref. [79] in the context of GUP in a crystal-like universe with lattice spacing of the order of Planck length.

Let us then consider the case $\beta < 0$. With this setting, from (5.8) and (5.9) we obtain that both the modified temperature and emission rate are well-defined even for $M < \sqrt{|\beta|}\, m_p$. For a sufficiently small M, in particular, the modified temperature in (5.8) can be approximated as

$$T_{\text{GUP}} \simeq \frac{m_p}{4\pi\sqrt{|\beta|}} < \infty. \tag{5.10}$$

Even though no lower bound on the black hole size arises in this framework, the Hawking temperature remains finite as the black hole evaporates to zero mass. From (5.10) we also deduce that the bound on the Hawking temperature is independent of the initial black hole mass.

5.1.2 Unruh effect

Together with the Hawking effect, the Unruh effect [117] is one of the most outstanding manifestations of the non-trivial nature of quantum vacuum. Indeed, its implication is that the zero-particle state for an inertial observer in Minkowski spacetime looks like a thermal state for a uniformly accelerating observer, with a temperature given by

$$T_{\text{U}} = \frac{a}{2\pi}\,, \tag{5.11}$$

where a is the magnitude of the acceleration.

The above relation can be rigorously derived within the framework of quantum field theory [117]. Following Refs. [75, 76], however, here we review a heuristic calculation based exclusively on the HUP. This procedure will be the starting point to compute GUP corrections to the Unruh temperature (5.11). Now, consider a gas of relativistic particles at rest in a uniformly accelerated frame. Assuming that the frame moves a distance δx, the kinetic energy acquired by each of these particles is

$$E_k = m\, a\, \delta x\,, \tag{5.12}$$

where m is the mass of the particles and a the acceleration of the frame. Suppose this energy is barely enough to create N particle-antiparticle pairs from the quantum vacuum, i.e. $E_k \simeq 2Nm$. Using (5.12), it follows that the minimal distance along which each particle must be accelerated reads

$$\delta x \simeq \frac{2\,N}{a}\,. \tag{5.13}$$

Now, since the whole system is localized inside a spatial region of width δx, the energy fluctuation of each single particle can be estimated from the HUP as

$$\delta E \simeq \frac{1}{2\,\delta x}\,, \tag{5.14}$$

where we have assumed $\delta E \simeq \delta p$. This gives

$$\delta E \simeq \frac{a}{4\,N}\,. \tag{5.15}$$

If we interpret this fluctuation as a thermal agitation effect, from the equipartition theorem we have

$$\frac{3}{2}\,T \simeq \delta E \simeq \frac{a}{4\,N}\,, \tag{5.16}$$

which can be easily inverted for T, yielding

$$T = \frac{a}{6N}\,. \tag{5.17}$$

Comparison with the Unruh temperature (5.11) allows us to set an effective number of pairs $N = \pi/3 \simeq 1$.

Let us now repeat similar calculations in the context of the GUP. From the uncertainty relation (5.2), we first note that the GUP version of the standard Heisenberg formula (5.14) is

$$\delta x \simeq \frac{1}{2\,\delta E} + 2\beta\,\ell_{\mathrm{p}}^2 \delta E\,. \tag{5.18}$$

Upon replacing (5.13) into (5.18), and using the same thermodynamic argument as in (5.16) for δE, we obtain

$$\frac{2N}{a} \simeq \frac{1}{3\,T} + 3\beta\,\ell_{\mathrm{p}}^2\, T\,. \tag{5.19}$$

Once again, by requiring that T equals the Unruh temperature (5.11) for $\beta \to 0$, we can fix $N = \pi/3$, so that

$$\frac{2\pi}{a} \simeq \frac{1}{T} + 9\beta\,\ell_{\rm p}^2\,T\,. \tag{5.20}$$

Solving for T, we obtain the the following expression for the modified Unruh temperature:

$$T \,=\, \frac{\pi}{9\beta\,\ell_p^2\,a}\left(1\,\pm\,\sqrt{1-9\beta\,\ell_p^2\,\frac{a^2}{\pi^2}}\right), \tag{5.21}$$

which agrees with the standard result (5.11) in the semiclassical limit $\beta\,\ell_p^2\,a^2 \ll 1$, provided that the negative sign is chosen, whereas the positive sign has no evident physical meaning.

At this point, as already highlighted in Sec. 1.3 and as we will see below, it is worth emphasizing that the magnitude of β is typically assumed to be of order unity, i.e. $\beta \simeq \mathcal{O}(1)$, and from a theoretical perspective this has been demonstrated in several papers [58, 60, 72, 76, 78]. However, we should also note that the current experimental constraints on β are by far less stringent than the order of magnitude exhibited here. For instance, gravitational tests give $\beta < 10^{78}$ from light deflection experiments [162], $\beta < 10^{60}$ from the spectrum of GW 150914 [163], and $\beta < 10^{21}$ from violation of equivalence principle on Earth [164]. Likewise, tests which do not involve the gravitational interaction lead to $\beta < 10^{39}$ from ^{87}Rb cold-atom-recoil experiments [165], $\beta < 10^{34}$ from electroweak measurement [166], $\beta < 10^{20}$ from Lamb shift experiments [166], and $\beta < 10^{18}$ from the evolution of micro and nano mechanical oscillators at Planck mass [167].

5.2 GUP and corpuscular gravity

In what follows, we will theoretically evaluate the magnitude of β by comparing GUP predictions of Sec. 5.1.1 with the ones deduced by the corpuscular picture of black holes, a model conceived for the first time in Ref. [152]. Clearly, before introducing the discussion, it is opportune to recall the main aspects of the aforesaid model.

5.2.1 Corpuscular gravity: an overview

In the corpuscular gravity picture black holes can be conceived as Bose–Einstein condensates of N interacting and non–propagating longitudinal gravitons, and thus as intrinsically quantum objects. Let us then consider a Bose–Einstein condensate of total mass M and radius R, which is made up of N weakly interacting gravitons. At low energy, we can define a quantum gravitational self–coupling for each single graviton of wavelength λ as follows [152]

$$\alpha_g \equiv \frac{G}{\lambda^2} = \frac{\ell_p^2}{\lambda^2}\,. \tag{5.22}$$

One of the main features of a Bose–Einstein condensate is that, due to the interaction, its constituents acquire a collective behavior, so that their wavelengths get increasingly larger and their masses smaller; strictly speaking, the constituents become *softer* bosons. In particular, most of the gravitons composing the gravitational system will have a wavelength of the order $\lambda \sim R$, namely of the order of the size of the system itself. Hence, similarly to (5.22), it is possible to define a collective quantum coupling as

$$N\alpha_g \equiv N\frac{G}{\lambda^2} \simeq N\frac{\ell_p^2}{R^2}. \tag{5.23}$$

We now seek the relation that links the total mass of a Bose–Einstein condensate and its radius to the number N of quanta composing the system. By performing a standard computation, one can show that the gravitational binding energy of the system is given by

$$E_g \simeq \frac{GM^2}{R}. \tag{5.24}$$

On the other hand, from a purely quantum point of view, the binding energy can be expressed as the sum of the energies associated to each single graviton, that is

$$E_g \simeq N\frac{1}{\lambda} \simeq N\frac{1}{R}. \tag{5.25}$$

Therefore, by comparing (5.24) and (5.25), we obtain

$$M \simeq \sqrt{N}m_p, \tag{5.26}$$

which also implies for the Schwarzschild radius

$$r_s \simeq \sqrt{N}\ell_p. \tag{5.27}$$

By assuming that the size of the condensate is $R \sim r_s$ (i.e. the overall gravitational system is a black hole) and using the expression in (5.27) for the Schwarzschild radius, we notice that the collective quantum coupling defined in (5.23) is always of order unity in the case of a black hole

$$N\alpha_g \simeq 1. \tag{5.28}$$

In condensed matter physics, it is well known that the inequality $N\alpha_g < 1$ corresponds to a phase in which a Bose–Einstein condensate is weakly interacting. On the other hand, the equality $N\alpha_g = 1$ represents a critical point at which a phase transition occurs, thus letting the condensate become strongly interacting, whereas for $N\alpha_g > 1$ it is possible to observe only a strongly interacting phase [168]. Thus, in this quantum corpuscular picture, a black hole can be defined as a Bose–Einstein condensate of gravitons stuck at the critical point of a quantum phase transition [153].

Thermodynamic properties of corpuscular black holes

We now analyze some thermodynamic aspects of quantum corpuscular black holes, and in particular we show that gravitons can escape from the considered system. Such a phenomenon represents the corpuscular counterpart of the black hole radiation emission [152].

First of all, we need to compute the probability for a graviton to escape from a gravitational bound state, namely we have to determine the so–called *escape energy* and *escape wavelength* of a single graviton. To this aim, observe that, for N weakly interacting quanta composing a condensate of radius R and mass M, a quantum gravitational interaction strength can be defined as [152]

$$\hbar N\alpha_g \equiv N\frac{L_p^2}{\lambda^2}, \tag{5.29}$$

so that each graviton is subject to the following binding potential

$$E_{\text{esc}} = \frac{N\alpha_g}{R}, \tag{5.30}$$

which is the threshold to exceed in order to escape. The corresponding escape wavelength is defined as

$$\lambda_{\text{esc}} = \frac{1}{E_{\text{esc}}}\,. \tag{5.31}$$

If we now employ (5.27) and (5.28) for the case of a black hole, we obtain

$$E_{\text{esc}} \simeq \frac{1}{\sqrt{N}\ell_p}\,, \qquad \lambda_{\text{esc}} \simeq \sqrt{N}\ell_p\,. \tag{5.32}$$

This means that, although N gravitons of wavelength $\lambda \sim \sqrt{N}\ell_p$ can form a gravitational bound state, at the same time a depletion process is present, which is traduced in a leakage of the constituents of the condensate for any N. Clearly, this is related to the fact that λ_{esc} coincides with the wavelength of each graviton belonging to the condensate, that is, $\sqrt{N}\ell_p$.

In terms of scattering amplitudes, the above picture can be regarded as a $2 \to 2$ scattering process, in which one of the two gravitons is energetic enough to be able to exceed the threshold given by $E_{\rm esc}$. We can also obtain an estimation for the depletion rate Γ of such a process. As usual, this should be given by a product involving the squared coupling constant α_g^2, the characteristic energy scale of the process $E_{\rm esc}$ and a combinatoric factor $N(N-1)$, which can be approximated by N^2 for a very large number of constituents [152]

$$\Gamma \simeq \alpha_g^2 N^2 E_{\rm esc} \simeq \frac{1}{\sqrt{N}\ell_p}. \tag{5.33}$$

From the above relation, we can easily obtain the corresponding time scale of the considered process, which is given by $\Delta t = 1/\Gamma \simeq \sqrt{N}\ell_p$.

On the other hand, Eq. (5.33) allows us to infer the mass decrease over time of the condensate

$$\frac{dM}{dt} = -\frac{\Gamma}{\lambda_{\rm esc}} \simeq -\frac{1}{N\ell_p^2} \simeq -\frac{m_p^4}{M^2}\,, \tag{5.34}$$

which can be cast in terms of the rate of emitted gravitons by use of (5.26),

$$\frac{dN}{dt} \simeq -\frac{1}{\sqrt{N}\ell_p}. \tag{5.35}$$

We stress that, up to the factor $1/[60(16)^2\pi]$, Eq. (5.34) reproduces the thermal evaporation rate of a black hole in (5.7), assuming the Hawking temperature in the corpuscular model to be given by [152]

$$T_{\rm H} \simeq \frac{1}{\sqrt{N}\ell_p} \simeq \frac{m_p^2}{M}. \tag{5.36}$$

We have seen that the black hole quantum N-portrait manages to reproduce the semiclassical result, according to which a black hole emits a thermal radiation with temperature given by the Hawking formula (5.5). However, from a more scrupulous investigation, one can see that such a result holds true only to the leading order, since in general there will be higher-order corrections which scale as negative powers of the number of gravitons [152, 153].

In this connection, notice that, in the computation of the depletion rate Γ (5.33), we have only considered the simplest kind of interaction (i.e. a tree-level scattering diagram with two vertices); nevertheless, one expects that even higher-order processes provide Γ with contributions that induce gravitons to escape. For instance, the next relevant $2 \to 2$ scattering process would possess three vertices, thus contributing with terms proportional to α_g^3. Therefore, up to first-order corrections, the depletion rate would take the form

$$\Gamma \simeq \alpha_g^2 N^2 E_{\rm esc} + \mathcal{O}\left(\alpha_g^3 N^2 E_{\rm esc}\right) \simeq \frac{1}{\sqrt{N}\ell_p} + \mathcal{O}\left(\frac{1}{\ell_p}\frac{1}{N^{3/2}}\right). \tag{5.37}$$

As for the zeroth-order in (5.34), the mass decrease of the black hole can be now estimated from the modified depletion rate (5.37), yielding [153]

$$\frac{dM}{dt} \simeq -\frac{m_p^4}{M^2} + \mathcal{O}\left(\frac{m_p^6}{M^4}\right). \tag{5.38}$$

We emphasize that such a result only describes the qualitative behavior of the evaporation rate in the CG framework. In the next Section, we shall consider the exact expression of dM/dt in order to make a quantitative comparison with the corresponding GUP result.

5.2.2 GUP and corpuscular gravity: a quantitative comparison

In the previous Sections, the evaporation rate of a black hole has been computed within both the GUP and CG frameworks. Here, we compare the two expressions: as it will be shown, this allows us to set the value of the GUP deformation parameter β for which the GUP and CG treatments are consistent [83].

For this purpose, let us consider the GUP–modified expressions of the emission rate (5.9) expanded up to the order $\mathcal{O}(1/M^4)$ and the CG outcome (5.38). We have

$$\left(\frac{dM}{dt}\right)_{\rm GUP} \simeq \frac{-1}{60(16)^2\pi}\left(\frac{m_p^4}{M^2} \pm |\beta|\frac{m_p^6}{M^4}\right), \tag{5.39}$$

and

$$\left(\frac{dM}{dt}\right)_{\rm CG} \simeq \frac{-1}{60(16)^2\pi}\left[\frac{m_p^4}{M^2} + \mathcal{O}\left(\frac{m_p^6}{M^4}\right)\right], \tag{5.40}$$

where we recall that the sign $\pm$ in (5.39) corresponds to a positive/negative value of the deformation parameter β. Note that in (5.40) the correct numerical factor has been restored by requiring that the Hawking formula (5.7) is consistently recovered to the zeroth order.

By comparing (5.39) and (5.40), it follows that, at least up to the first-order, the GUP- and CG-induced corrections exhibit the same functional dependence on the black hole mass. Furthermore, since the coefficient in front of the correction in (5.40) is predicted to be of order unity [152, 153], numerical consistency between the two expressions automatically leads to

$$|\beta| \sim \mathcal{O}(1), \tag{5.41}$$

which is in agreement with the predictions of other models of quantum gravity. Therefore, in spite of their completely different underlying backgrounds, the GUP and CG approaches are found to be compatible with each other.

However, the result (5.41) does not give any specific information about the sign of β. Since a full-fledged analytic derivation of (5.40) including also higher-order scattering processes is still lacking, a definitive conclusion on this issue cannot be reached. On the one hand, relying on basic considerations on the nature of the scattering amplitudes, we would naively expect the correction in (5.40) to contribute with the same sign as the zeroth-order term, since we are only adding higher-order diagrams describing the probability for a graviton to escape from the condensate. This would yield a positive value for the deformation parameter.

On the other hand, there are different claims which assert that the above correction should be opposite to the zeroth-order term, in such a way to slightly decrease the evaporation rate of the black hole. This was shown, for example, within the framework of Horizon Quantum Mechanics [169], where the depletion rate reads [170]

$$\Gamma \simeq \frac{1}{\sqrt{N}\ell_p} - \frac{3\gamma^2 N_H^2}{\ell_p N^{3/2}} \left(6\,\zeta(3) - \frac{\pi^4}{15}\right), \tag{5.42}$$

which would imply the following formula for the evaporation rate

$$\left(\frac{dM}{dt}\right)_{\mathrm{CG}} \simeq \frac{-1}{60(16)^2\pi} \left[\frac{m_p^4}{M^2} - 3\,\gamma^2 N_H^2 \left(6\,\zeta(3) - \frac{\pi^4}{15}\right)\frac{m_p^6}{M^4}\right], \tag{5.43}$$

where $N_H \equiv \sqrt{3}/\sqrt{\pi^2 - 6\zeta(3)} \simeq 1.06$ and $\zeta(x)$ is the Riemann zeta function. Note also that the constant factor in (5.43) is given by $3\gamma^2 N_H^2\,(6\zeta(3) - \pi^4/15) \simeq 2.4\gamma^2$ and γ may be of order one [170]. With such a setting, the comparison of (5.39) and (5.43) would further confirm the result of (5.41), since

$$|\beta| \simeq 2.4\,, \tag{5.44}$$

but it would lead to a negative value for the deformation parameter,

$$\beta < 0\,. \tag{5.45}$$

Moreover, we remark that positive corrections to the evaporation rate of a black hole are required by the principle of energy conservation [171], thus enforcing the validity of (5.45).

However, as already mentioned before, there is still an ongoing debate in literature on the sign of the deformation parameter β. As a matter of fact, the upcoming analysis involving the maximal acceleration model [156] will agree on the magnitude of β, but it will predict a different sign for the deformation parameter of GUP.

5.3 GUP and maximal acceleration

Since the modification to the Unruh temperature due to GUP has already been discussed in Sec. 5.1.2, here we summarize how to obtain the corrections to T_U in (5.11) attributable to the maximal acceleration theory and then compare the two outcomes. To this aim, we first sketch the main features of the maximal acceleration model as conceived in Refs. [156].

5.3.1 Maximal acceleration theory

In a series of works [156] it has been shown that the one-particle quantum mechanics acquires a geometric interpretation if one incorporates quantum aspects into the geometric structure of spacetime. Such an outcome is achieved by treating the momentum and position operators as covariant derivatives with a proper connection in an eight-dimensional manifold. As a result, the usual quantization procedure can be viewed as the curvature of the phase space.

The above geometric picture allows for the emergence of a maximal acceleration A that massive particles can undergo [156]. In principle, this new parameter should be regarded as a mass-dependent quantity, since it varies according to

$$A = \frac{2mc^3}{\hbar} \equiv 2m, \tag{5.46}$$

where m is the rest mass of the particle. On the other side, however, some authors interpret A as a universal constant [157, 172]. In particular, this would happen at energies of the order of Planck scale, where the definition (5.46) is usually rewritten in terms of the Planck mass as [157, 172]

$$A = \frac{m_p c^3}{\hbar} \equiv m_p. \tag{5.47}$$

In order to build the aforementioned eight-dimensional manifold, we basically start from the four-dimensional spacetime $\mathcal{M}$ on which the metric tensor $g_{\mu\nu}$ is defined and then enlarge it with the tangent bundle, so that $\mathcal{M}_8 = \mathcal{M} \otimes T\mathcal{M}$. After performing this, the line element on $\mathcal{M}_8$ becomes

$$d\tau^2 = g_{AB} d\xi^A d\xi^B, \quad A, B = 1, \ldots, 8, \tag{5.48}$$

where the coordinates and the metric in the above equation can be expressed in terms of the corresponding four-dimensional ones by [157]

$$\xi^A = \left(x^\mu, \frac{\dot{x}^\mu}{A}\right), \quad g_{AB} = g_{\mu\nu} \otimes g_{\mu\nu}, \quad \mu, \nu = 1, \ldots, 4. \tag{5.49}$$

Here, the dot represents a derivative with respect to the proper time s defined on $\mathcal{M}$.

From the above considerations, it is straightforward to check that

$$d\tau^2 = \left(1 - \frac{|\ddot{x}^\mu \ddot{x}_\mu|}{A^2}\right) ds^2 \equiv \left(1 - \frac{a^2}{A^2}\right) ds^2, \tag{5.50}$$

with a being the squared length of the spacelike four-acceleration.

With the aid of (5.50), in what follows we derive the modification to the Unruh temperature due to the presence of an upper limit for the acceleration. For this purpose, we employ the Unruh-DeWitt particle detector method as explained in Ref. [99].

5.3.2 Unruh temperature from Maximal Acceleration

Consider a massless scalar field ϕ interacting with a particle detector with internal energy levels by means of a monopole interaction. The Lagrangian related to this process can be sketched as [99]

$$\mathcal{L}_{int} = \chi M(s) \phi(x(s)), \tag{5.51}$$

where χ is a small coupling constant and M is the monopole moment operator of the detector, which travels along a world line with proper time s. Let us further assume that the scalar field is initially in the Minkowski vacuum $|0_M\rangle \equiv |0\rangle$ and the detector in its ground state with energy E_0. Since we do not impose any restriction to the detector's trajectory, it is possible that these initial conditions vary along the world line due to the interaction, thus allowing the scalar field to reach an excited state $|\lambda\rangle$ and the detector to undergo a transition to an energy level $E > E_0$.

By resorting to a first order perturbation theory, the transition amplitude for the process $|E_0, 0\rangle \to |E, \lambda\rangle$ reads [99]

$$\mathcal{A} = i\chi\langle E, \lambda|\left(\int M(s)\phi(x(s))ds\right)|E_0, 0\rangle, \tag{5.52}$$

or

$$\mathcal{A} = i\chi\langle E|M(0)|E_0\rangle \int e^{i(E-E_0)s}\langle\lambda|\phi(x(s))|0\rangle ds. \tag{5.53}$$

where the integral extends over all the real axis.

We stress that the equality between the above relations is guaranteed by the time evolution equation of the operator $M(s)$. By squaring the modulus of $\mathcal{A}$ and summing over all the complete set of values for E and λ, we obtain the transition probability $\mathcal{P}$ related to any possible excitation of the analyzed system. In the case of a trajectory lying on Minkowski background, it is possible to write the transition probability per unit proper time, $\Gamma \equiv \mathcal{P}/T$, as follows

$$\Gamma = -\frac{\chi^2 \sum_E |\langle E|M(0)|E_0\rangle|^2}{4\pi^2} \int \frac{e^{-i(E-E_0)\Delta s}\, d(\Delta s)}{(t-t'-i\varepsilon)^2 - |\mathbf{x}-\mathbf{x}'|^2}\,. \tag{5.54}$$

At this point, we must select the parametrization for the trajectory we mean to study. In order to derive the modified expression of the Unruh temperature, we require the particle detector to move along a hyperbola in the (t, x) plane. This indeed corresponds to the characteristic worldline of a relativistic uniformly accelerated (Rindler) motion with proper acceleration a. Such a trajectory can be parametrized as[1]

$$t = 1/a \sinh(as)\,, \tag{5.55}$$

$$x = 1/a \cosh(as)\,. \tag{5.56}$$

Using (5.50), we can now rewrite the above relations in terms of the parameter τ, so as to make the dependence on the maximal acceleration A manifest. We then obtain

$$t = 1/a \sinh(a\gamma\tau)\,, \tag{5.57}$$

$$x = 1/a \cosh(a\gamma\tau)\,, \tag{5.58}$$

where we have defined $\gamma \equiv 1/\sqrt{1-a^2/A^2}\,$.

[1] For simplicity, we assume that the acceleration is directed along the x-axis.

With the above setting, one can check that (5.54) takes the form

$$\Gamma = \gamma\chi^2 \sum_E |\langle E|M(0)|E_0\rangle|^2 \int e^{-i\gamma(E-E_0)\Delta\tau}\, W(\Delta\tau)\, d(\Delta\tau)\,, \tag{5.59}$$

where $\Delta\tau \equiv \tau - \tau' > 0$, and

$$\begin{aligned} W &= -\left\{\frac{16\pi^2}{a^2}\left[\sinh^2\left(a\frac{\gamma\Delta\tau}{2}\right) - i\varepsilon a \sinh\left(a\frac{\gamma\Delta\tau}{2}\right)\right]\right\}^{-1} \\ &= -\left[\frac{16\pi^2}{a^2}\sinh^2\left(a\,\frac{\gamma\Delta\tau - 2i\varepsilon}{2}\right)\right]^{-1}, \end{aligned} \tag{5.60}$$

is the positive-frequency Wightman Green function [99]

$$W\left(s, s'\right) = \langle 0|\phi(x(s))\phi(x(s'))|0\rangle\,. \tag{5.61}$$

Note that, in the second step of (5.60), we have redefined ε by extracting the positive function $2\cosh\left(a\gamma\Delta\tau/2\right)$. We further emphasize that the particular dependence of W on $\Delta\tau$ (rather than τ and τ' separately) reflects the fact that our system is invariant under time translations in the reference frame of the detector[2].

Now, using for $W(\Delta\tau)$ the identity

$$\mathrm{cosec}^2(\pi x) = \pi^{-2} \sum_{k=-\infty}^{\infty} (x-k)^2\,, \tag{5.62}$$

and replacing into (5.59), we obtain

$$\Gamma = \frac{\chi^2}{2\pi}\sum_{\tilde{E}} \frac{\left(\tilde{E} - \tilde{E}_0\right)\left|\langle\tilde{E}|M(0)|\tilde{E}_0\rangle\right|^2}{e^{2\pi(\tilde{E}-\tilde{E}_0)/a\gamma} - 1}\,, \tag{5.63}$$

where the Fourier transform has been performed by means of a contour integral [99]. Moreover, we have absorbed a factor γ into the definition of ε introduced in (5.60) and $\tilde{E} \equiv \gamma E$ is the energy defined with respect to the detector proper time τ.

Because of the appearance of the Planck factor in (5.63), the rate of absorption of the accelerated detector due to the interaction with the field in its ground state is the same as the one we would obtain if the detector were static, but immersed in

[2] In other terms, we can say that the detector is in equilibrium with the field ϕ, so that the rate of absorbed quanta is constant.

a thermal bath at the temperature

$$T = \frac{a\gamma}{2\pi} \equiv T_{\mathrm{U}} \left(1 - \frac{a^2}{A^2}\right)^{-1/2}. \tag{5.64}$$

We remark that this result is in agreement with the one of Ref. [99], where the correction induced by the existence of a maximal acceleration has been derived by employing the time-dependent Doppler effect approach proposed in Ref. [173].

5.3.3 GUP and maximal acceleration: a quantitative comparison

In Ref. [157], it was argued that the geometrical interpretation of QM through a quantization model that implies the existence of a maximal acceleration naturally leads to a generalization of the uncertainty principle similar to the one in (5.2). Thus, one may wonder which is the value of the parameter β that allows the GUP-deformed and the metric-deformed Unruh temperatures in (5.21) and (5.64) to coincide. Clearly, given that the regime of validity of (5.2) is at Planck scale, we should consider the maximal acceleration A as depending on the quantity m_p (see (5.47)) in order to compare the two expressions.

Since we are only interested in small (i.e. linear in β) corrections to the Unruh temperature, we can expand (5.21) as

$$T \simeq T_{\mathrm{U}} \left(1 + \frac{9\,\beta}{4}\,\frac{\ell_{\mathrm{p}}^2\, a^2}{\pi^2}\right), \tag{5.65}$$

which obviously recovers the standard Unruh result (5.11) for $\beta \to 0$.

Likewise, for realistic values of the acceleration, we have $a << A \sim 10^{51}\mathrm{m/s}^2$, so that (5.64) becomes (to the leading order)

$$T \approx T_{\mathrm{U}} \left(1 + \frac{1}{2}\frac{a^2}{A^2}\right) = T_{\mathrm{U}} \left(1 + 2\,\ell_p^2 a^2\right), \tag{5.66}$$

where we have used the definition (5.47) of the maximal acceleration. By requiring the GUP-deformed Unruh temperature to be equal to the corresponding geometric-corrected formula, we then obtain

$$\beta = \frac{8\pi^2}{9}\,, \tag{5.67}$$

which is of the order of unity, in agreement with the general belief and with several models of string theory. We stress that such a result is perfectly consistent with the outcome of Ref. [157], where it has been shown that the generalized uncertainty principle of string theory is recovered (up to a free parameter) by taking into account the existence of an upper limit on the acceleration.

Concluding remarks

This Chapter has dealt with the consequences of working with a generalized version of the Heisenberg uncertainty relations that account for the presence of gravity. By means of heuristic considerations, in Sec. 5.1 we have shown how to properly consider the implications of GUP in the context of the Hawking and the Unruh effect. This has represented the starting point for the subsequent investigations that have aimed at illustrating the consistency between GUP predictions and other models that merge quantum and gravitational effects.

- In Sec. 5.2, we have analyzed to what extent black hole thermodynamics gets modified both in the presence of a generalized uncertainty principle and in the corpuscular gravity theory. In particular, we have focused on the computation of the temperature and the evaporation rate of a black hole. By comparing the expressions derived within the two frameworks, we have finally managed to estimate the GUP deformation parameter β. The obtained result shows that, in order for the GUP and CG predictions to be consistent, β must be of order unity.

  Furthermore, we have speculated on the sign of β. Although on this matter we are still far from the definitive solution, a preliminary analytic evaluation of the evaporation rate within the framework of Horizon Quantum Mechanics and some considerations related to the conservation of energy, suggest that the most plausible picture is the one with a negative deformation parameter, $\beta < 0$. In this connection, we emphasize that a similar result would not be surprising in the context of a corpuscular (i.e discrete) description of black holes; in Ref. [79], indeed, it was shown that a GUP with $\beta < 0$ can be derived assuming that the universe has an underlying crystal lattice-like structure.

- In Sec. 5.3, we have precisely calculated the deformation parameter β appearing in the GUP. A specific numerical value has been obtained by computing the Unruh temperature for a uniformly accelerated observer in two different ways. In the first case, the GUP (instead of the usual HUP) has been used to derive the Unruh formula. The resulting temperature (5.65) exhibits a (first-order) correction that explicitly depends on β. The second calculation has been performed within the framework of Caianiello's quantum geometry model. By rewriting the line element of a uniformly accelerated observer in such a way to include an upper limit on the acceleration, the Unruh temperature turns out to be accordingly modified (see (5.64)). Then, if we demand the GUP-deformed and the metric-deformed Unruh temperatures to be equal, we obtain the numerical value $\beta = 8\pi^2/9$ for the GUP parameter.

In this connection, we emphasize that, although a variety of experiments have been proposed to test GUP effects in laboratory, to the best of our knowledge there are only few theoretical studies which aim to fix the deformation parameter β in contexts other than string theory. In this regard, the pioneering analysis has been carried out in Ref. [72], where the conjecture that the GUP-deformed temperature of a Schwarzschild black hole coincides with the modified Hawking temperature of a quantum-corrected Schwarzschild black hole yields $\beta = 82\pi/5$. Developments of this result have been obtained in Ref. [174], where the parameter α_0 appearing in the GUP with both a linear and quadratic term in momentum has been expressed in terms of the dimensionless ratio m_p/M, with M being the mass of the considered black hole. Along this line, in Ref. [78] a possible link between the GUP parameter β and the deformation parameter Υ arising in the framework of noncommutative geometry has been discussed in Schwarzschild spacetime. In particular, it has been argued that setting Υ of the order of Planck scale would lead to $|\beta| = 7\pi^2/2$.

Chapter 6

Conclusions and future perspectives

The previous Chapters have offered an overview on how to rely on fundamental principles in Physics. Indeed, we have first pedagogically discussed the significance of general covariance, equivalence principle and Heisenberg uncertainty relations in Chapter 1. Therefore, we have investigated a viable way to probe several implications of the aforementioned principles by means of the Casimir effect in Chapter 2. After this, Chapter 3 has been devoted to illustrate the perspectives that general covariance fulfillment entail by studying the properties of the decay of an accelerated proton. Then, Chapter 4 has explored some conditions under which the weak equivalence principle is violated. Finally, the generalization of Heisenberg uncertainty relations that takes into account gravitational effects and its consequences have been tackled in Chapter 5.

For each principle that has been mentioned up to now there is a vast literature, which thus confirms the existence of a vibrant interest of the scientific community towards such topics. The content of the present thesis is only a minuscule tile of the whole puzzle, but we hope to have at least conveyed the importance that the analysis of fundamental principles in Physics possesses. In this sense, the historical development of physical models supports the previous statement. As a matter of fact, a sharp and fine working hypothesis such as the ones that have been examined all along may lead to ground-breaking theoretical achievement, as it has been for the case of GR and QM, which have naturally stemmed from GC, EP and HUP. In our opinion, a similar path shall guide physicists towards a successful and unique theory for quantum gravity, as already suggested in the Introduction.

Apart from these considerations which should be clear by now to the reader, it must be said that there are a plethora of directions one can follow starting from the arguments contained in the previous Chapters, as there is still a huge amount of work to be done. To give an insight on the further developments that can be performed, we will concisely exhibit some future perspectives related to each topic separately.

- **Chapter 2:** as demonstrated with relevant examples, the Casimir effect turns out to be a valuable resource for the detection of new physics phenomenology. For this reason, it would be natural to try to extend the reasoning of Chapter 2 to other gravitational models and quantify the corrections that they would add to the standard results. In principle, this could also lead to the requirement of strict bounds for the free parameters of such theories, as seen in Sec. 2.2 for the framework of SME. However, this would necessarily entail an improvement in the sensitivity of experimental devices. In addition to the above arguments, it is worthy to stress that one should also perform these computations by taking into account more realistic physical fields, thus replacing a massless scalar field with either a massless vector field (i.e. photons) or a massive spin-1/2 field (the constituents of matter).

- **Chapter 3:** although it may appear that the analysis of the inverse β-decay is completed by the introduction of neutrino oscillations, there are still many issues yet to be unraveled. In this connection, we would like to recall that all the outcomes of Refs. [123] have been derived by resorting to a simplified two-flavor model. By moving on, we believe that general covariance fulfillment for the case of three flavors could settle the dispute revolving around the correct description for neutrinos. Unquestionably, this subject requires more attention and a greater effort, but the envisaged goal is extremely crucial for particle physics and, more specifically, for physics beyond the Standard Model. Moreover, the same calculations performed for the case of an accelerated proton can be extended to a more general scenario in which the particle is non-uniformly accelerating due to the presence of gravity. For instance, in proximity of a black hole, the "counterpart" of the Unruh radiation can be identified with the Hawking radiation, and similar circumstances would inevitably lead to intriguing observations (i.e. see Refs. [175] for a preliminary investigation in this direction).

- **Chapter 4:** the equation for non-relativistic neutrinos in the flavor basis must be studied with greater concern. In particular, we shall look at the implications arising from WEP violation from a quantum mechanical point of view, thus for instance seeking solutions for (4.25). With that knowledge, it would be possible to analyze also cosmological consequences, and specifically the ones inherent to relic neutrinos, as argued at the beginning of Sec. 4.1. On the other hand, the

WEP violation induced by the existence of a thermal bath is an elegant way to check that EP violation may occur even in a general relativistic framework. Such a result can be the starting point for several experimental tests aiming at verifying the tiny deviation from unity of the ratio (4.36). However, this is still not the end of the story, since the same considerations can be carried out in the context of extended theories of gravity, as shown in Sec. 4.2.2 with the Brans-Dicke model. Last but not least, one may also evaluate the error committed in the derivation of (4.36) by neglecting the influence of a non-vanishing temperature on the background geometry. A similar study can be useful to find a realistic geodesic equation that can be applied in any study involving the motion of test particles near gravitational sources, given that the temperature of our universe is about 2.73 K.

- **Chapter 5:** even though impossible to detect in the next years, GUP implications are still regarded as one of the most promising candidates for quantum gravity phenomenology. As a matter of fact, many QG models predict a modification of HUP that accounts for the presence of gravity. Motivated by such ideas, a significant number of tests (both gravitational and quantum ones) have tried to put an upper bound to the deformation parameter β, as argued in Sec. 5.1.2. Nevertheless, we are still far from achieving an experimental constraint that is close to the theoretical predictions. Despite this, investigations on GUP still continue to flourish, and in light of the contents of Chapter 5 it must be said that the comparison between different QG models is tracing the path towards the discovery of the crucial properties a genuine theory of quantum gravity should own. Therefore, the research on the consequences of GUP must be enhanced, so that in the (hopefully) not-so-remote future Physics will have all the indispensable tools to build a solid and consistent theory for the quantum description of gravity.

Appendix A

Geodesic deviation

In order to convey the concepts related to locality expressed in Sec. 1.2 in a more rigorous way, we will rely on a pedagogical book on GR [176] and in particular we will insist on the meaning of geodesic deviation.

With reference to Fig. 1.5, consider that the two apples are moving along different geodesics parametrized by the proper time τ, namely $x^\mu(\tau)$ and $y^\mu(\tau)$. Suppose that the deviation between these two geodesics is small, and represented as

$$y^\mu(\tau) - x^\mu(\tau) = \xi^\mu(\tau)\,. \tag{A.1}$$

Therefore, the geodesic equations are given by

$$\ddot{x}^\mu + \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 0\,, \tag{A.2}$$

$$\ddot{x}^\mu + \ddot{\xi}^\mu + \Gamma^\mu_{\alpha\beta}\left(\dot{x}^\alpha + \dot{\xi}^\alpha\right)\left(\dot{x}^\beta + \dot{\xi}^\beta\right) = 0\,. \tag{A.3}$$

Since ξ^μ is regarded as an infinitesimal parameter, we can expand (A.3) up to the first order in ξ. In this view, we point out that in the second equation $\Gamma^\mu_{\alpha\beta}(y) \equiv \Gamma^\mu_{\alpha\beta}(x+\xi)$, and hence we can expand also this quantity. After simple computations, we can then subtract (A.2) from (A.3), thus remaining with

$$\ddot{\xi}^\mu + 2\Gamma^\mu_{\alpha\beta}\dot{x}^\alpha\dot{\xi}^\beta + \dot{x}^\alpha\dot{x}^\beta\xi^\nu\partial_\nu\Gamma^\mu_{\alpha\beta} = 0\,. \tag{A.4}$$

At this point, we need to introduce the covariant derivative along a given curve $\gamma^\mu(\tau)$; for the case of a generic four-vector A^μ, we observe that [176]

$$\frac{DA^\mu}{D\tau} = \dot{A}^\mu + \Gamma^\mu_{\alpha\beta}A^\alpha\dot{x}^\beta\,. \tag{A.5}$$

Starting from (A.5), it is straightforward to deduce that we have to compute the second covariant derivative of ξ^μ along x^μ in such a way to evaluate the relative acceleration between the geodesics y^μ and x^μ. Should we perform the same analysis in absence of a gravitational field, all Christoffel symbols would be vanishing, thus ending with no relative departure of the two apples in Fig. 1.5.

Instead, if we perform the aforementioned calculation in the presence of gravity and we resort to (A.4) we obtain

$$\frac{D^2\xi^\mu}{D\tau^2} = -\,\xi^\nu R^\mu{}_{\beta\nu\alpha}\dot{x}^\alpha\dot{x}^\beta\,, \tag{A.6}$$

with $R^\mu{}_{\beta\nu\alpha}$ being the Riemann tensor. Clearly, this quantity encloses all the information related to the gravitational field, and hence it is non-vanishing. However, as expressed by (1.11), under certain conditions the effects of gravity can be locally "ignored", in the sense that the relative departure of two close geodesics (A.6) can be safely neglected to some extent. This is the important consideration underlying the concept according to which gravitational effects can be locally (and only locally) eliminated.

Appendix B

Linearized gravitational sector of SME

For the sake of completeness, in this Appendix we show all the conceptual considerations that led us to define the metric (2.12) in Sec. 2.2 by closely following the arguments contained in Ref. [97]. In so doing, we will investigate the characteristic features of the gravitational sector of SME in detail.

If we vary the action (2.8) with respect to $g_{\mu\nu}$ while keeping the Lorentz-violating fields u, $s^{\mu\nu}$ and $t^{\alpha\beta\gamma\delta}$ fixed, we obtain the following field equations:

$$G^{\mu\nu} - T^{\mu\nu}_{(LV)} = \kappa\, T^{\mu\nu}_{(m)}\,, \tag{B.1}$$

where $G^{\mu\nu}$ is the Einstein tensor, $T^{\mu\nu}_{(m)}$ is the usual stress-energy tensor derived from the matter action, whereas

$$\begin{aligned} T^{\mu\nu}_{(LV)} = & - \frac{1}{2}\nabla^\mu\nabla^\nu u - \frac{1}{2}\nabla^\nu\nabla^\mu u + g^{\mu\nu}\nabla_\alpha\nabla^\alpha u + G^{\mu\nu} u \\ & + \frac{1}{2}g^{\mu\nu}s^{\alpha\beta}R_{\alpha\beta} + \frac{1}{2}\nabla_\alpha\nabla^\mu s^{\alpha\nu} + \frac{1}{2}\nabla_\alpha\nabla^\nu s^{\alpha\mu} \\ & - \frac{1}{2}\nabla_\alpha\nabla^\alpha s^{\mu\nu} - \frac{1}{2}g^{\mu\nu}\nabla_\alpha\nabla_\beta s^{\alpha\beta} + \frac{1}{2}t^{\alpha\beta\gamma\mu}R_{\alpha\beta\gamma}{}^{\nu} \\ & + \frac{1}{2}t^{\alpha\beta\gamma\nu}R_{\alpha\beta\gamma}{}^{\mu} + \frac{1}{2}g^{\mu\nu}t^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} \\ & - \nabla_\alpha\nabla_\beta\left(t^{\mu\alpha\nu\beta} + t^{\nu\alpha\mu\beta}\right)\,. \end{aligned} \tag{B.2}$$

In the linearization procedure, the authors of Ref. [97] make several assumptions. The first two of them are related to the decomposition of Lorentz-violating fields as in (2.11). Indeed, it is claimed that

1) the vacuum values of the Lorentz-violating terms are constant in asymptotically inertial Cartesian coordinates;

2) the most relevant contributions of Lorentz violation are linear in the vacuum values of u, $s^{\mu\nu}$ and $t^{\alpha\beta\gamma\delta}$.

By virtue of these assumptions, it is possible to reformulate (B.1) in terms of linear combinations of $\tilde{u}$, $\tilde{s}^{\mu\nu}$, $\tilde{t}^{\alpha\beta\gamma\delta}$ and $h_{\mu\nu}$, where the last quantity represents the fluctuation around the flat metric $\eta_{\mu\nu}$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,. \tag{B.3}$$

The trace-reversed field equations derived from (B.1) can then be cast in the form

$$R_{\mu\nu} = \kappa\, E_{\mu\nu} + A_{\mu\nu} + B_{\mu\nu}\,, \tag{B.4}$$

where

$$E_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T^{\alpha}{}_{\alpha}\,, \tag{B.5}$$

with $T_{\mu\nu}$ being the linearized stress-energy tensor that contains the information on both the matter and fluctuations, whereas

$$\begin{aligned} A_{\mu\nu} = \; & - \;\; \partial_\mu\partial_\nu\tilde{u} - \frac{1}{2}\eta_{\mu\nu}\Box\tilde{u} + \partial_\alpha\partial_{(\mu}\tilde{s}^{\alpha}{}_{\nu)} - \frac{1}{2}\Box\tilde{s}_{\mu\nu} \\ & + \;\; \frac{1}{4}\eta_{\mu\nu}\Box\tilde{s}^{\alpha}{}_{\alpha} - 2\partial_\alpha\partial_\beta\tilde{t}_{\mu}{}^{\alpha}{}_{\nu}{}^{\beta} + \eta_{\mu\nu}\partial_\alpha\partial_\beta\tilde{t}^{\alpha\gamma\beta}{}_{\gamma} \\ & + \;\; \bar{s}^{\alpha}{}_{(\mu}\partial_\alpha\Gamma^{\beta}{}_{\nu)\beta} + \bar{s}^{\alpha\beta}\partial_\alpha\Gamma_{(\mu\nu)\beta} - \bar{s}^{\alpha}{}_{(\mu}\partial^{\beta}\Gamma_{\nu)\beta\alpha} \\ & + \;\; \frac{1}{2}\eta_{\mu\nu}\bar{s}^{\alpha}{}_{\beta}\partial^{\gamma}\Gamma^{\beta}{}_{\gamma\alpha} - 4\,\bar{t}_{(\mu}{}^{\alpha}{}_{\nu)}{}^{\beta}\partial_\alpha\Gamma^{\gamma}{}_{\gamma\beta}\,, \end{aligned} \tag{B.6}$$

and

$$\begin{aligned} B_{\mu\nu} = \; & - \;\; \frac{1}{2}\eta_{\mu\nu}\bar{s}^{\alpha\beta}R_{\alpha\beta} + \bar{u}\,R_{\mu\nu} + \bar{s}^{\alpha}{}_{(\mu}R_{\nu)\alpha} \\ & + \;\; 2\,\bar{t}^{\alpha\beta\gamma}{}_{(\mu}R_{\alpha\beta\gamma\nu)} - \frac{3}{2}\eta_{\mu\nu}\bar{t}^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} \\ & - \;\; 2\,\bar{t}_{(\mu}{}^{\alpha}{}_{\nu)}{}^{\beta}R_{\alpha\beta}\,, \end{aligned} \tag{B.7}$$

with the brackets for indexes denoting a sum over all possible permutations of them. Clearly, all the quantities related to the metric (such as Christoffel symbols and curvature tensors) are taken to be linearly expanded in $h_{\mu\nu}$. At this point, the third assumption of Ref. [97] states that

3) the fluctuations $\tilde{u}$, $\tilde{s}^{\mu\nu}$ and $\tilde{t}^{\alpha\beta\gamma\delta}$ are not coupled to conventional matter.

Such a choice immediately allows us to write $E_{\mu\nu}$ as

$$E_{\mu\nu} = E^{(m)}_{\mu\nu} + E^{(LV)}_{\mu\nu} \,. \tag{B.8}$$

In order to explicitly provide the latter term in the r.h.s. of the previous equation, we make use of the contracted Bianchi identity (namely $\nabla_\mu G^{\mu\nu} = 0$), which in the linearized version yields the following condition:

$$\kappa \partial_\mu T^{\mu\nu} = -\bar{s}_{\alpha\beta} \partial^\beta R^{\alpha\nu} - 2\,\bar{t}_{\alpha\beta\gamma\delta} \partial^\delta R^{\alpha\beta\gamma\nu} \,. \tag{B.9}$$

However, since $T^{\mu\nu}_{(m)}$ is separately conserved, Eq. (B.9) is satisfied if $E^{(LV)}_{\mu\nu}$ is equal to[1]

$$\begin{aligned} \kappa\, E^{(LV)}_{\mu\nu} = & - 2\,\bar{s}^{\alpha}{}_{(\mu} R_{\nu)\alpha} + \frac{1}{2} \bar{s}_{\mu\nu} \mathcal{R} + \eta_{\mu\nu} \bar{s}^{\alpha\beta} R_{\alpha\beta} \\ & - 4\,\bar{t}^{\alpha\beta\gamma}{}_{(\mu} R_{\alpha\beta\gamma\nu)} + 2\,\eta_{\mu\nu} \bar{t}_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} \\ & + 4\,\bar{t}^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu} R_{\alpha\beta} \,. \end{aligned} \tag{B.10}$$

The knowledge of the above term would permit us to analyze (B.4), but let us focus for a moment on the second term of the r.h.s. of the aforementioned equation, namely $A_{\mu\nu}$. Apparently, it seems that field equations still show a dependence on the fluctuations because of the tensor $A_{\mu\nu}$. Nevertheless, assumption 3 claims that these fluctuations only couple to gravity, which means that they can be related to $h_{\mu\nu}$ via their equations of motion. In order to have access to the complete picture, the last assumption plays a crucial role; in fact, it says that

4) the undetermined terms of $A_{\mu\nu}$ are linear combination of two partial derivatives of $h_{\mu\nu}$ and of the vacuum values $\eta_{\mu\nu}$, $\bar{u}$, $\bar{s}^{\mu\nu}$ and $\bar{t}^{\alpha\beta\gamma\delta}$.

If we apply this notion together with the diffeomorphism invariance of the action (2.8), it is possible to prove that [97]

$$\begin{aligned} A_{\mu\nu} = & - 2\,a\,\bar{u}\,R_{\mu\nu} + \bar{s}^{\alpha\beta} R_{\alpha(\mu\nu)\beta} - \bar{s}^{\alpha}{}_{(\mu} R_{\nu)\alpha} \\ & - b\,\bar{t}^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu} R_{\alpha\beta} - \frac{1}{4}\, b\, \eta_{\mu\nu} \bar{t}_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} \\ & - b\,\bar{t}^{\alpha\beta\gamma}{}_{(\mu} R_{\nu)\gamma\alpha\beta} \,, \end{aligned} \tag{B.11}$$

[1]In this step, we have neglected an assumption of Ref. [97] which is irrelevant for our purposes.

where a and b are arbitrary scaling factors. From the previous reasoning, we conclude that fluctuations are completely absent in the linearized regime, and hence the Post-Newtonian metric only depends on the vacuum values of the Lorentz-violating fields.

In order to reach the final formulation of the trace-reversed field equations, we must plug the expressions of $E_{\mu\nu}^{(LV)}$, $A_{\mu\nu}$ and $B_{\mu\nu}$ in (B.4). The outcome of this process gives

$$R_{\mu\nu} = \kappa\, E_{\mu\nu}^{(m)} + \bar{u}\, R_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}\bar{s}_{\alpha\beta}R^{\alpha\beta} - 2\,\bar{s}^{\alpha}{}_{(\mu}R_{\alpha\nu)} + \frac{1}{2}\bar{s}_{\mu\nu}\mathcal{R} + \bar{s}^{\alpha\beta}R_{\alpha\mu\nu\beta}\,. \quad \text{(B.12)}$$

Note that in (B.12) the information related to the vacuum value of $t^{\alpha\beta\gamma\delta}$ vanishes. This occurrence is also encountered whenever one attempts to study phenomenology attributable to the gravity sector of SME, and it is addressed as *t puzzle* in literature [177]. Moreover, the term related to the vacuum value $\bar{u}$ only appears near the Ricci tensor, which means that, if $\bar{u} \neq 0$, it acts as a mere scaling parameter for the Post-Newtonian metric we want to derive. For this reason, in accordance with Ref. [97], it can be safely neglected (namely, we can set $\bar{u} = 0$).

All the observations performed up to now explain why the metric (2.12) does not exhibit a dependence on the factors $\bar{u}$ and $\bar{t}^{\alpha\beta\gamma\delta}$. On the other hand, our choice of dealing with a point-like source of gravity further simplifies calculations, since among the potentials for a perfect fluid contained in Ref. [97] with which the Post-Newtonian expansion is carried out, only one is non-vanishing for a non-extended object, and it is the potential that allows for the appearance of the 00−component of $\bar{s}^{\mu\nu}$ alone.

Appendix C

Quadratic theories of gravity

In order to better describe the features of the quadratic models of gravity presented in Sec. 2.3, we will resort to a unified treatment already present in literature [94, 110]

To this aim, let us consider the most general gravitational action which is quadratic in the curvature, parity-invariant and torsion-free [110]

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left\{\mathcal{R} + \frac{1}{2}\left[\mathcal{R}\,\mathcal{F}_1(\Box)\,\mathcal{R} + R_{\mu\nu}\mathcal{F}_2(\Box)R^{\mu\nu} + R_{\mu\nu\rho\sigma}\mathcal{F}_3(\Box)R^{\mu\nu\rho\sigma}\right]\right\}, \tag{C.1}$$

where $\kappa := \sqrt{8\pi G}$, $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the d'Alembert operator in curved spacetime and the form-factors $\mathcal{F}_i(\Box)$ are generic operators of $\Box$ that can be either local or non-local

$$\mathcal{F}_i(\Box) = \sum_{n=0}^{N} f_{i,n}\Box^n, \qquad i = 1, 2, 3. \tag{C.2}$$

In principle, one can deal with both positive and negative powers of the d'Alembertian, which means that we can consider both ultraviolet and infrared modifications of GR. Note that, if $n > 0$ and N is finite (namely, $N < \infty$), we have a local theory of gravity of order $2N+2$ in derivatives, whereas if $N = \infty$ and/or $n < 0$ we have a non-local theory of gravity whose form-factors $\mathcal{F}_i(\Box)$ are *not* polynomials of $\Box$.

Since the interest in Sec. 2.3 is devoted to the study of quadratic theories of gravity in the weak-field approximation, we can work with the linearized regime of (C.1) around the Minkowski background $\eta_{\mu\nu}$

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}\,, \tag{C.3}$$

where $h_{\mu\nu}$ is the linearized metric perturbation.

At the linearized level, the relevant contribution coming from the action is of the order $\mathcal{O}(h^2)$; in such a regime, the term $R_{\mu\nu\rho\sigma}\mathcal{F}_3(\Box)R^{\mu\nu\rho\sigma}$ in (C.1) can be safely neglected. Indeed, if we do not exceed the aforementioned order of expansion, it is always possible to rewrite the Riemann squared contribution in terms of Ricci scalar and Ricci tensor squared by virtue of the following identity:

$$R_{\mu\nu\rho\sigma}\Box^n R^{\mu\nu\rho\sigma} = 4R_{\mu\nu}\Box^n R^{\mu\nu} - \mathcal{R}\Box^n\mathcal{R} + \mathcal{O}(\mathcal{R}^3) + \text{div}, \tag{C.4}$$

where div stands for total derivatives and $\mathcal{O}(\mathcal{R}^3)$ only contributes at order $\mathcal{O}(h^3)$. Hence, in the linearized regime we can set $\mathcal{F}_3(\Box) = 0$ without loss of generality.

We now want to linearize the action (C.1) and analyze the corresponding linearized field equations. By expanding the spacetime metric around the Minkowski background as in (C.3), the quadratic gravitational action up to the order $\mathcal{O}(h^2)$ reads [110]

$$\begin{aligned} S &= \frac{1}{4}\int d^4x \left\{\frac{1}{2}h_{\mu\nu}a(\Box)\Box h^{\mu\nu} - h^\sigma_\mu a(\Box)\partial_\sigma\partial_\nu h^{\mu\nu} + hc(\Box)\partial_\mu\partial_\nu h^{\mu\nu}\right. \\ &- \left.\frac{1}{2}hc(\Box)\Box h + \frac{1}{2}h^{\lambda\sigma}\frac{a(\Box)-c(\Box)}{\Box}\partial_\lambda\partial_\sigma\partial_\mu\partial_\nu h^{\mu\nu}\right\}, \end{aligned} \tag{C.5}$$

where $h \equiv \eta_{\mu\nu}h^{\mu\nu}$ and

$$\begin{aligned} a(\Box) &= 1 + \frac{1}{2}\mathcal{F}_2(\Box)\Box\,, \\ c(\Box) &= 1 - 2\mathcal{F}_1(\Box)\Box - \frac{1}{2}\mathcal{F}_2(\Box)\Box\,. \end{aligned} \tag{C.6}$$

The related linearized field equations are represented by

$$\begin{aligned} 2\kappa^2 T_{\mu\nu} &= f(\Box)\left(\Box h_{\mu\nu} - \partial_\sigma\partial_\nu h^\sigma_\mu - \partial_\sigma\partial_\mu h^\sigma_\nu\right) + g(\Box)\left(\eta_{\mu\nu}\partial_\rho\partial_\sigma h^{\rho\sigma} + \partial_\mu\partial_\nu h - \eta_{\mu\nu}\Box h\right) \\ &+ \frac{f(\Box)-g(\Box)}{\Box}\partial_\mu\partial_\nu\partial_\rho\partial_\sigma h^{\rho\sigma}, \end{aligned} \tag{C.7}$$

where

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}}\,, \tag{C.8}$$

is the stress-energy tensor generating the gravitational field, with S_m being the matter action.

We are interested in finding the expression for the linearized metric generated by a static point-like source[1]

$$ds^2 = (1+2\Phi)dt^2 - (1-2\Psi)(dr^2 + r^2 d\Omega^2), \tag{C.9}$$

where Φ and Ψ are the metric potentials generated by

$$T_{\mu\nu} = m\delta^0_\mu \delta^0_\nu \delta^{(3)}(\mathbf{r}). \tag{C.10}$$

By using $\kappa h_{00} = 2\Phi$, $\kappa h_{ij} = 2\Psi\delta_{ij}$, $\kappa h = 2(\Phi - 3\Psi)$ and assuming the source to be static, that is $\Box \simeq -\nabla^2$, $T = \eta_{\rho\sigma}T^{\rho\sigma} \simeq T_{00} = \rho$, the field equations for the two metric potentials read[2]

$$\begin{aligned} \frac{a(a-3c)}{a-2c}\nabla^2\Phi(r) &= 8\pi G\rho(r), \\ \frac{a(a-3c)}{c}\nabla^2\Psi(r) &= -8\pi G\rho(r), \end{aligned} \tag{C.11}$$

where $a \equiv a(\nabla^2)$, $c \equiv c(\nabla^2)$ and $\rho(r) = m\delta^{(3)}(\mathbf{r})$.

We can solve the two differential equations (C.11) by employing Fourier transform and then anti-transform to coordinate space. Thus, we obtain

$$\Phi(r) = -8\pi Gm \int \frac{d^3k}{(2\pi)^3}\frac{1}{k^2}\frac{a-2c}{a(a-3c)}e^{i\mathbf{k}\cdot\mathbf{r}} = -\frac{4Gm}{\pi r}\int_0^\infty dk \frac{a-2c}{a(a-3c)}\frac{\sin(kr)}{k},$$

$$\Psi(r) = 8\pi Gm \int \frac{d^3k}{(2\pi)^3}\frac{1}{k^2}\frac{c}{a(a-3c)}e^{i\mathbf{k}\cdot\mathbf{r}} = \frac{4Gm}{\pi r}\int_0^\infty dk \frac{c}{a(a-3c)}\frac{\sin(kr)}{k}, \tag{C.12}$$

where $a \equiv a(k^2)$ and $c \equiv c(k^2)$.

It is immediate to observe that, if $a = c$, the two metric potentials coincide, $\Phi = \Psi$. Therefore, as a special case we recover general relativity

$$a = c = 1 \implies \Phi(r) = \Psi(r) = -\frac{Gm}{r}, \tag{C.13}$$

as expected.

According to the choice of the form-factors appearing in (C.1), we have different values for a and c, thus resulting in a precise quadratic model of gravity. In order to account for all the theories analyzed in Sec. 2.3, in the following table we will report all the form-factors from which the aforementioned models stem.

[1]Note that the linearized metric in (C.9) is expressed in isotropic coordinates, where $dr^2 + r^2d\Omega^2 = dx^2 + dy^2 + dz^2$.

[2]In order to obtain the differential equations (C.11), we have considered and combined the trace and (00)-component of the field equations (C.7).

Table C.1: The form-factors for the considered quadratic theories of gravity.

Model	Form-factors
$f(\mathcal{R})$	$\mathcal{F}_1 = \alpha \quad \mathcal{F}_2 = 0$
Fourth order gravity	$\mathcal{F}_1 = \alpha \quad \mathcal{F}_2 = \beta$
Sixth order gravity	$\mathcal{F}_1 = \alpha\Box \quad \mathcal{F}_2 = \beta\Box$
Infinite derivative gravity	$\mathcal{F}_1 = -\mathcal{F}_2/2 = (1 - e^{-\Box/M_s^2})/2\,\Box$
Non-local gravity	$\mathcal{F}_1 = \alpha/\Box \quad \mathcal{F}_2 = 0$

Appendix D

Neutrino mixing and flavor states

In this Appendix, we will elucidate the main aspects related to the standard treatment of neutrino mixing together with its extension in the language of QFT. For the sake of conciseness, we will deal with a simplified two-flavor model, but an analogous reasoning holds also in the case of three flavors.

As already discussed in Sec. 3.2, it is well-known that neutrino phenomenology can be described with an excellent degree of precision by Pontecorvo flavor states [127] introduced in (3.33), namely

$$\begin{aligned} |\nu_e(x)\rangle &= \cos\theta\,|\nu_1(x)\rangle + \sin\theta\,|\nu_2(x)\rangle\,, \\ |\nu_\mu(x)\rangle &= -\sin\theta\,|\nu_1(x)\rangle + \cos\theta\,|\nu_2(x)\rangle\,, \end{aligned} \tag{D.1}$$

where $|\nu_i(x)\rangle$, $i = 1, 2$ represent states with definite mass, whose evolution is governed by the usual formula

$$|\nu_i(x)\rangle = e^{-i(E_i t - \mathbf{k}_i\cdot\mathbf{x})}|\nu_i(0)\rangle\,. \tag{D.2}$$

In the above framework, it is not difficult to observe that the flavor and mass basis are not conceptually different, in that they are linked by a mere rotation (see (3.33)). In other words, this implies that the vacuum states $|0\rangle_{e,\mu}$ and $|0\rangle_{1,2}$ are completely equivalent, $|0\rangle_{e,\mu} \equiv |0\rangle_{1,2}$.

With the knowledge of (D.1) and (D.2), one can evaluate the transition probabilities for the flavor conversion. In particular, the probability to pass from an α to a β flavor neutrino with $\alpha, \beta = e, \mu$ is

$$\mathcal{P}_{\alpha\to\beta} = \left|\langle\nu_\beta(x)|\nu_\alpha(0)\rangle\right|^2\,. \tag{D.3}$$

If we now assume that the distance traveled by the flavor neutrino is L, we set $\alpha = \mu$

and $\beta = e$ and we require to work in the ultra-relativistic regime, so that

$$E_i = \sqrt{|\mathbf{k}_i|^2 + m_i^2} \simeq |\mathbf{k}_i| + \frac{m_i^2}{2|\mathbf{k}_i|} \simeq E + \frac{m_i^2}{2E}, \tag{D.4}$$

with E being the total energy of the particle, Eq. (D.2) can be rephrased as

$$|\nu_i(x)\rangle = e^{-i\frac{m_i^2 L}{2E}} |\nu_i(0)\rangle . \tag{D.5}$$

In light of the aforesaid considerations, it is an easy task to prove that

$$\mathcal{P}_{\mu\to e} = \sin^2(2\theta)\sin^2\left(\frac{\Delta m^2 L}{4E}\right), \tag{D.6}$$

with $\Delta m^2 = m_2^2 - m_1^2$. The above formula is the renowned flavor oscillation probability for a two-flavor model. However, for future convenience it is opportune to evaluate the above probability in the case of equal momenta $|\mathbf{k}_1| = |\mathbf{k}_2|$; such a procedure yields

$$\mathcal{P}_{\mu\to e} = \sin^2(2\theta)\sin^2\left[\frac{(E_2 - E_1)\,t}{2}\right], \tag{D.7}$$

which will be recalled later on.

On the other hand, if we want to tackle neutrino mixing from a field theoretical point of view, we should start from the following Lagrangian for flavor fields [127, 137]:

$$\mathcal{L} = \overline{\psi}_e\left(i\partial\!\!\!/ - m_e\right)\psi_e + \overline{\psi}_\mu\left(i\partial\!\!\!/ - m_\mu\right)\psi_\mu - m_{e\mu}\left(\overline{\psi}_e\psi_\mu + \overline{\psi}_\mu\psi_e\right), \tag{D.8}$$

that can be diagonalized by imposing the same rotation of (D.1), but at the level of fields

$$\begin{aligned} \psi_e(x) &= \cos\theta\,\psi_1(x) + \sin\theta\,\psi_2(x)\,, \\ \psi_\mu(x) &= -\sin\theta\,\psi_1(x) + \cos\theta\,\psi_2(x)\,, \end{aligned} \tag{D.9}$$

which thus gives the Lagrangian for two free spinor fields

$$\mathcal{L} = \overline{\psi}_1\left(i\partial\!\!\!/ - m_1\right)\psi_1 + \overline{\psi}_2\left(i\partial\!\!\!/ - m_2\right)\psi_2\,, \tag{D.10}$$

with the mass terms that can be rewritten as a function of m_e, m_μ and $m_{e\mu}$ according to (4.15). In so doing, it is possible to verify [137] that the flavor and mass Fock spaces one can build from the respective vacuum states are *unitarily in-*

equivalent [178], since $|0\rangle_{e,\mu}$ is indeed a condensate of different fermion-antifermion pairs

$$\begin{aligned}|0\rangle_{e,\mu} &= \prod_{k,\sigma}\Big[\big(1-\sin^2\theta|V_k|^2\big) - \varepsilon^\sigma \sin\theta\cos\theta\, V_k\, (A_k^\sigma + B_k^\sigma) \\ &+ \varepsilon^\sigma \sin^2\theta\, (U_k^*\, C_k^\sigma - U_k\, D_k^\sigma) + \sin^2\theta|V_k|^2\, A_k^\sigma\, B_k^\sigma\Big]\, |0\rangle_{1,2}\,, \qquad (D.11)\end{aligned}$$

where $\varepsilon^\sigma = (-1)^\sigma$ and

$$A_k^\sigma \equiv b_{k,1}^{\sigma\,\dagger}\, d_{-k,2}^{\sigma\,\dagger}, \quad B_k^\sigma \equiv b_{k,2}^{\sigma\,\dagger}\, d_{-k,1}^{\sigma\,\dagger}, \quad C_k^\sigma \equiv b_{k,1}^{\sigma\,\dagger}\, d_{-k,1}^{\sigma\,\dagger}, \quad D_k^\sigma \equiv b_{k,2}^{\sigma\,\dagger}\, d_{-k,2}^{\sigma\,\dagger}, \qquad (D.12)$$

with $b_{k,j}^\sigma$ ($d_{k,j}^\sigma$), $j = 1, 2$ being the annihilators for neutrinos (antineutrinos) of mass m_j, momentum k and polarization σ. These operators are related to the corresponding annihilators for neutrinos (antineutrinos) with definite flavor as follows:

$$\begin{aligned} b_{k,e}^\sigma &= \cos\theta\, b_{k,1}^\sigma + \sin\theta\left(U_k^*\, b_{k,2}^\sigma + \varepsilon^\sigma\, V_k\, d_{-k,2}^{\sigma\,\dagger}\right), \\ b_{k,\mu}^\sigma &= \cos\theta\, b_{k,2}^\sigma - \sin\theta\left(U_k\, b_{k,1}^\sigma - \varepsilon^\sigma\, V_k\, d_{-k,1}^{\sigma\,\dagger}\right), \\ d_{-k,e}^\sigma &= \cos\theta\, d_{-k,1}^\sigma + \sin\theta\left(U_k^*\, d_{-k,2}^\sigma - \varepsilon^\sigma\, V_k\, b_{k,2}^{\sigma\,\dagger}\right), \\ d_{-k,\mu}^\sigma &= \cos\theta\, d_{-k,2}^\sigma - \sin\theta\left(U_k\, d_{-k,1}^\sigma + \varepsilon^\sigma\, V_k\, b_{k,1}^{\sigma\,\dagger}\right). \qquad (D.13)\end{aligned}$$

The above expressions are the combination of a rotation and a Bogoliubov transformation. The Bogoliubov coefficients are defined as

$$U_k = u_{k,2}^{\sigma\,\dagger}\, u_{k,1}^\sigma = v_{-k,1}^{\sigma\,\dagger} v_{-k,2}^\sigma, \qquad V_k = \varepsilon^\sigma u_{k,1}^{\sigma\,\dagger}\, v_{-k,2}^\sigma = -\varepsilon^\sigma u_{k,2}^{\sigma\,\dagger}\, v_{-k,1}^\sigma, \qquad (D.14)$$

where $u_{k,i}^\sigma$ ($v_{-k,i}^\sigma$) are the field modes for fermions (antifermions). By explicit calculation, one can show that [137]

$$U_k = |U_k|\; e^{i(E_2-E_1)t}, \qquad V_k = |V_k|\; e^{i(E_2+E_1)t}, \qquad (D.15)$$

with

$$\begin{aligned} |U_k| &= \left(\frac{E_1+m_1}{2E_1}\right)^{\frac{1}{2}}\left(\frac{E_2+m_2}{2E_2}\right)^{\frac{1}{2}}\left(1+\frac{k^2}{(E_1+m_1)(E_2+m_2)}\right), \qquad (D.16) \\ |V_k| &= \left(\frac{E_1+m_1}{2E_1}\right)^{\frac{1}{2}}\left(\frac{E_2+m_2}{2E_2}\right)^{\frac{1}{2}}\left(\frac{k}{(E_2+m_2)} - \frac{k}{(E_1+m_1)}\right), \qquad (D.17)\end{aligned}$$

and

$$|U_k|^2 + |V_k|^2 = 1. \tag{D.18}$$

In this field theoretical context, an accurate analysis of the flavor transition probability $\mathcal{P}_{\mu\to e}$ in the limit $|\mathbf{k}_1| = |\mathbf{k}_2|$ gives [137]

$$\mathcal{P}_{\mu\to e} = \sin^2(2\theta)\left\{|U_k|^2 \sin^2\left[\frac{(E_2 - E_1)\,t}{2}\right] + |V_k|^2 \sin^2\left[\frac{(E_2 + E_1)\,t}{2}\right]\right\}. \tag{D.19}$$

The above expression is the QFT generalization of (D.7), which is recovered in the ultra-relativistic limit, for which $|U_k| \to 1$ and $|V_k| \to 0$.

Appendix E

Brans-Dicke model

As already anticipated in Sec. 4.2.2, Brans-Dicke model is the most famous scalar-tensor theory of gravity. In order to properly discuss it, it is opportune to briefly introduce the main ideas of the scalar-tensor models.

For this purpose, let us consider the action [144]

$$S_J = \int d^4x\sqrt{-g}\left[\varphi_J^{\gamma}\left(R - \omega_J \frac{1}{\varphi_J^2} g^{\mu\nu}\partial_\mu\varphi_J\partial_\nu\varphi_J\right) + \mathcal{L}_{\text{matter}}\left(\varphi_J, \psi\right)\right], \tag{E.1}$$

where φ_J is the scalar field, γ and ω_J are constants and ψ contains the contribution of matter fields. We immediately note that:

- $-\omega_J\,\varphi_J^{-2}\,g^{\mu\nu}\partial_\mu\varphi_J\partial_\nu\varphi_J$ can be correctly interpreted as the kinetic contribution related to the scalar field;

- $\varphi_J^{\gamma}R$ is a non-minimal coupling term;

- the Lagrangian density $\mathcal{L}_{\text{matter}}$ depends not only on the matter fields, but in principle also on the scalar field.

The above expression describes a conspicuous number of models, according to the choice for the free parameters. However, we are mainly concerned with the one developed by Brans and Dicke [143], which is examined in the following.

Let us introduce the Brans-Dicke action [143], which is similar to the one showed in (E.1), but with several differences

$$S_{BD} = \int d^4x\sqrt{-g}\left(\varphi R - \omega\frac{1}{\varphi}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + \mathcal{L}_{\text{matter}}\left(\psi\right)\right). \tag{E.2}$$

In (E.2), the matter Lagrangian density does not depend on the scalar field and $\gamma = 1$. This is crucial, because it means that ω is the only parameter of the theory.

Moreover, it is clear that

$$\varphi = \frac{1}{16\pi G_{eff}}, \tag{E.3}$$

and such a result is traduced in the introduction of a new "effective" gravitational constant that has to be identified with the scalar field. This consideration requires some restrictions. In particular, it is essential that φ is spatially uniform, and it must vary slowly with cosmic time. If these characteristics are not possessed by φ, the theory cannot be consistent with experimental data, since they clearly support the presence of a gravitational constant that enters field equations as predicted by GR. On the other hand, experiments may put a constraint on the only free parameter of the theory, namely ω. In this sense, the Brans-Dicke gravity can be genuinely regarded as an extended theory of gravity which generalizes results of GR.

Field equations derived from (E.2) are given by

$$2\varphi G_{\mu\nu} = T_{\mu\nu} + T^{\varphi}_{\mu\nu} - 2\left(g_{\mu\nu}\Box - \nabla_\mu \nabla_\nu\right)\varphi, \tag{E.4}$$

that can be obtained by means of a variation with respect to $g^{\mu\nu}$, and

$$\Box\varphi = \zeta^2 T, \tag{E.5}$$

deduced by a variation with respect to φ, where $\zeta^{-2} = 6 + 4\omega$ and $T = g^{\mu\nu}T_{\mu\nu}$. In (E.4), $T_{\mu\nu}$ and $T^{\varphi}_{\mu\nu}$ are extracted by varying $\mathcal{L}_{matter}$ and the kinetic term of S_{BD}, respectively. As expected, field equations for the metric tensor becomes the ones derived by GR in the limit $\varphi = \text{const} = 1/16\pi G$.

If a static and isotropic solution is now sought (in order to fit the description of Sec. 4.2.2), it is possible to find an expression for the line element

$$ds^2 = e^v dt^2 - e^u \left[dr^2 + r^2\left(d\vartheta^2 + \sin^2\vartheta d\Phi^2\right)\right], \tag{E.6}$$

where

$$e^v = e^{2\alpha_0}\left(\frac{1-\frac{B}{r}}{1+\frac{B}{r}}\right)^{\frac{2}{\lambda}}, \qquad e^u = e^{2\beta_0}\left(1+\frac{B}{r}\right)^4\left(\frac{1-\frac{B}{r}}{1+\frac{B}{r}}\right)^{\frac{2(\lambda-C-1)}{\lambda}}, \tag{E.7}$$

with α_0, β_0, B, C and λ being constants that can be connected to the free parameter of the theory ω. Since it is a scalar-tensor theory, a solution for φ must also be found;

in the considered case, the outcome turns out to be

$$\varphi = \varphi_0 \left(\frac{1 - \frac{B}{r}}{1 + \frac{B}{r}} \right)^{-\frac{C}{\lambda}}, \tag{E.8}$$

where φ_0 is another constant.

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