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The use of technologies in Mathematics Education

research paths in the "Mathematical High School" Project

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Sono stati tre anni ricchi, intensi ed impegnativi.

Non è stato un corso di studi, è stato un indimenticabile percorso di vita.

CHAPTER 1

INTRODUCTION

In the last twenty years the school has been hit by a succession of disruptive and unknown innovations. In particular, there is institutional talk for the first time of ascertaining competences in DPR 323-98 (reform of the state examinations of secondary school). Leaving aside the various indications that have followed one another very quickly, in the Regulations for the Reorganization of Higher Institutes (DPR 87, 88 and 89 of 2010) a disciplinary system centered on competences is envisaged, followed by the relative ministerial indications. Finally, we come to the much-discussed Law 107 of 2015 in which it is provided that the activity of School-Work Alternation is extended to all three-year courses of high schools.

The didactics “skills’ oriented”, strongly solicited on the theoretical level for the evident positive effects on the dynamics of learning, still highlights, in particular in secondary school, some ostracisms for an effective and complete implementation, due to the difficulty of transporting theoretical indications to the classroom teaching.

In recent years, teachers have been invited to participate in many refresher courses to adapt their activities to the new teaching but if on the one hand there is a rich literature on the pedagogical-didactic approach for skills and on reality tests, on the other hand it must be recognized that the production of materials actually useful to teachers as they can be used directly in the classroom is very scarce.

I personally viewed texts and exercises with a skills’ setting and I found that at the moment the books are moving in that direction, but not all the texts have an adequate structure in the organization of the topics. Even with regard to reality tests, they are only few "problems" differently set at the end of each chapter.

The lack of reference material inevitably ends up slowing down the processes of educational innovation in the school.

To this, should also be added all the school-work alternation paths that scientific lyceums try to set up with universities and research centers in a unitary vision of scientific knowledge. Also, the Scientific Degree Plans (PLS) provides for research-action activities on unconventional levels of learning.

Finally, in order of time, there is to underline the project “Mathematical High School” that is expanding like wildfire throughout Italy: particularly motivated and brilliant students deepen their knowledge in rich and engaging transcurricular educational solicitations designed and presented by researchers in mathematics education (Liceo Matematico).

In the face of these new challenges and educational frontiers, the school responds too often, unfortunately, with obsolete laboratories and teachers who, although willing and interested, remain confused about implementation practices.

To pretend to be up to date, schools find themselves "changing everything without anything changing": the evaluation grids become evaluative rubrics or evaluation portfolios, the tasks in the classroom become evidence of reality. But concretely, the operation is a change of name, not a change of structure.

Internationally, contemporary educational systems showed difficulties in interfacing with our highly technologized society, in particular in digital technologies, because of the rapid and profound changes (Collins, Halverson 2009), specifically with the need to overcome the gap in digital skills that has accumulated over the years (Avvisati, Kotzman, Hennessy, Vincent-Lacrin 2013) to enhance teaching practice and optimize training (Pellerey 2015).

In Italy in the recent years, in order to reduce to narrow the gap between the high-tech real world and the school with its modest resources, the Ministry of Education approved the realization of the “National Digital School Plan” (PNSD) dedicated to the acquisition of new hardware and software and to the professional updating of teachers to raise their degree of digital skills (Avvisati, Kotzman, Hennessy, Vincent-Lacrin 2013) and thousands of teachers participated to these activities.

The PNSD states that “digital technologies intervene to support all dimensions of transversal skills. But they also fit vertically, as part of the literacy of our time and fundamental skills for a full, active and informed citizenship”, as anticipated by the Recommendation of the European Parliament and the Council of Europe and as even better emphasized by frameworks such as the 21st Century Skills, promoted by the "World Economic Forum" (Bellanca 2010).

In fact, the S.T.E.M. curriculum incorporates the "four C's" of 21st-century skills: creativity, critical thinking, collaboration and communication. Students work together to create innovative solutions to real-world problems and communicate their solutions with others. As they carry out their investigations and projects, they must access, analyze, and use the information they need to complete the learning tasks. While working through the task, students build important life and career skills by learning to manage their time, to become self-directed workers and to collaborate effectively with others. Using appropriate technology tools to complete their task, students discover the most effective and efficient ways to access and manage the world of digital information that is available to them (Beers 2011).

The importance of new skills not included in the traditional school curricula was highlighted in numerous teaching conferences where it was emphasized that "Exemplary science education can offer a rich context for developing many 21st-century skills, such as critical thinking, problem-solving, and information literacy. These skills not only contribute to a well-prepared workforce of the future but also give all individuals life skills that help them succeed." (NSTA 2011)

Based on my experience both as a trainer and as a trainee and the various interviews organized with the teachers, it was observed that even high-profile and good quality training courses are not satisfactory for teachers because the topics covered in the courses do not reflect the topics they are doing in their classrooms and the delay between when they learn and when they are able to put into practice is too long and part of the acquired content is lost (Bologna, Perrotta, Rogora, Veronesi 2020). Teachers are actually asking for practical guides, operational models of teaching by skills and problems placed in real contexts.

To meet this need, high schools are developing collaboration paths with universities to enrich the curriculum of students through high-profile technological and scientific activities.

On the other hand, the students' families show a strong desire for quality in the teaching-learning processes. Proof of this is the strong demand for enrollment of their sons and daughters in the Mathematical High Schools, schools that participate in The Mathematical High School Project, a research project that offers some further in-depth courses organized by the Department of Mathematics of the University of Salerno, aimed at expanding the training of students, with the aim of developing their critical skills and aptitude for scientific research by always placing the student with his or her styles, times and learning methods at the center of every didactic activity.

1.1 My research in the "Mathematical High School" Project

The choice of the research group in mathematics education of the Department of Mathematics of the University of Salerno to develop educational paths that exploit the potential of technological tools for the training of teachers and laboratory activities for students of upper secondary school, is the natural evolution of an educational Didactic Design aimed at developing models and paths that can be proposed in the classroom thanks to the help of new technologies.

The research group has been involved for years in updating the teaching of mathematics by bringing to the center of educational choices the close interconnection between the various disciplines and in particular the role of mathematics as a bridge between the two cultures, the scientific and the humanistic. In this interdisciplinary key that aims to become transdisciplinary, new technologies have the important task of allowing the reworking of in-depth paths on the various curricular themes through a laboratory activity and in particular experimental research.

This vision of mathematics as a link between the various cultural areas and new technologies as the key to activating these interconnections, led the didactic research group in mathematics of Salerno to develop numerous training proposals at various levels. The will of the construction of educational paths was therefore to provide a substantial cultural and notional baggage to make a conscious and reasoned choice about students' future studies thanks to the enrichment of technology in teaching (Rabardel 2002) exploiting the positive motivation of studying mathematics in terms of needs and skills (Gura, Maschler 2008), (Waege 2009).

My research project in mathematics teaching aimed to

- deepen the interrelationships of mathematics with other disciplinary areas,
- prepare both informative and technical teaching material, both in the traditional format of dossiers and handouts
- organize and structure educational paths for skills that provide for the presence of "mathematics with...", or not monothematic paths where mathematics is linked to physics and chemistry, but transdisciplinar paths that tie mathematics to all the disciplines of the curriculum. In particular, the importance of mathematics in various historical contexts, in philosophical processes, interconnections with literature, art, music, religions, sport...
- implement the use of new technologies and new learning processes also thanks to dedicated hardware and software. Young people are commonly called "digital natives", they are used to living with computer technology much more than previous generations and it is appropriate that they also find it in study paths, in order to learn that even non-"dedicated" tools can be portable laboratories (exactly as twenty years ago scientific calculators supplanted logarithmic and trigonometric tables ...). Inevitably the future will lead to an increasingly pushed and exasperated use. It therefore becomes essential that the world of education uses and makes its own the available tools and transforms them into an "accelerator of interest": only if teachers and students speak the same language, they can communicate.

The extraordinary upheaval that the school world has undergone in the last two years in light of the effects of the Covid-19 pandemic and the consequent lockdowns, has strengthened and accelerated the affirmation of an unprecedented level of distance learning. The lockdowns have further pushed my research in a direction of predisposition to design a series of activities developed on-line for teaching implemented with the use of technologies, also thanks to the aid of e-learning platforms for distance learning. The closure of all educational institutions led to the modification of some of the paths designed to be developed in presence and made it almost impossible to have observations "on the field" in presence, where many ideas could have been provided thanks also to those non-verbal indicators in terms of appreciation and / or difficulty, as looks, gestures, pauses and uncertainties even in writing. Distance learning has prevented (due to the "closed microphones" to avoid background noise and "cameras off" not to weaken the signal) to fully experience the teaching activities, but "making a virtue by the necessity", we transformed the verbal brain storming into modules to be discussed in real time, the laboratories in presence into activities on remote platforms with screen sharing, the paper satisfaction questionnaires answered in presence into moodles, the group activities around the tables in breakout rooms of the various platforms.

Since my path involves the use of technology in teaching, in fact the crisis has been a natural driving force to "force" the school to implement towards the future.

From teaching mathematics with the use of technologies, my path has expanded to teaching with technologies, because I found myself supporting schools in this swirling and immediate process from which there will be no going back.

The activities have been structured in order to be a real teaching tool and to avoid a fragmentation of knowledge, contents and methods. They have been projected for

- different level of classes of the Mathematical High Schools
- refresher courses and training courses for teachers
- collaborative projects between upper secondary schools and the University such as scientific degree plans (PLS) or school and work alternation (ASL initially, then PCTO).

In my PhD course I have dealt with three effective applications of different technological contexts and applications and repercussions in teaching and learning processes. My research spaces have been the training-updating activities of secondary school teachers and the activities of Mathematical High School, Scientific Degree Plan of secondary school students as well as the presentation and dissemination with communications and interventions in national and international conferences dedicated to teaching.

Specifically, during my PhD I have been involved in many activities carried out by the research group in mathematics teaching of the DIPMAT of UNISA. The paths that I have developed in my doctoral course have been inserted curricularly into the activities that are carried out in the Mathematical High School, implement the activities already structured with new paths strongly based on the use of technologies in teaching and are currently in the teaching calendar of schools. With a view to sharing good practices to develop a collaborative network, some aspects of the activities that will be described and analyzed in depth in this thesis have been presented in national and international mathematics education conferences and are published in the proceedings.

I mainly dealt with three research subjects:

- mathematics with graphic calculators, a path of heuristic exploration of mathematical concepts through different learning channels, the numerical, graphic, analytical, tabular approach developed in the classes of the Mathematical High Schools with observations on the impact on students' learning levels thanks to an active teachers' training model in a research project DIPMAT UNISA-CASIO ITALIA (chapter 3)
- mathematics and economics as a language to understand and explain a real-life problem, in a collaboration DIPMAT-DISES UNISA. In particular, the research deal with the issue of solving an economic problem using not only real analysis instruments but also geometrical topics concerning in particular Euclidean geometry and topology implemented by the use of online platforms and dedicated mathematical software such as dynamic geometry software and computer simulations (chapter 4)
- the role of semiotic mediators of mathematical knowledge played by the tools in the astroparticles physics research, and the preparation of paths dedicated to teachers in the OCRA project of INFN which was followed by a path dedicated to the analysis of cosmic rays in the classes of "Mathematical High School" Project with the use of the CRC (cosmic Ray Cube) in collaboration INFN Naples-DIPMAT UNISA (chapter 5)

CHAPTER 2

THE IMPORTANCE OF ARTIFACTS IN MATHEMATICS

This chapter will be dedicated to the importance of artifacts in mathematics both to provide a theoretical framework for the use of artifacts for the teaching of mathematics and to highlight how this discipline is constantly evolving and can take advantage of new technologies in a positive and fruitful way to enhance new learning models.

Behind the construction of an educational project there is a vision of the theories of learning to which researchers, that built theoretical frameworks for didactic research activities, refer.

In particular, with regard to the projects developed by the mathematics research group of the Department of Mathematics of the University of Salerno, we must focus attention on an essential aspect: the idea of a didactic path that develops within laboratories through the use of artifacts in an interaction between peers between the students themselves in the group activity and in the teacher-students dynamics.

The Vigotskian approach to the dynamics of learning through the use of artifacts highlights that they become semiotic mediators of knowledge for students through teaching-learning processes in which teachers assume the role of cultural mediators.

In this chapter this methodological approach will be developed with reference to the learning of mathematics, but the vision of teaching that will be presented is related to all areas of knowledge. The constructivist approach to learning and the importance of artifacts in teaching practices extend to all disciplines, in fact it can be emphasized how it represents the paradigm of teaching in a transdisciplinary key, that is, the artifact is seen as a vehicle for learning in an approach that goes beyond disciplinary fragmentation.

The National Indications concerning the specific learning objectives concerning the activities and courses included in the study plans envisaged for the high school courses of the decree of the President of the Republic of 15 March 2010, currently in force, which are the guidelines of the curricula of high schools in Italy issued by the Ministry of Education, underline the importance for students of the maths' laboratory that represents a natural context to stimulate the ability to argue and stimulate peer comparison:

"(...) In mathematics, as in other scientific disciplines, the laboratory is a fundamental element, both as a physical place and as a moment in which the student is active, formulates his own hypotheses and checks the consequences, designs and experiments, discusses and argues his own choices, learns to collect data, negotiates and constructs meanings, leads to temporary conclusions and new openings to the building of personal and collective knowledge."

National indications also highlight the importance of technological tools

"The computer tools available today offer suitable contexts for representing and manipulating mathematical objects. (...) The path, when this proves appropriate, will favor the use of these tools, also in view of their use for data processing in other scientific disciplines. The use of IT tools is an important resource that will be introduced critically, without creating the illusion that it is an automatic means of solving problems"

Even the UMI, (the association of mathematicians "Italian Mathematical Union"), which is a priority reference of interface with the Ministry of Education, has published in Mathematics 2003 that:

"The mathematics laboratory presents itself as a series of transversal methodological indications, certainly based on the use of tools, technological and otherwise, but mainly aimed at the construction of thematic meanings.

The construction of meanings, in the mathematics laboratory, is closely linked, on the one hand, to the use of the tools in the various activities, on the other, to the interactions that develop between people in the exercise of these activities".

The word "laboratory" suggests classrooms or dedicated spaces, suitably equipped, in which activities distinct from those traditionally carried out in the classroom are carried out. The laboratory, on the other hand, is to be understood in a broader sense, not as a physical place distinct from the classroom but as a unique project of activities aimed at building mathematical meanings through appropriately structured experimental activities. The laboratory is therefore a broader concept that involves teachers, students, researchers and technicians and all the figures involved, concerns the spaces (whether they are the classroom or other dedicated ones) with the related tools and time management and above all concerns the whole projects related to activities, organization, experimentation. Speaking of laboratory, we are defining not to a "where" but to a "how". Referring to a representation that is often made when talking about Mathematical High School, the mathematics laboratory is somewhat reminiscent of the Renaissance laboratory, in which apprentices learned by seeing doing and doing, thanks to the interaction between those who were learning and with the experts.

In this regard, it is impossible not to mention Emma Castelnuovo. In her work, it is vital the effort to consider the student as "a being full of curiosity, an investigator, a miniature scientist" to whom one can attribute "the same way of proceeding, the same passages of the scientist's mind"(Castelnuovo 1957).

The class itself can be a "mathematics laboratory" (Castelnuovo 1963), for example when in geometry to introduce conics the teacher develops an activity of observation of the projection of the light of a lamp from various inclinations on the wall to show the different types of conics, or when he introduces the concept of isoperimetric figures by building several rectangles with a rope with his hands. The didactic choice is to build dynamic experiences to make the abstract concept their own.

Emma Castelnuovo understood the importance of concrete experiences linked to observations of everyday life and the use of everyday objects for the construction of mathematical meanings. The constructive method has a fundamental characteristic, "the freedom to conceive and interpret, equally within the reach of the teacher and the student": the stimulus to the mathematical imagination.

In particular, nowadays, digital technologies, online platforms and technological tools support the world of the school system allowing the improvement of teaching activities, enhancing the targeted paths and allowing the continuation of the students' educational and training objectives. Currently the learning environment no longer coincides with the only physical space that delimits the school classroom according to the traditional conception of the school, the virtual and digital worlds have become extraordinarily useful cultural mediators for teaching and learning activities.

When the information concepts of knowledge are introduced through the functionality or characteristics of an object, the object itself is invested with the important role of mediator of knowledge as it conveys information from the teacher to the learner through a didactic strategy previously outlined by the teacher. The object is intentionally transformed in its use, it becomes an artifact.

2.1 Artifact

On the vocabulary (for example the Treccani Vocabulary, but it is almost identical in many other Italian vocabularies) the artifact is defined as the "Work that derives from an intentional transformative process of man".

The term artifact is composed of the words "art" and "fact" whose etymology refers to the Latin words "artis" and "factum". The meaning recalls "made with art", where "art" means the human activity in its original meaning and not in its current meaning linked to activities aimed at aesthetic satisfaction.

It is therefore the activity of human design based on experience and study regulated by appropriate protocols (even if only experiential) that make an "object" (even non-material) an artifact. Animals are also able to produce useful or necessary objects for their survival, think of burrows and nests, and developed skills in using simple objects for a given use, think of the use of stones to break a shell or sticks to catch fish, but man is the only animal species that has reached high levels of complexity in design.

Since ancient times, in fact, man invented and improved devices that would allow him to improve his performances, both physical with objects that increased his strength or endurance or speed, and intellectual ones to improve his cognitive abilities. The ability to design increasingly articulated artifacts, and therefore the consequent technological progress, developed very slowly over the centuries and is associated with the levels of importance of the societies in which they were designed. For example, already in ancient times it was an advantage for a community to possess more powerful weapons or tools to transport water more efficiently. With an expansive effect then, the possession of more effective instrumentation implements the advantage in designing and producing another even more performing one, thus determining the enrichment of the fields of knowledge necessary for the design and deriving from the use of the new products of those communities.

This knowledge has been handed down from generation to generation, both through experience, sharing, use of artifacts, either through written description or graphic representations and therefore the use of signs. Man's ability to build artifacts has determined the structuring of an ever-changing society (Norman 1991).

The primary artifacts, which were also the first to be intentionally produced by man, are those physical devices that are used directly for human activities and that serve to interact between individuals and between individuals and the environment and constitute the material culture. Among them we can include both the oldest tools some of which are still in use (for example among the tools produced by and for mathematics we remember the compass, the abacus, the prospectus) is the most modern means of communication.

systems of signs universally recognized in communities such as natural language, writing, painting, graphics, formulas. they serve to exchange information in social interaction, to coordinate and organize events and life within a group, to share a collective memory, to determine the identity of the community, to create an ideal culture. The symbolic artifacts constitute the mental models of the primary artifacts and represent their cognitive patterns of modes of use, they are the necessary objects as a means to share information about primary artifacts.

Tertiary artifacts are the systems of formal rules, abstract and no longer directly linked to the instruments, they are "creative expressions of the world of imagination, like fantasy" (Wartofsky 1979), they are forms of expressive culture. Among the tertiary artifacts we can include the various artistic phenomena and processes, but also the mathematical theories. Tertiary artifacts organize constructed models as secondary artifacts.

2.2 Technological artifacts in teaching

Technologies are material artifacts that played and play a fundamental role in the development of contemporary society. Initially, they were designed to help man perform certain actions or to allow him greater speed of the development, subsequently technological development allowed the production of machines produced at an industrial level, therefore widespread and within the reach of many individuals, which replace men in performing such actions, think for example of household appliances, assembly lines.

Thinking about the use of artifacts in teaching we observe that we can classify them according to the reason why they were designed, those designed specifically for teaching, and we think of symbolic algebraic calculation software, dynamic geometry, graphing calculators, apps... and those that have not been built specifically but are currently in common use in educational activities, for example vocabularies, translators, microprocessors and microcontrollers, the control units, virtual classrooms, word processing, spreadsheet, presentations, ...

With the development of automatic information, there has been a widespread diffusion of electronic computers and in general of many electronic and digital tools that are part of the technological material artifacts and are able to process signs and sign systems by implementing the skills of man even at the most abstract level of symbolic processing. and are acquiring an increasingly central role in the various areas and teaching Although with a certain delay compared to the technological development of society more generally. in mathematics in particular technology allows you to develop efficient paths is richer in information for students.

In these artifacts it is particularly interesting to observe that the primary artifact coexists, understood as a material component, or the hardware that is the physical support, the structure of the artifact itself, both the secondary artifact, the software, which determines the actions to be performed by establishing the priorities and the rules of application, which turns out to be the symbolic component. Precisely because the "material" technological artifact operates according to the user's inputs and the programs that move it, it is not static and repetitive and its dynamism allows to expand the horizons of teaching and knowledge that allow other references to metaphorical worlds, the cognitive component of the artifacts raises the orientation towards the outside, the interaction with the environment and therefore the experiential aspect and the orientation towards the self that allows individuals to develop intelligence (Norman 1993),

Electronic-digital artifacts can be read differently depending on whether you interact as material artifacts, observing their functioning and the logic with which this happens, or as cognitive artifacts, depending on the methods of interaction offered to the user. Evidently the two levels of reading are closely connected.

Technological artifacts allow you to solve no longer individual problems but large and diversified classes of problems because in a very short time they compare a number of data unthinkable to be obtained without technology, they are more performing because they improve the effectiveness and efficiency of the performance of the user or the system, favor the work for objectives and can be used for an improvement in daily life, understood in the sense of well-being or social and economic advantage and favor the creation of new paradigms of social and cultural value. With specific reference to teaching, we observe that they stimulate and facilitate learning processes thanks to the possibility of integrating information by interfacing different artifacts with specific technological characteristics.

But the changes brought about by digital technologies are even more profound: it is in fact a profound change in time management and information processing capacity; having more information available quickly means that a quantitative difference becomes a change in quality. It also creates a deep gap between what is learned at school and what is known outside the school walls School and university are therefore faced today with the need to synergistically interact the properties of learning based on verbal language with other languages, linked to know-how and informal knowledge that especially digital natives possess at their entry into the education system.

The evident richness of technological innovation requires that the entire education system as a whole strive to make the most of its potential and develop a new teaching approach enriched by technology, as well as a modification of textbooks to a modification of the methodologies for the examination tests and for the texts of these tests, teachers must be able to manipulate technologies with mastery and develop new paths taking advantage of the possibility of really managing reality problems since there is the possibility of operating on a large number of data with "uncomfortable" numbers without technology and not simulating reality problems with "easy numbers" or few data. In the description of Wolfram's book "The Math(s) Fix: An Education Blueprint for the AI Age" the author sublines that math's education is in crisis worldwide and the only solution is a fundamentally new mainstream subject. He argues that today's math's education isn't working to elevate society with modern computation, data science and AI. Instead, students are subjugated to compete with what computers do best, and lose (Wolfram 2020).

The changes that technology brings in teaching are some aspects of a phenomenon in general as technology affects the dynamics of social relationships through, for example, communication, information, social networks or virtual platforms or mapping software, recognition, therefore has a strong impact on the lives of individuals by modifying their mental patterns and enhancing cognitive modalities.

Technology is not neutral in society and in schools.

2.3 Artifacts in teaching

Changes in education are therefore the result of a more general change.

The use of new media has influenced the way of communicating and the way of thinking about communication and consequently new interactions can be ambiguous and difficult to place within categories already elaborated culturally.

Just as in everyday life technology does not play a neutral role, even in teaching the use of new tools or new technologies does not automatically determine an improvement in students' learning or in the didactic effectiveness of teachers' strategies. The new does not automatically replace the old, and so the new technologies determine an acceleration in the development of processes by implementing the effectiveness of the old technologies. For example, in geometry it is still essential to use a line and compass to manipulate and make mathematical objects their own, but the teaching action of the teachers is enriched by the dynamic geometry software currently also present in graphic calculators that speed up the vision of movements of objects in space, facilitating the acquisition of otherwise more difficult content for students.

The introduction of new teaching tools and new technologies in teaching reopens every time the debate related to the efficiency, effectiveness and needs of this novelty because we try to find what are the most appropriate contexts and the correct approaches to these tools is only over time with the experimentation in the classroom if they evaluate their actual repercussions in curricular teaching and the opportunity to integrate them stably. Thus, we observe a cyclicality in teaching that the introduction of new tools and artifacts is followed by new tasks because being able to perform traditional tasks more efficiently leads with it over time to the transformation of the tasks themselves. Think for example how the mathematics tests of the state exams are evolving after the introduction of graphing calculators in 2017.

But the introduction of new technologies in teaching always brought with it different opinions, sometimes even very distant on the efficiency and effectiveness of the same. This divergence of opinions is not limited to the comment of the teachers that start from the totally contrary ones or for inefficiency of technology or for a personal evaluation about the relapse of the use of the tools themselves, to the super technological teachers who flood their activities in the classroom with experiments and artifacts, but crossing all the intermediate positions of those who are more or less skeptical, more or less inclined to experiment with the use of technology in the classroom.

The staunchest proponents of technological determinism, including Marshall McLuhan, assert convincingly that technologies influence the human experience (but believe the other way around). However, there are many intermediate positions that dampen the most radical judgments and that believe that the relationship between individual artifacts and society is circular and that technology no longer occupies the focus of communication but interacts together with other factors to the realization of everyday reality. So, a technology linked to the context in which it is used, because it is part of it but it is also a builder. Of this constructivist approach that belongs to Vygotsky and we will talk later (De Piano, Ganino 2016).

It is quite evident and natural that the introduction of a new tool within a community determines within it an initial disorientation because there is a lack of previous knowledge or not recognized, its characteristics are not recognized and previous experiences with the instrument itself are not remembered. However, as technology spreads, this initial disorientation is replaced by normality with the introduction of rules and methods, so a sense of habit and familiarity is reached and the instrument becomes in effect part of everyday life and at the same time modifies the previous classifications. With land technologies it is therefore possible to promote contextualized learning and enhance and respect the differences of each incentive and the participation of all.

Teaching-learning practices are not only changed by introducing these new technologies into schools. It is necessary to use them correctly, in fact in an inadequate use there is an inherent risk that the mathematical meanings that should be evoked and facilitated by the artifact are opaque in the eyes of the student and remain only evident to the mediator teacher (Mariotti 2003). If on the one hand the artifact is a very powerful teaching tool because it conveys high-level signs and meanings, on the other hand it has the limit of being effectively efficient only if it becomes the fulcrum and engine of a suitably designed and structured didactic action. In this regard, constructivism has offered multiple observations on user-centred learning processes with a semiotic double link between the artifact and knowledge. It is observed how the signs produced through the activities with a certain artifact, are interpreted by the student and the trainer in a different way, the artifact highlights its polysemic potential because on the one hand it is linked to the task accessible to the student and on the other to the knowledge of the trainer.

2.4 Instrumental Approach - Rabardel

The discipline that studies how the human mind interacts with the tools of environments and cognitive ergonomics that defines how to design tools whose interaction is effective satisfactory and efficient. The researcher French Pierre Rabardel works in this field is precise and defines the distinction between artifact and instrument. For Rhubarb the artifact is the material or symbolic object that does not depend on the observer or user, the instrument instead is seen as an intermediary between the object and the subject and includes within it both the components and characteristics of the artifact, and the subjective sphere that refers to the patterns of use used by the individual to perform a task with the use of the artifact. Usage patterns are the processes used by each individual to solve a task and are characterized by certain variables: the artifact itself, the task that is assigned, the individual who goes to solve it.

Rhubarb defines instrumental Genesis as the process of individual construction of an instrument from an Artifact that develops into two sub-processes:

- the "instrumentalization", which concerns the different components of the artifact, the study of its evolution, the analysis of its limits and its potential: it is a process directed towards the outside, towards the artifact
- the "instrumentation", which concerns the study of the evolution and development of the patterns of use, in this process the personal use schemes are built according to appropriate tasks to be performed: it is directed inwards and relative to the subject in learning (Rabardel 2002).

The task of the teacher is to choose the artifact and to guide the students in the process of instrumental genesis so that they build the meanings of knowledge.

Each artifact can therefore become a tool through a process of instrumental genesis through the appropriation of already known patterns of use or through the elaboration of new protocols by the subject. Just as we said earlier that technology is not neutral, we observe that even the use of a tool is not because it requires the aggregation of knowledge and skills and the reorganization of the cognitive structures of the subject in which psychological aspects and social relationships also converge (Rabardel, Samurcay 2001).

Rabardel's cognitive-instrumental approach has not been oriented by indications of an educational nature, but his elaborations can provide rich indications in the design, administration and analysis of educational activities.

2.5 Approach to artifacts - Vygotsky

Lev Semyonovich Vygotsky (1896-1934) worked in the early decades of the twentieth century and died in 1934 at the age of 38. It is obligatory to reference to the work of Vygotsky for those who work in the educational field, with attention to the role of social interaction and cultural mediation of the teachers in the acquisition of knowledge. In addition to general elaborations, we are interested in the attention that Vygotsky devotes to the function of technical and psychological tools, transmitted socio-culturally, whose role is crucial in development processes.

The Vygotskian framework is characterized by three fundamental constructs:

– The zone of proximal development.

– Internalization or internalization.

– Semiotic mediation.

The process of *internalization* determines cognitive development within the limits of what Vygotsky defines as a "*zone of proximal development*". In this metaphorical space, learning is realized through social interaction between more and less "experienced" individuals.

"The zone of proximal development (or area of potential development) is the distance between the actual level of development as determined by autonomous problem-solving and

the level of potential development as it is determined through problem-solving under the guidance of an adult or in collaboration with one's most capable peers." (Vygotsky 1987).

In the *zone of proximal development,* the problem-solving activity is developed with the cooperation of the trainer with the learner who, through the help provided to him, is able to face and solve problems that he would not be able to solve independently. The zone of proximal development is therefore a metaphorical space in which collaboration is expressed in different forms, both through guided and participatory lessons, both with the use of tools (for example in mathematics we think of line and compass, curvigraph, scientific or graphic calculator, dynamic geometry software or symbolic algebraic calculation) and with the transmission of shared signs (from verbal or written language, to microlanguae, to representation systems, to gestures) (Bartolini Bussi, Maschietto 2006).

Vygotsky assesses that the process of internalization develops on two levels: the external process that is governed essentially by interpersonal communication is a fundamentally social aspect and the process of internalization that is directed by semiotic processes and is characterized by the reworking and internal reconstruction of an external activity. The latter process is evolutionary because it consists in the restructuring and/or reconstruction of internal individual knowledge through complex sequential passages that take place in contexts of a social nature and that determine the production and interpretation of tools and signs (Bartolini Bussi, Mariotti 2009).

In the process of semiotic mediation, artifacts assume different roles than the learner, physical artifacts are "extensions" of the self and are oriented outwards, cultural artifacts are instead projected inwards and play the role of "controls".

2.6 Artifacts and Signs

Vygotsky emphasizes the analogy that exists between technical tools and psychological tools, in fact just as artifacts support practical activities, signs serve to develop mental activities and acting as mediators between the individual and the environment favor the process of internalization. Just as tools are ancillary to the development of a job, signs also play the role of an instrument of psychological activity and the invention and use of signs to respond to a stimulus or solve a problem correspond to the invention of a tool for such an activity (Bartolini Bussi, Mariotti 2009).

The tools and systems of signs produced by man, which act on the external world and determine an internal reworking in the individual who uses them (instrumentation process) are therefore the result of a high level of refinement of social activities (process of instrumentalization) and contain within them (sometimes in a way that is not immediately understandable) essential contents of knowledge (Bartolini Bussi, Maschietto 2006).

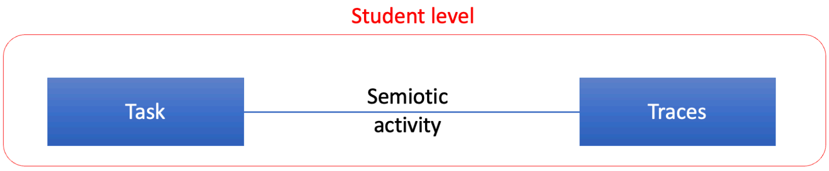
In the process of semiotic mediation the expert can "drag" (the term is Vygotskian) a tool and thus inhibit a response based on previous knowledge or automatisms to develop processes that to information and experiences not known to the learner who uses it: in the process it is the tool itself that determines a new context that develops both in its exploration, both in the research and knowledge that are produced with its use, opening new spaces and new scenarios related to the accuracy of the explorations.

2.7 Semiotic Mediation

The linguist and natural language scholar Hasan, to describe Semiotic Mediation states that "The noun mediation comes from the verb mediate, which refers to a process with a complex semantic structure that includes the following participants and circumstances that are potentially relevant in this process: someone who mediates, the mediator; something that is mediated, the content/force/energy released by mediation; someone/something subject to mediation, the recipient to whom mediation makes some difference; the circumstance of the mediation; the means of mediation, the modality; the place, the site where mediation can take place." (Bartolini Bussi, Mariotti 2009).

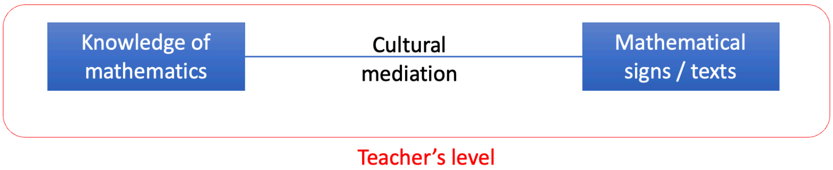
Therefore, in the process of semiotic mediation all the actors who intervene in the action are relevant: the mediator, the receiver, the content and the ways in which the dynamics take place. We move from an educational process of direct stimulus-feedback (question-answer, listening-repetition) to a much more articulated and complex process in which the mediation carried out by the mediator guides the recipient to research that leads to higher levels of the organicity of mental contents and operational skills. (Bartolini Bussi, Mariotti 2009).

If we analyze the process of semiotic mediation on the student's level, we can observe how the task assigned by the teacher determines in the student a semiotic activity useful for solving and completing the check.



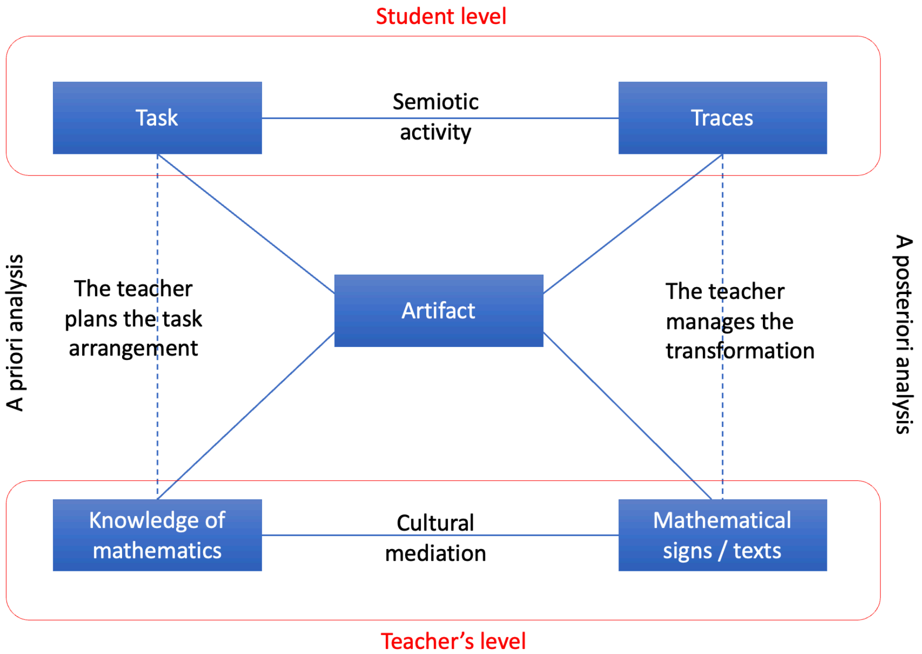
This activity can initially be extremely individualizing, in fact it often happens to observe on the notes strange signs or graphisms or unconventional representative graphs and closely linked for example to the recommended methodological approaches of the teachers or to the most "comfortable" ways for students to represent the data in their possession but then evolves in the structuring of the task through a conventional language whose symbolism is universally recognized in the mathematical field.

If, on the other hand, we analyze the process of cultural mediation on the level of the teacher in his capacity as a cultural mediator



It is important to underline how he is aware of the dual role that the artifact (instrument of cultural mediation) plays towards the task and towards the knowledge to be transmitted.

Going to analyze the process of semiotic mediation in its complexity of the interrelationships between the subjects and the contents and also adding the temporal axes in the teaching-learning processes, we can observe how important and decisive the use of signs is. (Bartolini Bussi 2011)



Analyzing the relationships that exist between the actors of semiotic mediation activities, we can describe that the relationship between mathematical knowledge and the artifact takes place through signs whose coding is uniquely recognized and is of reference in the context in which it developed, uses a synthetic language and is descriptive of the procedural operations with the artifact. The relationship between the delivery to the learner and the artifact can instead be encoded with contextualized language and can use located signs that have a precise meaning related to the use of the artifact in the community in which they are used. The relationship between the teacher's plan and the student's plan is not immediate but takes place through the artifact, the social relationship therefore becomes the fulcrum of the educational activities that must have as a priority objective the transformation of the signs of contextualized language express the relationship between the delivery and the artifact into uniquely recognized signs that express the relationship between the artifact and mathematical knowledge.

2.8 The Didactic Cycle

According to the theory of semiotic mediation, the evolution of the personal signs of the students towards the desired mathematical signs is favored by the iteration of the didactic cycles.

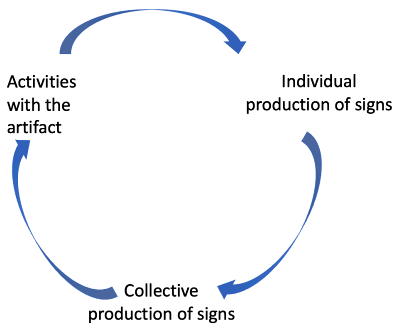
The didactic cycle can be considered the representation of a process in which the artifact becomes the means to achieve a delivery and the semiotic mediation tool to achieve a didactic goal thanks also to the use of signs and this activity opens up new didactic scenarios that lead to new deliveries in a circular process in which the various moments influence each other and the results are the result of a fusion of theory and practice.

The didactic cycle includes the following semiotic activities:

• activities with the artifact for the execution of an assigned task: students work in pairs or in small groups and are invited to produce common solutions. This involves the production of shared signs;

• individual production by students of reports on class activity that involves a personal and delayed rethinking of activity with the artifact and individual production of signs;

• collective discussion in the classroom orchestrated by the teacher



2.9 Role of the teacher

Therefore, the teacher who knows the effectiveness of the artifact and has prepared, and therefore knows, the task assigned to the students, consciously adopts the artifact as a tool of semiotic mediation, through a didactic action consciously aimed at stimulating the learning of the contents. The teacher assumes the role of cultural mediator that makes evident the knowledge hidden in the artifact and that stimulates the discovery of new meanings thanks to the use of the same by voluntarily and deliberately moving forward the zone of proximal development of his students.

The action of the teacher is crucial at every stage of the teaching cycle. In fact, the teacher must design tasks aimed at favoring the development of the semiotic potential of the artifact, observe the activity of the students with the artifact, collect and analyze the written solutions of the students and domestic relationships, in particular paying attention to the signs that emerge in the resolutive process, therefore, based on his analysis of the written productions of the students, must design and manage the class in the discussion in order to favor the evolution towards the desired higher level mathematical signs.

The theory of semiotic mediation offers not only a framework for the design of educational interventions based on the use of ICT, but also a lens through which it is possible to analyze semiotic processes, which take place in the classroom (Bartolini Bussi, Mariotti 2009).

2.10 Mathematics laboratory

Within the theoretical framework of semiotic mediation, the mathematics laboratory, which is not necessarily a physical place different from the classroom, plays a leading role for the importance that the use of tools and social interactions assume in it because it involves people (students and teachers), structures (classrooms, tools, organization of spaces and times), ideas (projects, experiments...) to the point of recalling a "Renaissance workshop" in which apprentices learned from direct confrontation with expert subjects.

CHAPTER 3

THE GRAPHING CALCULATOR

As part of the research activities on the potential of the use of the graphing calculator in secondary school, teachers and students of institutions that share educational paths and projects with UNISA were involved and four macro areas of intervention, collaboration and research were designed.

1. For the upper secondary schools in which the "Mathematical High School" project is active, which was developed in Salerno 6 years ago and which has become a national reality that involves over 160 institutes and numerous universities from all over Italy, expert trainers of the university propose experimental laboratory teaching modules that deepen in a transdisciplinary key the themes traditionally treated by school curricula by implementing them both in content and in teaching methodologies. Since last school year, a didactic module has been developed and prepared focused precisely on mathematics with the graphing calculator: in this activity the graphing calculator is seen as a Black Box, that is, a black box whose functions are not known but which is used to observe the results and build theoretical models.
2. For upper secondary schools, as part of the Scientific Degrees Plan, it was decided to develop a path in a PCTO perspective focused on the development of heuristic research paths of confirmations of mathematical models of reality problems with the use of the graphic calculator, used as an "accelerant" of numerical and graphic experiments since it allows students to focus their attention on the analysis of the model and the formulation of possible paths decisive without the mere computational operation.
3. As part of the training and updating of teachers in service, a specialization course entitled "mathematics between the two cultures" has been developed dedicated to teachers of all disciplinary fields to share transdisciplinary thematic nuclei, tools and paths that can be immediately spent in the classroom with transversal research ideas. Among the various courses, mathematics and new technologies have also been

included to provide teachers, who often have many years of experience but consequently have a more traditional methodological and content training, all the most modern information and tools.

4. in the narrowest area of mathematics teachers who relate and collaborate with the research group, the LiC-CG project is being developed (acronym for Laboratory in the classroom with graphic calculators), in the laboratory activities the use of the graphing calculator is envisaged from the classes of the two-year period in order to make students safe in the use of such a manageable and powerful calculation tool and to familiarize them with the calculator admitted to the State Exams, both (and this is the peculiarity that this device offers us) to design paths of experimentation and research in mathematics that allow students to focus on the methods used and on the results obtained regardless of all the technicalities that sometimes the themes impose so that students can focus on the heart of the contents made free from the difficulties of calculation. The aim of the project is precisely to present a cross-section of the activities developed ranging from simpler activities designed by teachers who for the first time experimented with graphic graphing calculators to more articulated and complex learning teaching units designed and elaborated by teachers who instead have made graphic calculators a teaching tool for some time now.

3.1 The graphing calculator in teaching mathematics

The introduction of the graphic calculator to the state exams for upper secondary schools in Italy has inevitably changed the relationship between mathematics teachers and this tool. In fact, if in other countries the graphing calculator is a tool used in the classroom for over 40 years now, in Italy it has remained little used until the modification of the state exams of 2017. In a few months, the school world has thus interfaced with a tool that was been neglected for a long time. If, on the one hand, the methodological approach of the students - who are defined as digital natives because they now interface with technologies from an early age - was quite natural and the students asked and ask to be able to use the calculator convinced that it can help them (perhaps without knowing how) in carrying out the activities, on the other hand teachers have a diversified behavior towards this teaching aid: some of them are refractory to digital technologies, others, although in favor of the use of technology, are against graphic calculators because they can lazy students or because they consider them ineffective, others are still afraid of the use of digital technologies and finally there are teachers in favor of the use of technology and eager to learn how to use graphing calculators.

Precisely counting on energy, skills and the sharing of good practices, it was decided to develop the project with the graphic calculator, counting on the fact that, as a few decades ago scientific calculators have supplanted the much suffered logarithmic and trigonometric tables by enhancing and improving the didactic approach to those themes, so today graphic calculators can support students curricularly to allow them an experiment that was not foreseeable before.

The decision to develop a project using the graphing calculator in the classroom is therefore aimed at designing new didactic paths and new ways of dealing with mathematics, exploiting the potential that this tool offers. It is not intended the preparation of activities that are already "ready to be used", for this reason there are manuals produced by the various manufacturers. For mathematics teachers, the real goal is to study strategies and educational paths aimed at ensuring that this automatic calculation support enhances with a Research-action approach and does not limit the development of students' operational calculation skills. The aim is therefore to allow students to use the calculator in daily mathematical teaching activities by refining with an independent path the ability to operate the calculations independently. The calculator therefore becomes an aid in four fundamental areas: after acquiring the skills to solve a certain model of problems, the calculator allows them to focus on the development of similar models without getting lost in pages of calculations, calculator is an ethical tool because it allows to deal with deeper and abstract mathematics issues even those with learning difficulties such as dyscalculia neglecting the operational executive. the possibility of using various interfaces, graphics, programming, algebraic, analytical, tabular, favors the construction of contents in their connections and representative systems, the possibility of managing important numbers facilitates the study of real problems with real numbers giving an effective answer to the requests of the ministry that solicits a mathematics that interfaces with reality problems. too often the need not to burden the students' computational effort excessively exemplifies real models making them lose the more generalized vision. All these possible openings to a conscious strategic use of efficient and powerful mathematical tools are often linked to the skills and will of the teachers and are found that they do not find on school textbooks activities consistent with this didactic approach.

In developing the didactic activities related to the use of the graphic calculator in the classroom we decided to apply the "collaborative training model" (Bologna, Rogora, Veronesi 2019), a training model tailored to the needs of teachers with the idea that successful training should adequately recognize teachers' teaching and methodological skills and should be based on their needs.

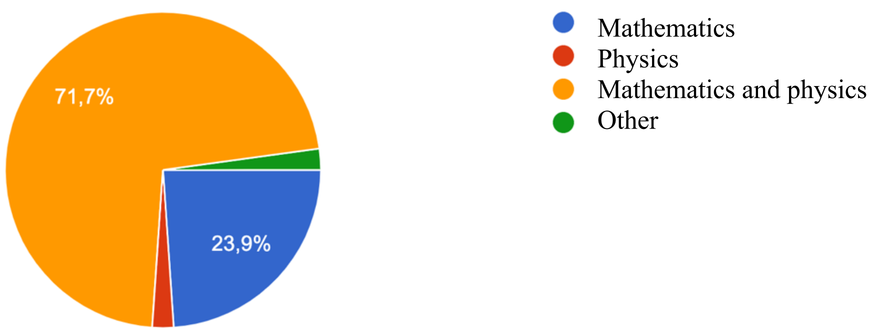
The methodology envisaged is that of Cooperative Learning, in which the teachers and students participating in the activities work in small groups with common objectives, mutually improving their learning, and Peer-to-Peer education, in fact, cooperate to acquire knowledge. in the first training meeting, the teachers also work with some of their students. The purpose is to change the relationship of educational dialogue between teachers and students, as they work with their teachers during the training and then become trainers of their classmates, supporting the teachers in the activities and cooperating in guiding their companions.

The training activity is developed in three meetings that take place over a period of approximately two months. The first meeting is divided into two parts, in the first part the trainer meets the teachers and students and introduces them to the use of the graphing calculator by exploring the menus and the most important functions by solving simple exercises commonly faced in class in collective and cooperative activities between teachers and students. In the second part of the meeting the teachers agree with the trainer on the common themes that they intend to develop in the following weeks in the classroom and some activities are chosen that can be developed with the graphing calculator. In the period between the two meetings the teachers prepare the cards of the activities to be carried out with the calculator. in class the teachers bring the calculators and propose to the class the activities seen in training with the help of the students who participated. In the second meeting, after discussing the experiences of the teachers in the classroom, the forms for classroom teaching are analyzed, implemented and defined. In the following weeks the teachers propose the works produced in class. in the third meeting, after an extensive discussion on what the teachers found in class, the trainer suggests further instructions and / or solutions for the proposed exercises. Observations and reflections of teachers and their suggestions for the organization of future activities are collected in the form of individual written interviews.

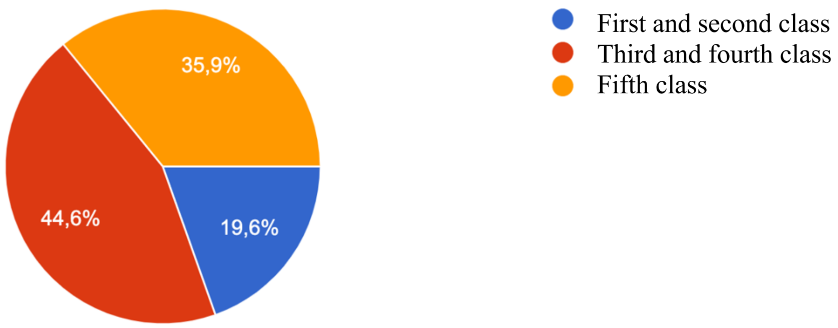
Before developing an educational research path in the classroom with graphing calculators, it was decided to carry out a survey on this didactic tool by proposing to the teachers who interface with the didactic research group of the Department of Mathematics, a questionnaire to understand the levels of competence, trust and experience that they have towards graphing calculators, both in relation to the ability to use them and for a theoretical opinion regardless of their skills. 92 teachers took part in the survey.

First of all, it was noted:

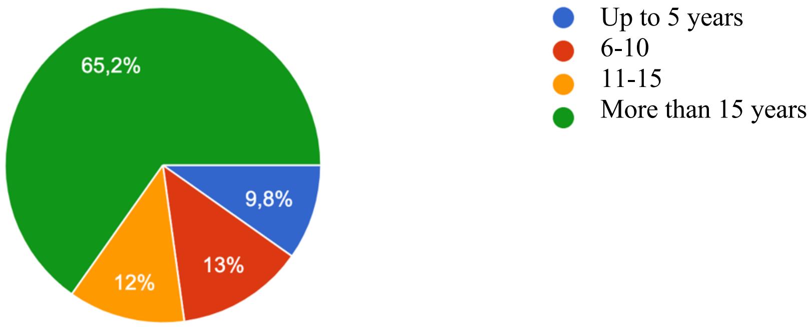
That almost all of the participating teachers teach mathematics or mathematics and physics



That all the classes are equally represented

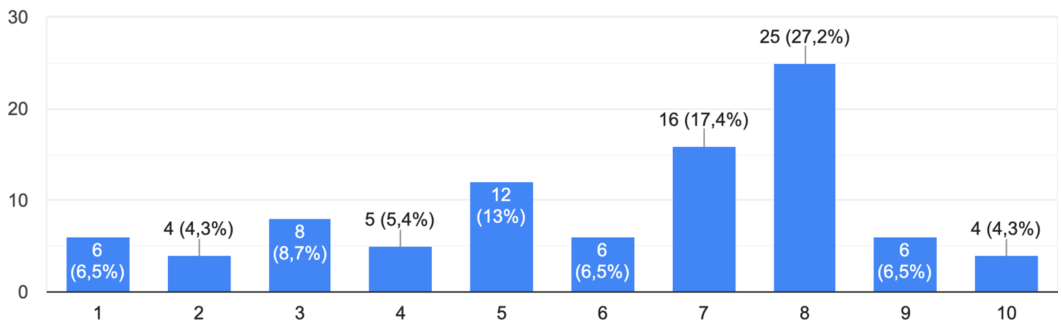


And that most teachers have a career of several years of teaching



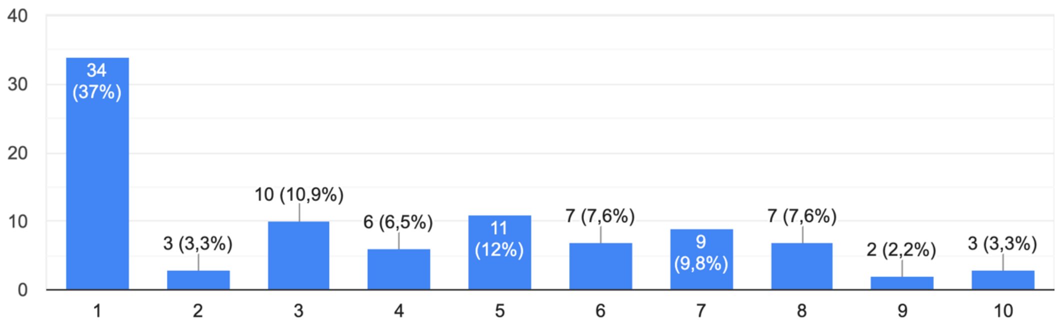
There are few teachers who make sporadic or none use of technologies in teaching activities

Level of use of dynamic and non-dynamic software (Geogebra, Desmos, ...) in the everyday teaching of mathematics (1 none - 10 excellent)



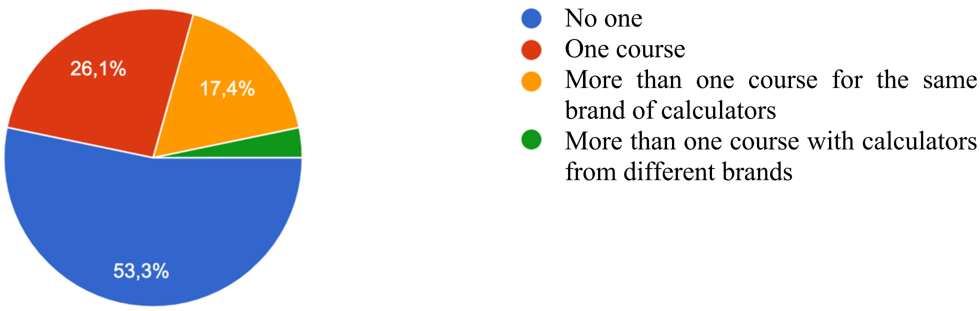
On the other hand, however, it is observed that only 30% of teachers use the graphing calculator in daily teaching with an evaluation at least "sufficient"

Level of use of graphing calculators in everyday math teaching (1 none - 10 excellent)



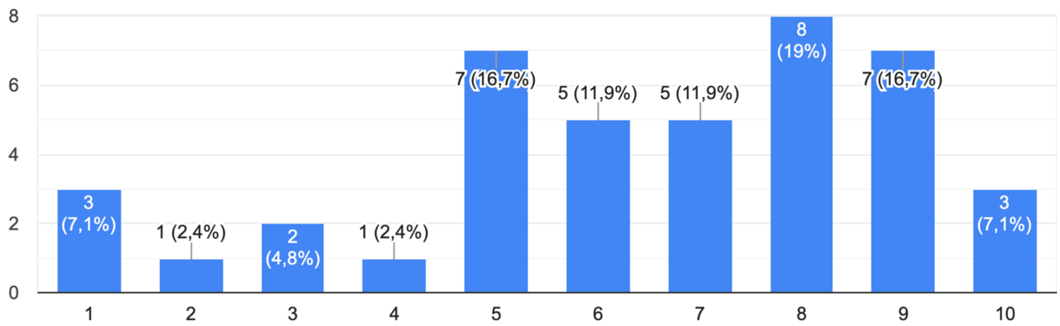
More than half of the teachers have never attended a training course on graphing calculators

Have you attended training courses on graphing calculators? how many?



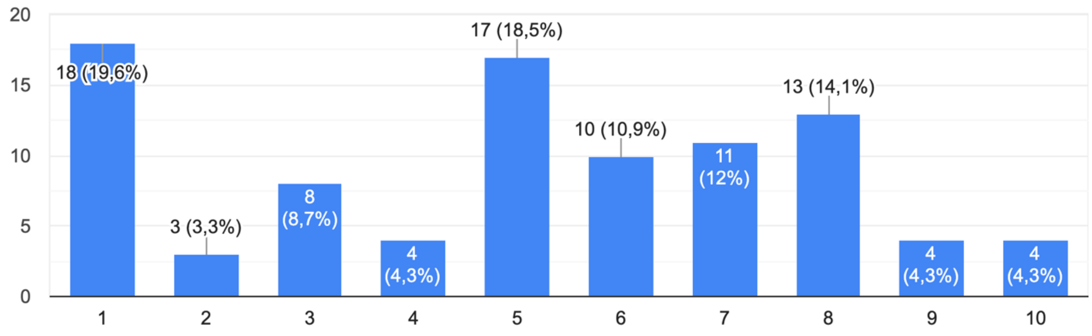
And those who have followed a training course have obtained an improvement in the use of the graphing calculator (expected result)

If you have attended training courses, how do you perceive your improvement in the use of the graphing calculator after following the course? (42 answers up to 92 teachers)



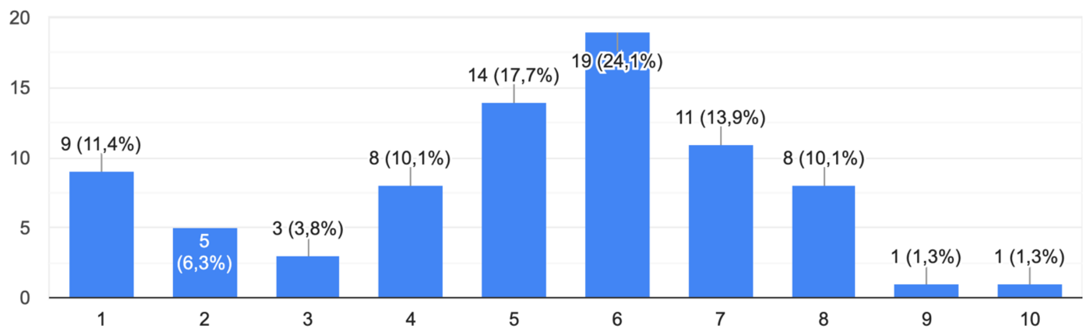
With a level of acquired competence that for about 20% is none (it is assumed that they are teachers who have never attended courses) and that for the majority settles starting from almost sufficient values

Level of general proficiency in the use of graphing calculators (1 none - 10 excellent)



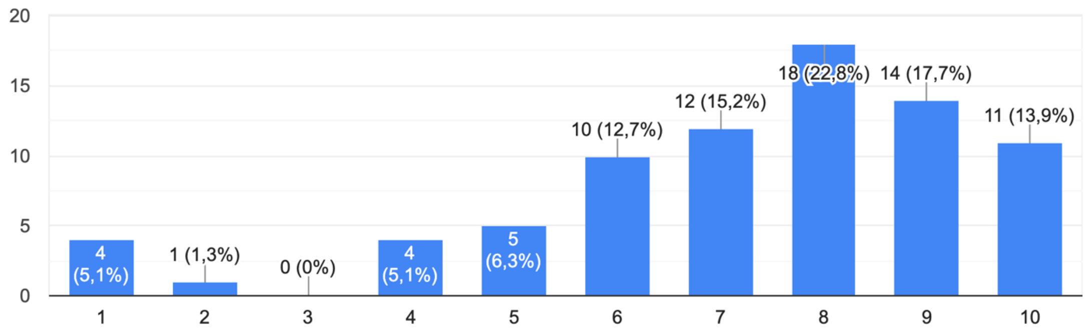
The level of confidence of teachers in the use of graphing calculators is distributed around the sufficiency for students of the first two years

Level of confidence in the use of graphing calculators for students in the first two years of high school (1 none - 10 excellent)



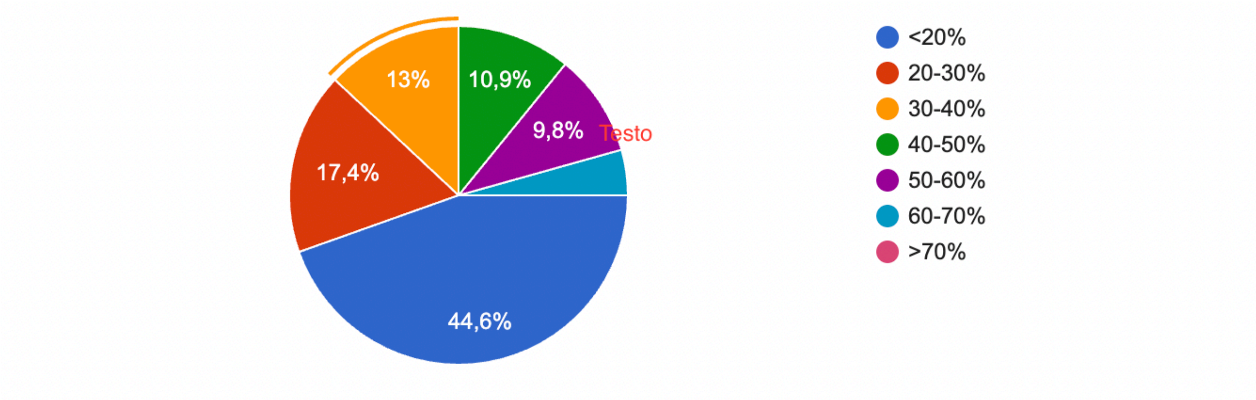
And is around the value of "good" for students of the last three years

Level of confidence in the use of graphing calculators for third, fourth- and fifth-year high school students (1 none - 10 excellent)



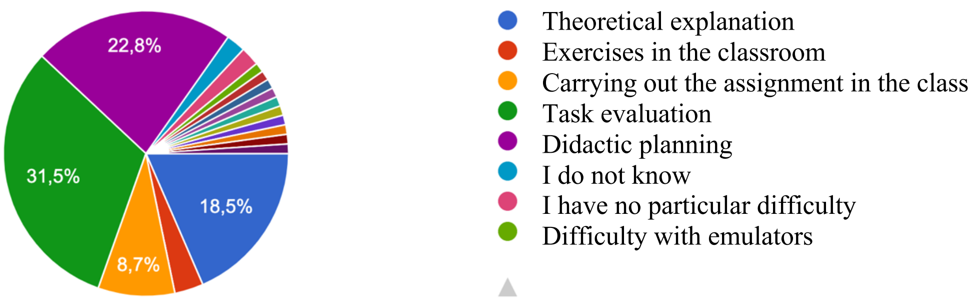
The majority of teachers estimate that they use the calculator for less than 30% of curricular activities

Give an estimate in percentage terms of the use of the calculator in your curriculum activities



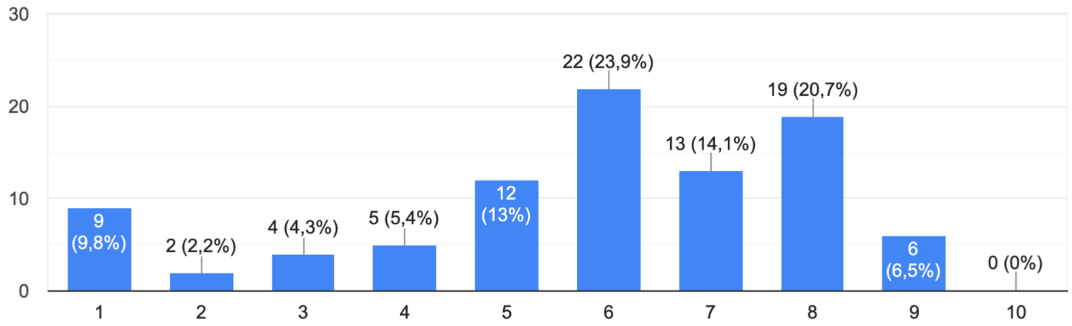
The next sheet is particularly interesting because it highlights the difficulties encountered by teachers in using the graphing calculator in the classroom, specifically the teachers highlight more critical issues in the evaluation of a task carried out with the graphing calculator, in the design of activities with the calculator and in the explanation classroom theoretical activity with the calculator.

Which activity highlights the greatest criticality and/or difficulty in teaching activities with the graphing calculator?



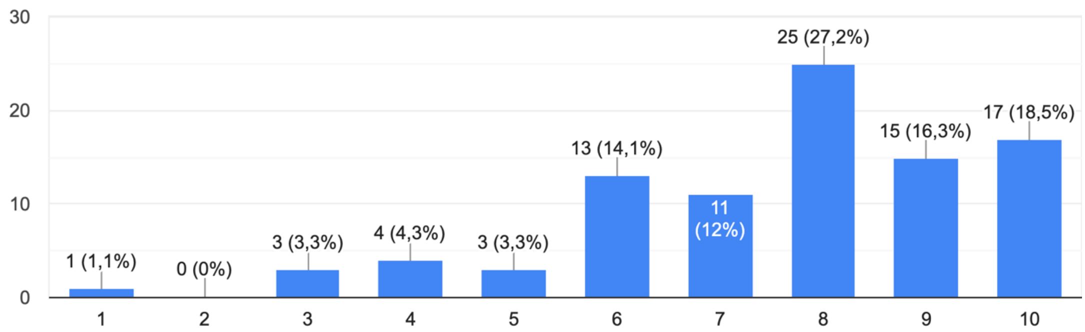
The teachers found an improvement in the students' results in terms of academic performance thanks to the use of the graphing calculator

Students who use the calculator have an improvement in performance ... (choose the grade option)



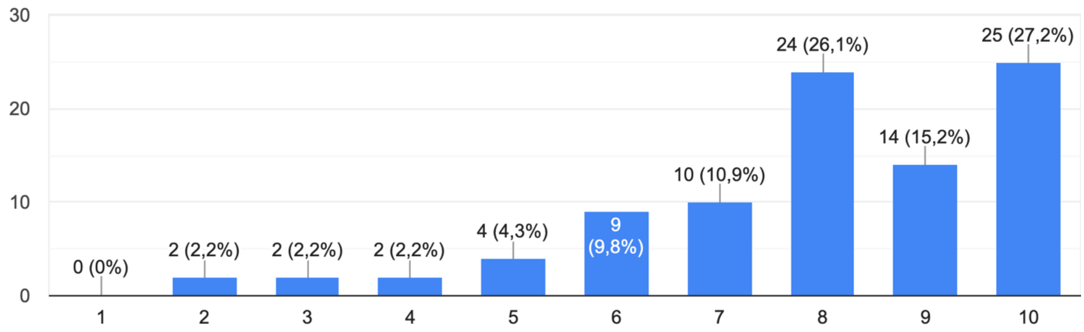
And they consider it extremely important to receive specific training on the functions of the instrument

What is the importance of specific training only related to the functions of the instrument before a didactic approach?



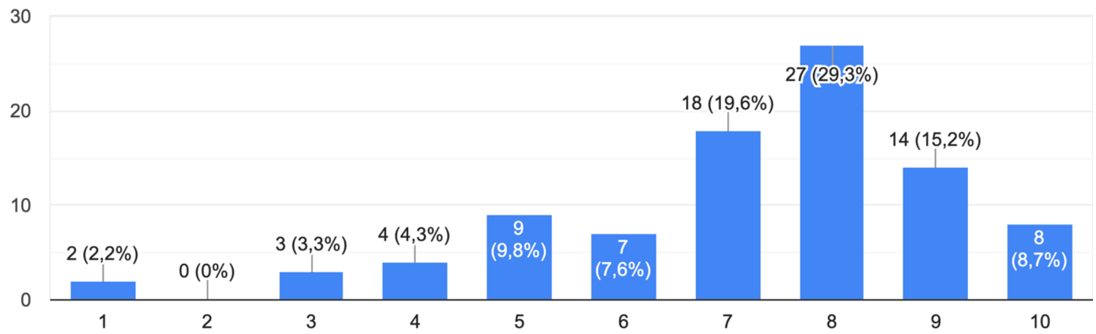
Followed by specific training linked to the methodological-didactic approach

What is the importance of specific training always linked to the didactic methodological approach and not only to the instrument?



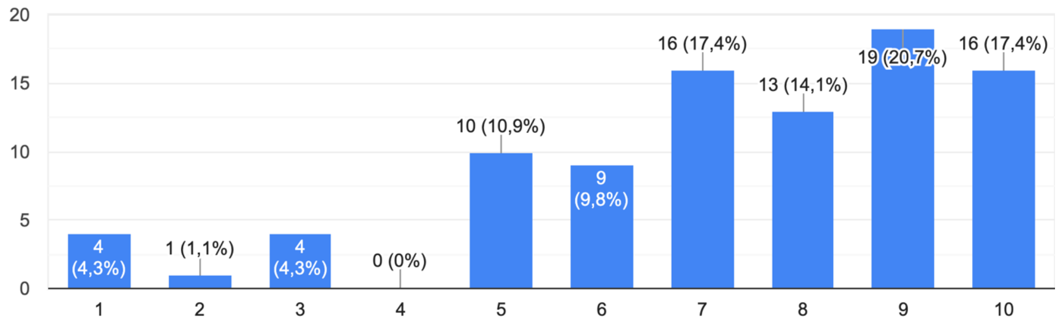
And they attribute a grade of "8" to the confidence of the graphical calculator in research and analysis activities

What is the degree of confidence in the tool "graphing calculator" as a research and mathematical analysis of students?



Stressing the importance of involving students in training courses for the use of graphing calculators

Do you find it useful for students to participate in training courses for the use of graphing calculators?



3.2 LiC-CG PROJECT: The Classroom Laboratory with Graphing Calculators

3.2.1 Introduction

The "LiC-CG Project" (Laboratory in the Classroom with Graphic Calculators) build up by the research group in Mathematics Teaching of the Department of Mathematics of the University of Salerno has improved within the Project "Mathematical High School".

This didactic experimentation plan, dedicated to students of Scientific High Schools, divided into afternoon activities with extracurricular in-depth courses, is aimed at developing the aptitude for scientific research, expanding the cultural training of students and enhancing their critical skills.

In the paths of the Mathematical High School, transversal themes have always been privileged, involving both the field of scientific culture and that of humanistic culture through the lenses of mathematics that, like a bridge, build dialogical paths between the various disciplinary areas. The focal points of the activities are the inter-disciplinary aspect of the topics covered and the involvement of new technologies, all aimed at allowing students to better orient themselves in the contemporary reality so rapidly evolving (Rogora, Tortoriello 2021).

3.2.2 Contextualization

In the field of teaching, there are numerous publications focused on the use and effectiveness of graphic calculators. Among the first published research works on the use of this tool for the teaching and learning of mathematics, we find the work of Penglase and Arnold in 1996 in which the potential revolution in mathematics teaching is outlined both from the point of view of content and curricula (Penglase, Arnold 1996). Since the end of the 80s, many jobs have followed. Some of them focus on teaching learning teaching models and on the way of constructing mathematical meanings thanks to the use of this technology, among others (Burrill 1996), (Drijvers, Doorman 1996), (Ellington 2003), (Robutti 2010), (Trouche, Drijvers 2010). Many others, including those mentioned below but many can be listed, are oriented to the analysis of the effects of technology and behavior by teachers (Lee 2005), (Ozgun-Koca, Asli 2009), and / or students (Nzuki 2016), (Parrot, Leong 2014), (Tan, Choo-Kim, Madhubala Bava Harji, Siong-Hoe Lau. 2011) , (McCulloch 2009), (Leng 2011), (Ng 2011), (Mitchelmore, Cavanagh 2000), (Quesada, Maxwell 1994), (Quesada, Einsporn, Wiggins 2008). These works concern presentation and description of activities and feedback from teachers and/or students, very often a class group, that deals with a single activity that is described throughout its path and a prevalence can be observed in dealing with the functions from different mathematical, heuristic, graphic, analytical points of view. In many of them it is emphasized positive influence of the use of the graphing calculator in the approach to the study of mathematics, the observations are based on qualitative reports compiled by the students and/or on the observations of the results of the activities carried out on two groups of students, the research one that uses the graphing calculator and the control one that does not use it. On the other hand, courses aimed at measuring the performance of the same students with and without the calculator and quantitative analysis on the evaluations expressed by the teachers with shared evaluation methods have not been developed.

With the Ministerial Ordinance no. 257/2017 Article 18 paragraph 8 (Ref. "For the purpose of carrying out the second written test in scientific high schools, the use of scientific and / or graphic calculators is allowed, provided they are not equipped with symbolic calculation capacity, CAS – Computer Algebra System") and in subsequent regulatory developments, the then Ministry of Education and University and Research (M.I.U.R) (today Ministry of Education, M.I.), allowed the use of the graphing calculator in the second written test of the State Exams of scientific high schools, bringing back to the current research topics of Didactics of Mathematics on the use of Graphic Calculators.

In this regard, the researchers of the Research Group in Didactics of Mathematics of DIPMAT UNISA have decided to dedicate specific paths.

Although in fact there are much more advanced technologies that allow particularly complex operations (cf. tablets, smartphones, netbooks, ...), the exclusive use of the graphic calculator as a tool allowed for the State Exams places a different educational level that deserves and needs new insights.
As reported in the National Indications in which reference is made to chapter 2 of this thesis about the role of mathematics with technological instrumentation.

In the National *Indications* we therefore read of a mathematics strongly connected to experientiality where the laboratory is both physical space, to experiment, measure, evaluate, and metaphorical space, for mental laboratory activities; we read of a mathematics enriched by the use of technologies to be interpreted in a critical sense that should not be seen as a panacea for students' problems, but must be exploited for the immense potential of calculation and to lighten the operating procedures in order to focus on the abstract manipulation of mathematical concepts.
So, emerges the strong desire to orient the mathematics curricula towards a laboratory direction to favor the acquisition of a mathematical *"consciousness"* that guides students to a conscious use of technologies: a didactic choice that pushes to overcome a vision of traditional mathematics, developed in a logical and formal way through the process of axiomatic-deductive argumentation, to recover in school learning paths the heuristic pleasure of discovery by highlighting the importance of informal exploration of mathematical contents favored by new technologies before proceeding to a rigorous formalization.
In this sense, the graphic calculator is the technology that, in addition to being the only one allowed for the State Exams, over a few decades has been designed and implemented with particular reference to secondary school students and that is well suited to develop experimental paths in the school because, in the face of a modest cost compared to the disbursement necessary for specially built laboratories, offers students a new perspective that allows them to focus their observations on the mathematical aspect of the study phenomenon by delegating, at least at first, to the machine the performance of all the calculations, thus favoring even the weakest students in learning and in the operational phases.

In particular, the possibility of developing a topic by combining numerical representation with symbolic and graphic representation, helps to give an overview of the mathematical perspectives involved. The interactive activities between the various representations use different communicative registers and favor a complete and mature learning also allowing to treat reality problems with real data, not simplified or reduced in number as is necessarily the case when carrying out such exercises in the traditional way, because the management of data and related calculations are entrusted, at least in a first phase, to the machine.

3.2.3 Methodology

The constructivist learning model, headed by Vygotsky, well represents the basic idea of the Mathematical High School based on the active participation of students in *problem-solving* activities and on the development of paths aimed at enhancing critical thinking in a shared and therefore engaging and motivating learning.
The experimental laboratory paths of L.M., and therefore also the activities of the LiC-CG project, are grafted into the cultural substratum of teaching by skills and aim to promote a coordinated system of *knowledge, skills and attitudes* that are mobilized by the student in relation to a purpose (a task, a set of tasks or an action) close to his interests that therefore favors and good *internal* motivational and affective dispositions (Pellerey 2015). Teachers in their role as cultural mediators have the responsibility to involve the emotional aspects and to solicit the motivational ones of the students by enhancing the didactic effectiveness of their procedural choices with tools that can also be technological and that act as semiotic mediators of knowledge as they can favor knowledge processes that convey mathematical knowledge through experientiality.
Great attention is therefore paid to the introduction of the most modern technologies in laboratory activities because they allow, thanks to the enormous potential of calculation, the management and processing of a large amount of data. Through the activities designed by the teachers, developed in the classroom in small groups to solicit a peer-to-peer education, followed by the collective discussion in the classroom orchestrated by the teacher, students acquire new skills that redefine the entire cognitive scheme of the students.
In the LiC-CG project, the semiotic mediator par excellence was the graphic calculator which, through the activities designed by the teachers in synergy with the researchers

involved in the project, provided students with the opportunity to address some curricular topics in laboratory mode through different mathematical representations to promote the acquisition of a full competence that goes beyond the fragmented theoretical knowledge.

3.2.4 The LiC-CG project

The Group in Didactic Research of Mathematics of the Department of Mathematics of the University of Salerno in collaboration with the teachers, trainers of the educational staff of Casio Italia, from the academic year 2018/2019 has started and developed, as part of the activities of the Mathematical High School, the Didactic Research project "LiC-CG Laboratory in the Classroom with the Graphic Calculator" which involved 15 teachers, each of whom participated with their own class. This project aims to introduce the use of these tools into curricular activities in mathematics hours and to verify their didactic effectiveness. In particular, in the second classes the intent was also to compare the results of the Invalsi tests of the students involved in the project with the results of the peers of the other second classes for an evaluation of the relapse of the use of the graphic calculator (although for the space of two laboratory activities) in curricular teaching.
The project aimed to follow up on the recent educational experiments carried out on groups of students in the high schools in Campania during the "Summer School of introduction to the use of graphic calculators" (Bologna, ROgora, Tortoriello).
In this perspective, the intent is to develop the research activities of discovery of mathematics through the graphic calculator, which allows to place the emphasis on the modeling evidence that emerges in mathematical problems by delegating, at least in the first phase of heuristic research, all the operating procedures to the calculator. (Bologna, Tortoriello, Veronesi 2021)
It therefore makes sense to design teaching activities and learning models in order to develop a profitable use by students, but in order for this design to have a positive impact on the curricular teaching activity, with a view to sharing good practices, the collaboration of teachers is necessary to develop modules to be included in the daily school path.
For the above, the training of teachers followed the training model (Bologna, Rogora, Veronesi 2019) based on the global interaction between trainer and teachers and between

teachers and students, both in training meetings and during classroom activities designed by teachers.

In the various meetings, attended by DIPMAT researchers, Casio trainers and project teachers, training in the use of the tool and in the development of general models of use was provided (Vérillon, Rabardel 1995), the learning objectives specific for teaching were outlined and the methodologies to be used during classroom activities have been identified and shared, in line with the theoretical system of the Mathematical High School. The evaluation form of the written tests was also prepared in collaboration between the teachers and researchers: a grid with score, with precise descriptors to measure the degree of correctness and precision of the answers, unique for all teachers and for all activities. Finally, "mathematics in reality" was chosen as the unifying theme of all the activities so that all teachers could prepare the paths with the use of the graphing calculators contextualized in the areas of daily life.

Each teacher designed the activities to be carried out in the classroom in a teaching lasting one hour, structuring the action with didactic forms that flanked the theoretical instruction to the commands to be assigned to the calculator in a *learning-by-doing* perspective and preparing a written test to evaluate the skills acquired to be submitted to students at the end of the activity. To measure objectively the benefits of these teaching aids, the structuring of tests to be administered to the students divided into two groups, one with and the other without the graphing calculator, was chosen.

In particular, for the proposed activity, two verification tests were structured by the teachers (we call for convenience test X and test Y), different in form but related to the same contents, which were solvable even without the use of the graphing calculator. The students were therefore divided into two groups and the X verification test was proposed, one group was instructed to use the graphing calculator while the other was not allowed to use it. The students then carried out the Y test with the opposite methodology to the one indicated in the X test: those who had used the calculator in X, did without in Y and vice versa. The grid described in the following paragraph has been used for the evaluation of the tests. Table-sheets were then built to summarize all the information of the test scores and the outcome of the test with the graphing calculator and without it for an easy data analysis.

After completing the training of teachers and the design of the experiments to be developed with the students, the realization of the activity was planned in the 2019/2020 school year.

The thematic choices developed by the teachers led most of them to foresee the development of the LiC-CG project in the second part of the school year with the exception of three teachers who had designed and structured the activities to carry out the first quarter.
The health emergency linked to the Covid-19 pandemic has led to the issuance of regulatory measures that have imposed the adoption of integrated digital teaching (DAD) throughout the national territory. This led to the postponement of some of the activities and only a few classes completed the experimentation of one of the two planned activities: two second classes and one fourth class.
With regard to the activities carried out, each teacher:

- designed the activities to be carried out in the classroom in a lesson lasting one hour;
- drafted didactic schedules that combined theoretical instruction with the commands to be assigned to the calculator in a learning-by-doing perspective.
- prepared the two written tests to be solved in class and delivered, one with the calculator and one without.

Each teacher who has completed the educational path also

- carried out the lesson in the classroom in the presence of a DIPMAT researcher (me);
- carried out the cross-verification test in the presence of a DIPMAT researcher (me);
- corrected the two tests (one with and one without calculator) for each student by assigning a score for each descriptor of the grid;
- delivered the correct evidence to the DIPMAT Research Group.

The researchers subsequently developed the analysis of the data emerged from the evaluation of the tests carried out by the teachers for their class, to observe whether or not there was the presence of a positive curvature in the evaluation of the test carried out with the calculator compared to the test without.
Given the prolongation of the epidemiological phase of COVID-19, it was decided to present the analysis of the data emerging from the activities and tests that took place in the three classes in which the LiC-CG project was completed since they allow interesting reflections and hypotheses, to be verified with the data that will be obtained at the end of the entire educational project. Due to the successive waves of covid-19 it is currently not possible to determine when the activities can be completed in full, since for this path it is planned to be carried out in the classroom by the teacher with the presence of a researcher for monitoring. Despite their incompleteness, the collected data, even if partial, show a positive curvature of

the results of the tests taken by the students thanks to the use of the graphing calculator.

3.2.5 Evaluation of evidence

In order for the attribution of the evaluation to be authentic and to allow to analyze the results of tasks carried out in different classes while maintaining the parameters considered constant, an evaluation grid was prepared by the research group which was then shared with the teachers involved in the project and appropriately revised according to their indications. The grid, which strongly recalls the one proposed by the Ministry of Education for the evaluation of the second tests of the State Exams of Scientific High Schools, is suitable for the correction of any mathematics test and has been supplemented by specific references to the evaluation of the use of the graphing calculator and transformed for convenience of use into decimal grades.

It consists of four indicators: understand, identify, develop the resolution process, argue, each of which specified by descriptors to determine how much the students' performance responds to the demands of the indicator and is presented below:

Indicator	Descriptors	score
UNDERSTAND *Understanding of requests, data, their interpretation. Clarity of the theoretical references and the chosen procedures. Correctness of the necessary grapho-symbolic codes.*	Complete with analytical capabilities	2,5
	Appropriate	2
	Essential	1,5
	Incomplete	1
	Fragmentary	0,5
	Uncertain/undetectable	0
LOCATE *Organization and use of knowledge and skills to analyze, decompose, process. Logical sequence and order of drafting. Choosing effective procedures with the calculator. Effectiveness of the resolution strategy. Choice of best practices and non-standard.*	Complete, original and safe	3
	Widespread and articulated	2,5
	Organized and appropriate	2
	Consistent and essential	1,5

	Not always clear	1
	Fragmentary	0,5
	Incongruent/undetectable	0
DEVELOP THE RESOLUTION PROCESS *Correctness in calculations, in the application of techniques and procedures. Correctness and precision in the execution of geometric representations and graphs. Correctness and precision in the execution of the calculation procedures adopted with the calculator.*	Complete with executive mastery	2,5
	Appropriate	2
	Essential	1,5
	Incomplete	1
	Fragmentary	0,5
	Uncertain/undetectable	0
ARGUE *Properties of language, communication and comment of the solution punctual and logically rigorous. Clarity in the description of the calculation procedures adopted with the calculator. Consistency of results with the context of the problem*	correct, complete, coherent and formulated test with specific vocabulary	2
	correct and complete test, formulated with some inaccuracy in the specific lexicon	1,5
	incomplete evidence and /or with some inconsistencies and / or with some inaccuracy in the specific lexicon	1
	inconsistent, incomprehensible evidence	0,5
	absent	0

3.2.6 Data analysis

3.2.6.1 Tests carried out in the LiC-CG project

In this article we considered it significant to analyze the evaluations achieved by the students in the three classes in which the first activity of the project was completed and in the following tables the data relating to the tests of each teacher (for a total of 66 students) are summarized.

Brief remarks:

- The class of experimentation of prof. A.is a second class of the Mathematical High School and consists of 18 students who, according to the teacher who is their teacher from the first class, have a good basic preparation.
- The class of experimentation of prof. B. is a second class of the Mathematical High School and consists of 23 students who, according to the teacher who is their teacher from the first class, have a good basic preparation.
- The class of experimentation of prof. C. is a fourth class of the Scientific High School with applied sciences and consists of 23 students who, according to the teacher who is their teacher only from the fourth class and who over the years has seen various teachers succeed each other, have a modest preparation.
- The approach proposed in the second classes concerned the resolution of linear systems, a theme deepened by the teachers with a heuristic approach through the use of the graphic calculator developing both the algebraic and the Cartesian geometric resolution and also the graphic one.
- The theme introduced with the graphing calculator in the fourth class was that of successions, addressed through the construction of tables and analysis and experimentation also thanks to recursion; the analysis of the graphic view led to the experimental deduction and, later, to the rigorous definition of the limit of a sequence and allowed to deepen the number *and* as a limit of a convergent sequence.
- In the courses developed in all three classes, the teachers presented the students with the possibility in mathematics of being able to undertake a plurality of solution paths for each problem faced and allowed the students to choose the methodological approach that best suits their learning style.

In the vision of the tests, it was noted that Prof.C requested precise explanations of the steps taken and corrected with punctuality even minor errors in the improper use of mathematical terms. Prof. A and Prof. B evaluated more generously, not requiring a technical explanation of the steps taken. A different value to the theoretical justification of the steps taken and to the use of mathematical terminology is to be attributed to the levels of learning and re-elaboration of students who are of two different school years and therefore have different maturity of study. In fact, Prof. C carried out her activity in a fourth class, the professors Prof. A and Prof.B instead in second classes.

3.2.6.2 Analysis of the evaluation in the two tests

In the analysis of the data, it will be compared the evaluation that the students had in the test developed with and without the calculator for each of the three classes that completed the project (the evaluation was carried out by the teachers of the class according to the grid described in paragraph 3.2.5 which had been previously shared, the analysis of the evaluation results will be discussed later in the text).

It will also be highlighted the difference between the two votes and, for ease of reading the table, it will be highlighted:

- in yellow the positive deviation, that is, the cases in which the evaluation with the calculator is greater than the one without
- in blue the negative deviation, i.e. cases in which the evaluation with the calculator is less than the evaluation without.

For each class, at the end of the list, a summary prospectus is presented in which the numbers of students with

- evaluation
- higher rating with calculator
- lower rating with calculator

and in the latter two cases all the scrap points are added together.

There is common evidence in all classes that the gap of those who had a higher rating with the calculator (difference between the grade with the calculator and the grade without the calculator) far exceeds the gap of those who had a lower rating (difference between the grade without the calculator and the grade with the calculator). The data was expected, but there was no confirmation in the literature of measurements that allow to quantify the improvement in performance during the verification phase, it was therefore intended to respond to this hypothesis with the analysis of the evidence.

3.2.6.3 Description of information contained in the columns:

Tables 1, 4 and 7 collect the data of the three classes involved and indicate:

- in column 1 references to students
- in column 2 the mark obtained in the test with the calculator
- in column 3 the mark obtained in the test without calculator
- in column 4 the change in valuation
  - in yellow if evaluation with calculator > evaluation without calculator
  - in blue if evaluation with calculator < evaluation without calculator

Tables 2, 5 and 8 summarize the results:

- in column 1 the ratios between the results in the two tests:
  - same assessment,
  - better rating with the calculator,
  - best rating without calculator,
- in column 2 the number of students corresponding to each result ratio
- in column 3 the sum of the differences between the two votes (with reference to column 4 of Table 1)

The ribbon diagrams in Tables 3, 6, and 9 allow an immediate view of the data from the previous tables for each class.

3.2.6.3.1 Second class – Prof. A

SECOND CLASS Prof. A			
student	with calc. CG50	withouta CG50	D

A1	6	6	0
A2	6	5	1
A3	6	6,5	-0,5
A4	6	8,5	-2,5
A5	5	5	0
A6	6	6	0
A7	6	8,5	-2,5
A8	8,5	5	3,5
A9	8,5	6	2,5
A10	3	4,5	-1,5
A11	5	4,5	0,5
A12	8,5	4,5	4
A13	8,5	2,5	6
A14	8,5	8	0,5
A15	8,5	7,5	1
A16	8,5	8,5	0
A17	8,5	8	0,5
A18	8,5	8	0,5

Table 1. Grades class prof. A

evaluation of the two tests	number of students	sum of waste
same rating	4	
higher rating with calculator	10	points 20
lower rating with calculator	4	points -7

Table 2. Evaluation analysis of the two tests

With reference to the class, it is noted that:

- 4 students obtained the same evaluation in the test with and without the calculator,
- 10 students obtained a higher rating in the test with the calculator than in the test without and adding the positive difference of all 10 students it is observed that they obtained a total of 20 points more, or an average rating higher of 2 points,
- 4 students obtained with the calculator a lower evaluation than the test carried out without, the negative difference is 7 points overall, which corresponds to an evaluation with the calculator on average lower than 1.75 points.

As a percentage, the data can be enclosed in the following figure

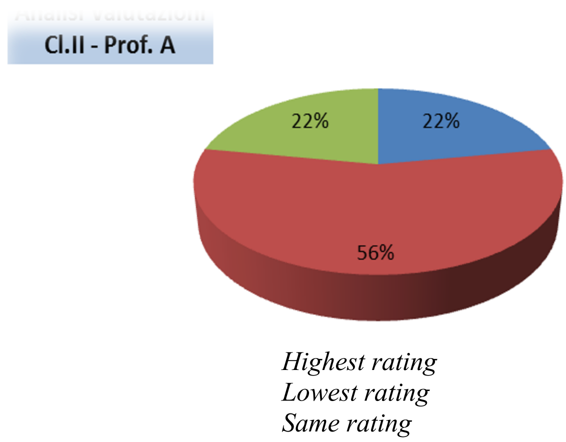


Table 3. Evaluation analysis

3.2.6.3.2 Second class – Prof.B

SECOND CLASS Prof.B			
student	with calc. CG50	without CG50	D
B1	9	8	1
B2	9	9	0

B3	9	9	0
B4	9	9	0
B5	9	9	0
B6	9	9	0
B7	8,5	9	-0,5
B8	8,5	9	-0,5
B9	8	9	-1
B10	8,5	9	-0,5
B11	7,5	9	-1,5
B12	8	9	-1
B13	9	9	0
B14	9	9	0
B15	9	9	0
B16	9	9	0
B17	9	8,5	0,5
B18	9	8,5	0,5
B19	8,5	8,5	0
B20	9	8,5	0,5
B21	9	8,5	0,5
B22	9	7	2
B23	6,5	6	0,5

Table 4. Grades class prof. B

evaluation of the two tests	number of students	sum of waste
same rating	10	
top rating calculator	7	points 5,5
lower rating with calculator	6	points -5

Table 5. Evaluation analysis of the two tests

With reference to the class, it is noted that:

- 10 students obtained the same evaluation in the test with and without the calculator,
- 7 students obtained a higher rating in the test with the calculator than in the test without a calculator and adding the positive difference of all 7 students it is observed that they obtained a total of 5.5 points more, or an average evaluation higher at 0.8 points,
- 6 students obtained with the calculator a lower evaluation than the test carried out without a calculator, the negative shot is 5 points overall which corresponds to an evaluation with the calculator on average lower than 0.8 votes.

As a percentage, the data can be summarized with the following figure:

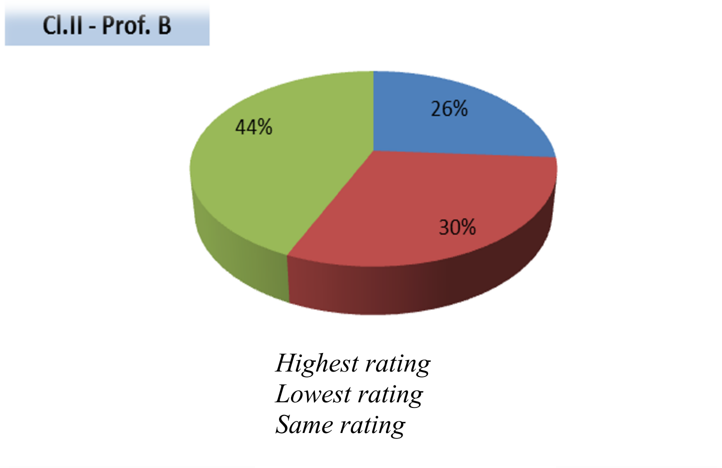


Table 6. Evaluation analysis

3.2.6.3.1 Fourth class – Prof.C

FOURTH CLASS Prof.C			
student	with calc. CG50	without CG50	D
C1	4	4	0
C2	8	5	3
C3	4	4	0
C4	6	5	1
C5	6,5	8,5	-2
C6	5,5	5,5	0
C7	5	7	-2
C8	8,5	5	3,5
C9	5	4	1
C10	8,5	7,5	1
C11	5	4	1
C12	8	8	0
C13	6	4	2
C14	6,5	6,5	0
C15	4,5	4	0,5
C16	7,5	7,5	0
C17	4	4,5	-0,5
C18	5	5,5	-0,5
C19	8,5	5,5	3
C20	5	4	1
C21	4,5	4,5	0
C22	4	4	0
C23	6,5	6,5	0
C24	5,5	5,5	0
C25	4	4	0

Table 7. Grades class prof.C

evaluation of the two tests	number of students	sum of waste
same rating	11	
top rating calculator	10	points 17
lower rating with calculator	4	points -5

Table 8. Evaluation analysis of the two tests

With reference to the class, it is noted that:

- 11 students obtained the same evaluation in the test with and without the calculator,
- 10 students obtained a higher rating in the test with the calculator than in the test without a calculator and adding the positive difference of all 10 students it is observed that they obtained a total of 17 points more, or an average evaluation higher at 1.7 points,
- 4 students obtained with the calculator a lower evaluation than the test carried out without a calculator, the negative shot is 5 points overall which corresponds to an evaluation with the calculator on average lower than 1.25 points.

As a percentage, the data can be summarized with the following figure:

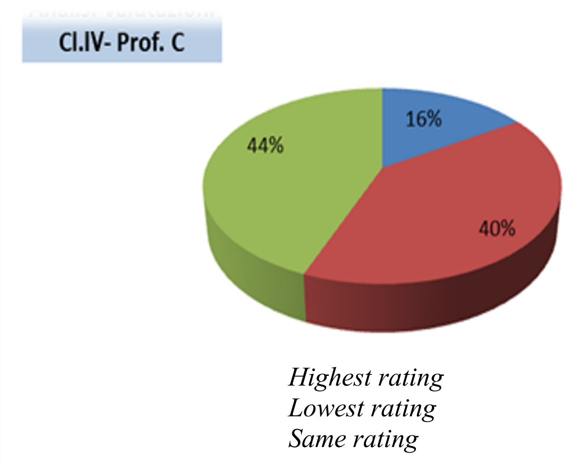


Table 9. Evaluation analysis

3.2.7 Data collection

In the following tables, in which we will analyze the score attributed to the boys in each indicator, it must be understood that the four consecutive numbers correspond to the scores obtained in the four items.

Description of the information contained in the columns:

- first column name of the student,
- second column: scores of each descriptor of the evaluation sheet in the test taken with the calculator
- third column: scores of each descriptor of the evaluation sheet in the test taken without the calculator
- fourth column: ratio of assessments in tests with and without the calculator considering all four indicators of the scoreboard
- fifth column: reporting if the evaluation of the tests changes excluding the fourth indicator "ARGUE".

For an easier reading, we thought it appropriate to color the lines that have some particular characteristic useful for the purposes of the investigation:

- cases where ratings with and without calculator are the same have colored lines in orange,
- the cases in which the scores attributed to the two tests are the same but the values assigned to the descriptors change have the lines of a brighter shade. (It is noted that Prof.C has some cases in deep orange, Prof.B only one case). In most cases, if the tests have the same evaluation, the scores attributed to the descriptors of the evaluation sheets of the two tests are the same.

SECOND CLASS Prof. A				
student	with calc. CG50	without CG50	with < without with > without with = without	without the 4th indicator

A1	1,5-1,5-1,5-1,5	1,5-1,5-1,5-1,5	=	
A2	1,5-1,5-1,5-1,5	1,5-1-1,5-1	>	
A3	1,5-1,5-1,5-1,5	1,5-1,5-2-1,5	<	
A4	1,5-1,5-1,5-1,5	2-2,5-2,5-1,5	<	
A5	1,5-1-1,5-1	1,5-1-1,5-1	=	
A6	1,5-1,5-1,5-1,5	1,5-1,5-1,5-1,5	=	
A7	1,5-1,5-1,5-1,5	2-2,5-2,5-1,5	<	
A8	2-2,5-2,5-1,5	1,5-1,5-1-1	>	
A9	2-2,5-2,5-1,5	1,5-1,5-1,5-1,5	>	
A10	1-1-0,5-0,5	1,5-1-1-1	<	
A11	1,5-1,5-1-1	1,5-1-1-1	>	
A12	2-2,5-2,5-1,5	1,5-1-1-1	>	
A13	2-2,5-2,5-1,5	1-0,5-0,5-0,5	>	
A14	2-2,5-2,5-1,5	2-2-2,5-1,5	>	
A15	2-2,5-2,5-1,5	2-2-2-1,5	>	
A16	2-2,5-2,5-1,5	2-2,5-2,5-1,5	=	
A17	2-2,5-2,5-1,5	2-2-2,5-1,5	>	
A18	2-2,5-2,5-1,5	2-2-2,5-1,5	>	

Table 10. Evaluation with the four class A indicators

SECOND CLASS Prof.B				
student	with calc. CG50	without CG50	with < without with > without with = without	without the 4th indicator
B1	2-3-2,5-1,5	2-2,5-2-1,5	>	
B2	2-3-2,5-1,5	2-3-2,5-1,5	=	
B3	2-3-2,5-1,5	2-3-2,5-1,5	=	
B4	2-3-2,5-1,5	2-3-2,5-1,5	=	

B5	2-3-2,5-1,5	2-3-2,5-1,5	=	
B6	2-3-2,5-1,5	2-3-2,5-1,5	=	
B7	2-3-2-1,5	2-3-2,5-1,5	<	
B8	2-3-2-1,5	2-3-2,5-1,5	<	
B9	2-3-2,5-0,5	2-3-2,5-1,5	<	=
B10	2-3-2-1,5	2-3-2,5-1,5	<	
B11	2-2-2-1,5	2-3-2,5-1,5	<	
B12	2-3-2,5-0,5	2-3-2,5-1,5	<	
B13	2-3-2,5-1,5	2-3-2,5-1,5	=	
B14	2-3-2,5-1,5	2-3-2,5-1,5	=	
B15	2-3-2,5-1,5	2-3-2,5-1,5	=	
B16	2-3-2,5-1,5	2-3-2,5-1,5	=	
B17	2-3-2,5-1,5	2-3-2-1,5	>	
B18	2-3-2,5-1,5	2-3-2-1,5	>	
B19	2-3-2,5-1	2-3-2-1,5	=	>
B20	2-3-2,5-1,5	2-3-2-1,5	>	
B21	2-3-2,5-1,5	2-3-2-1,5	>	
B22	2-3-2,5-1,5	2-2,5-1,5-1	>	
B23	2-2,5-1-1	2-2-1-1	>	

Table 11. Evaluation with the four class B indicators

FOURTH CLASS Prof.C				
student	with calc. cg50	without cg50	with < without with > without with = without	without the 4th indicator
C1	1-1,5-0,5-1	1-1,5-0,5-1	=	
C2	2-2,5-2,5-1	1,5-1-1,5-1	>	
C3	1,5-1-0,5-1	1,5-1-0,5-1	=	
C4	2-1-2-1	2-1-1-1	>	

C5	2-1-2,5-1	2-2-2,5-2	<	
C6	2-1,5-1-1	2-1,5-1-1	=	
C7	1-1,5-1,5-1	2-1-2,5-1,5	<	
C8	2,5-2,5-2,5-1	1,5-1,5-1-1	>	
C9	1,5-1,5-1-1	2-0,5-0,5-1	>	
C10	2,5-2,5-2,5-1	2-2-1,5-2	>	
C11	2-0,5-1,5-1	1,5-1-0,5-1	>	
C12	2,5-2,5-2-1	2-2,5-2-1,5	=	>
C13	1-2-2-1	1,5-0,5-1-1	>	
C14	1,5-2-2-1	2-2-1-1,5	=	>
C15	1,5-1,5-0,5-1	2-0,5-0,5-1	>	
C16	2-2-2,5-1	2-1,5-2-2	=	>
C17	1-1-1-1	1,5-1-1-1	<	
C18	1-1-2-1	2-1,5-1-1	<	
C19	2,5-2,5-2,5-1	1,5-1,5-1-1,5	>	
C20	1,5-0,5-2-1	1-1-1-1	>	
C21	2-1-0,5-1	1,5-1-1-1	=	
C22	1-1,5-0,5-1	1,5-1-0,5-1	=	
C23	1,5-2,5-1,5-1	2-1-1,5-2	=	>
C24	2-1-1,5-1	2-1-1-1,5	=	>
C25	1,5-1-0,5-1	1,5-0,5-1-1	=	

Table 12. Evaluation with the four class C indicators

3.2.8 Comments on the data collected

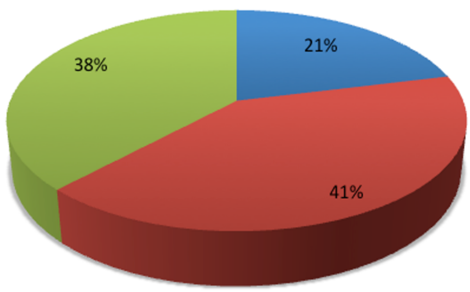
In the summary sheet of the data (Table1 3) it is clear that 41% of students in the test with the graphing calculator have a better performance in terms of evaluation, 38% get the same evaluation, 21% with the calculator get a lower rating. The negative difference of those who do worse with the calculator is less than the positive difference of those who do better with.

In particular, with reference to the grade of the tests, 62% of students have a non-negative evaluation following the use of the graphing calculator.

	Prof.C	Prof. A	Prof.B		total	percentage	
Same rating	=11	= 4	= 10		25	37,9%	
Better performance with the graphing calculator	> 10	> 10	> 7		27	40,9%	
Worst performance with the graphing calculator	< 4	< 4	< 6		14	21,2%	
waste sum	Prof.C	Prof. A	Prof.B		total		point each student
Positive	+17 points	20 points	5,5 points		42,5 points		+1,6
Negative	-5 points	-7 points	-5 points		-17 points		-1,2

Table 13. Data summary sheet with four indicators

Overall analysis of the four indicators
(understand, identify, develop the resolution process, argue)



Highest rating
Lowest rating
Same rating

Table 14. Results with four indicators

The summary of the data contains the information of the previous paragraphs and leads to underline a positive curvature of the students' school performance thanks to the use of the calculator for more than 40% of them. For about 20% of the students, on the other hand, a slight worsening of the results of the test with the calculator was observed.

Analyzing the data, it emerged that in various cases the score of the fourth indicator "ARGUE" significantly changed the evaluation always in a pejorative key of the task carried out with the calculator.

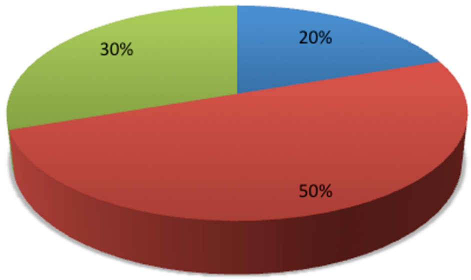
This element is significant and will be the subject of further study also because it is to be considered that the motivation of the negative curvature can be attributed to the lack of time or experience to elaborate new schemes and pedagogical tools for evaluating the argumentation of the resolutive processes implemented with the use of artifacts. In this case, in fact, part of the processing is carried out by the machine and, of course, it is necessary to specify how specific descriptions must be developed.

In this phase, to assess the impact of this criticality, it was decided to reformulate the analysis table considering only the first three indicators "UNDERSTAND", "IDENTIFY" and "DEVELOP THE RESOLUTION PROCESS", which represent the evaluation component linked to the correctness of the resolution of the proposed financial year.

Prof. A	Prof.B	Prof.C		total	percentage
= 4	=10	=6		20	30,3%
> 10	> 8	> 15		33	50,0%
< 4	< 5	< 4		13	19,7%

Table 15. Data summary sheet with three indicators

Overall analysis of the three indicators
(understand, identify, develop the resolution process)



Highest rating
Lowest rating
Same rating

Table 16. Results with three indicators

From the analysis, it can be observed that 50% of students score higher in the test with the calculator than in the one without, there remain about 20% of students who scored lower with the calculator and 30% of students who obtained the same scores. In the activities carried out and monitored in the project, there is a positive or in any case non-worsening impact for about 80% of the students.

In particular, it is noted that when the fourth indicator was excluded in which the professors expressed an evaluation on the argumentation of the procedures carried out, that is, they evaluated the argumentative component of the task and how the processes adopted were described by the students, six boys obtained a higher evaluation in the verification with the graphing calculator, that is, about 10% of the students. It also emerged that 5 of the 6 boys who without the 4th indicator went from having the same score to having a higher score in the test with the calculator, are of the fourth class, in which the teacher has exercised a rigorous correction on the part of description of the procedures implemented in the writings (as already presented). In this regard, two hypotheses can be put forward:

- the tasks of the fourth class had a greater degree of theoretical depth and the students have sometimes described with less rigor the steps developed with the calculator, considering them perhaps a common cultural baggage as sequences of instructions studied and made their own
- the teacher expected a more articulated and in-depth description that went beyond the simple reporting of the path implemented.

In conclusion it can be noted that with the exclusion of the evaluation of the argumentative component of the task, 10% of the students obtained a higher evaluation in the test with the calculator than in the one without.
This undoubtedly confirms the observations that teachers often make about the complexity of assessing the quality of a task carried out with the graphing calculator. This probably happens because students and teachers have a different way of interpreting the need to describe and argue the processes carried out with the calculator and it is an aspect on which it is important to reflect to design paths that align the requests of the teachers with the products of the students.

3.2.9 Conclusion

Due to the interviews that took place between the researchers and the teachers who developed the activities in the classroom, it emerged that the latter observed how the graphing calculator was very useful and effective for the weakest students in mathematics who obtained better ratings in the test with the graphing calculator than the one without a calculator.
In the analysis of the teachers' interviews, some qualitative indicators related to the activities carried out in the classroom emerged.
Analyzing the observations of the teachers regarding the activities carried out in the classroom it emerges:

- the interest and passion for this type of laboratory activity;
- the difficulties related mainly to the respect of the temporal scan of the development of the program;
- the need for continuous updating for an effective didactic action;
- the will to repeat in class other activities similar to those addressed in the project;
- the will to collaborate again in research paths with the graphing calculator.

The teachers observed the activities of the students and noted that students:

- were personally involved in carrying out the activities;
- helped each other in the passages in which there were difficulties with the calculator during the activities in the classroom;

- Even when they did not know the exact procedure for developing a calculation or a process, helping each other, they were able to solve the impasse and find the solution with the graphing calculator.

From the interviews carried out with the students, analyzing comments and observations, it is noted that they

- were particularly involved in the activity
- wished they could reply the approach to mathematical issues with the graphing calculator also with other activities
- highlighted
  - their interest in the use of the graphing calculator also aimed at state exams
  - their greater involvement in a less transmissive and more laboratory teaching activity
  - to have acquired the contents more easily thanks to the comparison of different approaches (numeric, tabular, graph) through the graphing calculator.

Evidently the data discussed and analyzed in this research are only partial, as previously pointed out, and can also be taken for granted for those who know the tool and have evaluated its effectiveness. The possibility of inserting the graphic calculator in daily teaching, however, meets teachers who dispute its value as a semiotic mediator, confining it to a mere automatic calculation tool.

The data collected bring to light some important aspects to highlight because they do not refer exclusively to the state of the art but can be interpreted in a broader key in the analysis of the activities that take place in the classroom with the graphing calculator:

- a positive and proactive attitude on the part of the students towards the use of the graphic calculator and more generally of the laboratory activities in the classroom;
- an improvement of the weakest students in the outcome of the tests and more generally in the participation in school activities with the use of the calculator;
- great interest, participation and willingness to collaborate also with the sharing of materials and experiences by teachers who use the graphing calculators in teaching;

- the need to deepen the development of activities also aimed at evaluating tasks carried out with the graphic calculator and defining the argumentative processes required of students and to be considered exhaustive for teachers.

CHAPTER 4

GAME THEORY LAB

My research activity has been oriented towards the development and deepening of interdisciplinary courses in economic mathematics, dedicated to upper secondary school students because statistics highlight that about 30% of students enroll in university faculties with an economic orientation even though they have never completed an hour of economics lesson in the entire school curriculum. The goal of my research was to build activities that focus on students as subjects in continuing education, integrating multimedia culture and digital culture with school and family cultures through a global training simulating real scenarios to provide students with mathematical and economic models that allow them to understand and analyze the dynamics of historical, political and economic decisions in real contexts. The idea of the Lab was to engage students in active learning, exploiting their natural curiosity about economic and social issues, and to make them think about the questions before trying to give them answers based on theory. The workshop was designed to be enjoyed in person and was reworked for online distance learning, due to restrictions related to the COVID-19 pandemic.

The activities were projected to aim

- at producing "Game Theory" and "Decision Theory" paths on online-platforms
- in deepening of the optimal Nash equilibrium problems using a topological approach.

In particular, thanks to a dynamic geometry software, students applied the Voronoi tessellation on a plane and, through the Delaunay triangulation they found the optimum of a multi-variable function, an unsuitable argument about analysis to the skills of high school students.

4.1 Background

The Ministry of Education in Italy, MIUR, has been encouraging the use of gaming and gamification at school for several years. The National Digital School Plan (PNSD), for example, includes the use of gaming and gamification in supporting the use of digital technologies and promotes innovative projects in the field of education.

On February 22, 2018, the document NATIONAL INDICATIONS AND NEW SCENARIOS was presented to MIUR that references to the 17 objectives of the 2030 agenda for sustainable development (Agenda-2030), in particular Objective 4 requires to "Provide quality, fair education and inclusive, and learning opportunities for all "and about mathematics it is written:

“Mathematics provides tools to investigate and explain many phenomena of the world around us, promoting a rational approach to the problems that reality poses and therefore providing an important contribution to the construction of a conscious citizenship. Mathematics (...) also allows you to develop important transversal skills through activities that enhance the typical processes of the discipline: "In particular, mathematics (...) contributes to developing the ability to communicate and discuss, to argue correctly, to understand the views and arguments of others. " These skills are relevant for the formation of an active and aware citizenship.”

And again, the mathematics laboratory represents a natural context to stimulate the ability to argue and stimulate peer comparison: (…)

“In mathematics, as in other scientific disciplines, the laboratory is fundamental, intended both as a physical place and as a moment in which the student is active, formulates his own hypotheses and controls their consequences, designs and experiments, discusses and argues his choices, learns to collect data, negotiates and builds meanings, leads to temporary conclusions and new openings the construction of personal knowledge and collective.”

Following the Indications, the laboratory becomes gym to learn how to make informed choices, to evaluate the consequences and therefore to take responsibility for them. These aspects are essential in education for an active and responsible citizenship.

Starting from what emerges in the new scenarios regarding national indications, in the context of the activities of the “Mathematical High School Project”, has been decided to introduce the Game Theory educational module on strongly impacting themes in the life of youngers creating moments of reflection on daily reality, where mathematics is closely interconnected with the experiences of daily life presenting themes usually not addressed in regular school curricula.

Game theory is a branch of mathematics widely applied in economics, engineering, political science, and biology, and can be seen as a study of how to mathematically determine the best strategy under certain conditions to optimize an outcome. It deals with interactive decision-making models, i.e. situations in which two or more individuals (called players) make decisions that influence each other and is based on some assumptions that characterize the fundamental way in which individuals think and act: those who interact in a decision problem should be intelligent and rational. The intelligent individual has the ability to understand the situation in front of him and is aware that other individuals participate with their own mental schemes on that situation. The rational individual in the strict sense is able to order his preferences on a set of outcomes; moreover, his preferences satisfy the set of von Neumann and Morgenstern axioms of rationality.

Every day we find ourselves in situations in which decisions must be made, using the available information, evaluating and interpreting the outcomes of possible choices, in order to select an action among several alternatives: for example, selecting a candidate in a vote, participating in auction, decide where to place a shop. In economic and social analysis, Decision Theory deals with an agent's choices starting from the definition of the set of possible choices; a decision is the selection of one of these options. Decision analysis consists in the application of a scientific method to problems in order to provide useful solutions to agents to make better decisions and therefore to make rational choices thanks to tools, methodologies and software. The Expected Utility Theory of Von Neumann and Morgenstern (Von Neumann, Morgenstern 1947) considers the decisions of an individual based on the assumption of the rationality of individuals. The individual, by solving a simple "algebraic algorithm" structured with the information in his possession, evaluates all the possible alternatives, orders his preferences and builds a real choice function. Rubinstein states that Game Theory is the study of a set of considerations used by people in strategic situations (Rubinstein 1999). His goal as a teacher is to see game theoretical models and predictions of strategic behavior in real life separately. He conducted a pilot experiment with a group of graduate students at Tel Aviv University in which he found that the results before and after the Game Theory course did not differ.

4.2 Gamification

This contingent educational scenario of a liquid society in constant change (Baumann 2000) is changing rapidly and is characterized by a very high degree of technology. Consequently, the school has to evolve rapidly and students are forced to constantly get involved in order to change their learning schemes to realign them and adapt them in a distance learning perspective. It is the task of the school to identify the well-being of all people as a main topic and place it at the center of social and economic dynamics that have to be oriented in terms of economic stability and above all on the plan of psychological serenity (Baumann 2000). Starting from this point of view, it has been decided to build and outline a didactic path tailored on the model of a video game, not in a literal sense but an action-oriented path in which students take on the role of protagonists of a story in which they interact with the situations they encounter and must decide the actions to be taken to optimize their profit / score. *It is well known that students get excited when they have to solve problems proposed in the form of competitive games. To exploit this tendency and introduce students to solve complex problems in real situations, the authors designed an educational project to help students understand an abstract model, which requires the use of mathematical analysis in economics* The choice can be particularly effective, as it is akin to the world of students, and significant thanks to the evident motivational potential in teaching and training in general because it *improves students*' *performances, encourages competition, and rewards productive and cooperative behavior*.

Gamification can be seen as a set of activities and processes to solve problems using or applying the characteristics of the elements of a game and *may be applied to the construction of a reward-based system that acts as an incentive for students to engage in class-discussion*. Gamification can be a method of engaging learners and improving motivation through a set of activities and processes in which players solve specific real-life problems. The relationship between Gamification, mathematics and economics is interdisciplinary. Gamification can be used to introduce economic concepts and real-life problems in which analysis through strategic interaction models from Game Theory are the fundamental mechanism. On the other hand, economics is closely intertwined with mathematics because a mathematical infrastructure is needed to support the construction and elaboration of economic interpretations of solutions to real-life problems. It is no coincidence that many Nobel Prizes in Economics have been awarded to mathematicians. Mathematics and economics make it possible to deal with many subjects studied at school in an interdisciplinary key, such as history, philosophy, literature, science and physics, and help to interpret with a unified vision the historical and cultural events of the past and present. As Morin points out, it is necessary to know all the dynamics in order to produce effective forecasts to make an informed and reasoned choice (Morin 1999). Contemporary society is a complex reality in which changes' dynamics require complex thinking capable of making decisions that can be favorable to private life but also to ethical or social professional choices and of interpreting the wide impact of choices on the community. Gamification is designed to promote social behaviors and aimed at obtaining complex learning outcomes, through non-traditional processes, it is demanding and structured, more suitable for older students able to use even elaborate strategies.

Gamification is the application of game mechanisms in a non-game context where students work in cooperative groups (whose members have different roles, tasks and tools), live a play adventure, a mission that uses game strategies to carry out academic tasks and reach the achievement of a complex goal.

Supporting each other and working as a team, students carry out together or individually educational activities that usually have an engaging context and are set in a fantasy world, close to the world of role-playing video games with which they interface every day reaching increasingly higher levels, up upon completion of the mission. Gamification, in fact, uses strategies and mechanisms typical of video games to obtain the learning of complex and structured contents through activities with increasing levels of difficulty until the completion of the missions. This student-centered approach fosters active participation because it stimulates emotional intelligence (Damon 1984), (Goleman 1995).

Many researchers have revealed that gamification activities in teaching and education can improve learning outcomes, higher-order thinking skills, declarative and procedural knowledge (Sitzmann 2011) and tests' performances. Furthermore Randel et al. state that subjects with specific contents, such as mathematics, may show more positive effects of gamification than others (Randel, Morris, Wetzel, Whitehill, 1992).

The use of play as an access key to learning was theorized in 1975 by the psychologist Csikszentmihalyi who describes with the term "flow" a subjective psychological state of maximum positivity and gratification, which can be experienced during the performance of activities and which corresponds to "complete immersion in the task". The "flow experience" is the sense of total rapture or absorption when the goal is clear, achievable and controllable, has immediate feedback and an ascending climax that seduces students and does not register too high a gap between the purpose and the real possibilities of the participants. The situation that allows one to get in touch with this state of being is characterized by the perception, by the individual, of sufficient and adequate opportunities for action (challenges) on the part of the environment and, consequently, of adequate personal abilities to being able to act (skill). The entry into the flow therefore depends on the balance between these two components, subjectively assessed (Csikszentmihalyi 1975).

Already in the pre-digital era the evolutionary and evocative role of play and its formative function were defined by pedagogists Vygotsky and Piaget, who were the firsts to understand and pedagogize the importance of play for the construction of a learning process centered on the student. The game, through physical and mental stimulation, activates understanding processes that help to acquire disciplinary and socio-emotional knowledge, improves the climate of the learning environment, thus increasing the pleasure of learning.

Since 2010 Gamification is an internationally recognized teaching methodology in the world of education and training. It is a constructivist model in which the student, at the center of the training process, builds his own learning through direct experience that allows him to memorize better and longer-term knowledge thanks to the much more rewarding self-motivation of the vote booster.

Gamification is a learning process divided into consecutive phases based on the personalization of the path, which enhances and gives particular prominence to socialization and, when combined with the metacognitive component, becomes a tool for learning to learn, making strategies effective for students. Studies show that through exciting challenges such as in video games, awarding progress badges and visibility of performance graphs, educational goals are transformed into concrete promotional goals that satisfy the need for competence and increase the perceived significance of the task (Sailer, Hense, Mayr, Mandl 2017).

Learning is a continuous process of confirmations and adjustments: it is always based on the comparison between the output expectations and the one actually obtained and it is only through immediately visible feedback that the path and methodologies used can be adapted, just like in the game having a score visible satisfies the typical challenge of the human being and the sociability deriving from cooperative games (VanSickle 1978). Furthermore, the key elements of gamification are a compelling storyline, risk, uncertainty about what could happen, and are very useful tools to increase emotional, relational and social skills (Lazzaro 2005).

4.3 Methodology

In the 'Game Theory Lab', students have been introduced to real life problems with a Game Theory approach, conducting experiments in the classroom before introducing theoretical elements and models. The class experiments involved students, either individually or in groups, and were conducted with very brief introductory discussions, a few explanations during the student choice phase, and in-depth theoretical discussions showing the theoretical models and expected results in the literature with commentary on the results at the end of the proposed activities to analyze the interrelation between their own decision and those of other actors involved. In this way, students were able to actively learn about social choices, game theory and, more generally, mathematical economics (Bimonte, Tortoriello, Veronesi 2021, I), (Bimonte, Tortoriello, Veronesi 2021, II), (Bimonte, Tortoriello, Veronesi 2021, III), (Bimonte, Tortoriello, Veronesi 2020, I), (Bimonte, Tortoriello, Veronesi 2020, II).

The students were asked*: How can a group of individuals choose a winning outcome (e.g. a policy, an electoral candidate) from a given set of options? What are the properties of different voting systems? Is it possible to construct coherent collective preferences on the basis of individual preferences? If so, how?*
The aim of the workshop was to answer these questions by observing concrete examples taken from reality.

For all activities, students were divided into teams and were ranked according to their scores. The score was not visible to the teams until the end of the game. In some cases, the teams remained unknown to the other players. In each team, decisions were discussed intensively and taken on a majority basis or by one player (dictator). Rewards were given at the end of each round, about the partial results, and at the end of the lab.

In the classroom experiments the researcher's role was that facilitator asking, guiding questions and drawing attention to interesting results that emerged at the end of the choice activities. Next, students were led to discover the historical and theoretical framework through rigorous mathematical formalization. In classroom experiments, students were involved in commenting on the collected data and observing the phenomena and features present and were asked to make predictions and reflect on their observations and the motivations behind the choices of the other students. In some experiments, the result was 'surprising' but convincing, so students were able to build ownership of the new idea that emerged and use it to support their learning.
Much recent innovation in teaching has focused on the use of technology for active learning. Ball developed research demonstrating how the difficulties encountered in teaching economics can be partially overcome through the use of wireless devices technology known to students (Ball, Eckel, Rojas, 2006). They measured student learning relapse of the use of technology in teaching and found significantly better teacher ratings in the experimental class compared to classes with a traditional approach to the subject. In the game theory laboratory, following Ball, experiments were designed with the use of wireless devices, with which the students are very familiar. they actively participated in the proposed games, and were able to communicate with each other in the groups and directly with the game administrators (researchers).

For all activities it has used the same development scheme:

1. the storytelling approach has been used to create a defined context in the setting of a kingdom in medieval times (in the style of Fantasy video games currently present on the market), to define the roles, tasks and skills of each student within their group and to enhance the exchange of information, interpersonal relationships and the development of group activities

2. the feasible choices and possible actions have been analyzed: students are presented in a clear and complete way with the challenges, proposals or actions they can take, leaving them free to choose but having provided them with all the information necessary for an informed decision

3. Peer discussion is solicited, within the groups, space is left for comparison between the various choices or positions of the individual components, then asking to report at the end how the choice of the group's action finally took place

4. Students face the game and try to maximize their results and efforts and this is certainly the most exciting part of the activity for students, but later they are curious to verify the effectiveness of their evaluations and what, if any, was the strategic choice

5. A detailed written description of the choices is required: to enhance logical reasoning, to develop descriptive communication and narrative skills and integrate students' “literacy” skills with “numeracy” skills, teachers ask for a written report of the choices and motivations that led to these choices.

4.4 Decision Theory activities

A game in normal form, Γ, is defined by the n-tuple

$$\Gamma=(N; X_1,\dots,X_n; f_1,\dots, f_n)$$

where

N:= $\{1,\dots,n\}$ is the set of players;

$a_i \in X_i$ is a possible action for player i

A:= $(a_i)_{i\in N}$ is a profile of actions for the game, one for each player;

$f_i = \prod_{i=1}^{N} X_i \to \Re$ is the pay-off function.

The main assumptions and equilibrium notion of Game Theory, as in Patrone (2006), can be summarized by the following:

1. **Individual rationality**: human beings are rational beings, always looking for the best alternative in a set of possible choices so that each player maximizes his ‘utility’.

2. **Stability**: all players will "respect" a given strategy profile $x = (x_1,...,x_n)$ if it is a profit maximizing choice of player i when the rest of the players choose exactly the given profile. No player wants to deviate from x.
3. **Nash equilibrium**: is a profile of strategies such that each player's strategy is an optimal response to the other players' strategies. A Nash Equilibrium is a strategy profile $a^*=(a_1^*, a_2^*,...,a_n^*)$ such that for any $i \in N$ and any $ai \in Xi$ we have that

$$f_i(a^*) \geq f_i(a_1^*, \ldots, a_i, \ldots, a_n^*);$$

that is,

$$f_i(a^*) = max_{a_i \in X_i} f_i(a^*, a_i)$$

From Individual to Social Choices

The activities organized in the course of mathematics and economics were oriented towards two topics with different relevance compared to the school curriculum. The first path concerned Decision Theory, while the second concerned Competitive Location Models. These two themes were addressed by developing various activities within a path in which the students were actors in a simulated game. The idea was to try to build collective choices and to influence collective choices from individual choices. The topic was briefly introduced with the help of story-telling. The setting chosen was a medieval court, where the presence of the king, knights, acrobats and taverns, the students came to terms with rationality. The students, who played the role of the wisest men of the two kingdoms, were asked to make a choice, alternating between the individual optimum choice (the alternative they preferred most, according to their own criteria) and the social optimum choice (what would be best for the good of the two kingdoms). Measurable learning objectives related to the experiment were selected, in particular how individual choices are made, how collective preferences are aggregated, how rationality underlies choices and how it can sometimes be ignored.

Students were confronted with results that they were not always able to understand easily, followed by an explanation using the literature and the underlying mathematical models. They observed the collected data and tried to explain the motivations behind the choices of the other players. Finally, they had to motivate their own choices and those of the group they belonged to.

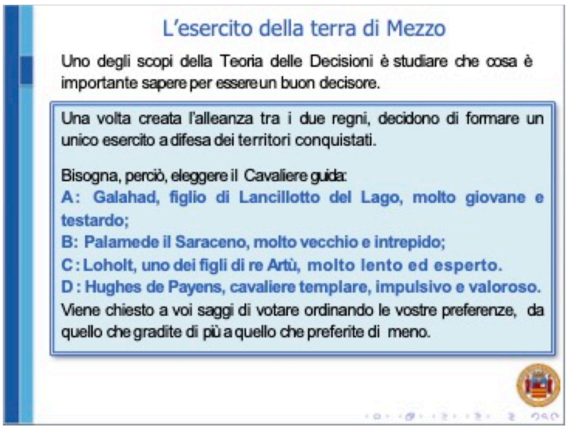
Online decision forms were constructed to keep the students' actions private, so as not to influence their choices.

Sorting preferences and individual choices

The first activities involved sorting preferences on multiple alternatives first on unimodal features and then on alternatives defined by multiple features (vectors). In particular, the first activity proposed was a coordination game on 4 alternatives and the result, consistent with the literature, is that players manage to coordinate on the salient option. Michael Bacharach's conjecture predicts that when each alternative is described in terms of a number of features, the salient option is the one that stands out in most of the features. The following activities, also involving coordination, concerned the choice between several options described by a vector of features. In this case, students experimented that it is not so easy to coordinate in order to make the choice converge on a single option.

Figure 1. Construction of the Individual Choice Function

Who will be the knight who will lead the army of Middle-earth?



Collective choices and voting methods

The following activities focused on the construction of collective choices from individual choices. Students had to select the knight who would be able to lead the army common to the two kingdoms.

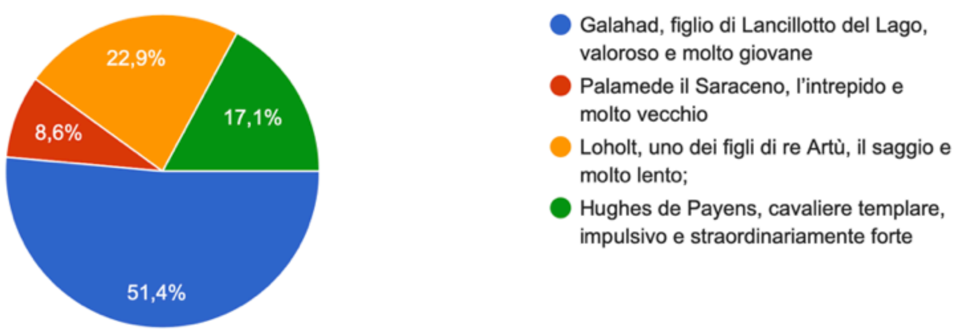
Each alternative was defined by two characteristics constructed in such a way that one was perceived as positive and the other as negative, according to common sense. In the text, in order not to influence the students' choice, no adversative conjunctions (such as but, instead, etc.) were used which could make one of the two aspects stand out more. Modules were created using Google's Moodle application to allow students to play, and each alternative (knight) was associated with a reference image reflecting the characteristics of the knights.

The first activity, centered on voting according to the Plurality and Majority Rules, required students to choose the knight they preferred most, selecting him according to the characteristics highlighted: Galahad, son of Lancelot of the Lake, very young and stubborn; Palamedes the Saracen, very old and intrepid; Loholt, one of King Arthur's sons, very slow and experienced; Hughes de Payens, Knight Templar, impulsive and valiant. The names echoed those of certain characters from the stories and legends of the Knights Templar and the well-known story of King Arthur. Among the alternatives, candidates linked to the two most important figures in the history of the Round Table were included in order to make the context more familiar and appealing to the students' imagination. We believe that this choice could have conditioned the choices in some way, making the students identify with the characters, preferring the characteristics most akin to them (youth, wisdom).

Activity 1 - Individual choice:

"What should be in your opinion the knight at the head of the Army of the two Kingdoms?"

Figure 2. Results on the Individual Choice Function



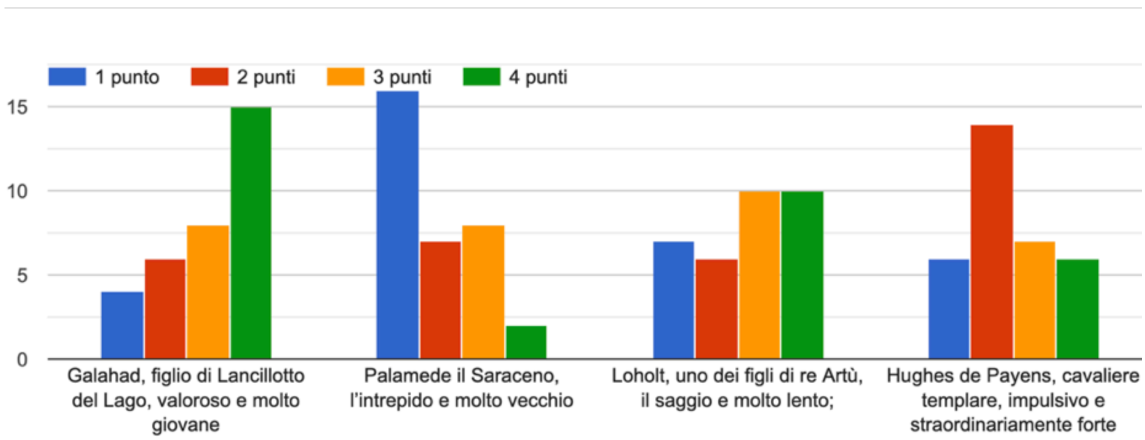
Among the alternatives, candidates linked to the two most important figures in the history of the round table were included to make the context more familiar and engaging to the students' imagination. We believe that this characterization may have somehow conditioned the choices, making the students identify with the characters, preferring the characteristics that are more akin to them (youth, wisdom).

In order to introduce Borda's criterion, the students ordered their preferences by assigning a decreasing score to the four alternatives

Activity 2 - Sorting of individual preferences:

"Sort your preferences from the one you prefer the most (4 points) to the one you prefer the least (1 point)"

Figure 3. Ordering of individual preferences



Does the winner always exist?

In the specific laboratory activity, a winner was declared, according to both the Plurality rule and the Majority rule. After displaying the results, a discussion was opened on the reasons for each student's choice. Next, the students were asked whether they considered their preferences to be "comparable", referring to the possibility of ordering one-dimensional elements and elements defined by a vector of characteristics.

The result of individual and collective choices, therefore, depended on how much of the information input "enriched" by the elements inserted for each alternative the students, individually first and then in groups, decided to use in determining the ordering of preferences: more formally, the result depended on the assumption of the interpersonal measurability and comparability of the "optimal" choices of the agents involved. To introduce Borda's criterion, students sorted their preferences by assigning descending scores to the four alternatives. The winner of Borda turned out to be the same as the winner of Plurality and Majority voting. The possibility that there was no Borda winner and the manipulability of the voting system were discussed with the students. After observing the distribution of the aggregate preferences, the students tried to formulate hypotheses on the score to be assigned in the voting phase, in order to be able to induce one choice rather than another. Students found the possibility of conditioning the outcome of a vote very interesting; very intuitively, students perceived how aggregations and agreements in the pre-voting phase can lead to a biased ordering of preferences in favour of a common candidate (violation of the Strategy-proofness condition).

To determine the Condorcet winner, students were asked to decide the winner in a pairwise comparison.

The order of the pairs put to the vote was randomized, to avoid any form of coordination between the students and to minimize the risk of automatic voting for the first alternative present. Comparison of the data on the ordering of individual preferences and the voting in pairs did not reveal any conflicting results: the students remained consistent with their initial ordering. The Condorcet winner, therefore, turned out to be the same as the other selection methods.

Choices in coalition

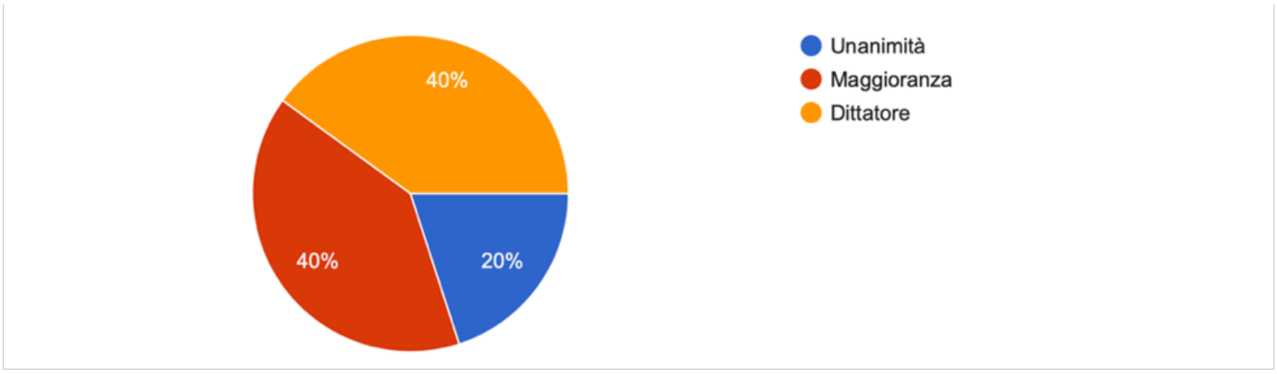
In the last activity, students formed 10 coalitions of 3-4 people. They first had to establish a decision rule within themselves in order to create the collective preference order, based on individual preferences. The result, surprising for the students, was that in 40% of the groups the decision was taken in a dictatorial way. The students, who belonged to the dictatorial groups, justified the decision as the result of a long decision-making process, from which a dictator imposing his choices emerged. It is easy to see, in this kind of attitude, the well-known result of Arrow's Impossibility Theorem.

The result of voting according to Condorcet's and Borda's method had the same winner.

Activity 4 - Aggregation of individual preferences:

"How did you make the decision in the group: unanimity, majority, dictator, other?"

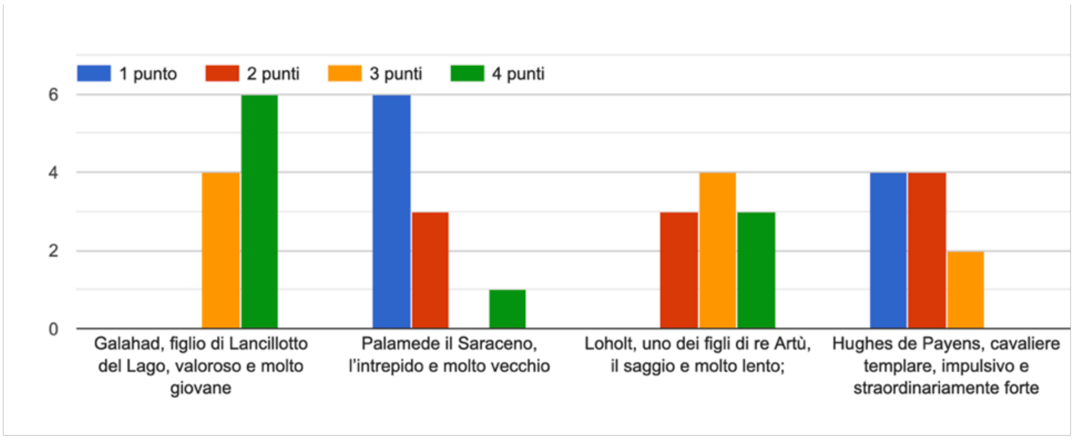
Figure 4. Collective preferences and choices – Modality



Activity 5 - Sorting of collective preferences:

"Sort your preferences from the one you prefer the most (4 points) to the one you prefer the least (1 point)"

Figure 5. Collective preferences and choices – results



Free educational games site: Economics-games.com

Economics-games.com is a free educational games site for teaching microeconomics, industrial organization and game theory. With this very comprehensive platform, we decided to select the multiplayer game for all students by getting them to play individually first and then in teams.
The activities have been focused on Game Theory simulation, by selecting some important games, among them the *Repeated Prisoner's Dilemma*, *the Beauty Contest*, and *the Chicken Game*.

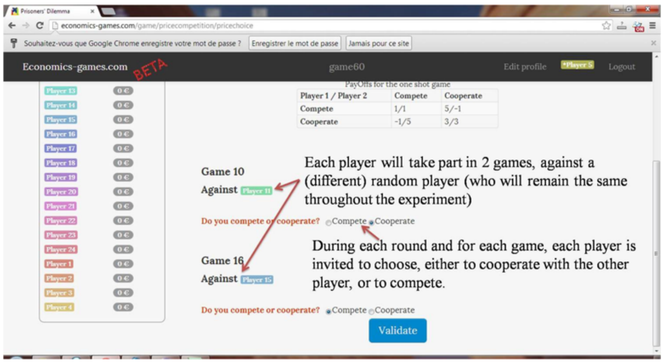
Repeated Prisoner's Dilemma
Each student played the Prisoner's Dilemma repeatedly. In this game, the players are company managers who must simultaneously decide whether to "cooperate" or "compete".
The corresponding payoffs are determined as follows: For one turn of the game, if both players compete, they both get a payoff equal to 1. If they both cooperate, they both get 3. If one cooperates and the other competes, the former gets -1 and the latter gets 5.

Table 1. Bi-matrix of the game

Player 1 / Player 2	Compete	Cooperate
Compete	1/1	5/-1
Cooperate	-1/5	3/3

Each player played two of these games at the same time, with random players of the same class, who remained the same for all repetitions. Each student knows that their action during one round may have some effect on the other player's actions in subsequent rounds.
The aim was to maximize one's own payoff, not just to be 'better' than the opposing players.

Figure 6. Economic-games.com instructions



All detailed and synthetic results and their evolution were not visible to the players but only to the game administrator.

Table 2. Synthetic Results after Round 6

Actions:

Round	6	5	4	3	2	1
Proportion of players playing Coop	*3.85%*	*13.46%*	*21.15%*	*21.15%*	*15.38%*	*34.62%*

Average Profit:

Round	6	5	4	3	2	*1*
Average profit	€1.08	€1.27	€1.42	€1.42	€1.31	€1.69

After six rounds, the proportion of cooperative choices has been drastically reduced. Cooperation problems arose because of a mismatch between individual incentives and collective goals. For the students, there was uncertainty about the end of the game: the administrator could decide to end the game after the second round, at an unspecified time. This uncertainty led the students to choose not to cooperate.

Direct reciprocity can support cooperation: in a Prisoner's Dilemma repeated an uncertain number of times (at most 10), a strategy that always cooperates will be overwhelmed by a strategy in which one always competes, because the latter always exploits one's opponent's first choice. The "Tit-for-Tat" (TFT) strategy cooperates on the first turn and repeats its opponent's previous move thereafter. Since the game in anticipation is repeated many times, the advantage of exploiting the other in the first round is more than offset by the disadvantage of receiving the mutual defection payoff instead of the mutual cooperation payoff in subsequent rounds. Thus, players using TFT will have little incentive to switch to a competitive strategy, and a mutually cooperative outcome becomes feasible.

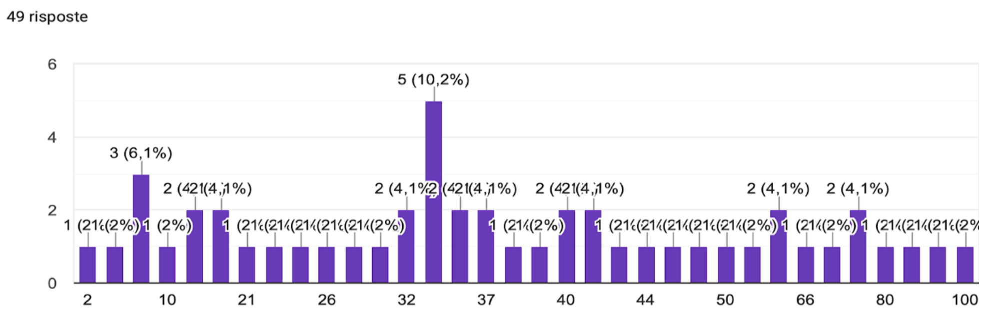
The Beauty Contest

The beauty contest is a game in which each student had to pick a number between 0 and 100. The winner is the participant whose number is closest to two-third of the average of all chosen numbers. The winner will receive 100 points, while other participants will receive nothing. In case of a tie, the 100 points will be equally divided between winners. In this simple guessing game repetition lets students react to other players' choices and converge iteratively to the equilibrium solution.

In *Figure 7*, we can note that the students did not understand the reasoning to be carried out.

Declare an integer between 0 and 100 to guess "2/3 of the average of all answers"

Figure 7. The Beauty Contest result after the first round



Students played the game several times and found that the result approached the theoretical expectation, through iterated elimination of dominated strategies. The result was not convincing students, their beliefs were conditioned by their probability bias.

The Chicken Game

The students in this game were faced with a choice problem requiring some effort, the cost of which fell only on the person making the effort, while the benefit was shared with everyone. For a project to be successful, a particularly painful task must be undertaken by at least one member of a team. Each team is constructed randomly, and the students do not know the other participants in the team. Team members simultaneously choose whether or not to undertake the task. If players behave according to Nash equilibrium in symmetrical mixed strategies of the game, the more players in the team, the less often the project is successful. The game was repeated for seven rounds and two different universe configurations (10 universes and 6 universes) and no player knew when the game would be stopped.

Table 3. Synthetic Results after Round 7

Proportion of Players choosing ***Effort****:*

Repartition/ Round	7	6	5	4	3	2	1	Over the game
First experiment (10 different universes)	*40.91%*	*31.82%*	*36.36%*	*36.36%*	*63,64%*	*63,64%*	*90.91%*	*45.45%*
Second experiment (6 different universe)	*13.64%*	*13.64%*	*36.36%*	*22.73%*	*40.91%*	*31.82%*	*59.09%*	*27.27%*

Proportion of ***successful projects****:*

Repartition/ Round	7	6	5	4	3	2	1	Over the game
First experiment (10 different universes)	*70%*	*50%*	*60%*	*70%*	*100%*	*100%*	*100%*	*68.75%*
Second experiment (6 different universe)	*50%*	*50%*	*100%*	*50%*	*100%*	*83.3%*	*100%*	*66.67%*

The conclusive analysis of the results of the various rounds made it possible to observe how the behavior of the students was consistent with the literature data from which it emerges that if the game is repeated, the number of players who choose "effort" decreases as the rounds increase.

4.5 Voronoi approach for discrete competitive facility location Model

The second part of the educational path developed the Competitive Localization Models which concern the fact that some facilities are already present in the market and that the new facility will compete for market share, in fact the students were sellers who had to open a new business by placing it within the kingdom to optimize their income compatibly with the shops that were already present in the area.

There are two main approaches to estimating and analyzing the market share captured by facilities such as retail establishments, restaurants, etc., as a function of their location.

The first approach was introduced by Hotelling (Hotelling 1929). Hotelling assumed that customers patronize the closest facility. This assumption leads to a partition of the plane by a Voronoi diagram (Okabe, Suzuki 1987), (Okabe, Boots, Sugihara 1992). The second approach was pioneered by Huff (Huff 1964), (Huff 1966), who suggested that the probability that a customer patronizes a facility is proportional to the facility's size and inversely proportional to a power of the distance to the facility.

After describing the two models to the students, it was decided, basing on the skills possessed by the students, to develop the Hotelling model which develops from a duopoly in a linear market (Hotelling 1929), and which to the students was represented by the positioning of two street vendors on a straight road.

Many extensions of the Hotelling model, such as Okabe, Suzuki 1987), assume that customers patronize the nearest facility and consider a class of continuous localization optimization problems that can be solved through the Voronoi diagram (Okabe, Suzuki 1997).

For Drezner (1994) the best position is found by maximizing the market share captured using the expected values for quality and distance.

4.5.1 The Basic Location Model

The Basic Location Model is defined as follows:

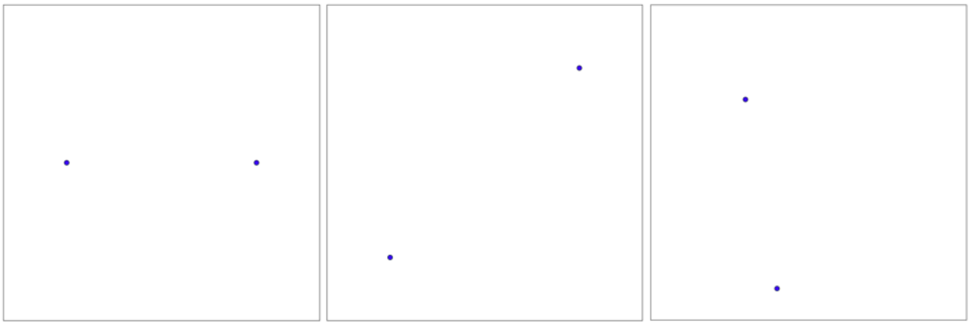
Consumers are distributed according to a measure λ on a compact Borel metric space (S ; λ) with S compact subset of R^2.

A finite set K = {1, ..., k} of retailers have located their facilities on S.

A new retailer wants to maximize her market share after locating a new facility, depending only on the "distance" variable.

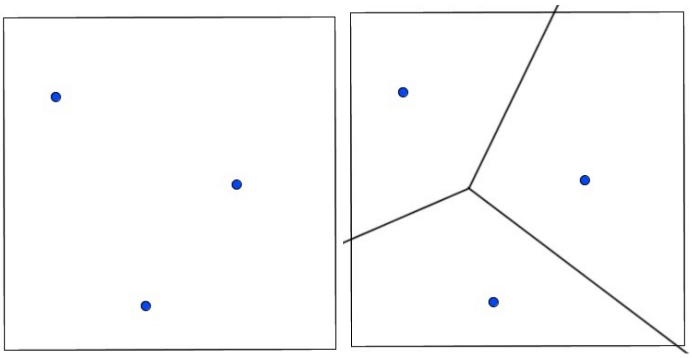
The activity started asking students to divide the space according to the rules mentioned above in the case of two facilities. Using first ruler and squares and then Geogebra software, students were guided in creating each retailer's attraction domains.

The "two points" cases (figures below) were quite easy to understand because students knew the definition of the axis of a segment in geometry and obtained the partition of the space by building the axis of the segment joining the two points that correspond to the two retailers. In the first two examples the evident symmetries of the figures have led to having equal areas and therefore equal payoffs. The third case was easy to build but led the students to observe the possibility that payoffs could change based on the allocation of retailers and this goes against a natural intuition of equal space division.

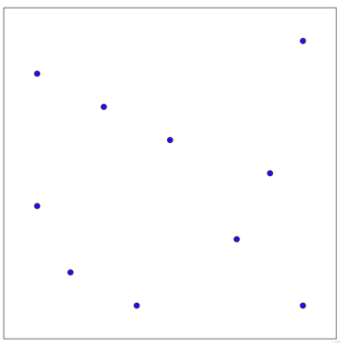


What if there are three facilities?

Even in the case of 3 points the students with lines and squares were able to divide the space again using the axes of the segments obtained by joining two points, but the difficulty in developing the activity was already increasing.



.. and with 10 points how could they proceed?



When the number of points becomes high it is convenient to introduce some geometric properties of the plane and some definition. To trace the dominant regions of the players (i.e. their payoffs) in the intent of solving this locational optimization problem, we have to resort to Competitive Location Models.

4.5.2 Competitive Location Models

Definition

On the space S is defined a tessellation:

Let

$$P_K := \{p_1, ..., p_k\} \subset S$$

be a finite collection of points in S (where retailers have just opened a store),

$$V_K(p_k) := \{y \in S : d(y, p_k) \leq d(y, p_j) \text{ for all } p_j \in P_K\}$$

is the Voronoi tessellation of S induced by P_K.

The cell $V_K(p_k)$ contains all points whose distance from p_k is not larger than the distance from the other points in P_K.

Call

$$V(P_K) := (V_K(p_k))_{k \in K}$$

the set of all Voronoi cells $V_K(p_k)$.

For

$$J \subset L \subset K$$

we have

$$V_J(p_j) \supset V_L(p_j) \text{ for every } j \in J.$$

We interpret $\lambda(V_K(p_k))$ as the mass of consumers who are weakly closer to pk than to any other point in P_K. These consumers will weakly prefer to shop at location pk rather than at other locations in P_K since we assume that all firms offer the same good at the same price.

S is a compact subset of some Euclidean space, that λ is absolutely continuous with respect to the Lebesgue measure on this space and

$$\lambda(V_K(p_j)) > 0 \text{ for all } p_j \in P_K.$$

This assumption implies that the set of consumers that belong to r different Voronoi cells V_K (p_{k1}), ..., V_K (p_{kr}) (i.e. are at the same distance of several points in P_K) is of zero measure.

Definition

Two Voronoi regions, VK (p_k) and VK (p_j), are adjacent if they share an edge, and pk is a neighbour of pj and viceversa.

The set of indexes of the Voronoi neighbours of pk is denoted by N (k).

Clearly, j ∈ N (k) if and only if k ∈ N (j). The face of ij is

$$\Delta_{ij} = V_K (p_i) \cap V_K (p_j).$$

Definition

The dominance region of p_k over p_j associated to the family (measure) λ is define as:

$$Dom(p_k , p_j) = \{x \in S \,|\lambda_k (x, p_k) \leq \lambda_j (x, p_j)\}$$

and the separator between pk and pj is

$$sep(p_k , p_j) = \{x \in S \,|\lambda_k (x, p_k) = \lambda_j (x, p_j)\}.$$

It is easy to verify that:

$$sep(p_k , p_j) = Dom(p_k , p_j) \cap Dom(p_j , p_k),$$

$$V_K (p_k) =\cap_{j=1,n} Dom(p_k , p_j),$$

$$S=\cup_{k=1,n} V_K (p_k).$$

On S compact subset of R^2 we define the Euclidean distance function ‖ · ‖. The dominance region is called the "ordinary Voronoi polygon" associated with p_k, and the partition V_K (P_K) is the planar ordinary Voronoi diagram generated by P_K . The edges of Voronoi polygons in R^2 are line segments.

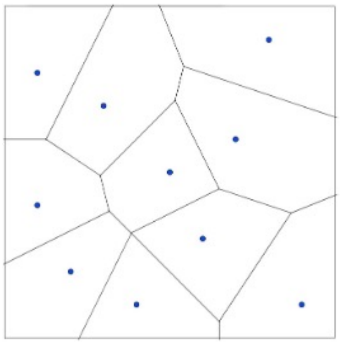
For each firm, action will be identified with a position (a point) in the plane.

The payoff of player i is

$$f_i(a)=\sum_{J\in K}\lambda(V_J(a_i))$$

That is the measure of the consumers that are closer to the location that they choose than to any other location chosen by any other player

Figure: the payoff of players (Voronoy tessellation)



Under these assumptions, the location of the new facility is determined by the maximization of the distance from other existing facilities: i.e., the firm i decides to locate the new store in the farthest point respect to all existing stores. The task of determining this location is the largest empty circle problem.

4.5.3 Delaunay triangulation of the convex hull of P_K

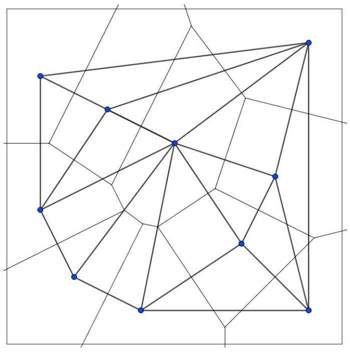
Definition

The Delaunay Triangulation, DT (P_K), for a set of points P_K is a triangulation such that no point in P_K is inside the circumcircle of any triangle in DT (P_K). The edges of DT (S) are called Delaunay edges.

Lemma 1

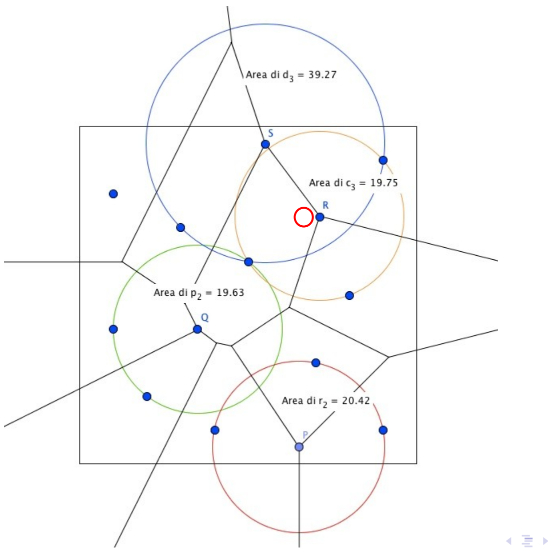
Any two points of S are joined by a Voronoi edge if and only if their Voronoi regions are edge-adjacent.

Figure: The Delaunay triangulation



Students, thanks to the software, drew the Delaunay triangulation and then they built the circumferences circumscribed to the triangles and they determined the circle of maximum area.

Figure: the circumferences circumscribed to the Delaunay's triangles



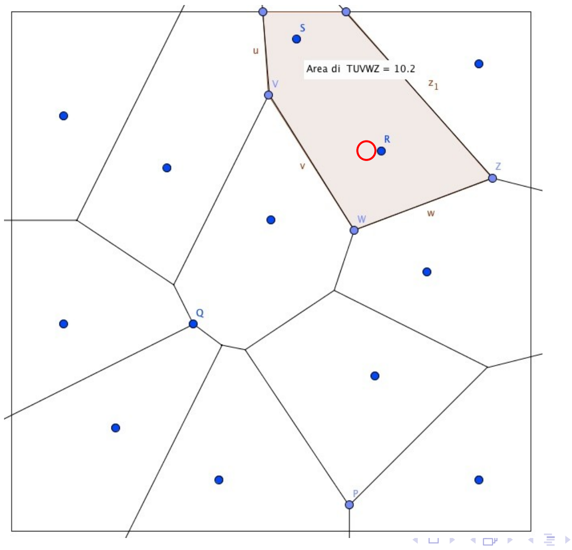
They verified that if a circle had the greatest area but part of it was external to the plane S (object of the search for the point of maximum gain), the area included in S was in any case smaller than the largest empty circle entirely contained in the space S.

The largest empty circle has these characteristics:

- its center is in the interior of S,
- the circle contains no points from P_K inside it,

- it is the largest in the sense that there is no other circle with strictly wider area contained in S,
- it is fully included in S.

Figure: the new configuration, under Voronoi solution, with the location of the new retailer with the maximum area



Students used the Voronoi diagram of P_K to solve this problem and find the location of the new facility adding to the points already known also R, the center of the widest circle.
It can be shown that if the center of a largest empty circle is strictly inside the convex hull of P_K, then the center coincides with a Voronoi vertex. However, not every Voronoi vertex is a good candidate.

Given the initial configuration P_K, the equilibrium of the spatial competition is a new configuration, under Voronoi solution:

Definition

The configuration $P^{*}=1,\ldots,k+1=P_k \cup p_{(k+1)}$ is a global equilibrium if for all $\bar{P}$,

$$H\ (P^{*},\ V(P^{*})) \geq H\ (\bar{P},\ V(\bar{P}))$$

Where *H* is the function "area" of the P* point that generates the V*(P*)* Voronoi tessellation.

Under compactness of S, the global equilibrium P^* exists, but may not be unique.
If the new retailer is interested in minimizing the "interference" of her facility, he will decide to locate it as far as possible from the other ones, i.e. he is practicing an isolation game.

After completing the geometric construction and solving the search for where to allocate the new facility as a function of distance only, it has been explained to students that the Voronoi problem can be approximated by an isolation game with a measure of distance and that the solutions of the isolation game correspond to the Nash equilibrium points obtained by solving the maximum distance problem analytically in multi-variable functions, a topic that would not have been possible to deal with in a high school since there are no mathematical prerequisites necessary for students. The students were then defined the configuration of the isolation game model and described that players' strategy set in the isolation game is the entire S.

Definition

A configuration of the isolation game is a vector

$$P_{(K+1)} = 1,\dots,k+1 = P_K \cup p_{(K+1)}$$

where $p_i \in S$ specifies the position of player i.

The utility function of player *i* is a (k − 1)-place function

$$f_i\,(\dots, \|p_i\,, p_{i-1}\|, \|p_i\,, p_{i+1}\|, \dots)$$

Definition

A best response of a player i in a configuration Pk is a new position $p_i/{=}p_j$ for all $j \in P_k$, that maximizes the utility of agent i while agents j remain in position j/=i. the new configuration $P_{(k+1)}$ is the result of Best Response Move of player i in configuration P_k (Zhao, Chen, Teng 2008).

Definition

A pure Nash equilibrium of an isolation game is a configuration in which no player has any better response in the configuration.

Hence, the Nash equilibrium exists but may not be unique in the finite set S.

4.6 Conclusion

Students' daily life is permeated with activities carried out thanks and through digital technologies, it would therefore be lacking to exclude them from the didactic paths and to think that playing is an activity that does not concern learning. Introducing gamification into teaching means a radical transformation of teaching-learning methodologies thanks to the implementation with a new language that is closer to students' world and to the way of interacting with the contemporary reality.

These reflections were confirmed by the analysis of the students' behaviors and reflections. In fact, at the end of the activities during the Game Theory Lab, students had a better attitude to the critical analysis of the economic phenomena with a widespread interest in issues they had never encountered before. They unhinged a number of beliefs held by the exact sciences which, when applied to the social sciences, lead to results that are not easily motivated. Decision Theory and Competitive Location Models confronted students with several questions: What should a good decision maker do? How can he choose an alternative, taking into account all those involved? Why doesn't the winner always exist?

In particular, students had to 'accept' that there can be no single solution to a problem and that sometimes the solution does not exist. They were also very intrigued by the theoretical assumptions and the admissible results, animating the lessons with extensive discussions on the subject.

The use of wireless technology, with which the students are very familiar, allowed the students to actively participate in the proposed games. In fact, they were able to answer multiple-choice quizzes and were able to communicate directly with the administrators of the games.

In this way, the proposed activities involved all the students, without any specific background, and allowed each of them to use their previous knowledge and unhinge their beliefs where necessary.

CHAPTER 5

MUONS' DETECTION

5.1 Introduction

The last research activity that is described in this PhD thesis, runs through the interdisciplinary mathematics activities with physics and sciences, which fits into my activities of the DipMat Research Group and develops with and for INFN, to which I have been associated for some years. In particular, my profile as a teacher of mathematics and physics in high schools and as a current PhD student in didactic research of mathematics with new technologies at DipMat UNISA, is perfectly inserted in the elaboration of the paths developed by INFN in the OCRA project (Outreach Cosmic Rays Activities) and led to the inclusion of a research path on astroparticles in the mathematics and physics modules of the Project "Mathematical High School". The modules have been developed by the INFN researchers and coordinated and monitored from an educational and didactic point of view by DipMat UNISA in collaboration with INFN Naples to observe and evaluate the students' involvement and the didactic impact of the activity (Aramo, Tortoriello, Veronesi 2019), (Aramo, Colalillo, Tortoriello, Veronesi, 2020), (Aramo, Colalillo, Tortoriello, Veronesi, 2021), (Veronesi, Aramo, Colalillo, Tortoriello 2021).

Within the INFN Section of Naples, I have been involved in the numerous outreach and didactic research activities planned with/for schools. In particular, the most relevant tasks in terms of scientific dissemination in which I participated are:

- the OCRA Outreach Cosmic Race Activities project a great "collector" of all the dissemination activities organized by INFN at national level,
    - the International Cosmic Day (ICD), an international day dedicated to research activities on cosmic rays,
    - the Science Camp, an internship for high school students at the National Laboratories of Gran Sasso to experiment data collection on muons

  - the collaboration for the preparation of online activities on the OCRA platform
  - speaker in the professional refresher course for teachers in service
- laboratory activities and analysis of educational data in research path on astroparticles in the mathematics and physics modules of the Project "Mathematical High School".

The paragraphs 5.3 and 5.4 summarize the peculiarities of the aforementioned events.

In the OCRA project, my role is to focus attention on the world of teachers by providing methodological-didactic indications and mathematical insights also with interventions in the professional refresher meetings for teachers organized on the Sofia platform. Similarly to what I did during the past editions of ICD and during the Science Camp held in 2019 at the Gran Sasso Laboratories, in which I collaborated with a group of INFN researchers at national level in path proposed to 32 high school students from all over Italy in which were carried out the measurements of counting the number of muons detected by the CRC cube, a muon detector and counter, in a defined time at various altitudes: at the top of the Gran Sasso mountain, at the arrival of the ski slopes, in the middle of the mountain, at the foot of the Gran Sasso (at the University of L'Aquila) and inside the LNGS laboratories inside the mountain. Since these data are confidential as subject of an article to be published in the future by the INFN, it is not possible to use such data in this thesis. However, to highlight the mathematical effectiveness of cosmic ray analysis with the CRC cube, it will be presented as a model the activity proposed in the three-year classes of the MHS schools.

The description of the activities will be concluded with the description of a laboratory designed with graphical calculators to deal with the statistical analysis of the data deriving from the experiments of the study of cosmic rays with CRC.

5.2 Background

In the National Indications of the Ministry of Education, there are numerous requests and references to an interdisciplinary development of mathematics with the scientific field in the broadest sense. Below is an excerpt from the "General lines and skills" that determined the choices to dedicate courses to a theme apparently so niche as it is specific and sectoral which, however, brings a lot of interdisciplinary content described with topics regularly dealt with in school curricula.

National Indications for MATHEMATICS

"At the end of the scientific high school, the student will know the basic concepts and methods of mathematics, both internal to the discipline in itself, and relevant for the description and prediction of phenomena, in particular of the physical world.

The student will have acquired a historical-critical view of the relationships between the main themes of mathematical thought and the philosophical, scientific and technological context.

...

Hence the groups of concepts and methods that will be the objective of the study:

...

3) the basic mathematical tools for the study of physical phenomena, ...;

4) elementary knowledge of some developments in modern mathematics, in particular of the elements of probability and statistical analysis;

5) the concept of mathematical model and a clear idea of the difference between the vision of mathematization characteristic of classical physics (unique correspondence between mathematics and nature) and that of modeling (possibility of representing the same class of phenomena through different approaches);

6) construction and analysis of simple mathematical models of classes of phenomena, also using IT tools for description and calculation;

...

This articulation of themes and approaches will form the basis for establishing conceptual and methodical links and comparisons with other disciplines such as physics, natural and social sciences, philosophy and history.

...

The computer tools available today offer suitable contexts for representing and manipulating mathematical objects. The teaching of mathematics offers numerous opportunities to become familiar with these tools and to understand their methodological value. The path, when this proves appropriate, will favor the use of these tools, also in view of their use for data processing in other scientific disciplines."

National Indications for PHYSICS

"At the end of the high school course the student will have learned the fundamental concepts of physics, the laws and theories that make them explicit, acquiring awareness of the cognitive value of the discipline and of the link between the development of physical knowledge and the historical and philosophical context in which it occurs. is developed.

In particular, the student will have acquired the following skills: observe and identify phenomena; formulate explanatory hypotheses using models, analogies and laws; formalize a physics problem and apply the mathematical and disciplinary tools relevant for its resolution; gain experience and explain the meaning of the various aspects of the experimental method, where the experiment is intended as a reasoned interrogation of natural phenomena, choice of significant variables, collection and critical analysis of data and the reliability of a process of measurement, construction and / or model validation; understand and evaluate the scientific and technological choices that affect the society in which they live.

The freedom, competence and sensitivity of the teacher - who will evaluate from time to time the most suitable educational path for each class - will play a fundamental role in finding a connection with other teachings (in particular with those of mathematics, science, history and philosophy) and in promoting collaborations between his school and university, research institutions, science museums and the world of work, especially for the benefit of students of the last two years.

...

The experimental dimension can be further investigated with activities to be carried out not only in the school's didactic laboratory, but also in university laboratories and research institutions, also adhering to orientation projects. In this context, the student will be able to explore topics of interest to him, approaching the most recent discoveries in physics (for example in the field of astrophysics and cosmology, or in the field of particle physics)."

National Indications for NATURAL SCIENCE

"At the end of the high school course the student possesses the disciplinary knowledge and methodologies typical of the natural sciences, in particular of the Earth sciences, chemistry and biology. These different subject areas are characterized by their own concepts and

methods of investigation, but they are all based on the same strategy of scientific investigation which also refers to the dimension of "observation and experimentation".
The acquisition of this method, according to the particular declinations it has in the various fields, together with the possession of the fundamental disciplinary contents, constitutes the training and orientation aspect of the learning / teaching of science. This is the specific contribution that scientific knowledge can make to the acquisition of "cultural and methodological tools for an in-depth understanding of reality".
The student acquires critical awareness of the relationships between the development of knowledge within the subject areas of study and the historical, philosophical and technological context, as well as the mutual links and with the scientific field more generally.
In this process, the experimental dimension is of fundamental importance, a constitutive dimension of these disciplines and as such to always be kept in mind.
The laboratory is one of the most significant moments in which it is expressed, as a privileged circumstance of "doing science" through the organization and execution of experimental activities, which can still usefully take place in the classroom or in the field.
This dimension remains an indispensable aspect of scientific training and a guide for the entire training course, even when laboratory activities in the strict sense are not possible, for example through the presentation, discussion and processing of experimental data, the use of films, simulations, virtual models and experiments, the presentation - also through original pieces by scientists - of crucial experiments in the development of scientific knowledge. The experiment is in fact an indispensable moment in scientific training and must therefore be promoted in all years of study and in all disciplinary areas, because it educates the student to ask questions, to collect data and to interpret them, gradually acquiring the typical attitudes of scientific investigation.
... Insights of a disciplinary and multidisciplinary, scientific and technological nature, will also have an orientation value to the continuation of studies. In this context it is desirable to involve above all the students of the last two years, to establish a connection with the teachings of physics, mathematics, history and philosophy, and to activate, where possible, collaborations with universities, research institutions, science museums and the world of work.”

5.3 The OCRA Project

This paragraph briefly describes the salient data of the main INFN activities in which I participate and collaborate and on which the interdisciplinary path of mathematics and physics analyzed in this thesis has developed.

The OCRA (Outreach Cosmic Ray Activities) project was born in 2018 within the National Institute of Nuclear Physics INFN with the aim of gathering the many public engagement activities already developed in the local INFN divisions to disseminate them to national and international level in the field of astroparticle physics (Hemmer et al. for OCRA Collaboration 2021).

In this macro container of didactic paths of experimental laboratories and research activities prepared by the INFN researchers, it has been decided to give ample space to the preparation of methodological and pedagogical tools so that teachers can make the best use of the materials available for an interdisciplinary didactic impact in a global learning vision (Veronesi for OCRA Collaboration 2022).

Among the various activities within the project, a series of online workshops was developed. The various paths developed lead to the discovery of cosmic rays through the analysis of data from real scientific experiments. It is therefore possible for anyone to reproduce the analyzes conducted by the researchers with commonly used software.

The focus of the OCRA project is the scientific dissemination understood in terms of didactic fall-out of the proposed topics, in fact, particular attention was paid to the materials for teachers, focusing on the regulatory references of the Ministry of Education which solicits collaboration between the various areas of knowledge, school , universities and research, in order to enhance operational skills and deepen the characteristic procedures of scientific thought also thanks to the IT tools currently available. The desire to experimentally develop the activities aims to overcome the boundaries of the various disciplinary areas and to support students in the choice of future university paths.

The project presents technical sheets of all the courses so that the teachers can replicate them in full or redefine them according to the classes in which they are proposed. There is also a rich collection of bibliographical references to deepen the themes present in each activity. All the project sheets provide information and didactic ideas but a specific section is dedicated to the analysis of learning dynamics and teaching methodologies, to the importance of the use of tools in laboratory activities for the construction of new knowledge developed and through the tools used in the various activities and thanks to the interactions that develop in teaching among peers and between teacher and students.

The methodological approach used to elaborate the paths is constructivist, in which the construction of knowledge is determined by the interaction with other individuals in a reality that acquires meaning thanks to the active participation of the observer (Vygotsky 1987). The levels of involvement and autonomy achieved by students in the activities that develop thanks to the use of technologies are decidedly higher than those deriving from traditional teaching as they favor the development of metacognitive activities.

The physical and cultural artifacts, such as the CRC cube with its detectors, the hardware of the instruments and the software for data analysis, or even the physical and mathematical models, are semiotic mediators of knowledge as they convey the teaching-learning processes guided by researchers and teachers that play the role of cultural mediators.

In this context, constructivism has offered multiple observations on user-centered learning processes and the action of the teacher is crucial in every phase of the teaching cycle including the semiotic activities with the artifact for the performance of certain tasks to produce common solutions, the production by students of reports with the individual production of signs and the following collective discussion in the classroom orchestrated by the teacher.

Since its foundation, OCRA has made an effective contribution to the curricular inclusion of the theme of cosmic rays in outreach and education activities. To meet the specific educational needs of teachers and students, numerous online laboratories developed during the first spring lockdown of 2020 were also designed and presented to which the participants showed interest and enthusiasm.

In the spring of 2021, OCRA proposed a refresher course for teachers in the scientific field, focused precisely on OCRA activities. The over 70 teachers who participated then filled out a questionnaire to evaluate the teaching effectiveness of the proposals and the impact on teaching activities. From the analysis of the answers emerged the involvement in the proposed activities, the desire to reproduce them in the classroom and the very positive evaluation of the project.

5.3.1 International Cosmic Day (ICD)

The International Cosmic Day (ICD) is an astroparticle physics outreach event, it is dedicated to high schools and brings together various physical outreach projects from around the world. In November of each year, for one day, groups of researchers, scientists, teachers and students meet to learn about cosmic rays and perform an experiment with atmospheric muons (Aramo, Hemmer, for the OCRA Collaboration 2019).

In the period before the Covid, the students took part in the various INFN Sections in Italy, so the data collection activities and the analysis took place in presence with great participation of schools, while the training seminars of the ICD day were developed online with various connections between the various Sections. From 2020, due to the Covid lock-down, it was necessary to change the way the ICD was carried out by transforming it into an online activity. In 2021, in particular, it was decided to build a mixed project and in addition to all the schools connected online, small groups of students participated in presence in the various INFN Sections.

The heart of the ICD has always been the practical experience, the desire to make students live the research and analysis of data in first person, whether the day took place in presence or remotely. The muon detector Cosmic Ray Cube (CRC) made it possible to carry out the measurement campaign with the taking of data, that were shared with all the participants in both ways.

In particular, all the participating groups investigated the distribution of the zenith angle of atmospheric muons and tried to answer the following questions: is the number of air rain particles coming from the horizon the same as the zenith? What is the angular distribution of the cosmic flux of muons? Students experienced teamwork like a true international collaboration, discussing their results in joint video conferences at the end of the day. Finally, they were invited to publish the results of their measurements in the proceedings. The data collection session was intertwined with short presentations by INFN researchers, with a virtual visit to the LNGS underground laboratory and with the analysis and discussion of the data. During the event, participants were invited to ask questions in chats and the researchers responded in real time.

To make participation even more active and engaging, INFN choose to reward the students who have distinguished during the ICD activities. In 2019 the award was a Science Camp at the Gran Sasso Laboratories, described in the next paragraph. For 2020 the Science Camp was planned at the Frascati Laboratories but the advent of Covid prevented it from taking place and students were honored with informative material on cosmic rays. For the winners of the ICD 2021 the Science Camp in Frascati is scheduled in May 2022, Covid permitting.

Two short questionnaires, one for teachers and one for students, made it possible to evaluate the satisfaction of the participants in the OCRA ICD event. It turned out that many students expressed their satisfaction at having been able to make the measurements themselves, others appreciated the theoretical introduction on cosmic rays or the virtual visit to the LNGS laboratory. Teachers' questionnaires showed similar results to those of students, with overall satisfaction even higher. The constructivist approach with which the research activities have been structured has proven to be effective in terms of contextualized learning, the students working in a peer-to-peer education teaching have experimented and "touched" scientific concepts of extreme topicality.

5.3.2 OCRA “Science Camp” at Gran Sasso

In 2018, 14 of the INFN Sections currently adhering to the OCRA participated in the ICD and the 800 students were invited to participate in a competition for the OCRA Science Camp at Gran Sasso. For each OCRA ICD 2018 Section, two students were selected from all those who presented an article written in pairs about the event and their experience. All the winning teams participated to the OCRA Science Camp at Gran Sasso in Italy accompanied by INFN researchers and five professors from all over Italy, including myself, who supported them in the research activities during the camp.

The science camp took place from April 14 to 17, 2019. The local organizers were the Gran Sasso Science Institute (GSSI) based in L'Aquila and the Gran Sasso National Laboratories (LNGS), both members of the OCRA Collaboration. The activities involved introductory seminars, a series of measurements in the field, data analysis by students under the guidance of the researchers, and a visit to LNGS underground laboratories. The measurements aimed to determine the dependence of the cosmic ray flux on altitude by means of a series of measurements made in different places around the Gran Sasso massif in central Italy. The acquisition of all data was carried out with the Cosmic Ray Cube (CRC). As mentioned at the beginning of the chapter, a dedicated article is being prepared on the scientific results and the perception of students that have been evaluated by ante and post activity questionnaires.

5.3.3 Online activities on the OCRA platform

On the platform a space is dedicated to collect the laboratories of educational experiments where it is possible to analyze data from real experiments, both informative and dedicated to research. Each activity is described qualitatively and carried out on worksheets with the details of the analysis indicated step by step. Links are provided to explanatory videos, reconstructions and simulations, supplementary information documents, spreadsheets. In some laboratories, the Python language is used to analyze the data made available by the official websites of the collaborations (such as the Pierre Auger Observatory) and compare the results with the official ones.

Each online activity describes an experimental measure and, through data analysis, describes the measurement of:

- cosmic muon rate and track reconstruction
- muon angular distribution
- muon intensity as a function of Zenith angle
- muons as a function of height in the atmosphere
- muons as a function of depth in water

In the space of the laboratories in the platform, an in-depth study has been developed dedicated to the analysis of learning dynamics and teaching methodologies centered on the importance of the use of tools in laboratory activities. The target is to provide teachers with educational and methodological tools suitable for producing effective teaching in the development of the issues addressed with a constructivist approach. A container of information-instructions has also been set up to manage the structuring and development of educational paths from an organizational point of view to facilitate the activities that teachers must carry out in order to introduce projects into the school curriculum. In this macro container of educational contents about experimental laboratories and research activities it has been decided to give ample space to the elaboration of methodological and pedagogical tools so that teachers can make the best use of the available materials for an interdisciplinary didactic impact in a global learning vision. I developed this section of the website.

5.3.4 Professional refresher course for in-service teachers

With the aim of providing teachers with the most effective tools to use the materials available on the OCRA platform, in spring 2021 a professional refresher course was presented on the SOFIA platform dedicated to in-service science, mathematics and physics high school teachers. The lessons aimed to provide teachers with the cultural and methodological tools to bring the many activities present on the OCRA website into the classroom: through the "cosmic rays path" section they found out what cosmic rays are, where they come from and much more, also through the analysis of data real science experiments. The course was developed in 9 meetings with the participation of 71 teachers. I was the designer and speaker of the meeting dedicated to the presentation of teaching methodologies and the interdisciplinary interpretation of the proposed activities.

5.4 Activities with the CRC Cube in the MHS Schools

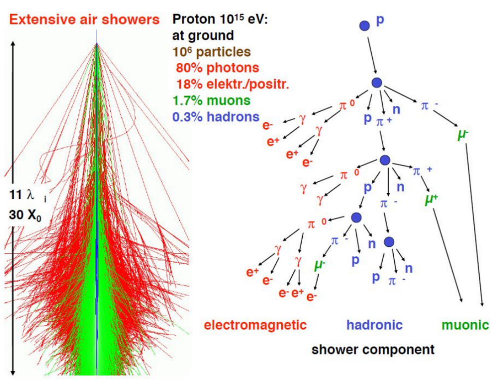
The experimental activities of data analysis on muons obtained with the CRC, were carried out in the physics modules of the third grades of the MHS schools. The students understood how the scientific world is closely interconnected and how indispensable mathematics is in the world of scientific research. In fact, they interacted with various mathematical fields such as Cartesian geometry, the theory of errors, linear regression, the adaptation of trajectories, the change of the numbering base. But above all, while they were carrying out the analysis of the data, they realized that the contents of the various fields were recalled, unlike how it was used to study mathematics by tackling problems independently of each other.

The teaching activities began in presence and the researchers brought the Cosmic Ray Cube (CRC) muon detector available thanks to the Gran Sasso National Laboratories and the OCRA-INFN Collaboration. The students observed and studied how a particle detector is built and understood its geometry, mathematical and physical significance of the collected data. Unfortunately, the COVID-19 pandemic necessarily led to the transformation of the path into activity on online platforms. Therefore, the students used the open-source app already prepared and available for data analysis. The entire laboratory activity was monitored both from the scientific point of view, because the data collected have experimental value, and from the didactic point of view.

5.4.1 A brief introduction to cosmic rays

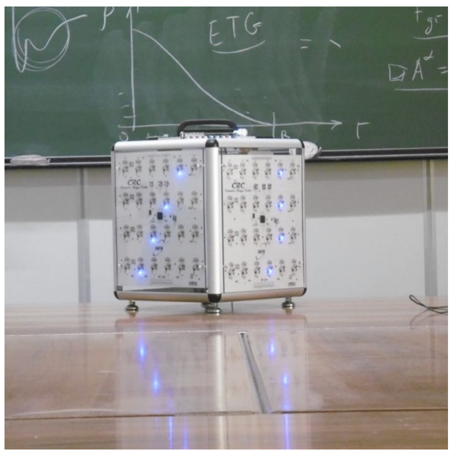
Cosmic rays are particles produced by energetic phenomena in the Universe. They continually hit the Earth. At the ground level, we do not directly detect the primary particles produced in the Universe, but the particle shower, namely Extensive Air Shower (EAS), produced by their interaction with atmospheric molecules. An EAS is shown in figure and is composed by three main parts: the hadronic, the electromagnetic, and the muonic component.

The particles which reach us are mainly muons, which can go through several layers of material without being absorbed. Their flux near the sea level is about 100 Hz/m^2.



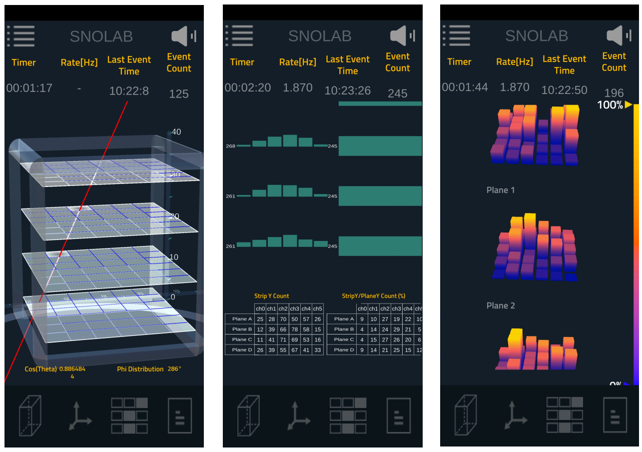
The first aim of this work is to measure the muon flux at different zenith angle, the angle formed by the vertical and the particle trajectory.

The muon component, in green in figure, is very narrow around the shower axis, the direction identified by the particle that started the shower. Considering the educational purpose of the measurements it can be assumed, as a first approximation, that the direction indicated by the trajectory of a shower muon is analogous to that indicated by the shower axis.



The experimental setup consists of a cosmic ray detector, the CRC, a counter and a tracker to study cosmic muons, whose measurements are recorded and analyzed by specific applications on PC that are also available through a mobile app.

The CRC is composed of 4 layers of plastic scintillator and each layer is divided in two sub-layers to ensure a two-dimensional reading of the particle trajectory. The light produced by the scintillator, when a charged particle passes through it, is collected by a Silicon Photomultiplier (SiPM), which transforms the light into an electrical impulse that allows the lighting of a LED connected to it.



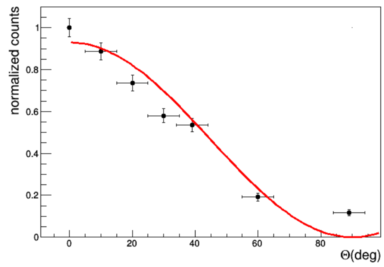
On the two lateral faces X-Z and Y-Z of the cube, lit LEDs show the projection of the particle trajectory.

To study the muon trace, students read on the telescope’s display the number of particles that passed through the four layers of the telescope in five minutes repeating the procedure by tilting the telescope at different angles with steps of about 10 degrees. The position of lit LEDs is the starting point for the reconstruction of the trajectories of the particles in the X-Z and Y-Z planes, using the least mean square method.

In this way the students follow the experiment from start to finish and can apply the geometric notions of the Cartesian plane and approach the geometry of space thanks to the apps. through real data they deepen the statistics and the concept of fit. they tackle the experimental problems in the data collection and analysis phases, understanding the meaning of measurement uncertainty. Furthermore, they understand how a physical phenomenon can be interpreted through the experimental data provided by a detection tool.

5.4.2 The muon trace

The expected muon flux at the ground level as a function of the zenith angle is described by the function $I(\theta)=I_0*\cos^2(\theta)$.



To verify this, students counted the number of particles that passed through the telescope and turned on at least one LED per layer in five minutes. They repeated the procedure by tilting the telescope at different angles, with steps of about 10 degrees. Data are normalized at the counts, the black dots represent the experimental points with their uncertainties, the red line, the fit of our points with the theoretical function) shows that our data follow the $cos^2(\theta)$ shape.

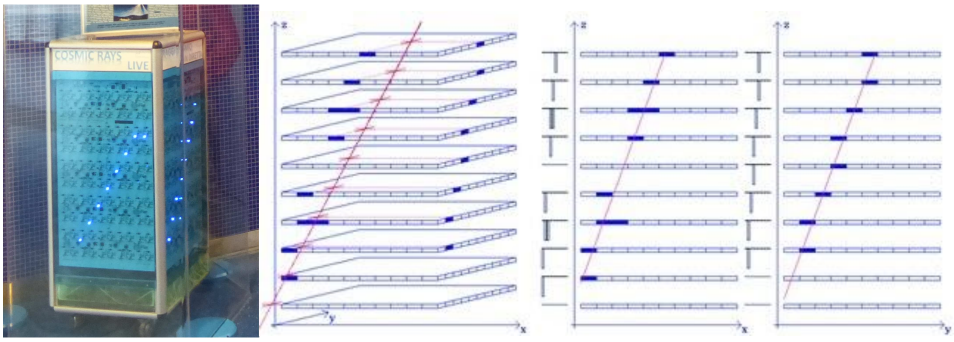
The uncertainty on the counts is evaluated considering a Poissonian distribution of the events, while the error on angles is equal to five degrees (because the support used to tilt the telescope does not have great accuracy. Data perfectly match the theoretical expectation.

The muon flux decreases as the zenith angle increases because the particles with inclined trajectories cross a greater thickness of the atmosphere in which they are absorbed.

5.4.3 The reconstruction of the trajectories

The data collected by a telescope located at the "Toledo" metro station in Naples have been used for the reconstruction of the particle trajectory in two- and three-dimensions. The Toledo telescope has the same characteristics of the CRC, but it has 10 planes and more layers and has an automatic reconstruction of the data that students could compare with their results.

The telescope is made up of a stack of ten detector planes built each with two planes of crossed scintillator strips that allow it to detect the passage of muons and to identify the point where they pass for each X-Y plane. By arranging the various detector planes in a stack, we obtain an apparatus capable of tracing and detecting the passage of an ionizing particle with light signals on the various planes, allowing us to reconstruct the direction of origin in a three-dimensional system.



As can be seen from the image of the telescope and from its schematization, it was absolutely natural to face the concept of Cartesian space in mathematics and the problem of choosing reference systems particularly favorable to analysis.

To memorize the coordinates of the points detected by the telescope in order to re-elaborate the data of the traces of the passage of muons, the INFN researchers used the hexadecimal and binary numbers, so that a new field of investigation was opened to the students, that of algebras with bases different from 10. The electrical signals generated in the telescope as each particle passes are coded in binary logic: 0 indicates an unaffected detector (led off) and 1 a detector passed through (led on). The sequences of 0 and 1 are then translated into hexadecimal code and stored in two strings that contain information on which x and y detectors are activated in each plane of the telescope. Upon the arrival of each event, the internal computer of the totem decodes the sequence of hexadecimal characters and reconstructs the event by showing the x and y projections and its origin from outer space.

And finally, the regression line is drawn in the graphs both on the X-Z plane and on the Y-Z plane.

The study of the issues related to this first mathematical phase of the project lasted about two months, during which we dealt with the theoretical level: the study of Cartesian space, points, lines and planes in their algebraic and geometric representation; the binary and hexadecimal numbering systems; the regression line.

Students, starting from the information saved in hexadecimal data, reconstructed the points for each plane, X-Z and Y-Z, obtained the points indicating the trajectory of the particle and tabulated and drew in a Cartesian diagram.

Finally, using the least mean square formulas they reconstructed the trajectory in each plane. The spreadsheet that students created is shown below in the case of the X-Z plane.

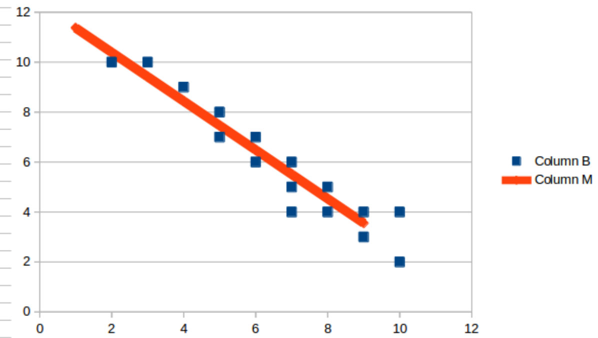
x	z		x-Mx	Z-Mz	(x-Mx)*(z-Mz)	(x-Mx)^2		
10	2		3.375	-3.875	-13.078125	11.390625		
9	3		2.375	-2.875	-6.828125	5.640625		
7	4		0.375	-1.875	-0.703125	0.140625		
8	4		1.375	-1.875	-2.578125	1.890625		
9	4		2.375	-1.875	-4.453125	5.640625		
10	4		3.375	-1.875	-6.328125	11.390625		
7	5		0.375	-0.875	-0.328125	0.140625		
8	5		1.375	-0.875	-1.203125	1.890625		
6	6		-0.625	0.125	-0.078125	0.390625		
7	6		0.375	0.125	0.046875	0.140625		
5	7		-1.625	1.125	-1.828125	2.640625		
6	7		-0.625	1.125	-0.703125	0.390625		
5	8		-1.625	2.125	-3.453125	2.640625		
4	9		-2.625	3.125	-8.203125	6.890625		
2	10		-4.625	4.125	-19.078125	21.390625		
3	10		-3.625	4.125	-14.953125	13.140625		
106	94	**sum**			-83.75	85.75	-0.976676385	**m**
6.625	5.875	**mean**					12.34548105	**q**

z = mx + q

$$m = \frac{\sum_i [x_i - M(x)][z_i - M(z)]}{\sum_i [x_i - M(x)]^2}$$

$$q = M(z) - m * M(x)$$

Where:
M(x) is the mean of x_i value
M(z) is the mean of z_i value



Students appreciated that their reconstruction was compatible with the Toledo telescope reconstruction (Veronesi et al. for OCRA Collaboration 2021) (Veronesi, Aramo, Colalillo, Tortoriello 2021)(Aramo, Colalillo, Tortoriello, Veronesi 2021).

5.4.4 Cognitive tests

During the didactic activity, the students experienced the activity of the researcher. From the point of view of educational research, it was important to monitor the idea that students had of the world of research, correlating it with their future study choices. Without the desire to find confirmation in a positive curve in the progress of the students' choices towards the study of physics, the research aimed to evaluate the impact in terms of acquired skills and research whether the activities carried out helped to choose or not. The students were offered questionnaires divided into two parts: some questions to investigate what was the thought on the figure of the physicist and research, other questions on specific issues concerning cosmic rays. The results of the questionnaire are summarized below with the average of the scores attributed to the students for each item.

Questionnaire:

Enter a score from 1 (disagree) to 5 (strongly agree) for each of the following statements.

Question		Answer before the beginning of the activities (grade average)	Answer after the data collection (grade average)
1	I have some idea of what a physicist does in his research	3,50	3,95
2	I have some idea what the aims of physics research are	3,60	3,95
3	I think the module is useful for my studies.	3,85	3,58
4	I think the module is useful for my future choices.	3,40	2,74
5	I think I will learn a lot of new things.	4,35	4,47

Questions 1 and 2 were intended to explore whether students knew what a physicist is, what his job is, and what the goals of scientific research are. It is observed that the students already had a sufficient idea based on previous laboratory experiences, and that after the laboratory comprehension has increased to a good level.

Questions 3 and 4 concern the prospects for future university choices. In this regard, the summary data is not immediately legible. The scores prior to the laboratory were distributed around the central score. After the laboratory, on the other hand, the average value of the calculated score is the result of polarized data between the students who expressed the adhesion of their future choices to the project (with a high score of 4-5) and those who they highlighted the non-compliance of the activities with the propensities for future study (with a very low score of 1-2). Therefore, the activities have at least partially clarified the ideas to the students.

In question 5, the students, who already had a very positive opinion on the expectations on this course even before starting, strengthened their judgment regardless of the answers given to questions 3 and 4, that is, regardless of their passions and predispositions.

In the second part of the questionnaire open questions were proposed to the students and they were asked for information on the topics of the course. In summary, the questions investigated the knowledge of the topic of the activities and asked what cosmic rays are, what are the components of an extensive air shower, what are muons and how they are measured, how they can reach us, how muography works.

As it was reasonable to imagine, in the questionnaire developed at the beginning of the course, almost all of the students were unable to answer the questions or at least tried to give fairly generic answers with the exception of two students who already knew the thematic of astroparticles. In the second questionnaire completed after the activities, all the students were able to give the answers to the questions, demonstrating that they understood the contents addressed.

The answers given by the students confirmed the expectations of the research group, highlighting, on the one hand, the acquisition of new mathematical skills applied to a physical research context and, on the other, the effectiveness of tutoring for the definition of future study choices.

5.4.5 Analyzing cosmic rays with graphing calculators

Starting from the open-source data that can be downloaded from the Internet on the OCRA (Outreach Cosmic Rays Activities) the frequency of cosmic rays intercepted by the portable telescope CRC (Cosmic Ray Cube) is studied with the use of the graphing calculator, depending on the inclination with respect to the Zenith. It may be observed that it is a Gaussian and its characteristic parameters are analyzed.
The activity satisfies the ministerial indications to propose problems of reality and does not require any theoretical study, teachers in fact they may decide to apply the data analysis as proposed without going further, they may use data as "numbers" without referring to physics theory or, finally, they can frame the activity in a multi-disciplinary or interdisciplinary educational path.

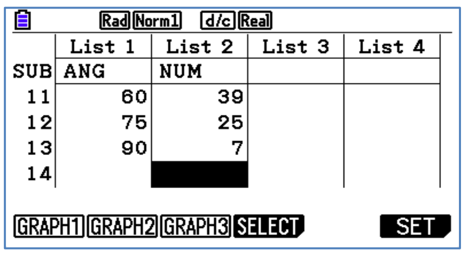
The measures described in the data analysis activity were performed during the INTERNATIONAL COSMIC DAY (ICD) 2020 organized by the INFN in collaboration with other research institutions and are available for outreach activities on the OCRA website (but the activity may be replicated with other data collections related to cosmic ray measurements or can similarly applied in different contexts).

The purpose is to analyze the intensity of the muon flux (secondary cosmic rays) that hit the CRC as function of the angle with the zenith. To study the muon flux, students read on the telescope's display the number of particles that passed through the four layers of the telescope in five minutes repeating the procedure by tilting the telescope at different angles with steps of about 15 degrees. The measurement of the ICD data is listed on the side.
The data show that the flow is maximum when the particles arrive perpendicular to the earth's surface and continuously decreases as the angle increases. Among the effects responsible for this behavior, there is the variation of the length of the path traveled by cosmic particles through the atmosphere.

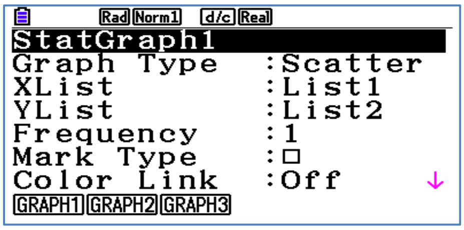
angle (degrees)	particles in 100 seconds
-90	11
-75	15
-60	21
-45	43
-30	44
-15	78
0	77
15	67
30	52
45	36
60	39
75	25
90	7

Finally, the characteristic parameters of the distribution that best represents the collected data, the Gaussian distribution, are studied and the reference parameters for data analysis are obtained from the graphical calculator.

The angular values ANG are entered in the first column (List1) by pressing EXE after each value. In the second column (List2) the values of the number of counted muons are entered, pressing EXE after each value.

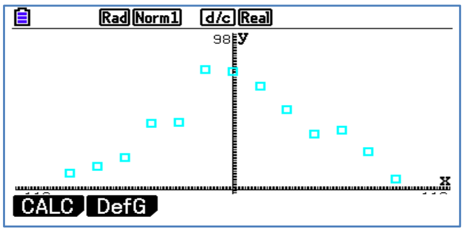


it is possible to move between the boxes and columns for any corrections with the cursor key.



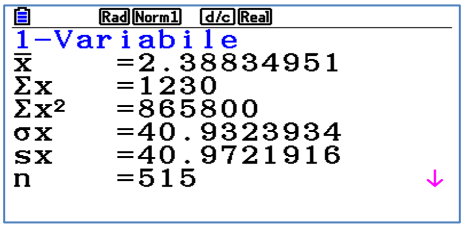
The commands GRAPH and then GRAPH1 build the scatter plot of the distribution.

It is advisable to let students experience how the graph changes as the choices made vary in order to introduce them to experimentation and research with the graphing calculator.

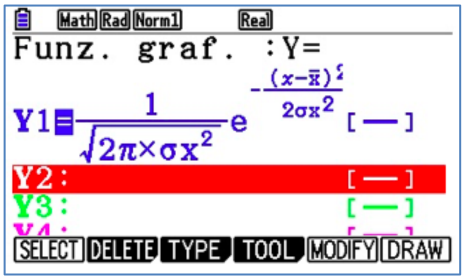


the data and frequencies stored in the lists are assigned to the variables to study the distribution and to obtain the statistical parameters

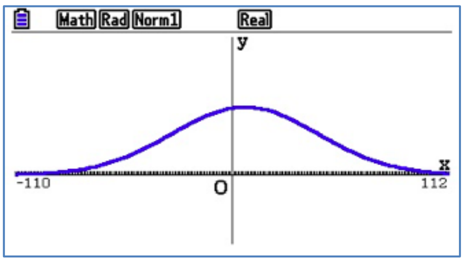
(mean, mode, median, gap, variance ... etc)



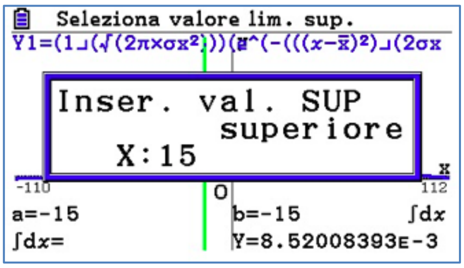
To study the probability distribution (the graph unequivocally indicates the bell curve), by entering the equation of the Gaussian function, the calculator will automatically insert the average value and variance calculated in the previous procedure.



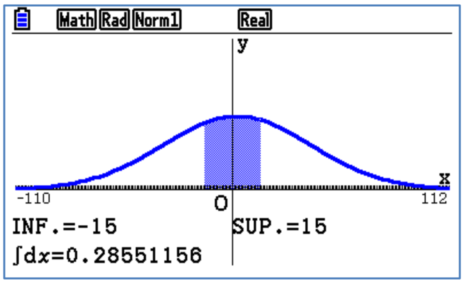
We can therefore observe the graph of the Gaussian probability distribution.



It is also possible to calculate the probability of having data between two values (for example between -15 ° and 15 °) with the calculation of the integral, by appropriately choosing the extremes of integration.



The area under the curve and the relative measure that corresponds to the probability that a ray will affect with an angle of incidence between -15 ° and + 15 ° is highlighted in blue.
In this example it is 0.2855 ... that is about 28.6% probability.



CHAPTER 6

CONCLUSIONS AND OBSERVATIONS

The research work presented in the present PhD Thesis intended to highlight how new technologies in teaching-learning mathematical processes can be effective semiotic mediation tools to contribute to the development of transversal skills in the scientific field and the design of dedicated paths can be the key to building interdisciplinary knowledge.

In my research, various spaces for didactic experimentation were presented even if, due to the long period of distance learning due to the Covid-19 pandemic, the analysis of the data has suffered a sharp slowdown because all the activities in presence with the related observations and recordings have failed.

Experimental research always has limits to be taken into account in order to correctly interpret the results and to be able to plan future interventions. In my research the great limit has been the impossibility of concretely and physically living teaching in the classroom.

The activities designed and elaborated to be carried out in presence were therefore reworked and calibrated in order to be administered online and the interviews and observations in the class were replaced with questionnaires proposed online.

But if on the one hand what happened can be seen as a limit, on the other it can be interpreted as a stimulus to embark on new research paths, using new methods thanks to technologies. In fact, creating new research ideas that were not preventable ex-ante.

The results made it possible to observe that the use of technological instruments within the didactic action favors on the one hand, that of the teachers, the design of innovative activities, on the other, that of the students, a positive attitude towards learning: direct involvement, as protagonists of the proposaed activities, makes them active builders of their own learning.

9.1 Possible future developments

The results emerged from the analysis of the partial data of the experimentation with the graphing calculators stimulate to continue the investigations with the aim of analyzing more deeply the positive curvature of the results of the students' tests with the graphing calculator, also following the interviews recorded before the interruption of face-to-face teaching.

The analysis of the data emerged from the activities of Game Theory and Mathematics suggest further experimentation, actions and discussions in groups during strategic choices with face-to-face monitoring through criteria for measuring behaviors in strategic interactions.

The research activities in the analysis of the data of the experiments on cosmic rays present rich ideas to deepen themes strongly current and present in everyday life but almost absent in the high school curricular courses such as statistics, programming, geometry in space.

Although since September 2021 the educational activities have returned to presence, the school world is still suffering in an attempt to recover what students have lost in the last two years both in terms of skills and in terms of socialization.

All the know-how acquired in the technological field, will allow to design new research projects starting from the results obtained here. The work has already begun.

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