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Some group properties associated with two-variable words

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Introduction

Given a class of groups $\mathscr{X}$, a group-theoretical property $\mathscr{P}$ is said to be *bigenetic* [26] in the class $\mathscr{X}$ if an $\mathscr{X}$-group G has property $\mathscr{P}$ whenever all two-generator subgroups of G have property $\mathscr{P}$. It is a well-known fact that nilpotency is bigenetic in the class of all finite groups, and in [34] J.G. Thompson also showed that a finite group is solvable whenever each pair of its elements generates a solvable group. More recently, S. Dolfi, R.M. Guralnick, M. Herzog and C.E. Praeger [10] proved that solvability of finite groups is ensured by a seemingly weaker condition, namely a finite group G is solvable if for every pair of elements $a, b \in G$ there exists an element $g \in G$ for which the subgroup $\langle a^g, b\rangle$ is solvable. Moreover, it is possible to obtain a similar criterion also for the class of nilpotent finite groups: a finite group G is nilpotent if for every pair of elements $a, b \in G$ there exists an element $g \in G$ for which $\langle a^g, b\rangle$ is nilpotent.

Let $\mathscr{Y}$ be the class of groups which cannot be covered by conjugates of any proper subgroup. This class was investigated by J. Wiegold in [38] and [39]. In particular, it contains all finite and solvable groups. In [19], M. Herzog, P. Longobardi and M. Maj observed that a group G which belongs to the class $\mathscr{Y}$ is abelian if for every $a, b \in G$ there exists $g \in G$ such that the subgroup $\langle a^g, b\rangle$ is abelian, or equivalently, if for every $a, b \in G$ there exists $g \in G$ for which $[a^g, b] = 1$.

In Chapter 2 of the thesis, which includes a work made in collaboration with C. Nicotera [29], we consider the following problem:

Problem 1. *Let G be a $\mathscr{Y}$-group and $w(x, y)$ be a word; if for every $a, b \in G$ there exists $g \in G$ such that $w(a^g, b) = 1$, then is it true that $w(a, b) = 1$ for all $a, b \in G$, i.e. does G belong to the variety determined by the law $w(x, y) = 1$?*

In Section 2.4 we introduce for every $g \in G$ the sets

$$W_L^w(g) = \{a \in G \mid w(g,a) = 1\}$$

and

$$W_R^w(g) = \{a \in G \mid w(a,g) = 1\},$$

where the letters L and R stand for left and right. Observe that if w is the commutator word $[x,y]$, then the set $W_L^w(g) = W_R^w(g)$ is the centralizer of g in G, and the result due to M. Herzog, P. Longobardi and M. Maj ensures that the answer to Problem 1 is affirmative. In Theorem 2.3.1 we show that, more generally, Problem 1 has a positive answer whenever each subset $W_L^w(g)$ is a subgroup of G, or equivalently, if each subset $W_R^w(g)$ is a subgroup of G.

The sets $W_L^w(g)$ and $W_R^w(g)$ can be called the *centralizer-like subsets* associated with the word w. They need not be subgroups in general. In Chapter 3 we examine some sufficient conditions on the group G ensuring that the sets $W_L^w(g)$ and $W_R^w(g)$ are subgroups of G for all g in G. We denote by $\mathscr{W}_L^w$ the class of all groups G for which the set $W_L^w(g)$ is a subgroup of G for every $g \in G$, and in Theorem 3.1.1 we show that a group G belongs to the class $\mathscr{W}_L^w$ if for every $g,h,k \in G$, $w(g,1) = 1$ and $w(g,hk)$ is the product of a conjugate of $w(g,h)$ and a conjugate of $w(g,k)$. A similar property holds for the class $\mathscr{W}_R^w$ of all groups G for which the set $W_R^w(g)$ is a subgroup of G for every $g \in G$.

In Section 2.4 we begin our investigation with the n-Engel word

$$w(x,y) = [x,_n y],$$

with $n \geq 2$. We say that a group G is in the class $\mathscr{C}_n$ if for every pair of elements $a, b \in G$ there exists $g \in G$ such that

$$[a^g,_n b] = 1.$$

L.-C. Kappe and P.M. Ratchford proved [24] that if G is a metabelian group, then the centralizer-like subsets associated with the second variable of w is a subgroup of G for every $g \in G$. From this property it follows Theorem 2.4.2, which states that every metabelian $\mathscr{C}_n$-group is n-Engel.

If $n = 2$, we extend Theorem 2.4.2 by considering other classes of groups which are contained in the class $\mathscr{Y}$. In particular, we prove that any solvable $\mathscr{C}_2$-group is 2-Engel. There are Tarski's monsters G such that for every $a, b \in G$ there exists $g \in G$ for which $[a^g, b] = 1$. For every $n \geq 2$, these are examples of non-solvable $\mathscr{C}_n$-groups which are not n-Engel. These examples also show that the class $\mathscr{C}_n$ does not coincide with the class of n-Engel groups. If $n > 2$, then we give a partial solution to the following problem:

Problem 2. *Is it true that every solvable $\mathscr{C}_n$-group is n-Engel?*

We prove in Theorem 2.4.4 that a finite solvable $\mathscr{C}_n$-group is nilpotent, and in Corollary 2.4.2 that every finitely generated solvable $\mathscr{C}_n$-group is nilpotent. It follows that a finitely generated solvable $\mathscr{C}_n$-group is m-Engel, for some non-negative integer m. But the following problem remains open:

Problem 3. *Is it true that every finitely generated solvable $\mathscr{C}_n$-group is n-Engel? At least, is it possible to find a function $f(n)$ such that every finitely generated solvable $\mathscr{C}_n$-group is $f(n)$-Engel?*

In Chapter 3 we consider the centralizer-like subsets associated with some commutator words in two variables. These results can be found in [22]. First we focus on two-variable words of the form

$$w(x, y) = C_n[y, x],$$

where C_n is a left-normed commutator of weight $n \geq 3$ with entries from the set $\{x, y, x^{-1}, y^{-1}\}$. N.D. Gupta [15] considered a number of group laws of the form

$$C_n = [x, y],$$

observing that any finite or solvable group satisfying such a law is abelian. The question arises whether each group satisfying a law of the form $C_n = [x, y]$ is abelian. L.-C. Kappe and M.J. Tomkinson [25] solved the problem in the case $n = 3$, by showing that the variety of groups satisfying one of the laws of the form $C_3 = [x, y]$ is the variety of the abelian groups. In [30], P. Moravec extended the result to the case $n = 4$.

We show in Corollary 3.2.1 that every locally nilpotent group belongs to the classes $\mathscr{W}_L^w$ and $\mathscr{W}_R^w$ associated with the word w. Moreover, if $w(x,y)$ is one of the 2^{n-1} words of the form

$$[y, x^{\alpha_1}, x^{\alpha_2}, \ldots, x^{\alpha_{n-1}}][y, x]$$

or one of the 2^{n-1} words of the form

$$[x^{\alpha_1}, y, x^{\alpha_2}, \ldots, x^{\alpha_{n-1}}][y, x],$$

where $\alpha_i \in \{-1, 1\}$ for every $i = 1, \ldots, n-1$, then any metabelian group belongs to the class $\mathscr{W}_L^w$. In metabelian groups a symmetry of the centralizer-like subsets associated with the words of the form $w(x,y) = C_n[y,x]$ holds: if

$$w(x,y) = [r_1, r_2, r_3, \ldots, r_n][y, x],$$

with $r_i \in \{x, y, x^{-1}, y^{-1}\}$ for every $i = 1, \ldots, n$, then for every element g in a metabelian group G we have

$$W_R^w(g) = W_L^{\overline{w}}(g)$$

and

$$W_L^w(g) = W_R^{\overline{w}}(g),$$

where $\overline{w}(y,x) = [r_2, r_1, r_3, \ldots, r_n][x, y]$.

In Section 3.2.1 we more specifically investigate the word w when $n = 3$. Excluding the trivial cases in which r and s are equal or inverses, there are thirty-two remaining non-trivial words of the form

$$[r, s, t][y, x],$$

with $r, s, t \in \{x, y, x^{-1}, y^{-1}\}$. In [25], L.-C. Kappe and M.J. Tomkinson observed that six of the thirty-two words are *strongly equivalent* to the simple commutator word $[x, y]$, i.e. for every $g, h \in G$ the value of these words at (g, h) is 1 if and only if the elements g and h commute. Therefore, if $w(x,y)$ is one of these six words, then for every element g in a group G the sets $W_L^w(g)$ and $W_R^w(g)$ are exactly the centralizer of g in G. If G is a metabelian group, for the case $n = 3$

we observe that G belongs to the class $\mathscr{W}_L^w$ for exactly eleven of the thirty-two words w, by exhibiting counterexamples for the remaining words.

We conclude Chapter 3 investigating the words of the form

$$w(x,y) = (xy)^n y^{-n} x^{-n},$$

for some integer n. The *Collection Formula* of Philip Hall (see Lemma 3.3.1) ensures that $w(x,y) = (xy)^n y^{-n} x^{-n}$ is a commutator word. It is also called the *n-commutator word.* In Theorem 3.3.2 and Theorem 3.3.3 we prove that $\mathscr{W}_L^w = \mathscr{W}_R^w$, and if the centralizer-like subsets $W_L^w(g)$ and $W_R^w(g)$ are both subgroups of G, then we also have $W_L^w(g) = W_R^w(g)$.

R. Baer introduced the *n-center* $Z(G,n)$ of a group G: it is defined as the set of all elements $g \in G$ which *n-commute* with every element $h \in G$, i.e.

$$(gh)^n = g^n h^n \quad \text{and} \quad (hg)^n = h^n g^n.$$

A group G which coincides with its n-center is said to be *n-abelian.* For every element g in a group G we define the *n-centralizer* $C_G(g,n)$ of g in G as the set of all elements of G which n-commute with g, namely, with our notation,

$$C_G(g,n) = W_L^w(g) \cap W_R^w(g)$$

when w is the n-commutator word. The n-centralizer $C_G(g,n)$ is not necessarily a subgroup, even if the group is metabelian. However, we prove that if a group G is 2-Engel, then $C_G(g,n) = W_L^w(g) = W_R^w(g)$ is a subgroup of G for every $g \in G$.

Chapter 1

Preliminaries

The purpose of this chapter is to recall some basic notions and to establish some of the notation and terminology which will be used in the sequel. In particular, we will mention briefly some results on Engel groups.

1.1 Basic concepts and definitions

Let n be a positive integer and let $x_1, x_2, \ldots, x_n$ be elements of a group G. We remind that the *commutator* $[x_1, x_2]$ of x_1 and x_2 is defined by

$$[x_1, x_2] = x_1^{-1}x_2^{-1}x_1x_2 = x_1^{-1}x_1^{x_2},$$

while for $n > 2$ a *left-normed commutator of weight* n is defined inductively by the rule

$$[x_1, \ldots, x_n] = [[x_1, \ldots, x_{n-1}], x_n].$$

By convention $[x_1, \ldots, x_n] = x_1$ if $n = 1$.

For every $x, y \in G$ we use the symbol

$$[x,_n y]$$

to denote the left-normed commutator of weight $n+1$ of x and y, where y appears n times on the right. We also assume $[x,_0 y] = x$.

In the following lemma we summarize the standard commutator identities, see [32], which will be used without further reference.

Lemma 1.1.1. *Let x, y, z be elements of a group. Then:*

1) $[xy, z] = [x, z]^y[y, z] = [x, z][x, z, y][y, z]$;

2) $[x, yz] = [x, z][x, y]^z = [x, z][x, y][x, y, z]$;

3) $[x, y^{-1}] = [x, y]^{-y^{-1}}$;

4) $[x^{-1}, y] = [x, y]^{-x^{-1}}$;

5) $[x, y^{-1}, z]^y[y, z^{-1}, x]^z[z, x^{-1}, y]^x = 1$ *(the Hall-Witt identity).*

6) $[x, y, z^x][z, x, y^z][y, z, x^y] = 1$ *(the Jacobi identity).*

Lemma 1.1.2. *Let x, y, z be elements of a group G. Then the following properties are satisfied:*

1) if $[x, z] = 1$ and $[x, y, z] = 1$, then

$$[x, yz, yz] = [x, y, y]^z;$$

2) if $[x, [y, z]] = 1$ and $[x, y, [y, z]] = 1$, then

$$[x, y^z, y^z] = [x, y, y]^{[y,z]};$$

3) if $[x, y, y] = 1$, then

$$[x, y^n] = [x, y]^n,$$

for every non-negative integer n.

Proof. *1)* If $[x, z] = 1$ and $[x, y, z] = 1$, then we have

$$[x, yz, yz] = [[x, z][x, y][x, y, z], yz] = [x, y, yz] = [x, y, z][x, y, y]^z = [x, y, y]^z.$$

2) Let $[x, [y, z]] = 1 = [x, y, [y, z]]$. Since $y[y, z] = y^z$, from property *1)* it follows

$$[x, y^z, y^z] = [x, y[y, z], y[y, z]] = [x, y, y]^{[y,z]}.$$

3) Let us use induction on n, the case $n \leq 1$ being clear. If $n > 1$ then we may assume $[x, y^{n-1}] = [x, y]^{n-1}$ by inductive hypothesis. If $[x, y, y] = 1$ then $[x, y]^y = [x, y]$, and so we have

$$[x, y^n] = [x, y^{n-1}y] = [x, y][x, y^{n-1}]^y = [x, y]\left([x, y]^{n-1}\right)^y = [x, y]^n.$$

□

If X and Y are nonempty subsets of a group G, then $[X, Y]$ denote the *commutator subgroup* of X and Y, namely the subgroup generated by the set of all commutators of elements of X with elements of Y. Moreover, we recall that $G' = [G, G]$ is the *derived subgroup* of the group G, and the sequence

$$G = G^{(0)} \geq G^{(1)} \geq G^{(2)} \geq \cdots ,$$

where $G^{(n)} = (G^{(n-1)})'$ for every $n > 0$, is termed the *derived series* of G. A group G is said to be *solvable* of *derived length* at most n if $G^{(n)} = 1$. In particular, a solvable group with derived length at most 2 is said to be *metabelian*.

Lemma 1.1.3. *Let G be a metabelian group, $x, y, z, g \in G$, and $c \in G'$. Then:*

1) $[x, y, z][z, x, y][y, z, x] = 1$ *(the Jacobi identity);*

2) $[c^g, z] = [c, z]^g$*;*

3) $[x^g, c, y^z] = [x, c, y]$.

Proof. If G is a metabelian group, then for every $x, y, z, g \in G$, $c \in G'$ we have

$$[c^g, z] = [c, z^{g^{-1}}]^g = [c, z[z, g^{-1}]]^g = [c, [z, g^{-1}]]^g[c, z]^g = [c, z]^g,$$

$$[x^g, c, y^z] = [x[x, g], c, y[y, z]] = [x, c, y],$$

and

$$[x, y, z^g] = [x, y, z[z, g]] = [x, y, z].$$

We also obtain

$$[x, y, z][z, x, y][y, z, x] = [x, y, z^x][z, x, y^z][y, z, x^y] = 1,$$

by property *6)* of Lemma 1.1.1. □

Lemma 1.1.4. *Let $n > 1$ an integer, G be a metabelian group and $x_i \in G$ for every $i = 1, \ldots, n$. Then we have*

$$[x_1, x_2, x_3, \ldots, x_n]^{-1} = [x_2, x_1, x_3, \ldots, x_n].$$

Proof. We induct on the integer n. Certainly $[x_1, x_2]^{-1} = [x_2, x_1]$. If $n > 2$, then by induction we obtain

$$\begin{aligned} [x_1, x_2, x_3, \ldots, x_n]^{-1} &= \left([x_1, x_2, x_3, \ldots, x_{n-1}]^{-1}[x_1, x_2, x_3, \ldots, x_{n-1}]^{x_n}\right)^{-1} \\ &= [x_1, x_2, x_3, \ldots, x_{n-1}][x_1, x_2, x_3, \ldots, x_{n-1}]^{-x_n} \\ &= [x_2, x_1, x_3, \ldots, x_{n-1}]^{-1}[x_2, x_1, x_3, \ldots, x_{n-1}]^{x_n} \\ &= [x_2, x_1, x_3, \ldots, x_n]. \end{aligned}$$

□

We remind that the *lower central series* of a group G is the descending series of subgroups

$$G = \gamma_1(G) \geq \gamma_2(G) \geq \cdots,$$

with $\gamma_n(G) = [\gamma_{n-1}(G), G]$ for every $n > 1$. Instead, the *upper central series* of a group G is the ascending sequence of subgroups

$$1 = Z_0(G) \leq Z_1(G) \leq Z_2(G) \leq \cdots,$$

defined by $Z_n(G)/Z_{n-1}(G) = Z(G/Z_{n-1}(G))$ for every $n > 0$. Of course $Z_1(G) = Z(G)$ is the center of G, whereas each $Z_n(G)$ is called the *nth center* of G. For infinite groups, one can extend the two series to infinite ordinal numbers via transfinite recursion: if α is a limit ordinal, then the subgroups $\gamma_\alpha(G)$ and $Z_\alpha(G)$ (also called the *α-center* of G) are defined by the rules

$$\gamma_\alpha(G) = \bigcap_{\lambda<\alpha} \gamma_\lambda(G)$$

and

$$Z_\alpha(G) = \bigcup_{\lambda<\alpha} Z_\lambda(G).$$

Since the cardinality of G cannot be exceeded, there exists a cardinal β at which the upper central series stabilizes. The terminal group $\overline{Z}(G) = Z_\beta$ is called the *hypercenter* of G.

In addition, the following property holds (see, for example, [32]).

Lemma 1.1.5. *If G is any group, then $G^{(i)} \leq \gamma_{2^i}(G)$ for every integer $i \geq 1$.*

Let X be an alphabet of letters $x_1, x_2, \dots$ to which we refer as *variables*, and let F be the free group having X as a free basis. An element $w = x_{i_1}^{l_1} \cdots x_{i_m}^{l_m} \in F$ is called a *word in n variables*, where $n \leq m$, if n distinct letters $x_{j_1}, \dots, x_{j_n}$ occur in w. In this case, we also denote the word w by

$$w(x_{j_1}, \dots, x_{j_n}).$$

An element of the commutator subgroup F' is termed a *commutator word*.

If $g_1, \dots, g_n$ are elements of a group G and $w(x_{j_1}, \dots, x_{j_n})$ is an n-variable word, then the element $w(g_1, \dots, g_n)$ in the group G, computed by substituting each g_i for the indeterminate x_{j_i}, is called the *value* of w at $(g_1, \dots, g_n)$. The word w is said to be a *law* in the group G if the only possible value of w in G is 1. Two words $w_1, w_2 \in F$ are said *equivalent* if w_1 is a law in a group G whenever w_2 is a law in G and vice versa. Moreover, if w_1 and w_2 are n-variable words, then we define the two words *strongly equivalent* if, for every $g_1, \dots, g_n$ in a group G, $w_1(g_1, \dots, g_n) = 1$ implies $w_2(g_1, \dots, g_n) = 1$ and vice versa.

1.2 Engel elements

An element x of a group G is called a *right Engel element* of G if for all $g \in G$ there exists a non-negative integer $n = n(g)$ such that $[x,_n g] = 1$. Instead x is said to be a *left Engel element* of G if for all $g \in G$ there exists a non-negative integer $n = n(g)$ such that $[g,_n x] = 1$. If in either case the integer n can be chosen independently of g then we talk about *right n-Engel* or *left n-Engel element* respectively.

We say that a group G is an *Engel group* if every element of G is a left Engel element or, equivalently, if every element of G is a right Engel element of G. By an *n-Engel group* we mean a group G satisfying the law $[x,_n y] = 1$, so that every element of G is both a left and right n-Engel element of G. Obviously a 0-Engel group has order 1 and the variety of 1-Engel groups is the variety of all abelian groups.

As an example, the variety of n-Engel groups contains every nilpotent group whose nilpotency class is bounded by n. It is clear that every locally nilpotent group is an Engel group, while the converse is not true in general. Indeed, E.S. Golod [13] constructed for every prime p and for every integer $d > 2$ a non-nilpotent d-generated p-group in which every $(d-1)$-generated subgroup is nilpotent. Every 2-generator subgroup of these groups is nilpotent; then, these are all examples of Engel groups which are not locally nilpotent. It has been a long-standing open question whether all n-Engel groups are locally nilpotent. During the international conference "Algebraic groups and related structures", held on 17-22 September 2012 in Saint-Petersburg, Russia, in honour of Nikolai Vavilov on the occasion of his 60th birthday, E. Plotkin announced that E. Rips has constructed, for n sufficiently large, examples of non-nilpotent finitely generated groups which are n-Engel.

The first main result on Engel groups is Zorn's Theorem:

Theorem 1.2.1 (M. Zorn [40])**.** *Every finite Engel group is nilpotent.*

Moreover, the following theorems hold (see, for instance, [32]):

Theorem 1.2.2 (K.W. Gruenberg)**.** *Every finitely generated solvable Engel group is nilpotent.*

Theorem 1.2.3 (R. Baer)**.** *Every Engel group satisfying max is nilpotent.*

Observe that if x is an element of the nth center $Z_n(G)$, then x is a right n-Engel element of G. Indeed, $x \in Z_n(G)$ if and only if $[x, a] \in Z_{n-1}(G)$ for every $a \in G$. Therefore, it follows $[x, a, a] \in Z_{n-2}(G)$ for every $a \in G$, and proceeding in the same way, we get $[x,_n a] = 1$ for every $a \in G$. Conversely, it is not true in general that right n-Engel elements need to be in the hypercenter. For instance, for every prime p the standard wreath product G of a group of order p and an infinite elementary abelian p-group is a $(p+1)$-Engel group whose center is trivial. Hence all elements of G are right $(p+1)$-Engel elements, but $\overline{Z}(G) = 1$.

However, the converse is true for a number of classes of groups:

Theorem 1.2.4 (R. Baer [1]). *In every finite group the right Engel elements belong to the hypercenter.*

Theorem 1.2.5 (C.J.B. Brookes [5]). *In every finitely generated solvable group the right Engel elements belong to the hypercenter.*

1.3 2-Engel groups

For ease of reference, in this section we recall some well-known results on 2-Engel groups. See [33] for a more detailed treatment.

A 2-Engel group satisfies the commutator law $[x, y, y] = 1$ or equivalently the law $[x^y, x] = 1$, i.e. any two conjugates commute.

Theorem 1.3.1 (F.W. Levi). *Let G be a 2-Engel group and let x, y, z, t be elements of G. Then:*

1) x^G *is abelian;*

2) $[x, y, z] = [z, x, y]$;

3) $[x, y, z]^3 = 1$;

4) $[x, y, z, t] = 1$.

From Theorem 1.3.1 it follows that a 2-Engel group is nilpotent of class at most 3. In particular every 2-Engel group without elements of order 3 is nilpotent of class at most 2.

Theorem 1.3.2 (F.W. Levi). *Every group of exponent 3 is a 2-Engel group. If G is a 2-Engel group, then $[G', G]^3 = 1$.*

If G is a 2-Engel group, then by Lemma 1.1.5 we have $G'' \leq \gamma_4(G) = 1$. Consequently, every 2-Engel group is metabelian.

In the following theorems we state some basic properties of right 2-Engel elements and 2-Engel groups:

Theorem 1.3.3 (F.W. Levi, W.P. Kappe)**.** *Let a be a right 2-Engel element and let x, y, z be elements of a group G. Then:*

1) a is a left 2-Engel element;

2) $[a, x, y] = [a, y, x]^{-1}$;

3) $[a, [x, y], z] = 1$;

4) $[a, x, y, z]^2 = 1$;

5) $[a, [x, y]] = [a, x, y]^2$.

Theorem 1.3.4 (W.P. Kappe)**.** *The right 2-Engel elements of a group form a characteristic subgroup.*

Theorem 1.3.5 (I.D. Macdonald, B.H. Neumann)**.** *In an arbitrary group a right 2-Engel element of odd order lies in the third term of the upper central series.*

Theorem 1.3.6 (W.P. Kappe)**.** *The following properties of a group G are equivalent:*

1) G is a 2-Engel group;

2) x^G is abelian for every $x \in G$;

3) if $x \in G$, then $C_G(x) \trianglelefteq G$;

4) every maximal abelian subgroup of G is normal;

5) each 2-generator subgroup of G is nilpotent of class at most 2;

6) the identity $[x, y, z] = [x, z, y]^{-1}$ holds in G.

1.4 n-Engel groups when $n \geq 3$

If n is greater than 2, then an n-Engel group need not be nilpotent. For example, the standard wreath product of the cyclic group of order 2 by an infinite elementary abelian 2-group is a 3-Engel group which is not nilpotent, since it is centerless. H. Heineken [18] proved that if G is a 3-Engel group, then G is locally nilpotent and $\gamma_5(G)$ is a torsion group in which the only primes that can occur as orders of elements are 2 and 5. Hence a 3-Engel group without elements of order 2 or 5 is nilpotent of class at most 4. G. Traustason [35] observed that if a locally nilpotent 4-Engel group has no elements of order 2, 3 or 5, then it is nilpotent of class at most 7. In 2005, G. Havas and M.R. Vaughan-Lee [17] showed that every 4-Engel group is locally nilpotent; thus, 4-Engel groups without elements of order 2, 3 or 5 are nilpotent of class at most 7.

Notice that a group G is 2-Engel if and only if the normal closure of every element of G is abelian, or equivalently 1-Engel. Moreover, L.-C. Kappe and W.P. Kappe [21] proved that a group is 3-Engel if and only if x^G is nilpotent of class at most 2, from which it follows that a group G is a 3-Engel group if and only if the normal closure of every element of G is 2-Engel. Recently, M.R. Vaughan-Lee [36] generalized this property to 4-Engel groups: a group G is a 4-Engel group if and only if x^G is 3-Engel for all $x \in G$.

Little is known about n-Engel groups when $n > 4$.

Chapter 2

On a property of two-variable laws

In this chapter we introduce a problem concerning laws in two variables in the class of groups which cannot be covered by conjugates of any proper subgroup. In particular, we focus on the case of the n-Engel word.

2.1 Bigenetic properties

The purpose of this section is essentially to illustrate some results which show how the structure of a group can be influenced by properties satisfied by its 2-generator subgroups. For a more detailed analysis, see [27].

A group-theoretical class $\mathscr{X}$ is said to be *closed under the formation of subgroups*, or equivalently *S-closed*, if every subgroup of an $\mathscr{X}$-group is still in the class $\mathscr{X}$. Obviously the classes of abelian groups, cyclic groups and finite groups are all S-closed. Also the class of all n-Engel groups (for every non-negative integer n) is closed under taking subgroups, while the class of all finitely generated groups is an example of class of groups which is not S-closed: if G is the free group on two generators a and b, and S is the subset consisting of all elements of G of the form a^{b^n}, with n a natural number, then $\langle S\rangle$ is isomorphic to the free group of countable rank and it cannot be finitely generated.

If a group G belongs to an S-closed class $\mathscr{X}$, then $\langle x, y\rangle \in \mathscr{X}$ for every pair of elements x, y in G. Therefore, one might ask whether it is true that a group

G is in the class $\mathscr{X}$ whenever the 2-generator subgroup $\langle x, y\rangle$ belongs to $\mathscr{X}$ for every $x, y \in G$.

A group-theoretical property $\mathscr{P}$ is said to be *bigenetic* if a group G has property $\mathscr{P}$, whenever all 2-generator subgroups of G have property $\mathscr{P}$. This terminology was first introduced by J.C. Lennox in [26]. For instance, the class of n-Engel groups is bigenetic and, more generally, every variety of groups defined by a word in two variables is bigenetic. The class of all cyclic groups, instead, is not bigenetic, since a locally cyclic group need not be cyclic.

Consider now, for every integer $n \geq 1$, the class $\mathscr{N}_n$ of all nilpotent groups whose nilpotency class is bounded by n and let $\mathscr{N} = \cup_{n\geq 1}\mathscr{N}_n$ be the class of all nilpotent groups. Clearly such classes are closed under the formation of subgroups. By Theorem 1.3.6 a group G is 2-Engel if and only if each of its 2-generator subgroups is in $\mathscr{N}_2$. As there exist 2-Engel groups which are nilpotent of class 3, the class $\mathscr{N}_2$ is not bigenetic. In [3], S. Bachmuth and H.Y. Mochizuki proved the existence of a non-nilpotent 3-Engel group of exponent 5 all of whose 2-generator subgroups are nilpotent of class at most 3; thus the class $\mathscr{N}_3$ is not bigenetic. Also nilpotency is not a bigenetic property: M.R. Vaughan-Lee and J. Wiegold [37] constructed, for each prime $p \geq 5$, a countable locally finite perfect group of exponent p for which $\langle x, y\rangle$ is nilpotent of bounded class for every pair of elements x, y; thus, considering that a perfect group is not solvable, such examples show that neither the class $\mathscr{N}$ nor the class of solvable groups is bigenetic.

If a group property $\mathscr{P}$ is not bigenetic, in some cases it is possible to determine some interesting class of groups $\mathscr{X}$ such that a group G is in $\mathscr{X}$ whenever $\langle x, y\rangle \in \mathscr{P}$ for every x, y in G. For example, if every 2-generator subgroup of a group G is cyclic, then G is abelian. Furthermore, if $\langle x, y\rangle \in \mathscr{N}_2$ for every x, y in G, then G is in $\mathscr{N}_3$ as a 2-Engel group is nilpotent of class ≤ 3.

In addition, when a property is not bigenetic one can try to establish if there exists a sufficiently large class of groups in which the property is bigenetic. Given a class of groups $\mathscr{X}$, a property $\mathscr{P}$ is called *bigenetic in the class* $\mathscr{X}$ if an $\mathscr{X}$-group G has property $\mathscr{P}$ whenever all 2-generator subgroups of G have property $\mathscr{P}$.

Let us now consider the class of finite groups.

Theorem 2.1.1. *Nilpotency is bigenetic in the class of all finite groups.*

Proof. If every pair of elements of a finite group G generates a nilpotent group, then there exists a natural number c such that every 2-generator subgroup of G is nilpotent of class bounded by c. It follows that $[x,_c y] = 1$ for every $x, y \in G$; hence G is a finite c-Engel group and so it is nilpotent. □

A *minimal simple group* is a non-abelian simple group all of whose proper subgroups are solvable. In [34], J.G. Thompson classified finite minimal simple groups and showed that every finite minimal simple group is generated by two elements. As a consequence, the author obtained that a finite group is solvable if and only if every pair of its elements generates a solvable group.

Theorem 2.1.2. *The property of solvability is bigenetic in the class of all finite groups.*

Proof. Suppose by contradiction that there exist finite non-solvable groups all of whose 2-generator subgroups are solvable and let G be a counterexample of minimal order. In particular, we may assume that G is not abelian. If G is not simple, then there exists a non-trivial normal subgroup N of G. Observe that every 2-generator subgroup of N and of G/N is still solvable. Therefore, N and the quotient G/N are both solvable by the minimality of the order of G. This means that G is solvable, a contradiction. If G is simple, then G is a finite non-abelian simple group with every proper subgroup solvable by the minimality of the order of G. Hence G is a minimal simple group and by Thompson's result it is 2-generated. Consequently G is solvable by hypothesis, but this is a contradiction. □

The result was later proved by P. Flavell [11] without using Thompson's classification of minimal simple groups. Also polycyclicity is bigenetic in the class of finite groups. In fact, a group is polycyclic if and only if it is solvable and satisfies the maximal condition. Thus for finite groups polycyclicity and solvability are equivalent. Moreover, the following theorem holds:

Theorem 2.1.3 (R.W. Carter, B. Fischer, T. Hawkes [6])**.** *The property of supersolvability is bigenetic in the class of all finite groups.*

The next results show that nilpotency and polycyclicity are bigenetic in the class of all finitely generated solvable groups.

Theorem 2.1.4. *Nilpotency is bigenetic in the class of all finitely generated solvable groups.*

Proof. Let g be an element of a finitely generated solvable group G. If every 2-generator subgroup of G is nilpotent, then for every $x \in G$ we have that $\langle g, x\rangle$ is nilpotent and so for every $x \in G$ there exists a natural number c such that $[g,_c x] = 1$. Thus g is a right Engel element of G. It follows that G is an Engel group and by Theorem 1.2.2 it is nilpotent. $\square$

Theorem 2.1.5 (J.C. Lennox [26])**.** *If every* 2*-generator subgroup of a solvable finitely generated group G is polycyclic, then G is polycyclic.*

2.2 Some solvability criteria for finite groups

Theorem 2.1.2 states that a finite group is solvable if and only if every pair of its elements generates a solvable group. Actually, several other weaker conditions ensuring solvability of a finite group have appeared in the literature recently. For instance, R.M. Guralnick and J.S Wilson [16] showed that solvability of a finite group is guaranteed by solvability of a sufficient proportion of its 2-generator subgroups. They proved that a finite group G is solvable if and only if more than $\frac{11}{30}$ of the pairs of its elements generates a solvable group.

Another solvability criterion for finite groups is due to N. Gordeev, F. Grunewald, B. Kunyavskiĭ and E. Plotkin [14]. It asserts that a finite group G is solvable if and only if in every conjugacy class of G each pair of elements generates a solvable subgroup.

The next extension of Theorem 2.1.2 was obtained in a paper published in 2012 [10].

Theorem 2.2.1 (S. Dolfi, R.M. Guralnick, M. Herzog, C.E. Praeger)**.** *Let G be a finite group. The following are equivalent:*

1) G is solvable;

2) for every pair of elements a and b in G there exists an element $g \in G$ for which $\langle a^g, b\rangle$ is solvable;

3) for every pair of conjugacy classes C and D of G there exist $x \in C$ and $y \in D$ for which $\langle x, y\rangle$ is solvable.

In the same paper [10] the authors generalized this property to every class $\mathscr{X}$ of finite groups which is closed under taking subgroups, quotient groups, and forming extensions: a finite group G is in $\mathscr{X}$ if and only if for every pair of conjugacy classes C and D of G there exist $x \in C$ and $y \in D$ for which the subgroup $\langle x, y\rangle$ is in $\mathscr{X}$. Moreover, from the following result it is possible to state that a similar criterion is true also for the class of finite nilpotent groups.

Theorem 2.2.2 (S. Dolfi, R.M. Guralnick, M. Herzog, C.E. Praeger)**.** *A finite group G is nilpotent if and only if for every pair of distinct primes p and q and for every pair of elements x and y in G with x a p-element and y a q-element, x and y^g commute for some $g \in G$.*

Corollary 2.2.1. *Let G be a finite group. The following are equivalent:*

1) G is nilpotent;

2) for every pair of elements a and b in G there exists an element $g \in G$ for which $\langle a^g, b\rangle$ is nilpotent;

3) for every pair of conjugacy classes C and D of G there exist $x \in C$ and $y \in D$ for which $\langle x, y\rangle$ is nilpotent.

Proof. The implications *1)* $\Rightarrow$ *2)* and *2)* $\Rightarrow$ *3)* are obvious.

3) $\Rightarrow$ *1).* Let p and q be distinct primes and consider a pair of elements x and y in G with x a p-element and y a q-element. In view of the former theorem, to prove that G is nilpotent it suffices to show that x^g and y commute for some element g in G. By hypothesis there exist $a, b \in G$ such that the 2-generator subgroup $\left\langle x^a, y^b\right\rangle$ is nilpotent. The conjugate of x by a and the conjugate of y by b have still orders which are powers respectively of p and q. Since a finite nilpotent group is the direct product of its nontrivial Sylow subgroups, the elements x^a and y^b are in two distinct factors of the direct product, and so they commute. It follows that the conjugate of x by ab^{-1} and y commute. $\square$

2.3 Groups which cannot be covered by conjugates of any proper subgroup

Denote by $\mathscr{Y}$ the class of all groups which cannot be written as the union of the conjugates of any proper subgroup, i.e. if H is a subgroup of a $\mathscr{Y}$-group G, then H is the whole group G when $G = \bigcup_{g \in G} H^g$. This class was investigated in [38] and [39] by J. Wiegold, who observed that a $\mathscr{Y}$-group can be characterized in the following way:

Observation 2.3.1. *Let G be a non-trivial group. The following statements are equivalent:*

1) G is a $\mathscr{Y}$-group;

2) for every transitive action of G on a set Ω, $|\Omega| > 1$, there is an element of G displacing every element of Ω;

3) if S is any subset of G containing at least one representative of every conjugacy class of G, then S generates G.

Proof. 1) $\Rightarrow$ 2). First consider the transitive action of G on the set of the right cosets of a proper subgroup H of G. If G belongs to the class $\mathscr{Y}$, then there is an element $x \in G$ which is not contained in the union of the conjugates of H. Since $x \in H^g$ if and only if $Hgx = Hg$, we have $Hgx \neq Hg$ for every $g \in G$. Hence, there is an element of G which displaces every right cosets of H.

The result follows from the fact that every transitive action of G is equivalent to an action on the right cosets of a subgroup of G.

2) $\Rightarrow$ 3). Let S be a subset of G which contains a representative of every conjugacy class of G. The subgroup H generated by S cannot be proper, otherwise, considering the transitive action of G on the set of the right cosets of H, by condition *2)* there would exist an element $x \in G$ for which $Hgx \neq Hg$ for every $g \in G$, i.e. $x \notin \cup_{g \in G} H^g$. This would be a contradiction because H contains a conjugate of x in G. Thus, G is generated by the subset S.

3) $\Rightarrow$ 1). Let G be the union of the conjugates of a subgroup H. Then for every $x \in G$ there exists $g \in G$ such that $x \in H^g$. In particular, for every element x of a conjugacy class C of G the element $x^{g^{-1}}$ of C is in H. Therefore,

H contains representatives of every conjugacy class of G, and it follows by *3)* that H is the whole group G. Thus, G is in $\mathscr{Y}$. $\square$

In [39], J. Wiegold showed that the class $\mathscr{Y}$ is not closed under subgroups by exhibiting an example of $\mathscr{Y}$-group whose commutator subgroup is not in $\mathscr{Y}$. Moreover, this class is closed under extensions and restricted direct products, but it is not closed under cartesian products.

Observation 2.3.2. *Every finite group belongs to the class $\mathscr{Y}$.*

Proof. Let H be a proper subgroup of a group G. If $r > 1$ is the index of H in G, then the index s of the normalizer of H in G is the number of the conjugates of H in G and it is bounded by r. As all the conjugates of H have the same order and contain the identity, we have

$$|\cup_{g\in G} H^g| \leq s(|H|-1)+1 \leq r(|H|-1)+1 = |G|-(r-1) < |G|\,.$$

It follows that $\bigcup_{g\in G} H^g \lneqq G$; hence $G \in \mathscr{Y}$. $\square$

It is also known [38] that the class $\mathscr{Y}$ contains every solvable group, as well as all hypercentral groups. In particular, $\mathscr{Y}$ contains the classes of groups considered in Theorem 2.2.1 and Corollary 2.2.1.

In [19], M. Herzog, P. Longobardi and M. Maj proved that in $\mathscr{Y}$ the following property holds:

Theorem 2.3.1 (M. Herzog, P. Longobardi, M. Maj)**.** *A $\mathscr{Y}$-group G is abelian if for every $a, b \in G$ there exists $g \in G$ for which $[a^g, b] = 1$.*

Notice that equivalently a $\mathscr{Y}$-group G is abelian if for every $a, b \in G$ there exists $g \in G$ such that $\langle a^g, b\rangle$ is abelian, or if for every pair of conjugacy classes C and D of G there exist $x \in C$ and $y \in D$ for which the 2-generator subgroup $\langle x, y\rangle$ is abelian. Therefore, this theorem may be restated in a form analogous to the statements of Theorem 2.2.1 and Corollary 2.2.1.

Theorem 2.3.1 is not true in general for a group which does not belong to the class $\mathscr{Y}$. G. Cutolo, H. Smith and J. Wiegold showed [7] that there are non-abelian groups of finitary permutations which are the union of the conjugates of an abelian subgroup. Moreover, there exist Tarski's monsters T_p, with p a large enough prime, in which all the subgroups of order p are conjugate. These are all examples of non-abelian groups G which do not belong to the class $\mathscr{Y}$ and in which for every pair of elements a, b there exists $g \in G$ such that a^g and b commute. In fact, if G is the union of the conjugates of an abelian subgroup H, then for every $a, b \in G$ there exist $g_1, g_2 \in G$ such that $a^{g_1}, b^{g_2} \in H$, and so $[a^{g_1}, b^{g_2}] = 1$ implies $[a^{g_1 g_2^{-1}}, b] = 1$.

2.4 On a property of two-variable laws

This section deals with some results recently obtained in collaboration with C. Nicotera [29]. The theorems of the previous sections (Theorem 2.2.1, Corollary 2.2.1 and Theorem 2.3.1) suggest the study of the following problem:

Problem 1. *Let G be a $\mathscr{Y}$-group and $w(x, y)$ be a word; if for every $a, b \in G$ there exists $g \in G$ such that $w(a^g, b) = 1$, then is it true that $w(a, b) = 1$ for all $a, b \in G$, i.e. does G belong to the variety determined by the law $w(x, y) = 1$?*

Let w a word in two variables, G a group, and $g \in G$. Then, we define

$$W_L^w(g) = \{a \in G \mid w(g, a) = 1\}$$

and

$$W_R^w(g) = \{a \in G \mid w(a, g) = 1\},$$

where the letters L and R stand respectively for left and right.

Theorem 2.3.1 guarantees that the answer to Problem 1 is affirmative when w is the commutator word. Notice that if $w(x, y) = [x, y]$, then the subset $W_L^w(g) = W_R^w(g)$ is the centralizer of g in G. More generally, the problem has a positive answer if each subset $W_L^w(g)$ is a subgroup of G, or if each subset $W_R^w(g)$ is a subgroup of G. In this case $W_L^w(g)$ and $W_R^w(g)$ can be called the *centralizer-like subgroups* associated with the word w.

Theorem 2.4.1. *Let G be a $\mathscr{Y}$-group and $w(x, y)$ be a word; assume that for all $a, b \in G$ there exists $g \in G$ such that $w(a^g, b) = 1$. If one of the conditions*

i) $W_L^w(g)$ is a subgroup of G for every $g \in G$, *ii)* $W_R^w(g)$ *is a subgroup of* G *for every* $g \in G$,

is satisfied, then G *belongs to the variety determined by the law* $w(x,y) = 1$.

Proof. For every pair of elements a, b of G we have $w(a^g, b) = 1$ for some $g \in G$. Thus, for all $a, b \in G$ there exists $g \in G$ such that $b \in W_L^w(a^g) = (W_L^w(a))^g$. But then

$$G = \bigcup_{g \in G} (W_L^w(a))^g \,.$$

If condition *i)* is satisfied, the hypothesis implies $G = W_L^w(a)$ for every $a \in G$. Hence $w(a,b) = 1$ for all $a, b \in G$.

Observe that the word $w(x,y)$ is a law in G if and only if the word $v(x,y) = w(y,x)$, which switches x and y, is a law. Since

$$W_L^v(g) = \{a \in G \mid v(g,a) = 1\} = W_R^w(g)$$

for every $g \in G$, the property is also true when condition *ii)* is satisfied. □

2.4.1 n-Engel law

Let us now consider the word $w(x,y) = [x,_n y]$, which determines the variety of n-Engel groups. While the right 2-Engel elements of a group form a subgroup, I.D. Macdonald has shown in [28] that there exists a finite 2-group in which the set of right 3-Engel elements is not a subgroup. The set of right n-Engel elements of a group G coincides with the intersection of all the sets

$$W_R^w(g) = \{a \in G \mid [a,_n g] = 1\}\,,$$

with $g \in G$. Then, when $n \geq 3$ the set $W_R^w(g)$ cannot be a subgroup of G in general, but property *ii)* of Theorem 2.4.1 holds in metabelian groups. Indeed, L.-C. Kappe and P.M. Ratchford proved [24] that if G is a metabelian group, then $W_R^w(g)$ is a subgroup of G for every $g \in G$. Moreover, they showed a slightly stronger result for $n = 2$: for every group G and for every $g \in G$ the set

$$W_R^w(g) = \{a \in G \mid [a,g,g] = 1\}$$

is a subgroup of G if and only if $[a, g, b, g] = 1$ for all $a, b \in W_R^w(g)$. Therefore, from Theorem 2.4.1 the next theorem follows.

Theorem 2.4.2. *Let G be a metabelian group. If $n \geq 2$ and for all $a, b \in G$ there exists $g \in G$ such that $[a^g,_n b] = 1$, then G is an n-Engel group.*

2.4.2 2-Engel law

The goal of this section is to generalize Theorem 2.4.2, in the case $n = 2$.

We say that a group G is in the class $\mathscr{C}_2$ if for every pair of elements $a, b \in G$ there exists $g \in G$ such that

$$[a^g, b, b] = 1.$$

Obviously the class $\mathscr{C}_2$ is closed under taking quotients. Instead, at least apparently, a subgroup of a $\mathscr{C}_2$-group is not necessarily a $\mathscr{C}_2$-group, even if it is a normal subgroup.

Theorem 2.4.2 states that a metabelian $\mathscr{C}_2$-group is 2-Engel. Our aim is to extend this result by considering other classes of groups which are contained in the class $\mathscr{Y}$. We begin by proving that nilpotent $\mathscr{C}_2$-groups are 2-Engel.

Observation 2.4.1. *A nilpotent $\mathscr{C}_2$-group is 2-Engel.*

Proof. Let G be a nilpotent $\mathscr{C}_2$-group and let

$$R = \{a \in G \mid [a, x, x] = 1 \ \forall x \in G\}$$

be the subgroup of the right 2-Engel elements of G. By contradiction, assume that R is a proper subgroup of G. The quotient G/R being nilpotent has a non-trivial center, and thus there exists $c \in G \setminus R$ such that $cR \in Z(G/R)$. Therefore, for every $g \in G$ we have $cR = (cR)^{gR} = c^g R$ and $[c, g] = c^{-1}c^g \in R$, and so there exists $r \in R$ such that $c^g = cr$. By hypothesis for every $x \in G$ there exists an element $g \in G$ for which $[c^g, x, x] = 1$. Hence we obtain

$$1 = [c^g, x, x] = [cr, x, x] = [[c, x]^r [r, x], x] = [[c, x]^r, x]^{[r,x]} [r, x, x] = [c, x, x^{r^{-1}}]^{r[r,x]},$$

from which it follows $[c, x, x^{r^{-1}}] = 1$. By properties of right 2-Engel elements we have

$$1 = \left[r^{-1}, [c, x], x\right] = \left[r^{-1}, x, [c, x]\right]^{-1} = \left[[c, x], [r^{-1}, x]\right].$$

Therefore $[c, x, [x, r^{-1}]] = 1$ and

$$1 = [c, x, x^{r^{-1}}] = \left[c, x, x[x, r^{-1}]\right] = \left[[c, x], [x, r^{-1}]\right] [c, x, x]^{[x,r^{-1}]} = [c, x, x]^{[x,r^{-1}]},$$

from which it follows $[c, x, x] = 1$. Hence $c \in R$, and this is a contradiction. $\square$

Using this result we can show that every solvable $\mathscr{C}_2$-group is 2-Engel. First, we need a few preliminary remarks.

Lemma 2.4.1. *Let G be a $\mathscr{C}_2$-group. If G'' is an abelian group which has no element of order 2 or 3, then G is nilpotent.*

Proof. Let G be a $\mathscr{C}_2$-group for which G'' is abelian. The quotient G/G'' is a metabelian $\mathscr{C}_2$-group. Then it is 2-Engel by Theorem 2.4.2, and thus it is nilpotent of class at most 3. By a well-known nilpotency criterion due to P. Hall (for instance, see [32]), to show that G is nilpotent it is enough to prove that G' is nilpotent. Since G/G'' is 2-Engel, we obtain $[G', G]^3 \leq G''$ and $[G', G, G] \leq G''$. For all $a \in [G', G]$ and $b \in G''$ we have $a^3 \in G''$, and so $[b, a^3] = 1$ by the commutativity of G''. By hypothesis there exists an element $g \in G$ such that $[b, a^g, a^g] = 1$. As $b, [b, a], [b, a, a]$ are all elements of G'', from $[a, g] \in [G', G, G] \leq G''$ it follows

$$[[a, g], b] = [[a, g], [b, a]] = [[a, g], [b, a, a]] = 1.$$

In light of property *2)* of Lemma 1.1.2 we have

$$1 = [b, a^g, a^g] = [b, a, a]^{[a,g]} = [b, a, a],$$

and property *3)* of Lemma 1.1.2 assures us that

$$[b, a]^3 = [b, a^3] = 1.$$

If G'' has no element of order 3, then $[b, a] = 1$ for every $b \in G''$, $a \in [G', G]$. This means that $G'' \leq Z([G', G])$, namely G'' lies in the center of $[G', G]$. Moreover, for every $b \in G''$ and $x \in G'$ there exists $y \in G$ such that $[b, x^y, x^y] = 1$. From $b, [b, x], [b, x, x] \in G''$ it follows $[b, [x, y]] = 1 = [b, x, [x, y]]$, and using property *2)* of Lemma 1.1.2 we get

$$1 = [b, x^y, x^y] = [b, x, x]^{[x,y]} = [b, x, x].$$

This shows that every element of G'' is a right 2-Engel element of G'. If G'' has no elements of order 2, then G'' lies in the third term of the upper central series of G' by Theorem 1.3.5, and thus G' is nilpotent. □

Lemma 2.4.2. *Let G be a $\mathscr{C}_2$-group. If G'' is an abelian 2-group, then G is nilpotent.*

Proof. Let G be a $\mathscr{C}_2$-group for which G'' is abelian. As G/G'' is a metabelian $\mathscr{C}_2$-group, by Theorem 2.4.2 it is 2-Engel. In particular, we have $[G', G]^3 \leq G''$ and $[G', G, G] \leq G''$. By P. Hall's criterion it suffices to show that G' is nilpotent. In the proof of Lemma 2.4.1 we have already observed that if G'' has no element of order 3, then $G'' \leq Z([G', G])$ and for every $b \in G''$, $x \in G'$ we have $[b, x, x] = 1$. Moreover, from $G'' \leq Z([G', G])$ it follows

$$\gamma_3([G', G]) \leq [G'', [G', G]] = 1,$$

and thus $[G', G]$ is nilpotent of class at most 2. Since $[G', G]^3$ is contained in the 2-group G'', we can deduce that $[G', G]$ is a torsion group, and being also nilpotent, we have $[G', G] = S \times T$, where S is a 3-group and T is a 2-group. Observe that S is a normal subgroup of G. Consequently, we have $[S, G] \leq S$. As $[S, G] \leq [G', G, G] \leq G''$ and G'' is a 2-group, we obtain $[S, G] \leq S \cap G'' = 1$, and therefore $S \leq Z(G)$. Besides, from $T^3 \leq [G', G]^3 \leq G''$ it follows $T \leq G''$, because T is a 2-group. Hence, for every $a \in [G', G]$, $x \in G'$ there exist $s \in S$, $t \in T$ such that $a = st$, and by previous observations we have

$$[a, x, x] = [st, x, x] = [[s, x]^t [t, x], x] = [t, x, x] = 1.$$

This shows that $[a, x, x] = 1$ for every $a \in [G', G]$ and $x \in G'$. Let now x and y be elements of G'. There exists an element $g \in G$ such that $[y^g, x, x] = 1$. Since $[y, g] \in [G', G]$, $x \in G'$ and $[y, x] \in G''$ we have $[y, g, x, x] = 1$ and

$[y,x]^{[y,g]} = [y,x]$, from which it follows

$$\begin{aligned} 1 &= [y^g, x, x] = [y[y,g], x, x] = [[y,x]^{[y,g]}[y,g,x], x] \\ &= [[y,x]^{[y,g]}, x]^{[y,g,x]}[y,g,x,x] = [y,x,x]^{[y,g,x]}, \end{aligned}$$

which implies $[y,x,x] = 1$. This means that G' is a 2-Engel group, and so it is nilpotent. □

Lemma 2.4.3. *Let G be a $\mathscr{C}_2$-group. If G'' is an abelian 3-group, then G is nilpotent.*

Proof. Let G be a $\mathscr{C}_2$-group for which G'' is an abelian 3-group. By Theorem 2.4.2 the quotient G/G'' is a 2-Engel group, and hence it is nilpotent and we have $[G', G, G] \leq G''$. Then by P. Hall nilpotency criterion all that remains is to establish that G' is nilpotent. Firstly, observe that G'' lies in the third term $Z_3([G', G])$ of the upper central series of $[G', G]$. Indeed, for every $a \in [G', G]$, $b \in G''$ there exists $g \in G$ such that $[b, a^g, a^g] = 1$, and from property *2)* of Lemma 1.1.2 it follows

$$1 = [b, a^g, a^g] = [b, a, a],$$

because $[a, g] \in [G', G, G] \leq G''$ and $[b, a], [b, a, a] \in G''$. This shows that every element of G'' is a right 2-Engel element of $[G', G]$. Since G'' does not have elements of even order, by Theorem 1.3.5 we obtain $G'' \leq Z_3([G', G])$. Consequently, for all $b \in G''$ and $a_1, a_2 \in [G', G]$ we have $[b, a_1, a_2] \in Z([G', G])$. Now for every $x_1 \in G'$ there exists an element g of G such that $[b, a_1, a_2, x_1^g, x_1^g] = 1$. From $[b, a_1, a_2] \in Z([G', G])$ it follows $[b, a_1, a_2, x_1], [b, a_1, a_2, x_1, x_1] \in Z([G', G])$, and since $[x_1, g] \in [G', G]$ we get

$$1 = [b, a_1, a_2, x_1^g, x_1^g] = [b, a_1, a_2, x_1, x_1]$$

by property *2)* of Lemma 1.1.2. This means that the element $[b, a_1, a_2]$ is a right 2-Engel element of G'. As $[b, a_1, a_2] \in G''$, which is a 3-group, we have $[b, a_1, a_2] \in Z_3(G')$ by Theorem 1.3.5, and thus modulo $Z_3(G')$ we have $[b, a_1, a_2] = 1$ for every $a_2 \in [G', G]$. It follows $[b, a_1] \in Z([G', G])$ modulo $Z_3(G')$. Proceeding in a similar manner as above, for every $x_2 \in G'$ we obtain $[b, a_1, x_2, x_2] = 1$ modulo $Z_3(G')$, which implies $[b, a_1] \in Z_6(G')$. Similarly, for every $x_3 \in G'$ we have $[b, x_3, x_3] = 1$ modulo $Z_6(G')$, and so $b \in Z_9(G')$. This shows that $G'' \leq Z_9(G')$, from which it follows $G' = Z_{10}(G')$, and hence G' is nilpotent. $\square$

Theorem 2.4.3. *Let G be a $\mathscr{C}_2$-group. If G'' is an abelian group, then G is 2-Engel.*

Proof. Let G be a $\mathscr{C}_2$-group for which G'' is abelian and denote by T the torsion-subgroup of G''. The quotient G''/T is a torsion-free abelian group and $G/T \in \mathscr{C}_2$. Hence G/T is nilpotent by Lemma 2.4.1 and it is 2-Engel by Observation 2.4.1. Since a 2-Engel group is metabelian, it follows $G'' = T$ and thus G'' is a torsion abelian group. Then G'' is the direct product of its primary components and we can write

$$G'' = A \times B \times C,$$

where A is a 2-group, B is a 3-group, and C is a $\{2,3\}'$-group. Now $G''/(A\times B) \simeq C$ has no element of order 2 or 3 and $G/(A \times B) \in \mathscr{C}_2$. Hence $G/(A \times B)$ is 2-Engel by Lemma 2.4.1 and Observation 2.4.1, and so it is metabelian. This means that $G'' = A \times B$. Since $G''/A \simeq B$ is a 3-group and $G''/B \simeq A$ is a 2-group, in light of Lemma 2.4.2 and Lemma 2.4.3 from $G/A, G/B \in \mathscr{C}_2$ it follows that G/A and G/B are both nilpotent. In particular, they are 2-Engel by Observation 2.4.1, and so for every $x, y \in G$ we have $[x,_2 y] \in A \cap B = 1$, which allows us to conclude that also G is 2-Engel. $\square$

We are now in position to extend Theorem 2.4.2 to solvable $\mathscr{C}_2$-groups.

Corollary 2.4.1. *Every solvable $\mathscr{C}_2$-group is 2-Engel.*

Proof. Let G be a solvable $\mathscr{C}_2$-group. Suppose that G is not metabelian. Then the solvability of G assures us that $G''' \lneq G'' \neq 1$. From $G/G''' \in \mathscr{C}_2$ with G''/G''' abelian it follows that G/G''' is 2-Engel by Theorem 2.4.3. Then it is metabelian and we have $G'' = G'''$, a contradiction. Therefore, G is a metabelian $\mathscr{C}_2$-group, and by Theorem 2.4.2 it is 2-Engel. $\square$

2.4.3 Generalization

Let n be an integer greater than 2. We say that a group G is in the class $\mathscr{C}_n$ if for every pair of elements $a, b \in G$ there exists $g \in G$ such that

$$[a^g,_n b] = 1.$$

We have already observed that there are Tarski's monsters G such that for every $a, b \in G$ there exists $g \in G$ for which $[a^g, b] = 1$. These are examples of non-solvable $\mathscr{C}_n$-groups, for every $n \geq 2$. In addition, these groups cannot be n-Engel, otherwise satisfying max they would be nilpotent by Theorem 1.2.3. Therefore, the class $\mathscr{C}_n$ does not coincide with the class of n-Engel groups.

A natural problem is to investigate whether it is possible to generalize Theorem 2.4.2 to the class of all solvable groups when $n > 2$.

Problem 2. *If $n > 2$, then is it true that every solvable $\mathscr{C}_n$-group is n-Engel?*

A partial solution to the problem is given by the next theorem. First we need a technical lemma.

Lemma 2.4.4. *Let G be a finite $\mathscr{C}_n$-group and let H be a subgroup of G such that $H \leq \gamma_{i+1}(G)$, for some integer $i \geq 0$. If for some integer j, with $2 \leq j \leq i+1$, there exists a non-negative integer r for which $H \leq Z_r(\gamma_j(G))$, then there exists a non-negative integer s for which $H \leq Z_s(\gamma_{j-1}(G))$.*

Proof. Let us argue by induction on r. Firstly, assume $r = 1$. For every $a \in H$, $b \in \gamma_{j-1}(G)$ there exists an element g of G such that $[a,_n b^g] = 1$. Since $[b, g] \in [\gamma_{j-1}(G), G] = \gamma_j(G)$ and $a \in H \leq Z(\gamma_j(G))$, by property *2)* of Lemma 1.1.2 we obtain

$$1 = [a,_n b^g] = [a,_n b].$$

This means that every element of H is a right n-Engel element of $\gamma_{j-1}(G)$, and as G is finite we have $H \leq \overline{Z}(\gamma_{j-1}(G))$ by Theorem 1.2.4. It follows the existence of an integer $s \geq 0$ such that $H \leq Z_s(\gamma_{j-1}(G))$.

Now assume $r > 1$. From $H \leq Z_r(\gamma_j(G))$ we get $[H, \gamma_j(G)] \leq Z_{r-1}(\gamma_j(G))$, and by inductive hypothesis we have $[H, \gamma_j(G)] \leq Z_t(\gamma_{j-1}(G))$, for some integer $t \geq 0$. Thus it follows $H \leq Z(\gamma_j(G))$ modulo $Z_t(\gamma_{j-1}(G))$. For every $a \in H$

and $b \in \gamma_{j-1}(G)$ there exists $g \in G$ for which $[a,_n b^g] = 1$. Then $[b, g] \in \gamma_j(G)$, $a \in H \leq Z(\gamma_j(G))$ modulo $Z_t(\gamma_{j-1}(G))$, and using Lemma 1.1.2 we obtain

$$1 = [a,_n b^g] = [a,_n b] \pmod{Z_t(\gamma_{j-1}(G))}.$$

This shows that modulo $Z_t(\gamma_{j-1}(G))$ every element of H is a right n-Engel element of $\gamma_{j-1}(G)$. By Theorem 1.2.4 it follows that there exists an integer $s \geq 0$ such that $H \leq Z_s(\gamma_{j-1}(G))$ modulo $Z_t(\gamma_{j-1}(G))$, and so $H \leq Z_{t+s}(\gamma_{j-1}(G))$. □

Theorem 2.4.4. *Let G be a finite solvable group. If $G \in \mathscr{C}_n$, then G is nilpotent.*

Proof. Use induction on the derived length d of G. If $d \leq 2$, then the result is true by Theorem 2.4.2. Suppose $d > 2$ and let $A = G^{(d-1)}$ be the last non-trivial term of the derived series of G. Then A is abelian and the quotient G/A is nilpotent by induction hypothesis. Hence there exists a positive integer i such that $\gamma_{i+1}(G) \leq A$. Consequently, $\gamma_{i+1}(G)$ is abelian and in particular we have $\gamma_{i+1}(G) \leq Z(\gamma_{i+1}(G))$. It follows from Lemma 2.4.4 the existence of an integer $r \geq 0$ for which $\gamma_{i+1}(G) \leq Z_r(\gamma_i(G))$. Again by Lemma 2.4.4 there exists an integer $s \geq 0$ such that $\gamma_{i+1}(G) \leq Z_s(\gamma_{i-1}(G))$, and applying $i-2$ more times Lemma 2.4.4 we get $\gamma_{i+1}(G) \leq Z_t(G)$, for some non-negative integer t. Then it follows $G \leq Z_{t+i}(G)$, which yields the nilpotency of G. □

Notice that Theorem 1.2.5 ensures that Lemma 2.4.4 is satisfied also in the class of finitely generated solvable groups, and hence in the same way we can show that a finitely generated solvable $\mathscr{C}_n$-group is nilpotent. This result can also be obtained as a corollary of Theorem 2.4.4.

Corollary 2.4.2. *Every finitely generated solvable $\mathscr{C}_n$-group is nilpotent.*

Proof. By contradiction, assume that G is a finitely generated solvable $\mathscr{C}_n$-group which is not nilpotent. It follows from a theorem due to D.J.S. Robinson and B.A.F. Wehrfritz (see [32, p. 477]) that there exists a finite quotient of G which is not nilpotent. This is a contradiction since Theorem 2.4.4 states that a finite solvable $\mathscr{C}_n$-group is nilpotent. □

Being nilpotent, a finitely generated solvable $\mathscr{C}_n$-group is m-Engel, for some non-negative integer m. But the following problem remains open.

Problem 3. *Is it true that every finitely generated solvable $\mathscr{C}_n$-group is n-Engel? At least, is it possible to find a function $f(n)$ such that every finitely generated solvable $\mathscr{C}_n$-group is $f(n)$-Engel?*

Chapter 3

Centralizer-like subsets associated with two-variable words

Let w be a word in two variables and let G be a group. In [24], L.-C. Kappe and P.M. Ratchford introduced for every element g in G certain *centralizer-like subgroups* of G associated with the word w. The terminology is justified by the fact that for $w(x, y) = [x, y]$ these subgroups coincide with the centralizer of g in G. In the following, we consider for every $g \in G$ the centralizer-like subsets

$$W_L^w(g) = \{a \in G \mid w(g, a) = 1\}$$

and

$$W_R^w(g) = \{a \in G \mid w(a, g) = 1\}$$

associated with the word w which have been introduced in Section 2.4. We have already observed that these subsets need not be subgroups in general. In this chapter we examine some sufficient conditions on the group G ensuring that the sets $W_L^w(g)$ and $W_R^w(g)$ are subgroups of G for all g in G. Hereinafter, we investigate whether the sets $W_L^w(g)$ and $W_R^w(g)$ are subgroups for some given commutator words in two variables. The results of this chapter can be found in [22]: it is a joint research project with Professor Luise-Charlotte Kappe. Most of this work was done while I was a visiting scholar in the Department of Mathematical Sciences at the State University of New York at Binghamton from January through May 2012.

3.1 The class $\mathscr{W}_L^w$

Let $w(x, y)$ be a two-variable word and denote by $\mathscr{W}_L^w$ the class of all groups G for which the set $W_L^w(g)$ is a subgroup of G for every $g \in G$. In light of Theorem 2.4.1, we are interested in finding conditions which guarantee that a group belongs to the class $\mathscr{W}_L^w$.

Example 3.1.1. Let $w(x, y) = [y,_n x]$, for some positive integer n. In [24], L.-C. Kappe and P.M. Ratchford proved that if G is a metabelian group, then $W_L^w(g)$ is a subgroup of G for every $g \in G$. We can observe by induction on n that if G is metabelian, then we get

$$[hk,_n g] = [h,_n g]^k [k,_n g],$$

for every $g, h, k \in G$. Obviously the property is true if $n = 1$. Let $n > 1$ and assume that $[hk,_{n-1} g] = [h,_{n-1} g]^k [k,_{n-1} g]$. By Lemma 1.1.3 we have

$$\begin{aligned} [hk,_n g] &= [[hk,_{n-1} g], g] = [[h,_{n-1} g]^k [k,_{n-1} g], g] \\ &= [[h,_{n-1} g]^k, g] [[k,_{n-1} g], g] = [[h,_{n-1} g], g]^k [k,_n g] \\ &= [h,_n g]^k [k,_n g]. \end{aligned}$$

Hence $w(g, hk) = w(g, h)^k w(g, k)$, for every $g, h, k \in G$.

More generally, the following sufficient condition holds.

Theorem 3.1.1. *Let w be a two-variable word and assume that the following two conditions are satisfied:*

i) $w(g, 1) = 1$, for every $g \in G$;

ii) for every $g, h, k \in G$ there exist $c_1, c_2 \in G$ for which

$$w(g, hk) = w(g, h)^{c_1} w(g, k)^{c_2}.$$

Then G belongs to the class $\mathscr{W}_L^w$.

Proof. By condition *i)* the identity is an element of $W_L^w(g)$ for every $g \in G$. Now let $a, b \in W_L^w(g)$. We obtain $w(g,a) = 1 = w(g,b)$, from which it follows $w(g,ab) = 1$, i.e. $ab \in W_L^w(g)$. Moreover, using conditions *i)* and *ii)* we get

$$1 = w(g,1) = w(g,a^{-1}a) = w(g,a^{-1})^c,$$

for some $c \in G$. It follows $w(g,a^{-1}) = 1$, and so we have $a^{-1} \in W_L^w(g)$. Hence $W_L^w(g)$ is a subgroup of G for every $g \in G$. $\square$

Observe that condition *ii)* is satisfied if, for every $g, h, k \in G$, there exist some $d_1, d_2 \in G$ for which

$$w(g,hk) = w(g,k)^{d_1} w(g,h)^{d_2}.$$

Indeed, for all $a, b \in G$ we have $ab = b^{a^{-1}}a$, and so there exists an element $c \in G$ such that $ab = b^c a$. Therefore, if $w(g,hk) = w(g,k)^{d_1} w(g,h)^{d_2}$, then we also have $w(g,hk) = w(g,h)^c w(g,k)^{d_1}$ for some $c \in G$.

The following example shows that the two conditions of Theorem 3.1.1 are not necessary for an arbitrary group to stay in the class $\mathscr{W}_L^w$.

Example 3.1.2. Consider the word $w(x,y) = x^6y^2x^6$ and let G be the semidirect product of C_3 and C_4, namely the cyclic groups respectively of order 3 and 4, the latter group acting on C_3 by inversion. Assume $C_3 = \langle r \rangle$ and $C_4 = \langle s \rangle$. Then the group $G = \langle s \rangle \ltimes \langle r \rangle$ is a non-abelian group of order 12, and we get $r^s = r^{-1}$, $sr = r^2s$ and $sr^2 = rs$. Moreover, the center of G is the subgroup generated by s^2, the only involution of G. Hence for every $g \in G$ we have $w(g,1) = g^{12} = 1$ and the set

$$W_L^w(g) = \{a \in G \mid g^6a^2g^6 = 1\} = \{a \in G \mid a^2 = 1\} = \langle s^2 \rangle$$

is a subgroup of G. Thus $G \in \mathscr{W}_L^w$. However, condition *ii)* of Theorem 3.1.1 is not satisfied because, for example, we have $w(1,s) = s^2$,

$$w(1,r^2s) = (r^2s)^2 = (sr)^2 = s(rs)r = s(sr^2)r = s^2,$$

and

$$w(1, s \cdot r^2 s) = w(1, rss) = (rs^2)^2 = rs^2rs^2 = rs^3r^ss = rs^{-1}r^{-1}s = r(r^s)^{-1} = r^2.$$

Since $Z(G) = \langle s^2 \rangle$ is a normal subgroup of G, the element r^2 cannot be a conjugate of s^2, and so $w(1, s \cdot r^2 s)$ does not belong to the normal closure of the set $\{w(1, s), w(1, r^2 s)\}$.

Assume that there exists a word u in three variables such that for all elements g, h, k in the group G there exist $c_1, c_2 \in G$ for which the following two conditions are satisfied:

i) $w(g, 1) = 1$,

ii) $w(g, hk) = w(g, h)^{c_1} w(g, k)^{c_2} u(g, h, k)$.

If the *residual* word $u(x, y, z)$ is a law in G, then G is in the class $\mathscr{W}_L^w$ by Theorem 3.1.1. This allows us to obtain a method to recognize groups which are in the class $\mathscr{W}_L^w$.

Example 3.1.3. Let G be a group and consider the word

$$w(x, y) = [x, y^2] = [x, y][x, y]^y.$$

Certainly $w(g, 1) = 1$ for all $g \in G$. By commutator expansion, for every g, h, k we have

$$\begin{aligned} w(g, hk) &= [g, hk][g, hk]^{hk} = [g, k][g, h]^k[g, k]^{hk}[g, h]^{khk} \\ &= [g, k][g, k]^k[g, k]^{-k}[g, h]^k[g, h]^{hk}[g, h]^{-hk}[g, k]^{hk}[g, h]^{khk} \\ &= w(g, k)[g, k]^{-k}w(g, h)^k[g, h]^{-hk}[g, k]^{hk}[g, h]^{khk} \\ &= w(g, k)w(g, h)^{k[g,k]^k}[g, k]^{-k}[g, h]^{-hk}[g, k]^{hk}[g, h]^{khk} \\ &= w(g, h)^{[g,k]kw(g,k)^{-1}}w(g, k)u(g, h, k), \end{aligned}$$

where

$$\begin{aligned} u(g,h,k) &= [g,k]^{-k}[g,h]^{-hk}[g,k]^{hk}[g,h]^{khk} \\ &= ([g,k]^{-1}[g,h]^{-h}[g,k]^{h}[g,h]^{kh})^{k} \\ &= ([g,k]^{-1}[g,k]^{h[g,h]^{h}}[g,h]^{-h}[g,h]^{kh})^{k} \\ &= [g,k,[g,h]h]^{k}[g,h,k]^{hk}. \end{aligned}$$

The word u is a law in G if and only if G is a 2-Engel group. Indeed, if G is a 2-Engel group, then it is metabelian and for every g, h, k we have

$$u(g,h,k)^{k^{-1}} = [g,k,h][g,h,k]^{h} = [g,h,k]^{-1}[g,h,k]^{h} = [g,h,k,h] = 1,$$

by property *6)* of Theorem 1.3.6 and property *4)* of Theorem 1.3.1. Hence the identity $u(x,y,z) = 1$ holds in G. Conversely, if $u(x,y,z) = 1$ for all $x,y,z \in G$, then for every $g,h \in G$ we obtain

$$u(g,h,g) = [g,h,g]^{hg} = 1,$$

from which it follows $[g,h,g] = 1$. Then the two-variable word $[x,y,x]$ is a law in G, and thus G is 2-Engel. Therefore, by the previous observations every 2-Engel group belongs to the class $\mathscr{W}_L^w$ when $w(x,y) = [x,y^2]$.

Notice that the word u is not univocally determined, and so there might be groups in $\mathscr{W}_L^w$ for which the word u is not a law.

Now we analyze the behaviour of the class $\mathscr{W}_L^w$ under some closure operations.

Theorem 3.1.2. *Let w be a two-variable word. Then:*

1) if $G \in \mathscr{W}_L^w$ and $H \leq G$, then $H \in \mathscr{W}_L^w$;

2) if $G/N, G/M \in \mathscr{W}_L^w$, then $G/(N \cap M) \in \mathscr{W}_L^w$;

3) if $H, K \in \mathscr{W}_L^w$, then $H \times K \in \mathscr{W}_L^w$.

Proof. *1)* If $G \in \mathscr{W}_L^w$, then for every $h \in H \leq G$ the set

$$W_L^w(h) = \{a \in G \mid w(h,a) = 1\}$$

is a subgroup of G. Therefore $W_L^w(h) \cap H$, namely the set of all elements $a \in H$ for which $w(h, a) = 1$, is a subgroup of H for every $h \in H$. Hence H belongs to $\mathscr{W}_L^w$.

2) Observe that if H is a normal subgroup of G, then $aH \in W_L^w(gH)$ if and only if $w(g, a) \in H$, because $w(gH, aH) = w(g, a)H$. Let N and M normal subgroups of G, and suppose that $G/N, G/M \in \mathscr{W}_L^w$. Then for every $g \in G$ we have that $W_L^w(gN)$ and $W_L^w(gM)$ are subgroups respectively of G/N and G/M. Thus, from $N \in W_L^w(gN)$ and $M \in W_L^w(gM)$ it follows $w(g, 1) \in N \cap M$, which means $N \cap M \in W_L^w(g(N \cap M))$. Now let $a(N \cap M)$ and $b(N \cap M)$ be elements of $W_L^w(g(N \cap M))$. Then $w(g, a), w(g, b) \in N \cap M$, and we obtain $aN, bN \in W_L^w(gN)$ and $aM, bM \in W_L^w(gM)$. Since we get $abN \in W_L^w(gN)$ and $abM \in W_L^w(gM)$, it follows $w(g, ab) \in N \cap M$, which implies that $ab(N \cap M)$ is an element of $W_L^w(g(N \cap M))$. Moreover, if $a(N \cap M) \in W_L^w(g(N \cap M))$, then from $w(g, a) \in N \cap M$ we obtain $aN \in W_L^w(gN)$ and $aM \in W_L^w(gM)$, from which we have $a^{-1}N \in W_L^w(gN)$ and $a^{-1}M \in W_L^w(gM)$. It follows $w(g, a^{-1}) \in N \cap M$, which means $a^{-1}(N \cap M) \in W_L^w(g(N \cap M))$. This shows that $W_L^w(g(N \cap M))$ is a subgroup of $G/(N \cap M)$ for every $g \in G$, and thus $G/(N \cap M) \in \mathscr{W}_L^w$.

3) This property follows immediately from property *2)* when $G = H \times K$. $\square$

The next example shows that the quotient of a $\mathscr{W}_L^w$-group need not be a $\mathscr{W}_L^w$-group.

Example 3.1.4. Let G be the group considered in Example 3.1.2 and let $w(x, y) = x^6y^2x^6$. Then $G = \langle s \rangle \ltimes \langle r \rangle \in \mathscr{W}_L^w$, with $r^s = r^{-1}$, and the center $Z(G) = \langle s^2 \rangle$ has order 2. Denote by N the center of G. As G/N is isomorphic with the symmetric group of degree 3, it contains three involutions. Hence the set

$$W_L^w(N) = \{aN \in G/N \mid w(N, aN) = N\} = \big\{aN \in G/N \mid a^2 \in N\big\}$$

has order 4, and so it is not a subgroup of G/N. Thus $G/N \notin \mathscr{W}_L^w$, although $G \in \mathscr{W}_L^w$.

However, if the two conditions of Theorem 3.1.1 are satisfied, then the class $\mathscr{W}_L^w$ is closed under homomorphic images, and since property *2)* of Theorem 3.1.2 holds, it is a formation of groups (see, for instance, [9]).

Theorem 3.1.3. *Let w be a two-variable word. Every quotient of a $\mathscr{W}_L^w$-group satisfying conditions i) and ii) of Theorem 3.1.1 is a $\mathscr{W}_L^w$-group.*

Proof. Let N be a normal subgroup of a group G which satisfies conditions *i)* and *ii)* of Theorem 3.1.1. Then for every $g \in G$ we have $w(g,1) = 1$, and thus $N \in W_L^w(gN)$. For all $a, b \in G$ there exist c_1, c_2 in G for which

$$w(g, ab) = w(g, a)^{c_1} w(g, b)^{c_2}.$$

Consequently, if $aN, bN \in W_L^w(gN)$, then $w(g,a), w(g,b) \in N$ and we obtain $w(g, ab) \in N$, from which it follows $abN \in W_L^w(gN)$. In addition, for every $aN \in W_L^w(gN)$ we have $w(a,g) \in N$, and so we get

$$1 = w(g,1) = w(g, aa^{-1}) = w(g,a)^{c_1} w(g, a^{-1})^{c_2},$$

for some $c_1, c_2 \in G$. Hence $w(g, a^{-1}) \in N$, and $a^{-1}N \in W_L^w(gN)$. This assures us that $W_L^w(gN)$ is a subgroup of G/N for every $g \in G$. □

If w is a word in two variables, denote by $\mathscr{W}_R^w$ the class of all groups G for which the set $W_R^w(g)$ is a subgroup of G for every $g \in G$. Properties analogous to those given in Theorem 3.1.1 and Theorem 3.1.2 hold for the class $\mathscr{W}_R^w$.

3.2 Some commutator words in two variables

Let G be a group, n an integer greater than 2, and x, y elements of G. In [15], N.D. Gupta considered group laws of the form

$$C_n = [x, y],$$

where C_n is a left-normed commutator of weight n with entries from the set consisting of x, y and their inverses. N.D. Gupta showed that any finite or solvable group satisfying such a law is abelian and he exhibited some examples of commutators C_n for which the restrictions on the structure of G are unnecessary. Clearly if $n \leq 2$ the group G might not be abelian.

L.-C. Kappe and M.J. Tomkinson [25] investigated the case $n = 3$, for which they completely solved the problem. They proved that the variety of groups satisfying one of the laws of the form $C_3 = [x, y]$ is the variety of the abelian groups. Moreover, they raised the following question.

Problem 4. *Is each group satisfying a law of the form $C_n = [x, y]$ abelian when $n > 3$? In case the answer is no, what is the smallest integer n for which one of the laws $C_n = [x, y]$ is not equivalent to the commutative law?*

In [30], P. Moravec extended the result to the case $n = 4$, by proving that also the laws of the form $C_4 = [x, y]$ imply the commutative law.

Now we return to the study of the centralizer-like subsets. Let

$$v(x, y) = C_n[y, x],$$

where C_n is a left-normed commutator of weight $n > 2$ with entries drawn from the set $\{x, y, x^{-1}, y^{-1}\}$.

Problem 5. *Under what conditions a group belongs to the classes $\mathscr{W}_L^v$ and $\mathscr{W}_R^v$ associated with the word $v(x, y) = C_n[y, x]$?*

We say that two words $w_1(x, y)$ and $w_2(x, y)$ are *strongly equivalent* in a group G if for every $g, h \in G$, $w_1(g, h) = 1$ if and only if $w_2(g, h) = 1$. P.M. Ratchford [31] observed, in his Ph.D. thesis, that if G is a nilpotent group, then the word v is strongly equivalent to the simple commutator word $[x, y]$, i.e. for every $g, h \in G$ the value of the word at (g, h) is 1 if and only if the elements g and h commute.

Theorem 3.2.1. *If G is a nilpotent group, then $v(x, y)$ is strongly equivalent to $[x, y]$ in G.*

Proof. Assume

$$v(x, y) = [r_1, r_2 \dots, r_n][y, x],$$

with $r_i \in \{x, y, x^{-1}, y^{-1}\}$, for every $i = 1, \dots, n$. Clearly if two elements g, h commute, then $v(g, h) = 1$. Now let g, h be elements of G for which $v(g, h) = 1$. We will argue by induction on the nilpotency class c of G. If $c < n$, then every commutator of weight n in G is trivial, and so $1 = v(g, h) = [h, g]$. Let $c \geq n$. Since $Z(G) \neq 1$, the quotient $G/Z(G)$ is nilpotent of class at most c, and by induction $[g, h] \in Z(G)$. By commutator identities the word $w(x, y) = [r_1, r_2]$ is conjugate to either $[x, y]$ or $[x, y]^{-1}$. Consequently, $w(g, h) \in Z(G)$. It follows that the value of the word $[r_1, r_2 \ldots, r_n]$ at (g, h) is 1, and so $1 = v(g, h) = [h, g]$. $\square$

Corollary 3.2.1. *Every locally nilpotent group belongs to the classes $\mathscr{W}_L^v$ and $\mathscr{W}_R^v$ associated with the word $v(x, y)$.*

Proof. Let G be a locally nilpotent group. Then for every pair of elements g, h in G the subgroup $\langle g, h\rangle$ is nilpotent, and it follows from the previous result that $v(g, h) = 1$ if and only if g and h commute. Hence, for every $g \in G$ the subset $W_L^v(g) = W_R^v(g)$ is the centralizer of g in G. $\square$

In particular, any 2-Engel, 3-Engel, or 4-Engel group belongs to the classes $\mathscr{W}_L^v$ and $\mathscr{W}_R^v$.

3.2.1 Centralizer-like subsets of words of the form $C_3[y, x]$

We now investigate more closely the centralizer-like subsets associated with two-variable words of the form $C_3[y, x]$. If $n = 3$, then there are thirty-two non-trivial laws of the form

$$[r, s, t] = [x, y],$$

where $r, s, t \in \{x, y, x^{-1}, y^{-1}\}$. Indeed, we can exclude the trivial cases in which $r = s$ or $r = s^{-1}$. For convenience of reference we list the possibilities for the commutator $[r, s, t]$ in the Table 3.1.

Table 3.1

1. $[x, y, y]$	9. $[x^{-1}, y, y]$	17. $[y, x, x]$	25. $[y^{-1}, x, x]$
2. $[x, y, y^{-1}]$	10. $[x^{-1}, y, y^{-1}]$	18. $[y, x, x^{-1}]$	26. $[y^{-1}, x, x^{-1}]$
3. $[x, y^{-1}, y]$	11. $[x^{-1}, y^{-1}, y]$	19. $[y, x^{-1}, x]$	27. $[y^{-1}, x^{-1}, x]$
4. $[x, y^{-1}, y^{-1}]$	12. $[x^{-1}, y^{-1}, y^{-1}]$	20. $[y, x^{-1}, x^{-1}]$	28. $[y^{-1}, x^{-1}, x^{-1}]$
5. $[x, y, x]$	13. $[x^{-1}, y, x]$	21. $[y, x, y]$	29. $[y^{-1}, x, y]$
6. $[x, y, x^{-1}]$	14. $[x^{-1}, y, x^{-1}]$	22. $[y, x, y^{-1}]$	30. $[y^{-1}, x, y^{-1}]$
7. $[x, y^{-1}, x]$	15. $[x^{-1}, y^{-1}, x]$	23. $[y, x^{-1}, y]$	31. $[y^{-1}, x^{-1}, y]$
8. $[x, y^{-1}, x^{-1}]$	16. $[x^{-1}, y^{-1}, x^{-1}]$	24. $[y, x^{-1}, y^{-1}]$	32. $[y^{-1}, x^{-1}, y^{-1}]$

In the following, we refer to the two-variable word $[r, s, t][y, x]$ as $v_i(x, y)$, where the integer i is the number of the list corresponding to the commutator $[r, s, t]$, so that for example $v_1(x, y) = [x, y, y][y, x]$, $v_2(x, y) = [x, y, y^{-1}][y, x]$ and so forth.

L.-C. Kappe and M.J. Tomkinson showed in [25] that six of the thirty-two laws $v_i(x, y) = 1$ directly imply the abelian law, namely six of the thirty-two words $v_i(x, y)$ are strongly equivalent to the commutator word $[x, y]$. Therefore, if $w(x, y)$ is one of these six words, then for every element g in a group G the sets $W_L^w(g)$ and $W_R^w(g)$ are exactly the centralizer of g in G; thus they are subgroups.

Observation 3.2.1. *If $w(x, y) = v_i(x, y)$, for $i \in \{17, 18, 19, 21, 22, 29\}$, then the word $w(x, y)$ is strongly equivalent to $[x, y]$.*

Proof. This follows immediately from the fact that the word $w(x, y)$ is a conjugate of the commutator $[y, x]$. Indeed, we have

$$\begin{aligned}
v_{17}(x, y) &= [y, x, x][y, x] = [y, x]^{-1}[y, x]^x[y, x] = [y, x]^{x[y,x]};\\
v_{18}(x, y) &= [y, x, x^{-1}][y, x] = [y, x]^{-1}[y, x]^{x^{-1}}[y, x] = [y, x]^{x^{-1}[y,x]};\\
v_{19}(x, y) &= [y, x^{-1}, x][y, x] = [y, x^{-1}]^{-1}[y, x^{-1}]^x[y, x]\\
&= [y, x]^{x^{-1}}[y, x]^{-1}[y, x] = [y, x]^{x^{-1}};\\
v_{21}(x, y) &= [y, x, y][y, x] = [y, x]^{-1}[y, x]^y[y, x] = [y, x]^{y[y,x]};\\
v_{22}(x, y) &= [y, x, y^{-1}][y, x] = [y, x]^{-1}[y, x]^{y^{-1}}[y, x] = [y, x]^{y^{-1}[y,x]};\\
v_{29}(x, y) &= [y^{-1}, x, y][y, x] = [y^{-1}, x]^{-1}[y^{-1}, x]^y[y, x]\\
&= [y, x]^{y^{-1}}[y, x]^{-1}[y, x] = [y, x]^{y^{-1}}.
\end{aligned}$$

Hence for every pair of elements g, h of a group G, we have $w(g, h) = 1$ if and only if g and h commute. □

In the other cases the centralizer-like subsets are not subgroups in general.

Example 3.2.1. Consider the word $w(x, y) = v_9(x, y)$. Setting $G = \mathcal{S}_3$, i.e. the symmetric group of degree 3, and $g = (12)$. Then the subsets

$$W_L^w(g) = \left\{a \in G \mid [g^{-1}, a, a] = [g, a]\right\} = \{1, (12), (23), (13)\}$$

and

$$W_R^w(g) = \{a \in G \mid [a^{-1}, g, g] = [a, g]\} = \{1, (12), (23), (13)\}$$

are not subgroups of G.

In addition, some of the conditions $v_i(x, y)$ are strongly equivalent, as our next result shows.

Observation 3.2.2. *The words $v_2(x, y)$ and $v_3(x, y)$ are strongly equivalent, as well as the words $v_6(x, y)$ and $v_{13}(x, y)$.*

Proof. It suffices to observe that we have

$$\begin{aligned}
v_2(x, y) &= [x, y, y^{-1}][y, x] = [x, y]^{-1}[x, y]^{y^{-1}}[y, x] = [y, x][x, y]^{y^{-1}}[y, x]; \\
v_3(x, y) &= [x, y^{-1}, y][y, x] = [x, y^{-1}]^{-1}[x, y^{-1}]^{y}[y, x] \\
&= [x, y]^{y^{-1}}[x, y]^{-1}[y, x] = [x, y]^{y^{-1}}[y, x]^2; \\
v_6(x, y) &= [x, y, x^{-1}][y, x] = [x, y^{-1}]^{-1}[x, y]^{x^{-1}}[y, x] = [y, x][x, y]^{x^{-1}}[y, x]; \\
v_{13}(x, y) &= [x^{-1}, y, x][y, x] = [x^{-1}, y]^{-1}[x^{-1}, y]^{x}[y, x] \\
&= [x, y]^{x^{-1}}[x, y]^{-1}[y, x] = [x, y]^{x^{-1}}[y, x]^2.
\end{aligned}$$

Hence, for every g, h in a group G, we obtain $v_2(g, h) = 1$ if and only if $v_3(g, h) = 1$, and $v_6(g, h) = 1$ if and only if $v_{13}(g, h) = 1$. $\square$

As a consequence, for every element g in a group G we get $W_L^{v_2}(g) = W_L^{v_3}(g)$ and $W_R^{v_2}(g) = W_R^{v_3}(g)$. Similarly, we have $W_L^{v_6}(g) = W_L^{v_{13}}(g)$ and $W_R^{v_6}(g) = W_R^{v_{13}}(g)$.

Now we restrict our attention to metabelian groups. Let G be a metabelian group, and let $r, s, t \in \{x, y, x^{-1}, y^{-1}\}$. By the Jacobi identity the law

$$[r, s, t][s, t, r][t, r, s] = 1$$

holds in G. Hence, by Lemma 1.1.4, if $r = t^\alpha$, with $\alpha \in \{-1, 1\}$, then the two-variable words $[r, s, t]$ and $[t, s, r]$ are equal in G; instead, if $s = t^\alpha$, with $\alpha \in \{-1, 1\}$, then the word $[r, s, t]$ is equal in G to the word $[r, t, s]$. It follows immediately that some of the words $v_i(x, y)$ are equal in metabelian groups. For instance, we have $v_8(x, y) = v_{15}(x, y)$ and $v_{10}(x, y) = v_{11}(x, y)$ as $[x, y^{-1}, x^{-1}] = [x^{-1}, y^{-1}, x]$ and $[x^{-1}, y, y^{-1}] = [x^{-1}, y^{-1}, y]$ in a metabelian group.

It was shown in [31] that a metabelian group lies in the class $\mathscr{W}_L^w$ when $w(x,y) = v_{20}(x,y)$ or $w(x,y)$ is one of the four words of the form

$$[x^\alpha, y, x^\beta][y,x],$$

where $\alpha, \beta \in \{-1, 1\}$.

Theorem 3.2.2. *Let $w(x,y) = v_i(x,y)$, for $i \in \{5, 6, 13, 14, 20\}$. Then any metabelian group belongs to the class $\mathscr{W}_L^w$.*

Proof. Let G be a metabelian group. First observe that if $w(x,y) = v_{20}(x,y)$, then for every $g, h, k \in G$ we get

$$\begin{aligned} w(g,hk) &= [hk, g^{-1}, g^{-1}][hk, g] = [[h, g^{-1}]^k[k, g^{-1}], g^{-1}][h,g]^k[k,g] \\ &= [[h, g^{-1}]^k, g^{-1}][k, g^{-1}, g^{-1}][h,g]^k[k,g] \\ &= [h, g^{-1}, g^{-1}]^k[k, g^{-1}, g^{-1}][h,g]^k[k,g] = w(g,h)^k w(g,k). \end{aligned}$$

Suppose now $w(x,y) = v_i(x,y)$, for $i \in \{5, 6, 13, 14\}$. Then

$$w(x,y) = [x^\alpha, y, x^\beta][y,x],$$

for some $\alpha, \beta \in \{-1, 1\}$, and for every $g, h, k \in G$ we have

$$\begin{aligned} w(g,hk) &= [g^\alpha, hk, g^\beta][hk, g] = [[g^\alpha, k][g^\alpha, h]^k, g^\beta][h,g]^k[k,g] \\ &= [g^\alpha, k, g^\beta][[g^\alpha, h]^k, g^\beta][h,g]^k[k,g] \\ &= [g^\alpha, k, g^\beta][g^\alpha, h, g^\beta]^k[h,g]^k[k,g] = w(g,h)^k w(g,k). \end{aligned}$$

Since $v_i(g,1) = 1$ for every $g \in G$ and for every i, the two conditions of Theorem 3.1.1 are satisfied. □

Observe that the five words considered in Theorem 3.2.2 are not strongly equivalent to the commutative word in a generic group. Hence in these cases the centralizer-like subgroup $W_L^w(g)$ does not coincide with the centralizer $C_G(g)$.

Example 3.2.2. Let $w(x,y) = v_i(x,y)$, for $i \in \{5, 6, 13, 14\}$. Setting $G = \mathcal{S}_3$ and $g = (12)$, we obtain $w(g,a) = [g,a,g][a,g] = [g,a]^g[g,a]^{-2}$ for all $a \in G$, and so we have

$$W_L^w(g) = \left\{a \in G \;\middle|\; [g,a]^g = [g,a]^2\right\}.$$

Since $G' = \langle (123) \rangle$, we get $[g,a]^g = [g,a]^{-1} = [g,a]^2$ for every $a \in G$. Then we obtain $W_L^w(g) = G$, while the centralizer $C_G(g) = \langle g \rangle$ is strictly contained in G. Similarly, if $G = \mathcal{A}_4$, i.e. the alternating group of degree 4, $g = (123)$, and we consider the word $w(x,y) = v_{20}(x,y)$, then the centralizer of g in G has order 3, whereas $W_L^w(g)$ is the whole group G. Indeed, for all $a \in G$ we have

$$\begin{aligned} v_{20}(g,a) &= [a, g^{-1}, g^{-1}][a,g] = [a,g^{-1}]^{-1}[a,g^{-1}]^{g^{-1}}[a,g] \\ &= [a,g]^{g^{-1}}[a,g]^{-g^{-2}}[a,g] = [a,g]^{g^2}[a,g]^{-g}[a,g]. \end{aligned}$$

As the commutator subgroup of G is the Klein four-group and $g = (123)$ does not commute with the products of two transpositions, we get

$$v_{20}(g,a) = [a,g]^{g^2}[a,g]^g[a,g] = 1,$$

from which it follows $W_L^{v_{20}}(g) = G$.

If $v_i(x,y)$ is not one of the five words of Theorem 3.2.2 or one of the six words strongly equivalent to the simple commutator word considered in Observation 3.2.1, then a metabelian group need not belong to the class $\mathscr{W}_L^{v_i}$.

Observation 3.2.3. *A metabelian group is not necessarily a $\mathscr{W}_L^w$-group when $w(x,y) = v_i(x,y)$, for $i \notin \{5,6,13,14,20\} \cup \{17,18,19,21,22,29\}$.*

Proof. We will find counterexamples of metabelian groups which are not in the class $\mathscr{W}_L^w$, when $w(x,y)$ is one of the remaining words $v_i(x,y)$. If $i \neq 32$, then these examples can be found considering the symmetric group of degree 3 or the alternating group of degree 4. First let $G = \mathcal{S}_3$ and $g = (12)$. If $w(x,y) = v_i(x,y)$, for $i \in \{1,2,3,4,9,10,11,12\}$, then for all $a \in G$ we have

$$w(g,a) = [g, a^\alpha, a^\beta][a,g],$$

where $\alpha, \beta \in \{-1,1\}$. Since $G' = \langle (123) \rangle$ is abelian, for every 3-cycle a in G we get $[[g,a^\alpha], a^\beta] = 1$ and $w(g,a) = [g,a^\alpha,a^\beta][a,g] = [a,g] \neq 1$. Instead, if $a^2 = 1$, then we have

$$w(g,a) = [g,a,a][a,g] = [g,a]^{-1}[g,a]^a[g,a]^{-1} = [g,a]^a[g,a]^{-2},$$

and as $[g,a] \in \langle (123) \rangle$, we obtain $[g,a]^a = [g,a]^{-1} = [g,a]^2$ and $w(g,a) = 1$. Hence

$$W_L^w(g) = \{1, (12), (23), (13)\}$$

is not a subgroup of G.

Let now $w(x,y) = v_i(x,y)$, for $i \in \{7, 8, 15, 16\}$. Setting $G = \mathcal{S}_3$ and $g = (12)$, for all $a \in G$ we have $[g,a]^g = [g,a]^{-1}$, and thus

$$w(g,a) = [g,a^{-1},g][a,g] = [g,a^{-1}]^{-1}[g,a^{-1}]^g[a,g] = [g,a^{-1}]^{-2}[a,g].$$

If a is a 3-cycle, then we have $[g,a^{-1}]^{-1} = [g,a]^{a^{-1}} = [g,a]$, which implies

$$w(g,a) = [g,a^{-1}]^{-2}[a,g] = [g,a]^2[a,g] = [g,a] \neq 1,$$

while if $a^2 = 1$, then from $[a,g] \in \langle (123) \rangle$ it follows

$$w(g,a) = [g,a^{-1}]^{-2}[a,g] = [g,a]^{-2}[a,g] = [a,g]^3 = 1.$$

Consequently, the centralizer-like subset

$$W_L^w(g) = \{1, (12), (23), (13)\}$$

is not a subgroup.

If $w(x,y) = v_i(x,y)$, for $i \in \{25, 26, 27, 28\}$, considering again $G = \mathcal{S}_3$ and $g = (12)$, for every $a \in G$ we obtain

$$w(g,a) = [a^{-1},g,g][a,g] = [a^{-1},g]^{-1}[a^{-1},g]^g[a,g] = [a^{-1},g]^{-2}[a,g]$$

because $[a^{-1},g]^g = [a^{-1},g]^{-1}$. For every $a \in G' = \langle (123) \rangle$ we have $[a^{-1},g]^{-1} = [a,g]^{a^{-1}} = [a,g]$, and so we get

$$w(g,a) = [a^{-1},g]^{-2}[a,g] = [a,g]^2[a,g] = [a,g]^3 = 1.$$

Certainly $w(g,g)=1$, whereas if a is a 2-cycle in $G \setminus \{g\}$, then $[a^{-1},g]^{-1} = [a,g]^{-1}$ and it follows

$$w(g,a) = [a^{-1},g]^{-2}[a,g] = [a,g]^{-2}[a,g] = [a,g] \neq 1.$$

Therefore, we obtain

$$W_L^w(g) = \{1, (12), (123), (132)\}.$$

Consider now the group $G = \mathcal{A}_4$. Let g be the element (123) and $w(x,y) = v_{23}(x,y) = [y, x^{-1}, y][y,x]$. For all elements a in the commutator subgroup V_4 of G we have $[[a,g^{-1}],a]=1$, and so if $a \neq 1$ we obtain

$$w(g,a) = [a, g^{-1}, a][a,g] = [a,g] \neq 1.$$

Observe that if $a = (124)$, then

$$\begin{aligned} w(g,a) &= [a, g^{-1}, a][a,g] = [(124),(132),(124)][(124),(123)] \\ &= [(13)(24),(124)](12)(34) = ((12)(34))^2 = 1. \end{aligned}$$

Clearly also g and g^{-1} are elements of $W_L^w(g)$. Since G has no proper subgroups of order greater than 4, $W_L^w(g)$ cannot be a subgroup.

Similarly, if $w(x,y) = v_{24}(x,y) = [y, x^{-1}, y^{-1}][y,x]$, setting $G = \mathcal{A}_4$ and $g = (123)$, for all $a \in V_4 \setminus \{1\}$ we have $[[a,g^{-1}],a^{-1}] = 1$ and

$$w(g,a) = [a, g^{-1}, a^{-1}][a,g] = [a,g] \neq 1.$$

If $a = (134)$, then

$$\begin{aligned} w(g,a) &= [a, g^{-1}, a^{-1}][a,g] = [(134),(132),(143)][(134),(123)] \\ &= [(13)(24),(143)](12)(34) = ((12)(34))^2 = 1. \end{aligned}$$

Being a proper subset of G which contains at least 4 elements, the centralizer-like subset $W_L^w(g)$ is not a subgroup.

We have already observed that the law $[y, x^{-1}, y^{-1}] = [y^{-1}, x^{-1}, y]$ holds in every metabelian group. It follows that the words $v_{24}(x, y)$ and $v_{31}(x, y)$ are equal in a metabelian group, and so $\mathcal{A}_4$ provides a counterexample also for the word $v_{31}(x, y)$.

Let now $w(x, y) = v_{30}(x, y) = [y^{-1}, x, y^{-1}][y, x]$. Considering again $G = \mathcal{A}_4$ and $g = (123)$, then for all $a \in V_4$ we have $[[a^{-1}, g], a^{-1}] = 1$ and if $a \neq 1$ we obtain

$$w(g, a) = [a^{-1}, g, a^{-1}][a, g] = [a, g] \neq 1.$$

Instead, if a is a 3-cycle, then we can observe that $[a, g]^{a^2}[a, g]^a[a, g] = 1$. Indeed, as $G' = V_4$ is the Klein four-group and a does not commute with the non-trivial elements of V_4, the elements $[a, g]$, $[a, g]^a$ and $[a, g]^{a^2}$ are the three products of transpositions. Hence we have

$$\begin{aligned} w(g, a) &= [a^{-1}, g, a^{-1}][a, g] = [a^{-1}, g]^{-1}[a^{-1}, g]^{a^{-1}}[a, g] \\ &= [a, g]^{a^{-1}}[a, g]^{-a^{-2}}[a, g] = [a, g]^{a^2}[a, g]^a[a, g] = 1, \end{aligned}$$

and thus $W_L^w(g)$ does not contain the elements of G which have order 2, but it contains all the 3-cycles. In particular, it is not a subgroup of G.

Finally, we give a detailed description of the counterexample of minimal order for $i = 32$. This group has been found with the help of GAP [12]. Consider the word $w(x, y) = v_{32}(x, y)$, and let

$$G = \left\langle c, e_1, e_2, e_3 \;\middle|\; \begin{array}{l} e_1^2 = e_2^2 = e_3^2 = c^7 = 1 = [e_i, e_j], 1 \leq i < j \leq 3, \\ e_1^c = e_2, e_2^c = e_3, e_3^c = e_1 e_2 \end{array} \right\rangle.$$

The group G can be seen as the semidirect product $N \rtimes \langle c \rangle$, where N is an elementary abelian 2-group of rank 3 generated by the elements e_1, e_2 and e_3, $\langle c \rangle$ is the cyclic group of order 7, and c induces an automorphism on the group N given by $e_1^c = e_2$, $e_2^c = e_3$ and $e_3^c = e_1 e_2$. G is a metabelian group of order 56 in which the equalities $e_1 c = c e_2$, $e_2 c = c e_3$, $e_3 c = c e_1 e_2$, and $c e_1 = e_3 c e_2$ hold. Let $g = c$ and $a = e_2 c$. Then we obtain $a^{-1} = e_3 c^6$, because

$$e_3 c^6 \cdot e_2 c = e_3 c^6 c e_3 = e_3^2 = 1.$$

A straightforward computation shows that

$$\begin{aligned}
w(g,a) &= [a^{-1},g^{-1},a^{-1}][a,g] = [e_3c^6,c^6,e_3c^6][e_2c,c] \\
&= [(e_3c^6)^{-1}c^{-6}e_3c^6c^6,e_3c^6](e_2c)^{-1}c^{-1}e_2cc = [e_2cce_3c^5,e_3c^6]e_3c^6c^{-1}e_2c^2 \\
&= [e_2ce_2cc^5,e_3c^6]e_3c^5ce_3c = [e_2e_1cc^6,e_3c^6]e_3c^6ce_1e_2 = [e_1e_2,e_3c^6]e_1e_2e_3 \\
&= (e_1e_2)^{-1}(e_3c^6)^{-1}e_1e_2e_3c^6e_1e_2e_3 = e_1e_2e_2ce_1e_2e_3c^6e_1e_2e_3 \\
&= e_1e_3ce_2e_2e_3c^6e_1e_2e_3 = e_1e_3ce_3c^6e_1e_2e_3 = e_1e_3e_2cc^6e_1e_2e_3 \\
&= e_1e_3e_2e_1e_2e_3 = 1
\end{aligned}$$

and

$$\begin{aligned}
w(g,a^{-1}) &= [a,g^{-1},a][a^{-1},g] = [e_2c,c^6,e_2c][e_3c^6,c] \\
&= [(e_2c)^{-1}c^{-6}e_2cc^6,e_2c](e_3c^6)^{-1}c^{-1}e_3c^6c = [e_3c^6ce_2cc^6,e_2c]e_2cc^6e_3 \\
&= [e_3e_2,e_2c]e_2e_3 = (e_3e_2)^{-1}(e_2c)^{-1}e_3e_2e_2ce_2e_3 \\
&= e_3e_2e_3c^6e_3ce_2e_3 = e_2c^6ce_1e_2e_2e_3 = e_2e_1e_3 = e_1e_2e_3 \neq 1.
\end{aligned}$$

It follows $a \in W_L^w(g)$, while $a^{-1} \notin W_L^w(g)$. Thus G does not belong to the class $\mathscr{W}_L^w$. □

Observe that the group G of order 56 considered in the proof of the previous Observation has a presentation similar to that of the alternating group of degree 4, namely

$$\mathcal{A}_4 = \left\langle c, e_1, e_2 \;\middle|\; \begin{array}{l} e_1^2 = e_2^2 = c^3 = 1 = [e_1,e_2]\,, \\ e_1^c = e_2, e_2^c = e_1e_2 \end{array} \right\rangle.$$

It was shown in [8] that a group of the form

$$G(p,n,k) = N \rtimes \langle c \rangle\,,$$

where N is an elementary abelian p-group of rank n and $\langle c \rangle$ is the cyclic group of order k, which operates faithful and irreducibly on N, is uniquely determined up to the choice of the parameters p, n and k, and the possible values for k are the divisors of $p^n - 1$ which are not divisors of $p^i - 1$, for $i < n$. In particular, the groups $\mathcal{A}_4$ and G coincide respectively with the group $G(2,2,3)$ and $G(2,3,7)$.

Symmetry of the centralizer-like subsets in metabelian groups

As regards the centralizer-like subsets $W_R^{v_i}(g)$ of a metabelian group G, with $g \in G$, the next property holds:

Theorem 3.2.3. *Consider one of the words $v_i(x,y) = [r,s,t][y,x]$. If G is a metabelian group, then for every $g \in G$ we have*

$$W_R^{v_i}(g) = W_L^{\overline{v}_i}(g)$$

and

$$W_L^{v_i}(g) = W_R^{\overline{v}_i}(g),$$

where $\overline{v}_i(y,x) = [s,r,t][x,y]$.

Proof. Since G is a metabelian group, by Lemma 1.1.4 we get

$$v_i(x,y) = 1 \Leftrightarrow [x,y] = [r,s,t] \Leftrightarrow [y,x] = [s,r,t] \Leftrightarrow \overline{v}_i(y,x) = 1.$$

Hence for every $a \in G$ we have that $v_i(a,g) = 1$ if and only if $\overline{v}_i(g,a) = 1$, and so $W_R^{v_i}(g) = W_L^{\overline{v}_i}(g)$. Clearly $\overline{\overline{v}}_i(x,y) = v_i(x,y)$, and so we have also $W_L^{v_i}(g) = W_R^{\overline{v}_i}(g)$. □

Therefore, in metabelian groups there is a kind of symmetry of the centralizer-like subsets associated with the words v_i and $\overline{v}_i$: for example, if $i = 1$, then we obtain $v_1(x,y) = [x,y,y][y,x]$ and $\overline{v}_1(y,x) = [y,x,y][x,y]$; thus $\overline{v}_1 = v_5$, and it follows $W_L^{v_1}(g) = W_R^{v_5}(g)$ and $W_R^{v_1}(g) = W_L^{v_5}(g)$ for every element $g \in G$. Bearing in mind that some of the words v_i coincide in metabelian groups, in Table 3.2 we list the pairs of words $(v_i, \overline{v}_i)$, or $(\overline{v}_i, v_i)$, for which this type of symmetry holds.

Observe that from Theorem 3.2.2 it follows that if $w(x,y) = v_i(x,y)$, for $i \in \{1,2,3,4,30\}$, then any metabelian group belongs to $\mathscr{W}_R^{w}$. In light of Observation 3.2.1 and Theorem 3.2.2, a metabelian group belongs to the class $\mathscr{W}_L^{v_i}$ or to the class $\mathscr{W}_R^{v_i}$ for sixteen of the thirty-two words v_i; hence by Theorem 2.4.1 we can state that Problem 1 has an affirmative answer for at least sixteen of the thirty-two conditions.

Table 3.2

v_i	$\overline{v}_i$
v_1	v_5
$v_2 = v_3$	$v_6 = v_{13}$
v_4	v_{14}
v_9	v_7
$v_{10} = v_{11}$	$v_8 = v_{15}$
v_{12}	v_{16}
v_{17}	v_{21}
$v_{18} = v_{19}$	$v_{22} = v_{29}$
v_{20}	v_{30}
v_{25}	v_{23}
$v_{26} = v_{27}$	$v_{24} = v_{31}$
v_{28}	v_{32}

Turning our attention to a generic group, the following property of the words v_i can be proved similarly to the case of metabelian groups.

Theorem 3.2.4. *Consider one of the words $v_i(x,y) = [r,s,t][y,x]$. Then for every element g in a group G we have*

$$W_R^{v_i}(g) = W_L^{\overline{v}_i}(g)$$

and

$$W_L^{v_i}(g) = W_R^{\overline{v}_i}(g),$$

where $\overline{v}_i(y,x) = [r,s,t]^{-1}[x,y]$.

Note that if G is a generic group, then the word $\overline{v}_i(y,x) = [r,s,t]^{-1}[x,y]$ is not in general one of the thirty-two of the form $C_3[y,x]$. However, the symmetry which holds in metabelian groups can be extended to a generic group for some of the words $v_i(x,y)$.

Theorem 3.2.5. *Let G be a group, and $g \in G$. Then the following equalities hold:*

1) $W_L^{v_1}(g) = W_R^{v_5}(g)$ and $W_R^{v_1}(g) = W_L^{v_5}(g)$;

2) $W_L^{v_i}(g) = W_R^{v_j}(g)$ and $W_R^{v_i}(g) = W_L^{v_j}(g)$, for $i \in \{2,3\}$, $j \in \{6,13\}$;

3) $W_L^{v_i}(g) = W_R^{v_j}(g)$, *for* $i, j \in \{17, 18, 19, 21, 22, 29\}$.

Proof. First observe that we have

$$\begin{aligned} v_1(x,y) &= [x,y,y][y,x] = [[y,x]^{-1}, y][y,x] \\ &= [y,x,y]^{-[y,x]^{-1}}[y,x] = [y,x][y,x,y]^{-1} \\ &= [x,y]^{-1}[y,x,y]^{-1} = v_5(y,x)^{-1} \end{aligned}$$

and

$$\begin{aligned} v_2(x,y) &= [x,y,y^{-1}][y,x] = [[y,x]^{-1}, y^{-1}][y,x] \\ &= [y,x,y^{-1}]^{-[y,x]^{-1}}[y,x] = [y,x][y,x,y^{-1}]^{-1} \\ &= [x,y]^{-1}[y,x,y^{-1}]^{-1} = v_6(y,x)^{-1}. \end{aligned}$$

This means that, for every $g, h \in G$, $v_1(g,h) = 1$ if and only if $v_5(h,g) = 1$, and $v_2(g,h) = 1$ if and only if $v_6(h,g) = 1$. Hence we obtain the equalities of property *1)*, and since by Observation 3.2.2 the words $v_2(x,y)$ and $v_3(x,y)$ are strongly equivalent, as well as the words $v_6(x,y)$ and $v_{13}(x,y)$, also property *2)* holds. Finally, the equalities of property *3)* are true by Observation 3.2.1. □

The class $\mathscr{X}$

Denote by $\mathscr{X}$ the variety of all groups which satisfy the law

$$[x,y,y]^2 = 1.$$

$\mathscr{X}$ contains every 2-Engel group, as well as the alternating group $\mathcal{A}_4$ and the metabelian group of order 56 considered in the proof of Observation 3.2.3. But it also contains some non-metabelian group: there are examples of $\mathscr{X}$-groups G in which $\gamma_3(G)$ has exponent 2 and G' is not abelian.

Theorem 3.2.6. *Let G be a group which belongs to the class $\mathscr{X}$. If $w(x,y) = v_i(x,y)$, for $i \in \{1,2,3,5,6,13\}$, then the word $w(x,y)$ is strongly equivalent to $[x,y]$ in G.*

Proof. Note that since

$$[x,y,x]^2 = [[y,x]^{-1},x]^2 = ([y,x,x]^2)^{-[y,x]} = 1,$$

in every $\mathscr{X}$-group G the laws $[x,y,y]^{-1} = [x,y,y]$ and $[x,y,x]^{-1} = [x,y,x]$ hold. It follows

$$\begin{aligned}
v_1(x,y) &= [x,y,y][y,x] = [x,y,y]^{-1}[y,x] = ([x,y]^{-1}[x,y]^y)^{-1}[y,x] = [x,y]^{-y};\\
v_2(x,y) &= [x,y,y^{-1}][y,x] = [x,y,y]^{-y^{-1}}[y,x] = [x,y,y]^{y^{-1}}[y,x]\\
&= ([x,y]^{-1}[x,y]^y)^{y^{-1}}[y,x] = [x,y]^{-y^{-1}};\\
v_5(x,y) &= [x,y,x][y,x] = [x,y,x]^{-1}[y,x] = ([x,y]^{-1}[x,y]^x)^{-1}[y,x] = [x,y]^{-x};\\
v_6(x,y) &= [x,y,x^{-1}][y,x] = [x,y,x]^{-x^{-1}}[y,x] = [x,y,x]^{x^{-1}}[y,x]\\
&= ([x,y]^{-1}[x,y]^x)^{x^{-1}}[y,x] = [x,y]^{-x^{-1}}.
\end{aligned}$$

Bearing in mind that by Observation 3.2.2 the word $v_3(x,y)$ is strongly equivalent to the word $v_2(x,y)$ and $v_{13}(x,y)$ is strongly equivalent to $v_6(x,y)$, we obtain that the word $v_i(x,y)$ is strongly equivalent to the commutator word in G, for each $i \in \{1,2,3,5,6,13\}$. □

Consequently, if $w(x,y) = v_i(x,y)$, for $i \in \{1,2,3,5,6,13\}$, then for every element g in a $\mathscr{X}$-group G the subset $W_L^{v_i}(g) = W_R^{v_i}(g)$ is the centralizer of g in G; so G belongs to the classes $\mathscr{W}_L^w$ and $\mathscr{W}_R^w$. Remember that the six words considered in Observation 3.2.1 are strongly equivalent to the commutative law in every group. Except for these twelve words, the centralizer-like subsets $W_L^{v_i}(g)$ and $W_R^{v_i}(g)$ associated with the remaining words $v_i(x,y)$ in a $\mathscr{X}$-group G do not coincide with the centralizer of g in G in general. However, we do not have counterexamples of $\mathscr{X}$-groups which do not belong to the class $\mathscr{W}_L^{v_i}$, for $i \in \{14,20\}$, nor counterexamples of $\mathscr{X}$-groups which are not in $\mathscr{W}_R^{v_i}$, for $i \in \{4,30\}$.

3.2.2 Centralizer-like subsets of words of the form $C_n[y,x]$ in metabelian groups

To conclude the section, we investigate the centralizer-like subsets associated with a two-variable word of the form $C_n[y,x]$, with n greater than 3, in a metabelian group. In particular, we can generalize some of the properties which hold in the case $n=3$.

Theorem 3.2.7. *Let $w(x,y)$ be one of the 2^{n-1} conditions of the form*

$$[y, x^{\alpha_1}, x^{\alpha_2}, \ldots, x^{\alpha_{n-1}}][y,x]$$

or one of the 2^{n-1} conditions of the form

$$[x^{\alpha_1}, y, x^{\alpha_2}, \ldots, x^{\alpha_{n-1}}][y,x],$$

where $\alpha_i \in \{-1,1\}$ for every $i=1,\ldots,n-1$. Then any metabelian group belongs to the class $\mathscr{W}_L^w$.

Proof. We will prove that for every $g,h,k \in G$,

$$w(g,hk) = w(g,h)^k w(g,k).$$

Since $w(g,1)=1$ for every $g\in G$, the result will follow from Theorem 3.1.1. Let $w(x,y)=[y,x^{\alpha_1},\ldots,x^{\alpha_{n-1}}][y,x]$. For every $g,h,k\in G$ we can observe, by applying induction on n, that

$$[hk, g^{\alpha_1}, \ldots, g^{\alpha_{n-1}}] = [h, g^{\alpha_1}, \ldots, g^{\alpha_{n-1}}]^k [k, g^{\alpha_1}, \ldots, g^{\alpha_{n-1}}].$$

For $n=3$ the property is true by Observation 3.2.1 and Theorem 3.2.2. If $n>3$, then by property *2)* of Lemma 1.1.3 we get

$$\begin{aligned}
[hk, g^{\alpha_1}, \ldots, g^{\alpha_{n-1}}] &= [[h, g^{\alpha_1}, \ldots, g^{\alpha_{n-2}}]^k [k, g^{\alpha_1}, \ldots, g^{\alpha_{n-2}}], g^{\alpha_{n-1}}] \\
&= [[h, g^{\alpha_1}, \ldots, g^{\alpha_{n-2}}]^k, g^{\alpha_{n-1}}][k, g^{\alpha_1}, \ldots, g^{\alpha_{n-1}}] \\
&= [h, g^{\alpha_1}, \ldots, g^{\alpha_{n-1}}]^k [k, g^{\alpha_1}, \ldots, g^{\alpha_{n-1}}],
\end{aligned}$$

from which it follows

$$\begin{aligned} w(g,hk) &= [hk, g^{\alpha_1}, \ldots, g^{\alpha_{n-1}}][hk, g] \\ &= [h, g^{\alpha_1}, \ldots, g^{\alpha_{n-1}}]^k [k, g^{\alpha_1}, \ldots, g^{\alpha_{n-1}}][h, g]^k [k, g] \\ &= w(g,h)^k w(g,k) \end{aligned}$$

as G is metabelian. Suppose now $w(x,y) = [x^{\alpha_1}, y, x^{\alpha_2}, \ldots, x^{\alpha_{n-1}}][y,x]$. Then we can show by induction that for every $g,h,k \in G$ we have

$$[g^{\alpha_1}, hk, g^{\alpha_2}, \ldots, g^{\alpha_{n-1}}] = [g^{\alpha_1}, h, g^{\alpha_2}, \ldots, g^{\alpha_{n-1}}]^k [g^{\alpha_1}, k, g^{\alpha_2}, \ldots, g^{\alpha_{n-1}}].$$

If $n = 3$, then the property follows from Theorem 3.2.2. If $n > 3$, then by property *2)* of Lemma 1.1.3 we obtain

$$\begin{aligned} [g^{\alpha_1}, hk, g^{\alpha_2}, \ldots, g^{\alpha_{n-1}}] &= [[g^{\alpha_1}, h, g^{\alpha_2}, \ldots, g^{\alpha_{n-2}}]^k [g^{\alpha_1}, h, g^{\alpha_2}, \ldots, g^{\alpha_{n-2}}], g^{\alpha_{n-1}}] \\ &= [[g^{\alpha_1}, h, g^{\alpha_2}, \ldots, g^{\alpha_{n-2}}]^k, g^{\alpha_{n-1}}][g^{\alpha_1}, k, g^{\alpha_2}, \ldots, g^{\alpha_{n-1}}] \\ &= [g^{\alpha_1}, h, g^{\alpha_2}, \ldots, g^{\alpha_{n-1}}]^k [g^{\alpha_1}, k, g^{\alpha_2}, \ldots, g^{\alpha_{n-1}}]; \end{aligned}$$

thus

$$\begin{aligned} w(g,hk) &= [g^{\alpha_1}, hk, g^{\alpha_2}, \ldots, g^{\alpha_{n-1}}][hk, g] \\ &= [g^{\alpha_1}, h, g^{\alpha_2}, \ldots, g^{\alpha_{n-1}}]^k [g^{\alpha_1}, k, g^{\alpha_2}, \ldots, g^{\alpha_{n-1}}][h, g]^k [k, g] \\ &= w(g,h)^k w(g,k). \end{aligned}$$

□

In the case $n = 4$, if $w(x,y) = C_4[y,x]$ is not one of the sixteen words considered in the previous Theorem, a metabelian group does not necessarily belong to the class $\mathscr{W}_L^w$: counterexamples can be found considering the symmetric group $\mathcal{S}_3$, the alternating group $\mathcal{A}_4$ or the dihedral group of order 10.

Furthermore, in light of Lemma 1.1.4, the symmetry of the centralizer-like subsets associated with the words $w(x,y) = C_n[y,x]$, which holds in metabelian groups for $n = 3$, continue to hold when $n > 3$.

Theorem 3.2.8. *Let* $w(x,y) = [r_1, r_2, r_3, \ldots, r_n][y,x]$, *with* $r_i \in \{x, y, x^{-1}, y^{-1}\}$

for every $i = 1, \dots, n$. If G is a metabelian group, then for every $g \in G$ we have

$$W_R^w(g) = W_L^{\overline{w}}(g)$$

and

$$W_L^w(g) = W_R^{\overline{w}}(g),$$

where $\overline{w}(y, x) = [r_2, r_1, r_3, \dots, r_n][x, y]$.

3.3 On the n-commutator word

Let n be an integer. In [1], R. Baer termed two elements g and h in a group *n-commutative* if

$$(gh)^n = g^n h^n \quad \text{and} \quad (hg)^n = h^n g^n,$$

and he defined the *n-center* $Z(G, n)$ of a group G as the set of all elements of G which n-commute with every element in the group. Moreover, a group G is said to be *n-abelian* if it coincides with its n-center. Obviously, every group is 0-abelian and 1-abelian. $Z(G, 2)$ is exactly the center of G, and it is contained in the n-center of G for any integer n.

The n-center $Z(G, n)$ is a characteristic subgroup of G which shares many properties with the center of G: for instance, if $G/Z(G, n)$ is cyclic then the group G is n-abelian, just as a group with a cyclic central quotient is abelian. For further properties on the n-center see [23].

In this section we consider the word

$$w(x, y) = (xy)^n y^{-n} x^{-n},$$

where n is an integer. We first observe that w is a commutator word. Since for all elements x, y in a group we have $xy = yx[x, y]$, then

$$x^2 y^2 = xxyy = xyx[x, y]y = (xy)^2 [x, y][x, y, y].$$

For every $n \geq 2$ the following *Collection Formula* of Philip Hall (also called the *Hall-Petrescu formula*) holds (see [4, Appendix 1]):

Lemma 3.3.1. *If x and y are elements of a group G and $n \geq 2$ is an integer, then*

$$x^n y^n = (xy)^n c_2^{\binom{n}{2}} c_3^{\binom{n}{3}} \cdots c_n^{\binom{n}{n}},$$

where $c_i \in \gamma_i(\langle x, y \rangle)$, for any $i = 2, \ldots, n$.

Observation 3.3.1. *The word $w(x, y) = (xy)^n y^{-n} x^{-n}$ is a commutator word for every integer n.*

Proof. Let F be the free group on x and y. We will prove that $w(x, y) \in F'$ for any n. Certainly, if $n \in \{0, 1\}$ we get $w(x, y) = 1 \in F'$. If $n = -1$ then $w(x, y) = (xy)^{-1} yx = [y, x] \in F'$. Let $n \geq 2$. By the Hall-Petrescu formula we have that $(xy)^n$ and $x^n y^n$ are equal modulo F'. It follows $x^n y^n = (xy)^n c$ for some $c \in F'$, and thus $w(x, y) = c \in F'$. Let now $n \leq -2$ and denote $-n$ by m. We obtain $w(x, y) = (xy)^n y^{-n} x^{-n} = (xy)^{-m} y^m x^m$. As $m \geq 2$, by the Hall-Petrescu formula we get $x^m y^m = (xy)^m c$ for some $c \in F'$. Hence we obtain $y^m x^m = x^m y^m [y^m, x^m] = (xy)^m c[y^m, x^m]$ and $w(x, y) = (xy)^{-m} y^m x^m = c[y^m, x^m] \in F'$. $\square$

The word $w(x, y) = (xy)^n y^{-n} x^{-n}$ is called the *n-commutator word.*

For every element g in a group G we define the *n-centralizer* $C_G(g, n)$ of g in G as the set of all elements of G which n-commute with g. Since for all elements $a \in G$ we have $(ga)^n = g^n a^n$ if and only if $w(g, a) = 1$, and $(ag)^n = a^n g^n$ if and only if $w(a, g) = 1$, with our notation we obtain

$$C_G(g, n) = W_L^w(g) \cap W_R^w(g).$$

Clearly, the 2-centralizer of g coincides with the centralizer $C_G(g)$, while $C_G(g, 0)$ and $C_G(g, 1)$ are the whole group G. Notice that the n-center of a group G is the intersection of the sets $C_G(g, n)$, with $g \in G$. But in general an n-centralizer $C_G(g, n)$ need not be a subgroup, as our next example shows for $n = 3$.

Example 3.3.1. Consider the 3-commutator word $w(x, y) = (xy)^3 y^{-3} x^{-3}$. Let G be the alternating group $\mathcal{A}_4$, $g = (123)$ and $a = (142)$. Since $ga = (123)(142) = (234)$ and $ag = (142)(123) = (143)$, we have $(ga)^3 = 1 = g^3 a^3$ and $(ag)^3 = 1 = a^3 g^3$, and thus $a \in C_G(g, 3)$. Instead, the element a^{-1} does not belong to

$C_G(g,3)$: from $ga^{-1} = (123)(124) = (14)(23)$ and $a^{-1}g = (124)(123) = (13)(24)$ it follows

$$(ga^{-1})^3 = (14)(23) \neq 1 = g^3a^{-3}$$

and

$$(a^{-1}g)^3 = (13)(24) \neq 1 = a^{-3}g^3.$$

This shows that neither the 3-centralizer $C_G(g,3)$ nor the centralizer-like subsets $W_L^w(g)$ and $W_R^w(g)$ are subgroups of G.

R. Baer observed that $Z(G,n) = Z(G,1-n)$ for any integer n. An analogous property holds for the n-centralizers.

Theorem 3.3.1. *Let g be an element of a group G. Then*

$$C_G(g,n) = C_G(g,1-n)$$

for any integer n.

Proof. Let n be an integer. Consider the n-commutator word $w_1(x,y) = (xy)^n y^{-n} x^{-n}$ and the $(1-n)$-commutator word $w_2(x,y) = (xy)^{1-n} y^{n-1} x^{n-1}$. Observe that for every $g, a \in G$, $w_1(g,a) = 1$ if and only if $w_2(a,g) = 1$. Indeed, $(ga)^n = g^n a^n$ if and only if $(ag)^{n-1} = g^{n-1}a^{n-1}$, which holds if and only if

$$(ag)^{1-n} = \left((ag)^{n-1}\right)^{-1} = \left(g^{n-1}a^{n-1}\right)^{-1} = a^{1-n}g^{1-n}.$$

In particular, it follows $W_L^{w_1}(g) = W_R^{w_2}(g)$ and $W_R^{w_1}(g) = W_L^{w_2}(g)$ for every $g \in G$. Hence, we get

$$C_G(g,n) = W_L^{w_1}(g) \cap W_R^{w_1}(g) = W_R^{w_2}(g) \cap W_L^{w_2}(g) = C_G(g,1-n).$$

□

Theorem 3.3.2. *Let $w(x,y) = (xy)^n y^{-n} x^{-n}$. Then $\mathscr{W}_L^w = \mathscr{W}_R^w$.*

Proof. Observe that for all elements g, a in a group G we have $(ga)^n = g^n a^n$ if and only if

$$(a^{-1}g^{-1})^n = (ga)^{-n} = (g^n a^n)^{-1} = a^{-n}g^{-n}.$$

It follows $w(g,a)=1$ if and only if $w(a^{-1},g^{-1})=1$; hence $a\in W_L^w(g)$ if and only if $a^{-1}\in W_R^w(g^{-1})$. In particular, if $W_L^w(g)$ is a subgroup, then $W_L^w(g)=W_R^w(g^{-1})$. Therefore, if $W_L^w(g)$ is a subgroup of G for every $g\in G$, then also $W_R^w(g)$ is a subgroup of G for every $g\in G$, and vice versa. $\square$

It was shown in [23] that

$$Z(G,n)=\{a\in G \mid (ga)^n=g^na^n \ \forall\ g\in G\}=\{a\in G \mid (ag)^n=a^ng^n \ \forall\ g\in G\}\,.$$

A similar argument shows that if the centralizer-like subsets $W_L^w(g)$ and $W_R^w(g)$ are both subgroups of G, then $C_G(g,n)=W_L^w(g)=W_R^w(g)$, namely

$$C_G(g,n)=\{a\in G \mid (ga)^n=g^na^n\}=\{a\in G \mid (ag)^n=a^ng^n\}\,.$$

This property follows from the next result.

Theorem 3.3.3. *Let $w(x,y)$ be the n-commutator word. For every element g in a group G the following properties hold:*

i) if $W_L^w(g)$ is a subgroup of G, then $C_G(g,n)=W_L^w(g)\subseteq W_R^w(g)$;

ii) if $W_R^w(g)$ is a subgroup of G, then $C_G(g,n)=W_R^w(g)\subseteq W_L^w(g)$.

Proof. We have already observed in the proof of Theorem 3.3.2 that if $W_L^w(g)$ is a subgroup, then $W_L^w(g)=W_R^w(g^{-1})$. Let $a\in W_L^w(g)=W_R^w(g^{-1})$. In order to show that $a\in W_R^w(g)$, we will prove that a^n and g^{n-1} commute. Since $g^{-1}\in W_L^w(g)$ we obtain $ag^{-1}\in W_L^w(g)$ and $(gag^{-1})^n=g^n(ag^{-1})^n$, from which it follows

$$ga^ng^{-1}=(gag^{-1})^n=g^n(ag^{-1})^n=g^na^ng^{-n}.$$

Therefore, we get $a^ng^{n-1}=g^{n-1}a^n$. Observing that $(ga)^n=g^na^n$ if and only if $(ag)^{n-1}=g^{n-1}a^{n-1}$, we obtain

$$a^ng^n=a^ng^{n-1}g=g^{n-1}a^ng=g^{n-1}a^{n-1}ag=(ag)^{n-1}ag=(ag)^n,$$

which demonstrates that $a\in W_R^w(g)$. In an analogous way it is possible to show property *ii)*. $\square$

The following example shows that the centralizer-like subsets $W_L^w(g)$ and $W_R^w(g)$ do not necessarily coincide if they are not both subgroups. Actually, we do not have examples in which only one of the two is a subgroup.

Example 3.3.2. Consider the 3-commutator word $w(x,y) = (xy)^3 y^{-3} x^{-3}$. Let $\langle r \rangle$ and $\langle s \rangle$ be the cyclic groups respectively of order 5 and 4, and let $G = \langle r \rangle \rtimes \langle s \rangle$, where $r^s = r^2$. In particular, it follows $rs = sr^2$,

$$r^2 s = r(rs) = rsr^2 = sr^2r^2 = sr^4,$$

and

$$r^3 s = r(r^2 s) = rsr^4 = sr^2r^4 = sr.$$

A straightforward computation shows that

$$\begin{aligned} w(s,r) &= (sr)^3 r^{-3} s^{-3} = s(rs)rs(r^3 s) = s(sr^2)rs(sr) = s^2r^3s^2r \\ &= (r^3)^{s^2} r = (r^6)^s r = r^s r = r^3 \neq 1, \end{aligned}$$

$$\begin{aligned} w(r,s) &= (rs)^3 s^{-3} r^{-3} = (rs)rsrs^2r^2 = (sr^2)rsrs^2r^2 = s(r^3s)rs^2r^2 \\ &= s(sr)rs^2r^2 = s^2r^2s^2r^2 = (r^2)^{s^2} r^2 = (r^4)^s r^2 = r^8 r^2 = 1, \end{aligned}$$

$$\begin{aligned} w(s,r^2s^3) &= (sr^2s^3)^3 (r^2s^3)^{-3} s^{-3} = sr^2s^3sr^2s^3sr^2s^3(s^{-3}r^{-2})^3 s \\ &= s(r^2)^3 s^3 (sr^3)^3 s = srs^3sr^3sr^3sr^3s = (sr^4)sr^3s(r^3s) \\ &= r^2s^2r^3s^2r = r^2(r^3)^{s^2} r = r^2(r^6)^s r = r^2 r^s r = r^5 = 1, \end{aligned}$$

$$\begin{aligned} w(r^2s^3,s) &= (r^2s^3s)^3 s^{-3} (r^2s^3)^{-3} = (r^2)^3 s^{-3} (s^{-3}r^{-2})^3 = rs(sr^3)^3 \\ &= rs^2r^3s(r^3s)r^3 = rs^2r^3s^2r^4 = r(r^3)^{s^2} r^4 = r(r^6)^s r^4 \\ &= rr^s r^4 = r^2 \neq 1. \end{aligned}$$

This assures us that if $g = s$, then $r^2s^3 \in W_L^w(g)$ and $r \in W_R^w(g)$, while $r \notin W_L^w(g)$ and $r^2s^3 \notin W_R^w(g)$. Therefore, the centralizer-like subsets $W_L^w(g)$ and $W_R^w(g)$ are distinct. Moreover, neither of the two is contained in the other: in light of Theorem 3.3.3 the two sets are not subgroups.

The following lemma, which can be found in [20], will be used to prove that a 2-Engel group belongs to $\mathscr{W}_L^w = \mathscr{W}_R^w$, for any n-commutator word $w(x,y)$.

Lemma 3.3.2. *If x and y are elements of a metabelian group and n is an integer, then*

$$(xy^{-1})^n = x^n \left(\prod_{0<i+j<n} [x,_i\, y,_j\, x]^{m_{i,j}} \right) y^{-n},$$

where $m_{i,j} = \binom{n}{i+j+1}$.

Theorem 3.3.4. *Let $w(x,y) = (xy)^n y^{-n} x^{-n}$. If G is a 2-Engel group, then for every $g \in G$ we obtain that $C_G(g,n) = W_L^w(g) = W_R^w(g)$ is a subgroup of G.*

Proof. Let $v(x,y) = (xy^{-1})^n y^n x^{-n}$. Since a 2-Engel group is metabelian, we can use the expansion formula stated in Lemma 3.3.2. We have

$$(xy^{-1})^n = x^n \left(\prod_{0<i+j<n} [x,_i\, y,_j\, x]^{m_{i,j}} \right) y^{-n},$$

where $m_{i,j} = \binom{n}{i+j+1}$. Considering that $[x,_0\, y,_j\, , x] = [x,_j\, x] = 1$ and G is 2-Engel, we can assume $i > 0$ and $i, j < 2$. Moreover, in every 2-Engel group the law $[x,y,x] = 1$ holds. Hence, from $0 < i+j \leq 2$ it follows that the only possible choice for the pair (i,j) is $(1,0)$. Denoting $m_{1,0} = \binom{n}{2}$ by p, in the group G we have

$$(xy^{-1})^n = x^n [x,y]^p y^{-n},$$

and so we get

$$v(x,y) = (xy^{-1})^n y^n x^{-n} = x^n [x,y]^p x^{-n} = ([x,y]^p)^{x^{-n}}.$$

Therefore, for every $g, h, k \in G$ we have $v(g,1) = 1$ and

$$\begin{aligned} v(g,hk) &= ([g,hk]^p)^{g^{-n}} = (([g,k][g,h]^k)^p)^{g^{-n}} = ([g,k]^p)^{g^{-n}} ([g,h]^p)^{kg^{-n}} \\ &= v(g,k) v(g,h)^{g^n k g^{-n}} = v(g,h)^{g^n k g^{-n}} v(g,k)^{v(g,h)^{g^n k g^{-n}}}. \end{aligned}$$

Then G belongs to $\mathscr{W}_L^v$ by Theorem 3.1.1.

For all elements $a, g \in G$ we have $v(g,a) = w(g,a^{-1})$, and thus $a \in W_L^v(g)$ if and only if $a^{-1} \in W_L^w(g)$. As $W_L^v(g)$ is a subgroup of G for every element

$g \in G$, we get $W_L^w(g) = W_L^v(g)$. It follows $G \in \mathscr{W}_L^w$, and by Theorem 3.3.2 and Theorem 3.3.3 we obtain

$$C_G(g, n) = W_L^w(g) = W_R^w(g)$$

for any $g \in G$. $\square$

In particular, it follows from the proof that the word $w(x, y) = (xy)^n y^{-n} x^{-n}$ is strongly equivalent to the word $[x, y]^{\binom{n}{2}}$ in a 2-Engel group.

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