UNIVERSITÀ DEGLI STUDI DI SALERNO

DIPARTIMENTO DI FISICA “E.R. CAIANIELLO”

DOTTORATO DI RICERCA IN FISICA

TESI DI DOTTORATO IN

CONSTRAING MODELS OF EXTENDED THEORIES OF GRAVITY WITH TERRESTRIAL AND ASTROPHYSICAL EXPERIMENTS

ANTONIO STABILE

Tutor

PROF GAETANO LAMBIASE

COORDINATORE

PROF CANIO NOCE

XIII CICLO – Nuova Serie

A.A. 2013/2014

To my family

né dolcezza di figlio, né la pieta
del vecchio padre, né 'l debito amore
lo qual dovea Penelopé far lieta,

vincer potero dentro a me l'ardore
ch'i' ebbi a divenir del mondo esperto,
e de li vizi umani e del valore;

ma misi me per l'alto mare aperto

Dante Alighieri, Inferno XXVI 94-100

Contents

Introduction

The General Relativity (GR) [1] is a theory of gravitation that was developed by Albert Einstein between 1907 and 1915. According to GR, the observed gravitational attraction between masses results from their warping of space and time. Up to the beginning of the 20th century, Newton's law of universal gravitation had been accepted for more than two hundred years as a valid description of the gravitational force between masses. In Newton's model, gravity is the result of an attractive force between massive objects. Although even Newton was bothered by the unknown nature of that force, the basic framework was extremely successful for describing motions. Experiments and observations show that Einstein's description of gravitation accounts for several effects that are unexplained by Newton's law, such as anomalies in the orbits of Mercury and other planets. GR also predicts novel effects of gravity, such as gravitational waves, gravitational lensing and an effect of gravity on time known as gravitational time dilation. Many of these predictions have been confirmed by experiments, while others are the subject of ongoing research. For example, although there is indirect evidence for gravitational waves, direct evidence of their existence is still being sought by several teams of scientists in experiments such as VIRGO and LIGO. GR has developed as an essential tool in modern astrophysics. It provides the foundation for the current understanding of black holes, regions of space where gravitational attraction is so strong that light can not escape. Their strong gravity is thought to be responsible for the intense radiation emitted by certain types of astronomical objects (such as active galactic nuclei or quasars). GR is also part of the framework of the standard Big Bang model of cosmology. Although GR is not the only relativistic theory of gravity, it is the simplest theory that is consistent with the experimental data.

In the last thirty years several shortcomings came out in the Einstein theory and people began to investigate whether GR is the only fundamental theory capable of explaining the gravitational interaction. Such issues come, essentially, from cosmology and quantum field theory. Many people will agree that modern physics is based on two main pillars: GR and Quantum Field Theory. Each of these two theories has been very successful in its own arena of physical phenomena: GR in describing gravitating systems and non-inertial frames from a classical point of view on large enough scales, and Quantum Field Theory at high energy or small scale regimes where a classical description breaks down. However, Quantum Field

Theory assumes that space-time is flat and even its extensions, such as Quantum Field Theory in curved space time, consider space-time as a rigid arena inhabited by quantum fields. GR, on the other hand, does not take into account the quantum nature of matter. Therefore, it comes naturally to ask what happens if a strong gravitational field is present at quantum scales. How do quantum fields behave in the presence of gravity? To what extent are these amazing theories compatible? The main difficulty for a true theory of quantum gravity, is that the gravitational interaction is so weak compared with other interactions that the characteristic scale under which one would expect to experience non-classical effects relevant to gravity, the Planck scale, is 10^{-33} cm. Such a scale is not of course accessible by any current experiment and it is doubtful whether it will ever be accessible to future experiments either. However, if we consider the Big Bang scenario the Universe inevitably goes through an era in which its dimensions are smaller than the Planck scale (Planck era). On the other hand, space-time in GR is a continuum and so in principle all scales are relevant. Therefore, only a theory of quantum gravity may be the right tool for the investigation early Universe. From this perspective, in order to derive conclusions about the nature of space-time one has to answer the question of what happens on very small scales (ultra-violet scales) and very large scales (infra-red scales).

In this scenario we have the Extended Theories of Gravity. These are theories describing gravity, which are metric theory, "a linear connection" or related affine theories, or metric-affine gravitation theory. Rather than trying to discover correct calculations for the matter side of the Einstein field equations; which include inflation, dark energy, dark matter, large-scale structure, and possibly quantum gravity; it is proposed, instead, to change the gravitational side of the equation.

0.1 Extended Theories of Gravity

The study of possible modifications of Einstein's theory of gravitation has a long history which reaches back to the early 1920s [7, 8, 317, 318, 11, 12]. While the proposed early amendments of Einstein's theory were aimed toward the unification of gravity with the other interactions of physics, like electromagnetism and whether GR is the only fundamental theory capable of explaining the gravitational interaction, the recent interest in such modifications comes from cosmological observations (for a comprehensive review, see [13]). Such issues come, essentially, from Cosmology and Quantum Field Theory. In the first case, the presence of the Big Bang singularity, the flatness and horizon problems [323] led to the statement that Cosmological Standard Model, based on GR and Standard Model of Particle Physics, is inadequate to describe the Universe at extreme regimes. These observations usually lead to the introduction of additional *ad-hoc* concepts like dark energy/matter if interpreted within Einstein's theory. On the other hand, the emergence of such stopgaps could be interpreted as a first signal of a breakdown of GR at astrophysical and cosmological scales [15, 16], and led to the proposal of several alternative modifications of the underlying gravity theory (see [17] for a review). Besides from Quantun Field Theory point view, GR is a *classical* theory which does not work as a fundamental theory, when one wants to achieve a full quantum description of spacetime (and then of gravity).

While it is very natural to extend Einstein's gravity to theories with additional geometric degrees of freedom, (see for example [18, 19, 20] for general surveys on this subject as well as [21] for a list of works in a cosmological context), recent attempts focused on the old idea of modifying the gravitational Lagrangian in a purely metric framework, leading to higher order field equations. Such an approach is the so-called *Extended Theories of Gravity* which have become a sort of paradigm in the study of gravitational interaction. They are based on corrections and enlargements of the Einstein theory. The paradigm consists, essentially, *in adding higher order curvature invariants and minimally or non-minimally coupled scalar fields into dynamics which come out from the effective action of quantum gravity* [22].

The idea to extend Einstein's theory of gravitation is fruitful and economic also with respect to several attempts which try to solve problems by adding new and, most of times, unjustified ingredients in order to give a self-consistent picture of dynamics. The today observed accelerated expansion of the Hubble flow and the missing matter of astrophysical large scale structures, are primarily enclosed in these considerations. Both the issues could be solved changing the gravitational sector, *i.e.* the *l.h.s.* of field equations. The philosophy is alternative to add new cosmic fluids (new components in the *r.h.s.* of field equations) which should give rise to clustered structures (dark matter) or to accelerated dynamics (dark energy) thanks to exotic equations of state. In particular, relaxing the hypothesis that gravitational Lagrangian has to be a linear function of the Ricci curvature scalar R, like in the Hilbert-Einstein formulation, one can take into account an effective action where the gravitational Lagrangian includes other scalar invariants.

In summary, the general features of Extended Theories of Gravity are that the Einstein field equations result to be modified in two senses: $i)$ geometry can be non-minimally coupled to some scalar field, and / or $ii)$ higher than second order derivative terms in the metric come out. In the former case, we generically deal with scalar-tensor theories of gravity; in the latter, we deal with higher order theories. However combinations of non-minimally coupled and higher-order terms can emerge as contributions in effective Lagrangians. In this case, we deal with higher-order-scalar-tensor theories of gravity.

Due to the increased complexity of the field equations in this framework, the main amount of works dealt with some formally equivalent theories, in which a reduction of the order of the field equations was achieved by considering the metric and the connection as independent fields [326, 325, 327, 328, 329]. In addition, many authors exploited the formal relationship to scalar-tensor theories to make some statements about the weak field regime, which was already worked out for scalar-tensor theories more than ten years ago [330].

Other motivations to modify GR come from the issue of a full recovering of the Mach principle which leads to assume a varying gravitational coupling. The principle states that the local inertial frame is determined by some average of the motion of distant astronomical objects [29]. This fact implies that the gravitational coupling can be scale-dependent and related to some scalar field. As a consequence, the concept of "inertia" and the Equivalence Principle have to be revised. For example, the Brans-Dicke theory [30] is a serious attempt to define an alternative theory to the Einstein gravity: it takes into account a variable Newton gravitational coupling, whose dynamics is governed by a scalar field non-minimally coupled to the geometry. In such a way, Mach's principle is better implemented [30, 31, 32].

As already mentioned, corrections to the gravitational Lagrangian, leading to higher order field equations, were already studied by several authors [8, 11, 12] soon after the GR was proposed. Developments in the 1960s and 1970s [33, 34, 35, 36, 236], partially motivated by the quantization schemes proposed at that time, made clear that theories containing *only* a R^2 term in the Lagrangian were not viable with respect to their weak field behavior. Buchdahl, in 1962 [33] rejected pure R^2 theories because of the non-existence of asymptotically flat solutions.

Another concern which comes with generic *Higher Order Gravity* (HOG) theories is linked to the initial value problem. It is unclear if the prolongation of standard methods can be used in order to tackle this problem in every theory. Hence it is doubtful that the Cauchy problem could be properly addressed in the near future, for example within $1/R$ theories, if one takes into account the results already obtained in fourth order theories stemming from a quadratic Lagrangian [38, 39]. Starting from the Hilbert-Einstein lagrangian $\mathcal{L}_{GR} = \sqrt{-g}R$, the following terms $\mathcal{L}_1 = \sqrt{-g}R^2$, $\mathcal{L}_2 = \sqrt{-g}R_{\alpha\beta}R^{\alpha\beta}$, $\mathcal{L}_3 = \sqrt{-g}R_{\alpha\beta\gamma\delta}R^{\alpha\beta\alpha\delta}$, and combinations of them, represent a first obvious choices for an extended gravity theory with improved dynamics with respect to GR. Since the variational derivative of $\mathcal{L}_3$ can be linearly expressed [318, 40] via the variational derivatives of $\mathcal{L}_1$ and $\mathcal{L}_2$, one can omit $\mathcal{L}_3$ in the final Lagrangian of a HOG without loss of generality.

In summary, higher order terms in curvature invariants (such as R^2, $R_{\alpha\beta}R^{\alpha\beta}$, $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$, $R\Box R$, or $R\Box^k R$) or non-minimally coupled terms between scalar fields and geometry (such as $\phi^2 R$) have to be added to the effective Lagrangian of gravitational field when quantum corrections are considered. For instance, one can notice that such terms occur in the effective Lagrangian of strings or in Kaluza-Klein theories, when the mechanism of dimensional reduction is used [43].

On the other hand, from a conceptual viewpoint, there are no *a priori* reason to restrict the gravitational Lagrangian to a linear function of the Ricci scalar R, minimally coupled with matter [325]. More precisely, higher order terms appear always as contributions of order two in the field equations. For example, a term like R^2 gives fourth order equations [44], $R\Box R$ gives sixth order equations [45, 46], $R\Box^2 R$ gives eighth order equations [47] and so on. By a conformal transformation, any 2nd order derivative term corresponds to a scalar field[1]: for example, fourth order gravity gives Einstein plus one scalar field, sixth order gravity gives Einstein plus two scalar fields and so on [45, 48]. Furthermore, the idea that there are no "exact" laws of physics could be taken into serious account: in such a case, the effective Lagrangians of physical interactions are generic functions.

0.2 Issues from dark matter and dark energy

Beside fundamental physics motivations, the Extended Theories of Gravity have acquired a huge interest in Cosmology due to the fact that they "naturally" exhibit inflationary behaviors able to overcome the shortcomings of Cosmological Standard Model (based on GR). The related cosmological models seem realistic and capable of matching with the Cosmic Microwave Background Radiation (CMBR) observations [39, 50, 51]. Furthermore, it is possible to show that, via conformal transformations, the higher order and non-minimally coupled terms always correspond to the Einstein gravity plus one or more than one minimally coupled scalar fields [38, 45, 52, 53, 54].

Furthermore, it is possible to show that the $f(R)$-gravity (f-gravity) is equivalent not only to a scalar-tensor one but also to the Einstein theory plus an ideal fluid [55]. This feature results very interesting if we want to obtain multiple inflationary events since an early stage could select "very" large-scale structures (clusters of galaxies today), while a late stage could select "small" large-scale structures (galaxies today) [46]. The philosophy is that each inflationary era is related to the dynamics of a scalar field. Finally, these extended schemes could naturally solve the problem of "graceful exit" bypassing the shortcomings of former inflationary models [51, 56].

In recent years, the efforts to give a physical explanation to the today observed cosmic acceleration [57, 58, 59, 60] have attracted a good amount of interest in f-gravity, considered as a viable mechanism to explain the cosmic acceleration by extending the geometric sector of field equations [61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 327, 75, 76, 308, 78]. There are several physical and mathematical motivations to enlarge GR by these theories. For comprehensive reviews, see [79, 17, 80].

Specifically, cosmological models coming from f-gravity were firstly introduced by Starobinsky [50] in the early 80'ies to build up a feasible inflationary model where geometric degrees of freedom had the role of the scalar field ruling the inflation and the structure formation.

[1]The dynamics of such scalar fields is usually given by the corresponding Klein-Gordon equation, which is second order.

In addition to the revision of Standard Cosmology at early epochs (leading to the Inflation), a new approach is necessary also at late epochs. Extended Theories of Gravity could play a fundamental role also in this context. In fact, the increasing bulk of data that have been accumulated in the last few years have paved the way to the emergence of a new cosmological model usually referred to as the *Cosmological Concordance Model* (Λ *Cold Dark Matter*: ΛCDM).

After these observational evidences, an overwhelming flood of papers has appeared: they present a great variety of models trying to explain this phenomenon. In any case, the simplest explanation is claiming for the well known cosmological constant Λ [86]. Although it is the best fit to most of the available astrophysical data [308, 78, 189], the ΛCDM model fails in explaining why the inferred value of Λ is so tiny (120 orders of magnitude lower than the value of quantum gravity vacuum state!) if compared with the typical vacuum energy values predicted by particle physics and why its energy density is today comparable to the matter density (the so called *coincidence problem*).

Although the cosmological constant [87, 88, 89] remains the most relevant candidate to interpret the accelerated behavior, several proposals have been suggested in the last few years: quintessence models, where the cosmic acceleration is generated by means of a scalar field, in a way similar to the early time inflation [50], acting at large scales and recent epochs [90, 91]; models based on exotic fluids like the Chaplygin-gas [92, 93, 94], or non-perfect fluids [95]); phantom fields, based on scalar fields with anomalous signature in the kinetic term [218, 97, 98, 99], higher dimensional scenarios (braneworld) [100, 101, 102, 103]. These results can be achieved in metric and Palatini approaches [327, 62, 69, 70, 72, 73, 71, 74, 327, 104]. In addition, reversing the problem, one can reconstruct the form of the gravity Lagrangian by observational data of cosmological relevance through a "back scattering" procedure. All these facts suggest that the function f should be more general than the linear Hilbert-Einstein one implying that Extended Theories of Gravity could be a suitable approach to solve GR shortcomings without introducing mysterious ingredients as dark energy and dark matter (see e.g. [105, 106]).

Actually, all of these models, are based on the peculiar characteristic of introducing new sources into the cosmological dynamics, while it would be preferable to develop scenarios consistent with observations without invoking further parameters or components non-testable (up to now) at a fundamental level.

Moreover, it is not clear where this scalar field originates from, thus leaving a great uncertainty on the choice of the scalar field potential. The subtle and elusive nature of dark energy has led many authors to look for completely different scenarios able to give a quintessential behavior without the need of exotic components. To this aim, it is worth stressing that the acceleration of the Universe only claims for a negative pressure dominant component, but does not tell anything about the nature and the number of cosmic fluids filling the Universe.

Actually, there is still a different way to face the problem of cosmic acceleration. As stressed in [107], it is possible that the observed acceleration is not the manifestation of another ingredient in the cosmic pie, but rather the first signal of a breakdown of our understanding of the laws of gravitation (in the infra-red limit).

It is evident, from this short overview, the large number of cosmological models which are viable candidates to explain the observed accelerated expansion. This abundance of models is, from one hand, the signal of the fact that we have a limited number of cosmological tests to discriminate among rival theories, and, from the other hand, that a urgent degeneracy problem has to be faced.

The resort to modified gravity theories, which extend in some way the GR, allows to pursue this different approach (no further unknown sources) giving rise to suitable cosmological models where a late time accelerated expansion naturally arises.

The idea that the Einstein gravity should be extended or corrected at large scales (infrared limit) or at high energies (ultraviolet limit) is suggested by several theoretical and observational issues. Quantum field theories in curved spacetimes, as well as the low energy limit of string theory, both imply semi - classical effective Lagrangians containing higher-order curvature invariants or scalar-tensor terms. In addition, GR has been tested only at solar system scales while it shows several shortcomings if checked at higher energies or larger scales.

Summarizing, almost 95% of matter-energy content of the universe is unknown in the framework of Standard Cosmological Model while we can experimentally probe only gravity and ordinary (baryonic and radiation) matter. Considering another point of view, anomalous acceleration (Solar System), dark matter (galaxies and galaxy clusters), dark energy (cosmology) could be nothing else but the indications that shortcomings are present in GR and gravity is an interaction depending on the scale. The assumption of a linear Lagrangian density in the Ricci scalar R for the Hilbert-Einstein action could be too simple to describe gravity at any scale and more general approaches should be pursued to match observations. Among these schemes, several motivations suggest to generalize GR by considering gravitational actions where generic functions of curvature invariants are present.

0.3 Issues from quantum theory of gravitation

One of the main challenges of modern physics is to construct a theory able to describe the fundamental interactions of nature as different aspects of the same theoretical construct. This goal has led, in the past decades, to the formulation of several unification schemes which attempt to describe gravity by putting it on the same footing as the other interactions. All these schemes try to describe the fundamental fields in terms of the conceptual apparatus of Quantum Mechanics. One the main conceptual problem is that the gravitational field describes simultaneously the gravitational degrees of freedom and the background space-time in which these degrees of freedom live. Besides, due to the Uncertainty Principle, in non-relativistic Quantum Mechanics, particles do not move along well-defined trajectories and one can only calculate the probability amplitude $\psi(t, x)$ that a measurement at time t detects a particle around the spatial point x. Similarly, in Quantum Gravity, the evolution of an initial state does not provide a specific space-time. In the absence of a space-time, how is it possible to introduce basic concepts such as causality, and time.

Owing to the difficulties of building a complete theory unifying interactions and particles, during the last decades the two fundamental theories of modern physics, GR and Quantum Mechanics, have been critically re-analyzed. On the one hand, one assumes that the matter fields (bosons and fermions) come out from superstructures (*e.g.* Higgs bosons or superstrings) that, undergoing certain phase transitions, have generated the known particles. On the other hand, it is assumed that the geometry (*e.g.* the Ricci tensor or the Ricci scalar) interacts directly with quantum matter fields which back-react on it. This interaction necessarily modifies the standard gravitational theory, that is, the Lagrangian of gravity plus the effective fields is modified with respect to the Hilbert-Einstein one, and this fact can directly lead to the Extended Theories of Gravity. From the point of view of cosmology, the modifications of standard gravity provide inflationary scenarios of interest. In any case, a condition that must be satisfied in order for such theories to be physically acceptable is that GR is recovered in the low-energy limit. Although remarkable conceptual progress has been made following the introduction of generalized gravitational theories, at the same time the mathematical difficulties have increased. The corrections introduced into the Lagrangian augment the (intrinsic) non-linearity of the Einstein equations, making them more difficult to study because differential equations of higher order than second are often obtained and because it is impossible to separate the geometric from the matter degrees of freedom.

0.4 Plan of Thesis

The layout of the PhD thesis is organized as follows. In the first chapter we report a general review of Extended Theories of Gravity and the fundamental aspects of GR. In particular we display all fundamental tools: Einstein Equation, Metric and Palatini formalism, Extended Theories of Gravity (Scalar-tensor, HOG theories and so on), Coordinates system transformations and the relations between them (for example standard, isotropic coordinates etc).

In the second chapter, we show the technicality of development of field equation with respect to Newtonian and Post-Newtonian approach [**D**]. Finally we perform the post-Minkowskian limit: the gravitational waves [**E**]. In the end, we want to address the problem of how conformally transformed models behave in the weak field limit approximation. This issue could be extremely relevant in order to select conformally invariant physical quantities [**C**].

In the third chapter we investigate the equation for the photon deflection considering the Newtonian Limit of a general class of $f(X,Y,Z)$-Gravity where f is an unspecific function of $X = R$ (Ricci scalar), $Y = R_{\alpha\beta}R^{\alpha\beta}$ (Ricci tensor square) and $Z = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ (Riemann tensor square) [**A**].

In the fourth one we consider models of Extended Gravity and in particular, generic models containing scalar-tensor and higher-order curvature terms, as well as a model derived from noncommutative spectral geometry. Studying, in the weak-field approximation (the Newtonian and Post-Newtonian limit of the theory), the geodesic and Lense-Thirring processions, we impose constraints on the free parameters of such models by using the recent experimental results of the Gravity Probe B (GPB) and LARES satellites [**D**]. The imposed constraint by GPB and LARES is independent of the torsion-balance experiment, though it is much weaker [**B**].

In the fifth chapter we investigate the propagation of gravitational waves in the context of fourth order gravity nonminimally coupled to a massive scalar field. Using the damping of the orbital period of coalescing stellar binary systems, we impose constraints on the free parameters of extended gravity models [**E**]. In particular, we find that the variation of the orbital period is a function of three *mass* scales which depend on the free parameters of the model under consideration; we can constrain these mass scales from current observational data.

This thesis is the result of the work I did during my Ph.D. at the University of Salerno (Italy), under the supervision of the prof Gaetano Lambiase. The Thesis is based on the papers:

A A. Stabile, **An. Stabile**, *Weak Gravitational Lensing in Fourth Order Gravity* Phys. Rev. D 85, 044014 (2012)

B G. Lambiase, M. Sakellariadou, **An. Stabile**, *Constraints on NonCommutative Spectral Action from Gravity Probe B and Torsion Balance Experiments* JCAP 12, 020 (2013)

C A. Stabile, **An. Stabile**, S. Capozziello, *Conformal Transformations and Weak Field Limit of Scalar-Tensor Gravity* Phys. Rev. D 88, 124011 (2013)

D S. Capozziello, G. Lambiase, M. Sakellariadou, A. Stabile, **An. Stabile** *Constraining Models of Extended Gravity using Gravity Probe B and LARES experiments* Phys. Rev. D 91, 044012 (2015)

E G. Lambiase, M. Sakellariadou, A. Stabile, **An. Stabile** *Astrophysical constraints on extended gravity models* arXiv:1503.08751 (Submitted to JCAP)

Chapter 1

Extended Theories of Gravity a short review

1.1 General Relativity and its extensions

Any relativistic theory of gravity has to match some minimal requirements to address gravitational dynamics. First of all, it has to explain issues coming from Celestial Mechanics as the planetary orbits, the potential of self-gravitating systems, the Solar System stability.

This means that it has to reproduce the Newtonian dynamics in the weak field limit and then it has to pass the Solar System experiments which are all well founded and constitute the test bed of GR [131].

Besides, any theory of gravity has to be consistent with stellar structures and galactic dynamics considering the observed baryonic constituents (e.g. luminous components as stars, sub-luminous components as planets, dust and gas), radiation and Newtonian potential which is, by assumption, extrapolated to galactic scales.

The third step is cosmology and large scale structure which means to reproduce, in a self-consistent way, the cosmological parameters as the expansion rate, the Hubble constant, the density parameter and the clustering of galaxies. Observations probe the standard baryonic matter, the radiation and an attractive overall interaction, acting at all scales and depending on distance. From a phenomenological point of view this is *gravity*.

GR is the simplest theory which partially satisfies the above requirements [161]. It is based on the assumption that space and time are entangled into a single spacetime structure, which, in the limit of no gravitational forces, has to reproduce the Minkowski spacetime. Besides, the Universe is assumed to be a curved manifold and the curvature depends on mass-energy distribution [162]. In other words, the distribution of matter influences point by point the local curvature of the spacetime structure.

Furthermore, GR is based on three first principles that are *Relativity*, *Equivalence*, and

General Covariance. Another requirement is the *Principle of Causality* that means that each point of spacetime admits a notion of past, present and future.

Let us also recall that the Newtonian theory, the weak field limit of GR, requires absolute concepts of space and time, that particles move in a preferred inertial frame following curved trajectories function of the sources (*i.e.*, the "forces").

On these bases, GR postulates that gravitational forces have to be expressed by the curvature of a metric tensor field $ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$ on a four-dimensional spacetime manifold, having the same signature of Minkowski metric, here assumed to be $(+ - - -)$. Curvature is locally determined by the distribution of the sources, that is, being the spacetime a continuum, it is possible to define a stress-energy tensor $T_{\mu\nu}$ which is the source of the curvature.

Once a metric $g_{\mu\nu}$ is given, the inverse $g^{\mu\nu}$ satisfies the condition[1]

$$g^{\mu\alpha}g_{\alpha\beta} = \delta^\mu_\nu\,. \tag{1.1}$$

Its curvature is expressed by the *Riemann tensor* (curvature)

$$R^\alpha{}_{\mu\beta\nu} = \Gamma^\alpha_{\mu\nu,\beta} - \Gamma^\alpha_{\mu\beta,\nu} + \Gamma^\sigma_{\mu\nu}\Gamma^\alpha_{\sigma\beta} - \Gamma^\alpha_{\sigma\nu}\Gamma^\sigma_{\mu\beta} \tag{1.2}$$

where the comas are partial derivatives. The $\Gamma^\alpha_{\mu\nu}$ are the Christoffel symbols given by

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\sigma}(g_{\mu\sigma,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma})\,, \tag{1.3}$$

if the Levi-Civita connection is assumed. The contraction of the Riemann tensor (1.2)

$$R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu} = \Gamma^\sigma_{\mu\nu,\sigma} - \Gamma^\sigma_{\mu\sigma,\nu} + \Gamma^\sigma_{\mu\nu}\Gamma^\rho_{\sigma\rho} - \Gamma^\rho_{\sigma\nu}\Gamma^\sigma_{\mu\rho}, \tag{1.4}$$

is the *Ricci tensor* and the scalar

$$R = g^{\sigma\tau}R_{\sigma\tau} = R^\sigma{}_\sigma = g^{\tau\xi}\Gamma^\sigma_{\tau\xi,\sigma} - g^{\tau\xi}\Gamma^\sigma_{\tau\sigma,\xi} + g^{\tau\xi}\Gamma^\sigma_{\tau\xi}\Gamma^\rho_{\sigma\rho} - g^{\tau\xi}\Gamma^\rho_{\tau\sigma}\Gamma^\sigma_{\xi\rho} \tag{1.5}$$

is called the *scalar curvature* of $g_{\mu\nu}$. The Riemann tensor (1.2) satisfies the so-called *Bianchi identities*:

[1]The Greek index runs between 0 and 3; the Latin index between 1 and 3.

$$\begin{aligned}
&R_{\alpha\mu\beta\nu;\delta} + R_{\alpha\mu\delta\beta;\nu} + R_{\alpha\mu\nu\delta;\beta} = 0 \\
&R_{\alpha\mu\beta\nu}{}^{;\alpha} + R_{\mu\beta;\nu} - R_{\mu\nu;\beta} = 0 \\
&2R_{\alpha\beta}{}^{;\alpha} - R_{;\beta} = 0 \\
&2R_{\alpha\beta}{}^{;\alpha\beta} - \Box R = 0
\end{aligned} \tag{1.6}$$

where the covariant derivative is $A^{\alpha}{}_{\beta\,;\,\mu} \equiv \nabla_{\mu}A^{\alpha}{}_{\beta} = A^{\alpha}{}_{\beta\,,\,\mu} + \Gamma^{\alpha}_{\sigma\mu}A^{\sigma}{}_{\beta} - \Gamma^{\beta}_{\sigma\mu}A^{\alpha}{}_{\sigma}$ and $\nabla_{\alpha}\nabla^{\alpha} = \Box = \frac{\partial_{\alpha}(\sqrt{-g}g^{\alpha\beta}\partial_{\beta})}{\sqrt{-g}}$ is the d'Alembert operator with respect to the metric $g_{\mu\nu}$.

Einstein was led to postulate the following equations for the dynamics of gravitational forces

$$R_{\mu\nu} = \mathcal{X}\,T_{\mu\nu} \tag{1.7}$$

where $\mathcal{X} = 8\pi G$ is a coupling constant (we will use the convention $c = 1$). These equations turned out to be physically and mathematically unsatisfactory.

As Hilbert pointed out [163], they have not a variational origin, *i.e.* there was no Lagrangian able to reproduce them exactly (this is slightly wrong, but this remark is unessential here). Einstein replied that he knew that the equations were physically unsatisfactory, since they were contrasting with the continuity equation of any reasonable kind of matter. Assuming that matter is given as a perfect fluid, that is

$$T_{\mu\nu} = (p+\rho)u_{\mu}u_{\nu} - pg_{\mu\nu} \tag{1.8}$$

where $u_{\mu}u_{\nu}$ define a comoving observer, p is the pressure and ρ the density of the fluid, then the continuity equation requires $T_{\mu\nu}$ to be covariantly constant, i.e. to satisfy the conservation law

$$T^{\mu\sigma}{}_{;\sigma} = 0\,. \tag{1.9}$$

In fact, it is not true that $R^{\mu\sigma}{}_{;\sigma}$ vanishes (unless $R = 0$). Einstein and Hilbert reached independently the conclusion that the wrong field equations (1.7) had to be replaced by the correct ones

$$G_{\mu\nu} = \mathcal{X} T_{\mu\nu} \tag{1.10}$$

where

$$G_{\mu\nu} = R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} \tag{1.11}$$

that is currently called the "*Einstein tensor*" of $g_{\mu\nu}$. These equations are both variational and satisfy the conservation laws (1.9) since the following relation holds

$$G^{\mu\sigma}{}_{;\sigma} = 0 \,, \tag{1.12}$$

as a byproduct of the so-called *Bianchi identities* that the curvature tensor of $g_{\mu\nu}$ has to satisfy [353, 348].

The Lagrangian that allows to obtain the field equations (1.10) is the sum of a *matter Lagrangian* $\mathcal{L}_m$ and of the Ricci scalar:

$$\mathcal{L}_{HE} = \sqrt{-g}(R + \mathcal{X}\mathcal{L}_m) \,, \tag{1.13}$$

where $\sqrt{-g}$ denotes the square root of the value of the determinant of the metric $g_{\mu\nu}$. The action of GR is

$$\mathcal{S}[g] = \int d^4x \sqrt{-g}(R + \mathcal{X}\mathcal{L}_m) \,. \tag{1.14}$$

From the action principle, we get the field equations (1.10) by the variation:

$$\begin{aligned}\delta\mathcal{S}[g] = \quad & \delta\int d^4x\sqrt{-g}(R+\mathcal{X}\mathcal{L}_m) = \int d^4x\sqrt{-g}\Big[R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} + \\ & -\mathcal{X}T_{\mu\nu}\Big]\delta g^{\mu\nu} + \int d^4x\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} = 0 \,, \end{aligned} \tag{1.15}$$

where $T_{\mu\nu}$ is energy momentum tensor of matter:

$$T_{\mu\nu} = -\frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}\,. \tag{1.16}$$

The last term in (1.15) is a 4-divergence

$$\int d^4x\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} = \int d^4x\sqrt{-g}[(-\delta g^{\mu\nu})_{;\mu\nu} - \Box(g^{\mu\nu}\delta g_{\mu\nu})] \tag{1.17}$$

then we can neglect it and we get the field equation (1.10). For the variational calculus (1.15) we used the following relations

$$\begin{cases} \delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\,g_{\alpha\beta}\,\delta g^{\alpha\beta} \\ \\ \delta\, R = R_{\alpha\beta}\,\delta g^{\alpha\beta} + g^{\alpha\beta}\,\delta R_{\alpha\beta} \\ \\ \delta\, R_{\alpha\beta} = \frac{1}{2}(\delta g^{\rho}{}_{\alpha;\beta\rho} + \delta g^{\rho}{}_{\beta;\alpha\rho} - \Box\,\delta g_{\alpha\beta} - g^{\rho\sigma}\delta\, g_{\rho\sigma;\alpha\beta}) \end{cases} \tag{1.18}$$

The choice of Hilbert and Einstein was completely arbitrary (as it became clear a few years later), but it was certainly the simplest one both from the mathematical and the physical viewpoint. As it was later clarified by Levi-Civita in 1919, curvature is not a "purely metric notion" but, rather, a notion related to the "linear connection" to which "parallel transport" and "covariant derivation" refer [165].

It was later clarified that the three principles of relativity, equivalence and covariance, together with causality, just require that the spacetime structure has to be determined by either one or both of two fields, a Lorentzian metric g and a linear connection Γ, assumed to be torsionless for the sake of simplicity.

The metric g fixes the causal structure of spacetime (the light cones) as well as its metric relations (clocks and rods); the connection Γ fixes the free-fall, i.e. the locally inertial observers. They have, of course, to satisfy a number of compatibility relations which amount to require that photons follow the null geodesics of Γ, so that Γ and g can be independent, *a priori*, but constrained, *a posteriori*, by some physical restrictions. These, however, do not impose that Γ has necessarily to be the Levi Civita connection of g [168].

This justifies - at least on a purely theoretical basis - the fact that one can envisage the so-called "alternative theories of gravitation", that we prefer to call "*Extended Theories of Gravitation*" since their starting points are exactly those considered by Einstein and Hilbert: theories in which gravitation is described by either a metric (the so-called "purely metric theories"), or by a linear connection (the so-called "purely affine theories") or by both fields (the so-called "metric-affine theories", also known as "first order formalism theories"). In these theories, the Lagrangian is a scalar density of the curvature invariants constructed out of both g and Γ.

The choice (1.13) is by no means unique and it turns out that the Hilbert-Einstein Lagrangian is in fact the only choice that produces an invariant that is linear in second derivatives of the metric (or first derivatives of the connection). A Lagrangian that, unfortunately, is rather singular from the Hamiltonian viewpoint, in much than same way as Lagrangians, linear in canonical momenta, are rather singular in Classical Mechanics (see e.g. [169]).

A number of attempts to generalize GR (and unify it to Electromagnetism) along these lines were followed by Einstein himself and many others (Eddington, Weyl, Schrodinger, just to quote the main contributors; see, e.g., [170]) but they were eventually given up in the fifties of XX Century, mainly because of a number of difficulties related to the definitely more complicated structure of a non-linear theory (where by "non-linear" we mean here a theory that is based on non-linear invariants of the curvature tensor), and also because of the new understanding of physics that is currently based on four fundamental forces and requires the more general "gauge framework" to be adopted (see [171]).

Still a number of sporadic investigations about "alternative theories" continued even after 1960 (see [131] and refs. quoted therein for a short history). The search of a coherent quantum theory of gravitation or the belief that gravity has to be considered as a sort of low-energy limit of string theories (see, e.g., [172]) - something that we are not willing to enter here in detail - has more or less recently revitalized the idea that there is no reason to follow the simple prescription of Einstein and Hilbert and to assume that gravity should be classically governed by a Lagrangian linear in the curvature.

Further curvature invariants or non-linear functions of them should be also considered, especially in view of the fact that they have to be included in both the semi-classical expansion of a quantum Lagrangian or in the low-energy limit of a string Lagrangian.

Moreover, it is clear from the recent astrophysical observations and from the current cosmological hypotheses that Einstein equations are no longer a good test for gravitation at Solar System, galactic, extra-galactic and cosmic scale, unless one does not admit that the matter side of Eqs.(1.10) contains some kind of exotic matter-energy which is the "dark matter" and "dark energy" side of the Universe.

The idea which we propose here is much simpler. Instead of changing the matter side of Einstein Equations (1.10) in order to fit the "missing matter-energy" content of the currently observed Universe (up to the 95% of the total amount!), by adding any sort of inexplicable and strangely behaving matter and energy, we claim that it is simpler and more convenient to change the gravitational side of the equations, admitting corrections coming from non-linearities in the Lagrangian. However, this is nothing else but a matter of taste and, since it is possible, such an approach should be explored. Of course, provided that the Lagrangian can be conveniently tuned up (i.e., chosen in a huge family of allowed Lagrangians) on the basis of its best fit with all possible observational tests, at all scales (solar, galactic, extragalactic and cosmic).

1.2 The Extended Theories of Gravity models

Historically there are two ways to modify GR: the first, we have the *Scalar Tensor Theories of Gravity*, where the geometry can couple non-minimally to some scaler field; the second, we have the *Higher Order Theories*, where the derivatives of the metric components of order higher than second may appear. Combinations of non-minimally coupled and higher order terms can also emerge in effective Lagrangians, producing mixed *Higher Order Scalar Tensor Gravity*.

A general class of Higher Order Scalar Tensor Gravity theories in four dimensions given by the action

$$\mathcal{S}[g,\,\phi] = \int d^4x\sqrt{-g}[\mathcal{F}(R, \Box R, \Box^2 R, ..., \Box^k R, \phi) + \omega(\phi)\phi_{;\alpha}\phi^{;\alpha} + \mathcal{X}\mathcal{L}_m]\,, \tag{1.19}$$

where $\mathcal{F}$ is an unspecified function of curvature invariants and of a scalar field ϕ. The term $\mathcal{L}_m$, as above, is the minimally coupled ordinary matter contribution and $\omega(\phi)$ is a generic function of the scalar field ϕ. For example its values could be $\omega(\phi) = \pm 1, 0$ fixing the nature and the dynamics of the scalar field which can be a standard scalar field, a phantom field or a field without dynamics (see [115, 173] for details).

In the metric approach, the field equations are obtained by varying (1.19) with respect to $g_{\mu\nu}$. We get

$$\begin{aligned}\hat{\mathcal{F}}G_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(\mathcal{F} - \hat{\mathcal{F}}R) - \hat{\mathcal{F}}_{;\mu\nu} + g_{\mu\nu}\Box\hat{\mathcal{F}} + g_{\mu\nu}\left[(\Box^{j-1}R)^{;\alpha}\Box^{i-j}\frac{\partial\mathcal{F}}{\partial\Box^i R}\right]_{;\alpha} + \\ -\frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{i}\left[g_{\mu\nu}(\Box^{j-i})^{;\alpha}\left(\Box^{i-j}\frac{\partial\mathcal{F}}{\partial\Box^i R}\right)_{;\alpha} + (\Box^{j-i})_{;\nu}\left(\Box^{i-j}\frac{\partial\mathcal{F}}{\partial\Box^i R}\right)_{;\mu}\right] + \\ -\frac{\omega(\phi)}{2}\phi_{;\alpha}\phi^{;\alpha}g_{\mu\nu} + \omega(\phi)\phi_{;\mu}\phi_{;\nu} = \mathcal{X}\,T_{\mu\nu}\end{aligned} \tag{1.20}$$

where $G_{\mu\nu}$ is the above Einstein tensor (1.11) and

$$\hat{\mathcal{F}} = \sum_{j=0}^{n}\Box^j\frac{\partial\mathcal{F}}{\partial\Box^j R}\,. \tag{1.21}$$

The differential Equations (1.20) are of order $(2k+4)$. The (eventual) contribution of a potential

$V(\phi)$ is contained in the definition of $\mathcal{F}$. By varying with respect to the scalar field ϕ, we obtain the Klein - Gordon equation

$$\Box\phi = \frac{1}{2}\frac{\delta \ln\omega(\phi)}{\delta\phi}\phi_{;\alpha}\phi^{;\alpha} + \frac{1}{2\omega(\phi)}\frac{\delta\mathcal{F}(R, \Box R, \Box^2 R, ..., \Box^k R, \phi)}{\delta\phi} \,. \tag{1.22}$$

Several approaches can be used to deal with such equations. For example, as we said, by a conformal transformation (see in the next chapter), it is possible to reduce an Extended Theories of Gravity to a (multi) scalar - tensor theory of gravity [330, 45, 53, 134, 174].

1.2.1 Scalar Tensor Gravity: Bran-Dicke Theory

From the action (1.19), it is possible to obtain the Scalar Tensor Gravity by choosing:

$$\mathcal{F} \,=\, F(\phi)R + V(\phi)\,, \tag{1.23}$$

where $V(\phi)$ and $F(\phi)$ are generic functions describing respectively the potential and the coupling of a scalar field ϕ. In this case, we get

$$\mathcal{S}^{ST}[g,\,\phi] \,=\, \int d^4x\sqrt{-g}[F(\phi)R + \omega(\phi)\phi_{;\alpha}\phi^{;\alpha} + V(\phi) + \mathcal{X}\mathcal{L}_m]\,. \tag{1.24}$$

The Brans-Dicke theory of gravity is a particular case of the action (1.24) in which we have $V(\phi) = 0$ and $\omega(\phi) = -\frac{\omega_{BD}}{\phi}$. In fact we have

$$\mathcal{S}^{BD}[g\,,\phi] = \int d^4x\sqrt{-g}\left[\phi R - \omega_{BD}\frac{\phi_{;\alpha}\phi^{;\alpha}}{\phi} + \mathcal{X}\mathcal{L}_m\right]\,. \tag{1.25}$$

The factor ϕ in the denominator of the kinetic term of ϕ in the action (1.25) is purely conventional and has the only purpose of making ω_{BD} dimensionless. Matter does not couple directly to ϕ, *i.e.*, the Lagrangian density $\mathcal{L}_m$ is independent of ϕ ("minimal coupling" of matter). However, ϕ couples directly to the Ricci scalar. The gravitational field is described by both the metric tensor $g_{\mu\nu}$ and the Brans-Dicke scalar ϕ which, together with the matter variables, constitute the degrees of freedom of the theory. As usual for scalar fields, the potential $V(\phi)$ generalizes the cosmological constant and may reduce to a constant, or to a mass term.

The variation of (1.24) with respect to $g_{\mu\nu}$ and ϕ gives the second-order field equations

$$
\begin{aligned}
& F(\phi)G_{\mu\nu} - \frac{1}{2}V(\phi)g_{\mu\nu} + \omega(\phi)\left[\phi_{;\mu}\phi_{;\nu} - \frac{1}{2}\phi_{;\alpha}\phi^{;\alpha}g_{\mu\nu}\right] - F(\phi)_{;\mu\nu} + \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + g_{\mu\nu}\Box F(\phi) = \mathcal{X}\,T_{\mu\nu} \\
& 2\omega(\phi)\Box\phi - \omega_{,\phi}(\phi)\phi_{;\alpha}\phi^{;\alpha} - [F(\phi)R + V(\phi)]_{,\phi} = 0 \\
& 3\Box F(\phi) - F(\phi)R - 2V(\phi) - \omega(\phi)\phi_{;\alpha}\phi^{;\alpha} = \mathcal{X}\,T \\
& 2\omega(\phi)\Box\phi + 3\Box F(\phi) - [\omega_{,\phi}(\phi) + \omega(\phi)]\phi_{;\alpha}\phi^{;\alpha} + F(\phi)R - 2V(\phi) + \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad - [F(\phi)R + V(\phi)]_{,\phi} = \mathcal{X}\,T
\end{aligned}
\tag{1.26}
$$

1.2.2 Fourth Order Gravity: $f(R\,,R_{\alpha\beta}R^{\alpha\beta}\,,R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta})$

The simplest extension of GR is achieved assuming

$$
\mathcal{F} \,=\, f(R)\,, \qquad \omega(\phi) = 0\,, \tag{1.27}
$$

in the action (1.19); f is an arbitrary (analytic) function of the Ricci curvature scalar R. Then

$$
\mathcal{S}^f[g] \,=\, \int d^4x\sqrt{-g}\Big[f + \mathcal{X}\mathcal{L}_m\Big] \tag{1.28}
$$

where the standard Hilbert-Einstein action is, of course, recovered for $f \,=\, R$. By varying the action (1.28) and by using the properties (1.18) we get the field equations:

$$
\begin{aligned}
\delta\mathcal{S}[g] &= \delta\int d^4x\sqrt{-g}[f+\mathcal{X}\,\mathcal{L}_m] \\
&= \int d^4x\sqrt{-g}\Big[\Big(f'R_{\mu\nu}-\frac{f}{2}g_{\mu\nu}-\mathcal{X}\,T_{\mu\nu}\Big)\delta g^{\mu\nu}+g_{\mu\nu}f'\delta R^{\mu\nu}\Big] \\
&= \int d^4x\sqrt{-g}\Big\{\Big(f'R_{\mu\nu}-\frac{f}{2}g_{\mu\nu}-\mathcal{X}\,T_{\mu\nu}\Big)\delta g^{\mu\nu}+ \\
&\qquad\qquad\qquad +f'[-(\delta g^{\mu\nu})_{;\mu\nu}-\Box(g^{\mu\nu}\delta g_{\mu\nu})]\Big\} \\
&\sim \int d^4x\sqrt{-g}\Big\{f'R_{\mu\nu}-\frac{f}{2}g_{\mu\nu}-\mathcal{X}\,T_{\mu\nu}-f'_{;\mu\nu}+g_{\mu\nu}\Box f'\Big\}\delta g^{\mu\nu} \\
&= \int d^4x\sqrt{-g}(H_{\mu\nu}-\mathcal{X}\,T_{\mu\nu})\delta g^{\mu\nu}=0
\end{aligned}
$$

where the symbol $\sim$ means that we neglected a pure divergence; then we obtain the field equation (2.152). Eq. (2.152).

We get the field equations

$$
G^f_{\mu\nu} \doteq f'R_{\mu\nu}-\frac{1}{2}fg_{\mu\nu}-f'_{;\mu\nu}+g_{\mu\nu}\Box f'=\mathcal{X}\,T_{\mu\nu}\,, \tag{1.29}
$$

where $G^f_{\mu\nu}$ is the Einstein Tensor modified. The (2.152) are fourth-order equations due to the terms $f'_{;\mu\nu}$ and $\Box f'$; the prime indicates the derivative with respect to R. The trace of (2.152) is

$$
G^f \,=\, g^{\alpha\beta}G^f_{\alpha\beta} \,=\, 3\Box f'+f'R-2f=\mathcal{X}\,T\,. \tag{1.30}
$$

The Eq. (2.152) satisfies the condition $G^{f\,\alpha\mu}{}_{;\alpha} \,=\, \mathcal{X}\,T^{\alpha\mu}{}_{;\alpha} \,=\, 0$. In fact it is easy to check that

$$
\begin{aligned}
G^{f\,\alpha\mu}{}_{;\alpha} &= f'_{;\alpha}R^{\alpha\mu}+f'R^{\alpha\mu}{}_{;\alpha}-\frac{1}{2}f^{\prime;\mu}-f^{\prime;\alpha\mu}{}_{\alpha}+f^{\prime;\alpha}{}_{\alpha}{}^{\mu} \\
&= f''R^{\alpha\mu}R_{;\alpha}-f^{\prime;\alpha\mu}{}_{\alpha}+f^{\prime;\alpha}{}_{\alpha}{}^{\mu} \\
&= f''R^{\alpha\mu}R_{;\alpha}-f^{\prime;\alpha}R_{\alpha}{}^{\mu} \\
&= f''R^{\alpha\mu}R_{;\alpha}-f''R^{;\alpha}R_{\alpha}{}^{\mu} \,=\, 0\,;
\end{aligned}
$$

where we used the properties $G^{\alpha\mu}{}_{;\alpha} \,=\, 0$ and $[\nabla^\mu,\nabla_\alpha]f^{\prime;\alpha} \,=\, -f^{\prime;\alpha}R_\alpha{}^\mu$. If we develop the covariant derivatives in (2.152) and in (1.30) we obtain the complete expression for a generic f-theory

$$
\begin{aligned}
G^f_{\mu\nu} &= f'R_{\mu\nu} - \frac{1}{2} f g_{\mu\nu} + \mathcal{G}_{\mu\nu} \,=\, \mathcal{X}\, T_{\mu\nu} \\
G^f &= f'R - 2f + \mathcal{G} \,=\, \mathcal{X}\, T
\end{aligned}
\tag{1.31}
$$

where the two quantities $\mathcal{G}_{\mu\nu}$ and $\mathcal{G}$ read:

$$
\begin{aligned}
\mathcal{G}_{\mu\nu} &= -f''\Big\{R_{,\mu\nu} - \Gamma^{\sigma}_{\mu\nu}R_{,\sigma} - g_{\mu\nu}\Big[\Big(g^{\sigma\tau}{}_{,\sigma} + g^{\sigma\tau}\ln\sqrt{-g}_{,\sigma}\Big)R_{,\tau} + \\
&\qquad + g^{\sigma\tau}R_{,\sigma\tau}\Big]\Big\} - f'''\Big(R_{,\mu}R_{,\nu} - g_{\mu\nu}g^{\sigma\tau}R_{,\sigma}R_{,\tau}\Big) \\
\mathcal{G} &= 3f''\Big[\Big(g^{\sigma\tau}{}_{,\sigma} + g^{\sigma\tau}\ln\sqrt{-g}_{,\sigma}\Big)R_{,\tau} + g^{\sigma\tau}R_{,\sigma\tau}\Big] + 3f'''g^{\sigma\tau}R_{,\sigma}R_{,\tau}
\end{aligned}
\tag{1.32}
$$

$\Gamma^{\alpha}_{\mu\nu}$ are the standard Christoffel's symbols defined by (1.3). We conclude, then, this paragraph having shown the most general expression of field equations of f-gravity in metric formalism.

The field equations for the $R_{\alpha\beta}R^{\alpha\beta}$ and $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ − invariants

We start from the action principle for a Lagrangian densities $\sqrt{-g}\, R_{\alpha\beta}R^{\alpha\beta}$ and $\sqrt{-g}\, R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ and we get their field equations:

$$
\begin{aligned}
\delta\mathcal{S}[g] &= \delta\int d^4x\sqrt{-g}[R_{\alpha\beta}R^{\alpha\beta} + \mathcal{X}\,\mathcal{L}_m] \\
&= \delta\int d^4x\sqrt{-g}[R_{\alpha\beta}g^{\alpha\rho}g^{\beta\sigma}R_{\rho\sigma} + \mathcal{X}\mathcal{L}_m] \\
&= \int d^4x\sqrt{-g}\Big[\Big(2R_{\mu}{}^{\alpha}R_{\alpha\nu} - \frac{R_{\alpha\beta}R^{\alpha\beta}}{2}g_{\mu\nu} - \mathcal{X}\,T_{\mu\nu}\Big)\delta g^{\mu\nu} \\
&\qquad + 2R^{\mu\nu}\delta R_{\mu\nu}\Big] \\
&= \int d^4x\sqrt{-g}\Big[\Big(2R_{\mu}{}^{\alpha}R_{\alpha\nu} - \frac{R_{\alpha\beta}R^{\alpha\beta}}{2}g_{\mu\nu} - \mathcal{X}\,T_{\mu\nu}\Big)\delta g^{\mu\nu} + \\
&\qquad + R^{\mu\nu}(2g^{\rho\sigma}\delta g_{\rho(\mu;\nu)\sigma} - \Box\,\delta g_{\mu\nu} - g^{\rho\sigma}\delta g_{\rho\sigma;\mu\nu})\Big]
\end{aligned}
$$

$$
\begin{aligned}
&\sim \int d^4x\sqrt{-g}\Bigg[\left(2R_{\mu}{}^{\alpha}R_{\alpha\nu} - \frac{R_{\alpha\beta}R^{\alpha\beta}}{2}g_{\mu\nu} - \mathcal{X}\,T_{\mu\nu}\right)\delta g^{\mu\nu} \\
&\qquad -2R^{\sigma}{}_{(\mu;\nu)\sigma}\delta g^{\mu\nu} + \Box R_{\mu\nu}\delta g^{\mu\nu} + R^{\sigma\tau}{}_{;\sigma\tau}g_{\mu\nu}\delta g^{\mu\nu}\Bigg] \\
&= \int d^4x\sqrt{-g}\Bigg[2R_{\mu}{}^{\alpha}R_{\alpha\nu} - \frac{R_{\alpha\beta}R^{\alpha\beta}}{2}g_{\mu\nu} - 2R^{\sigma}{}_{(\mu;\nu)\sigma} + \Box R_{\mu\nu} \\
&\qquad + g_{\mu\nu}R^{\sigma\tau}{}_{;\sigma\tau} - \mathcal{X}\,T_{\mu\nu}\Bigg]\delta g^{\mu\nu} = 0\,.
\end{aligned}
$$

Then, the field equations are

$$
G^{Ric}_{\mu\nu} = 2R_{\mu}{}^{\alpha}R_{\alpha\nu} - \frac{R_{\alpha\beta}R^{\alpha\beta}}{2}g_{\mu\nu} - 2R^{\sigma}{}_{(\mu;\nu)\sigma} + \Box R_{\mu\nu} + g_{\mu\nu}R^{\sigma\tau}{}_{;\sigma\tau} = \mathcal{X}\,T_{\mu\nu} \tag{1.33}
$$

and the trace is

$$
G^{Ric} = 2\Box R = \mathcal{X}\,T\,, \tag{1.34}
$$

where we used the Bianchi identity contracted (1.6).

Let us calculate the field equations for the $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ - invariant:

$$
\begin{aligned}
\delta\mathcal{S} &= \delta\int d^4x\sqrt{-g}[R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} + \mathcal{X}\mathcal{L}_m] \\
&= \delta\int d^4x\sqrt{-g}[R_{\alpha\beta\gamma\delta}g^{\alpha\rho}g^{\beta\sigma}g^{\gamma\tau}g^{\delta\xi}R_{\rho\sigma\tau\xi} + \mathcal{X}\mathcal{L}_m] \\
&= \int d^4x\sqrt{-g}\Bigg[\left(4R_{\mu\alpha\beta\gamma}R_{\nu}{}^{\alpha\beta\gamma} - \frac{R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}}{2}g_{\mu\nu} - \mathcal{X}T_{\mu\nu}\right)\delta g^{\mu\nu} + \\
&\qquad + 2R^{\alpha\beta\gamma\delta}\delta R_{\alpha\beta\gamma\delta}\Bigg] \\
&= \int d^4x\sqrt{-g}\Bigg[\left(4R_{\mu\alpha\beta\gamma}R_{\nu}{}^{\alpha\beta\gamma} - \frac{R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}}{2}g_{\mu\nu} - \mathcal{X}T_{\mu\nu}\right)\delta g^{\mu\nu} + \\
&\qquad + R^{\alpha\beta\gamma\delta}(\delta g_{\alpha\beta;\delta\gamma} + \delta g_{\alpha\delta;\beta\gamma} - \delta g_{\beta\delta;\alpha\gamma} - \delta g_{\alpha\beta;\gamma\delta} - \delta g_{\alpha\gamma;\beta\delta} + \delta g_{\beta\gamma;\alpha\delta})\Bigg] \\
&\sim \int d^4x\sqrt{-g}\Bigg[2R_{\mu\alpha\beta\gamma}R_{\nu}{}^{\alpha\beta\gamma} - \frac{R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}}{2}g_{\mu\nu} - 4R_{\mu}{}^{\alpha\beta}{}_{\nu;\alpha\beta} + \\
&\qquad - \mathcal{X}T_{\mu\nu}\Bigg]\delta g^{\mu\nu} = 0
\end{aligned} \tag{1.35}
$$

We used the expressions

$$\begin{aligned}\delta\, R_{\alpha\beta\gamma\delta} &= \delta(g_{\alpha\sigma}R^{\sigma}{}_{\beta\gamma\delta}) = R^{\sigma}{}_{\beta\gamma\delta}\delta\, g_{\alpha\sigma} + g_{\alpha\sigma}\delta\, R^{\sigma}{}_{\beta\gamma\delta} \\ \delta R^{\sigma}{}_{\beta\gamma\delta} &= \frac{1}{2}(\delta g^{\alpha}{}_{\beta;\delta\gamma} + \delta g^{\alpha}{}_{\delta;\beta\gamma} - \delta g_{\beta\delta}{}^{;\alpha}{}_{\gamma} - \delta g^{\alpha}{}_{\beta;\gamma\delta} - \delta g^{\alpha}{}_{\gamma;\beta\delta} + \delta g_{\beta\gamma}{}^{;\alpha}{}_{\delta})\end{aligned} \tag{1.36}$$

Then, the field equations, from (1.35), are

$$G^{Rie}_{\mu\nu} = 2R_{\mu\alpha\beta\gamma}R_{\nu}{}^{\alpha\beta\gamma} - \frac{R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}}{2}g_{\mu\nu} - 4R_{\mu}{}^{\alpha\beta}{}_{\nu;\alpha\beta} = \mathcal{X}T_{\mu\nu} \tag{1.37}$$

and the trace is

$$G^{Rie} = -4R_{\gamma}{}^{\alpha\beta\gamma}{}_{;\alpha\beta} = \mathcal{X}T\,. \tag{1.38}$$

Finally, the more general action of the Fourth Order we can write as:

$$\mathcal{S}[g] = \int d^4x\sqrt{-g}\Big[f(X,Y,Z) + \mathcal{X}\mathcal{L}_m\Big] \tag{1.39}$$

where $X = R$ (Ricci scalar), $Y = R_{\alpha\beta}R^{\alpha\beta}$ (Ricci tensor square) and $Z = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ (Rieman tensor square). In the metric approach, the field equations are obtained by varying (1.19) with respect to $g_{\mu\nu}$. We get

$$\begin{aligned}f_XR_{\mu\nu} - \frac{f}{2}g_{\mu\nu} - f_{X;\mu\nu} + g_{\mu\nu}\Box f_X + 2f_YR_{\mu}{}^{\alpha}R_{\alpha\nu} - 2[f_YR^{\alpha}{}_{(\mu}]_{;\nu)\alpha} + \Box[f_YR_{\mu\nu}] + \\ +[f_YR_{\alpha\beta}]^{;\alpha\beta}g_{\mu\nu} + 2f_ZR_{\mu\alpha\beta\gamma}R_{\nu}{}^{\alpha\beta\gamma} - 4[f_ZR_{\mu}{}^{\alpha\beta}{}_{\nu}]_{;\alpha\beta} = \mathcal{X}\,T_{\mu\nu}\end{aligned} \tag{1.40}$$

where $f_X = \frac{df}{dX}$, $f_Y = \frac{df}{dY}$, $f_Z = \frac{df}{dZ}$, $\Box = {}_{;\sigma}{}^{;\sigma}$.

1.2.3 Scalar-Tensor-Forth-Order Gravity: Noncommutative Spectral Geometry

The following action defines the Scalar-Tensor-Forth-Order Gravity:

$$\mathcal{S}[g,\phi] = \int d^4x\sqrt{-g}\Big[f(R, R_{\alpha\beta}R^{\alpha\beta},\phi) + \omega(\phi)\phi_{;\alpha}\phi^{;\alpha} + \mathcal{X}\mathcal{L}_m\Big]\,, \tag{1.41}$$

where f is an unspecified function of the Ricci scalar R, the curvature invariant $R_{\alpha\beta}R^{\alpha\beta}$ where $R_{\alpha\beta}$ is the Ricci tensor, and a scalar field ϕ. Here $\mathcal{L}_m$ is the minimally coupled ordinary matter Lagrangian density, ω is a generic function of the scalar field. In the metric approach, namely when the gravitational field is fully described by the metric tensor $g_{\mu\nu}$ only[2], the field equations are obtained by varying the action (1.41) with respect to $g_{\mu\nu}$, leading to

$$\begin{aligned} & f_R R_{\mu\nu} - \frac{f + \omega(\phi)\phi_{;\alpha}\phi^{;\alpha}}{2} g_{\mu\nu} - f_{R;\mu\nu} + g_{\mu\nu}\Box f_R + 2 f_Y R_\mu{}^\alpha R_{\alpha\nu} \\ & -2[f_Y R^\alpha{}_{(\mu}]_{;\nu)\alpha} + \Box[f_Y R_{\mu\nu}] + [f_Y R_{\alpha\beta}]^{;\alpha\beta} g_{\mu\nu} + \omega(\phi)\phi_{;\mu}\phi_{;\nu} \;=\; \mathcal{X}\, T_{\mu\nu}\;. \end{aligned} \tag{1.42}$$

The trace of the field equation (1.42) above, reads

$$f_R R + 2 f_Y R_{\alpha\beta}R^{\alpha\beta} - 2f + \Box[3 f_R + f_Y R] + 2[f_Y R^{\alpha\beta}]_{;\alpha\beta} - \omega(\phi)\phi_{;\alpha}\phi^{;\alpha} \;=\; \mathcal{X}\, T\;, \tag{1.43}$$

where $T \;=\; T^\sigma{}_\sigma$ is the trace of energy-momentum tensor.

By varying the action (1.41) with respect to the scalar field ϕ, we obtain the Klein-Gordon field equation

$$2\omega(\phi)\Box\phi + \omega_\phi(\phi)\phi_{;\alpha}\phi^{;\alpha} - f_\phi \;=\; 0\;, \tag{1.44}$$

where $\omega_\phi \;=\; \frac{d\omega}{d\phi}$ and $f_\phi \;=\; \frac{df}{d\phi}$.

A a particular model of Scalar-Tensor-Forth-Order Gravity, derived by a fundamental theory, is the Noncommutative Spectral Geometry [264, 265].

Noncommutative Spectral Geometry

Running backwards in time the evolution of our universe, we approach extremely high energy scales and huge densities within tiny spaces. At such extreme conditions, GR can no longer describe satisfactorily the underlined physics, and a full Quantum Gravity Theory has to be invoked. Different Quantum Gravity approaches have been worked out in the literature; they should all lead to GR, considered as an effective theory, as one reaches energy scales much below the Planck scale.

Even though Quantum Gravity may imply that at Planck energy scales spacetime is a widly noncommutative manifold, one may safely assume that at scales a few orders of magnitude below the Planck scale, the spacetime is only mildy noncommutative. At such intermediate scales, the algebra of coordinates can be considered as an almost-commutative algebra of matrix valued functions, which if appropriately chosen, can lead to the Standard Model of particle physics.

[2]It is worth noticing that in metric-affine theories, the gravitational field is completely assigned by the metric tensor $g_{\mu\nu}$, while the affinity connections $\Gamma^\alpha_{\mu\nu}$ are considered as independent fields [257].

The application of the spectral action principle [266] to this almost-commutative manifold led to the NonCommutative Spectral Geometry (NCSG) [267, 268, 269], a framework that offers a purely geometric explanation of the Standard Model of particles coupled to gravity [270, 271].

For almost-commutative manifolds, the geometry is described by the tensor product $\mathcal{M}\times\mathcal{F}$ of a four-dimensional compact Riemannian manifold $\mathcal{M}$ and a discrete noncommutative space $\mathcal{F}$, with $\mathcal{M}$ describing the geometry of spacetime and $\mathcal{F}$ the internal space of the particle physics model. The noncommutative nature of $\mathcal{F}$ is encoded in the spectral triple $(\mathcal{A}_{\mathcal{F}}, \mathcal{H}_{\mathcal{F}}, D_{\mathcal{F}})$. The algebra $\mathcal{A}_{\mathcal{F}} = C^\infty(\mathcal{M})$ of smooth functions on $\mathcal{M}$, playing the rôle of the algebra of coordinates, is an involution of operators on the finite-dimensional Hilbert space $\mathcal{H}_{\mathcal{F}}$ of Euclidean fermions. The operator $D_{\mathcal{F}}$ is the Dirac operator $\partial\!\!\!/_{\mathcal{M}} = \sqrt{-1}\gamma^\mu\nabla^s_\mu$ on the spin manifold $\mathcal{M}$; it corresponds to the inverse of the Euclidean propagator of fermions and is given by the Yukawa coupling matrix and the Kobayashi-Maskawa mixing parameters.

The algebra $\mathcal{A}_{\mathcal{F}}$ has to be chosen so that it can lead to the Standard Model of particle physics, while it must also fulfill noncommutative geometry requirements. It was hence chosen to be [272, 273, 274]

$$\mathcal{A}_{\mathcal{F}} = M_a(\mathbb{H}) \oplus M_k(\mathbb{C}) \ ,$$

with $k = 2a$; $\mathbb{H}$ is the algebra of quaternions, which encodes the noncommutativity of the manifold. The first possible value for k is 2, corresponding to a Hilbert space of four fermions; it is ruled out from the existence of quarks. The minimum possible value for k is 4 leading to the correct number of $k^2 = 16$ fermions in each of the three generations. Higher values of k can lead to particle physics models beyond the Standard Model [275, 276]. The spectral geometry in the product $\mathcal{M}\times\mathcal{F}$ is given by the product rules:

$$\begin{aligned}
\mathcal{A} &= C^\infty(\mathcal{M}) \oplus \mathcal{A}_{\mathcal{F}} \ ,\\
\mathcal{H} &= L^2(\mathcal{M}, S) \oplus \mathcal{H}_{\mathcal{F}} \ ,\\
\mathcal{D} &= \mathcal{D}_{\mathcal{M}} \oplus 1 + \gamma_5 \oplus \mathcal{D}_{\mathcal{F}} \ ,
\end{aligned} \tag{1.45}$$

where $L^2(\mathcal{M}, S)$ is the Hilbert space of L^2 spinors and $\mathcal{D}_{\mathcal{M}}$ is the Dirac operator of the Levi-Civita spin connection on $\mathcal{M}$. Applying the spectral action principle to the product geometry $\mathcal{M}\times\mathcal{F}$ leads to the NCSG action

$$\mathrm{Tr}(f(D_{\mathcal{A}}/\Lambda)) + (1/2)\langle J\psi, D\psi\rangle \ ,$$

splitted into the bare bosonic action and the fermionic one. Note that $D_{\mathcal{A}} = D + \mathcal{A} + \epsilon' J\mathcal{A}J^{-1}$ are uni-modular inner fluctuations, f is a cutoff function and Λ fixes the energy scale, J is the real structure on the spectral triple and ψ is a spinor in the Hilbert space $\mathcal{H}$ of the quarks and leptons. In what follows we concentrate on the bosonic part of the action, seen as the bare action at the mass scale Λ which includes the eigenvalues of the Dirac operator that are smaller than the cutoff scale Λ, considered as the grand unification scale. Using heat kernel methods, the trace $\mathrm{Tr}(f(\mathcal{D}_A/\Lambda)$ can be written in terms of the geometrical Seeley-de Witt coefficients a_n known for any second order elliptic differential operator, as $\Sigma_{n=0}^{\infty} F_{4-n}\Lambda^{4-n}a_n$ where the function F is defined such that $F(\mathcal{D}_A^2) = f(\mathcal{D}_A)$. Considering the Riemannian geometry to be four-dimensional, the asymptotic expansion of the trace reads [277, 278]

$$\mathrm{Tr}(f(\mathcal{D}_{\mathcal{A}}/\Lambda)) \sim 2\Lambda^4 f_4 a_0 + 2\Lambda^2 f_2 a_2 + f_0 a_4 + \cdots + \Lambda^{-2k} f_{-2k} a_{4+2k} + \cdots \ , \tag{1.46}$$

where f_k are the momenta of the smooth even test (cutoff) function which decays fast at infinity, and only enters in the multiplicative factors:

$$\begin{aligned} f_0 &= f(0) \ , \\ f_2 &= \int_0^\infty f(u) u \mathrm{d}u \ , \\ f_4 &= \int_0^\infty f(u) u^3 \mathrm{d}u \ , \\ f_{-2k} &= (-1)^k \frac{k!}{(2k)!} f^{(2k)}(0) \ . \end{aligned}$$

Since the Taylor expansion of the f function vanishes at zero, the asymptotic expansion of the spectral action reduces to

$$\mathrm{Tr}(f(\mathcal{D}_{\mathcal{A}}/\Lambda)) \sim 2\Lambda^4 f_4 a_0 + 2\Lambda^2 f_2 a_2 + f_0 a_4 \ . \tag{1.47}$$

Hence, the cutoff function f plays a rôle only through its momenta f_0, f_2, f_4, three real parameters, related to the coupling constants at unification, the gravitational constant, and the cosmological constant, respectively.

The NCSG model lives by construction at the grand unification scale, hence providing a framework to study early universe cosmology [279, 280, 281, 282]. The gravitational part of the asymptotic expression for the bosonic sector of the NCSG action[3], including the coupling between the Higgs field ϕ and the Ricci curvature scalar R, in Lorentzian signature, obtained

[3]Note that the obtained action does not suffer fot negative energy massive graviton modes [283].

through a Wick rotation in imaginary time, reads [270]

$$
\begin{aligned}
\mathcal{S}_{\rm NCSG} \;=\; & \int d^4x\sqrt{-g}\Bigg[\frac{R}{2{\kappa_0}^2} + \alpha_0\, C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \gamma_0 + \tau_0\, R^\star R^\star + \frac{G^i_{\mu\nu}G^{i\mu\nu}}{4} + \\
& + \frac{F^\alpha_{\mu\nu}F^{\alpha\mu\nu}}{4} + \frac{B^{\mu\nu}B_{\mu\nu}}{4} + \frac{\mathbf{H}_{;\alpha}\mathbf{H}^{;\alpha}}{2} - {\mu_0}^2\,\mathbf{H}^2 - \frac{R\,\mathbf{H}^2}{12} + \lambda_0\,\mathbf{H}^4\Bigg],
\end{aligned}
\tag{1.48}
$$

where $\mathbf{H} = (\sqrt{af_0}/\pi)\phi$ is a rescaling of the Higgs field ϕ with a a parameter related to fermion and lepton masses and lepton mixing, $R^\star$ is the topological term, while $G^i_{\mu\nu}$, $F^i_{\mu\nu}$ and $B_{\mu\nu}$ are the gauge fields. At unification scale (setup by Λ), $\kappa_0^2 = \frac{12\pi^2}{96f_2\Lambda^2 - f_0\mathfrak{c}}$, $\alpha_0 = -\frac{3f_0}{10\pi^2}$, $\tau_0 = \frac{11f_0}{60\pi^2}$, $\xi_0 = \frac{1}{12}$, $\mu_0^2 = 2\Lambda^2\frac{f_2}{f_0} - \frac{\mathfrak{e}}{\mathfrak{a}}$, $\gamma_0 = \frac{1}{\pi^2}\left(48f_4\Lambda^4 - f_2\Lambda^2\mathfrak{c} + \frac{f_0}{4}\mathfrak{d}\right)$, and $\lambda_0 = \frac{\pi^2\mathfrak{b}}{2f_0\mathfrak{a}^2}$. The geometric parameters $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{e}$ describe the possible choices of Dirac operators on the finite noncommutative space. These parameters correspond to the Yukawa parameters of the particle physics model and the Majorana terms for the right-handed neutrinos [268, 270]. The square of the Weyl tensor can be expressed in terms of R^2 and $R_{\alpha\beta}R^{\alpha\beta}$ as

$$
C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} \;=\; 2R_{\alpha\beta}R^{\alpha\beta} - \frac{2}{3}R^2.
$$

The above action (1.48) is clearly a particular case of the action (1.41) describing a general model of an Extended Theory of Gravity.

1.3 The Palatini Approach

As we said, the Palatini approach, considering g and Γ as independent fields, is "intrinsically" bi-metric and capable of disentangling the geodesic structure from the chronological structure of a given manifold. Starting from these considerations, conformal transformations assume a fundamental role in defining the affine connection which is merely "Levi - Civita" only for the Hilbert-Einstein theory.

In this section, we work out examples showing how conformal transformations assume a fundamental physical role in relation to the Palatini approach in Extended Theories of Gravity [178].

Let us start from the case of fourth-order gravity where Palatini variational principle is straightforward in showing the differences with Hilbert-Einstein variational principle, involving only metric. Besides, cosmological applications of f-gravity have shown the relevance of Palatini formalism, giving physically interesting results with singularity - free solutions [327, 69, 70, 71, 72, 73, 74]. This last nice feature is not present in the standard metric approach.

An important remark is in order at this point. The Ricci scalar entering in f is $\mathcal{R} \equiv$

$\mathcal{R}(g\,,\Gamma) \;=\; g^{\alpha\beta}\mathcal{R}_{\alpha\beta}(\Gamma)$ that is a *generalized Ricci scalar* and $\mathcal{R}_{\mu\nu}(\Gamma)$ is the Ricci tensor of a torsion-less connection Γ, which, *a priori*, has no relations with the metric g of spacetime. The gravitational part of the Lagrangian is controlled by a given real analytical function of one real variable f, while $\sqrt{-g}$ denotes a related scalar density of weight 1. Field equations, deriving from the Palatini variational principle are:

$$f_{\mathcal{R}}\mathcal{R}_{(\mu\nu)}(\Gamma)-\frac{1}{2}fg_{\mu\nu}\;=\;\mathcal{X}\,T_{\mu\nu} \tag{1.49}$$

$$\nabla^{\Gamma}_{\alpha}(\sqrt{-g}f_{\mathcal{R}}g^{\mu\nu})=0 \tag{1.50}$$

where ∇^{Γ} is the covariant derivative with respect to Γ and $f_{\mathcal{R}}\;=\;\frac{\partial f(\mathcal{R})}{\partial\mathcal{R}}$. We shall use the standard notation denoting by $\mathcal{R}_{(\mu\nu)}$ the symmetric part of $\mathcal{R}_{\mu\nu}$, *i.e.* $\mathcal{R}_{(\mu\nu)}\equiv\frac{1}{2}(\mathcal{R}_{\mu\nu}+\mathcal{R}_{\nu\mu})$.

In order to get the first one of (1.49), one has to additionally assume that $\mathcal{L}_m$ is functionally independent of Γ; however it may contain metric covariant derivatives $\overset{g}{\nabla}$ of fields. This means that the matter stress-energy tensor $T_{\mu\nu}=T_{\mu\nu}(g,\Psi)$ depends on the metric g and some matter fields denoted here by Ψ, together with their derivatives (covariant derivatives with respect to the Levi - Civita connection of g). From the second one of (1.49) one sees that $\sqrt{-g}f_{\mathcal{R}}g^{\mu\nu}$ is a symmetric twice contravariant tensor density of weight 1. As previously discussed in [178, 179], this naturally leads to define a new metric $h_{\mu\nu}$, such that the following relation holds:

$$\sqrt{-g}f_{\mathcal{R}}g^{\mu\nu}=\sqrt{-h}h^{\mu\nu}\,. \tag{1.51}$$

This *ansatz* is suitably made in order to impose Γ to be the Levi - Civita connection of h and the only restriction is that $\sqrt{-g}f_{\mathcal{R}}g^{\mu\nu}$ should be non-degenerate. The Eq (1.50) can be put in the following form $\nabla^{\Gamma}_{\alpha}g^{\mu\nu}=\partial_{\alpha}\ln\big(f_{\mathcal{R}}\big)\,g^{\mu\nu}$. Is easy to see that if $f_{\mathcal{R}}=1$ (GR Theory) we get $\nabla^{\Gamma}_{\alpha}g^{\mu\nu}=0$. Then, the connection Γ correspond with the Christoffel symbols.

Eq.(1.51) imposes that the two metrics h and g are conformally equivalent. The corresponding conformal factor can be easily found to be $f_{\mathcal{R}}$ (in dim $\mathcal{M}=4$) and the conformal transformation results to be ruled by:

$$h_{\mu\nu}=f_{\mathcal{R}}g_{\mu\nu} \tag{1.52}$$

Therefore, as it is well known, Eq.(1.49) implies that $\Gamma=\Gamma_{LC}(h)$ and $\mathcal{R}_{(\mu\nu)}(\Gamma)=\mathcal{R}_{\mu\nu}(h)\equiv\mathcal{R}_{\mu\nu}$. Field equations can be supplemented by the scalar-valued equation obtained by taking the trace of (1.49)

$$f_{\mathcal{R}}\mathcal{R} - 2f = \mathcal{X}T \tag{1.53}$$

which controls solutions of (1.49).

We shall refer to this scalar-valued equation as *the structural equation* of the spacetime. In the vacuum case (and spacetimes filled with radiation, such that $T = 0$) this scalar-valued equation admits constant solutions, which are different from zero only if one add a cosmological constant. This means that the universality of Einstein field equations holds [179], corresponding to a theory with cosmological constant [86].

In the case of interaction with matter fields, the structural equation (1.52), if explicitly solvable, provides an expression of $\mathcal{R} = \mathcal{R}(T)$ and consequently both f and $f_{\mathcal{R}}$ can be expressed in terms of T. The matter content of spacetime thus rules the bi-metric structure of spacetime and, consequently, both the geodesic and metric structures which are intrinsically different. This behavior generalizes the vacuum case and corresponds to the case of a time-varying cosmological constant. In other words, due to these features, conformal transformations, which allow to pass from a metric structure to another one, acquire an intrinsic physical meaning since "select" metric and geodesic structures which, for a given Extended Theories of Gravity, in principle, *do not* coincide.

Let us now try to extend the above formalism to the case of non-minimally coupled scalar-tensor theories. The effort is to understand if and how the bi-metric structure of spacetime behaves in this cases and which could be its geometric and physical interpretation.

We start by considering scalar-tensor theories in the Palatini formalism, calling $\mathcal{S}_1[g,\Gamma]$ the action functional. After, we take into account the case of decoupled non-minimal interaction between a scalar-tensor theory and a f-theory, calling $\mathcal{S}_2[g,\Gamma]$ this action functional. We finally consider the case of non-minimal-coupled interaction between the scalar field ϕ and the gravitational fields (g,Γ), calling $\mathcal{S}_3[g,\Gamma]$ the corresponding action functional. Particularly significant is, in this case, the limit of low curvature $\mathcal{R}$. This resembles the physical relevant case of present values of curvatures of the Universe and it is important for cosmological applications.

The action (1.24) for scalar-tensor gravity can be generalized, in order to better develop the Palatini approach, as:

$$\mathcal{S}_1[g,\Gamma] = \int d^4x\sqrt{-g}[F(\phi)\mathcal{R} + \omega(\phi) \overset{g}{\nabla}_\mu \phi \overset{g}{\nabla}{}^\mu \phi + V(\phi) + \mathcal{X}\mathcal{L}_m(\Psi, \overset{g}{\nabla} \Psi)] \tag{1.54}$$

As above, the values of $\omega(\phi) = \pm 1$ selects between standard scalar field theories and quintessence (phantom) field theories. The relative "signature" can be selected by conformal transformations. Field equations for the gravitational part of the action are, respectively for the metric g and the connection Γ:

$$
\begin{aligned}
F(\phi)[\mathcal{R}_{(\mu\nu)} - \frac{\mathcal{R}}{2} g_{\mu\nu}] &= \mathcal{X} T_{\mu\nu} + \frac{1}{2}\omega(\phi) \overset{g}{\nabla}_\mu \phi \overset{g}{\nabla}{}^\mu \phi g_{\mu\nu} + \frac{1}{2} V(\phi) g_{\mu\nu} \\
\nabla^\Gamma_\alpha (\sqrt{-g} F(\phi) g^{\mu\nu}) &= 0
\end{aligned}
\tag{1.55}
$$

$\mathcal{R}_{(\mu\nu)}$ is the same defined in (1.49). For matter fields we have the following field equations:

$$
2\omega(\phi) \overset{g}{\Box} \phi + \omega_{,\phi}(\phi) \overset{g}{\nabla}_\alpha \phi \overset{g}{\nabla}{}^\alpha \phi + V_{,\phi}(\phi) + F_{,\phi}(\phi)\mathcal{R} = 0
$$

$$
\frac{\delta \mathcal{L}_m}{\delta \Psi} = 0
$$

In this case, the structural equation of spacetime implies that:

$$
\mathcal{R} = -\frac{\mathcal{X} T + 2\omega(\phi) \overset{g}{\nabla}_\alpha \phi \overset{g}{\nabla}{}^\alpha \phi + 2V(\phi)}{F(\phi)} \tag{1.56}
$$

which expresses the value of the Ricci scalar curvature in terms of the traces of the stress-energy tensors of standard matter and scalar field (we have to require $F(\phi) \neq 0$). The bi-metric structure of spacetime is thus defined by the ansatz:

$$
\sqrt{-g} F(\phi) g^{\mu\nu} = \sqrt{-h} h^{\mu\nu} \tag{1.57}
$$

such that g and h result to be conformally related

$$
h_{\mu\nu} = F(\phi) g_{\mu\nu} \tag{1.58}
$$

The conformal factor is exactly the interaction factor. From (1.56), it follows that in the vacuum case $T = 0$ and $\omega(\phi) \overset{g}{\nabla}_\alpha \phi \overset{g}{\nabla}{}^\alpha \phi + V(\phi) = 0$: this theory is equivalent to the standard Einstein one without matter. On the other hand, for $F(\phi) = F_0$ we recover the Einstein theory plus a minimally coupled scalar field: this means that the Palatini approach intrinsically gives rise to the conformal structure (1.58) of the theory which is trivial in the Einstein, minimally coupled case.

As a further step, let us generalize the previous results considering the case of a non-minimal coupling in the framework of f-theories. The action functional can be written as:

$$\mathcal{S}_2[g,\Gamma] = \int d^4x\sqrt{-g}[F(\phi)f(\mathcal{R}) + \omega(\phi)\stackrel{g}{\nabla}_\alpha \phi \stackrel{g}{\nabla}{}^\alpha \phi + V(\phi) + 2\mathcal{X}\mathcal{L}_m(\Psi, \stackrel{g}{\nabla}\Psi)] \quad (1.59)$$

where f is, as usual, any analytical function of the Ricci scalar $\mathcal{R}$. Field equations (in the Palatini formalism) for the gravitational part of the action are:

$$\begin{aligned} F(\phi)[f_{\mathcal{R}}\mathcal{R}_{(\mu\nu)} - \frac{f}{2}g_{\mu\nu}] &= \mathcal{X}T_{\mu\nu} + \frac{1}{2}\omega(\phi)\stackrel{g}{\nabla}_\alpha \phi \stackrel{g}{\nabla}{}^\alpha \phi\, g_{\mu\nu} + \frac{1}{2}V(\phi)g_{\mu\nu} \\ \nabla^\Gamma_\alpha(\sqrt{-g}F(\phi)f_{\mathcal{R}}g^{\mu\nu}) &= 0 \end{aligned}$$

For scalar and matter fields we have, otherwise, the following field equations:

$$\begin{aligned} &2\omega(\phi)\stackrel{g}{\Box}\phi + \omega_{,\phi}(\phi)\stackrel{g}{\nabla}_\alpha \phi \stackrel{g}{\nabla}{}^\alpha \phi + V_{,\phi}(\phi) + F_{,\phi}(\phi)f(\mathcal{R}) = 0 \\ &\frac{\delta\mathcal{L}_m}{\delta\Psi} = 0 \end{aligned}$$

where the non-minimal interaction term enters into the modified Klein-Gordon equations. In this case the structural equation of spacetime implies that:

$$f_{\mathcal{R}}\mathcal{R} - 2f = \frac{\mathcal{X}T + 2\omega(\phi)\stackrel{g}{\nabla}_\alpha \phi \stackrel{g}{\nabla}{}^\alpha \phi + 2V(\phi)}{F(\phi)} \quad (1.60)$$

We remark again that this equation, if solved, expresses the value of the Ricci scalar curvature in terms of traces of the stress-energy tensors of standard matter and scalar field (we have to require again that $F(\phi) \neq 0$). The bi-metric structure of spacetime is thus defined by the ansatz:

$$\sqrt{-g}F(\phi)f_{\mathcal{R}}g^{\mu\nu} = \sqrt{-h}h^{\mu\nu} \quad (1.61)$$

such that g and h result to be conformally related by:

$$h_{\mu\nu} = F(\phi)f_{\mathcal{R}}g_{\mu\nu} \quad (1.62)$$

Once the structural equation is solved, the conformal factor depends on the values of the matter fields (ϕ, Ψ) or, more precisely, on the traces of the stress-energy tensors and the value of ϕ. From equation (1.60), it follows that in the vacuum case, *i.e.* both $T = 0$ and $\omega(\phi) \overset{g}{\nabla}_\alpha \phi \overset{g}{\nabla}{}^\alpha \phi + V(\phi) = 0$, the universality of Einstein field equations still holds as in the case of minimally interacting f-theories [179]. The validity of this property is related to the decoupling of the scalar field and the gravitational field.

Let us finally consider the case where the gravitational Lagrangian is a general function of ϕ and R. The action functional can thus be written as:

$$\mathcal{S}_3[g,\Gamma] = \int d^4x\sqrt{-g}[K(\phi\,,\mathcal{R}) + \omega(\phi) \overset{g}{\nabla}_\alpha \phi \overset{g}{\nabla}{}^\alpha \phi + V(\phi) + \mathcal{X}\mathcal{L}_m(\Psi, \overset{g}{\nabla} \Psi)] \tag{1.63}$$

Field equations for the gravitational part of the action are:

$$\begin{aligned}
\frac{\partial K(\phi,\mathcal{R})}{\partial \mathcal{R}}\mathcal{R}_{(\mu\nu)} - \frac{K(\phi,\mathcal{R})}{2}g_{\mu\nu} &= \mathcal{X}T_{\mu\nu} + \frac{1}{2}\omega(\phi) \overset{g}{\nabla}_\alpha \phi \overset{g}{\nabla}{}^\alpha \phi\; g_{\mu\nu} + \frac{1}{2}V(\phi)g_{\mu\nu} \\
\nabla^\Gamma_\alpha\left[\sqrt{-g}\left(\frac{\partial K(\phi,\mathcal{R})}{\partial \mathcal{R}}\right)g^{\mu\nu}\right] &= 0
\end{aligned}$$

For matter fields, we have:

$$\begin{aligned}
&2\omega(\phi) \overset{g}{\Box} \phi + \omega_{,\phi}(\phi) \overset{g}{\nabla}_\alpha \phi \overset{g}{\nabla}{}^\alpha \phi + V_{,\phi}(\phi) + \frac{\partial K(\phi,\mathcal{R})}{\partial \phi} = 0 \\
&\frac{\delta \mathcal{L}_m}{\delta \Psi} = 0
\end{aligned}$$

The structural equation of spacetime can be expressed as:

$$\frac{\partial K(\phi,\mathcal{R})}{\partial \mathcal{R}}\mathcal{R} - 2K(\phi,\mathcal{R}) = \mathcal{X}T + 2\omega(\phi) \overset{g}{\nabla}_\alpha \phi \overset{g}{\nabla}{}^\alpha \phi + 2V(\phi) \tag{1.64}$$

This equation, if solved, expresses again the form of the Ricci scalar curvature in terms of traces of the stress-energy tensors of matter and scalar field (we have to impose regularity conditions and, for example, $K(\phi,\mathcal{R}) \neq 0$). The bi-metric structure of spacetime is thus defined by the ansatz:

$$\sqrt{-g}\frac{\partial K(\phi,\mathcal{R})}{\partial \mathcal{R}}g^{\mu\nu} = \sqrt{-h}h^{\mu\nu} \tag{1.65}$$

such that g and h result to be conformally related by

$$h_{\mu\nu} = \frac{\partial K(\phi, \mathcal{R})}{\partial \mathcal{R}} g_{\mu\nu} \,. \tag{1.66}$$

Again, once the structural equation is solved, the conformal factor depends just on the values of the matter fields and (the trace of) their stress energy tensors. In other words, the evolution, the definition of the conformal factor and the bi-metric structure is ruled by the values of traces of the stress-energy tensors and by the value of the scalar field ϕ. In this case, the universality of Einstein field equations does not hold anymore in general. This is evident from (1.64) where the strong coupling between $\mathcal{R}$ and ϕ avoids the possibility, also in the vacuum case, to achieve simple constant solutions.

We consider, furthermore, the case of small values of $\mathcal{R}$, corresponding to small curvature spacetimes. This limit represents, as a good approximation, the present epoch of the observed Universe under suitably regularity conditions. A Taylor expansion of the analytical function $K(\phi, \mathcal{R})$ can be performed:

$$K(\phi, \mathcal{R}) = K_0(\phi) + K_1(\phi)\mathcal{R} + \mathcal{O}(\mathcal{R}^2) \tag{1.67}$$

where only the first leading term in $\mathcal{R}$ is considered and we have defined:

$$\begin{aligned} K_0(\phi) &= K(\phi, \mathcal{R})_{\mathcal{R}=0} \\ K_1(\phi) &= \left(\frac{\partial K(\phi, \mathcal{R})}{\partial \mathcal{R}}\right)_{\mathcal{R}=0} \end{aligned}$$

Substituting this expression in (1.64) and (1.66) we get (neglecting higher order approximations in $\mathcal{R}$) the structural equation and the bi-metric structure in this particular case. From the structural equation, we get:

$$\mathcal{R} = -\frac{T + 2\omega(\phi) \stackrel{g}{\nabla}_\alpha \phi \stackrel{g}{\nabla}{}^\alpha \phi + 2V(\phi) + 2K_0(\phi)}{K_1(\phi)} \tag{1.68}$$

such that the value of the Ricci scalar is always determined, in this first order approximation, in terms of $\omega(\phi) \stackrel{g}{\nabla}_\alpha \phi \stackrel{g}{\nabla}{}^\alpha \phi + V(\phi)$, T e ϕ. The bi-metric structure is, otherwise, simply defined by means of the first term of the Taylor expansion, which is

$$h_{\mu\nu} = K_1(\phi) g_{\mu\nu} \,. \tag{1.69}$$

It reproduces, as expected, the scalar-tensor case (1.58). In other words, scalar-tensor theories can be recovered in a first order approximation of a general theory where gravity and non-minimal couplings are any (compare (1.68) with (1.56)). This fact agrees with the above considerations where Lagrangians of physical interactions can be considered as stochastic functions with local gauge invariance properties.

1.4 Equivalence between $f(R)$ and scalar-tensor gravity

Metric and Palatini $f(R)$ gravities are equivalent to scalar-tensor theories with the derivative of the function $f(R)$ playing the role of the Brans-Dicke scalar. We illustrate this equivalence beginning with the metric formalism.

In metric $f(R)$-gravity, we introduce the scalar $\phi \equiv R$; then the action

$$\mathcal{S}[g] = \int d^4x \sqrt{-g}\, [\, f(R) + \mathcal{X}\mathcal{L}_m] \tag{1.70}$$

is rewritten in the form

$$\mathcal{S}[g] = \int d^4x \sqrt{-g}\, [\psi(\phi)R - V(\phi) + \mathcal{X}\mathcal{L}_m] \tag{1.71}$$

when $f''(R) \neq 0$, where

$$\psi = f_\chi(\phi) \,, \qquad\qquad V(\phi) = \phi f_\chi(\phi) - f(\phi) \,. \tag{1.72}$$

It is trivial to see that the action (1.71) coincides with (1.70) if $\phi = R$. *Vice-versa*, let us vary the action (1.71) with respect to ϕ, which leads to

$$R\,\frac{d\psi}{d\phi} - \frac{dV}{d\phi} = (R - \phi)\, f''(R) = 0 \,. \tag{1.73}$$

Eq. (1.73) implies that $\phi = R$ when $f''(R) \neq 0$. The action (1.71) has the Brans-Dicke form

$$\mathcal{S}[g] = \int d^4x \sqrt{-g} \left[\psi R - \frac{\omega_{BD}}{2}\, \nabla^\mu \psi \nabla_\mu \psi - U(\psi) + \mathcal{X}\mathcal{L}_m\right] \tag{1.74}$$

with Brans-Dicke field ψ, Brans-Dicke parameter $\omega_{BD} = 0$, and potential $U(\psi) = V\,[\phi(\psi)]$. An $\omega_{BD} = 0$ Brans-Dicke theory was originally studied for the purpose of obtaining a Yukawa correction to the Newtonian potential in the weak-field limit [183] and called "O'Hanlon theory" or "massive dilaton gravity". The variation of the action (1.71) yields the field equations

$$G_{\mu\nu} = \frac{\kappa}{\psi} T^{(m)}_{\mu\nu} - \frac{1}{2\psi} U(\psi) g_{\mu\nu} + \frac{1}{\psi} \left(\nabla_\mu \nabla_\nu \psi - g_{\mu\nu} \Box \psi \right) , \tag{1.75}$$

$$3\Box\psi + 2U(\psi) - \psi \frac{dU}{d\psi} = \kappa T^{(m)} . \tag{1.76}$$

Palatini $f(R)$-gravity is also equivalent to a special Brans-Dicke theory with a scalar field potential. The Palatini action

$$\mathcal{S}[g,\Gamma] = \int d^4x \sqrt{-g} [f(\mathcal{R}) + \mathcal{X}\mathcal{L}_m] \tag{1.77}$$

is equivalent to

$$\mathcal{S}[g,\Gamma] = \int d^4x \sqrt{-g} \left[f(\chi) + f_\chi(\chi) \left(\mathcal{R} - \chi \right) + \mathcal{X}\mathcal{L}_m \right] . \tag{1.78}$$

It is straightforward to see that the variation of this action with respect to χ yields $\chi = \mathcal{R}$. We can now use the field $\phi \equiv f_\chi(\chi)$ and the fact that the curvature $\mathcal{R}$ is the (metric) Ricci curvature of the new metric $h_{\mu\nu} = f_{\mathcal{R}}(\mathcal{R})\, g_{\mu\nu}$ conformally related to $g_{\mu\nu}$, as already explained. Using now the well known transformation property of the Ricci scalar under conformal rescalings

$$\mathcal{R} = R + \frac{3}{2\phi} \nabla^\alpha \phi \nabla_\alpha \phi - \frac{3}{2} \Box \phi \tag{1.79}$$

and discarding a boundary term, the action (1.78) can be presented in the form

$$\mathcal{S}[g,\Gamma] = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \left[\phi R + \frac{3}{2\phi} \nabla^\alpha \phi \nabla_\alpha \phi - V(\phi) + \mathcal{X}\mathcal{L}_m \right] , \tag{1.80}$$

where

$$V(\phi) = \phi \chi(\phi) - f\left[\chi(\phi) \right] . \tag{1.81}$$

This action is clearly that of a Brans-Dicke theory with Brans-Dicke parameter $\omega = -3/2$ and a potential.

Chapter 2

Weak field Limit and Conformal Transformations of Extended Theories of Gravity

At Galactic and Solar System scales, Extended Theories of Gravity exhibit gravitational potentials with non-Newtonian corrections [131, 330]. This feature was discovered long ago [236], and recent interest arises from the possibility of explaining the flatness of the rotation curves of spiral galaxies without huge amounts of dark matter.

In this chapter we discuss the weak-field limit of Extended Theories of Gravity without specifying the form of the theory and highlighting the differences and similarities with the post-Newtonian [**D**] and post-Minkowskian limits of GR [**E**]. Weak-field experiments such as light bending, the perihelion shift of planets, and gravito-electro-magnetism experiments are valuable tests of Extended Theories of Gravity. There are sufficient theoretical predictions to state that certain higher order theories of gravity can be compatible with Newtonian and post-Newtonian experiments.

Subsequently, the weak field limit of scalar tensor theories of gravity is discussed in view of conformal transformations [**C**]. Specifically, we consider how physical quantities, like gravitational potentials derived in the Newtonian approximation for the same scalar-tensor theory, behave in the Jordan and in the Einstein frame. The approach allows to discriminate features that are invariant under conformal transformations and gives contributions in the debate of selecting the true physical frame. As a particular example, the case of $f(R)$ gravity is considered.

2.1 Newtonian and post $-$ Newtonian approximation

In this section, we provide the explicit form of the various quantities needed to compute the Newtonian and post $-$ Newtonian approximation approximations in the field equations in GR theory and any metric theory of gravity.

If one consider a system of gravitationally interacting particles of mass $\bar{M}$, the kinetic energy $\frac{1}{2}\bar{M}\bar{v}^2$ will be, roughly, of the same order of magnitude as the typical potential energy $U = G\bar{M}^2/\bar{r}$, with $\bar{M}$, $\bar{r}$, and $\bar{v}$ the typical average values of masses, separations, and velocities of these particles. As a consequence:

$$\bar{v}^2 \sim \frac{G\bar{M}}{\bar{r}}\,, \tag{2.1}$$

(for instance, a test particle in a circular orbit of radius r about a central mass M will have velocity v given in Newtonian mechanics by the exact formula $v^2 = GM/r$.)

The post-Newtonian approximation can be described as a method for obtaining the motion of the system to higher approximations than the first order (approximation which coincides with the Newtonian mechanics) with respect to the quantities $G\bar{M}/\bar{r}$ and $\bar{v}^2$ assumed small with respect to the squared light speed. This approximation is sometimes referred to as an expansion in inverse powers of the light speed.

The typical values of the Newtonian gravitational potential Φ are nowhere larger (in modulus) than 10^{-5} in the Solar System (in geometrized units, Φ is dimensionless). On the other hand, planetary velocities satisfy the condition $\bar{v}^2 \lesssim -\Phi$, while the matter pressure p experienced inside the Sun and the planets is generally smaller than the matter gravitational energy density $-\rho\Phi$, in other words [1] $p/\rho \lesssim -\Phi$. Furthermore one must consider that even other forms of energy in the Solar System (compressional energy, radiation, thermal energy, etc.) have small intensities and the specific energy density Π (the ratio of the energy density to the rest-mass density) is related to U by $\Pi \lesssim U$ (Π is $\sim 10^{-5}$ in the Sun and $\sim 10^{-9}$ in the Earth [131]). As matter of fact, one can consider that these quantities, as function of the velocity, give second order contributions :

$$-\Phi \sim v^2 \sim p/\rho \sim \Pi \sim \mathcal{O}(2)\,. \tag{2.2}$$

Therefore, the velocity v gives $\mathcal{O}(1)$ terms in the velocity expansions, U^2 is of order $\mathcal{O}(4)$, Uv of $\mathcal{O}(3)$, $U\Pi$ is of $\mathcal{O}(4)$, and so on. Considering these approximations, one has

$$\frac{\partial}{\partial t} \sim \mathbf{v}\cdot\nabla\,, \tag{2.3}$$

and

[1] Typical values of p/ρ are $\sim 10^{-5}$ in the Sun and $\sim 10^{-10}$ in the Earth [131].

$$\frac{|\partial/\partial t|}{|\nabla|} \sim \mathcal{O}(1)\,. \tag{2.4}$$

Now, particles move along geodesics :

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\sigma\tau}\frac{dx^\sigma}{ds}\frac{dx^\tau}{ds} = 0\,, \tag{2.5}$$

which can be written in details as

$$\frac{d^2x^i}{dt^2} = -\Gamma^i_{tt} - 2\Gamma^i_{tm}\frac{dx^m}{dt} - \Gamma^i_{mn}\frac{dx^m}{dt}\frac{dx^n}{dt} + \left[\Gamma^t_{tt} + 2\Gamma^t_{tm}\frac{dx^m}{dt} + 2\Gamma^t_{mn}\frac{dx^m}{dt}\frac{dx^n}{dt}\right]\frac{dx^i}{dt}\,. \tag{2.6}$$

In the Newtonian approximation, that is vanishingly small velocities and only first-order terms in the difference between $g_{\mu\nu}$ and the Minkowski metric $\eta_{\mu\nu}$, one obtains that the particle motion equations reduce to the standard result :

$$\frac{d^2x^i}{dt^2} \simeq -\Gamma^i_{tt} \simeq -\frac{1}{2}\frac{\partial g_{tt}}{\partial x^i}\,. \tag{2.7}$$

The quantity $1 - g_{tt}$ is of order $G\bar{M}/\bar{r}$, so that the Newtonian approximation gives $\dfrac{d^2x^i}{dt^2}$ to the order $G\bar{M}/\bar{r}^2$, that is, to the order $\bar{v}^2/r$. As a consequence if we would like to search for the post-Newtonian approximation, we need to compute $\dfrac{d^2x^i}{dt^2}$ to the order $\bar{v}^4/\bar{r}$. Due to the Equivalence Principle and the differentiability of spacetime manifold, we expect that it should be possible to find out a coordinate system in which the metric tensor is nearly equal to the Minkowski one $\eta_{\mu\nu}$, the correction being expandable in powers of $G\bar{M}/\bar{r} \sim \bar{v}^2$. In other words one has to consider the metric developed as follows :

$$\begin{aligned} g_{tt}(t,\mathbf{x}) &\simeq 1 + g^{(2)}_{tt}(t,\mathbf{x}) + g^{(4)}_{tt}(t,\mathbf{x}) + \mathcal{O}(6) \\ g_{ti}(t,\mathbf{x}) &\simeq g^{(3)}_{ti}(t,\mathbf{x}) + \mathcal{O}(5) \\ g_{ij}(t,\mathbf{x}) &\simeq -\delta_{ij} + g^{(2)}_{ij}(t,\mathbf{x}) + \mathcal{O}(4) \end{aligned} \tag{2.8}$$

where δ_{ij} is the Kronecker delta, and for the controvariant form of $g_{\mu\nu}$, one has

$$
\begin{aligned}
g^{tt}(t,\mathbf{x}) &\simeq 1+g^{(2)tt}(t,\mathbf{x})+g^{(4)tt}(t,\mathbf{x})+\mathcal{O}(6)\\
g^{ti}(t,\mathbf{x}) &\simeq g^{(3)ti}(t,\mathbf{x})+\mathcal{O}(5)\\
g^{ij}(t,\mathbf{x}) &\simeq -\delta_{ij}+g^{(2)ij}(t,\mathbf{x})+\mathcal{O}(4)
\end{aligned}
\tag{2.9}
$$

The inverse of the metric tensor (2.8) is defined by (1.1). The relations among the higher than first order terms turn out to be

$$
\begin{aligned}
g^{(2)tt}(t,\mathbf{x}) &= -g^{(2)}_{tt}(t,\mathbf{x})\\
g^{(4)tt}(t,\mathbf{x}) &= g^{(2)}_{tt}(t,\mathbf{x})^2-g^{(4)}_{tt}(t,\mathbf{x})\\
g^{(3)ti} &= g^{(3)}_{ti}\\
g^{(2)ij}(t,\mathbf{x}) &= -g^{(2)}_{ij}(t,\mathbf{x})
\end{aligned}
\tag{2.10}
$$

In evaluating $\Gamma^{\mu}_{\alpha\beta}$ we must take into account that the scale of distance and time, in our systems, are respectively set by $\bar{r}$ and $\bar{r}/\bar{v}$, thus the space and time derivatives should be regarded as being of order

$$
\frac{\partial}{\partial x^i}\sim\frac{1}{\bar{r}}\,,\qquad\frac{\partial}{\partial t}\sim\frac{\bar{v}}{\bar{r}}\,.
\tag{2.11}
$$

Using the above approximations (2.8), (2.9) and (2.10) we have, from the definition (1.3),

$$\Gamma^{(3)t}_{\ \ tt} = \tfrac{1}{2} g^{(2)}_{tt,t} \qquad \Gamma^{(2)i}_{\ \ tt} = \tfrac{1}{2} g^{(2)}_{tt,i}$$

$$\Gamma^{(2)i}_{\ \ jk} = \tfrac{1}{2}\left(g^{(2)}_{jk,i} - g^{(2)}_{ij,k} - g^{(2)}_{ik,j} \right) \qquad \Gamma^{(3)t}_{\ \ ij} = \tfrac{1}{2}\left(g^{(3)}_{ti,j} + g^{(3)}_{jt,i} - g^{(2)}_{ij,t} \right)$$

$$\Gamma^{(3)i}_{\ \ tj} = \tfrac{1}{2}\left(g^{(3)}_{tj,i} - g^{(3)}_{it,j} - g^{(2)}_{ij,t} \right) \qquad \Gamma^{(4)t}_{\ \ ti} = \tfrac{1}{2}\left(g^{(4)}_{tt,i} - g^{(2)}_{tt} g^{(2)}_{tt,i} \right) \tag{2.12}$$

$$\Gamma^{(4)i}_{\ \ tt} = \tfrac{1}{2}\left(g^{(4)}_{tt,i} + g^{(2)}_{im} g^{(2)}_{tt,m} - 2 g^{(3)}_{it,t} \right) \qquad \Gamma^{(2)t}_{\ \ ti} = \tfrac{1}{2} g^{(2)}_{tt,i}$$

The Ricci tensor components (1.4) are

$$\begin{aligned}
R^{(2)}_{tt} &= \frac{1}{2} g^{(2)}_{tt,mm} \\
R^{(4)}_{tt} &= \frac{1}{2} g^{(4)}_{tt,mm} + \frac{1}{2} g^{(2)}_{mn,m} g^{(2)}_{tt,n} + \frac{1}{2} g^{(2)}_{mn} g^{(2)}_{tt,mn} + \frac{1}{2} g^{(2)}_{mm,tt} - \frac{1}{4} g^{(2)}_{tt,m} g^{(2)}_{tt,m} + \\
&\qquad - \frac{1}{4} g^{(2)}_{mm,n} g^{(2)}_{tt,n} - g^{(3)}_{tm,tm} \\
R^{(3)}_{ti} &= \frac{1}{2} g^{(3)}_{ti,mm} - \frac{1}{2} g^{(2)}_{im,mt} - \frac{1}{2} g^{(3)}_{mt,mi} + \frac{1}{2} g^{(2)}_{mm,ti} \\
R^{(2)}_{ij} &= \frac{1}{2} g^{(2)}_{ij,mm} - \frac{1}{2} g^{(2)}_{im,mj} - \frac{1}{2} g^{(2)}_{jm,mi} - \frac{1}{2} g^{(2)}_{tt,ij} + \frac{1}{2} g^{(2)}_{mm,ij}
\end{aligned} \tag{2.13}$$

and the Ricci scalar (1.5) is

$$\begin{aligned}
R^{(2)} &= R^{(2)}_{tt} - R^{(2)}_{mm} = g^{(2)}_{tt,mm} - g^{(2)}_{nn,mm} + g^{(2)}_{mn,mn} \\
R^{(4)} &= R^{(4)}_{tt} - g^{(2)}_{tt} R^{(2)}_{tt} - g^{(2)}_{mn} R^{(2)}_{mn} \\
&= \frac{1}{2} g^{(4)}_{tt,mm} + \frac{1}{2} g^{(2)}_{mn,m} g^{(2)}_{tt,n} + \frac{1}{2} g^{(2)}_{mn} g^{(2)}_{tt,mn} + \frac{1}{2} g^{(2)}_{mm,tt} - \frac{1}{4} g^{(2)}_{tt,m} g^{(2)}_{tt,m} + \\
&\quad - \frac{1}{4} g^{(2)}_{mm,n} g^{(2)}_{tt,n} - g^{(3)}_{tm,tm} - \frac{1}{2} g^{(2)}_{tt} g^{(2)}_{tt,mm} - \frac{1}{2} g^{(2)}_{mn} \Big(g^{(2)}_{mn,ll} - g^{(2)}_{ml,ln} + \\
&\qquad - g^{(2)}_{nl,lm} - g^{(2)}_{tt,mn} + g^{(2)}_{ll,mn} \Big)
\end{aligned} \tag{2.14}$$

The Einstein tensor components (1.11) are

$$
\begin{aligned}
G^{(2)}_{tt} &= R^{(2)}_{tt} - \frac{1}{2}R^{(2)} = \frac{1}{2}g^{(2)}_{mm,nn} + \frac{1}{2}g^{(2)}_{mn,mn} \\
G^{(4)}_{tt} &= R^{(4)}_{tt} - \frac{1}{2}R^{(4)} - \frac{1}{2}g^{(2)}_{tt}R^{(2)} \\
G^{(3)}_{ti} &= R^{(3)}_{ti} = \frac{1}{2}g^{(3)}_{ti,mm} - \frac{1}{2}g^{(2)}_{im,mt} - \frac{1}{2}g^{(3)}_{mt,mi} + \frac{1}{2}g^{(2)}_{mm,ti} \\
G^{(2)}_{ij} &= R^{(2)}_{ij} + \frac{\delta_{ij}}{2}R^{(2)} = \frac{1}{2}g^{(2)}_{ij,mm} - \frac{1}{2}g^{(2)}_{im,mj} - \frac{1}{2}g^{(2)}_{jm,mi} - \frac{1}{2}g^{(2)}_{tt,ij} + \frac{1}{2}g^{(2)}_{mm,ij} + \\
&\qquad + \frac{\delta_{ij}}{2}\left[g^{(2)}_{tt,mm} - g^{(2)}_{nn,mm} + g^{(2)}_{mn,mn}\right]
\end{aligned}
\tag{2.15}
$$

By assuming the harmonic gauge [2]

$$g^{\rho\sigma}\Gamma^{\mu}_{\rho\sigma} \,=\, 0 \tag{2.16}$$

it is possible to simplify the components of Ricci tensor (2.13). In fact for $\mu = 0$ one has

$$2g^{\sigma\tau}\Gamma^{t}_{\sigma\tau} \approx g^{(2)}_{tt,t} - 2g^{(3)}_{tm,m} + g^{(2)}_{mm,t} = 0\,, \tag{2.17}$$

and for $\mu = i$

$$2g^{\sigma\tau}\Gamma^{i}_{\sigma\tau} \approx g^{(2)}_{tt,i} + 2g^{(2)}_{mi,m} - g^{(2)}_{mm,i} = 0\,. \tag{2.18}$$

Differentiating Eq.(2.17) with respect to t, x^j and (2.18) and with respect to t, one obtains

$$g^{(2)}_{tt,tt} - 2g^{(3)}_{tm,mt} + g^{(2)}_{mm,tt} = 0\,, \tag{2.19}$$

$$g^{(2)}_{tt,tj} - 2g^{(3)}_{mt,jm} + g^{(2)}_{mm,tj} = 0\,, \tag{2.20}$$

$$g^{(2)}_{tt,ti} + 2g^{(2)}_{mi,tm} - g^{(2)}_{mm,ti} = 0\,. \tag{2.21}$$

[2]The gauge transformation is $\tilde{h}_{\mu\nu} = h_{\mu\nu} - \zeta_{\mu,\nu} - \zeta_{\nu,\mu}$ when we perform a coordinate transformation as $x'^{\mu} = x^{\mu} + \zeta^{\mu}$ with O(ζ^2)$\ll 1$. To obtain our gauge and the validity of field equation for both perturbation $h_{\mu\nu}$ and $\tilde{h}_{\mu\nu}$ the ζ_{μ} have satisfy the harmonic condition $\Box\zeta^{\mu} = 0$.

On the other side, combining Eq.(2.20) and Eq.(2.21), we get

$$g^{(2)}_{mm,ti} - g^{(2)}_{mi,tm} - g^{(3)}_{mt,mi} = 0\,. \tag{2.22}$$

Finally, differentiating Eq.(2.18) with respect to x^j, one has :

$$g^{(2)}_{tt,ij} + 2g^{(2)}_{mi,jm} - g^{(2)}_{mm,ij} = 0 \tag{2.23}$$

and redefining indexes as $j \to i$, $i \to j$ since these are mute indexes, we get

$$g^{(2)}_{tt,ij} + 2g^{(2)}_{mj,im} - g^{(2)}_{mm,ij} = 0\,. \tag{2.24}$$

Combining Eq.(2.23) and Eq.(2.24), we obtain

$$g^{(2)}_{tt,ij} + g^{(2)}_{mi,jm} + g^{(2)}_{mj,im} - g^{(2)}_{mm,ij} = 0\,. \tag{2.25}$$

Relations (2.19), (2.22), (2.25) guarantee us to rewrite Eqs. (2.13) as

$$\begin{aligned}
R^{(2)}_{tt}|_{HG} &= \frac{1}{2}\triangle g^{(2)}_{tt} \\
R^{(4)}_{tt}|_{HG} &= \frac{1}{2}\triangle g^{(4)}_{tt} + \frac{1}{2}g^{(2)}_{mn}g^{(2)}_{tt,mn} - \frac{1}{2}g^{(2)}_{tt,tt} - \frac{1}{2}|\nabla g^{(2)}_{tt}|^2 \\
R^{(3)}_{ti}|_{HG} &= \frac{1}{2}\triangle g^{(3)}_{ti} \\
R^{(2)}_{ij}|_{HG} &= \frac{1}{2}\triangle g^{(2)}_{ij}
\end{aligned} \tag{2.26}$$

and Eqs. (2.14) becomes

$$\begin{aligned}
R^{(2)}|_{HG} &= \frac{1}{2}\triangle g^{(2)}_{tt} - \frac{1}{2}\triangle g^{(2)}_{mm} \\
R^{(4)}|_{HG} &= \frac{1}{2}\triangle g^{(4)}_{tt} + \frac{1}{2}g^{(2)}_{mn}g^{(2)}_{tt,mn} - \frac{1}{2}g^{(2)}_{tt,tt} - \frac{1}{2}|\nabla g^{(2)}_{tt}|^2 - \frac{1}{2}g^{(2)}_{tt}\triangle g^{(2)}_{tt} - \frac{1}{2}g^{(2)}_{mn}\triangle g^{(2)}_{mn}
\end{aligned} \tag{2.27}$$

where ∇ and $\triangle$ are, respectively, the gradient and the Laplacian in flat space. The Einstein tensor components (1.11) in the harmonic gauge are

$$\begin{array}{rcl} G^{(2)}_{tt}|_{HG} & = & \frac{1}{4}\triangle g^{(2)}_{tt}+\frac{1}{4}\triangle g^{(2)}_{mm} \\ G^{(4)}_{tt}|_{HG} & = & ... \\ G^{(3)}_{ti}|_{HG} & = & \frac{1}{2}\triangle g^{(3)}_{ti} \\ G^{(2)}_{ij}|_{HG} & = & \frac{1}{2}\triangle g^{(2)}_{ij}+\frac{\delta_{ij}}{4}\left[\triangle g^{(2)}_{tt}-\triangle g^{(2)}_{mm}\right] \end{array} \tag{2.28}$$

On the matter side, i.e. right-hand side of the field equations (1.10), we start with the general definition of the energy-momentum tensor of a perfect fluid

$$T_{\alpha\beta}=(\rho+\Pi\rho+p)\,u_\alpha u_\beta-pg_{\alpha\beta}\,. \tag{2.29}$$

Following the procedure outlined in [157], we derive the explicit form of the energy-momentum as follows

$$\begin{array}{rcl} T_{tt} & = & \rho+\rho(v^2-2U+\Pi)+\rho\left[v^2\left(\frac{p}{\rho}+v^2+2V+\Pi\right)+\sigma-2\Pi U\right] \\ T_{ti} & = & -\rho v^i+\rho\left[-v^i\left(\frac{p}{\rho}+2V+v^2+\Pi\right)+h_{ti}\right] \\ T_{ij} & = & \rho v^iv^j+p\delta_{ij}+\rho\left[v^iv^j\left(\Pi+\frac{p}{\rho}+4V+v^2+2U\right)-2v^c\delta_{c(i}h_{0|j)}+2\frac{p}{\rho}V\delta_{ij}\right] \end{array} \tag{2.30}$$

We are now ready to make use of Einstein field equations (1.10), which we assume in the form

$$R_{\mu\nu}=\mathcal{X}\left[T_{\mu\nu}-\frac{T}{2}g_{\mu\nu}\right]. \tag{2.31}$$

From their interpretation as the energy density, momentum density and momentum flux, then, we have T_{tt}, T_{ti} and T_{ij} at various order

$$
\begin{aligned}
T_{tt} &= T_{tt}^{(0)} + T_{tt}^{(2)} + \mathcal{O}(4) \\
T_{ti} &= T_{ti}^{(1)} + \mathcal{O}(3) \\
T_{ij} &= T_{ij}^{(2)} + \mathcal{O}(4)
\end{aligned}
\tag{2.32}
$$

where $T_{\mu\nu}^{(N)}$ denotes the term in $T_{\mu\nu}$ of order $\bar{M}/\bar{r}^3\ \bar{v}^N$. In particular $T_{tt}^{(0)}$ is the density of rest-mass, while $T_{tt}^{(2)}$ is the non-relativistic part of the energy density. What we need is

$$S_{\mu\nu} = T_{\mu\nu} - \frac{T}{2} g_{\mu\nu}\,. \tag{2.33}$$

But $G\bar{M}/\bar{r}$ is of order $\bar{v}^2$, so (2.8) and (2.32) give

$$
\begin{aligned}
S_{tt} &= S_{tt}^{(0)} + S_{tt}^{(2)} + \mathcal{O}(6) \\
S_{ti} &= S_{ti}^{(1)} + \mathcal{O}(3) \\
S_{ij} &= S_{ij}^{(0)} + \mathcal{O}(2)
\end{aligned}
\tag{2.34}
$$

where $S_{\mu\nu}^{(N)}$ denotes the term in $S_{\mu\nu}$ of order $\bar{M}/\bar{r}^3\ \bar{v}^N$. In particular

$$
\begin{aligned}
S_{tt}^{(0)} &= \frac{1}{2} T_{tt}^{(0)} \\
S_{tt}^{(2)} &= \frac{1}{2} T_{tt}^{(2)} + \frac{1}{2} T_{mm}^{(2)} \\
S_{ti}^{(1)} &= T_{ti}^{(1)} \\
S_{ij}^{(0)} &= \frac{1}{2} \delta_{ij} T_{tt}^{(0)}
\end{aligned}
\tag{2.35}
$$

Using the (2.26) and (2.34) in the field equation (2.31) we find that the field equations in harmonic coordinates are indeed consistent with the expansions we are using, and give

$$
\begin{aligned}
R^{(2)}_{tt} &= \mathcal{X} S^{(0)}_{tt} \\
R^{(4)}_{tt} &= \mathcal{X} S^{(2)}_{tt} \\
R^{(3)}_{ti} &= \mathcal{X} S^{(0)}_{ti} \\
R^{(2)}_{ij} &= \mathcal{X} S^{(0)}_{ij}
\end{aligned}
$$

and in particular

$$
\begin{aligned}
\triangle g^{(2)}_{tt} &= \mathcal{X}\, T^{(0)}_{tt} \\
\triangle g^{(4)}_{tt} &= \mathcal{X}\left[T^{(2)}_{tt} + T^{(2)}_{mm}\right] - g^{(2)}_{mn} g^{(2)}_{tt,mn} + g^{(2)}_{tt,tt} + |\bigtriangledown g^{(2)}_{tt}|^2 \\
\triangle g^{(3)}_{ti} &= 2\,\mathcal{X}\, T^{(1)}_{ti} \\
\triangle g^{(2)}_{ij} &= \mathcal{X}\,\delta_{ij}\, T^{(0)}_{tt}
\end{aligned}
\tag{2.36}
$$

From the first one of (2.36), we find, as expected, the Newtonian mechanics:

$$
g^{(2)}_{tt} = -\frac{\mathcal{X}}{4\pi}\int d^3\mathbf{x}'\frac{T^{(0)}_{tt}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} = -2G\int d^3\mathbf{x}'\frac{T^{(0)}_{tt}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \doteq 2\Phi(\mathbf{x}) \tag{2.37}
$$

where $\Phi(\mathbf{x})$ is the gravitational potential which, in the case of point-like source with mass M, is

$$
\Phi(\mathbf{x}) = -\frac{GM}{|\mathbf{x}|}\,. \tag{2.38}
$$

From the third and fourth equations of (2.36) we find that

$$
\begin{aligned}
g_{ti}^{(3)} &= -\frac{\mathcal{X}}{2\pi}\int d^3\mathbf{x}'\frac{T_{ti}^{(1)}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} = 2\,A_i(\mathbf{x}) \\
g_{ij}^{(2)} &= -\frac{\mathcal{X}}{4\pi}\,\delta_{ij}\int d^3\mathbf{x}'\frac{T_{tt}^{(0)}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} = 2\delta_{ij}\Phi(\mathbf{x})
\end{aligned}
$$

The second equation of (2.36) can be rewritten as follows

$$
\triangle\left[g_{tt}^{(4)} - 2\Phi^2\right] = \mathcal{X}\left[T_{tt}^{(2)} + T_{mm}^{(2)}\right] - 8\Phi\triangle\Phi + 2\Phi_{,tt} \tag{2.39}
$$

and the solution for $g_{tt}^{(4)}$ is

$$
\begin{aligned}
g_{tt}^{(4)} = \quad & 2\Phi^2 - \frac{\mathcal{X}}{4\pi}\int d^3\mathbf{x}'\frac{T_{tt}^{(2)}(\mathbf{x}') + T_{mm}^{(2)}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} + \frac{2}{\pi}\int d^3\mathbf{x}'\frac{\Phi(\mathbf{x}')\triangle_{\mathbf{x}'}\Phi(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \\
& -\frac{1}{2\pi}\int d^3\mathbf{x}'\frac{\Phi_{,tt}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \doteq 2\,\Xi(\mathbf{x})\,.
\end{aligned} \tag{2.40}
$$

By using the equations at second order we obtain the final expression for the correction at fourth order in the time-time component of the metric:

$$
\begin{aligned}
\Xi(\mathbf{x}) = \quad & \Phi(\mathbf{x})^2 - \frac{\mathcal{X}}{8\pi}\int d^3\mathbf{x}'\frac{T_{tt}^{(2)}(\mathbf{x}') + T_{mm}^{(2)}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} + \frac{\mathcal{X}}{\pi}\int d^3\mathbf{x}'\frac{\Phi(\mathbf{x}')\,T_{tt}^{(0)}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \\
& -\frac{1}{4\pi}\partial_{tt}^2\int d^3\mathbf{x}'\frac{\Phi(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|},\,.
\end{aligned} \tag{2.41}
$$

We can rewrite the metric expression (2.8) as follows

$$
g_{\mu\nu} \sim \begin{pmatrix} 1 + 2\Phi + 2\,\Xi & 2\,A_i \\ 2\,A_i & -\delta_{ij} + 2\Psi\,\delta_{ij} \end{pmatrix} \tag{2.42}
$$

Finally the Lagrangian of a particle in presence of a gravitational field can be expressed as proportional to the invariant distance $ds^{1/2}$, thus we have :

$$\begin{aligned} L = \left(g_{\rho\sigma}\frac{dx^\rho}{dt}\frac{dx^\sigma}{dt}\right)^{1/2} &= \left(g_{tt} + 2g_{tm}v^m + g_{mn}v^m v^n\right)^{1/2} \\ &= \left(1 + g^{(2)}_{tt} + g^{(4)}_{tt} + 2g^{(3)}_{tm}v^m - \mathbf{v}^2 + g^{(2)}_{mn}v^m v^n\right)^{1/2}, \end{aligned}$$

which, to the $\mathcal{O}(2)$ order, reduces to the classic Newtonian Lagrangian of a test particle $L_{\text{New}} = \left(1 + 2\Phi - \mathbf{v}^2\right)^{1/2}$, where $v^m = \frac{dx^m}{dt}$ and $|\mathbf{v}|^2 = v^m v^m$. As matter of fact, post-Newtonian physics has to involve higher than $\mathcal{O}(2)$ order terms in the Lagrangian. In fact we obtain

$$L \sim 1 + \left[\Phi - \frac{1}{2}\mathbf{v}^2\right] + \frac{3}{4}\left[\Xi + 2\,A_i v^i + \Phi\,\mathbf{v}^2\right]. \tag{2.43}$$

An important remark concerns the odd-order perturbation terms $\mathcal{O}(1)$ or $\mathcal{O}(3)$. Since, these terms contain odd powers of velocity $\mathbf{v}$ or of time derivatives, they are related to the energy dissipation or absorption by the system. Nevertheless, the mass-energy conservation prevents the energy and mass losses and, as a consequence, prevents, in the Newtonian limit, terms of $\mathcal{O}(1)$ and $\mathcal{O}(3)$ orders in the Lagrangian. If one takes into account contributions higher than $\mathcal{O}(4)$ order, different theories give different predictions. GR, for example, due to the conservation of post-Newtonian energy, forbids terms of $\mathcal{O}(5)$ order; on the other hand, terms of $\mathcal{O}(7)$ order can appear and are related to the energy lost by means of the gravitational radiation.

2.1.1 The Newtonian Limit of Fourth Order Gravity

Let us consider the general class of Fourth Order Gravity given by the action (1.39)[3]

$$\mathcal{S} = \int d^4x\sqrt{-g}\left[f(X, Y, Z) + \mathcal{X}\mathcal{L}_m\right]$$

In the metric approach, the field equations (1.40) are obtained by varying (1.39) with respect to $g_{\mu\nu}$.

The paradigm of the Newtonian limit starts from the development of the metric tensor (and of all additional quantities in the theory) with respect to the dimensionless quantity v but considering only first term of tt- and ij-component of metric tensor $g_{\mu\nu}$ (for details, see [231]). The develop of the metric is the following

[3]We remember that $X = R$, $Y = R_{\alpha\beta}R^{\alpha\beta}$ and $Z = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$

$$ds^2 = (1+2\Phi)dt^2 - (1-2\Psi)\delta_{ij}dx^i dx^j \tag{2.44}$$

where Φ and Ψ are proportional to v^2. The set of coordinates adopted is $x^\mu = (t, x^1, x^2, x^3)$. The curvature invariants X, Y, Z become

$$\begin{aligned} X &\sim X^{(2)} + \ldots \\ Y &\sim Y^{(4)} + \ldots \\ Z &\sim Z^{(4)} + \ldots \end{aligned}$$

and the function f and its partial derivatives (f_X, f_{XX}, f_Y and f_Z) can be substituted by their corresponding Taylor develop. In the case of f we have

$$f(X,Y,Z) \sim f(0) + f_X(0)X^{(2)} + \frac{1}{2}f_{XX}(0){X^{(2)}}^2 + f_X(0)X^{(4)} + f_Y(0)Y^{(4)} + f_Z(0)Z^{(4)} + \ldots$$

and analogous relations for derivatives are obtained.

From the lowest order of field equations (1.40) we have

$$f(0) = 0 \tag{2.45}$$

while in the Newtonian Limit ($\propto v^2$) we have[4]

$$\begin{aligned} &(\triangle - {m_2}^2)\triangle\Phi + \left[{m_2}^2 - \frac{{m_1}^2 + 2{m_2}^2}{3{m_1}^2}\triangle\right]X^{(2)} = -2\,{m_2}^2\mathcal{X}\,\rho \\ &\triangle\Psi = \int d^3\mathbf{x}'\mathcal{G}_2(\mathbf{x},\mathbf{x}')\left(\frac{{m_2}^2}{2} - \frac{{m_1}^2 + 2{m_2}^2}{6{m_1}^2}\triangle_{\mathbf{x}'}\right)X^{(2)}(\mathbf{x}') \\ &(\triangle - {m_1}^2)X^{(2)} = {m_1}^2\mathcal{X}\,\rho \end{aligned} \tag{2.46}$$

where $X^{(2)}$ is the Ricci scalar at Newtonian order, ρ is the matter density and $\mathcal{G}_2$ is the Green function of field operator $\triangle - {m_2}^2$. The quantities ${m_i}^2$ are linked to derivatives of f with

[4]in this chapter we assume always $f_X(0) > 0$, and therefore we may set $f_X(0) = 1$ without loss of generality.

respect to the curvature invariants X, Y and Z

$$\begin{aligned} {m_1}^2 &\doteq -\frac{f_X(0)}{3f_{XX}(0)+2f_Y(0)+2f_Z(0)} \\ {m_2}^2 &\doteq \frac{f_X(0)}{f_Y(0)+4f_Z(0)} \end{aligned} \tag{2.47}$$

By solving the field equations (2.46), if ${m_i}^2 > 0$ for $i = 1, 2$, the proper time interval, generated by a point-like source with mass M, is (for details, see [231, 232])

$$\begin{aligned} ds^2 = & \biggl[1-r_g\biggl(\frac{1}{|\mathbf{x}|}+\frac{1}{3}\frac{e^{-\mu_1|\mathbf{x}|}}{|\mathbf{x}|}-\frac{4}{3}\frac{e^{-\mu_2|\mathbf{x}|}}{|\mathbf{x}|}\biggr)\biggr]dt^2+ \\ & -\biggl[1+r_g\biggl(\frac{1}{|\mathbf{x}|}-\frac{1}{3}\frac{e^{-\mu_1|\mathbf{x}|}}{|\mathbf{x}|}-\frac{2}{3}\frac{e^{-\mu_2|\mathbf{x}|}}{|\mathbf{x}|}\biggr)\biggr]\delta_{ij}dx^idx^j \end{aligned} \tag{2.48}$$

where $r_g = 2GM$ is the Schwarzschild radius and $\mu_i \doteq \sqrt{|{m_i}^2|}$. The field equations (2.46) are valid for any values of quantities ${m_i}^2$, while the Green functions of field operator $\triangle - {m_i}^2$ admit two different behaviors if ${m_i}^2 > 0$ or ${m_i}^2 < 0$. The possible choices of Green function, for spherically symmetric systems (*i.e.* $\mathcal{G}_i(\mathbf{x},\mathbf{x}') = \mathcal{G}_i(|\mathbf{x}-\mathbf{x}'|)$), are the following

$$\mathcal{G}_i(\mathbf{x},\mathbf{x}') = \begin{cases} -\frac{1}{4\pi}\frac{e^{-\mu_i|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} & \text{if} \quad {m_i}^2 > 0 \\ \\ -\frac{1}{4\pi}\frac{\cos\mu_i|\mathbf{x}-\mathbf{x}'|+\sin\mu_i|\mathbf{x}-\mathbf{x}'|}{|\mathbf{x}-\mathbf{x}'|} & \text{if} \quad {m_i}^2 < 0 \end{cases} \tag{2.49}$$

The first choice in (2.49) corresponds to Yukawa-like behavior, while the second one to the oscillating case. Both expressions are a generalization of the usual gravitational potential ($\propto |\mathbf{x}|^{-1}$), and when ${m_i}^2 \rightarrow \infty$ (*i.e.* $f_{XX}(0)\,, f_Y(0)\,, f_Z(0) \rightarrow 0$ from the (2.47)) we recover the field equations of GR. Independently of algebraic sign of ${m_i}^2$ we can introduce two scale lengths ${\mu_i}^{-1}$. We note that in the case of $f(X)$-Gravity we obtained only one scale length (${\mu_1}^{-1}$ with $f_Y(0) = f_Z(0) = 0$) on the which the Ricci scalar evolves [231, 232], but in $f(X, Y, Z)$-Gravity we have an additional scale length ${\mu_2}^{-1}$ on the which the Ricci tensor evolves.

Often for spherically symmetric problems it is convenient rewriting the metric (2.48) in the so-called standard coordinates system[5] (the usual form in which we write the Schwarzschild solution). By introducing a new radial coordinate $\tilde{r} = |\tilde{\mathbf{x}}|$ as follows

[5]Generally the set of coordinates (t, r, θ, ϕ) are called standard coordinates if the metric is expressed as $ds^2 = g_{tt}(t,r)\,dt^2+g_{rr}(t,r)dr^2-r^2d\Omega$ while if one has $ds^2 = g_{tt}(t,\mathbf{x})\,dt^2+g_{ij}(t,\mathbf{x})dx^idx^j$ (like the solution (2.48)) the set (t, x^1, x^2, x^3) is called isotropic coordinates [353].

$$\left[1 - 2\Psi(r)\right] r^2 \,=\, \tilde{r}^2 \tag{2.50}$$

the relativistic invariant (2.48) becomes[6]

$$\begin{aligned} ds^2 \;\;=\;\; & \left[1 - \frac{r_g}{r}\left(1 + \frac{1}{3}\,e^{-\mu_1 r} - \frac{4}{3}\,e^{-\mu_2 r}\right)\right] dt^2 + \\ & -\left[1 + \frac{r_g}{r}\left(1 - \frac{\mu_1 r + 1}{3}\,e^{-\mu_1 r} - \frac{2(\mu_2 r + 1)}{3}\,e^{-\mu_2 r}\right)\right] dr^2 - r^2 d\Omega \end{aligned} \tag{2.51}$$

where $d\Omega \,=\, d\theta^2 + \sin^2\theta\, d\phi^2$ is the solid angle and we renamed the radial coordinate $\tilde{r}$.

2.1.2 Newtonian and post Newtonian limit of Scalar-Tensor-Forth-Order Gravity

Now we study, in the weak-field approximation, the Newtonian and post-Newtonian limits of Scalar-Tensor-Forth-Order Gravity (1.41). To do this, we can set as a perturbative scheme for the metric tensor (2.42) and the scalar field as follows:

$$\phi \,\sim\, \phi^{(0)} + \phi^{(2)} + \ldots \,=\, \phi^{(0)} + \varphi\,, \tag{2.52}$$

where Φ, Ψ, φ are proportional to the power c^{-2} (Newtonian limit) while A_i is proportional to c^{-3} and Ξ to c^{-4} (post-Newtonian limit). The function f, up to the c^{-4} order, can be developed as

$$\begin{aligned} f(R, R_{\alpha\beta}R^{\alpha\beta}, \phi) \;=\; \quad & f_R(0,0,\phi^{(0)})\, R + \frac{f_{RR}(0,0,\phi^{(0)})}{2}\, R^2 + \frac{f_{\phi\phi}(0,0,\phi^{(0)})}{2}(\phi - \phi^{(0)})^2 \\ & + f_{R\phi}(0,0,\phi^{(0)}) R\,\phi + f_Y(0,0,\phi^{(0)}) R_{\alpha\beta}R^{\alpha\beta}\,, \end{aligned} \tag{2.53}$$

[6]The metrics (2.51) and (2.48) represent the same space-time at first order of r_g/r.

while all other possible contributions in f are negligible [231, 232, 233]. The field equations (1.42), (1.43) and (1.44) hence read

$$f_R(0,0,\phi^{(0)})\left[R_{tt}-\frac{R}{2}\right]-f_Y(0,0,\phi^{(0)})\triangle R_{tt}-[f_{RR}(0,0,\phi^{(0)})+\frac{f_Y(0,0,\phi^{(0)})}{2}]\triangle R+ \\ -f_{R\phi}(0,0,\phi^{(0)})\triangle\varphi \,=\, \mathcal{X}\,T_{tt}\,,$$

$$f_R(0,0,\phi^{(0)})\left[R_{ij}+\frac{R}{2}\delta_{ij}\right]-f_Y(0,0,\phi^{(0)})\triangle R_{ij}-f_{RR}(0,0,\phi^{(0)})R_{,ij}+\left[f_{RR}(0,0,\phi^{(0)})+ \\ +\frac{f_Y(0,0,\phi^{(0)})}{2}\right]\delta_{ij}\triangle R-2f_Y(0,0,\phi^{(0)})R^\alpha{}_{(i,j)\alpha}-f_{R\phi}(0,0,\phi^{(0)})(\partial^2_{ij}-\delta_{ij}\triangle)\varphi \,=\, \mathcal{X}\,T_{ij}\,,$$

$$f_R(0,0,\phi^{(0)})\,R_{ti}-f_Y(0,0,\phi^{(0)})\triangle R_{ti}-f_{RR}(0,0,\phi^{(0)})R_{,ti}-2f_Y(0,0,\phi^{(0)})R^\alpha{}_{(t,i)\alpha}+ \\ -f_{R\phi}(0,0,\phi^{(0)})\,\varphi_{,ti} \,=\, \mathcal{X}\,T_{ti}\,, \tag{2.54}$$

$$f_R(0,0,\phi^{(0)})\,R+[3f_{RR}(0,0,\phi^{(0)})+2f_Y(0,0,\phi^{(0)})]\triangle R+3f_{R\phi}(0,0,\phi^{(0)})\triangle\varphi \,=\, -\mathcal{X}\,T\,,$$

$$2\omega(\phi^{(0)})\triangle\varphi+f_{\phi\phi}(0,0,\phi^{(0)})\varphi+f_{R\phi}(0,0,\phi^{(0)})R \,=\, 0\,,$$

where $\triangle$ is the Laplace operator in the flat space. The geometric quantities $R_{\mu\nu}$ and R are evaluated at the first order with respect to the metric potentials Φ, Ψ and A_i. By introducing the quantities[7]

$$\begin{aligned} {m_R}^2 &\doteq -\tfrac{f_R(0,0,\phi^{(0)})}{3f_{RR}(0,0,\phi^{(0)})+2f_Y(0,0,\phi^{(0)})}\,, \\ {m_Y}^2 &\doteq \tfrac{f_R(0,0,\phi^{(0)})}{f_Y(0,0,\phi^{(0)})}\,, \\ {m_\phi}^2 &\doteq -\tfrac{f_{\phi\phi}(0,0,\phi^{(0)})}{2\omega(\phi^{(0)})}\,, \end{aligned} \tag{2.55}$$

and setting $f_R(0,0,\phi^{(0)}) \,=\, 1$, $\omega(\phi^{(0)}) \,=\, 1/2$ for simplicity[8], we get the complete set of differential equations

[7] In the Newtonian and post-Newtonian limits, we can consider as Lagrangian in the action (1.41), the quantity $f(X,Y) \,=\, a\,R+b\,R^2+c\,R_{\alpha\beta}R^{\alpha\beta}$ [232]. Then the masses (2.55) become ${m_R}^2 \,=\, -\frac{a}{2(3b+c)}$, ${m_Y}^2 \,=\, \frac{a}{c}$. For a correct interpretation of these quantities as real masses, we have to impose $a \,>\, 0$, $b \,<\, 0$ and $0 \,<\, c \,<\, -3b$.

[8] We can define a new gravitational constant: $\mathcal{X} \;\rightarrow\; \mathcal{X}\,f_R(0,0,\phi^{(0)})$ and $f_{R\phi}(0,0,\phi^0) \;\rightarrow\; f_{R\phi}(0,0,\phi^0)\,f_R(0,0,\phi^{(0)})$.

$$
\begin{aligned}
&(\triangle - m_Y{}^2)R_{tt} + \left[\tfrac{m_Y{}^2}{2} - \tfrac{m_R{}^2+2m_Y{}^2}{6m_R{}^2}\triangle\right]R + m_Y{}^2\, f_{R\phi}(0,0,\phi^{(0)})\,\triangle\varphi \,=\, -m_Y{}^2\mathcal{X}\,T_{tt}\;,\\
&(\triangle - m_Y{}^2)R_{ij} + \left[\tfrac{m_R{}^2-m_Y{}^2}{3m_R{}^2}\,\partial^2_{ij} - \delta_{ij}\left(\tfrac{m_Y{}^2}{2} - \tfrac{m_R{}^2+2m_Y{}^2}{6m_R{}^2}\triangle\right)\right]R+\\
&\qquad\qquad +m_Y{}^2\, f_{R\phi}(0,0,\phi^{(0)})\,(\partial^2_{ij} - \delta_{ij}\triangle)\varphi \,=\, -m_Y{}^2\,\mathcal{X}\,T_{ij}\;, \qquad (2.56)\\
&(\triangle - m_Y{}^2)R_{ti} + \tfrac{m_R{}^2-m_Y{}^2}{3m_R{}^2}R_{,ti} + m_Y{}^2\, f_{R\phi}(0,0,\phi^{(0)})\,\varphi_{,ti} \,=\, -m_Y{}^2\mathcal{X}\,T_{ti}\;,\\
&(\triangle - m_R{}^2)R - 3m_R{}^2\, f_{R\phi}(0,0,\phi^{(0)})\,\triangle\varphi \,=\, m_R{}^2\,\mathcal{X}\,T\;,\\
&(\triangle - m_\phi{}^2)\varphi + f_{R\phi}(0,0,\phi^{(0)})\,R \,=\, 0\;.
\end{aligned}
$$

The components of the Ricci tensor in Eq. (2.85) in the weak-field limit read

$$
\begin{aligned}
R_{tt} &= \tfrac{1}{2}\triangle\, g^{(2)}_{tt} \,=\, \triangle\Phi\;,\\
R_{ij} &= \tfrac{1}{2}g^{(2)}_{ij,mm} - \tfrac{1}{2}g^{(2)}_{im,mj} - \tfrac{1}{2}g^{(2)}_{jm,mi} - \tfrac{1}{2}g^{(2)}_{tt,ij} + \tfrac{1}{2}g^{(2)}_{mm,ij} \,=\, \triangle\Psi\,\delta_{ij} + (\Psi-\Phi)_{,ij}\;, \qquad (2.57)\\
R_{ti} &= \tfrac{1}{2}g^{(3)}_{ti,mm} - \tfrac{1}{2}g^{(2)}_{im,mt} - \tfrac{1}{2}g^{(3)}_{mt,mi} + \tfrac{1}{2}g^{(2)}_{mm,ti} \,=\, \triangle A_i + \Psi_{,ti}\;.
\end{aligned}
$$

The energy momentum tensor $T_{\mu\nu}$ can be also expanded. For a perfect fluid, when the pressure is negligible with respect to the mass density ρ, it reads $T_{\mu\nu} = \rho\, u_\mu u_\nu$ with $u_\sigma u^\sigma = 1$. However, the development starts form the zeroth order[9], hence $T_{tt} = T^{(0)}_{tt} = \rho$, $T_{ij} = T^{(0)}_{ij} = 0$ and $T_{ti} = T^{(1)}_{ti} = \rho\, v_i$, where ρ is the density mass and v^i is the velocity of the source. Thus, $T_{\mu\nu}$ is independent of metric potentials and satisfies the ordinary conservation condition

[9]This formalism descends from the theoretical setting of Newtonian mechanics which requires the appropriate scheme of approximation when obtained from a more general relativistic theory. This scheme coincides with a gravity theory analyzed at the first order of perturbation in a curved spacetime metric.

$T^{\mu\nu}{}_{,\mu} = 0$. Equations (2.85) thus read

$$\begin{aligned}
&(\triangle - {m_Y}^2)\triangle\Phi + \left[\frac{{m_Y}^2}{2} - \frac{{m_R}^2 + 2{m_Y}^2}{6{m_R}^2}\triangle\right]R + {m_Y}^2\, f_{R\phi}(0,0,\phi^{(0)})\,\triangle\varphi \,=\, -{m_Y}^2\mathcal{X}\,\rho\,,\\
&\left\{(\triangle - {m_Y}^2)\triangle\Psi - \left[\frac{{m_Y}^2}{2} - \frac{{m_R}^2 + 2{m_Y}^2}{6{m_R}^2}\triangle\right]R - {m_Y}^2\, f_{R\phi}(0,0,\phi^{(0)})\,\triangle\varphi\right\}\delta_{ij} +\\
&\qquad\qquad + \left\{(\triangle - {m_Y}^2)(\Psi - \Phi) + \frac{{m_R}^2 - {m_Y}^2}{3{m_R}^2}\,R + {m_Y}^2\, f_{R\phi}(0,0,\phi^{(0)})\,\varphi\right\}_{,ij} \,=\, 0\,,\\
&\left\{(\triangle - {m_Y}^2)\triangle A_i + {m_Y}^2\mathcal{X}\,\rho\, v_i\right\} + \left\{(\triangle - {m_Y}^2)\Psi + \frac{{m_R}^2 - {m_Y}^2}{3{m_R}^2}R\right.\\
&\qquad\qquad\qquad\qquad \left. + {m_Y}^2\, f_{R\phi}(0,0,\phi^{(0)})\,\varphi\right\}_{,ti} \,=\, 0\,,\\
&(\triangle - {m_R}^2)R - 3{m_R}^2\, f_{R\phi}(0,0,\phi^{(0)})\,\triangle\varphi \,=\, {m_R}^2\,\mathcal{X}\,\rho\,,\\
&(\triangle - {m_\phi}^2)\varphi + f_{R\phi}(0,0,\phi^{(0)})\,R \,=\, 0\,.
\end{aligned} \tag{2.58}$$

In the following we consider the Newtonian and Post-Newtonian limits.

2.1.3 The Newtonian limit: solutions of the fields Φ, φ and R

Equations (2.87a) and (2.87b) are coupled system and, for a point-like source $\rho(\mathbf{x}) = M\,\delta(\mathbf{x})$, admit the solutions:

$$\begin{aligned}
\varphi(\mathbf{x}) &= \sqrt{\tfrac{\xi}{3}}\,\tfrac{r_g}{|\mathbf{x}|}\,\tfrac{e^{-m_+|\mathbf{x}|} - e^{-m_-|\mathbf{x}|}}{w_+ - w_-}\,,\\
R(\mathbf{x}) &= -{m_R}^2\,\tfrac{r_g}{|\mathbf{x}|}\,\tfrac{(w_+ - \eta^2)\,e^{-m_+|\mathbf{x}|} - (w_- - \eta^2)\,e^{-m_-|\mathbf{x}|}}{w_+ - w_-}\,,
\end{aligned} \tag{2.59}$$

where r_g is the Schwarzschild radius and we have [233][10] the others parameters:

$$\begin{aligned}
&m_\pm^2 \equiv {m_R}^2\, w_\pm\big(\xi,\eta\big)\,, \qquad && w_\pm(\xi,\eta) \,=\, \frac{1 - \xi + \eta^2 \pm \sqrt{(1 - \xi + \eta^2)^2 - 4\eta^2}}{2}\,,\\
&\xi \,=\, 3 f_{R\phi}(0,0,\phi^{(0)})^2, && \eta \,=\, \frac{m_\phi}{m_R}\,.
\end{aligned} \tag{2.60}$$

[10] The parameter ξ is defined generally as $\frac{3 f_{R\phi}(0,0,\phi^{(0)})^2}{2\, f_R(0,0,\phi^{(0)})\,\omega(\phi^{(0)})}$.

Moreover ξ and η satisfy the condition $(\eta-1)^2-\xi\;>\;0$. The formal solution of the gravitational potential Φ, derived from Eq. (2.87a), reads

$$\Phi(\mathbf{x})\;=\;\frac{-1}{16\pi^2}\int\frac{d^3\mathbf{x}'d^3\mathbf{x}''}{|\mathbf{x}-\mathbf{x}'|}\frac{e^{-m_Y|\mathbf{x}'-\mathbf{x}''|}}{|\mathbf{x}'-\mathbf{x}''|}\biggl[\frac{4{m_Y}^2-{m_R}^2}{6}\mathcal{X}\,\rho(\mathbf{x}'')-\frac{{m_R}^4\eta^2}{2\sqrt{3}}\,\xi^{1/2}\,\varphi(\mathbf{x}'')+ \tag{2.61}$$

$$+\frac{{m_Y}^2-{m_R}^2(1-\xi)}{6}R(\mathbf{x}'')\biggr]\,,$$

which for a point-like source is

$$\Phi(\mathbf{x})\;=\;-\frac{GM}{|\mathbf{x}|}\left[1+k(\xi,\eta)\,e^{-m_+|\mathbf{x}|}+[\frac{1}{3}-k(\xi,\eta)]\,e^{-m_-|\mathbf{x}|}-\frac{4}{3}\,e^{-m_Y|\mathbf{x}|}\right]\,, \tag{2.62}$$

where

$$k(\xi,\eta)\;=\;\frac{1-\eta^2+\xi+\sqrt{\eta^4+(\xi-1)^2-2\eta^2(\xi+1)}}{6\sqrt{\eta^4+(\xi-1)^2-2\eta^2(\xi+1)}}\;.$$

Note that for $f_Y\;\rightarrow\;0$ *i.e.* $m_Y\;\rightarrow\;\infty$, we obtain the same outcome for the gravitational potential as in Ref. [233] for a $f(R,\phi)$-theory. The absence of the coupling term between the curvature invariant Y and the scalar field ϕ, as well as the linearity of the field equations (2.87) guarantee that the solution (2.62) is a linear combination of solutions obtained within an $f(R,\phi)$-theory and an $R+Y/{m_Y}^2$-theory.

2.1.4 The Post-Newtonian limit: solutions of the fields Ψ and A_i

Equation (2.87b) can be formally solved as

$$\Psi(\mathbf{x})\;=\;\Phi(\mathbf{x})+\frac{{m_R}^2-{m_Y}^2}{12\pi{m_R}^2}\int d^3\mathbf{x}'\frac{e^{-m_Y|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\,R(\mathbf{x}')+\frac{{m_Y}^2\xi^{1/2}}{4\sqrt{3}\pi}\int d^3\mathbf{x}'\frac{e^{-m_Y|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\,\varphi(\mathbf{x}')\,,$$

which for a point-like source reads

$$\Psi(\mathbf{x})\;=\;-\frac{GM}{|\mathbf{x}|}\left[1-k(\xi,\eta)\,e^{-m_+|\mathbf{x}|}-[1/3-k(\xi,\eta)]\,e^{-m_-|\mathbf{x}|}-\frac{4}{3}\,e^{-m_Y|\mathbf{x}|}\right]\,, \tag{2.63}$$

obtained by setting $\{\dots\}_{,ij}\;=\;0$ in Eq. (2.87b), while one also has $\{\dots\}\delta_{ij}\;=\;0$ leading to

$$\Psi(\mathbf{x}) = -\frac{1}{16\pi^2}\int d^3\mathbf{x}'d^3\mathbf{x}''\frac{e^{-m_Y|\mathbf{x}'-\mathbf{x}''|}}{|\mathbf{x}-\mathbf{x}'||\mathbf{x}'-\mathbf{x}''|}\left[\frac{{m_R}^2+2{m_Y}^2}{6}\mathcal{X}\rho(\mathbf{x}'')+\right.$$
$$\left.-\frac{{m_Y}^2-{m_R}^2(1-\xi)}{6}R(\mathbf{x}'')+\frac{{m_R}^4\eta^2}{2\sqrt{3}}\xi^{1/2}\varphi(\mathbf{x}'')\right], \tag{2.64}$$

which is however equivalent to solution (2.63). The solutions (2.62) and (2.63) generalize the outcomes of the theory $f(R,\, R_{\alpha\beta}R^{\alpha\beta})$ [232].

From Eq. (2.87c), we immediately obtain the solution for A_i, namely

$$A_i(\mathbf{x}) = -\frac{{m_Y}^2\mathcal{X}}{16\pi^2}\int d^3\mathbf{x}'d^3\mathbf{x}''\,\frac{e^{-m_Y|\mathbf{x}'-\mathbf{x}''|}}{|\mathbf{x}-\mathbf{x}'||\mathbf{x}'-\mathbf{x}''|}\,\rho(\mathbf{x}'')\,v''_i\,. \tag{2.65}$$

In Fourier space, solution (2.65) presents the massless pole of General Relativity, and the massive one[11] is induced by the presence of the $R_{\alpha\beta}R^{\alpha\beta}$ term. Hence, the solution (2.65) can be rewritten as the sum of General Relativity contributions and massive modes. Since we do not consider contributions inside rotating bodies, we obtain

$$A_i(\mathbf{x}) = -\frac{\mathcal{X}}{4\pi}\int d^3\mathbf{x}'\frac{\rho(\mathbf{x}')\,v'_i}{|\mathbf{x}-\mathbf{x}'|}+\frac{\mathcal{X}}{4\pi}\int d^3\mathbf{x}'\frac{e^{-m_Y|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\,\rho(\mathbf{x}')\,v'_i\,. \tag{2.66}$$

For a spherically symmetric system ($|\mathbf{x}| = r$) at rest and rotating with angular frequency $\mathbf{\Omega}(r)$, the energy momentum tensor T_{ti} is

$$T_{ti} = \rho(\mathbf{x})\,v_i = T_{tt}(r)\,[\mathbf{\Omega}(r)\times\mathbf{x}]_i = \frac{3M}{4\pi\mathcal{R}^3}\Theta(\mathcal{R}-r)\,[\mathbf{\Omega}(r)\times\mathbf{x}]_i\,, \tag{2.67}$$

where $\mathcal{R}$ is the radius of the body and Θ is the Heaviside function. Since only in General Relativity and Scalar Tensor Theories the Gauss theorem is satisfied, here we have to consider the potentials Φ, Ψ generated by the ball source with radius $\mathcal{R}$, while they also depend on the shape of the source. In fact for any term $\propto \frac{e^{-mr}}{r}$, there is a geometric factor multiplying the Yukawa term, namely $F(m\mathcal{R}) = 3\frac{m\mathcal{R}\cosh m\mathcal{R}-\sinh m\mathcal{R}}{m^3\mathcal{R}^3}$. We thus get

[11] Note that Eq. (2.87c) in Fourier space becomes $|\mathbf{k}|^2(|\mathbf{k}|^2+{m_Y}^2)\tilde{A}_i = -{m_Y}^2\mathcal{X}\tilde{T}_{ti}$ and its solution reads $\tilde{A}_i = -\mathcal{X}\tilde{T}_{ti}\left[\frac{1}{|\mathbf{k}|^2}-\frac{1}{|\mathbf{k}|^2+m_Y^2}\right]$.

$$\Phi_{\rm ball}(\mathbf{x}) \;=\; -\frac{GM}{|\mathbf{x}|}\biggl[1+k(\xi,\eta)\,F(m_+\mathcal{R})\,e^{-m_+|\mathbf{x}|}-\frac{4\,F(m_Y\mathcal{R})}{3}\,e^{-m_Y|\mathbf{x}|}+ +\bigl[\frac{1}{3}-k(\xi,\eta\bigr]\,F(m_-\mathcal{R})\,e^{-m_-|\mathbf{x}|}\biggr]\,,$$

$$\Psi_{\rm ball}(\mathbf{x}) \;=\; -\frac{GM}{|\mathbf{x}|}\biggl[1-k(\xi,\eta)\,F(m_+\mathcal{R})\,e^{-m_+|\mathbf{x}|}-\frac{2\,F(m_Y\mathcal{R})}{3}\,e^{-m_Y|\mathbf{x}|}+ -\bigl[\frac{1}{3}-k(\xi,\eta)\bigr]\,F(m_-\mathcal{R})\,e^{-m_-|\mathbf{x}|}\biggr]\,. \tag{2.68}$$

For $\mathbf{\Omega}(r) \;=\; \mathbf{\Omega}_0$, the metric potential (2.66) reads

$$\mathbf{A}(\mathbf{x}) \;=\; -\frac{3MG}{2\pi\mathcal{R}^3}\mathbf{\Omega}_0\times\int d^3\mathbf{x}'\frac{1-e^{-m_Y|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\,\Theta(\mathcal{R}-r')\,\mathbf{x}'\,. \tag{2.69}$$

Making the approximation

$$\frac{e^{-m_Y|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\;\sim\;\frac{e^{-m_Y\,r}}{r}+\frac{e^{-m_Y\,r}(1+m_Y\,r)\cos\alpha}{r}\frac{r'}{r}+\mathcal{O}\biggl(\frac{r'^2}{r^2}\biggr)\,, \tag{2.70}$$

where α is the angle between the vectors $\mathbf{x}, \mathbf{x}'$, with $\mathbf{x} \;=\; r\,\hat{\mathbf{x}}$ where $\hat{\mathbf{x}} \;=\; (\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta)$ and considering only the first order of r'/r, we can evaluate the integration in the vacuum ($r \;>\; \mathcal{R}$) as

$$\int d^3\mathbf{x}'\,\frac{e^{-m_Y|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\,\Theta(\mathcal{R}-r')\,\mathbf{x}' \;=\; \frac{4\pi}{15}\,\frac{(1+m_Y\,r)\,e^{-m_Y\,r}\,\mathcal{R}^5}{r^3}\,\mathbf{x}\,. \tag{2.71}$$

Thus, the field $\mathbf{A}$ outside the sphere is

$$\mathbf{A}(\mathbf{x}) \;=\; \frac{G}{|\mathbf{x}|^2}\biggl[1-(1+m_Y|\mathbf{x}|)\,e^{-m_Y|\mathbf{x}|}\biggr]\,\hat{\mathbf{x}}\times\mathbf{J}\,, \tag{2.72}$$

where $\mathbf{J} \;=\; 2M\mathcal{R}^2\mathbf{\Omega}_0/5$ is the angular momentum of the ball.

The modification with respect to General Relativity has the same feature as the one generated by the point-like source [285]. From the definition of m_R and m_Y (2.55), we note that the presence of a Ricci scalar function ($f_{RR}(0) \;\neq\; 0$) appears only in m_R. Considering only $f(R)$-gravity ($m_Y \;\rightarrow\; \infty$), the solution (2.72) is unaffected by the modification in the Hilbert-Einstein action.

In the following, we apply the above analysis in the case of bodies moving in the gravitational field.

2.2 Post – Minkowskian approximation

General Relativity

We suppose the metric to be close to the Minkowski metric $\eta_{\mu\nu}$:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{2.73}$$

with $h_{\mu\nu}$ small quantities ($O(h)^2 \ll 1$). To first order in h, the Christoffel symbols (1.3) are

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}\eta^{\alpha\sigma}(h_{\mu\sigma,\nu} + h_{\nu\sigma,\mu} - h_{\mu\nu,\sigma})\,. \tag{2.74}$$

As long as we restrict ourselves to first order in h, we must raise and lower all indices using $\eta_{\mu\nu}$, not $g_{\mu\nu}$; that is

$$\eta^{\sigma\tau}h_{\sigma\tau} = h^{\sigma}{}_{\sigma} = h, \qquad \eta^{\sigma\tau}\frac{\partial}{\partial x^{\sigma}} = \frac{\partial}{\partial x_{\tau}}, \qquad etc. \tag{2.75}$$

With this assumptions, the Ricci tensor and scalar (1.4) - (1.5) are then

$$\begin{aligned} R^{(1)}_{\mu\nu} &= h^{\sigma}_{(\mu,\nu)\sigma} - \frac{1}{2}\Box_\eta h_{\mu\nu} - \frac{1}{2}h_{,\mu\nu} \\ R^{(1)} &= h_{\sigma\tau}{}^{,\sigma\tau} - \Box_\eta h \end{aligned} \tag{2.76}$$

where $\nabla^\alpha\nabla_\alpha \sim {}_{,\sigma}{}^{,\sigma} = \Box_\eta$ is the d'Alembertian operator in the flat space. The field equation (1.10) becomes

$$G^{(1)}_{\mu\nu} = R^{(1)}_{\mu\nu} - \frac{1}{2}R^{(1)}\eta_{\mu\nu} = \mathcal{X}\,T^{(0)}_{\mu\nu} \tag{2.77}$$

where $T_{\mu\nu}$ is fixed at zero-order in (2.77) since in this perturbation scheme the first order on Minkowski space has to be connected with the zero order of the standard matter energy momentum tensor[12]. Eqs. (2.77) in terms of $h_{\mu\nu}$ are

$$h^{\sigma}_{(\mu,\nu)\sigma} - \frac{1}{2}\Box_\eta h_{\mu\nu} - \frac{1}{2}h_{,\mu\nu} - \frac{1}{2}[h_{\sigma\tau}{}^{,\sigma\tau} - \Box_\eta h]\eta_{\mu\nu} = \mathcal{X}\,T^{(0)}_{\mu\nu}\,. \tag{2.78}$$

Since $T_{\mu\nu}$ is taken to the lowest order in $h_{\mu\nu}$, so it is independent of $h_{\mu\nu}$, it has to satisfies the ordinary conservation conditions:

$$T^{\sigma\mu}{}_{,\sigma} \,=\, 0\,. \tag{2.79}$$

Note that it is this form of the conservation law that is needed for the consistency of (2.78), because (2.79) implies

$$G^{(1)\mu\sigma}{}_{,\sigma} = 0 \tag{2.80}$$

whereas the linearized Ricci tensor satisfies Bianchi identities (1.6) of the form

$$R^{(1)\sigma\mu}{}_{,\sigma} = \frac{1}{2}\left[h^{\alpha\beta}{}_{,\alpha\beta} - \Box_\eta h\right]^{,\mu} = \frac{1}{2}R^{(1),\mu}\,. \tag{2.81}$$

By choosing the transformation $\tilde{h}_{\mu\nu} = h_{\mu\nu} - \frac{h}{2}\eta_{\mu\nu}$ and the gauge condition $\tilde{h}^{\mu\nu}{}_{,\mu} = 0$ (harmonic gauge (2.16)) one obtains that field equations read

$$\Box\tilde{h}_{\mu\nu} = -2\mathcal{X}\,T^{(0)}_{\mu\nu}\,. \tag{2.82}$$

One solution is the *retarded potential*

$$\tilde{h}_{\mu\nu}(t,\mathbf{x}) = 4G\int d^3\mathbf{x}'\,\frac{T^{(0)}_{\mu\nu}(\mathbf{x}', t - |\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|} \tag{2.83}$$

or in terms of perturbation $h_{\mu\nu}$

[12] In this perturbation scheme the first order on Minkowski space has to be connected with the zero order of the standard matter energy momentum tensor. This formalism descends from the theoretical setting of Newtonian mechanics which requires the appropriate scheme of approximation and coincides with a gravity theory analyzed at the first order of perturbations in the curved spacetime metric.

$$h_{\mu\nu}(t,\mathbf{x}) = 4G\int d^3\mathbf{x}'\,\frac{S^{(0)}_{\mu\nu}(\mathbf{x}',t-|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|}\,. \tag{2.84}$$

The propagation of $h_{\mu\nu}$ is possible with a particle massless.

Scalar Tensor Fourth Order Gravity

We will analyze the field equations (1.42), (1.43), (1.44) within the *weak-field approximation* in a Minkowski background (2.73). We consider the expansion for the scalar field (2.52) and develop the function f as (2.53). Then, the field equations (1.42), (1.43) and (1.44) then read

$$\begin{aligned}
&(\Box_\eta + {m_Y}^2)R_{\mu\nu} - \left[\tfrac{{m_R}^2-{m_Y}^2}{3{m_R}^2}\,\partial^2_{\mu\nu} + \eta_{\mu\nu}\left(\tfrac{{m_Y}^2}{2} + \tfrac{{m_R}^2+2{m_Y}^2}{6{m_R}^2}\Box_\eta\right)\right]R \\
&\qquad\qquad -{m_Y}^2\,f_{R\phi}(0,0,\phi^{(0)})\,(\partial^2_{\mu\nu}-\eta_{\mu\nu}\Box_\eta)\varphi \;=\; {m_Y}^2\,\mathcal{X}\,T_{\mu\nu}\,, \\
&(\Box_\eta + {m_R}^2)R - 3{m_R}^2\,f_{R\phi}(0,0,\phi^{(0)})\,\Box_\eta\varphi \;=\; -{m_R}^2\,\mathcal{X}\,T\,, \\
&(\Box_\eta + {m_\phi}^2)\varphi - f_{R\phi}(0,0,\phi^{(0)})\,R \;=\; 0\,,
\end{aligned} \tag{2.85}$$

where $\Box_\eta$ is the D'Alambertian operator in flat space and we have used the definitions of the masses (2.55).

The geometric quantities $R_{\mu\nu}$ and R are evaluated to the first order with respect to the perturbation $h_{\mu\nu}$. Note that for simplicity[13] we set $f_R(0,0,\phi^{(0)}) \;=\; 1$ and $\omega(\phi^{(0)}) \;=\; 1/2$. The Ricci tensor in Eq. (2.85), in the weak-field limit, reads

$$R_{\mu\nu} \;=\; {h^\sigma}_{(\mu,\nu)\sigma} - \frac{1}{2}\Box_\eta\,h_{\mu\nu} - \frac{1}{2}h_{,\mu\nu}\,, \tag{2.86}$$

where $h \;=\; {h^\sigma}_\sigma$. Using the harmonic gauge condition (2.16), hence the Ricci tensor becomes

[13] We can define a new gravitational constant: $\mathcal{X} \;\to\; \mathcal{X}\,f_R(0,0,\phi^{(0)})$ and $f_{R\phi}(0,0,\phi^0) \;\to\; f_{R\phi}(0,0,\phi^0)\,f_R(0,0,\phi^{(0)})$.

$R_{\mu\nu} = -\frac{1}{2}\Box_\eta h_{\mu\nu}$. Equation (2.85) then reads

$$
\begin{aligned}
&(\Box_\eta + {m_Y}^2)\Box_\eta h_{\mu\nu} - \left[\frac{{m_R}^2 - {m_Y}^2}{3{m_R}^2}\,\partial^2_{\mu\nu} + \eta_{\mu\nu}\left(\frac{{m_Y}^2}{2} + \frac{{m_R}^2 + 2{m_Y}^2}{6{m_R}^2}\Box_\eta\right)\right]\Box_\eta h \\
&\qquad\qquad +2\,{m_Y}^2\,f_{R\phi}(0,0,\phi^{(0)})\,(\partial^2_{\mu\nu} - \eta_{\mu\nu}\Box_\eta)\varphi \;=\; -2\,{m_Y}^2\,\mathcal{X}\,T_{\mu\nu}\;, \\
&(\Box_\eta + {m_R}^2)\Box_\eta h + 6{m_R}^2\,f_{R\phi}(0,0,\phi^{(0)})\,\Box_\eta\varphi \;=\; 2{m_R}^2\,\mathcal{X}\,T\;, \\
&(\Box_\eta + {m_\phi}^2)\varphi + \frac{f_{R\phi}(0,0,\phi^{(0)})}{2}\Box_\eta h \;=\; 0\;.
\end{aligned}
\tag{2.87}
$$

The field equations (2.87) generalize those of Ref. [350], since in the latter there was no scalar field component. Moreover, these equations are the weak-field limit of the model discussed in Ref. [351].

To solve Eq. (2.87) we introduce the auxiliary field $\gamma_{\mu\nu}$ such that

$$
\begin{aligned}
(\Box_\eta + {m_Y}^2)\Box_\eta\gamma_{\mu\nu} \;=\; &(\Box_\eta + {m_Y}^2)\Box_\eta h_{\mu\nu} - \left[\eta_{\mu\nu}\left(\frac{{m_Y}^2}{2} + \frac{{m_R}^2 + 2\,{m_Y}^2}{6{m_R}^2}\Box_\eta\right) + \right. \\
&\left. + \frac{{m_R}^2 - {m_Y}^2}{3{m_R}^2}\,\partial^2_{\mu\nu}\right]\Box_\eta h + 2\,{m_Y}^2\,f_{R\phi}(0,0,\phi^{(0)})\,(\partial^2_{\mu\nu} - \eta_{\mu\nu}\Box_\eta)\varphi
\end{aligned}
\tag{2.88}
$$

leading to

$$
\begin{aligned}
h_{\mu\nu} \;=\; &\gamma_{\mu\nu} + \left[\frac{{m_R}^2 - {m_Y}^2}{3{m_R}^2}\,\partial^2_{\mu\nu} + \eta_{\mu\nu}\left(\frac{{m_Y}^2}{2} + \frac{{m_R}^2 + 2{m_Y}^2}{6{m_R}^2}\Box_\eta\right)\right](\Box_\eta + {m_Y}^2)^{-1}h \\
&-2\,{m_Y}^2\,f_{R\phi}(0,0,\phi^{(0)})\,(\partial^2_{\mu\nu} - \eta_{\mu\nu}\Box_\eta)(\Box_\eta + {m_Y}^2)^{-1}\Box_\eta^{-1}\varphi.
\end{aligned}
\tag{2.89}
$$

Since the trace h is

$$
h = -\frac{m_R^2}{m_Y^2}(\Box_\eta + {m_Y}^2)(\Box_\eta + {m_R}^2)^{-1}\gamma - 6\,{m_R}^2\,f_{R\phi}(0,0,\phi^{(0)})\,(\Box_\eta + {m_R}^2)^{-1}\varphi\;,
\tag{2.90}
$$

Equation (2.89) can be written as

$$
\begin{aligned}
h_{\mu\nu} \;=\; &\gamma_{\mu\nu} - \frac{m_R^2}{m_Y^2}\left[\frac{{m_R}^2 - {m_Y}^2}{3{m_R}^2}\,\partial^2_{\mu\nu} + \eta_{\mu\nu}\left(\frac{{m_Y}^2}{2} + \frac{{m_R}^2 + 2{m_Y}^2}{6{m_R}^2}\Box_\eta\right)\right](\Box_\eta + {m_R}^2)^{-1}\gamma \\
&-2\,{m_R}^2\,f_{R\phi}(0,0,\phi^{(0)})\left[\Box_\eta^{-1}(\Box_\eta + {m_R}^2)^{-1}\partial^2_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}(\Box_\eta + {m_R}^2)^{-1}\right]\varphi\;.
\end{aligned}
\tag{2.91}
$$

Using Eqs. (2.89), (2.91), Eq. (2.87) reads

$$(\Box_\eta + {m_Y}^2)\Box_\eta \gamma_{\mu\nu} = -2\,{m_Y}^2\,\mathcal{X}\,T_{\mu\nu}\,,$$

$$(\Box_\eta + {m_Y}^2)\Box_\eta \gamma = -2\,{m_Y}^2\,\mathcal{X}\,T\,, \tag{2.92}$$

$$(\Box_\eta + {m_\phi}^2)\varphi - 3{m_R}^2 f^2_{R\phi}(0,0,\phi^{(0)})\,(\Box_\eta + {m_R}^2)^{-1}\Box_\eta\varphi = -{m_R}^2 f_{R\phi}(0,0,\phi^{(0)})\,(\Box_\eta + {m_R}^2)^{-1}\mathcal{X}\,T\,.$$

Hence, Eqs. (2.87b) and (2.87c) have been decoupled. Let us now rewrite Eq. (2.92c) in the form:

$$(\Box_\eta + m_+^2)(\Box_\eta + m_-^2)\varphi \;=\; -{m_R}^2 f_{R\phi}(0,0,\phi^{(0)})\,\mathcal{X}\,T\,, \tag{2.93}$$

where m_+ and m_- are defined by (2.60).

Let us now introduce the new auxiliary fields Γ, Ψ, Ξ defined through

$$(\Box_\eta + {m_R}^2)\Gamma \;=\; -{m_R}^2\,\gamma\,,$$

$$(\Box_\eta + {m_R}^2)\Box_\eta\Psi \;=\; -2\,{m_R}^2\,\varphi\,, \tag{2.94}$$

$$(\Box_\eta + {m_R}^2)\Xi \;=\; -2\,{m_R}^2\,\varphi\,,$$

so that the solution (2.91) can be expressed as

$$h_{\mu\nu} \;=\; \gamma_{\mu\nu} + \frac{1}{m_Y^2}\left[\frac{{m_R}^2 - {m_Y}^2}{3{m_R}^2}\,\partial^2_{\mu\nu} + \eta_{\mu\nu}\left(\frac{{m_Y}^2}{2} + \frac{{m_R}^2 + 2{m_Y}^2}{6{m_R}^2}\Box_\eta\right)\right]\Gamma + \\ + f_{R\phi}(0,0,\phi^{(0)})\left[\partial^2_{\mu\nu}\,\Psi + \frac{\Xi}{2}\eta_{\mu\nu}\right]\,. \tag{2.95}$$

Notice that in the limit $m_R \to \infty$, $m_Y \to \infty$ and for vanishing $f_{R\phi}(0,0,\phi^{(0)})$ we recover the standard results of General Relativity since Eq. (2.95) reduces to $h_{\mu\nu} = \gamma_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\gamma$.

To solve these equations we need the Green's functions (see Appendix 6.1), and hence we consider the distributions $\mathcal{G}_{\rm KG,m}$, $\mathcal{G}_{\rm GR}$ which satisfy the equations

$$(\Box_\eta + m^2)\mathcal{G}_{\rm KG,m}(x,x') \;=\; \delta^4(x - x')\,,$$

$$\Box_\eta\mathcal{G}_{\rm GR}(x,x') \;=\; \delta^4(x - x')\,, \tag{2.96}$$

where $\mathcal{G}_{\mathrm{KG,m}}$ and $\mathcal{G}_{\mathrm{GR}}$ are, respectively, the Green functions of Klein-Gordon field with mass m and the one for massless modes.

Due to causality, we are only interested in the retarded Green's functions, hence

$$\begin{aligned}
\mathcal{G}^{\mathrm{ret}}_{\mathrm{KG,m}}(x,x') &= \frac{\Theta\Big(t-t'\Big)}{4\pi}\left[\frac{\delta(t-t'-|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|}-\frac{m\,\mathcal{J}_1(m\,\tau_{xx'})}{\tau_{xx'}}\Theta\Big(\frac{\tau^2_{xx'}}{2}\Big)\right],\\
\mathcal{G}^{\mathrm{ret}}_{\mathrm{GR}}(x,x') &= \frac{\Theta\Big(t-t'\Big)}{4\pi}\frac{\delta(t-t'-|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|},
\end{aligned} \tag{2.97}$$

where

$$\tau^2_{xx'}=(x-x')^2=(t-t')^2-|\mathbf{x}-\mathbf{x}'|^2\,. \tag{2.98}$$

The terms with the Dirac distribution describe the dynamics on the light cone, i.e. $t-t'=|\mathbf{x}-\mathbf{x}'|$, while the ones with the Bessel function of the first kind $\mathcal{J}_1(x)$ describe the dynamics interior to the light cone, i.e. $t-t'>|\mathbf{x}-\mathbf{x}'|$. We can now build the Green's functions for the auxilarly fields $\gamma_{\mu\nu}$, Γ, Ψ, Ξ and φ as particular combinations of the $\mathcal{G}^{\mathrm{ret}}_{\mathrm{KG,m}}$ and $\mathcal{G}^{\mathrm{ret}}_{\mathrm{GR}}$:

$$\begin{aligned}
\mathcal{G}^{\mathrm{ret}}_{\gamma}(x,x') &= \Theta\Big(t-t'\Big)\,\Theta\Big(\frac{\tau^2_{xx'}}{2}\Big)\frac{\mathcal{J}_1(m_Y\,\tau_{xx'})}{4\pi\,m_Y\,\tau_{xx'}}\,,\\
\mathcal{G}^{\mathrm{ret}}_{\varphi}(x,x') &= \frac{\Theta\Big(t-t'\Big)\,\Theta\Big(\frac{\tau^2_{xx'}}{2}\Big)}{4\pi\,(m_+^2-m_-^2)\,\tau_{xx'}}\left[m_+\,\mathcal{J}_1(m_+\,\tau_{xx'})-m_-\,\mathcal{J}_1(m_-\,\tau_{xx'})\right],\\
\mathcal{G}^{\mathrm{ret}}_{\Gamma}(x,x') &= \frac{\Theta\Big(t-t'\Big)\,\Theta\Big(\frac{\tau^2_{xx'}}{2}\Big)}{4\pi\,(m_R^2-m_Y^2)\,\tau_{xx'}}\left[\frac{\mathcal{J}_1(m_Y\,\tau_{xx'})}{m_Y}-\frac{\mathcal{J}_1(m_R\,\tau_{xx'})}{m_R}\right],\\
\mathcal{G}^{\mathrm{ret}}_{\Psi}(x,x') &= \frac{\Theta\Big(t-t'\Big)\,\Theta\Big(\frac{\tau^2_{xx'}}{2}\Big)}{4\pi\,\tau_{xx'}}\left[\frac{\mathcal{J}_1(m_+\,\tau_{xx'})}{m_+(\,m_-^2\,-\,m_+^2)(\,m_R^2\,-\,m_+^2)}+\right.\\
&\left.+\frac{\mathcal{J}_1(m_-\,\tau_{xx'})}{m_-(\,m_+^2\,-\,m_-^2)(\,m_R^2\,-\,m_-^2)}+\frac{\mathcal{J}_1(m_R\,\tau_{xx'})}{m_R(\,m_+^2\,-\,m_R^2)(\,m_-^2\,-\,m_R^2)}\right],\\
\mathcal{G}^{\mathrm{ret}}_{\Xi}(x,x') &= -\frac{\Theta\Big(t-t'\Big)\,\Theta\Big(\frac{\tau^2_{xx'}}{2}\Big)}{4\pi\,\tau_{xx'}}\left[\frac{m_+\mathcal{J}_1(m_+\,\tau_{xx'})}{(\,m_-^2\,-\,m_+^2)(\,m_R^2\,-\,m_+^2)}+\right.\\
&\left.+\frac{m_-\mathcal{J}_1(m_-\,\tau_{xx'})}{(\,m_+^2\,-\,m_-^2)(\,m_R^2\,-\,m_-^2)}+\frac{m_R\mathcal{J}_1(m_R\,\tau_{xx'})}{(\,m_+^2\,-\,m_R^2)(\,m_-^2\,-\,m_R^2)}\right].
\end{aligned} \tag{2.99}$$

Using Eq. (2.99) we then derive the particular solution for the field $\gamma_{\mu\nu}$:

$$\gamma_{\mu\nu}(t,\,\mathbf{x}) \;=\; -\frac{m_Y\,\mathcal{X}}{2\,\pi}\int d^3x'\int_{-\infty}^{t-|\mathbf{x}-\mathbf{x}'|}dt'\,\frac{\mathcal{J}_1\big(m_Y\sqrt{(t-t')^2-|\mathbf{x}-\mathbf{x}'|^2}\,\big)}{\sqrt{(t-t')^2-|\mathbf{x}-\mathbf{x}'|^2}}\,T_{\mu\nu}\big(t',\,\mathbf{x}'\big) \tag{2.100}$$

where $t-|\mathbf{x}-\mathbf{x}'|$ is the retarded time. Introducing the variable $\tau = m_Y\sqrt{(t-t')^2-|\mathbf{x}-\mathbf{x}'|^2}$, Eq. (2.100) takes the form

$$\gamma_{\mu\nu}(t,\,\mathbf{x}) \;=\; -\frac{m_Y\,\mathcal{X}}{2\,\pi}\int d^3x'\int_0^{\infty}d\tau\,\frac{\mathcal{J}_1(\tau)}{\sqrt{\tau^2+m_Y^2|\mathbf{x}-\mathbf{x}'|^2}}\,T_{\mu\nu}\left(t-\frac{\sqrt{\tau^2+m_Y^2|\mathbf{x}-\mathbf{x}'|^2}}{m_Y},\,\mathbf{x}'\right)$$

in the limit $m_Y\to\infty$ we obtain the standard results of General Relativity (2.83).

2.3 Gravitational waves emitted by a quadrupole source

Let us assume that the sources are localized in a limited portion of space within the neighborhood of the origin of the coordinates, namely the sources have a maximal spatial extension $|\mathbf{x}'_{\rm max}|$ (where $T_{\mu\nu}\neq 0$ if $|\mathbf{x}|<|\mathbf{x}'_{\rm max}|$). If we consider the far zone limit, or radiation zone limit, i.e. $|\mathbf{x}|>>\lambda>>|\mathbf{x}'_{\rm max}|$ where λ is the gravitational wavelength of the waves emitted, we can consider the solution (2.100) at a great distance from the source. In this limit (i.e. radiation zone), we set $|\mathbf{x}-\mathbf{x}'|\approx|\mathbf{x}|$ and the solution can be approximated by plane waves that have only the spatial components being nonzero, i.e $\gamma_{tt}=\gamma_{ti}=0$ and $\gamma_{ij}\neq 0$. Note that in modified theories of gravity one has in general six different polarization states [350]. The spatial components of $\gamma_{\mu\nu}$ can be written as

$$\gamma_{ij}(t,\,\mathbf{x})\approx -\frac{m_Y\,\mathcal{X}}{2\,\pi}\int_0^{\infty}d\tau\,\frac{\mathcal{J}_1(\tau)}{\sqrt{\tau^2+m_Y^2|\mathbf{x}|^2}}\int d^3x'\,T_{ij}\big(t-\tau_{m_Y},\,\mathbf{x}'\big)\;, \tag{2.101}$$

where in general

$$\tau_m \;=\; \frac{\sqrt{\tau^2+m^2|\mathbf{x}|^2}}{m}\,. \tag{2.102}$$

The spatial components of the energy-momentum tensor T_{ij} are related to the quadrupole moment

$$Q_{ij}(t) \;=\; 3\int d^3\mathbf{x}'\,T_{tt}(t,\,\mathbf{x}')\,x'_i\,x'_j \;=\; 3\int d^3\mathbf{x}'\,\rho(t,\,\mathbf{x})\,x'_i\,x'_j\;, \tag{2.103}$$

through the relation

$$\int d^3\mathbf{x}'\,T_{ij}(t,\,\mathbf{x}') \;=\; \frac{1}{6}\frac{d^2}{dt^2}Q_{ij}(t) \;=\; \frac{\ddot{Q}_{ij}(t)}{6}\,. \tag{2.104}$$

Equation (2.101) can be casted in the form

$$\gamma_{ij}(\mathbf{x},t) \;=\; -2\,m_Y\,\Upsilon^{m_Y}_{ij}(|\mathbf{x}|,t)\,, \tag{2.105}$$

where

$$\Upsilon^m_{ij}(t,\,|\mathbf{x}|) \;=\; \frac{\mathcal{X}}{24\pi}\int_0^\infty d\tau\,\frac{\mathcal{J}_1(\tau)}{\sqrt{\tau^2+m^2|\mathbf{x}|^2}}\,\ddot{Q}_{ij}\big(t-\tau_m\big)\,. \tag{2.106}$$

Considering the trace of Eq. (2.100) in the radiation zone limit

$$\gamma(t,\,\mathbf{x}) \approx -\frac{m_Y\,\mathcal{X}}{2\,\pi}\int_0^\infty d\tau\,\frac{\mathcal{J}_1(\tau)}{\sqrt{\tau^2+m_Y^2|\mathbf{x}|^2}}\int d^3x'\,T(t-\tau_{m_Y},\,\mathbf{x}')\,, \tag{2.107}$$

we get

$$\int d^3\mathbf{x}'\,T(t,\,\mathbf{x}') \;=\; \eta^{\mu\nu}\int d^3\mathbf{x}'\,T_{\mu\nu}(t,\,\mathbf{x}',) \;=\; M_0+\frac{\ddot{Q}(t)}{6}\,, \tag{2.108}$$

where M_0 is the mass of the source and $Q(t)=\eta^{ij}Q_{ij}(t)$ is the trace of the quadrupole moment (2.103). Hence, Eq. (2.107) becomes

$$\begin{aligned}\gamma(\mathbf{x},t) \quad = \quad & -\frac{m_Y\,M_0\,\mathcal{X}}{2\,\pi}\int_0^\infty d\tau\,\frac{\mathcal{J}_1(\tau)}{\sqrt{\tau^2+m_Y^2|\mathbf{x}|^2}}+ \\ & -\frac{m_Y\,\mathcal{X}}{12\,\pi}\int_0^\infty d\tau\,\frac{\mathcal{J}_1(\tau)}{\sqrt{\tau^2+m_Y^2|\mathbf{x}|^2}}\,\ddot{Q}(t-\tau_{m_Y})\,.\end{aligned} \tag{2.109}$$

The first term on the r.h.s. of the equation above reads

$$\int_0^\infty d\tau\,\frac{\mathcal{J}_1(\tau)}{\sqrt{\tau^2+m^2|\mathbf{x}|^2}} = \frac{1}{m|\mathbf{x}|}\left[1-e^{-m\mathbf{x}}\right], \tag{2.110}$$

hence, it does not depend explicitly on time, so that it does not contribute to energy loss of the system. Introducing the spatial trace of Eq. (2.106)

$$\Upsilon^m(t,\,|\mathbf{x}|) = \eta^{ij}\Upsilon^m_{ij}(t,\,|\mathbf{x}|) \;=\; \frac{\mathcal{X}}{24\pi}\int_0^\infty d\tau\,\frac{\mathcal{J}_1(\tau)}{\sqrt{\tau^2+m^2|\mathbf{x}|^2}}\,\ddot{Q}(t-\tau_m)\,, \tag{2.111}$$

the solution for the trace γ (see Eq. (2.109)) takes the form

$$\gamma(t,\,|\mathbf{x}|)\;=\;-2\,m_Y\,\Upsilon^{m_Y}(t,\,|\mathbf{x}|)\,. \tag{2.112}$$

The solutions for all fields then read

$$\gamma_{ij}(t,\,|\mathbf{x}|)\;=\;-2\,m_Y\,\Upsilon^{m_Y}_{ij}(t,\,|\mathbf{x}|)\,,$$

$$\gamma(t,\,|\mathbf{x}|)\;=\;-2\,m_Y\,\Upsilon^{m_Y}(t,\,|\mathbf{x}|)\,,$$

$$\varphi(t,\,|\mathbf{x}|)\;=\;-\frac{{m_R}^2f_{R\phi}(0,0,\phi^{(0)})}{(m_+^2-m_-^2)}\left[m_+\,\Upsilon^{m_+}(t,\,|\mathbf{x}|)-m_-\Upsilon^{m_-}(t,\,|\mathbf{x}|)\right]\,,$$

$$\Gamma(t,\,|\mathbf{x}|)\;=\;\frac{2\,{m_R}^2\,{m_Y}^2}{(m_R^2-m_Y^2)}\left[\frac{\Upsilon^{m_Y}(t,\,|\mathbf{x}|)}{m_Y}-\frac{\Upsilon^{m_R}(t,\,|\mathbf{x}|)}{m_R}\right]\,,$$

$$\begin{aligned}\Psi(t,\,|\mathbf{x}|)\;=\;2\,{m_R}^4f_{R\phi}(0,0,\phi^{(0)})\Bigg[&\frac{\Upsilon^{m_+}(t,\,|\mathbf{x}|)}{m_+(\,m_-^2-\,m_+^2)(\,m_R^2-\,m_+^2)}+\\&+\frac{\Upsilon^{m_-}(t,\,|\mathbf{x}|)}{m_-(\,m_+^2-\,m_-^2)(\,m_R^2-\,m_-^2)}+\frac{\Upsilon^{m_R}(t,\,|\mathbf{x}|)}{m_R(\,m_+^2-\,m_R^2)(\,m_-^2-\,m_R^2)}\Bigg]\,,\end{aligned}$$

$$\begin{aligned}\Xi(t,\,|\mathbf{x}|)\;=\;-2\,{m_R}^4f_{R\phi}(0,0,\phi^{(0)})\Bigg[&\frac{m_+\Upsilon^{m_+}(t,\,|\mathbf{x}|)}{(\,m_-^2-\,m_+^2)(\,m_R^2-\,m_+^2)}+\\&+\frac{m_-\Upsilon^{m_-}(t,\,|\mathbf{x}|)}{(\,m_+^2-\,m_-^2)(\,m_R^2-\,m_-^2)}+\frac{m_R\Upsilon^{m_R}(t,\,|\mathbf{x}|)}{(\,m_+^2-\,m_R^2)(\,m_-^2-\,m_R^2)}\Bigg]\,.\end{aligned}$$

After laborious mathematical calculations (see Appendix 6.2 for details) we can rewrite the spatial components of the perturbation $h_{\mu\nu}$ given in Eq. (2.95), as (from Eq. (6.10))

$$\begin{aligned}h_{ij}(|\mathbf{x}|,t)\;&=\;-2\,m_Y\,\Upsilon^{m_Y}_{ij}+\eta_{ij}\Bigg\{m_Y\,g_Y\,\Upsilon^{m_Y}(|\mathbf{x}|,t)-m_R\,g_R\,\Upsilon^{m_R}(|\mathbf{x}|,t)+\\&-\sum_{S=\pm}\frac{m_S\,g_S(\xi,\eta)}{3}\,\Upsilon^{m_S}(|\mathbf{x}|,t)+g_R\left[\frac{B^{m_Y}(|\mathbf{x}|,t)}{m_Y}-\frac{B^{m_R}(|\mathbf{x}|,t)}{m_R}\right]\Bigg\}+\\&+\frac{2}{3}\left[\frac{D^{m_Y}_{ij}(|\mathbf{x}|,t)}{m_Y}+\sum_{S=\pm}g_S(\xi,\eta)\,\frac{D^{m_S}_{ij}(|\mathbf{x}|,t)}{m_S}\right]\,.\end{aligned} \tag{2.113}$$

This is the main result of our analysis which we will use in the following to constrain the free parameters of extended gravity models found in the literature.

2.4 Conformal Transformation

In the theories of the gravitation we have two separate classes of theories: minimally and non-minimally coupled theories. In general, also higher-order theories of gravity can be reduced to the non-minimally coupled standard (see [312] for details). In the first case, the gravitational coupling is the Newton constant. The scalar fields are added to the Ricci scalar R in the gravitational Lagrangian. In this case, we are dealing with the so-called *Einstein frame*. In the second case, the gravitational coupling is a a function of space and time and it is dynamically related to the scalar fields. It consists of a scalar field ϕ non-minimally coupled to R and a kinetic term for the scalar field into the gravitational action. As a result, the coupling is non-minimal and the gravitational interaction changes with distance and time according to the Mach principle. The straightforward generalization is to take into account theories where also a self-interacting potential or more scalar fields are present. Furthermore, gravitational theories non-linear in the Ricci scalar R or containing other curvature invariants can be reduced to scalar-tensor ones. In general, when we take into account non-minimal couplings or higher-order terms, we are dealing with the *Jordan frame*.

The Einstein and Jordan frames are related by geometrical maps that are the *conformal transformations* and the question is whether such frames are only mathematically equivalent or also physically equivalent. The problem of identifying the physical frame has been longly debated and nowadays strongly emerges in order to address the problem of ”dark sector” either from a geometrical or a material viewpoint [224].

An important example is related to the geodesic motion. In the Jordan frame, in vacuum, neutral massive test particles fall along time-like geodesics. This is not true in the Einstein frame where they deviate from geodesic motion due to a force coming from the conformal scalar field gradient. As a consequence, from conformal transformations point of view, the Equivalence Principle holds only in the Jordan frame. It is important to stress that such a Principle is the basic foundation of relativistic theories of gravity. Then, a representation-independent formulation should physically discriminate between frames. No final result holds in this sense and the violation of the Equivalence Principle (in the Einstein frame) could be interpreted as the fact that frames are not physically equivalent. On the other hand, if the Equivalence Principle holds in a given frame and not in any frame means that it is not a covariant feature but only a kinematical one. In other words, Equivalence Principle is not sufficient to discriminate between conformal frames.

Furthermore, there are results where exact cosmological solutions accelerate in one frame but not in the other. This fact could mean that, for an astronomer attempting to fit observations, the two frames are not physically equivalent [55, 225]. In these situations, one must state precisely what the physical equivalence is and the concept is not obvious at all. In a naive formulation, such an equivalence could be related to the fact that it should be possible to select a set of physically invariant quantities that can be conformally transformed.

As we said, conformal transformations allow to disentangle the further gravitational degrees of freedom coming from general actions [312, 177]. The idea is to perform a conformal rescaling of the space-time metric $g_{\mu\nu} \to \tilde{g}_{\mu\nu}$ and a redefinition of the scalar field ϕ as $\phi \to \tilde{\phi}$. New dynamical variables $\left\{\tilde{g}_{\mu\nu}, \tilde{\phi}\right\}$ are thus obtained. The scalar field redefinition allows, for example, to cast the kinetic energy density of this field in a canonical form. The new set of variables $\left\{\tilde{g}_{\mu\nu}, \tilde{\phi}\right\}$ defines the *Einstein conformal frame*, while $\{g_{\mu\nu}, \phi\}$ constitutes the *Jordan frame*. When a scalar degree of freedom ϕ is present in the theory, as in scalar tensor or $f(R)$ gravity, it generates the transformation to the Einstein frame in the sense that the rescaling is completely determined by a function of ϕ. In principle, infinite conformal frames could be introduced, giving rise to many representations of the theory.

Let the pair $\{\mathcal{M}, g_{\mu\nu}\}$ be a space-time, with $\mathcal{M}$ a smooth manifold of dimension $n \geq 2$ and $g_{\mu\nu}$ a (pseudo)-Riemannian metric on $\mathcal{M}$. The point-dependent rescaling of the metric tensor

$$g_{\mu\nu} \longrightarrow \tilde{g}_{\mu\nu} = \Omega^2 \, g_{\mu\nu} \tag{2.114}$$

where $\Omega = \Omega(x)$ is a nowhere vanishing, regular function, called a *Weyl* or *conformal* transformation. Obviously the transformation rule for the controvariant metric tensor is $\tilde{g}^{\mu\nu} = \Omega^{-2} g^{\mu\nu}$.

Due to this metric rescaling, the lengths of space-like and time-like intervals and the norms of space-like and time-like vectors change, while null vectors and null intervals of the metric $g_{\mu\nu}$ remain null in the rescaled metric $\tilde{g}_{\mu\nu}$ (in this sense, they are conformally invariant quantities). The light cones are left unchanged by the transformation (2.114) and the space-times $\{\mathcal{M}, g_{\mu\nu}\}$ and $\{\mathcal{M}, \tilde{g}_{\mu\nu}\}$ exhibit the same causal structure; the converse is also true [227]. A vector that is time-like, space-like, or null with respect to the metric $g_{\mu\nu}$ has the same character with respect to $\tilde{g}_{\mu\nu}$, and *vice-versa.*

In this section, we want to address the problem of how conformally transformed models behave in the weak field limit approximation. This issue could be extremely relevant in order to select conformally invariant physical quantities.

2.4.1 Scalar tensor gravity in the Jordan frame

The action of a scalar tensor theory of gravity in 4 dimensions is

$$\mathcal{S}^{JF} = \int d^4x\sqrt{-g}\left[\phi\, R + V(\phi) + \omega(\phi)\, \phi_{;\alpha}\, \phi^{;\alpha} + \mathcal{X}\mathcal{L}_m\right]. \tag{2.115}$$

The term $\mathcal{L}_m$ is the minimally coupled ordinary matter contribution considered as a *perfect fluid*; $\omega(\phi)$ is a function of the scalar field and $V(\phi)$ is its potential which specifies the dynamics. Actually if $\omega(\phi) = \pm 1, 0$ the nature and the dynamics of the scalar field is fixed. It can be a canonical scalar field, a phantom field or a field without dynamics (see e.g. [98, 229] for details). In the metric approach, the field equations are obtained by varying the action (2.115) with respect to $g_{\mu\nu}$ and ϕ. The field equations are

$$\phi\, R_{\mu\nu} - \frac{\phi\, R + V(\phi) + \omega(\phi)\, \phi_{;\alpha}\, \phi^{;\alpha}}{2}\, g_{\mu\nu} + \omega(\phi)\, \phi_{;\mu}\, \phi_{;\nu} - \phi_{;\mu\nu} + g_{\mu\nu} \Box\, \phi \;=\; \mathcal{X}\, T_{\mu\nu} \tag{2.116}$$

$$2\, \omega(\phi)\, \Box\, \phi + \omega_{\phi}(\phi)\, \phi_{;\alpha} \phi^{;\alpha} - R - V_{\phi}(\phi) \;=\; 0$$

and the trace equation is

$$\phi\, R + 2V(\phi) + \omega(\phi)\, \phi_{;\alpha} \phi^{;\alpha} - 3\, \Box\, \phi \;=\; -\mathcal{X}\, T \tag{2.117}$$

Here we introduced, respectively, the energy-momentum tensor of matter and the d'Alembert operator

$$T_{\mu\nu} \;=\; -\frac{1}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\, \mathcal{L}_m)}{\delta g^{\mu\nu}}\,, \qquad \Box\, (\cdot) \;=\; \frac{\partial_\sigma(\sqrt{-g}\, g^{\sigma\tau} \partial_\tau(\cdot))}{\sqrt{-g}} \tag{2.118}$$

$T \;=\; T^{\sigma}{}_{\sigma}$ is the trace of energy-momentum tensor and $V_\phi \;=\; \frac{dV}{d\phi}$, $\omega_\phi(\phi) \;=\; \frac{d\omega(\phi)}{d\phi}$. If we assume that the Lagrangian density $\mathcal{L}_m$ of matter depends only on the metric components $g_{\mu\nu}$ and not on its derivatives, we obtain $T_{\mu\nu} \;=\; 1/2\, \mathcal{L}_m\, g_{\mu\nu} - \delta \mathcal{L}_m / \delta g^{\mu\nu}$. Let us consider a source with mass M. The energy-momentum tensor is

$$T_{\mu\nu} \;=\; \rho\, u_\mu u_\nu\,, \qquad T \;=\; \rho \tag{2.119}$$

where ρ is the mass density, u_μ satisfies the condition $g^{00} {u_0}^2 \;=\; 1$, and $u_i \;=\; 0$. Here, we are not interested to the internal structure. It is useful to get the expression of $\mathcal{L}_m$. In fact from the definition (2.118), we have

$$\delta \int d^4x \sqrt{-g}\, \mathcal{L}_m \;=\; -\int d^4x \sqrt{-g}\, T_{\mu\nu}\, \delta g^{\mu\nu} \;=\; -\int d^4x \sqrt{-g}\, \rho\, u_\mu u_\nu\, \delta g^{\mu\nu} \tag{2.120}$$

From the mathematical properties of metric tensor we have

$$\delta(\sqrt{-g}\, \rho) \;=\; 1/2\, \sqrt{-g}\, \rho\, u^\mu u^\nu\, \delta g_{\mu\nu} \;=\; -1/2\, \sqrt{-g}\, \rho\, u_\mu u_\nu\, \delta g^{\mu\nu} \tag{2.121}$$

then we find

$$\mathcal{L}_m = 2\,\rho \tag{2.122}$$

The variation of density is given by

$$\delta\rho = \frac{\rho}{2}(g_{\mu\nu} - u_\mu u_\nu)\,\delta g^{\mu\nu} \tag{2.123}$$

order to deal with standard self-gravitating systems, any theory of gravity has to be developed in its Newtonian or post-Newtonian limit depending on the order of approximation in terms of squared velocity v^2 [230, 231]. The Newtonian limit starts from developing the metric tensor (and other additional quantities in the theory) with respect to the dimensionless velocity[14] v of the moving massive bodies embedded in the gravitational potential. The perturbative development takes only first term of $(0,0)$- and (i,j)-component of metric tensor $g_{\mu\nu}$ (for details, see [231, 232]). The metric assumes the form

$$ds^2 = (1+2\Phi)\,dt^2 - (1-2\Psi)\,\delta_{ij}dx^i dx^j \tag{2.124}$$

where the gravitational potentials Φ, $\Psi < 1$ are proportional to v^2. The Ricci scalar is approximated as $R = R^{(1)} + R^{(2)} + \ldots$ where $R^{(1)}$ is proportional to Φ, and Ψ, while $R^{(2)}$ is proportional to Φ^2, Ψ^2 and $\Phi\Psi$. In this context, also the scalar field ϕ is approximated as the Ricci scalar. In particular we get $\phi = \phi^{(0)} + \phi^{(1)} + \ldots$ while the functions $V(\phi)$ and $\omega(\phi)$ can be substituted by their corresponding Taylor series. From the lowest order of field Eqs. (2.116) we have

$$\begin{aligned} V(\phi^{(0)}) &= 0\,, \\ V_\phi(\phi^{(0)}) &= 0 \end{aligned} \tag{2.125}$$

and also in the scalar tensor gravity a missing cosmological component in the action (1) implies that the space-time is asymptotically Minkowskian; moreover the ground value of scalar field

[14] The velocity v is here expressed in light speed units.

ϕ must be a stationary point of potential. In the Newtonian limit, we have

$$\begin{aligned}
&\triangle\left[\Phi-\frac{\phi^{(1)}}{\phi^{(0)}}\right]-\frac{R^{(1)}}{2}\,=\,\frac{\mathcal{X}\,\rho}{\phi^{(0)}}\\
&\left\{\triangle\left[\Psi+\frac{\phi^{(1)}}{\phi^{(0)}}\right]+\frac{R^{(1)}}{2}\right\}\delta_{ij}+\left\{\Psi-\Phi-\frac{\phi^{(1)}}{\phi^{(0)}}\right\}_{,ij}\,=\,0\\
&\triangle\phi^{(1)}+\frac{V_{\phi\phi}(\phi^{(0)})}{2\,\omega(\phi^{(0)})}\,\phi^{(1)}+\frac{R^{(1)}}{2\,\omega(\phi^{(0)})}\,=\,0\\
&R^{(1)}+3\,\frac{\triangle\phi^{(1)}}{\phi^{(0)}}\,=\,-\frac{\mathcal{X}\,\rho}{\phi^{(0)}}
\end{aligned}\tag{2.126}$$

where $\triangle$ is the Laplacian in the flat space. These equations are not simply the merging of field equations of GR and a further massive scalar field, but come out to the fact that the scalar tensor gravity generates a coupled system of equations with respect to Ricci scalar R and scalar field ϕ. The gravitational potentials Φ, Ψ and the Ricci scalar $R^{(1)}$ are given by

$$\begin{aligned}
\Phi(\mathbf{x})\,&=\,-\frac{\mathcal{X}}{4\pi\,\phi^{(0)}}\int d^3\mathbf{x}'\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}-\frac{1}{8\pi}\int d^3\mathbf{x}'\frac{R^{(1)}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}+\frac{\phi^{(1)}(\mathbf{x})}{\phi^{(0)}}\\
\Psi(\mathbf{x})\,&=\,\Phi(\mathbf{x})+\frac{\phi^{(1)}(\mathbf{x})}{\phi^{(0)}}\\
R^{(1)}(\mathbf{x})\,&=\,-\frac{\mathcal{X}\,\rho(\mathbf{x})}{\phi^{(0)}}-3\,\frac{\triangle\phi^{(1)}(\mathbf{x})}{\phi^{(0)}}
\end{aligned}\tag{2.127}$$

and supposing that $2\,\omega(\phi^{(0)})\,\phi^{(0)}-3\,\neq\,0$ we find for the scalar field $\phi^{(1)}$ the Yukawa-like field equation

$$\left[\triangle-{m_\phi}^2\right]\phi^{(1)}\,=\,\frac{\mathcal{X}\,\rho}{2\,\omega(\phi^{(0)})\phi^{(0)}-3}\tag{2.128}$$

where we introduced the mass definition

$${m_\phi}^2\,\doteq\,-\frac{\phi^{(0)}\,V_{\phi\phi}(\phi^{(0)})}{2\,\omega(\phi^{(0)})\,\phi^{(0)}-3}\,.\tag{2.129}$$

It is important to stress that the potential Ψ can be found also as

$$\Psi(\mathbf{x})\,=\,\frac{1}{8\pi}\int d^3\mathbf{x}'\frac{R^{(1)}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}-\frac{\phi^{(1)}(\mathbf{x})}{\phi^{(0)}}\tag{2.130}$$

see for example [233].

By using the Fourier transformation, the solution of Eq. (2.128) has the following form

$$\phi^{(1)}(\mathbf{x}) = -\frac{\mathcal{X}}{2\,\omega(\phi^{(0)})\phi^{(0)}-3}\int\frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\frac{\tilde{\rho}(\mathbf{k})\,e^{i\mathbf{k}\cdot\mathbf{x}}}{\mathbf{k}^2+m_\phi{}^2} \tag{2.131}$$

The expressions (2.127) and (2.131) represent the most general solution of any scalar-tensor gravity in the Newtonian limit. Since the superposition principle is yet valid (the field Eqs. (2.126) are linear), it is sufficient to consider the solutions generated by a point-like source with mass M. Then if we consider $\rho = M\,\delta(\mathbf{x})$ the solutions are [231, 232, 233]

$$\begin{aligned}
\phi^{(1)}(\mathbf{x}) &= -\frac{1}{2\,\omega(\phi^{(0)})\,\phi^{(0)}-3}\frac{r_g}{|\mathbf{x}|}\,e^{-m_\phi|\mathbf{x}|}\\
R^{(1)}(\mathbf{x}) &= -\frac{4\pi\,r_g}{\phi^{(0)}}\,\delta(\mathbf{x})+\frac{3\,m_\phi{}^2}{[2\,\omega(\phi^{(0)})\,\phi^{(0)}-3]\,\phi^{(0)}}\frac{r_g}{|\mathbf{x}|}\,e^{-m_\phi|\mathbf{x}|}\\
\Phi(\mathbf{x}) &= -\frac{GM}{\phi^{(0)}|\mathbf{x}|}\left\{1-\frac{e^{-m_\phi|\mathbf{x}|}}{2\,\omega(\phi^{(0)})\,\phi^{(0)}-3}\right\}\\
\Psi(\mathbf{x}) &= -\frac{GM}{\phi^{(0)}|\mathbf{x}|}\left\{1+\frac{e^{-m_\phi|\mathbf{x}|}}{2\,\omega(\phi^{(0)})\,\phi^{(0)}-3}\right\}
\end{aligned} \tag{2.132}$$

where $r_g = 2GM$ is the Schwarzschild radius. In the case $V(\phi) = 0$, the scalar field is massless and $\omega(\phi) = -\omega_0/\phi$, we obtain

$$\begin{aligned}
\Phi(\mathbf{x}) = \Phi_{BD}(\mathbf{x}) &= -\frac{GM}{\phi^{(0)}|\mathbf{x}|}\left[\frac{2(2+\omega_0)}{2\,\omega_0+3}\right] = -\frac{G^*M}{|\mathbf{x}|}\\
\Psi(\mathbf{x}) = \Psi_{BD}(\mathbf{x}) &= -\frac{G^*M}{|\mathbf{x}|}\left(\frac{1+\omega_0}{2+\omega_0}\right)
\end{aligned} \tag{2.133}$$

the well-known Brans-Dicke solutions [30] with Eddington's parameter $\gamma = \frac{1+\omega_0}{2+\omega_0}$ [131] where the gravitational constat is defined as $G \rightarrow G^* = \frac{G}{\phi^{(0)}}\frac{2(2+\omega_0)}{2\,\omega_0+3}$.

2.4.2 Scalar tensor gravity in the Einstein frame

Let us now introduce the conformal transformation (2.114) to show that scalar-tensor theories are, in general, conformally equivalent to the Einstein theory plus minimally coupled scalar fields. However if standard matter is present, the conformal transformation generates the non-minimal coupling between the matter component and the scalar field.

By applying the transformation (2.114), the action in (2.115) can be reformulated as follows

$$\mathcal{S}^{EF} = \int d^4x\sqrt{-\tilde{g}}\Big[\Xi\,\tilde{R} + W(\tilde{\phi}) + \tilde{\omega}(\tilde{\phi})\tilde{\phi}_{;\alpha}\tilde{\phi}^{;\alpha} + \mathcal{X}\tilde{\mathcal{L}}_m\Big] \tag{2.134}$$

in which $\tilde{R}$ is the Ricci scalar relative to the metric $\tilde{g}_{\mu\nu}$ and Ξ is a generic constant. The two actions (2.115) and (2.134) are mathematically equivalent. In fact the conformal transformation is given by imposing the condition

$$\sqrt{-g}\Big[\phi\,R + V(\phi) + \omega(\phi)\,\phi_{;\alpha}\,\phi^{;\alpha} + \mathcal{X}\mathcal{L}_m\Big] = \sqrt{-\tilde{g}}\Big[\Xi\,\tilde{R} + W(\tilde{\phi}) + \tilde{\omega}(\tilde{\phi})\tilde{\phi}_{;\alpha}\tilde{\phi}^{;\alpha} + \mathcal{X}\tilde{\mathcal{L}}_m\Big] \tag{2.135}$$

The relations between the quantities in the two frames are

$$\begin{aligned}
\tilde{\omega}(\tilde{\phi})\,d\tilde{\phi}^2 &= \frac{\Xi}{2}\,[2\,\phi\,\omega(\phi) - 3]\left(\frac{d\phi}{\phi}\right)^2 \\
W(\tilde{\phi}) &= \frac{\Xi^2}{\phi(\tilde{\phi})^2}\,V(\phi(\tilde{\phi})) \\
\tilde{\mathcal{L}}_m &= \frac{\Xi^2}{\phi(\tilde{\phi})^2}\,\mathcal{L}_m\left(\frac{\Xi\,\tilde{g}_{\rho\sigma}}{\phi(\tilde{\phi})}\right) \\
\phi\,\Omega^{-2} &= \Xi
\end{aligned} \tag{2.136}$$

The field equations for the new fields $\tilde{g}_{\mu\nu}$ and $\tilde{\phi}$ are

$$\begin{aligned}
&\Xi\,\tilde{R}_{\mu\nu} - \frac{\Xi\,\tilde{R} + W(\tilde{\phi}) + \tilde{\omega}(\tilde{\phi})\,\tilde{\phi}_{;\alpha}\tilde{\phi}^{;\alpha}}{2}\,\tilde{g}_{\mu\nu} + \tilde{\omega}(\tilde{\phi})\,\tilde{\phi}_{;\mu}\tilde{\phi}_{;\nu} = \mathcal{X}\,\tilde{T}_{\mu\nu} \\
&2\,\tilde{\omega}(\tilde{\phi})\,\tilde{\Box}\tilde{\phi} + \tilde{\omega}_{\tilde{\phi}}(\tilde{\phi})\,\tilde{\phi}_{;\alpha}\tilde{\phi}^{;\alpha} - W_{\tilde{\phi}}(\tilde{\phi}) - \mathcal{X}\frac{\delta\,\tilde{\mathcal{L}}_m}{\delta\,\tilde{\phi}} = 0 \\
&\Xi\,\tilde{R} + 2W(\tilde{\phi}) + \tilde{\omega}(\tilde{\phi})\,\tilde{\phi}_{;\alpha}\tilde{\phi}^{;\alpha} = -\mathcal{X}\,\tilde{T}
\end{aligned} \tag{2.137}$$

where $\tilde{T}_{\mu\nu}$ and $\tilde{\Box}$ are the re-definition of the quantities (2.118) with respect to the metric $\tilde{g}_{\mu\nu}$. The field Eqs. (2.137) can be obtained from (2.116) by substituing all geometrical and physical quantities in terms of conformally transformed ones. In particular we have

$$
\begin{aligned}
R_{\mu\nu} &= \tilde{R}_{\mu\nu} + 2\ln\Omega_{;\tilde{\mu}\nu} + 2\ln\Omega_{;\mu}\ln\Omega_{;\nu} + [\tilde{\Box}\ln\Omega - 2\ln\Omega^{\tilde{;\sigma}}\ln\Omega_{;\sigma}]\,\tilde{g}_{\mu\nu} \\
R &= \Omega^2\left[\tilde{R} + 6\tilde{\Box}\ln\Omega - 3\ln\Omega^{\tilde{;\sigma}}\ln\Omega_{;\sigma}\right] \\
\phi_{;\mu\nu} &= \phi_{;\tilde{\mu}\nu} + 2\,\phi_{;\mu}\phi_{;\nu} - \ln\Omega^{\tilde{;\sigma}}\phi_{;\sigma}\,\tilde{g}_{\mu\nu} \\
\Box(\cdot) &= \Omega^2\,\tilde{\Box}(\cdot) - 2\ln\Omega^{\tilde{;\sigma}}\partial_\sigma(\cdot)
\end{aligned}
\tag{2.138}
$$

The integration of field Eqs. (2.137) is only formal because we do not know the analytical expression of the coupling function between the matter and the scalar field $\tilde{\phi}$ (see the third line of (2.136)). We can make some assumptions on the parameter Ξ and the function $\tilde{\omega}(\tilde{\phi})$ in the minimally coupled Lagrangian (2.134) and on the function $\omega(\phi)$ in the nonminimally coupled Lagrangian (2.115). If we choose $\tilde{\omega}(\tilde{\phi}) = -1/2$, $\Xi = 1$ and $\omega(\phi) = -\omega_0/\phi$, the transformation between the scalar fields ϕ and $\tilde{\phi}$ is given by the first line in (2.136), that is

$$
\tilde{\phi}(\phi) = \tilde{\phi}_0 + \sqrt{2\omega_0 + 3}\,\ln\phi \qquad \phi(\tilde{\phi}) = \exp\left(\frac{\tilde{\phi} - \tilde{\phi}_0}{\sqrt{2\omega_0 + 3}}\right) \tag{2.139}
$$

where obviously $\omega_0 > -3/2$ and $\tilde{\phi}_0$ is an integration constant[15]. The potential W and the matter Lagrangian $\tilde{\mathcal{L}}_m$ are

$$
W(\tilde{\phi}) = \exp\left(-\frac{2\tilde{\phi}}{\sqrt{2\omega_0 + 3}}\right) V\left(e^{\frac{\tilde{\phi}}{\sqrt{2\omega_0+3}}}\right) \qquad \tilde{\mathcal{L}}_m = 2\,\rho\exp\left(-\frac{2\tilde{\phi}}{\sqrt{2\omega_0 + 3}}\right) \tag{2.140}
$$

In both frames, the scalar fields are expressed as perturbative contributions on the cosmological background ($\phi^{(0)}$, $\tilde{\phi}^{(0)}$) with respect to the dimensionless quantity v^2. Then also for the scalar field $\tilde{\phi}$, we can consider the develop $\tilde{\phi} = \tilde{\phi}^{(0)} + \tilde{\phi}^{(1)} + \ldots$. Such a develop can be applied to the transformation rule (2.139) and we obtain

[15] Without losing generality, we can set $\tilde{\phi}_0 = 0$.

$$\tilde{\phi}(\phi) \,=\, \sqrt{2\omega_0+3}\,\ln\phi \,=\, \sqrt{2\omega_0+3}\,\ln\phi^{(0)} + \frac{\sqrt{2\omega_0+3}}{\phi^{(0)}}\,\phi^{(1)} + \ldots \,\doteq\, \tilde{\phi}^{(0)} + \tilde{\phi}^{(1)} + \ldots \tag{2.141}$$

$$\phi(\tilde{\phi}) \,=\, e^{\frac{\tilde{\phi}}{\sqrt{2\omega_0+3}}} \,=\, e^{\frac{\tilde{\phi}^{(0)}}{\sqrt{2\omega_0+3}}} + \frac{e^{\frac{\tilde{\phi}^{(0)}}{\sqrt{2\omega_0+3}}}}{\sqrt{2\omega_0+3}}\,\tilde{\phi}^{(1)} + \ldots \,\doteq\, \phi^{(0)} + \phi^{(1)} + \ldots$$

Since we are interested in the Newtonian limit of field Eqs. (2.137), we can assume, for the conformally transformed metric $\tilde{g}_{\mu\nu}$, an expression as (2.124) but with some differences. In fact from the conformal transformation (2.114) and from the last line of (2.136), we have

$$\tilde{g}_{\mu\nu} \,=\, \phi\, g_{\mu\nu} \,=\, \phi^{(0)}\eta_{\mu\nu} + [\phi^{(0)}g^{(1)}_{\mu\nu} + \phi^{(1)}\eta_{\mu\nu}] + \ldots \,=\, \tilde{\eta}_{\mu\nu} + \tilde{g}^{(1)}_{\mu\nu} + \ldots \tag{2.142}$$

then the conformally transformed metric becomes

$$ds^2 \,=\, (\phi^{(0)} + 2\tilde{\Phi})\,dt^2 - (\phi^{(0)} - 2\tilde{\Psi})\,\delta_{ij}dx^i dx^j \tag{2.143}$$

and the relation between the gravitational potentials in the two frames is

$$\tilde{\Phi} - \phi^{(0)}\,\Phi \,=\, \frac{\phi^{(1)}}{2}\,, \qquad \tilde{\Psi} - \phi^{(0)}\,\Psi \,=\, -\frac{\phi^{(1)}}{2} \tag{2.144}$$

Then the field Eqs. (2.137) become[16]

$$\begin{aligned}
&\frac{\triangle\tilde{\Phi}}{\phi^{(0)}} - \frac{\tilde{R}^{(1)}}{2}\,\phi^{(0)} \,=\, \mathcal{X}\,\tilde{T}^{(1)}_{00} \\
&\left\{\frac{\triangle\tilde{\Psi}}{\phi^{(0)}} + \frac{\tilde{R}^{(1)}}{2}\phi^{(0)}\right\}\delta_{ij} + \frac{(\tilde{\Psi}-\tilde{\Phi})_{,ij}}{\phi^{(0)}} \,=\, 0 \\
&\frac{\triangle\tilde{\phi}^{(1)}}{\phi^{(0)}} - W_{\tilde{\phi}\tilde{\phi}}(\tilde{\phi}^{(0)})\,\tilde{\phi}^{(1)} - \mathcal{X}\left[\frac{\delta\,\tilde{\mathcal{L}}_m}{\delta\,\tilde{\phi}}\right]^{(1)} \,=\, 0 \\
&\tilde{R}^{(1)} \,=\, -\mathcal{X}\,\tilde{T}^{(1)}
\end{aligned} \tag{2.145}$$

[16] With the assumptions of the metric (2.143) the Ricci tensor $\tilde{R}_{\mu\nu}$ in the Newtonian limit has the form $\frac{\triangle\tilde{\Phi}}{\phi^{(0)}}$ (a similar behaviour for $\tilde{R}^{(1)}_{ij}$), where the Ricci scalar is scaled by the factor ${\phi^{(0)}}^2$. The same scaling occurs for the Laplacian: $\triangle \to \frac{\triangle}{\phi^{(0)}}$.

where also in this case we have $W(\tilde{\phi}^{(0)}) = 0$ and $W_{\tilde{\phi}}(\tilde{\phi}^{(0)}) = 0$. However these conditions are an obvious consequence of the conformal transformation of conditions $V(\phi^{(0)}) = 0$ and $V_\phi(\phi^{(0)}) = 0$. In fact we can figure out that $V(\phi) \propto (\phi - \phi^{(0)})^2$ and then

$$W(\tilde{\phi}) \propto \left(e^{\frac{\tilde{\phi}}{\sqrt{2\omega_0+3}}} - \phi^{(0)}\right)^2$$

which, by using relations (2.141), satisfies the above conditions. Finally, we note that

$$W_{\tilde{\phi}\tilde{\phi}}(\tilde{\phi}^{(0)}) = \frac{V_{\phi\phi}\left(e^{\frac{\tilde{\phi}^{(0)}}{\sqrt{2\omega_0+3}}}\right)}{2\omega_0 + 3} = \frac{V_{\phi\phi}(\phi^{(0)})}{2\omega_0 + 3}$$

and by the definition of mass ${m_\phi}^2$, given in Eq. (2.129), we obtain

$$W_{\tilde{\phi}\tilde{\phi}}(\tilde{\phi}^{(0)}) = {m_\phi}^2/\phi^{(0)}\,.$$

Finally, the energy-momentum tensor $\tilde{T}_{\mu\nu}$ is given by the following expression

$$\tilde{T}_{\mu\nu} = \rho \exp\left(-\frac{2\,\tilde{\phi}}{\sqrt{2\,\omega_0 + 3}}\right)\tilde{u}_\mu\tilde{u}_\nu \tag{2.146}$$

where $\tilde{g}_{\mu\nu}\tilde{u}^\mu\tilde{u}^\nu = 1$ then $\tilde{u}_0 = \sqrt{\phi^{(0)} + 2\,\tilde{\Phi}}$. In the Newtonian limit, we find $\tilde{T}^{(1)}_{00} = \rho/\phi^{(0)}$ and $\tilde{T}^{(1)} = \rho/{\phi^{(0)}}^2$. It remains only to calculate the source term $\delta\tilde{\mathcal{L}}_m/\delta\tilde{\phi}$ of the scalar field $\tilde{\phi}^{(1)}$. From the third line of (2.136) and, by using the transformation rules (2.139), we find the coupling between the scalar field and the ordinary matter

$$
\begin{aligned}
\frac{\delta\tilde{\mathcal{L}}_m}{\delta\tilde{\phi}} &= \frac{\delta}{\delta\tilde{\phi}}\biggl\{e^{-\frac{2\tilde{\phi}}{\sqrt{2\omega_0+3}}}\mathcal{L}_m\biggl(e^{-\frac{\tilde{\phi}}{\sqrt{2\omega_0+3}}}\,\tilde{g}_{\rho\sigma}\biggr)\biggr\} = \frac{-2\,e^{-\frac{2\,\tilde{\phi}}{\sqrt{2\omega_0+3}}}}{\sqrt{2\omega_0+3}}\,\mathcal{L}_m(\cdot)+e^{-\frac{2\,\tilde{\phi}}{\sqrt{2\omega_0+3}}}\,\frac{\delta\mathcal{L}_m(\cdot)}{\delta g^{\mu\nu}}\,\frac{\delta g^{\mu\nu}}{\delta\tilde{\phi}} \\
&= \frac{-2\,e^{-\frac{2\,\tilde{\phi}}{\sqrt{2\omega_0+3}}}}{\sqrt{2\omega_0+3}}\,\mathcal{L}_m(\cdot)+e^{-\frac{2\,\tilde{\phi}}{\sqrt{2\omega_0+3}}}\,\frac{\mathcal{L}_m(\cdot)}{2}(g_{\mu\nu}-u_\mu u_\nu)\,\frac{\tilde{g}^{\mu\nu}\delta\,\phi(\tilde{\phi})}{\delta\,\tilde{\phi}} \qquad (2.147) \\
&= \frac{-2\,e^{-\frac{2\,\tilde{\phi}}{\sqrt{2\omega_0+3}}}}{\sqrt{2\omega_0+3}}\,\mathcal{L}_m(\cdot)+e^{-\frac{2\,\tilde{\phi}}{\sqrt{2\omega_0+3}}}\,\frac{3\,\mathcal{L}_m(\cdot)}{2}\frac{1}{\sqrt{2\omega_0+3}} = -\frac{1}{2}\frac{e^{-\frac{2\,\tilde{\phi}}{\sqrt{2\omega_0+3}}}}{\sqrt{2\omega_0+3}}\,\mathcal{L}_m(\cdot) \\
&= -\frac{e^{-\frac{2\,\tilde{\phi}}{\sqrt{2\omega_0+3}}}}{\sqrt{2\omega_0+3}}\,\rho
\end{aligned}
$$

Then the system of Eqs. (2.145) becomes

$$
\begin{aligned}
&\triangle\tilde{\Phi} = \frac{\mathcal{X}\,\rho}{2} \qquad \tilde{\Psi} = \tilde{\Phi} \\
&\biggl[\triangle - {m_\phi}^2\biggr]\tilde{\phi}^{(1)} = -\frac{\mathcal{X}\,\rho}{\phi^{(0)}\sqrt{2\omega_0+3}}
\end{aligned} \qquad (2.148)
$$

and their solutions in the case of pointlike source are

$$
\tilde{\Phi} = -\frac{G\,M}{|\mathbf{x}|} \qquad \tilde{\Psi} = \tilde{\Phi} \qquad \tilde{\phi}^{(1)} = \frac{1}{\phi^{(0)}\sqrt{2\omega_0+3}}\frac{r_g}{|\mathbf{x}|}\,e^{-m_\phi|\mathbf{x}|} \qquad (2.149)
$$

The difference in Eqs.(2.144) between the gravitational potentials is satisfied by using the expression of scalar field in the Jordan frame (first line of (2.132)) where, obviously, we set $\omega(\phi) = -\omega_0/\phi$. In fact we find

$$
\tilde{\Phi} - \phi^{(0)}\Phi = \frac{GM}{2\,\omega_0+3}\frac{e^{-m_\phi|\mathbf{x}|}}{|\mathbf{x}|} = \frac{\phi^{(1)}}{2} \qquad (2.150)
$$

and an analogous relations is found also for the couples Ψ, $\tilde{\Psi}$. Furthermore we can check also the transformation rules (2.139) and (2.141) for the solutions (2.132) and (2.149) of the scalar fields ϕ, $\tilde{\phi}$.

The redefinition of the gravitational constant G (as performed in the Jordan frame $G \rightarrow G^*$ in the case of Brans-Dicke theory [30]) is not available when we are interested to compare the outcomes in both frame. In fact the couple of potentials $\Phi, \tilde{\Phi}$ differs not only from the dynamical contribution of the scalar field ($\phi^{(1)}$) but also from the definition of the gravitational constant. Furthermore, in the Einstein frame, the scalar field $\tilde{\phi}$ does not contribute (the coupling constant between R and $\tilde{\phi}$ is vanishing), then we find the same outcomes of General Relativity with ordinary matter. However by supposing the Jordan frame as starting point and coming back via conformal transformation, we find that the gravitational constant is not invariant and depends on the background value of the scalar field in the Einstein frame, that is $G \rightarrow G_{eff} \propto e^{-\tilde{\phi}^{(0)}}G$.

2.4.3 The case of $f(R)$-gravity

Recently, several authors claimed that higher-order theories of gravity and among them, $f(R)$ gravity, are characterized by an ill defined behavior in the Newtonian regime. In particular, it is discussed that Newtonian corrections of the gravitational potential violate experimental constraints since these quantities can be recovered by a direct analogy with Brans-Dicke gravity simply supposing the Brans-Dicke characteristic parameter ω_0 vanishing (see [137] for a discussion). Actually, the calculations of the Newtonian limit of $f(R)$-gravity, directly performed in a rigorous manner, have showed that this is not the case [230, 231, 232, 234, 235] and it is possible to discuss also the analogy with Brans-Dicke gravity. The issue is easily overcome once the correct analogy between $f(R)$-gravity and the corresponding scalar-tensor framework is taken into account. It is worth noticing that several results already achieved in the Newtonian regime, see e.g.[237, 236], are confirmed by the present approach.

In literature, it is shown that $f(R)$ gravity models can be rewritten in term of a scalar-field Lagrangian non-minimally coupled with gravity but without kinetic term implying that the Brans-Dicke parameter is $\omega(\phi) = 0$. This fact is considered the reason for the ill-definition of the weak field limit that should be $\omega \rightarrow \infty$ inside the Solar System.

Let us deal with the $f(R)$ gravity formalism in order to set correctly the problem. The action is

$$\mathcal{S}^{JF}_{f(R)} = \int d^4x\sqrt{-g}\Big[f(R) + \mathcal{X}\mathcal{L}_m\Big] \tag{2.151}$$

and the field equations are

$$f_R R_{\mu\nu} - \frac{f}{2}g_{\mu\nu} - f_{R;\mu\nu} + g_{\mu\nu}\Box f_R = \mathcal{X}\,T_{\mu\nu} \tag{2.152}$$

with the trace

$$3\,\Box f' + f_R R - 2f \;=\; \mathcal{X}T \tag{2.153}$$

where $f_R \;=\; \frac{df}{dR}$. These equations can be recast in the framework of scalar-tensor gravity as son as we select a particular expression for the free parameters of the theory. The result is the so-called O'Hanlon theory [239] which can be written as

$$\mathcal{S}^{JF}_{OH} = \int d^4x\sqrt{-g}\Big[\phi R + V(\phi) + \mathcal{X}\mathcal{L}_m\Big] \tag{2.154}$$

The field equations are obtained by starting from Eqs. (2.116)

$$\begin{aligned}
&\phi\, R_{\mu\nu} - \frac{\phi\, R + V(\phi)}{2}\, g_{\mu\nu} - \phi_{;\mu\nu} + g_{\mu\nu}\Box\,\phi \;=\; \mathcal{X}\, T_{\mu\nu}\\
&R + V_\phi(\phi) \;=\; 0\\
&\phi\, R + 2V(\phi) - 3\,\Box\,\phi \;=\; -\mathcal{X}\, T
\end{aligned} \tag{2.155}$$

By supposing that the Jacobian of the transformation $\phi \;=\; f_R$ is non-vanishing, the two representations can be mapped one into the other considering the following equivalence

$$\begin{aligned}
&\omega(\phi) \;=\; 0\\
&V(\phi) \;=\; f - f_R R\\
&\phi\, V_\phi(\phi) - 2V(\phi) \;=\; f_R R - 2f
\end{aligned} \tag{2.156}$$

From the definition of the mass (2.129) we have $\phi\, V_\phi(\phi) - 2V(\phi) \;=\; 3\, {m_\phi}^2 \phi^{(1)}$, then we have also $f_R R - 2f \;=\; 3\, {m_\phi}^2\phi^{(1)}$ and by performing the Newtonian limit on the function f [231], we get $f_R(0)R^{(1)} \;=\; -3\, {m_\phi}^2\phi^{(1)}$. The spatial evolution of Ricci scalar is obtained by solving the field Eq.(2.152)

$$R^{(1)} \;=\; -\frac{3\, {m_\phi}^2\phi^{(1)}}{f_R(0)} \;=\; -\frac{{m_\phi}^2\, r_g}{f_R(0)}\frac{e^{-m_\phi|\mathbf{x}|}}{|\mathbf{x}|} \tag{2.157}$$

without using the conformal transformation [231, 232]. The solution for the potentials Φ, Ψ

are obtained simply by setting $\omega(\phi) = 0$ in Eqs. (2.132) and $\phi^{(0)} = f_R(0)$. In the case $f(R) \to R$, from the second line of (2.156), $V(\phi) = 0 \to m_\phi = 0$ and the solutions (2.132) become the standard s Schwarzschild solution in the Newtonian limit.

Finally, we can consider a Taylor expansion [17] of the form $f = f_R(0)\, R^{(1)} + \frac{f_{RR}(0)}{2}\, {R^{(1)}}^2$ so that the associated scalar field reads $\phi = f_R(0) + f_{RR}(0)\, R^{(1)}$. The relation between ϕ and $R^{(1)}$ is $R^{(1)} = \frac{\phi - f_R(0)}{f_{RR}(0)}$ while the self-interaction potential (second line of (2.156)) turns out the be $V(\phi) = -\frac{(\phi - f_R(0))^2}{2\, f_{RR}(0)}$ satisfying the conditions $V(f_R(0)) = 0$ and $V_\phi(f_R(0)) = 0$. In relation to the definition of the scalar field, we can opportunely identify $f_R(0)$ with a constant value $\phi^{(0)} = f_R(0)$ which justifies the previous ansatz for matching solutions in the limit of General Relativity. Furthermore, the mass of the scalar field can be expressed in term of the Lagrangian parameters as ${m_\phi}^2 = \frac{1}{3}\phi^{(0)}\, V_{\phi\phi}(\phi^{(0)}) = -\frac{f_R(0)}{3 f_{RR}(0)}$. Also in this case the value of mass is the same obtained by solving the problem without invoking the scalar tensor analogy [231, 232]. However with this last remark, it is clear the analogy between $f(R)$-gravity and a particular class of scalar tensor theories [239].

[17] The terms resulting from R^n with $n \geq 3$ do *not* contribute at the Newtonian order.

Chapter 3

Lensing

For a general class of analytic functions $f(R, R_{\alpha\beta}R^{\alpha\beta}, R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta})$ we discuss the gravitational lensing in the Newtonian Limit of theory [**A**]. From the properties of Gauss Bonnet invariant it is enough to consider only one curvature invariants between the Ricci and Riemann tensor. Then we analyze the dynamics of photon embedded in a gravitational field of a generic $f(R, R_{\alpha\beta}R^{\alpha\beta})$-Gravity. The metric is time independent and spherically symmetric. The metric potentials are Schwarzschild-like, but there are two additional Yukawa terms linked to derivatives of f with respect to two curvature invariants. Considering first the case of a point-like lens, and after the one of a generic matter distribution of lens, we study the deflection angle and the angular position of images. Though the additional Yukawa terms in the gravitational potential modifies dynamics with respect to the General Relativity, the geodesic trajectory of photon is unaffected by the modification if we consider only $f(R)$-Gravity. While we find different results (deflection angle smaller than one of General Relativity) only thank to introduction of a generic function of Ricci tensor square. Finally we can affirm the lensing phenomena for all $f(R)$-Gravities are equal to the ones known of General Relativity. We conclude the chapter showing and comparing the deflection angle and position of images for $f(R, R_{\alpha\beta}R^{\alpha\beta})$-Gravity with respect to the ones of General Relativity.

3.1 The Gravitational Lensing by $f(X, Y, Z)$-Gravity

Point-Like Source

The Lagrangian of photon in the gravitational field with metric (2.51) is

$$\mathcal{L} = \frac{1}{2}\biggl[\biggl(1 - \frac{r_g}{r}\,\Xi(r)\biggr)\dot{t}^2 - \biggl(1 + \frac{r_g}{r}\,\Lambda(r)\biggr)\dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}^2\biggr] \tag{3.1}$$

where $\Xi(r) \doteq 1+\hat{\Xi}(r) \doteq 1+\frac{1}{3}\,e^{-\mu_1 r}-\frac{4}{3}\,e^{-\mu_2 r}$, $\Lambda(r) \doteq 1+\hat{\Lambda}(r) \doteq 1-\frac{\mu_1 r+1}{3}\,e^{-\mu_1 r}-\frac{2(\mu_2 r+1)}{3}\,e^{-\mu_2 r}$ and the dot represents the derivatives with respect to the affine parameter λ. Since the variable θ does not have dynamics ($\ddot{\theta} = 0$) we can choose for simplicity $\theta = \pi/2$. By applying the Euler-Lagrangian equation to Lagrangian (3.1) for the cyclic variables t, ϕ we find two motion constants

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial\dot{t}} &= \left(1-\frac{r_g}{r}\,\Xi(r)\right)\dot{t} \doteq \mathcal{T} \\ \frac{\partial\mathcal{L}}{\partial\dot{\phi}} &= -r^2\dot{\phi} \doteq -J \end{aligned} \tag{3.2}$$

and respect to λ we find the "energy" of Lagrangian[1]

$$\mathcal{L} = 0 \tag{3.3}$$

By inserting the equations (3.2) into (3.3) we find a differential equation for $\dot{r}$

$$\dot{r}_\pm = \pm\mathcal{T}\sqrt{\frac{1}{1+\frac{r_g}{r}\,\Lambda(r)}\left[\frac{1}{1-\frac{r_g}{r}\,\Xi(r)}-\frac{J^2}{r^2}\right]} \tag{3.4}$$

$\dot{r}_+$ is the solution for leaving photon, while $\dot{r}_-$ is one for incoming photon. Let r_0 be a minimal distance from the lens center (Fig. 3.3). We must impose the condition $\dot{r}_\pm(r_0) = 0$ from the which we find

$$J^2 = \frac{{r_0}^2\mathcal{T}^2}{1-\frac{r_g}{r_0}\,\Xi(r_0)} \tag{3.5}$$

Now the deflection angle α (Fig. 3.3) is defined by following relation

$$\begin{aligned} \alpha &= -\pi+\phi_{fin} = -\pi+\int_0^{\phi_{fin}} d\phi = -\pi+\int_{\lambda_{in}}^{\lambda_{fin}}\dot{\phi}\,d\lambda \\ &= -\pi+\int_{\lambda_{in}}^{\lambda_0}\dot{\phi}\,d\lambda+\int_{\lambda_0}^{\lambda_{fin}}\dot{\phi}\,d\lambda \\ &= -\pi+\int_\infty^{r_0}\frac{\dot{\phi}}{\dot{r}_-}\,dr+\int_{r_0}^\infty\frac{\dot{\phi}}{\dot{r}_+}\,dr = -\pi+2\int_{r_0}^\infty\frac{\dot{\phi}}{\dot{r}_+}\,dr \end{aligned} \tag{3.6}$$

[1] The (3.1) is a quadratic form, so it corresponds to its Hamiltonian.

where λ_0 is the value of λ corresponding to the minimal value (r_0) of radial coordinate r. By putting the expressions of J, $\dot{\phi}$ and $\dot{r}_+$ into (3.6) we get the deflection angle

$$\alpha = -\pi + 2\int_{r_0}^{\infty} \frac{dr}{r\sqrt{\frac{1}{1+\frac{r_g}{r}\,\Lambda(r)}\left[\frac{1-\frac{r_g}{r}\,\Xi(r_0)}{1-\frac{r_g}{r}\,\Xi(r)}\frac{r^2}{{r_0}^2}-1\right]}} \tag{3.7}$$

which in the case $r_g/r \ll 1$[2] becomes

$$\alpha = 2\,r_g\left[\frac{1}{r_0} + \mathcal{F}_{\mu_1,\,\mu_2}(r_0)\right] \tag{3.8}$$

where

$$\mathcal{F}_{\mu_1,\,\mu_2}(r_0) \doteq \frac{1}{2}\int_{r_0}^{\infty} \frac{r_0 r^2[\hat{\Lambda}(r)-\hat{\Xi}(r)] + r^3\hat{\Xi}(r_0) - {r_0}^3\hat{\Lambda}(r)}{r^3(r^2-{r_0}^2)\sqrt{1-\frac{{r_0}^2}{r^2}}}\,dr \tag{3.9}$$

From the definition of $\hat{\Xi}$ and $\hat{\Lambda}$ we note that in the case $f(X,Y,Z) \to X$ we obtain $\mathcal{F}_{\mu_1,\,\mu_2}(r_0) \to 0$. In a such way we extended and contemporarily recovered the outcome of GR.

The analytical dependence of function $\mathcal{F}_{\mu_1,\,\mu_2}(r_0)$ from the parameters μ_1 and μ_2 is given by evaluating the integral (3.9). A such as integral is not easily evaluable from the analytical point of view. However this aspect is not fundamental, since we can numerically appreciate the deviation from the outcome of GR. In fact in Fig. 3.1 we show the plot of deflection angle (3.8) by $f(X,Y,Z)$-Gravity for a given set of values for μ_1 and μ_2. The spatial behavior of α is ever the same if we do not modify μ_2. This outcome is really a surprise: by the numerical evaluation of the function $\mathcal{F}_{\mu_1,\,\mu_2}(r_0)$ one notes that the dependence of μ_1 is only formal. If we solve analytically the integral we must find a μ_1 independent function. However, this statement should not be justified only by numerical evaluation but it needs an analytical proof. For these reasons in the next section we reformulate the theory of Gravitational Lensing generated by a generic matter distribution and demonstrate that for $f(X)$-Gravity one has the same outcome of GR.

Extended Matter Source

In this section we want to recast the framework of Gravitational Lensing for a generic matter source distribution $\rho(\mathbf{x})$ so the photon can undergo many deviations. In this case we leave the

[2] We do not consider the Gravitational Lensing generated by a black hole.

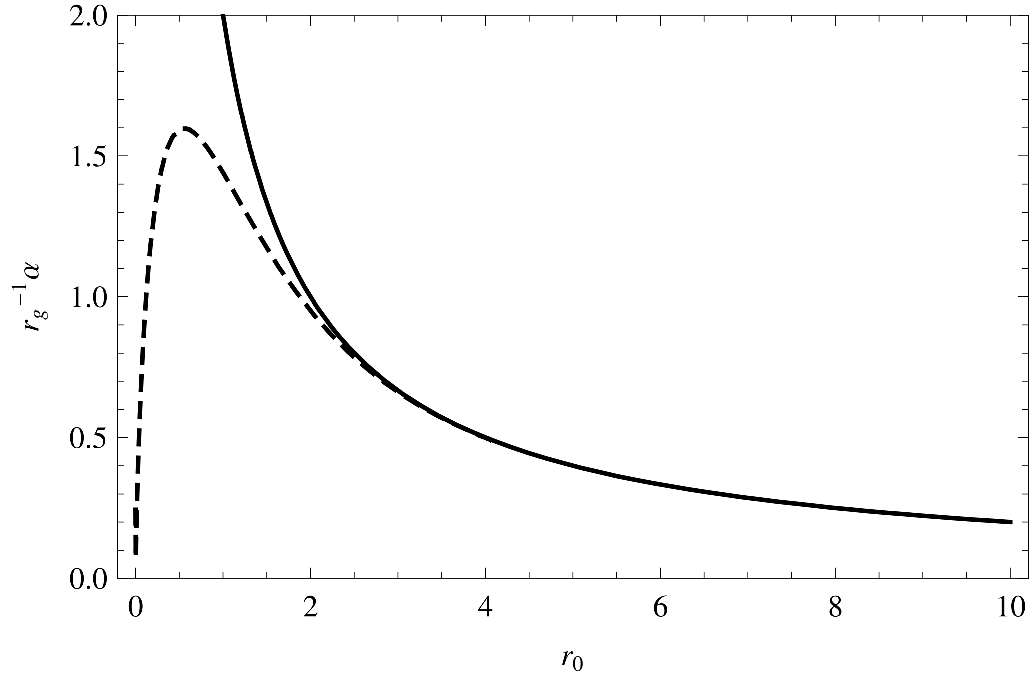

Figure 3.1: Comparison between the deflection angle of GR (solid line) and one of $f(X, Y, Z)$-Gravity (dashed line) (3.8) for a fixed value $\mu_2 = 2$ and any μ_1.

hypothesis that the flight of photon belongs always to the same plane, but we consider only the deflection angle as the angle between the directions of incoming and leaving photon. Finally we find the generalization of Gravitational Lensing in $f(X, Y, Z)$-Gravity including the previous outcome of deflecting point-like source (and resolving the integral (3.9)).

The relativistic invariant (2.44) is yet valid since we consider the superposition of point-like solutions. Indeed we can generalize the metric (2.48) by the following substitution

$$\begin{aligned}\Phi &= -G\int d^3\mathbf{x}'\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\left[1+\frac{1}{3}\,e^{-\mu_1|\mathbf{x}-\mathbf{x}'|}-\frac{4}{3}\,e^{-\mu_2|\mathbf{x}-\mathbf{x}'|}\right]\\ \Psi &= -G\int d^3\mathbf{x}'\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\left[1-\frac{1}{3}\,e^{-\mu_1|\mathbf{x}-\mathbf{x}'|}-\frac{2}{3}\,e^{-\mu_2|\mathbf{x}-\mathbf{x}'|}\right]\end{aligned} \tag{3.10}$$

This approach is correct only in the Newtonian limit since a such limit correspond also to the linearized version of theory. Obviously the $f(X, Y, Z)$-Gravity (like GR) is not linear, then we should have to solve the field equations (1.42) with a given ρ.

By introducing the four velocity $u^\mu = \dot{x}^\mu = (u^0, \mathbf{u})$ the flight of photon, from the metric (2.44), is regulated by the condition

$$g_{\alpha\beta}u^\alpha u^\beta = (1+2\Phi){u^0}^2-(1-2\Psi)|\mathbf{u}|^2 = 0 \tag{3.11}$$

then u^μ is given by

$$u^\mu = \left(\sqrt{\frac{1-2\Psi}{1+2\Phi}}\,|\mathbf{u}|, \mathbf{u}\right) \tag{3.12}$$

In the Newtonian limit we find that the geodesic motion equation becomes

$$\dot{u}^\mu + \Gamma^\mu_{\alpha\beta}u^\alpha u^\beta = 0 \rightarrow \dot{\mathbf{u}} + |\mathbf{u}|^2\nabla(\Phi+\Psi) - 2\mathbf{u}\nabla\Psi\cdot\mathbf{u} = 0 \tag{3.13}$$

and by supposing $|\mathbf{u}|^2 = 1$ we can recast the equation in a more known aspect

$$\dot{\mathbf{u}} = -2\left[\nabla_\perp\Psi + \frac{1}{2}\nabla(\Phi-\Psi)\right] \tag{3.14}$$

where $\nabla_\perp = \nabla - \left(\frac{\mathbf{u}}{|\mathbf{u}|}\cdot\nabla\right)\frac{\mathbf{u}}{|\mathbf{u}|}$ is the two dimensions nabla operator orthogonal to direction of vector $\mathbf{u}$. In GR we would had only $\dot{\mathbf{u}} = -2\,\nabla_\perp\Phi$ since we have $\Psi = \Phi$. In fact the field equations (2.46) are corrects [232] if we satisfy a constraint condition among the metric potentials Φ, Ψ as follows

$$\triangle(\Phi-\Psi) = \frac{{m_1}^2 - {m_2}^2}{3{m_1}^2}\int d^3\mathbf{x}'\mathcal{G}_2(\mathbf{x},\mathbf{x}')\,\triangle_{\mathbf{x}'}X^{(2)}(\mathbf{x}') \tag{3.15}$$

We can affirm, then, that only in GR the metric potentials are equals (or more generally their difference must be proportional to function $|\mathbf{x}|^{-1}$). The constraint (3.15) has been found also many times in the context of cosmological perturbation theory [120, 248, 249, 250].

The deflection angle (3.6) is now defined by equation

$$\vec{\alpha} = -\int_{\lambda_i}^{\lambda_f}\frac{d\mathbf{u}}{d\lambda}d\lambda \tag{3.16}$$

where λ_i and λ_f are the initial and final value of affine parameter [255]. For a generic matter distribution we can not *a priori* claim that the deflection angle belongs to lens plane (as point-like source), but we can only link the deflection angle to the difference between the initial and final velocity $\mathbf{u}$. So we only analyze the directions of photon before and after the interaction with the gravitational mass. Then the (3.16) is placed by assuming $\vec{\alpha} = \Delta\mathbf{u} = \mathbf{u}_i - \mathbf{u}_f$. From the geodesic equation (3.14) the deflection angle becomes

$$\vec{\alpha} = 2\int_{\lambda_i}^{\lambda_f}\left[\nabla_{\perp}\Psi + \frac{1}{2}\nabla(\Phi - \Psi)\right]d\lambda \tag{3.17}$$

The formula (3.17) represents the generalization of deflection angle in the framework of GR. By considering the photon incoming along the z-axes we can set $\mathbf{u}_i = (0,0,1)$. Moreover we decompose the general vector $\mathbf{x}\,\epsilon\,\mathbb{R}^3$ in two components: $\vec{\xi}\,\epsilon\,\mathbb{R}^2$ and $z\,\epsilon\,\mathbb{R}$. The differential operator now can be decomposed as follows $\nabla = \nabla_{\perp} + \hat{z}\,\partial_z = \nabla_{\vec{\xi}} + \hat{z}\,\partial_z$, while the modulus of distance is $|\mathbf{x} - \mathbf{x}'| = \sqrt{|\vec{\xi} - \vec{\xi}'|^2 + (z - z')^2} \doteq \Delta(\vec{\xi}, \vec{\xi}', z, z')$. Since the potentials $\Phi\,,\Psi \ll 1$, around the lens, the solution of (3.14) with the initial condition $\mathbf{u}_i = (0,0,1)$ can be expressed as follows

$$\mathbf{u} = (\mathcal{O}(\Phi,\Psi), \mathcal{O}(\Phi,\Psi), 1 + \mathcal{O}(\Phi,\Psi)) \tag{3.18}$$

and we can substitute the integration with respect to the affine parameter λ with z. In fact we note

$$d\lambda = \frac{dz}{dz/d\lambda} = \frac{dz}{1 + \mathcal{O}(\Phi,\Psi)} \sim dz \tag{3.19}$$

and the deflection angle (3.17) becomes

$$\vec{\alpha} = \int_{z_i}^{z_f}\left[\nabla_{\vec{\xi}}(\Phi + \Psi) + \hat{z}\,\partial_z(\Phi - \Psi)\right]dz \tag{3.20}$$

From the expression of potentials (3.10) we find the relations

$$\begin{aligned}
\Phi + \Psi &= -2\,G\int d^2\vec{\xi}'dz'\frac{\rho(\vec{\xi}', z')}{\Delta(\vec{\xi}, \vec{\xi}', z, z')} + 2\,G\int d^2\vec{\xi}'dz'\frac{\rho(\vec{\xi}', z')}{\Delta(\vec{\xi}, \vec{\xi}', z, z')}\,e^{-\mu_2\Delta(\vec{\xi},\vec{\xi}',z,z')} \\
\Phi - \Psi &= -\frac{2G}{3}\int d^2\vec{\xi}'dz'\frac{\rho(\vec{\xi}', z')}{\Delta(\vec{\xi}, \vec{\xi}', z, z')}\left[e^{-\mu_1\Delta(\vec{\xi},\vec{\xi}',z,z')} - e^{-\mu_2\Delta(\vec{\xi},\vec{\xi}',z,z')}\right]
\end{aligned} \tag{3.21}$$

and the equation (3.17) becomes

$$\begin{aligned}
\vec{\alpha} &= 2G\int_{z_i}^{z_f} d^2\vec{\xi'}dz'\,dz\frac{\rho(\vec{\xi'},z')(\vec{\xi}-\vec{\xi'})}{\Delta(\vec{\xi},\vec{\xi'},z,z')^3} \qquad (3.22)\\
&-2G\int_{z_i}^{z_f} d^2\vec{\xi'}dz'\,dz\frac{\rho(\vec{\xi'},z')[1+\mu_2\Delta(\vec{\xi},\vec{\xi'},z,z')]}{\Delta(\vec{\xi},\vec{\xi'},z,z')^3}e^{-\mu_2\Delta(\vec{\xi},\vec{\xi'},z,z')}(\vec{\xi}-\vec{\xi'})+\\
&+\frac{2G}{3}\hat{z}\int_{z_i}^{z_f} d^2\vec{\xi'}dz'\,dz\frac{\rho(\vec{\xi'},z')(z-z')}{\Delta(\vec{\xi},\vec{\xi'},z,z')^3}\biggl[\Bigl(1+\mu_1\Delta(\vec{\xi},\vec{\xi'},z,z')\Bigr)e^{-\mu_1\Delta(\vec{\xi},\vec{\xi'},z,z')}\\
&-\Bigl(1+\mu_2\Delta(\vec{\xi},\vec{\xi'},z,z')\Bigr)e^{-\mu_2\Delta(\vec{\xi},\vec{\xi'},z,z')}\biggr]
\end{aligned}$$

In the case of hypothesis of thin lens belonging to plane (x, y) we can consider a weak dependence of modulus $\Delta(\vec{\xi},\vec{\xi'},z,z')$ into variable z' so there is only a trivial error if we set $z' = 0$. With this hypothesis the integral into z' is incorporated by definition of two dimensional mass density $\Sigma(\vec{\xi'}) = \int dz'\rho(\vec{\xi'},z')$. Since we are interesting only to the Gravitational Lensing performed by one lens we can extend the integration range of z between $(-\infty,\infty)$. Now the deflection angle is the following

$$\vec{\alpha} = 4G\int d^2\vec{\xi'}\Sigma(\vec{\xi'})\biggl[\frac{1}{|\vec{\xi}-\vec{\xi'}|}-|\vec{\xi}-\vec{\xi'}|\,\mathcal{F}_{\mu_2}(\vec{\xi},\vec{\xi'})\biggr]\frac{\vec{\xi}-\vec{\xi'}}{|\vec{\xi}-\vec{\xi'}|} \qquad (3.23)$$

where

$$\mathcal{F}_{\mu_2}(\vec{\xi},\vec{\xi'}) = \int_0^\infty dz\frac{(1+\mu_2\Delta(\vec{\xi},\vec{\xi'},z,0))}{\Delta(\vec{\xi},\vec{\xi'},z,0)^3}e^{-\mu_2\Delta(\vec{\xi},\vec{\xi'},z,0)} \qquad (3.24)$$

The last integral in (3.22) is vanishing because the integrating function is odd with respect to variable z. The expression (3.23) is the generalization of outcome (3.8) and mainly we found a correction term depending only on the μ_2 parameter.

In the case of point-like source $\Sigma(\vec{\xi'}) = M\,\delta^{(2)}(\vec{\xi'})$ we find

$$\vec{\alpha} = 2\,r_g\biggl[\frac{1}{|\vec{\xi}|}-|\vec{\xi}|\,\mathcal{F}_{\mu_2}(\vec{\xi},0)\biggr]\frac{\vec{\xi}}{|\vec{\xi}|} \qquad (3.25)$$

and in the case of $f(X, Y, Z) \to f(X)$ (*i.e.* $\mu_2 \to \infty$ and $\mathcal{F}_{\mu_2}(\vec{\xi},\vec{\xi'}) \to 0$) we recover the outcome of GR $\vec{\alpha} = 2\,r_g\,\vec{\xi}/|\vec{\xi}|^2$. From the theory of Gravitational Lensing in GR we know that the deflection angle $2\,r_g/r_0$ is formally equal to $2\,r_g/|\vec{\xi}|$ if we suppose $r_0 = |\vec{\xi}|$. Besides both r_0, $|\vec{\xi}|$ are not practically measurable, while it is possible to measure the so-called impact parameter b (see Fig. 3.3). But only in the first approximation these three quantities are equal.

In fact when the photon is far from the gravitational source we can parameterize the trajectory as follows

$$\begin{cases} t = \lambda \\ x = -t \\ y = b \end{cases} \rightarrow \begin{cases} r = \sqrt{t^2+b^2} \\ \phi = -\arctan\frac{b}{t} \end{cases} \tag{3.26}$$

and from the definition of angular momentum (3.2) in the case of $t \gg b$ we have

$$J = \dot{\phi}\, r^2 = \frac{b/t^2}{1+b^2/t^2}(t^2+b^2) \sim b \tag{3.27}$$

By using the condition (3.5) $\dot{r}_\pm(r_0) = 0$ we find the relation among b and r_0[3]

$$b = \frac{r_0\,\mathcal{T}}{\sqrt{1-\frac{r_g}{r_0}\,\Xi(r_0)}} \sim r_0 \tag{3.28}$$

justifying then the position $r_0 = |\vec{\xi}|$ in the limit $r_g/r \ll 1$ (but also $r_g/r_0 \ll 1$).

In Fig. 3.2 we report the plot of deflection angle (3.25). The behaviors shown in figure are parameterized only by μ_2 and we note an equal behavior shown in Fig. 3.1. With the expression (3.25) we have the analytical proof of statement at the end of previous section. In fact in the equation (3.25) we have not any information about the correction induced in the action (1.19) by a generic function of Ricci scalar ($f_{XX} \neq 0$). This result is very important if we consider only the class of theories $f(X)$-Gravity. In this case, since $\mu_2 \rightarrow \infty$, we found the same outcome of GR. From the behavior in Fig. 3.2 we note that the correction to outcome of GR is deeply different for $r_0 \rightarrow 0$, while for $r_0 \rightarrow \infty$ the behavior (3.25) approaches the outcome of GR, but the deviations are smaller. This difference is given by the repulsive correction to the gravitational potential (see metric (2.48)) induced by $f(Y, Z)$. Only by leaving the thin lens hypothesis (the lens does not belong to plane $z = 0$) we can have the deflection angle depending by μ_1 (3.22). In fact in this case the third integral in (3.22) is not zero. Then in the case of thin lens we have a complete degeneracy of outcomes in the $f(X)$-Gravity: *all $f(X)$-Gravities are equivalent to the GR*. If we want to find some differences we must to include the

[3]The constant $\mathcal{T}$ is dimensionless if we consider that λ is the length of trajectory of photon. In this case without losing the generality we can choose $\mathcal{T} = 1$.

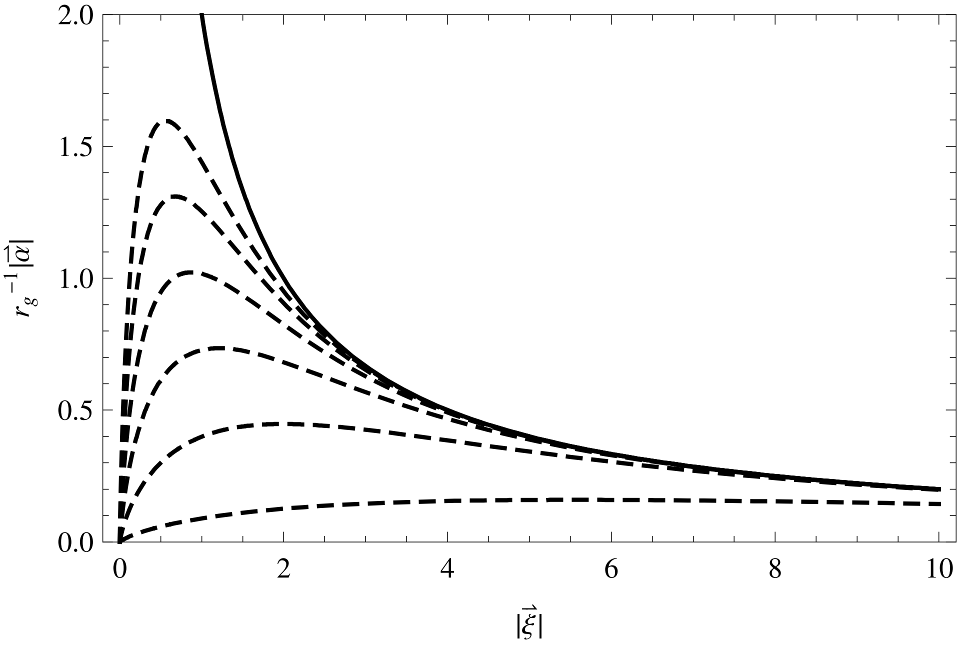

Figure 3.2: Comparison between the deflection angle of GR (solid line) and of $f(X, Y, Z)$-Gravity (dashed line) (3.25) for $0.2 < \mu_2 < 2$.

contributions generated by the Ricci tensor square. But also in this case we do not have the right behavior: the deflection angle is smaller than one of GR: $f(X, Y, Z)$-Gravity does not mimic the Dark Matter component if we assume the thin lens hypothesis.

3.1.1 Lens equation

To demonstrate the effect of a deflecting mass we show in Fig. 3.3 the simplest Gravitational Lensing configuration. A point-like mass is located at distance D_{OL} from the observer O. The source is at distance D_{OS} from the observer, and its true angular separation from the lens L is β, the separation which would be observed in the absence of lensing ($r_g = 0$). The photon which passes the lens at distance $r_0 \sim b$ is deflected with an angle α.

Since the deflection angle (3.8) is equal to (3.25), for sake of simplicity we will use the "vectorial" expression. Then the expression (3.25), by considering the relation (3.28), becomes

$$\alpha = 2\,r_g\left[\frac{1}{b} - b\,\mathcal{F}_{\mu_2}(b, 0)\right] \tag{3.29}$$

The condition that this photon reach the observer is obtained from the geometry of Fig. 3.3. In fact we find

$$\beta = \theta - \frac{D_{LS}}{D_{OS}}\,\alpha \tag{3.30}$$

Here D_{LS} is the distance of the source from the lens. In the simple case with a Euclidean background metric here, $D_{LS} = D_{OS} - D_{OL}$; however, since the Gravitational Lensing occurs in the Universe on large scale, one must use a cosmological model [255]. Denoting the angular separation between the deflecting mass and the deflected photon as $\theta = b/D_{OL}$ the lens equation for $f(X, Y, Z,)$-Gravity is the following

$$[1 + {\theta_E}^2\,\mathcal{F}(\theta)]\,\theta^2 - \beta\,\theta - {\theta_E}^2 = 0 \tag{3.31}$$

where $\theta_E = \sqrt{\frac{2\,r_g\,D_{LS}}{D_{OL}D_{OS}}}$ is the Einstein angle and

$$\mathcal{F}(\theta) = \int_0^\infty dz \frac{(1 + \mu_2 D_{OL}\sqrt{\theta^2 + z^2})}{\sqrt{(\theta^2 + z^2)^3}}\, e^{-\mu_2 D_{OL}\sqrt{\theta^2+z^2})} \tag{3.32}$$

Since we have $0 < \theta^2\mathcal{F}(\theta) < 1$ (Fig. 3.4) we can find a perturbative solution of (3.31) by starting from one in GR, $\theta_{\pm}^{GR} = \frac{-\beta\pm\sqrt{\beta^2+4{\theta_E}^2}}{2}$. In fact by assuming $\theta = \theta_{\pm}^{GR} + \theta^*$ and neglecting ${\theta^*}^2\mathcal{F}(\theta^*)$ in (3.31) we find

$$\theta = \theta_{\pm}^{GR} \mp \frac{{\theta_E}^2}{\sqrt{\beta^2 + 4\,{\theta_E}^2}}\,\mathcal{F}(\theta_{\pm}^{GR})\,\theta_{\pm}^{GR2} \tag{3.33}$$

and in the case of $\beta = 0$ we find the modification to the Einstein ring

$$\theta = \pm\theta_E\left[1 - \frac{{\theta_E}^2}{2}\,\mathcal{F}(\theta_E)\right] \tag{3.34}$$

In Fig. 3.5 we show the angular position of images with respect to the Einstein ring. Both the deflection angle and the position of images assume a smaller value than ones of GR. Then the corrections to the GR quantities are found only for the introduction in the action (1.19) of curvature invariants Y (or Z), while there are no modifications induced by adding a generic function of Ricci scalar X. The algebraic signs of terms concerning the parameter μ_2 are ever different with respect to the terms of GR in (2.48) and they can be interpreted as a "repulsive force" giving us a minor curvature of photon. The correction terms concerning the parameter μ_1 have opposite algebraic sign in the metric component g_{tt} and g_{ij} (2.48) and we lose their information in the deflection angle (3.20).

In both approaches we find the same outcomes μ_1-independent because the matter source (in our case it is a point-like mass) is symmetric with respect to z-axes and we neglect the second integral in (3.22). Obviously for a generic matter distribution the deflection angle is

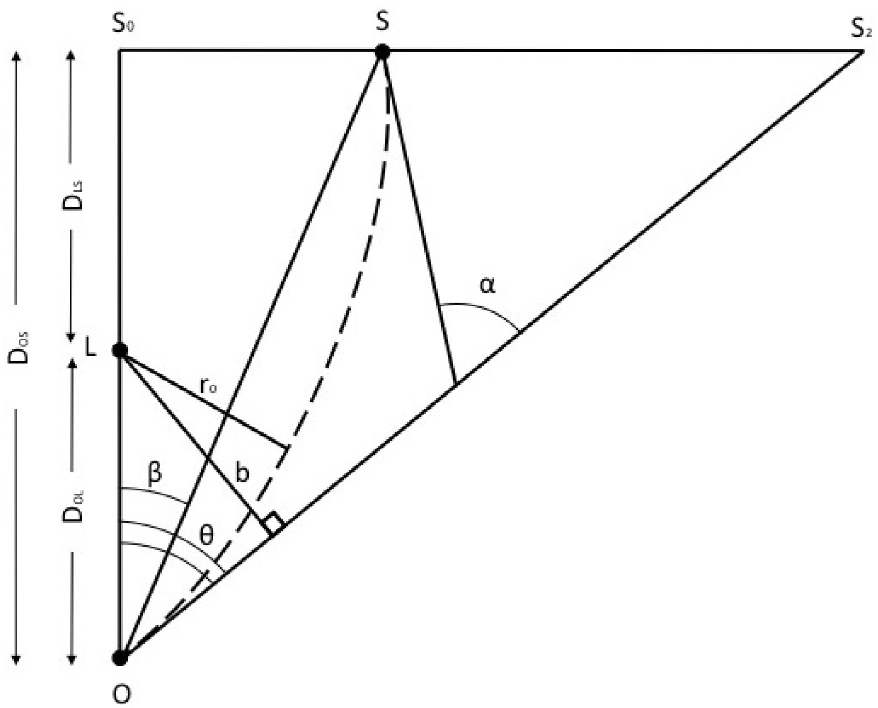

Figure 3.3: The gravitational lensing geometric for a point-like source lens L at distance D_{OL} from observer O. A source S at distance D_{OS} from O has angular position β from the lens. A light ray (dashed line) from S which passes the lens at minimal distance r_0 is deflected by α; the observer sees an image of the source at angular position $\theta = b/D_{OL}$ where b is the impact factor. D_{LS} is the distance lens - source.

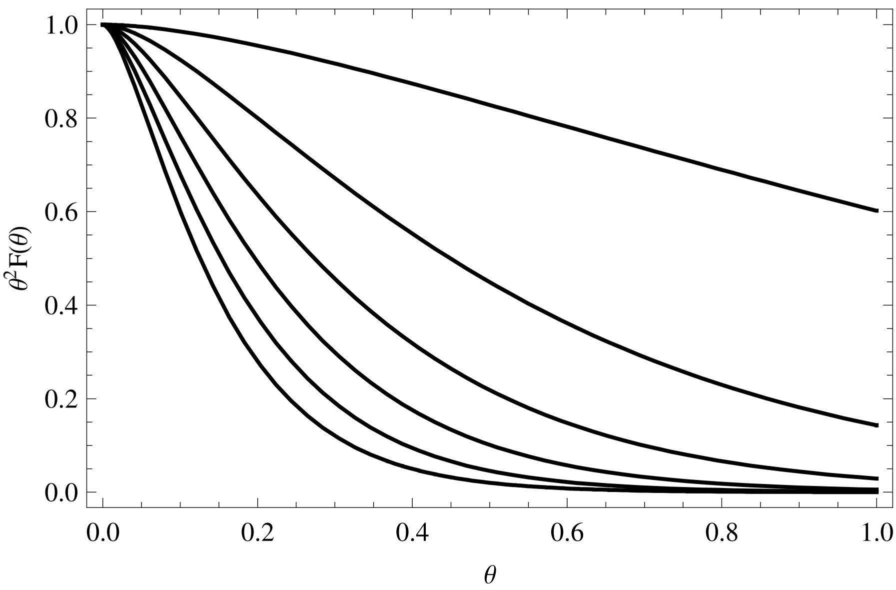

Figure 3.4: Plot of function $\theta^2\mathcal{F}(\theta)$ (3.32) for $1 < \mu_2 D_{OL} < 10$.

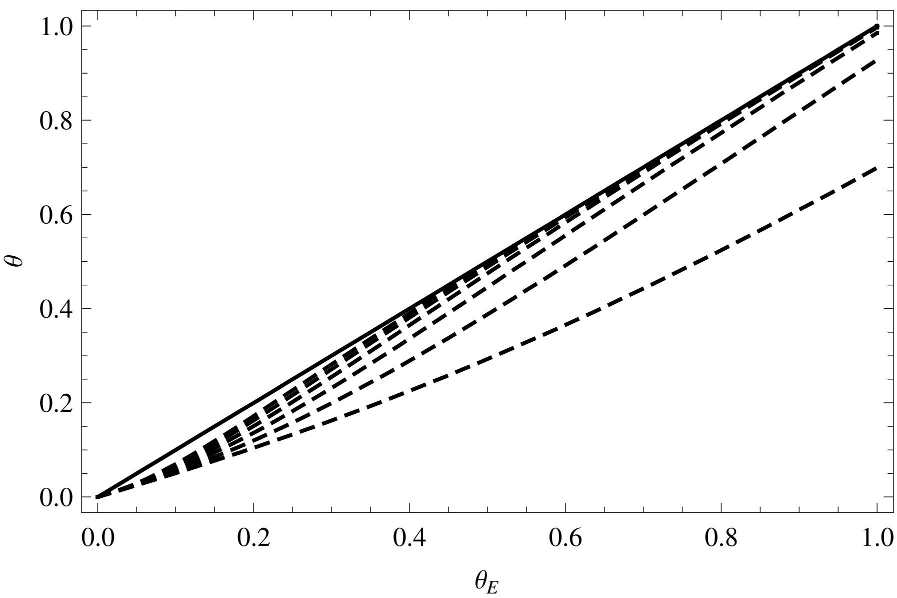

Figure 3.5: Plot of the Einstein ring (solid line) and its modification (3.34) in the $f(X, Y, Z)$-Gravity for $1 < \mu_2 D_{OL} < 10$ (dashed line).

defined by (3.22) and the choice of second derivative of function of Ricci scalar is not arbitrary anymore.

Chapter 4

Terrestrial Test of Scalar Tensor Forth Order Gravity: Gravity Probe B and LARES

We consider models of Extended Gravity and in particular, generic models containing scalar-tensor and higher-order curvature terms, as well as a model derived from noncommutative spectral geometry. Studying, in the weak-field approximation (the Newtonian and Post-Newtonian limit of the theory), the geodesic and Lense-Thirring processions, we impose constraints on the free parameters of such models by using the recent experimental results of the Gravity Probe B (GPB) and LARES satellites [**D**]. The imposed constraint by GPB and LARES is independent of the torsion-balance experiment, though it is much weaker [**B**].

4.1 The body motion in the weak gravitational field for a Scalar-Tensor-Forth-Order Gravity

In the second chapter, we have seen the Newtonian and post-Newtonian limit for a Scalar Tensor Forth Order Gravity (1.41). In terms of the potentials generated by the ball source with radius $\mathcal{R}$, the components of the metric $g_{\mu\nu}$ (2.68), (2.72) read

$$
\begin{aligned}
g_{tt} \,&=\, 1+2\Phi_{ball}(\mathbf{x}) \,=\, 1-\frac{2GM}{|\mathbf{x}|}\biggl[1+k(\xi,\eta)\,F(m_+\mathcal{R})\,e^{-m_+|\mathbf{x}|}-\frac{4\,F(m_Y\mathcal{R})}{3}\,e^{-m_Y|\mathbf{x}|}+\\
&\qquad +[1/3-k(\xi,\eta)]\,F(m_-\mathcal{R})\,e^{-m_-|\mathbf{x}|}\biggr]\,,\\
g_{ti} \,&=\, 2A_i(\mathbf{x}) \,=\, \frac{2G}{|\mathbf{x}|^2}\biggl[1-(1+m_Y|\mathbf{x}|)\,e^{-m_Y|\mathbf{x}|}\biggr]\,\hat{\mathbf{x}}\times\mathbf{J}\,, \qquad (4.1)\\
g_{ij} \,&=\, -\delta_{ij}+2\Psi_{ball}(\mathbf{x})\delta_{ij} \,=\, -\delta_{ij}\,-\,\frac{2GM}{|\mathbf{x}|}\biggl[1-k(\xi,\eta)\,F(m_+\mathcal{R})\,e^{-m_+|\mathbf{x}|}+\\
&\qquad -[1/3-k(\xi,\eta)]\,F(m_-\mathcal{R})\,e^{-m_-|\mathbf{x}|}-\frac{2\,F(m_Y\mathcal{R})}{3}\,e^{-m_Y|\mathbf{x}|}\biggr]\delta_{ij}\,,
\end{aligned}
$$

and the non-vanishing Christoffel symbols read

$$
\begin{aligned}
\Gamma^t_{ti} \,&=\, \Gamma^i_{tt} = \partial_i\Phi_{\rm ball}\,,\\
\Gamma^i_{tj} \,&=\, \frac{\partial_iA_j-\partial_jA_i}{2}\,, \qquad (4.2)\\
\Gamma^i_{jk} \,&=\, \delta_{jk}\partial_i\Psi_{\rm ball}-\delta_{ij}\partial_k\Psi_{ball}-\delta_{ik}\partial_j\Psi_{\rm ball}\,.
\end{aligned}
$$

Let us consider the geodesic equations

$$
\frac{d^2\,x^\mu}{ds^2}+\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} \,=\, 0\,, \qquad (4.3)
$$

where $ds\,=\,\sqrt{g_{\alpha\beta}dx^\alpha dx^\beta}$ is the relativistic distance.

Now we analyze some specific motion of bodies inside the weak gravitational field (4.1).

Circular rotation curves in a spherically symmetric field

In the Newtonian limit, Eq.(4.3), neglecting the rotating component of the source, leads to the usual equation of motion of bodies

$$
\frac{d^2\,\mathbf{x}}{dt^2} \,=\, -\nabla\Phi_{\rm ball}(\mathbf{x})\,, \qquad (4.4)
$$

where the gravitational potential is given by Eq. (2.68). The study of motion is very simple considering a particular symmetry for mass distribution ρ, otherwise analytical solutions are not available. However, our aim is to evaluate the corrections to the classical motion in the easiest situation, namely the circular motion, in which case we do not consider radial and vertical motions. The condition of stationary motion on the circular orbit reads

$$v_{\mathrm{c}}(r) = \sqrt{r \, \frac{\partial \Phi(r)}{\partial r}} \,, \tag{4.5}$$

where v_{c} denotes the velocity.

A further remark on Eq. (2.62) is needed. The structure of solutions is mathematically similar to the one of fourth-order gravity $f(R, R_{\alpha\beta}R^{\alpha\beta})$, however there is a fundamental difference regarding the algebraic signs of the Yukawa corrections. More precisely, whilst the Yukawa correction induced by a generic function of the Ricci scalar leads to an attractive gravitational force, and the one induced by Ricci tensor squared leads to a repulsive one [286], here the Yukawa corrections induced by a generic function of Ricci scalar and a nonminimally coupled scalar field, have both a positive coefficient (see for details Ref. [233]). Hence the scalar field gives rise to a stronger attractive force than in $f(R)$-gravity, which may imply that $f(R, \phi)$-gravity is a better choice than $f(R, R_{\alpha\beta}R^{\alpha\beta})$-gravity. However, there is a problem in the limit $|\mathbf{x}| \to \infty$: the interaction is scale-depended (the scalar fields are massive) and, in the vacuum, the corrections turn off. Thus, at large distances, we recover only the classical Newtonian contribution. In conclusion, the presence of scalar fields makes the profile smooth, a behavior which is apparent in the study of rotation curves.

For an illustration, let us consider the phenomenological potential $\Phi_{\mathrm{SP}}(r) = -\frac{GM}{r}\biggl[1 + \alpha\, e^{-m_{\mathrm{S}}\, r}\biggr]$, with α and m_{S} free parameters, chosen by Sanders [287] in an attempt to fit galactic rotation curves of spiral galaxies in the absence of dark matter, within the MOdified Newtonian Dynamics (MOND) proposal of Milgrom [288], was further accompanied by a relativistic partner known as Tensor-Vector-Scalar (TeVES) model [289][1]. The free parameters selected by Sanders were $\alpha \simeq -0.92$ and $1/m_{\mathrm{S}} \simeq 40\,\mathrm{Kpc}$. Note that this potential were recently used for elliptical galaxies [296]. In both cases, assuming a negative value for α, an almost constant profile for rotation curve is recovered, however there are two issues. Firstly, an $f(R, \phi)$-gravity does not lead to that negative value of α, and secondly the presence of Yukawa-like correction with negative coefficient leads to a lower rotation curve and only by resetting G one can fit the experimental data.

Only if we consider a massive, non minimally coupled scalar-tensor theory, we get a po-

[1]Note that the validity of MOND [290] and TeVeS [291, 292, 293] models of modified gravity were tested by using gravitational lensing techniques, with the conclusion that a non-trivial component in the form of dark matter has to be added to those models in order to match the observations. However, there are proposals of modified gravity, as for instance the string inspired model studied in Ref. [294], leading to an action that includes, apart from the metric tensor field, also scalar (dilaton) and vector fields, which may be in agreement with current observational data. Note that this model, based on brane universes propagating in bulk space-times populated by point-like defects does have dark matter components, while the rôle of extra dark matter is also provided by the population of massive defects [295].

tential with negative coefficient in Eq. (2.62) [233]. In fact setting the gravitational constant equal to $G_0 = \frac{2\,\omega(\phi^{(0)})\,\phi^{(0)} - 4}{2\,\omega(\phi^{(0)})\,\phi^{(0)} - 3}\frac{G_\infty}{\phi^{(0)}}$, where G_∞ is the gravitational constant as measured at infinity, and imposing $\alpha^{-1} = 3 - 2\,\omega(\phi^{(0)})\,\phi^{(0)}$, the potential (2.62) becomes $\Phi(r) = -\frac{G_\infty M}{r}\left\{1 + \alpha\, e^{-\sqrt{1-3\alpha}\,m_\phi r}\right\}$ and then the Sanders potential can be recovered.

In Fig. 4.1 we show the radial behaviour of the circular velocity induced by the presence of a ball source in the case of the Sanders potential and of potentials shown in Table 4.1.

Case	Theory	Gravitational potential	Free parameters
A	$f(R)$	$-\frac{GM}{\lvert\mathbf{x}\rvert}\left[1 + \frac{1}{3}\,e^{-m_R\lvert\mathbf{x}\rvert}\right]$	$m_R{}^2 = -\frac{f_R(0)}{3f_{RR}(0)}$
B	$f(R,\, R_{\alpha\beta}R^{\alpha\beta})$	$-\frac{GM}{\lvert\mathbf{x}\rvert}\left[1 + \frac{1}{3}\,e^{-m_R\lvert\mathbf{x}\rvert} - \frac{4}{3}\,e^{-m_Y\lvert\mathbf{x}\rvert}\right]$	$m_R{}^2 = -\frac{f_R(0,0)}{3f_{RR}(0,0)+2f_Y(0,0)}$ $m_Y{}^2 = \frac{f_R(0,0)}{f_Y(0,0)}$
C	$f(R,\,\phi) + \omega(\phi)\phi_{;\alpha}\phi^{;\alpha}$	$-\frac{GM}{\lvert\mathbf{x}\rvert}\Big[1 + k(\xi,\eta)\,e^{-m_+\lvert\mathbf{x}\rvert}$ $+[1/3 - k(\xi,\eta)]\,e^{-m_-\lvert\mathbf{x}\rvert}\Big]$	$m_R{}^2 = -\frac{f_R(0,0)}{3f_{RR}(0,\phi^{(0)})}$ $m_\phi{}^2 = -\frac{f_{\phi\phi}(0,\phi^{(0)})}{2\omega(\phi^{(0)})}$ $m_+ = m_R\sqrt{w_+}$ $m_- = m_R\sqrt{w_-}$ $\xi = \frac{3f_{R\phi}(0,\phi^{(0)})^2}{2\omega(\phi^{(0)})}$ $\eta = \frac{m_\phi}{m_R}$ $k(\xi,\,\eta) = \frac{1-\eta^2+\xi+\sqrt{\eta^4+(\xi-1)^2-2\eta^2(\xi+1)}}{6\sqrt{\eta^4+(\xi-1)^2-2\eta^2(\xi+1)}}$ $w_\pm = \frac{1-\xi+\eta^2\pm\sqrt{(1-\xi+\eta^2)^2-4\eta^2}}{2}$
D	$f(R,\, R_{\alpha\beta}R^{\alpha\beta},\,\phi) + \omega(\phi)\phi_{;\alpha}\phi^{;\alpha}$	$-\frac{GM}{\lvert\mathbf{x}\rvert}\Big[1 + k(\xi,\eta)\,e^{-m_+\lvert\mathbf{x}\rvert}$ $+[1/3 - k(\xi,\eta)]\,e^{-m_-\lvert\mathbf{x}\rvert} - \frac{4}{3}\,e^{-m_Y\lvert\mathbf{x}\rvert}\Big]$	$m_R{}^2 = -\frac{f_R(0,0,\phi^{(0)})}{3f_{RR}(0,0,\phi^{(0)})+2f_Y(0,0,\phi^{(0)})}$ $m_Y{}^2 = \frac{f_R(0,0,\phi^{(0)})}{f_Y(0,0,\phi^{(0)})}$ $m_\phi{}^2 = -\frac{f_{\phi\phi}(0,0,\phi^{(0)})}{2\omega(\phi^{(0)})}$ $m_+ = m_R\sqrt{w_+}$ $m_- = m_R\sqrt{w_-}$ $\xi = \frac{3f_{R\phi}(0,0,\phi^{(0)})^2}{2\omega(\phi^{(0)})}$ $\eta = \frac{m_\phi}{m_R}$ $k(\xi,\,\eta) = \frac{1-\eta^2+\xi+\sqrt{\eta^4+(\xi-1)^2-2\eta^2(\xi+1)}}{6\sqrt{\eta^4+(\xi-1)^2-2\eta^2(\xi+1)}}$ $w_\pm = \frac{1-\xi+\eta^2\pm\sqrt{(1-\xi+\eta^2)^2-4\eta^2}}{2}$

Table 4.1: Table of fourth order gravity models analyzed in the Newtonian limit for gravitational potentials generated by a point-like source Eq. (2.62). The range of validity of cases C, D is $(\eta-1)^2 - \xi > 0$.

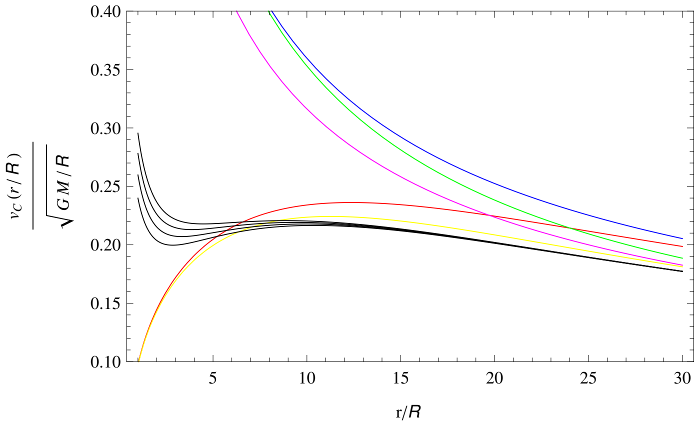

Figure 4.1: The circular velocity of a ball source of mass M and radius $\mathcal{R}$, with the potentials of Table 4.1. We indicate case A by green line, case B by yellow line, case D by red line, case C by blue line, and the GR case by magenta line. The black line correspond to the Sanders model for $-0.95 < \alpha < -0.92$. The values of free parameters are: $\omega(\phi^{(0)}) = -1/2$, $\xi = -5$, $\eta = .3$, $m_Y = 1.5 * m_R$, $m_S = 1.5 * m_R$, $m_R = .1 * \mathcal{R}^{-1}$

Rotating sources and orbital parameters

Considering the geodesic equations (4.3) with the Christoffel symbols given in Eq. (4.2), we obtain

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{tt} + 2\Gamma^i_{tj}\frac{dx^j}{ds} = 0\,, \tag{4.6}$$

which in the coordinate system $\mathbf{J} = (0, 0, J)$, reads

$$\begin{aligned}
\ddot{x} + \frac{GM}{r^3}x &= -\frac{GM\Lambda(r)}{r^3}x + \frac{2GJ}{r^5}\bigg\{\zeta(r)\bigg[\Big(x^2+y^2-2z^2\Big)\dot{y} + 3yz\dot{z}\bigg] + \\
&\qquad +2\Sigma(r)L_x z\bigg\}\,, \\
\ddot{y} + \frac{GM}{r^3}y &= -\frac{GM\Lambda(r)}{r^3}y - \frac{2GJ}{r^5}\bigg\{\zeta(r)\bigg[\Big(x^2+y^2-2z^2\Big)\dot{x} + 3xz\dot{z}\bigg] + \\
&\qquad -2\Sigma(r)L_y z\bigg\} \\
\ddot{z} + \frac{GM}{r^3}z &= -\frac{GM\Lambda(r)}{r^3}z + \frac{6GJ}{r^5}\bigg\{\zeta(r) + \frac{2}{3}\Sigma(r)\bigg\}L_z z\,,
\end{aligned} \tag{4.7}$$

where

$$\Lambda(r) \doteq k(\xi,\eta)\,F(m_+\mathcal{R})\,(1+m_+r)\,e^{-m_+r} - \frac{4\,F(m_Y\mathcal{R})}{3}\,(1+m_Yr)\,e^{-m_Yr} + \\ +[1/3-k(\xi,\eta)]\,F(m_-\mathcal{R})\,(1+m_-r)\,e^{-m_-r},$$

$$\zeta(r) \doteq 1-\left[1+m_Yr+(m_Yr)^2\right]e^{-m_Yr}\,,$$

$$\Sigma(r) \doteq (m_Yr)^2e^{-m_Yr}\,,$$

with L_x,L_y and L_z the components of the angular momentum.

The first terms in the right-hand-side of Eq. (4.7), depending on the three parameters m_R, m_Y and m_ϕ, represent the Extended Gravity modification of the Newtonian acceleration. The second terms in these equations, depending on the angular momentum J and the Extended Gravity parameters m_R, m_Y and m_ϕ, correspond to dragging contributions. The case $m_R \to \infty$, $m_Y \to \infty$ and $m_\phi \to 0$ (this implies $m_\pm \to \infty$) leads to $\Lambda(r) \to 0$, $\zeta(r) \to 1$ and $\Sigma(r) \to 0$, and hence one recovers the familiar results of GR [2]. These additional gravitational terms can be considered as perturbations of Newtonian gravity, and their effects on planetary motions can be calculated within the usual perturbative schemes assuming the Gauss equations [297]. We will follow this approach in what follows.

Let us consider the right-hand-side of Eq. (4.7) as the components (A_x, A_y, A_z) of the perturbing acceleration in the system (X, Y, Z) (see Fig. 4.2), with X the axis passing through the vernal equinox γ, Y the transversal axis, and Z the orthogonal axis parallel to the angular momentum **J** of the central body. In the system (S, T, W), the three components can be expressed as (A_s, A_t, A_w), with S the radial axis, T the transversal axis, and W the orthogonal one. We will adopt the standard notation: a is the semimajor axis; e is the eccentricity; $p = a(1-e^2)$ is the semilatus rectum; i is the inclination; Ω is the longitude of the ascending node N; $\tilde{\omega}$ is the longitude of the pericenter Π; $\mathcal{M}^0$ is the longitude of the satellite at time $t = 0$; ν is the true anomaly; u is the argument of the latitude given by $u = \nu + \tilde{\omega} - \Omega$; n is the mean daily motion equal to $n = (GM/a^3)^{1/2}$; and C is twice the velocity, namely $C = r^2\dot{\nu}a^2(1-e^2)^{1/2}$.

The transformation rules between the coordinates frames (X, Y, Z) and (S, T, W) are

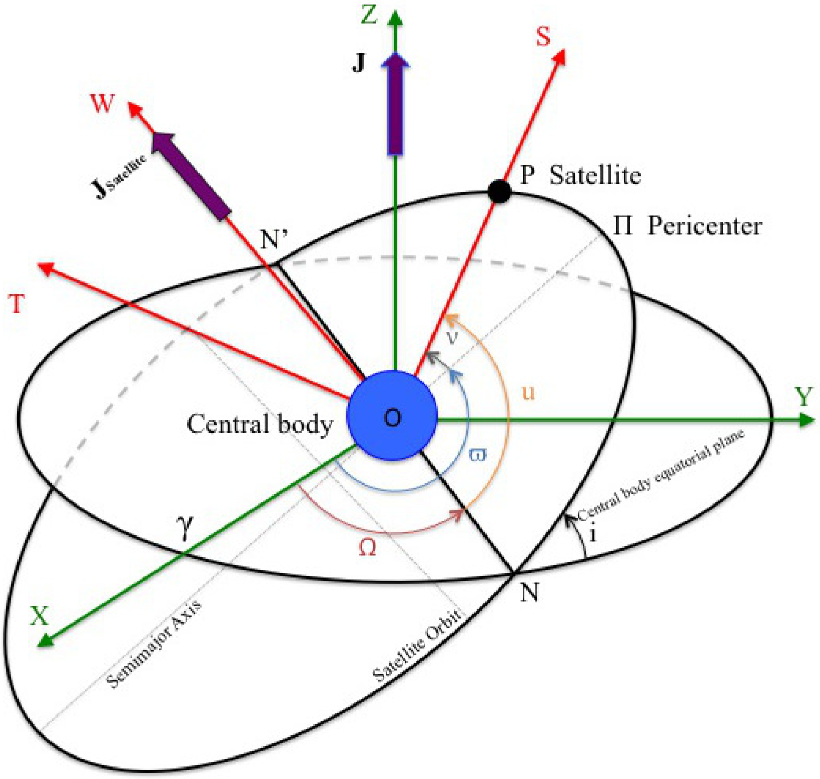

Figure 4.2: $i = \sphericalangle\, YN\Pi$ is the inclination; $\Omega = \sphericalangle\, XON$ is the longitude of the ascending node N; $\tilde{\omega}=$ broken$\sphericalangle\, XO\Pi$ is the longitude of the pericenter Π; $\nu = \sphericalangle\, \Pi OP$ is the true anomaly; $u = \sphericalangle\, \Omega OP = \nu + \tilde{\omega} - \Omega$ is the argument of the latitude; **J** is the angular momentum of rotation of the central body; and $\mathbf{J}_{\mathrm{Satellite}}$ is the angular momentum of revolution of a satellite around the central body.

$$
\begin{aligned}
x &= r(\cos u \cos\Omega - \sin u \sin\Omega \cos i) \,, \\
y &= r(\cos u \sin\Omega + \sin u \cos\Omega \cos i) \,, \\
z &= r \sin u \sin i \\
r &= \frac{p}{1 + e\cos\nu} \,,
\end{aligned}
$$

and the components of the angular momentum obey the equations

$$
\begin{aligned}
L_x &= y\dot{z} - z\dot{y} = C \sin i \, \sin\Omega \,, \\
L_y &= z\dot{x} - x\dot{z} = -C \cos\Omega \, \sin i \,, \\
L_z &= x\dot{y} - y\dot{x} = C \cos i \,.
\end{aligned}
$$

The components of the perturbing acceleration in the (S, T, W) system read

$$
\begin{aligned}
A_s &= -\frac{G\,M\,\Lambda(r)}{r^2} + \frac{2\,G\,J\,C\cos i}{r^4}\zeta(r) \,, \\
A_t &= -\frac{2\,G\,J\,C\,e\cos\, i \sin\nu}{p\,r^3}\zeta(r) \,, \\
A_w &= \frac{2\,G\,J\,C\sin i}{r^4}\left[\left(\frac{r\,e\sin\nu\cos u}{p} + 2\sin u\right)\zeta(r) + 2\sin u\,\Sigma(r)\right] \,.
\end{aligned}
\tag{4.8}
$$

The A_s component has two contributions: the former one results from the modified Newtonian potential $\Phi_{\rm ball}(\mathbf{x})$, while the latter one results from the gravito-magnetic field A_i and it is a higher order term than the first one. Note that the components A_t and A_w depend only on the gravito-magnetic field. The Gauss equations for the variations of the six orbital parameters, resulting from the perturbing acceleration with components A_x, A_y, A_z, read

$$
\begin{aligned}
\frac{da}{dt} &= \dot{a}_{\mathrm{EG}} = \frac{2e\,G\,M\,\Lambda(r)\,\sin\nu}{n\sqrt{1-e^2}\,C}\dot{\nu}\,,\\
\frac{de}{dt} &= \dot{e}_{\mathrm{GR}}+\dot{e}_{\mathrm{STFOG}} = \frac{\sqrt{1-e^2}\,G\,M\,\Lambda(r)\,\sin\nu}{n\,a\,C}\dot{\nu}+\\
&\qquad +\dot{e}_{\mathrm{GR}}\left[1-\,e^{-m_Y r}\Big(1+m_Y r+(m_Y r)^2\Big)\right]\,,\\
\frac{d\Omega}{dt} &= \dot{\Omega}_{\mathrm{GR}}+\dot{\Omega}_{\mathrm{STFOG}} = \dot{\Omega}_{\mathrm{GR}}\left\{1-\,e^{-m_Y r}\left[1+m_Y r+\big(1+f(\nu,u,e)\big)(m_Y r)^2\right]\right\}\,,\\
\frac{di}{dt} &= \dot{i}_{\mathrm{GR}}+\dot{i}_{\mathrm{STFOG}} = \dot{i}_{\mathrm{GR}}\left\{1-\,e^{-m_Y r}\left[1+m_Y r+\big(1+f(\nu,u,e)\big)(m_Y r)^2\right]\right\}\,,\\
\frac{d\tilde{\omega}}{dt} &= \dot{\tilde{\omega}}_{\mathrm{GR}}+\dot{\tilde{\omega}}_{\mathrm{EG}} = -\frac{\sqrt{1-e^2}\,G\,M\,\Lambda(r)\,\cos\nu}{n\,a\,e\,C}\dot{\nu}-2\sin^2\frac{i}{2}\,\dot{\Omega}_{\mathrm{GR}}f(\nu,u,e)\Sigma(r)+\\
&\qquad +\dot{\tilde{\omega}}_{\mathrm{GR}}\left[1-\,e^{-m_Y r}\Big(1+m_Y r+(m_Y r)^2\Big)\right]\,,\\
\frac{dM^0}{dt} &= \dot{M}^0{}_{\mathrm{GR}}+\dot{M}^0{}_{\mathrm{STFOG}} = -\frac{G\,M\,\Lambda(r)}{n\,a\,C}\left[\frac{2r}{a}+\frac{e\,\sqrt{1-e^2}}{1+\sqrt{1-e^2}}\cos\nu\right]\dot{\nu}+\dot{M}^0{}_{\mathrm{GR}}\Bigg[1+\\
&\qquad -\,e^{-m_Y r}\Big(1+m_Y r+(m_Y r)^2\Big)\Bigg]-2\sin^2\frac{i}{2}\,\dot{\Omega}_{\mathrm{GR}}f(\nu,u,e)\Sigma(r)\,,
\end{aligned}
\tag{4.9}
$$

where

$$
\begin{aligned}
\dot{e}_{\mathrm{GR}} &= \frac{2\,G\,J\cos i\sin\nu}{a\,C}\dot{\nu}\,,\\
\dot{\Omega}_{\mathrm{GR}} &= \frac{2\,G\,J\sin u}{p\,C}\left[e\sin\nu\cos u+2\Big(1+e\cos\nu\Big)\sin u\right]\dot{\nu}\,,\\
\dot{i}_{\mathrm{GR}} &= \frac{2\,G\,J\cos u\sin i}{C\,p}\left[e\sin\nu\cos u+2\Big(1+e\cos\nu\Big)\sin u\right]\dot{\nu}\,,\\
\dot{\tilde{\omega}}_{\mathrm{GR}} &= -\frac{2\,G\,J\cos i}{a\,C}\left(2+\frac{1+e^2}{e}\cos\nu\right)\dot{\nu}+2\sin^2\frac{i}{2}\,\dot{\Omega}_{\mathrm{GR}}\,,\\
\dot{\mathcal{M}}^0{}_{\mathrm{GR}} &= -\frac{4\,G\,J\cos i}{n\,a^2\,p}\Big(1+e\cos\nu\Big)\dot{\nu}+\frac{e^2}{1+\sqrt{1-e^2}}\dot{\tilde{\omega}}_{\mathrm{GR}}+2\sqrt{1-e^2}\sin^2\frac{i}{2}\,\dot{\Omega}_{\mathrm{GR}}\,,\\
f(\nu,u,e) &= \frac{1+e\cos\nu}{1+e\left(\frac{\sin\nu\cot u}{2}+\cos\nu\right)}\,.
\end{aligned}
$$

Hence, we have derived the corresponding equations of the six orbital parameters for Extended Gravity, with the dynamics of $a, e, \tilde{\omega}, \mathcal{M}^0$ depending mainly on the terms related to the modifications of the Newtonian potential, whilst the dynamics of Ω and i depending only on the dragging terms.

Considering an almost circular orbit ($e \ll 1$), we integrate the Gauss equations with respect to the only anomaly ν, from 0 to $\nu(t) = nt$, since all other parameters have a slower evolution than ν, hence they can be considered as constants with respect to ν. At first order we get

$$
\begin{aligned}
\Delta a(t) &= 0\,, \\
\Delta e(t) &= 0\,, \\
\Delta i(t) &= \frac{G\,J\,e^2\,\sin i}{na^3} e^{-m_Y p}(m_Y p)^2 \biggl[1 + \frac{(m_Y p)^2}{2}\biggl(m_Y p - 4\biggr)\biggr] \times \\
&\qquad \times \sin\biggl(\tilde{\omega}(t) - \Omega(t)\biggr)\nu(t) + \mathcal{O}(e^4)\,, \\
\Delta\Omega(t) &= \frac{2\,G\,J}{na^3}\biggl[1 - e^{-m_Y p}\biggl(1 + m_Y p + 2(m_Y p)^2\biggr)\biggr]\nu(t) + \mathcal{O}(e^2)\,, \\
\Delta\tilde{\omega}(t) &= \biggl\{\frac{\tilde{\Lambda}(p)}{2} - \frac{2\,G\,J}{na^3}\biggl[3\cos i - 1 + e^{-m_Y p}(1 + m_Y p + \frac{3}{2}(m_Y p)^2 + \\
&\qquad -(3 + 3m_Y p + 3(m_Y p)^2 + \frac{1}{12}(m_Y p)^3)\cos i)\biggr]\biggr\}\nu(t) + \mathcal{O}(e^2)\,, \\
\Delta\mathcal{M}^0(t) &= \biggl\{2\Lambda(p) - \frac{2\,G\,J}{na^3}\biggl[3\cos i - 1 - e^{-m_Y p}\biggl(1 + m_Y p + 2(m_Y p)^2\biggr) \times \\
&\qquad \times\biggl(\cos i - 1\biggr)\biggr]\biggr\}\nu(t) + \mathcal{O}(e^2)\,,
\end{aligned}
\tag{4.10}
$$

where

$$
\begin{aligned}
\tilde{\Lambda}(p) \doteq{}& k(\xi,\eta)\,F(m_+\mathcal{R})\,(m_+ p)^2\,e^{-m_+ p} + [1/3 - k(\xi,\eta)]\,F(m_-\mathcal{R})\,(m_- p)^2\,e^{-m_- p} + \\
& - \frac{4\,F(m_Y\mathcal{R})}{3}\,(m_Y p)^2\,e^{-m_Y p}\,.
\end{aligned}
$$

We hence notice that the contributions to the semimajor axis a and eccentricity e vanish, as in GR, whilst there are nonzero contributions to i, Ω, $\tilde{\omega}$ and $\mathcal{M}^0$. In particular, the contributions to the inclination i and the longitude of the ascending node Ω, depend only on the drag effects of the rotating central body; while the contributions to the pericenter longitude $\tilde{\omega}$ and mean longitude at time $t = 0$ $\mathcal{M}^0$, depend also on the modified Newtonian potential. Finally, note that in the Extended Gravity model we have considered here, the inclination i has a nonzero contribution, in contrast to the result obtained within GR, and also $\Delta\tilde{\omega}(t) \neq \Delta\mathcal{M}^0(t)$, given by

$$\Delta\tilde{\omega}(t) - \Delta\mathcal{M}^0(t) \simeq \left\{ \frac{\tilde{\Lambda}(p) - 4\Lambda(p)}{2} + \frac{2GJ}{na^3} e^{-m_Y p} \left[\frac{(m_Y p)^2}{2} + (2 + 2m_Y p + (m_Y p)^2 + \frac{(m_Y p)^3}{12}) \cos i \right] \right\} \nu(t) + \mathcal{O}(e^2) \; . \tag{4.11}$$

In the limit $m_R \to \infty, m_Y \to \infty$ and $m_\phi \to 0$ (this implies $m_\pm \to \infty$), we obtain the well-known results of GR.

In the following section, we use recent experimental results obtained from the Gravity Probe B and LARES satellites in order to constrain the free parameter m_Y which appears in the context of a specific model of extended gravity derived from a fundamental theory, namely noncommutative geometry. More precisely, we constrain the free parameter by demanding the deviation from the GR result to be within the accuracy of the measured effect.

4.2 Experimental constraints

The orbiting gyroscope precession can be split into a part generated by the metric potentials, Φ and Ψ, and one generated by the vector potential $\mathbf{A}$. The equation of motion for the gyro-spin three-vector $\mathbf{S}$ is

$$\frac{d\mathbf{S}}{dt} = \left.\frac{d\mathbf{S}}{dt}\right|_{\mathrm{G}} + \left.\frac{d\mathbf{S}}{dt}\right|_{\mathrm{LT}} \tag{4.12}$$

where the geodesic and Lense-Thirring precessions are

$$\begin{aligned} \left.\frac{d\mathbf{S}}{dt}\right|_{\mathrm{G}} &= \mathbf{\Omega}_{\mathrm{G}} \times \mathbf{S} \quad \text{with} \quad \mathbf{\Omega}_{\mathrm{G}} = \frac{\nabla(\Phi + 2\Psi)}{2} \times \mathbf{v} \, , \\ \left.\frac{d\mathbf{S}}{dt}\right|_{\mathrm{LT}} &= \mathbf{\Omega}_{\mathrm{LT}} \times \mathbf{S} \quad \text{with} \quad \mathbf{\Omega}_{\mathrm{LT}} = \frac{\nabla \times \mathbf{A}}{2} \, . \end{aligned} \tag{4.13}$$

The geodesic precession, $\mathbf{\Omega}_G$, can be written as the sum of two terms, one obtained with GR and the other being the Extended Gravity contribution. Then we have

$$\mathbf{\Omega}_{\rm G} \,=\, \mathbf{\Omega}_{\rm G}^{\rm (GR)} + \mathbf{\Omega}_{\rm G}^{\rm (STFOG)}\,, \tag{4.14}$$

where

$$\begin{aligned} \mathbf{\Omega}_{\rm G}^{\rm (GR)} \,&=\, \frac{3GM}{2|\mathbf{x}|^3}\,\mathbf{x}\times\mathbf{v}\,, \\ \mathbf{\Omega}_{\rm G}^{\rm (STFOG)} \,&=\, -\biggl[k(\xi,\eta)(m_+r+1)\,F(m_+\mathcal{R})\,e^{-m_+r}+\frac{8}{3}(m_Yr+1)\,F(m_Y\mathcal{R})\,e^{-m_Yr}+ \\ &\qquad +\biggl[\frac{1}{3}-k(\xi,\eta)\biggr](m_-r+1)F(m_-\mathcal{R})\,e^{-m_-r}\biggr]\frac{\mathbf{\Omega}_{\rm G}^{\rm (GR)}}{3}\,. \end{aligned} \tag{4.15}$$

where $|\mathbf{x}| = r$. Similarly one has

$$\mathbf{\Omega}_{\rm LT} \,=\, \mathbf{\Omega}_{\rm LT}^{\rm (GR)} + \mathbf{\Omega}_{\rm LT}^{\rm (STFOG)}\,, \tag{4.16}$$

with

$$\begin{aligned} \mathbf{\Omega}_{\rm LT}^{\rm (GR)} \,&=\, \frac{G}{2r^3}\,\mathbf{J}\,, \\ \mathbf{\Omega}_{\rm LT}^{\rm (STFOG)} \,&=\, -e^{-m_Yr}(1+m_Yr+{m_Y}^2r^2)\,\mathbf{\Omega}_{\rm LT}^{\rm (GR)}\,, \end{aligned} \tag{4.17}$$

where we have assumed that, on the average, $\langle(\mathbf{J}\cdot\mathbf{x})\,\mathbf{x}\rangle \,=\, 0$.

Gravity Probe B (GPB)

Gravity Probe B satellite was a relativity gyroscope experiment funded by NASA which launched on 20 April 2004 and and completed on 8 December 2010. The satellite contains a set of four gyroscopes and has tested two predictions of GR: the geodetic effect and frame-dragging (Lense-Thirring effect). This was to be accomplished by measuring, very precisely, tiny changes in the direction of spin of four gyroscopes contained in an Earth satellite orbiting at $h = 650km$ altitude, crossing directly over the poles. The values of the geodesic precession and the Lense-Thirring precession, measured by the Gravity Probe B satellite and those predicted by GR, are given in Table II. Imposing the constraint $|\mathbf{\Omega}_{\rm G}^{\rm (STFOG)}| \lesssim \delta\mathbf{\Omega}_{\rm G}$ and $|\mathbf{\Omega}_{\rm LT}^{\rm (STFOG)}| \lesssim \delta\mathbf{\Omega}_{\rm LT}$, [364], with $r^* \,=\, R_\oplus + h$ where $R_\oplus$ is the radius of the Earth and $h \,=\, 650\,km$ is the altitude of the satellite, we get

Table 4.2: The geodesic precession and Lense-Thitting (frame dragging) precession as predicted by GR and observed with the Gravity Probe B experiment [298].

Effect	Measured (mas/y)	Predicted (mas/y)
Geodesic precession	6602 ± 18	6606
Lense-Thirring precession	37.2 ± 7.2	39.2

$$\begin{aligned} & k(\xi,\eta)\big(m_+\, r^* + 1\big)\, F(m_+ R_\oplus)\, e^{-m_+\, r^*} + \frac{8}{3}(m_Y\, r^* + 1)\, F(m_Y R_\oplus)\, e^{-m_Y\, r^*} + \\ & \qquad + [1/3 - k(\xi,\eta)](m_-\, r^* + 1)\, F(m_- R_\oplus)\, e^{-m_-\, r^*} \lesssim \frac{3\,\delta|\mathbf{\Omega}_{\rm G}|}{|\mathbf{\Omega}_{\rm G}^{\rm (GR)}|} \simeq 0.008\,, \\ & (1 + m_Y r^* + {m_Y}^2 r^{*2})\, e^{-m_Y r^*} \lesssim \frac{\delta|\mathbf{\Omega}_{\rm LT}|}{|\mathbf{\Omega}_{\rm LT}^{\rm (GR)}|} \simeq 0.19\,, \end{aligned} \tag{4.18}$$

since, from the experiments, we have $|\mathbf{\Omega}_{\rm G}^{\rm (GR)}| = 6606\,{\rm mas}$ and $\delta|\mathbf{\Omega}_{\rm G}| = 18\,{\rm mas}$, $|\mathbf{\Omega}_{\rm LT}^{\rm (GR)}| = 37.2\,{\rm mas}$ and $\delta|\mathbf{\Omega}_{\rm LT}| = 7.2\,{\rm mas}$. From Eq. (4.18) we thus obtain that $m_Y \geq 7.3 \times 10^{-7} m^{-1}$.

LARES

The LAser RElativity Satellite (LARES) mission [300] of the Italian Space Agency scientific satellite launched on 13 February 2012.The satellite, completely passive, is made of tungsten alloy and houses 92 cube corner retro reflectors that are used to track the satellite via laser from stations on Earth. LARES's body has a diameter of about 36.4 cm and weighs about 400 Kg. LARES was inserted in an orbit with 1450 Km of perigee, an inclination of 69.5 degrees and reduced eccentricity $\sim 10^{-3}$. The satellite is tracked by the International Laser Ranging Service stations. The main scientific target of the LARES mission is the measurement of the frame-dragging, also known as Lense-Thirring effect, with an accuracy of about 1%. It allows us to obtain a stronger constraint for m_Y:

$$(1 + m_Y r^* + {m_Y}^2 r^{*2})\, e^{-m_Y r^*} \lesssim \frac{\delta|\mathbf{\Omega}_{\rm LT}|}{|\mathbf{\Omega}_{\rm LT}^{\rm (GR)}|} \simeq 0.01\,, \tag{4.19}$$

from the which we obtain $m_Y \geq 1.2 \times 10^{-6} m^{-1}$.

Noncommutative Spectral Geometry

In the specific case of the Noncommutative Spectral Geometry model (1.48), the quantities (2.55) become $m_R \to \infty$, $m_Y = \sqrt{\frac{5\pi^2\left(k_0^2\mathbf{H}^{(0)}-6\right)}{36 f_0 k_0^2}}$ and $m_\phi = 0$, implying that $\xi = \frac{af_0(\mathbf{H}^{(0)})^2}{12\pi^2}$, $\eta = 0$, $k(\xi,\eta) = \frac{af_0(\mathbf{H}^{(0)})^2+12\pi^2}{6|af_0(\mathbf{H}^{(0)})^2-12\pi^2|} + \frac{1}{6}$ and $\tilde{k}^2_{R,\phi} = 1 - \frac{af_0(\mathbf{H}^{(0)})^2}{12\pi^2}$, 0. The first relation (4.18) becomes

$$\frac{8}{3}(m_Y\, r^* + 1)\, F(m_Y R_\oplus)\, e^{-m_Y\, r^*} \lesssim 0.008\,,$$

hence the constraint on m_Y imposed from GBP is

$$m_Y > 7.1 \times 10^{-5}\mathrm{m}^{-1}\,,$$

whereas the LARES experiment (4.19) implies

$$m_Y > 1.2 \times 10^{-6}\mathrm{m}^{-1}\,,$$

a bound similar to the one obtained earlier on using binary pulsars [301], or the Gravity Probe B data [364].

However, a more stringent constraint has been obtained using torsion balance experiments. More precisely, as it has been shown in Ref. [364], using results from laboratory experiments design to test the fifth force, one arrives to the tightest constraint $m_Y > 10^4\mathrm{m}^{-1}$.

In conclusion, using data form Gravity Probe B and LARES missions, we obtain similar constraints on m_Y; a result that one could have anticipated since both these experiments are designed to test the same type of physical phenomenon. However, by using the stronger constraint for m_Y, namely $m_Y > 10^4\mathrm{m}^{-1}$, we observe that the modifications to the orbital parameters (4.9) induced by Noncommutative Spectral Geometry are indeed small, confirming the consistency between the predictions of NCSG as a gravitational theory beyond GR and the Gravity Probe B and LARES measurements. At this point let us stress that, in principle, space-based experiments can be used to test parameters of fundamental theories.

Chapter 5

Astrophysics Test of Scalar Tensor Fourth Order Gravity: binary systems

In this chapter we analyze, in the framework of post-Minkowskian approximation (weak-field limit) of a Scalar Tensor Fourth Order Gravity, the energy loss by a stellar binary system [**E**]. More specifically, by exploiting recent astrophysical data on the variation of the orbital period of binary systems, we will constrain the free parameters, namely the three *masses* $\{m_Y, m_R, m_\phi\}$ that characterize the scales on which higher order terms generated by the models of extended gravity become relevant.

5.1 Energy loss

In the second chapter, we discussed the gravitational wave emission from a quadrupole source and we calculated the spatial components of the perturbations h_{ij} (2.113). In general, the rate of energy loss from a binary system, in the far-field limit, reads

$$\frac{d\mathcal{E}}{dt} \approx -\frac{2\,\pi|\mathbf{x}|^2}{5\,\mathcal{X}}\dot{h}_{ij}\dot{h}^{ij}\ . \tag{5.1}$$

Let us make the choice

$$\begin{aligned} Q_{ij}(t) &= \mathcal{Q}_{ij}\cos\bigl(\omega_{(ij)}\,t+\vartheta_{(ij)}\bigr)+\mathcal{Q}^0_{ij}\ , \\ Q(t) &= \mathcal{Q}\cos\bigl(\nu\,t+\vartheta\bigr)+\mathcal{Q}^0\ , \end{aligned} \tag{5.2}$$

where $\mathcal{Q}_{ij}$ and its trace $\mathcal{Q}$ are the quadrupole oscillation amplitudes, $\mathcal{Q}^0_{ij}$ and its trace $\mathcal{Q}^0$ are constant terms, $\omega_{(ij)}$ and $\vartheta_{(ij)}$ are the frequencies of oscillations and the phases of the ij components respectively, while ν and ϑ are the frequency and phase of the trace, respectively. All these quantities are considered to be time independent.

From Eqs. (2.113) and (5.1), the energy loss reads

$$\frac{d\mathcal{E}}{dt} \approx -\frac{m_Y^2\,\omega_{(ij)}^6\,\mathcal{Q}_{ij}\mathcal{Q}^{ij}\,|\mathbf{x}|^2\,\mathcal{X}}{720\,\pi}F\big(|\mathbf{x}|;m_Y;m_Y;\omega_{(ij)}\big) + -\frac{\nu^6\,\mathcal{Q}^2\,|\mathbf{x}|^2\,\mathcal{X}}{2880\,\pi}\sum_{\{S,\,P\}}\zeta_{SP}\,m_S\,m_P\,F\big(|\mathbf{x}|;m_S;m_P;\nu\big)\,, \tag{5.3}$$

where we averaged over time neglecting higher order terms. Here with the notation $\sum_{\{S,\,P\}}$ we intend to extend the sum over all possible values belonging to set $\{S,\,P\} = \{Y,\,R,\,+,\,-\}$. Note that all quantities in Eq. (5.3) are defined in Appendix 6.2.

The model under consideration carries by itself a natural frequency scale $\omega_m^c \propto m$ linked to the masses m_R, m_Y, m_+ and m_-. The $F\big(|\mathbf{x}|;m_1;m_2;\omega\big)$ functions are highly oscillatory, with different behavior for $\omega > \omega^c$ and $\omega < \omega^c$, while for $\omega = \omega^c$ are highly resonant [354]. The $\omega > \omega^c$ case is excluded folowing a simple heuristic argument [355] we highlight below. A system with $\omega > \omega^c$ cannot decrease its orbital frequency across the lower boundary ω^c. Since one expects all astrophysical systems to have formed from the coalescence of relatively cold, slowly moving systems, it is reasonable to suppose that at some time in the past, all binary systems had $\omega < \omega^c$. Hence, we will only analyze frequencies lower than ω^c.

For $\omega < \omega^c$ the last function of Eq. (6.13), can be approximated by [354]

$$F\big(|\mathbf{x}|;\,m_1;\,m_2;\,\omega\big) \approx \frac{1+\Lambda\left(\sqrt{m_1\,m_2}|\mathbf{x}|;\,\frac{\omega}{C\,\sqrt{m_1\,m_2}}\right)}{m_1\,m_2\,|\mathbf{x}|^2}\,, \tag{5.4}$$

where $C \approx 0.175$ is approximately constant except for $\omega \to \omega^c$ and $\Lambda(x,\,y) = \frac{C\,\mathcal{J}_1(x-y)}{x(1-y)}$. Replacing Eq. (5.4) into Eq. (5.3), we obtain a contribution from General Relativity and one from Scalar Tensor Forth Order Gravity as

$$\frac{d\mathcal{E}}{dt} = -\dot{\mathcal{E}}_{\rm GR} - \dot{\mathcal{E}}_{\rm STFOG}\,, \tag{5.5}$$

where

$$\begin{aligned}\dot{\mathcal{E}}_{\rm GR} &= \frac{\mathcal{X}}{720\,\pi}\left(\omega_{(ij)}^6\,\mathcal{Q}_{ij}\mathcal{Q}^{ij} - \frac{\nu^6\mathcal{Q}^2}{4}\right),\\ \dot{\mathcal{E}}_{\rm STFOG} &= \frac{\mathcal{X}}{720\,\pi}\omega_{(ij)}^6\,\mathcal{Q}_{ij}\mathcal{Q}^{ij}\,\Lambda\left(m_Y\,|\mathbf{x}|;\,\frac{\omega_{(ij)}}{\omega_{YY}^c}\right) + \\ &\quad +\frac{\nu^6\,\mathcal{Q}^2\,\mathcal{X}}{2880\,\pi}\sum_{SP}^{Y,R,+,-}\zeta_{SP}\,\Lambda\left(\sqrt{m_S\,m_P}|\mathbf{x}|;\,\frac{\nu}{\omega_{SP}^c}\right),\end{aligned} \tag{5.6}$$

with

$$\zeta_{YY} \;=\; g_Y\,(3\,g_Y - 4)\;,\;\; \zeta_{RR} \;=\; 3\,g_R^2\;,\;\; \zeta_{SS} \;=\; \tfrac{g_S^2}{3}\;,\;\; \zeta_{YR} \;=\; (2 - 3\,g_Y)\,g_R\;,$$
$$\zeta_{YS} \;=\; \tfrac{1}{3}\,(2 - 3\,g_Y)\,g_S\;,\;\; \zeta_{RS} \;=\; g_R\,g_S\;,\;\; \zeta_{\pm} \;=\; \tfrac{1}{3}\,g_+\,g_- \tag{5.7}$$

for any values $\{S,\,P\} \;=\; \{Y,\,R,\,+,\,-\}$.

The correction to General Relativity, namely the term $\dot{\mathcal{E}}_{\rm STFOG}$, has ten characteristic frequencies (Table 5.1). We note however that $\omega^{\rm c}_{++} \;=\; \omega^{\rm c}_{R+}$ and $\omega^{\rm c}_{--} \;=\; \omega^{\rm c}_{R-}$, and in general $\omega^{\rm c}_{+-} > \omega^{\rm c}_{++} > \omega^{\rm c}_{--}$ and $\omega^{\rm c}_{Y+} > \omega^{\rm c}_{Y-}$. Since the binary systems cannot have more frequencies higher than those predicted from the theory, it follows that a Scalar Tensor Fourth Order Gravity model has at most only five characteristic frequencies, i.e. $\omega^{\rm c}_{YY}$, $\omega^{\rm c}_{RR}$, $\omega^{\rm c}_{YR}$, $\omega^{\rm c}_{--}$ and $\omega^{\rm c}_{Y-}$. We note also if the trace of the quadrupole Eq. (5.2) does not depend on time then the correction (second line of Eq. (5.6)) depends only on $\omega^{\rm c}_{YY}$. Therefore, theories constructed without the invariant $R_{\mu\nu}R^{\mu\nu}$ will not give different values from those of General Relativity.

$\omega^{\rm c}_{YY} \;= C\,m_Y$	$\omega^{\rm c}_{R-} \;= C\,m_R\sqrt{w_-}$
$\omega^{\rm c}_{RR} \;= C\,m_R$	$\omega^{\rm c}_{+-} \;= C\,m_R\sqrt{w_+\,w_-} \;= C\,\sqrt{m_R\,m_\phi}$
$\omega^{\rm c}_{++} \;= C\,m_R\sqrt{w_+}$	$\omega^{\rm c}_{YR} \;= c\,\sqrt{m_Y\,m_R}$
$\omega^{\rm c}_{--} \;= C\,m_R\sqrt{w_-}$	$\omega^{zrmc}_{Y+} \;= C\,\sqrt{m_Y\,m_R}\sqrt{w_+}$
$\omega^{\rm c}_{R+} \;= C\,m_R\sqrt{w_+}$	$\omega^{\rm c}_{Y-} \;= C\,\sqrt{m_Y\,m_R}\sqrt{w_-}$

Table 5.1: Ten characteristic frequencies for a Scalar Tensor Fourth Order Gravity.

The correction for the energy loss given by an Extended Gravity Model takes the form

$$\begin{aligned}\dot{\mathcal{E}}_{\rm STFOG} \;=\;& \frac{\mathcal{X}}{720\,\pi}\left[\omega^6_{(ij)}\,\mathcal{Q}_{ij}\,\mathcal{Q}^{ij} \;+\frac{\nu^6\,\mathcal{Q}^2}{4}\zeta_{YY}\right]\Lambda\left(m_Y\,|\mathbf{x}|;\,\frac{\omega_{(ij)}}{\omega^{\rm c}_{YY}}\right)+ \\ &+\frac{\nu^6\,\mathcal{Q}^2\,\mathcal{X}}{2880\,\pi}\left[\zeta_{RR}\,\Lambda\left(m_R\,|\mathbf{x}|;\,\frac{\nu}{\omega^{\rm c}_{RR}}\right)+\left(\zeta_{--}+\zeta_{R-}\right)\Lambda\left(m_-\,|\mathbf{x}|;\,\frac{\nu}{\omega^{\rm c}_{--}}\right)+\right. \\ &\left.+\zeta_{YR}\,\Lambda\left(\sqrt{m_Y\,m_R}|\mathbf{x}|;\,\frac{\nu}{\omega^{\rm c}_{YR}}\right)+\zeta_{Y-}\,\Lambda\left(\sqrt{m_Y\,m_-}|\mathbf{x}|;\,\frac{\nu}{\omega^{\rm c}_{Y-}}\right)\right].\end{aligned} \tag{5.8}$$

As an example let us consider a pair of masses m_1 and m_2 in an elliptic binary system. For such a system, orbiting in the (x, y) plane, the nonzero components of the quadrupole (5.2) are

$$\mathcal{Q}_{xx} \;=\; \frac{3}{2}\,\mu\,a^2\,,\;\; \mathcal{Q}_{yy} \;=\; -\frac{3}{2}\,\mu\,b^2\,,\;\; \mathcal{Q}_{xy} \;=\; \frac{3}{2}\,\mu\,a\,b\,,\;\; \mathcal{Q} \;=\; -\frac{3}{2}\,\mu\,a^2\,e^2\,, \tag{5.9}$$

with

$$\omega_{xx} \;=\; \omega_{yy} \;=\; \omega_{xy} \;=\; \nu \;=\; 2\,\Omega\,, \tag{5.10}$$

where $\mu = \frac{m_1 m_2}{m_1+m_2}$ is the reduced mass, a and b are the major and minor semiaxis, e is the eccentricity and Ω is the orbital frequency. The energy loss then reads

$$\begin{aligned}
\dot{\mathcal{E}}_{\rm GR} &= \frac{\Omega^6\,\mu^2\,\mathcal{X}}{20\,\pi}\biggl[4\bigl(a^2+b^2\bigr)^2-a^4\,e^2\biggr]\,, \\
\dot{\mathcal{E}}_{\rm STFOG} &= \frac{\Omega^6\,\mu^2\,\mathcal{X}}{20\,\pi}\Biggl[\Bigl(4\bigl(a^2+b^2\bigr)^2+a^4\,e^2\,\zeta_{YY}\Bigr)\,\Lambda\Biggl(m_Y\,|\mathbf{x}|;\frac{2\,\Omega}{\omega^{\rm c}_{YY}}\Biggr)+ \\
&\quad +a^4\,e^2\biggl[\zeta_{RR}\,\Lambda\Biggl(m_R\,|\mathbf{x}|;\frac{2\,\Omega}{\omega^{\rm c}_{RR}}\Biggr)+\bigl(\zeta_{--}+\zeta_{R-}\bigr)\,\Lambda\Biggl(m_-\,|\mathbf{x}|;\frac{2\,\Omega}{\omega^{\rm c}_{--}}\Biggr)+ \\
&\quad +\zeta_{YR}\,\Lambda\Biggl(\sqrt{m_Y\,m_R}|\mathbf{x}|;\frac{2\,\Omega}{\omega^{\rm c}_{YR}}\Biggr)+\zeta_{Y-}\,\Lambda\Biggl(\sqrt{m_Y\,m_-}|\mathbf{x}|;\frac{2\,\Omega}{\omega^{\rm c}_{Y-}}\Biggr)\biggr]\Biggr]\,.
\end{aligned} \tag{5.11}$$

These results will be used in the next section in order to constrain the three mass characterising the extended gravity model under consideration.

5.2 Observational constraints

One has to test the observational compatibility of an extended gravity model. Hence we will study the variation of the orbital period $\mathcal{P}$ for binary systems due to emission of gravitational waves. The relation between the time variation of period and the energy loss is

$$\frac{d\mathcal{P}}{dt} = \frac{\mathcal{P}^3}{4\pi I_{\rm PSR}}\frac{d\mathcal{E}}{dt}\,, \tag{5.12}$$

where $I_{\rm PSR}$ is the pulsar's moment of inertia, normally assumed to be 10^{45}g cm^2 [365]. For an elliptic binary system in the weak-field limit, one gets

$$\dot{\mathcal{P}} \approx \frac{P^3}{4\pi I_{\rm PSR}}\bigl(\dot{\mathcal{E}}_{\rm GR}+\dot{\mathcal{E}}_{\rm STFOG}\bigr)=\dot{\mathcal{P}}_{\rm GR}+\dot{\mathcal{P}}_{\rm STFOG}\,, \tag{5.13}$$

where

$$
\begin{aligned}
\dot{\mathcal{P}}_{\rm GR} &= -\frac{\Omega^6\,\mu^2\,\mathcal{P}^3\,\mathcal{X}}{80\,\pi^2\,I_{\rm PSR}}\left[4\big(a^2+b^2\big)^2 - a^4\,e^2\right],\\
\dot{\mathcal{P}}_{\rm STFOG} &= -\frac{\Omega^6\,\mu^2\,P^3\,\mathcal{X}}{80\,\pi^2\,I_{\rm PSR}}\Bigg\{\left[4\big(a^2+b^2\big)^2 + a^4\,e^2\,\zeta_{YY}\right]\Lambda\left(m_Y\,|\mathbf{x}|;\,\frac{2\,\Omega}{\omega^c_{YY}}\right)+\\
&\quad + a^4\,e^2\left[\zeta_{RR}\,\Lambda\left(m_R\,|\mathbf{x}|;\,\frac{2\,\Omega}{\omega^c_{RR}}\right) + \big(\zeta_{--}+\zeta_{R-}\big)\,\Lambda\left(m_-\,|\mathbf{x}|;\,\frac{2\,\Omega}{\omega^c_{--}}\right)+\right.\\
&\quad \left. +\zeta_{YR}\,\Lambda\left(\sqrt{m_Y\,m_R}|\mathbf{x}|;\,\frac{2\,\Omega}{\omega^c_{YR}}\right) + \zeta_{Y-}\,\Lambda\left(\sqrt{m_Y\,m_-}|\mathbf{x}|;\,\frac{2\,\Omega}{\omega^c_{Y-}}\right)\right\}.
\end{aligned}
$$

PSR J0348+0432 is a neutron star in a binary system with a white dwarf, with estimated massed $(2.01\pm0.04)\,M_\odot$ and $(0.172\pm0.003)\,M_\odot$, respectively [365]. This binary system has an almost circular orbit, a semi major axis $a = 8.3\times10^8$ m and a short orbital period $\mathcal{P} = 2.46$ hours orbit. For these values, General Relativity leads to a significant orbital decay. In particular, the authors of Ref. [365] obtained the constraint

$$
\dot{\mathcal{P}}_{\rm obs}/\dot{\mathcal{P}}_{\rm GR} = 1.05\pm0.18\,. \tag{5.14}
$$

Using this result and Eq. (5.13), we get

$$
-0.13 \leq \frac{\dot{\mathcal{P}}_{\rm STFOG}}{\dot{\mathcal{P}}_{\rm GR}} \leq 0.23 \quad\Rightarrow\quad -0.13 \leq \Lambda\left(m_Y\,|\mathbf{x}|;\,\frac{2\Omega}{\omega^c_{YY}}\right) \leq 0.23\,, \tag{5.15}
$$

$$
\text{hence} \qquad\qquad m_Y > 5\times10^{-11}\,m^{-1}\,.
$$

Thus, for a binary system with a negligible circular orbit ($e \ll 1$), one can always constrain the parameter m_Y. To constrain the other parameters, one has to consider elliptic systems, i.e. the eccentricity must not be negligible. For example, for the elliptic binary system PSR B1913+16 [366, 367] where the experimental eccentricity is $e = 0.6$, the semi major axis $a = 1.95\times10^9$ m, the orbital period $\mathcal{P} = 7.7$ hours orbit, and $\dot{\mathcal{P}}_{\rm obs}/\dot{\mathcal{P}}_{\rm GR} = 0.997\pm0.002$, one infers

$$
\begin{aligned}
-0.005 \leq & \frac{4\big(a^2+b^2\big)^2 + a^4\,e^2\,\zeta_{YY}}{4\big(a^2+b^2\big)^2 - a^4\,e^2}\,\Lambda\left(m_Y\,|\mathbf{x}|;\,\frac{2\,\Omega}{\omega^{\rm c}_{YY}}\right) \\
& + \frac{a^4\,e^2}{4\big(a^2+b^2\big)^2 - a^4\,e^2}\left[\zeta_{RR}\,\Lambda\left(m_R\,|\mathbf{x}|;\,\frac{2\,\Omega}{\omega^{\rm c}_{RR}}\right) + \big(\zeta_{--}+\zeta_{R-}\big)\,\Lambda\left(m_-\,|\mathbf{x}|;\,\frac{2\,\Omega}{\omega^{\rm c}_{--}}\right) + \right. \\
& \left. + \zeta_{YR}\,\Lambda\left(\sqrt{m_Y\,m_R}|\mathbf{x}|;\,\frac{2\,\Omega}{\omega^{\rm c}_{YR}}\right) + \zeta_{Y-}\,\Lambda\left(\sqrt{m_Y\,m_-}|\mathbf{x}|;\,\frac{2\,\Omega}{\omega^{\rm c}_{Y-}}\right)\right] \leq -0.001
\end{aligned}
\tag{5.16}
$$

We have thus obtained a relation between the characteristic masses and frequencies for a general extended gravity model. In what follows we will examine some particular extended gravity models studied in the literature.

5.3 Scalar Tensor Fourth Order Gravity models

Let us consider case A of Table 5.2; the only interesting quantity (see Eq. (2.55) is m_R. For the system PSR B1913+16, Eq. (5.16) implies

$$
-0.005 \leq \frac{a^4\,e^2\,\Lambda\left(m_R\,|\mathbf{x}|;\,\frac{2\,\Omega}{\omega^{\rm c}_{RR}}\right)}{4\big(a^2+b^2\big)^2 - a^4\,e^2} \leq -0.001\ \Rightarrow m_R\ \gtrsim\ 3\times 10^{-9}\,\mathrm{m}^{-1}\,. \tag{5.17}
$$

In general, one can consider the polynomial expression

$$
f(R)\ =\ R + \alpha\,R^2 + \sum_{n\,=\,3}^{N}\alpha_n\,R^n\,. \tag{5.18}
$$

Note however that the characteristic scale m_R is only generated by the R^2-term. An interesting model of $f(R)$-theories is that of Starobinsky $f(R)\ =\ R - R^2/R_0$ [360], for which $m_R^2\ =\ R_0/6$, hence using Eq. (5.17) we get $R_0 \gtrsim\ 5.4\times 10^{-19}\,\mathrm{m}^{-2}$.

To generalize the previously result we must include the curvature invariant $R_{\mu\nu}R^{\mu\nu}$. For case B of Table 5.2 we consider the general class of $f(R,\ R_{\alpha\beta}R^{\alpha\beta})$-theories and their characteristic scales m_R and m_Y. Using Eqs. (5.15) and (5.16) we obtain

$$
m_Y\ \gtrsim\ 5\times 10^{-11}\,\mathrm{m}^{-1}\ \ ,\ \ m_R\ \gtrsim\ 1.15\times 10^{-9}\,\mathrm{m}^{-1}\,. \tag{5.19}
$$

This class of theories includes the case of a Weyl square type model, *i.e.* $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}\ =\ 2R_{\mu\nu}R^{\mu\nu} - \frac{2}{3}R^2$, where there is only one characteristic scale $m_R\ \to\ \infty$.

The same argumentation is also valid for the scalar tensor case of theory, for which in the Newtonian limit (see case D in Table 5.2) the more general expression (2.53) becomes

$$\left(1-\phi^{(0)}\sqrt{\frac{\xi}{3}}\right)R+\sqrt{\frac{\xi}{3}}\,R\,\phi-\frac{{m_\phi}^2}{2}\,(\phi-\phi^{(0)})^2\,. \tag{5.20}$$

Thus, for the most general Scalar Tensor (ST) theory in the Newtonian limit, one can consider the model[1]

$$f_{\rm ST}(R,\phi)\;=\;\alpha_0\,R+\alpha_1\,R\,\phi-\frac{1}{2}{m_\phi}^2\,(\phi-\phi^{(0)})^2+\frac{1}{2}\phi_{,\alpha}\phi^{,\alpha}\,. \tag{5.21}$$

Since for this case $m_R\to\infty$, $m_Y\to\infty$, $\xi\;=\;3\,{\alpha_1}^2$, $\eta\;\to\;0$, $m_+\to\infty$ and $m_-\;=\;\frac{m_\phi}{\sqrt{1-\xi}}$, we obtain from Eq. (5.16)

$$-0.005\leq\frac{a^4\,e^2}{4\big(a^2+b^2\big)^2-a^4\,e^2}\,\frac{2\xi(3-2\xi)}{3(1-\xi)^2}\,\Lambda\!\left(\frac{m_\varphi\,|\mathbf{x}|}{\sqrt{1-\xi}};\,\frac{2\,\Omega}{\omega^{\rm c}_{--}}\right)\leq-0.001\,. \tag{5.22}$$

As a special case of a scalar-tensor fourth order gravity model (case E) we consider Non-Commutative Spectral Geometry (1.48) [361, 362][2]. We must impose the correspondence $\mathbf{H}\;\leftrightarrow\;\phi$ and further the conditions

$$\begin{aligned}
&f(0,\,0,\,\phi^{(0)})\;=\;\gamma_0-{\mu_0}^2\,{\mathbf{H}^{(0)}}^2+\lambda_0\,{\mathbf{H}^{(0)}}^4\;=\;0\\
&f_\phi(0,\,0,\,\phi^{(0)})\;=\;-{\mu_0}^2+2\lambda_0\,{\mathbf{H}^{(0)}}^2\;=\;0\\
&f_R(0,\,0,\,\phi^{(0)})\;=\;\tfrac{1}{{k_0}^2}-\tfrac{{\mathbf{H}^{(0)}}^2}{6}\;=\;1
\end{aligned} \tag{5.23}$$

From the mass definitions (2.55) and the auxiliary quantities (2.60), we get

[1]With the condition $\alpha_0+\alpha_1\,\phi^{(0)}\;=\;1$.

[2]Constraints on one of the three free parameters of this model have been set in Refs. [351, 364]

$$
\begin{aligned}
& m_R \to \infty && \xi = \tfrac{\mathbf{H}^{(0)2}}{12} \\
& m_Y = \tfrac{1}{\sqrt{2\alpha_0}} && \eta \to 0 \\
& m_\phi = \sqrt{2\mu_0{}^2 - 12\lambda_0\,\mathbf{H}^{(0)2}} && m_+ \to \infty \quad , \quad m_- \to \frac{m_\phi}{\sqrt{1-\frac{\mathbf{H}^{(0)2}}{12}}}
\end{aligned}
\tag{5.24}
$$

Using constraints (5.15) and (5.16), we get

$$
\begin{aligned}
-0.005 \leq \; & \frac{4\big(a^2+b^2\big)^2+15\,a^4\,e^2}{4\big(a^2+b^2\big)^2-a^4\,e^2}\,\Lambda\left(m_Y\,|\mathbf{x}|;\,\frac{2\,\Omega}{\omega^{\rm c}_{YY}}\right)+\frac{a^4\,e^2}{4\big(a^2+b^2\big)^2-a^4\,e^2}\left[\frac{2\xi(3-2\xi)}{3(1-\xi)^2}\,\times\right. \\
& \left.\times\,\Lambda\left(\frac{m_\phi}{\sqrt{1-\xi}}\,|\mathbf{x}|;\,\frac{2\,\Omega}{\omega^{\rm c}_{--}}\right)-\frac{7\xi}{3(1-\xi)}\,\Lambda\left(\frac{\sqrt{m_Y\,m_\phi}}{\sqrt{1-\xi}}|\mathbf{x}|;\,\frac{2\,\Omega}{\omega^{\rm c}_{Y-}}\right)\right]\;\leq\; -0.001\,,
\end{aligned}
\tag{5.25}
$$

with

$$
m_Y \gtrsim 5\times10^{-11}\,\mathrm{m}^{-1} \quad , \quad m_\phi \gtrsim 1.3\times10^{-11}\,\mathrm{m}^{-1}\,. \tag{5.26}
$$

Using equations (5.24) and (5.26), we can constrain the parameter α_0, which corresponds to a restriction on the particle physics at unification. We thus obtain

$$
\alpha_0 \leq 10^{20}\mathrm{m}^2\,, \tag{5.27}
$$

which is rather weak but can in principle be improved once further data of nearby pulsars are available.

The parameter α_0 has been constrained in the past using either pulsar measurements [355], Gravity Probe B or torsion balance experiments [364]. Here we have extended the original analysis of Ref. [355] for the case of pulsars with an elliptical orbit. Let us note that the strongest constraint on α_0, namely $\alpha_0 < 10^{-8}\,\mathrm{m}^2$, was obtained [364] using torsion balance measurements.

Case	STFOG	Mass definition
A	$f(R)$	$m_R{}^2 = -\frac{f_R(0)}{3f_{RR}(0)}$ $m_Y{}^2 \to \infty, \ m_\phi{}^2 = 0$ $\xi = 0, \ \eta = 0$ $m_+ = m_R, \ m_- = 0$
B	$f(R, \, R_{\alpha\beta}R^{\alpha\beta})$	$m_R{}^2 = -\frac{f_R(0,0)}{3f_{RR}(0,0)+2f_Y(0,0)}$ $m_Y{}^2 = \frac{f_R(0,0)}{f_Y(0,0)}, \ m_\phi{}^2 = 0$ $\xi = 0, \ \eta = 0$ $m_+ = m_R, \ m_- = 0$
C	$f(R, \, \phi) + \omega(\phi)\phi_{;\alpha}\phi^{;\alpha}$	$m_R{}^2 = -\frac{f_R(0, \phi^{(0)})}{3f_{RR}(0, \phi^{(0)})}$ $m_Y{}^2 \to \infty, \ m_\phi{}^2 = -f_{\phi\phi}(0, \, \phi^{(0)})$ $\xi = 3f_{R\phi}(0, \, \phi^{(0)})^2, \ \eta = \frac{m_\phi}{m_R}$ $m_\pm = \sqrt{\frac{1-\xi+\eta^2 \pm \sqrt{(1-\xi+\eta^2)^2-4\eta^2}}{2}}\, m_R$
D	$\alpha_0 R + \alpha_1 \, R\,\phi + f(\phi) + \omega(\phi)\phi_{;\alpha}\phi^{;\alpha}$	$m_R{}^2 \to \infty$, $m_Y{}^2 \to \infty, \ m_\phi{}^2 = -f_{\phi\phi}(\phi^{(0)})$ $\xi = 3\,\alpha^2, \ \eta \to 0$ $m_+ \to \infty, \ m_- \to \frac{m_\phi}{\sqrt{1-\xi}}$
E	$f(R, \, R_{\alpha\beta}R^{\alpha\beta}, \phi) + \omega(\phi)\phi_{;\alpha}\phi^{;\alpha}$	$m_R{}^2 = -\frac{f_R(0,0,\phi^{(0)})}{3f_{RR}(0,0,\phi^{(0)})+2f_Y(0,0,\phi^{(0)})}$ $m_Y{}^2 = \frac{f_R(0,0,\phi^{(0)})}{f_Y(0,0,\phi^{(0)})}, \ m_\phi{}^2 = -f_{\phi\phi}(0, \, 0, \, \phi^{(0)})$ $\xi = 3f_{R\phi}(0,0,\phi^{(0)})^2, \ \eta = \frac{m_\phi}{m_R}$ $m_\pm = \sqrt{\frac{1-\xi+\eta^2 \pm \sqrt{(1-\xi+\eta^2)^2-4\eta^2}}{2}}\, m_R$

Table 5.2: Here $f_R(0, \, 0, \, \phi^{(0)}) = 1$ and $\omega(\phi^{(0)}) = 1/2$ and for the case D we set also $\alpha_0 + \alpha_1\phi^{(0)} = 1$.

Chapter 6

Discussion and Conclusions

In this thesis models of Extended Gravity have been studied in the Newtonian limit (weak-field and small velocity), as well as in the Minkowskian limit (weak-field: gravitational waves). In the former one finds modifications of the gravitational potential, whilst in the latter one obtains massive gravitational wave modes. The weak-field limit of such proposals has to be tested against realistic self-gravitating systems. Galactic rotation curves, terrestrial experiments of gravitomagnetism (geodesic and Lense-Thirring effects), gravitational lensing and stellar binary system appear natural candidates as test-bed experiments.

We have also considered the problem of weak field limit of scalar-tensor theories of gravity showing how the Newtonian limit behaves in the Jordan and in the Einstein frame [**C**]. The general result is that Newtonian potentials, masses and other physical quantities can be compared in both frames once the perturbative analysis is performed. The main point is that if such an analysis is carefully developed in the same frame, the perturbative process can be controlled step by step leading to coherent results in both frames. In particular, it is important to fix the relation between conformally related potentials in order to understand how gravitational coupling and Yukawa-like corrections behave. Specifically, the potentials

$$
\begin{aligned}
\Phi(\mathbf{x}) &= -\frac{GM}{\phi^{(0)}|\mathbf{x}|}\left\{1-\frac{e^{-m_\phi|\mathbf{x}|}}{2\,\omega(\phi^{(0)})\,\phi^{(0)}-3}\right\} \\
\Psi(\mathbf{x}) &= -\frac{GM}{\phi^{(0)}|\mathbf{x}|}\left\{1+\frac{e^{-m_\phi|\mathbf{x}|}}{2\,\omega(\phi^{(0)})\,\phi^{(0)}-3}\right\} \\
\phi(\mathbf{x}) &= \phi^{(0)}-\frac{1}{2\,\omega(\phi^{(0)})\,\phi^{(0)}-3}\frac{r_g}{|\mathbf{x}|}\,e^{-m_\phi|\mathbf{x}|}
\end{aligned}
$$

achieved in the Jordan frame (see Eqs.(2.132) can be rigorously compared with their counterparts in the Einstein frame

$$\tilde{\Phi} = -\frac{GM}{|\mathbf{x}|} \qquad \tilde{\Psi} = \tilde{\Phi} \qquad \tilde{\phi} = \sqrt{2\omega_0+3}\,\ln\phi^{(0)} + \frac{1}{\phi^{(0)}\sqrt{2\omega_0+3}}\frac{r_g}{|\mathbf{x}|}\,e^{-m_\phi|\mathbf{x}|}$$

see Eqs.(2.149) when we set $\tilde{\omega}(\tilde{\phi}) = -1/2$, $\Xi = 1$ and $\omega(\phi) = -\omega_0/\phi$. This result, in principle, could constitute a paradigm to compare physical quantities in both frames. In this sense, the observable consequences of conformal transformations can be achieved.

About the gravitational Lensing the study has been evaluated on two steps [**A**]: in the first one we consider a point-like source and by analyzing the properties of Lagrangian of photon we obtain a correction to the outcome of GR depending apparently on two free parameters of theory. But by plotting, only numerically, the new angular behavior (3.8) with respect to the minimal distance r_0 we note that the correction term does not depend on the parameter μ_1 (Fig. 3.1). In the second step we start by more general geodesic motion and we reformulate the deflection angle for a generic matter distribution. In the case of an axially symmetric matter density we obtain the usual relation between the deflection angle and the orthogonal gradient of metric potentials (3.23). Otherwise we find that the angle is depending also onto the travel direction of photon (3.22). Particularly if there is a z-symmetry the deflection angle does not depend explicitly on the parameter μ_1 but we have only the correction term induced by μ_2 (3.24). From the definition of μ_1 and μ_2 (2.55) we note that the presence of function of Ricci scalar ($f_{RR}(0) \neq 0$) is only in μ_1. *Then if we consider only the $f(R)$-Gravity ($\mu_2 \rightarrow \infty$) the geodesic trajectory of photon is unaffected by the modification in the Hilbert-Einstein action.* Instead if we want to have the corrections to GR it needs to add a generic function of Ricci tensor square into Hilbert-Einstein action. But in this case we find the deflection angle smaller than one of GR (3.8) or (3.25). Obviously the same situation is present also in the Einstein ring (3.34), where the new angle is ever lower than the one of GR (Fig. 3.5). The mathematical motivation is a consequence of algebraic signs of terms containing the parameter μ_2 in the metric (2.48). In fact they are ever different with respect to the terms of GR in (2.48) and they can be interpreted as a "repulsive force" giving us a minor curvature of photon trajectory, instead the correction terms containing the parameter μ_1 have opposite algebraic sign in the metric components g_{tt} and g_{ij} (2.48) and we lose their information in the deflection angle (3.20). A similar outcome has been found for the galactic rotation curve, where the contribution of $f(R_{\alpha\beta}R^{\alpha\beta})$ in the action gives us a lower rotation velocity profile than the one of GR, but with a no trivial difference. In fact in galactic dynamics we are studying the motion of massive particles and in this case we find the corrections induced also by $f(R)$-Gravity. Then if we can estimate the weight of the corrections (induced by $f(R)$-Gravity) to the Ricci scalar for the galactic motion, from the point of view of Gravitational Lensing we have a perfect agreement with the GR. Only by adding $f(R_{\alpha\beta}R^{\alpha\beta})$ in the action we induce the modifications in both two frameworks, but we do not find the hoped behavior: the flat galactic rotation curve and a more strong deflection angle of photon. Also for a photon bending we need a Dark Matter component. *Moreover if we consider $f(R, R_{\alpha\beta}R^{\alpha\beta})$-Gravity for the Gravitational Lensing we need a bigger amount of Dark Matter then in GR.*

In the context of Scalar Tensor Forth Order we have studied the linearized field equations in the limit of weak gravitational fields and small velocities generated by rotating gravitational sources and the energy loss of stellar binary systems aiming at constraining the free parameters, which can be seen as effectives masses (or lengths), using recent terrestrial and astrophysical experimental results. In the first case [**B**] [**D**], we have studied the precession of spin of a gyroscope orbiting about a rotating gravitational source. Such a gravitational field gives rise, according to GR predictions, to geodesic and Lense-Thirring processions, the latter being strictly related to the off-diagonal terms of the metric tensor generated by the rotation of the source. We have focused in particular on the gravitational field generated by the Earth, and on the recent experimental results obtained by the Gravity Probe B satellite, which tested the geodesic and Lense-Thirring spin precessions with high precision. In particular, we have calculated the corrections of the precession induced by scalar, tensor and curvature corrections. Considering an almost circular orbit, we integrated the Gauss equations and obtained the variation of the parameters at first order with respect to the eccentricity. We have shown that the induced Extended Gravity effects depend on the effective masses m_R, m_Y and m_ϕ (4.10), while the nonvalidity of the Gauss theorem implies that these effects also depend on the geometric form and size of the rotating source. Requiring that the corrections are within the experimental errors, we then imposed constraints on the free parameters of the considered Extended Gravity model. Merging the experimental results of Gravity Probe B and LARES, our results can be summarized as follows:

$$k(\xi,\eta)\big(m_+\,r^*+1\big)\,F(m_+R_\oplus)\,e^{-m_+\,r^*}+\frac{8}{3}(m_Y\,r^*+1)\,F(m_YR_\oplus)\,e^{-m_Y\,r^*}+$$

$$+[1/3-k(\xi,\eta)](m_-\,r^*+1)\,F(m_-R_\oplus)\,e^{-m_-\,r^*}\lesssim 0.008\,,$$

and

$$m_Y\geq 1.2\times 10^{-6}m^{-1}\,.$$

It is interesting to note that the field equation for the potential A_i, Eq. (2.87c), is time-independent provided the potential Φ is time-independent. This aspect guarantees that the solution Eq. (2.72) does not depend on the masses m_R and m_ϕ and, in the case of $f(R,\,\phi)$ gravity, the solution is the same as in GR. In the case of spherical symmetry, the hypothesis of a radially static source is no longer considered, and the obtained solutions depend on choice of the $f(R,\,\phi)$ Extended Gravity Model, since the geometric factor $F(x)$ is time-dependent. Hence in this case, gravito-magnetic corrections to GR emerge with time-dependent sources.

In the second case [**E**], we have calculated the gravitational wave emission from a quadrupole source and the energy loss of a stellar binary system. Using astrophysical results on the orbital period damping, we infer lower limits on the free parameters $\{m_Y, m_R, m_\phi\}$ of Scalar-Tensor Fourth Order Gravity models studied in the literature. In particular to constrain experimentally the free parameters we considered the nearly circular binary system PSR J0348+0432 and found $m_Y > 5 \times 10^{-11}\, m^{-1}$. Considering the elliptic binary system PSR B1913+16 with eccentricity $e\,=\,0.6$ we have constrained all three free parameters. Choosing a particular Scalar Tensor Fourth Order Gravity scenario we were able to get a lower value for m_R and m_ϕ, namely $m_R \gtrsim 3\times 10^{-9}\,\mathrm{m}^{-1}$ and $m_\phi \gtrsim 1.3\times 10^{-11}\,\mathrm{m}^{-1}$, respectively. One may be able to set stronger constraints by considering systems which are closer. It is worth noting that for circular binary systems there are no corrections in the case of a pure $f(R)$ gravity with respect to General Relativity.

Apendix

6.1 Green functions for a Scalar Tensor Fourth Order Gravity

The complete set of equations substituing the field equations (2.87) for $h_{\mu\nu}$, φ and Eq. (2.94) for auxiliarly fields $\gamma_{\mu\nu}$, Γ, Ψ, Ξ are

$$
\begin{aligned}
&(\Box_\eta + m_Y{}^2)\Box_\eta\gamma_{\mu\nu} \,=\, -2\, m_Y{}^2\, \mathcal{X}\, T_{\mu\nu}\;, \\
&(\Box_\eta + m_Y{}^2)\Box_\eta\gamma \,=\, -2\, m_Y{}^2\, \mathcal{X}\, T\;, \\
&(\Box_\eta + m_+{}^2)(\Box_\eta + m_-{}^2)\varphi \,=\, -m_R{}^2 f_{R\phi}(0,0,\phi^{(0)})\, \mathcal{X}\, T\;, \\
&(\Box_\eta + m_R{}^2)(\Box_\eta + m_Y{}^2)\Box_\eta\Gamma \,=\, 2\, m_R{}^2\, m_Y{}^2\, \mathcal{X}\, T\;, \\
&(\Box_\eta + m_+{}^2)(\Box_\eta + m_-{}^2)(\Box_\eta + m_R{}^2)\Box_\eta\Psi \,=\, 2\, m_R{}^4 f_{R\phi}(0,0,\phi^{(0)})\, \mathcal{X}\, T\;, \\
&(\Box_\eta + m_+{}^2)(\Box_\eta + m_-{}^2)(\Box_\eta + m_R{}^2)\Xi \,=\, 2\, m_R{}^4 f_{R\phi}(0,0,\phi^{(0)})\, \mathcal{X}\, T\;,
\end{aligned}
\tag{6.1}
$$

where we have four characteristic lengths ($m_R{}^{-1}$, $m_Y{}^{-1}$, $m_+{}^{-1}$, $m_-{}^{-1}$), which we assume all different and real. The first two ($m_R{}^{-1}$, $m_Y{}^{-1}$) are generated by the geometry, while the last two ($m_+{}^{-1}$, $m_-{}^{-1}$) are lengths resulting from the interaction between geometry and the scalar field φ.

The solutions of Eq. (6.1) can be expressed in terms of Green functions as

$$
\begin{aligned}
\gamma_{\mu\nu}(x) &= -2\, m_Y{}^2\, \mathcal{X} \int d^4x' \mathcal{G}_\gamma(x,x')\, T_{\mu\nu}(x') \,, \\
\gamma(x) &= -2\, m_Y{}^2\, \mathcal{X} \int d^4x' \mathcal{G}_\gamma(x,x')\, T(x') \,, \\
\varphi(x) &= -\, m_R{}^2 f_{R\phi}(0,0,\phi^{(0)})\, \mathcal{X} \int d^4x' \mathcal{G}_\varphi(x,x')\, T(x') \,, \\
\Gamma(x) &= 2\, m_R{}^2\, m_Y{}^2\, \mathcal{X} \int d^4x' \mathcal{G}_\Gamma(x,x')\, T(x') \,, \\
\Psi(x) &= 2\, m_R{}^4 f_{R\phi}(0,0,\phi^{(0)})\, \mathcal{X} \int d^4x' \mathcal{G}_\Psi(x,x')\, T(x') \,, \\
\Xi(x) &= 2\, m_R{}^4 f_{R\phi}(0,0,\phi^{(0)})\, \mathcal{X} \int d^4x' \mathcal{G}_\Xi(x,x')\, T(x') \,,
\end{aligned}
\tag{6.2}
$$

with the Green functions fixed by

$$
\begin{aligned}
(\Box_\eta + m_Y{}^2)\Box_\eta \mathcal{G}_\gamma(x,x') &= \delta^4(x-x') \,, \\
(\Box_\eta + m_+^2)(\Box_\eta + m_-^2)\mathcal{G}_\varphi(x,x') &= \delta^4(x-x') \,, \\
(\Box_\eta + m_R{}^2)(\Box_\eta + m_Y{}^2)\Box_\eta \mathcal{G}_\Gamma(x,x') &= \delta^4(x-x') \,, \\
(\Box_\eta + m_+^2)(\Box_\eta + m_-^2)(\Box_\eta + m_R{}^2)\Box_\eta \mathcal{G}_\Psi(x,x') &= \delta^4(x-x') \,, \\
(\Box_\eta + m_+^2)(\Box_\eta + m_-^2)(\Box_\eta + m_R{}^2)\mathcal{G}_\Xi(x,x') &= \delta^4(x-x') \,,
\end{aligned}
\tag{6.3}
$$

where $\delta^4(x-x')$ is a four-dimensional Dirac distribution in flat space-time. To find the analytical dependence of the Green functions it can be shown that in Fourier space they are linear combination of only $\mathcal{G}_{\mathrm{KG,m}}$ and $\mathcal{G}_{\mathrm{GR}}$, which satisfy the second order equations

$$
\begin{aligned}
(\Box_\eta + m^2)\mathcal{G}_{\mathrm{KG,m}}(x,x') &= \delta^4(x-x') \,, \\
\Box_\eta \mathcal{G}_{\mathrm{GR}}(x,x') &= \delta^4(x-x') \,,
\end{aligned}
\tag{6.4}
$$

with solutions

$$
\begin{aligned}
\mathcal{G}^{\mathrm{ret}}_{\mathrm{KG,m}}(x,x') &= \frac{\Theta\Big(t-t'\Big)}{4\pi}\left[\frac{\delta(t-t'-|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|} - \frac{m\,\mathcal{J}_1(m\,\tau_{xx'})}{\tau_{xx'}}\Theta\Big(\frac{\tau^2_{xx'}}{2}\Big)\right] \,, \\
\mathcal{G}^{\mathrm{ret}}_{\mathrm{GR}}(x,x') &= \frac{\Theta\Big(t-t'\Big)}{4\pi}\frac{\delta(t-t'-|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|} \,,
\end{aligned}
$$

where $\tau^2_{xx'} = (x-x')^2 = (t-t')^2 - |\mathbf{x}-\mathbf{x}'|^2$.

Hence the Green's functions $\mathcal{G}^{\rm ret}_\gamma, \mathcal{G}^{\rm ret}_\varphi, \mathcal{G}^{\rm ret}_\Psi$ and $\mathcal{G}^{\rm ret}_\Xi$ are expressed as

$$\mathcal{G}^{\rm ret}_\gamma(x,x') = \tfrac{1}{m_Y^2}\biggl[\mathcal{G}^{\rm ret}_{\rm GR}(x,x') - \mathcal{G}^{\rm ret}_{{\rm KG},m_Y}(x,x')\biggr]$$

$$= \Theta\biggl(t-t'\biggr)\,\Theta\biggl(\tfrac{\tau^2_{xx'}}{2}\biggr)\tfrac{\mathcal{J}_1(m_Y\,\tau_{xx'})}{4\pi\,m_Y\,\tau_{xx'}}\,,$$

$$\mathcal{G}^{\rm ret}_\varphi(x,x') = \tfrac{1}{m_+^2-m_-^2}\biggl[\mathcal{G}^{\rm ret}_{{\rm KG},m_-^2}(x,x') - \mathcal{G}^{\rm ret}_{KG,m_+^2}(x,x')\biggr]$$

$$= \frac{\Theta\Bigl(t-t'\Bigr)\,\Theta\Bigl(\frac{\tau^2_{xx'}}{2}\Bigr)}{4\pi\,(m_+^2-m_-^2)\,\tau_{xx'}}\biggl[m_+\,\mathcal{J}_1(m_+\,\tau_{xx'}) - m_-\,\mathcal{J}_1(m_-\,\tau_{xx'})\biggr]\,,$$

$$\mathcal{G}^{\rm ret}_\Gamma(x,x') = \tfrac{1}{m_R^2-m_Y^2}\biggl[\tfrac{m_R^2-m_Y^2}{m_R^2\,m_Y^2}\mathcal{G}^{\rm ret}_{\rm GR}(x,x') + \tfrac{1}{m_R^2}\mathcal{G}^{\rm ret}_{{\rm KG},m_R}(x,x') - \tfrac{1}{m_Y^2}\mathcal{G}^{\rm ret}_{{\rm KG},m_Y}(x,x')\biggr]$$

$$= \frac{\Theta\Bigl(t-t'\Bigr)\,\Theta\Bigl(\frac{\tau^2_{xx'}}{2}\Bigr)}{4\pi\,(m_R^2-m_Y^2)\,\tau_{xx'}}\biggl[\frac{\mathcal{J}_1(m_Y\,\tau_{xx'})}{m_Y} - \frac{\mathcal{J}_1(m_R\,\tau_{xx'})}{m_R}\biggr]\,,$$

$$\mathcal{G}^{\rm ret}_\Psi(x,x') = \tfrac{\mathcal{G}^{\rm ret}_{\rm GR}(x,x')}{m_+^2\,m_-^2\,m_R^2} - \tfrac{\mathcal{G}^{\rm ret}_{KG,m_+}(x,x')}{m_+^2(\,m_-^2-m_+^2)(\,m_R^2-m_+^2)} - \tfrac{\mathcal{G}^{\rm ret}_{KG,m_-}(x,x')}{m_-^2(\,m_+^2-m_-^2)(\,m_R^2-m_-^2)} - \tfrac{\mathcal{G}^{\rm ret}_{{\rm KG},m_R}(x,x')}{m_R^2(\,m_+^2-m_R^2)(\,m_-^2-m_R^2)}$$

$$= \frac{\Theta\Bigl(t-t'\Bigr)\,\Theta\Bigl(\frac{\tau^2_{xx'}}{2}\Bigr)}{4\pi\,\tau_{xx'}}\biggl[\frac{\mathcal{J}_1(m_+\,\tau_{xx'})}{m_+(\,m_-^2-m_+^2)(\,m_R^2-m_+^2)} + \frac{\mathcal{J}_1(m_-\,\tau_{xx'})}{m_-(\,m_+^2-m_-^2)(\,m_R^2-m_-^2)} + \frac{\mathcal{J}_1(m_R\,\tau_{xx'})}{m_R(\,m_+^2-m_R^2)(\,m_-^2-m_R^2)}\biggr]$$

$$\mathcal{G}^{\rm ret}_\Xi(x,x') = \tfrac{\mathcal{G}^{\rm ret}_{KG,m_+}(x,x')}{(\,m_-^2-m_+^2)(\,m_R^2-m_+^2)} + \tfrac{\mathcal{G}^{\rm ret}_{{\rm KG},m_-}(x,x')}{(\,m_+^2-m_-^2)(\,m_R^2-m_-^2)} + \tfrac{\mathcal{G}^{\rm ret}_{{\rm KG},m_R}(x,x')}{(\,m_+^2-m_R^2)(\,m_-^2-m_R^2)}$$

$$= -\frac{\Theta\Bigl(t-t'\Bigr)\,\Theta\Bigl(\frac{\tau^2_{xx'}}{2}\Bigr)}{4\pi\,\tau_{xx'}}\biggl[\frac{m_+\mathcal{J}_1(m_+\,\tau_{xx'})}{(\,m_-^2-m_+^2)(\,m_R^2-m_+^2)} + \frac{m_-\mathcal{J}_1(m_-\,\tau_{xx'})}{(\,m_+^2-m_-^2)(\,m_R^2-m_-^2)} + \frac{m_R\mathcal{J}_1(m_R\,\tau_{xx'})}{(\,m_+^2-m_R^2)(\,m_-^2-m_R^2)}\biggr]\,.$$

6.2 Mathematical aspects of spatial metric components h_{ij}

The spatial components of the perturbation $h_{\mu\nu}$ (2.95) can be expressed as

$$h_{ij}(t,\,|\mathbf{x}|) = \quad -2\,m_Y\,\Upsilon^{m_Y}_{ij}(t,\,|\mathbf{x}|) + \frac{1}{3m_Y}\Big[2\,\partial^2_{ij} + \eta_{ij}\,\mathrm{H}\Big]\Upsilon^{m_Y}(t,\,|\mathbf{x}|) - \eta_{ij}\,\Pi\,\Upsilon^{m_R}(t,\,|\mathbf{x}|) + \\ + \sum_{S=\pm}\frac{g_S(\xi,\eta)}{3\,m_S}\Big[2\partial^2_{ij} - \eta_{ij}\,m^2_S\Big]\Upsilon^{m_S}(t,\,|\mathbf{x}|)\,, \tag{6.6}$$

where

$$\begin{aligned}\mathrm{H} &= \frac{\Big[9\,m_R^2 m_Y^2 + 3\,(2\,m_R^2 + m_Y^2)\Box\Big]}{(m_R^2 - m_Y^2)}\,,\\ \Pi &= \frac{\Big[(2\,m_R^2 + m_Y^2)(\Box + m_R^2)\Big]}{m_R\,(m_R^2 - m_Y^2)}\,,\\ g_\pm(\xi,\eta) &= \frac{\xi}{\Big[w_\mp(\xi,\eta) - w_\pm(\xi,\eta)\Big]\Big[1 - w_\pm(\xi,\eta)\Big]}\,,\end{aligned} \tag{6.7}$$

and $\Upsilon^m_{ij}(t,\,\mathbf{x})$ and $\Upsilon^m(t,\,\mathbf{x})$ are defined in Eqs. (2.106), (2.111). The derivatives of $\Upsilon^m(t,\,\mathbf{x})$ are

$$\begin{aligned}\partial_\mu\Upsilon^m(t,\,|\mathbf{x}|) &= \frac{\mathcal{X}}{24\pi}\int_0^\infty d\tau\,\mathcal{J}_1(\tau)\left[-\frac{m^2\,\partial_\mu(|\mathbf{x}|^2)\ddot{Q}(t-\tau_m)}{2\Big[\tau^2+m^2|\mathbf{x}|^2\Big]^{3/2}} + \frac{\partial_\mu\ddot{Q}(t-\tau_m)}{\sqrt{\tau^2+m^2|\mathbf{x}|^2}}\right],\\ \partial^2_{\mu\nu}\Upsilon^m(t,\,|\mathbf{x}|) &= \frac{\mathcal{X}}{24\pi}\int_0^\infty d\tau\,\mathcal{J}_1(\tau)\left\{\frac{3\,m^4}{4}\frac{\partial_\mu(|\mathbf{x}|^2)\partial_\nu(|\mathbf{x}|^2)\ddot{Q}(t-\tau_m)}{\Big[\tau^2+m^2|\mathbf{x}|^2\Big]^{5/2}} - \frac{m^2\,\partial^2_{\mu\nu}(|\mathbf{x}|^2)\ddot{Q}(t-\tau_m)}{2\,\Big[\tau^2+m^2|\mathbf{x}|^2\Big]^{3/2}}\right.\\ &\left.\quad -\frac{m^2}{2}\left[\frac{\partial_\mu(|\mathbf{x}|^2)\,\partial_\nu\ddot{Q}(t-\tau_m)+\partial_\nu(|\mathbf{x}|^2)\,\partial_\mu\ddot{Q}(t-\tau_m)}{\Big[\tau^2+m^2|\mathbf{x}|^2\Big]^{3/2}}\right] + \frac{\partial^2_{\mu\nu}\ddot{Q}(t-\tau_m)}{\sqrt{\tau^2+m^2|\mathbf{x}|^2}}\right\},\\ \partial^2_{ij}\Upsilon^m(t,\,|\mathbf{x}|) &= \frac{\mathcal{X}}{24\pi}\int_0^\infty d\tau\,\mathcal{J}_1(\tau)\left\{\frac{m^2\,\ddot{Q}(t-\tau_m)}{\Big[\tau^2+m^2|\mathbf{x}|^2\Big]^{3/2}}\left[\frac{3\,m^2\,x_i\,x_j}{\Big[\tau^2+m^2|\mathbf{x}|^2\Big]} - \delta_{ij}\right]\right.\\ &\left.\quad +\frac{m\,\ddot{Q}'(t-\tau_m)}{\Big[\tau^2+m^2|\mathbf{x}|^2\Big]}\left[\frac{2\,m^2\,x_i\,x_j}{\Big[\tau^2+m^2|\mathbf{x}|^2\Big]} - \delta_{ij}\right] + \frac{m^2\,x_i\,x_j\,\ddot{Q}''(t-\tau_m)}{\Big[\tau^2+m^2|\mathbf{x}|^2\Big]^{3/2}}\right\},\\ \Box\Upsilon^m(t,\,|\mathbf{x}|) &= \frac{\mathcal{X}}{24\pi}\int_0^\infty d\tau\,\mathcal{J}_1(\tau)\left[\frac{3\,m^2\,\ddot{Q}(t-\tau_m)\,\tau^2}{\Big[\tau^2+m^2|\mathbf{x}|^2\Big]^{5/2}} + \frac{3\,m\,\ddot{Q}'(t-\tau_m)\,\tau^2}{\Big[\tau^2+m^2|\mathbf{x}|^2\Big]^2} - \frac{m^2\,|\mathbf{x}|^2\,\ddot{Q}''(t-\tau_m)}{\Big[\tau^2+m^2|\mathbf{x}|^2\Big]^{3/2}} + \frac{\ddddot{Q}(t-\tau_m)}{\sqrt{\tau^2+m^2|\mathbf{x}|^2}}\right].\end{aligned} \tag{6.8}$$

Equations (6.8c) and (6.8d) can be approximated by considering only the terms scaling as $1/|\mathbf{x}|$; the other terms scale as $1/|\mathbf{x}|^n$ with $n > 1$. Thus, we have

$$\partial^2_{ij}\Upsilon^m(t,\,|\mathbf{x}|) \;\approx\; \tfrac{\mathcal{X}}{24\pi}\int_0^\infty d\tau\,\mathcal{J}_1\big(\tau\big)\,\tfrac{\partial^2_{ij}\ddot{Q}(t-\tau_m)}{\sqrt{\tau^2+m^2|\mathbf{x}|^2}} \approx \tfrac{m^2\,x_i\,x_j\,\mathcal{X}}{24\pi}\int_0^\infty d\tau\,\mathcal{J}_1\big(\tau\big)\;\tfrac{\ddot{Q}''(t-\tau_m)}{\left[\tau^2+m^2|\mathbf{x}|^2\right]^{3/2}} \;\equiv D^m_{ij}(|\mathbf{x}|,t)\;,$$

(6.

$$\Box\Upsilon^m(t,\,|\mathbf{x}|) \;\approx\; \tfrac{\mathcal{X}}{24\pi}\int_0^\infty d\tau\,\mathcal{J}_1\big(\tau\big)\tfrac{\ddddot{Q}(t-\tau_m)}{\sqrt{\tau^2+m^2|\mathbf{x}|^2}} \equiv B^m(|\mathbf{x}|,t)\;,$$

and Eq. (6.6) reads

$$\begin{aligned} h_{ij}(t,\,|\mathbf{x}|) \;=\; -2\,m_Y\,\Upsilon^{m_Y}_{ij} + \eta_{ij}\Bigg\{ & m_Y\,g_Y\,\Upsilon^{m_Y}(t,\,|\mathbf{x}|) - m_R\,g_R\,\Upsilon^{m_R}(t,\,|\mathbf{x}|) \\ & - \textstyle\sum_{S=\pm}\tfrac{m_S\,g_S(\xi,\eta)}{3}\,\Upsilon^{m_S}(t,\,|\mathbf{x}|) + g_R\left[\tfrac{B^{m_Y}(|\mathbf{x}|,t)}{m_Y} - \tfrac{B^{m_R}(|\mathbf{x}|,t)}{m_R}\right]\Bigg\} \\ & + \tfrac{2}{3}\left[\tfrac{D^{m_Y}_{ij}(t,|\mathbf{x}|)}{m_Y} + \textstyle\sum_{S=\pm} g_S(\xi,\eta)\,\tfrac{D^{m_S}_{ij}(t,|\mathbf{x}|)}{m_S}\right], \end{aligned} \tag{6.10}$$

where

$$g_Y \;=\; \frac{3\,m_R^2}{m_R^2 - m_Y^2} \quad,\quad g_R \;=\; \frac{2\,m_R^2 + m_Y^2}{m_R^2 - m_Y^2}\;. \tag{6.11}$$

The time derivatives of $\Upsilon^m_{ij}(t,\,|\mathbf{x}|)$, $B^m(t,\,|\mathbf{x}|)$, $D^m_{ij}(t,\,|\mathbf{x}|)$ needed to calculate the energy loss in Eq. (5.1) are

$$\begin{aligned}
\dot{\Upsilon}^m_{ij}(t,\,|\mathbf{x}|) \;&=\; \tfrac{\omega^3_{(ij)}\,\mathcal{Q}_{ij}\,\mathcal{X}}{24\,\pi}\left[\sin\!\big(\omega_{(ij)}\,t+\vartheta_{(ij)}\big)\,f^c_1\!\left(m|\mathbf{x}|;\,\tfrac{\omega_{(ij)}}{m}\right)-\cos\!\big(\omega_{(ij)}\,t+\vartheta_{(ij)}\big)\times\right.\\
&\qquad\qquad\left.\times f^s_1\!\left(m|\mathbf{x}|;\,\tfrac{\omega_{(ij)}}{m}\right)\right],\\
\dot{\Upsilon}^m(t,\,|\mathbf{x}|) \;&=\; \tfrac{\nu^3\,\mathcal{Q}\,\mathcal{X}}{24\,\pi}\left[\sin\!\big(\nu\,t+\vartheta\big)\,f^c_1\!\left(m|\mathbf{x}|;\,\tfrac{\nu}{m}\right)-\cos\!\big(\nu\,t+\vartheta\big)\,f^s_1\!\left(m|\mathbf{x}|;\,\tfrac{\nu}{m}\right)\right],\\
\dot{B}^m(t,\,|\mathbf{x}|) \;&=\; -\tfrac{\nu^5\,\mathcal{Q}\,\mathcal{X}}{24\,\pi}\left[\sin\!\big(\nu\,t+\vartheta\big)\,f^c_1\!\left(m|\mathbf{x}|;\,\tfrac{\nu}{m}\right)-\cos\!\big(\nu\,t+\vartheta\big)\,f^s_1\!\left(m|\mathbf{x}|;\,\tfrac{\nu}{m}\right)\right],\\
\dot{D}^m_{ij}(t,\,|\mathbf{x}|) \;&=\; -\tfrac{\nu^5\,m^2\,x_i\,x_j\,\mathcal{Q}\,\mathcal{X}}{24\,\pi}\left[\sin\!\big(\nu\,t+\vartheta\big)\,f^c_3\!\left(m|\mathbf{x}|;\,\tfrac{\nu}{m}\right)-\cos\!\big(\nu\,t+\vartheta\big)\times\right.\\
&\qquad\qquad\left.\times f^s_3\!\left(m|\mathbf{x}|;\,\tfrac{\nu}{m}\right)\right],
\end{aligned} \tag{6.12}$$

with the definitions

$$\begin{aligned}
f^c_n\big(x;z\big) \;&\equiv\; \int_0^\infty d\tau\,\frac{\mathcal{J}_1\big(\tau\big)\,\cos\!\big(z\sqrt{\tau^2+x^2}\big)}{\big(\sqrt{\tau^2+x^2}\big)^n}\,,\\
f^s_n\big(x;z\big) \;&\equiv\; \int_0^\infty d\tau\,\frac{\mathcal{J}_1\big(\tau\big)\,\sin\!\big(z\sqrt{\tau^2+x^2}\big)}{\big(\sqrt{\tau^2+x^2}\big)^n}\,,\\
F\big(|\mathbf{x}|;m_1;m_2;\omega\big) \;&\equiv\; f^c_1\!\left(m_1|\mathbf{x}|;\tfrac{\omega}{m_1}\right)f^c_1\!\left(m_2|\mathbf{x}|;\tfrac{\omega}{m_2}\right)+f^s_1\!\left(m_1|\mathbf{x}|;\tfrac{\omega}{m_1}\right)f^s_1\!\left(m_2|\mathbf{x}|;\tfrac{\omega}{m_2}\right).
\end{aligned} \tag{6.13}$$

Bibliography

[1] Einstein, A., *Die Feldgleichungen der Gravitation*, Sitzungsber. K. Preuss. Akad. Wiss., Phys.-Math. Kl., 1915, 844847, (1915)

[2] H. Thirring *Physikalische Zeitschrift* , **19**, 204, (1918).

[3] H. Thirring *Physikalische Zeitschrift*, **22**, 29, (1921).

[4] J. Lense, H. Thirring, *Physikalische Zeitschrift*, 19, 156, (1918).

[5] A. H. Chamseddine, A. Connes and M. Marcolli, *Gravity and the standard model with neutrino mixing*, Adv. Theor. Math. Phys. **11**, 991 (2007)

[6] Weinberg S., *Gravitation and Cosmology*, Wiley 1972, New York

[7] Weyl H., Math. Zeit. **2**, 384 (1918)

[8] Weyl H., *Raum-Zeit-Materie*, Springer 1921 Berlin

[9] Pauli W., Phys. Zeit. **20**, 457 (1919)

[10] Bach R., Math. Zeit. **9**, 110 (1921)

[11] Eddington A. S., *The mathematical theory of relativity*, Cambridge University Press 1924 London

[12] Lanczos C., Z. Phys. **73**, 147 (1931)

[13] Schmidt H. J., ArXiv: 0407095 [gr-qc]

[14] Guth A., Phys. Rev. D **23**, 347 (1981)

[15] Capozziello S., Cardone V. F., Troisi A., JCAP **0608**, 001 (2006)

[16] Capozziello S., Cardone V. F., Troisi A., Mon.Not. Roy. Astron. Soc. **375**, 1423 (2007)

[17] Nojiri S., Odintsov S. D., Int. Jou. Geom. Meth. Mod. Phys. **4**, 115 (2007)

[18] Hehl F. W., von der Heyde P., Kerlick G. D., Nester J. M., Rev. Mod. Phys. **48**, 393 (1976)

[19] Hehl F. W., Mccrea J. D., Mielke E. W., Neeman Y., Phys. Rep. **4258**, 1 (1995)

[20] Trautman A., Comp.rend. heb. sean. **257**, 617 (1963)

[21] Puetzfeld D., New Astron. Rev. **49**, 59 (2005)

[22] Buchbinder I. L., Odintsov S. D., Shapiro I. L., Effective Action in Quantum Gravity, IOP Publishing 1992 Bristol

[23] Amarzguioui M., Elgaroy O., Mota D. F., Multamaki T., Astron. and Astrophys **454**, 707 (2006)

[24] Magnano G., Ferraris M., Francaviglia M., Gen. Rel. Grav. **19**, 465 (1987)

[25] Allemandi G., Borowiec A., Francaviglia M., Phys. Rev. D **70**, 103503 (2004)

[26] Sotiriou T. P., Class. Quant. Grav. **23**, 1253 (2006)

[27] Sotiriou T. P., Liberati S., Ann. Phys. 322, 935 (2007)

[28] Damour T., Esposito - Farèse G., Class. Quantum Grav. **9**, 2093 (1992)

[29] Bondi H., *Cosmology*, Cambridge Univ. Press 1952, Cambridge

[30] Brans C., Dicke R. H., Phys. Rev. **124**, 925 (1961)

[31] Capozziello S., de Ritis R., Rubano C., Scudellaro P., La Rivista del Nuovo Cimento **4**,1 (1996)

[32] Sciama D. W., Mon. Not. R. Ast. Soc. **113**, 34 (1953)

[33] Buchdahl H. A., Nuovo Cim. **23**, 141 (1962)

[34] de Witt B. S., *Dynamical theory of groups and fields*, Gordon and Breach (1965) New York

[35] Bicknell G. V., Journ. phys. A **7**, 1061 (1974)

[36] Havas P., Gen. Rel. Grav. **8**, 631 (1977)

[37] Stelle K., Gen. Rel. Grav. **9**, 353 (1978)

[38] Teyssandier P., Tourrenc Ph., J. Math. Phys. **24**, 2793 (1983)

[39] Duruisseau J. P., Kerner R., Gen. Rel. Grav. **15**, 797 (1983)

[40] Lanczos C., Ann. Math. **39**, 842 (1938)

[41] Birrell N. D., Davies P.C.W., *Quantum Fields in Curved Space*, Cambridge Univ. Press 1982, Cambridge

[42] Vilkovisky G., Class. Quantum Grav. **9**, 895 (1992)

[43] Gasperini M., Veneziano G., Phys. Lett. B **277**, 256 (1992)

[44] Ruzmaikina T. V., Ruzmaikin A. A., JETP 30, 372 (1970)

[45] Gottlöber S., Schmidt H. J., Starobinsky A. A., Class. Quantum Grav. **7**, 893 (1990)

[46] Amendola L., Battaglia - Mayer A., Capozziello S., Gottlöber S., Müller V., Occhionero F., Schmidt H. J., Class. Quantum Grav. **10**, L43 (1993)

[47] Battaglia - Mayer A., Schmidt H. J., Class. Quantum Grav. **10**, 2441 (1993)

[48] Schmidt H. J., Class. Quantum Grav. **7**, 1023 (1990)

[49] Barrow J., Ottewill A. C., J. Phys. A: Math. Gen. **16**, 2757 (1983)

[50] Starobinsky A. A., Phys. Lett. B **91**, 99 (1980)

[51] La D., Steinhardt P. J., Phys. Rev. Lett. **62**, 376 (1989)

[52] Maeda K., Phys. Rev. D **39**, 3159 (1989)

[53] Wands D., Class. Quantum Grav. **11**, 269 (1994)

[54] S. Capozziello, R. de Ritis, A.A. Marino., Gen. Rel. Grav. **30**, 1247 (1998)

[55] Capozziello S., Nojiri S., Odintsov S. D., Phys. Lett. B **634**, 93 (2006)

[56] Amendola L., Capozziello S., Litterio M., Occhionero F., Phys. Rev. D **45**, 417 (1992)

[57] Riess A. G. et al., Astron. Journ. **116**, 1009 (1998)

[58] Riess A. G. et al., Astroph. Journ. **607**, 665 (2004)

[59] Perlmutter S. et al., Astroph. Journ. **517**, 565 (1999)

[60] Perlmutter S. et al. Astron. Astrophys. **447**, 31 (2006)

[61] Capozziello S., Int. J. Mod. Phys. D **11**, 483 (2002)

[62] Capozziello S., Carloni S., Troisi A., Rec. Res. Devel. Astronomy. & Astrophys. **1**, 625 (2003)

[63] Capozziello S., Cardone V. F., Carloni S., Troisi A., Int. J. Mod. Phys. D **12**, 1969 (2003)

[64] Carroll, S. M., Duvvuri, V., Trodden, M., Turner, M. S., Phys. Rev. D **70**, 043528 (2004)

[65] Nojiri S., Odintsov S. D., Phys. Rev. D **68**, 123512 (2003)

[66] Nojiri S., Odintsov S. D., Phys. Lett. B **576**, 5 (2003)

[67] Capozziello S., Cardone V. F., Troisi A., Phys. Rev. D **71**, 043503 (2005)

[68] Carloni S., Dunsby P. K. S., Capozziello S., Troisi A., Class. Quant. Grav. **22**, 4839 (2005)

[69] Vollick D. N., Phys. Rev. D **68**, 063510 (2003)

[70] Meng X. H., Wang P., Class. Quant. Grav. **20**, 4949 (2003)

[71] Meng X. H., Wang P., Class. Quant. Grav. **21**, 951 (2004)

[72] Flanagan E. E., Phys. Rev. Lett. **92**, 071101 (2004)

[73] Flanagan E. E., Class. Quant. Grav. **21**, 417 (2004)

[74] Kremer G. M., Alves D. S. M., Phys. Rev. D **70**, 023503 (2004)

[75] Capozziello S., Cardone V. F., Francaviglia M., Gen. Rel. Grav. **38**, 711 (2006)

[76] Cole S. et al., Mon. Not. Roy. Astron. Soc. **362**, 505 (2005)

[77] Spergel D. N. et al. Astroph Journ. Suppl. **148**, 175 (2003)

[78] Spergel D. N. et al., ApJS **170**, 377 (2007)

[79] Capozziello S., Francaviglia M., ArXiv: 0706.1146 [astro-ph]

[80] Farhoudi M., Gen. Relativ. Grav. **38**, 1261 (2006)

[81] Knop R. A. et al, Astroph. Journ. **598**, 102 (2003)

[82] Tonry J. L. et al., Astroph. Journ. **594**, 1 (2003)

[83] de Bernardis P. et al., Nature **404**, 955 (2000)

[84] Stompor R. et al., Astroph. Journ. **561**, L7 (2001)

[85] Hinshaw G. et al. ApJS **148**, 135 (2003)

[86] Sahni V., Starobinski A., Int. J. Mod. Phys. D **9**, 373 (2000)

[87] Padmanabhan T., Phys. Rev. D **66**, 021301 (2002)

[88] Peebles P. J. E., Rathra B., Rev. Mod. Phys., **75**, 559 (2003)

[89] Copeland E. J., Sami M., Tsujikawa S., Int. Journ. Mod. Phys. D **15**, 1753 (2006)

[90] Peebles P. J. E., Rathra B., Phys. Rev., **37**, 3406 (1998)

[91] Caldwell E. R., Steinhardt P. J., Phys. Rev. Lett., **80**, 1582 (1998)

[92] Kamenshchik A., Moschella U., Pasquier V., Phys. Lett. B **511**, 265 (2001)

[93] Bilić N., Tupper G., Viollier R. D., Phys. Lett. B **535**, 17 (2002)

[94] Bento M. C., Bertolami O., Sen A. A., Phys. Rev. D **67**, 063003 (2003)

[95] Cardone V. F., Tortora C., Troisi A., Capozziello S., Phys. Rev. D **73**, 043508 (2006)

[96] Caldwell R. R., Phys. Lett. B **545**, 23 (2002)

[97] Singh P., Sami M., Dadhich N., Phys. Rev. D **68**, 023522 (2003)

[98] Nojiri S., Odintsov S. D, Tsujikawa S., Phys. Rev. D **71**, 063004 (2005)

[99] Faraoni V., Class. Quant. Grav. **22**, 0607016 (2005)

[100] Dvali G. R., Gabadadze G., Porrati M., Phys. Lett. B **485**, 208 (2000)

[101] Dvali G. R., Gabadadze G., Kolanovic M., Nitti F., Phys. Rev. **64**, 084004 (2001)

[102] Dvali G. R., Gabadadze G., Kolanovic M., Nitti F., Phys. Rev. **64**, 024031 (2002)

[103] Maartens R., Living Rev. Rel. **7**, 7 (2004)

[104] Multamaki T., Vilja I., Phys. Rev. D **73**, 024018, (2006)

[105] Capozziello S., Cardone V. F., Troisi A., Phys. Rev. D **71**, 043503 (2005)

[106] Capozziello S., Cardone V. F., Carloni S., Troisi A., Phys. Lett. A **326**, 292 (2004)

[107] Lue A., Scoccimarro R., Starkman G., Phys. Rev. D **69**, 044005 (2004

[108] Freese K., Lewis M., Phys. Lett. B **540**, 1 (2002)

[109] Capozziello S., Cardone V. F., Piedipalumbo E., Sereno M., Troisi A., Int. J. Mod. Phys. D **12**, 381 (2003)

[110] Li B., Barrow J. D., Phys. Rev. D **75**, 084010 (2007)

[111] Nojiri S., Odintsov S. D., hep-th/06012113 (2006)

[112] Li B., Barrow J. D., Mota D. F., submitted to Phys. Rev. D, gr-qc/0705.3795

[113] Chen G., Rathra B., ApJ **582**, 586 (2003)

[114] Podariu S., Daly R. A., Mory M. P., Rathra B., ApJ **584**, 577 (2003)

[115] Faraoni V., Phys. Rev. D **72**, 124005 (2005)

[116] Cognola G., Zerbini S., J. Phys. A **39**, 6245 (2006)

[117] Cognola G., Gastaldi M., Zerbini S., ArXiv: 0701138 [gr-qc]

[118] Li B., Barrow J., ArXiv: 0701111 [astrp-ph]

[119] Carloni S., Dunsby P. K. S., Troisi A., ArXiv: 0707.0106 [gr-qc]

[120] Bean R., Bernat D., Pogosian L., Silvestri A., Trodden M., Phys. Rev. D **75**, 064020 (2007)

[121] Song Y. S., Hu W., Sawicki I., Phys. Rev. D **75**, 04404 (2007)

[122] Frigerio Martins C., Salucci P., ArXiv: 0703243 [astro-ph]

[123] Sobouti Y., A & A **464**, 921 (2007)

[124] Mendoza S., Rosas Guevara Y. M., A & A **472**, 367 (2007)

[125] Salucci P., Lapi A., Tonini C., Gentile G., Yegorova I., Klein U., ArXiv: 0703115 [astro-ph]

[126] Bertolami O., Boehmer Ch. G., Harko T., Lobo F. S. N., Phys.Rev. D **75**, 104016 (2007)

[127] Nojiri S., Odintsov S. D., Gen. Rel. Grav. **36**, 1765 (2004)

[128] Anderson J. D. et al., Phys. Rev. Lett. **81**, 2858 (1998)

[129] Anderson J. D. et al., Phys. Rev. D **65**, 082004 (2002)

[130] Bertolami O., Böhmer Ch. G., Lobo F. S. N., gr - qc/0704.1733

[131] Will C. M., *Theory and Experiments in Gravitational Physics* Cambridge Univ. Press 1993, Cambridge

[132] Sanders R. H., Ann. Rev. Astr. Ap. **2**, 1 (1990)

[133] Mannheim P. D., Kazanas D., Ap. J. **342**, 635 (1989)

[134] Quant I., Schmidt H. J., Astron. Nachr. **312**, 97 (1991)

[135] Schneider P., Ehlers J., Falco E. E., *Gravitational Lenses* Springer-Verlag 1992 Berlin

[136] Krauss L. M., White M., Ap. J. **397**, 357 (1992)

[137] Olmo G. J., Phys. Rev. Lett. **95**, 261102 (2005)

[138] Olmo G. J., Phys. Rev. D **72**, 083505 (2005)

[139] Capozziello S., Troisi A., Phys. Rev. D **72**, 044022 (2005)

[140] Allemandi G., Francaviglia M., Ruggiero M., Tartaglia A., Gen. Rel. Grav. **37**, 1891 (2005)

[141] Multamäki T., Vilja I. Phys. Rev. D **74**, 064022 (2006)

[142] Multamäki T., Vilja I. ArXiv: 0612775 [astro-ph]

[143] Sotiriou T. P., Faraoni V., Liberati S., arXiv: 0707.2748 [gr-qc]

[144] Sotiriou T. P., Gen. Rel. Grav. **38**, 1407 (2006)

[145] Chiba T., Phys. Lett. B **575**, 1 (2005)

[146] Erickcek A. L., Smith T. L., Kamionkowski M., Phys.Rev. D **74**, 121501 (2006)

[147] Clifton T., Barrow J. D., Phys. Rev. D **72**, 103005 (2005)

[148] Clifton T., Barrow J. D., Class. Quant. Grav. **23**, 2951 (2006)

[149] Clifton T., Barrow J. D., Class. Quant. Grav. **23**, L1 (2006)

[150] Faraoni V., ArXiv: 0607016 [gr-qc]

[151] Jin X. H., Liu D. J., Li X. Z., ArXiv: 0610854 [astro-ph]

[152] Chiba T., Erickeck A. L., smith T. L., ArXiv: 0611867 [astro-ph]

[153] Faraoni V., Phys. Rev. D **74**, 023529 (2006)

[154] Kainulainen K., Piilonen J., Reijonen V., Sunhede D., Phys. Rev. D **76**, 024020 (2007)

[155] Whitt B., Phys. Rev. D **38**, 3000 (1988)

[156] Barraco D. E., Hamity V. H., Phys. Rev. D **62**, 044027 (2000)

[157] Puetzfeld D., The Cosmological post-Newtonian equations of hydrodynamics in General Relativity, (submitted) (2006)

[158] Ince E. L., *Ordinary Differential Equations*, Dover (1956) New York

[159] Hawking S. W., Ellis G. F. R., 1973 T*he large scale structure of space-time*, Cambridge Univ. Press 1973 Cambridge

[160] Buchdahl H. A., J. Phys. A **12**, 1229 (1979)

[161] Einstein A., Ann. der Physik **49**, 769 (1916)

[162] Eisenhart E., *Riemannian Geometry*, Princeton Univ. Press 1955 Princeton

[163] Schrödinger E., *Space-Time Structure*, Cambridge Univ. Press 1960 Cambridge

[164] Landau Lev D., Lifšits E. M, *Theoretical physics* vol. II,

[165] Levi Civita T., *The Absolute Differential Calculus*, Blackie and Son 1929, London

[166] Klein O., New Theories in Physics **77**, Intern.Inst. of Intellectual Co-operation, League of Nations (1938)

[167] Cartan E., Ann. Ec. Norm. **42**, 17 (1925)

[168] Palatini A., Rend. Circ. Mat. Palermo **43**, 203 (1919)

[169] Arnold V. I., *Mathematical Methods of Classical Mechanics*, Springer Verlag 1978 Berlin

[170] Appelquist T., Chodos A., Freund P. G. O., *Modern Kaluza-Klein Theories* 1978 Addison-Wesley Reading

[171] Kaku M., Quantum Field Theory, 1933 Oxford Univ. Press, Oxford

[172] Green M. B., Schwarz J. H., Witten E., *Superstring Theory*, Cambridge Univ. Press 1987 Cambridge

[173] Rubano C., Scudellaro P., Gen. Rel. Grav. **37**, 521 (2005)

[174] Schmidt H. J., ArXiv: 0602017 [gr-qc]

[175] Ferraris M., Francaviglia M., Mechanics, Analysis and Geometry: 200 Years after Lagrange, Editor: M. Francaviglia, Elsevier Science Publishers 1991

[176] Capozziello S., de Ritis R., Class. Quant. Grav. **11**, 107 (1994)

[177] Magnano G., Sokołowski L. M., Phys. Rev. D **50**, 5039 (1994)

[178] Allemandi G., Capone M., Capozziello S., Francaviglia M., Gen. Rel. Grav. **38**, 33 (2006)

[179] Ferraris M, Francaviglia M., Volovich I., Class. Quantum Grav. **11**, 1505 (1994)

[180] Capozziello S., Lambiase G., Stornaiolo C., Ann. Phys. **10**, 713 (2001)

[181] Friedrichs K., Math. Ann. **98**, 566 (1927)

[182] Kilmister C. W., J. Math. Phys. **12**, 1 (1963)

[183] J. O'Hanlon, *J. Phys. A* **5**, 803, (1972).

[184] M. Tegmark, et al., Astrophys. J., **606**, 702, (2004).

[185] T.M. Davis *et al.*, Astrophys. J., **666**, 716, (2007).

[186] M. Kowalski, et al., Astrophys. J., **686**, 749, (2008).

[187] M. Hicken et al., Astroplys. J., **700**, 1097, (2009).

[188] E. Komatsu, et al., Astrophys. J. Suplem., **180**, 330, (2009).

[189] G. Hinshaw, et al., Astrophys. J. Suplem., **180**, 225, (2009).

[190] E. Komatsu, et al., Astrophys. J. Suplem., **192**, 18, (2011).

[191] J. A. S. Lima and J. S. Alcaniz, Mon. Not. Roy. Astron. Soc. **317**, 893 (2000).

[192] J. F. Jesus and J. V. Cunha, Astrophys. J. Lett. **690**, L85 (2009).

[193] S. Basilakos and M. Plionis, Astrophys. J. Lett, **714**, 185 (2010).

[194] B. Ratra and P. J. E. Peebles, Phys. Rev D., **37**, 3406 (1988).

[195] S. Capozziello, Int.J.Mod.Phys. **D 11**, 483 (2002).

[196] S. Capozziello, M. Francaviglia Gen.Rel.Grav. **40**, 357 (2008).

[197] S. Capozziello, M. De Laurentis, Phys. Rept. **509**, 167 (2011).

[198] S. Nojiri, S. D. Odintsov , Phys.Rept. **505**, 59 (2011).

[199] M. Ozer and O. Taha, Nucl. Phys. B **287**, 776 (1987).

[200] S. Weinberg, Rev. Mod. Phys., **61**, 1, (1989).

[201] W. Chen, Y-S. Wu, Phys. Rev. D **41**, 695 (1990).

[202] J. C. Carvalho, J. A. S. Lima and I. Waga, Phys. Rev. D **46**, 2404 (1992).

[203] J. A. S. Lima and J. M. F. Maia, Phys. Rev D **49**, 5597 (1994).

[204] J. A. S. Lima, Phys. Rev. D **54**, 2571 (1996), [gr-qc/9605055].

[205] A. I. Arbab and A. M. M. Abdel-Rahman, Phys. Rev. D **50**, 7725 (1994).

[206] J. M. Overduin and F. I. Cooperstock, Phys. Rev. D **58**, 043506 (1998).

[207] S. Basilakos, M. Plionis and S. Solà, Phys. Rev. D. **80**, 3511 (2009).

[208] C. Wetterich, Astron. Astrophys. **301**, 321 (1995).

[209] P. Brax, and J. Martin, Phys. Lett. **B468**, 40 (1999).

[210] A. Feinstein, Phys. Rev. D., **66**, 063511 (2002).

[211] M. C. Bento, O. Bertolami, and A.A. Sen, Phys. Rev. D., **70**, 083519 (2004).

[212] L. P. Chimento, and A. Feinstein, Mod. Phys. Lett. A, **19**, 761 (2004).

[213] E. V. Linder, Rep. Prog. Phys., **71**, 056901 (2008).

[214] J. A. S. Lima, F. E. Silva and R. C. Santos, Class. Quant. Grav. **25**, 205006 (2008).

[215] A. W. Brookfield, C. van de Bruck, D.F. Mota, and D. Tocchini-Valentini, Phys. Rev. Lett. **96**, 061301 (2006).

[216] C. G. Boehmer, and T. Harko, Eur. Phys. J. **C50**, 423 (2007).

[217] A. A. Starobinsky, JETP Lett. **86**, 157 (2007).

[218] R. R. Caldwell and M. Kamionkowski, Ann.Rev.Nucl.Part.Sci., **59**, 397, (2009), arXiv:0903.0866.

[219] I. Sawicki and W. Hu, Phys. Rev. D., **75**, 127502 (2007).

[220] L. Amendola and S. Tsujikawa, *Dark Energy Theory and Observations*, Cambridge University Press, Cambridge UK, (2010).

[221] K. Bamba, S. Capozziello, S. Nojiri, S. D. Odintsov, *Astrophys. Space Sci.* **342** 155 (2012), arXiv:1205.3421 [gr-qc].

[222] P. Jordan, *Naturwiss.* **26**, 417 (1938).

[223] P. Jordan, *Schwerkraft und Weltfall, Grundlagen der Theoretische Kosmologie*. Braunschweig: Vieweg und Sohns (1952).

[224] M. Tsamparlis, A. Paliathanasis, S. Basilakos, S. Capozziello, Gen. Rel. Grav. **45**, 2003 (2013).

[225] S. Capozziello, P. Martin-Moruno, C. Rubano, Phys. Lett. **B 689**, 117 (2010).

[226] H.-J. Schmidt, D. Singleton, J. Math. Phys. **54**, 062502 (2013).

[227] R.M. Wald, General Relativity (Chicago University Press, Chicago, 1984).

[228] S. Capozziello and M. De Laurentis, Found. Phys. **40** 867 (2010).

[229] S. Capozziello, M. De Laurentis, S. Nojiri, S.D. Odintsov, Phys.Rev. D **79**, 124007 (2009).

[230] S. Capozziello, A. Stabile, Class. Quant. Grav. **26**, 085019 (2009).

[231] A. Stabile, Phys. Rev. D **82**, 064021 (2010).

[232] A. Stabile, Phys. Rev. D **82**, 124026 (2010).

[233] A. Stabile, S. Capozziello, Phys. Rev. D **87**, 064002 (2013).

[234] S. Capozziello, A. Stabile, A. Troisi, Phys. Rev. **D 76**, 104019 (2007).

[235] R. Dick, Gen. Rel. Grav. **36**, 217 (2004).

[236] K. Stelle, Gen. Relat. Grav. **9**, 353 (1978).

[237] H.-J. Schmidt, Astron. Nachr. **307**, 339 (1986).

[238] I. Navarro, K. Van Acoleyen, Phys. Lett. **B 622**, 1 (2005).

[239] J. O'Hanlon, Phys. Rev. Lett. **29, 2**, 137 (1972).

[240] Astier P., et al., A&A **447**, 3 (2006)

[241] Clocchiati A., Ap. J. **642**, 1 (2006)

[242] Olmo G. J., Phys. Rev. D **72**, 083505 (2005)

[243] Olmo G. J., Phys. Rev. Lett. **95**, 261102 (2005)

[244] Olmo G. J., Phys. Rev. D **75**, 023511 (2007)

[245] Damour T., Esposito-Farèse G., Class. Quant. Grav. **9**, 2093 (1992)

[246] Clifton T., Phys. Rev. D **77**, 024041 (2008)

[247] Nojiri S., Odintsov S.D., Phys. Lett. B **631**, 1 (2005)

Pogosian L., Silvestri A., Phys. Rev. D **77**, 023503 (2008) [Erratum-ibid. D 81 (2010) 049901]

[248] Bertschinger E., Zukin P., Phys. Rev. D **78** 024015 (2008)

[249] Carloni S., Dunsby P.K.S., Troisi A., Phys. Rev. D **77** 024024 (2008)

[250] Nzioki A.M., Dunsby P.K.S., Goswami R., Carloni S., Phys. Rev. D **83** 024030 (2011)

[251] Capozziello S., Stabile A., Troisi A., International Journal of Theoretical Physics **49**, 1251 (2010)

[252] Boehmer C.G., Harko T:, Lobo F.S.N., Astropart. Phys. **29**, 386 (2008)

[253] Boehmer C.G., Harko T:, Lobo F.S.N., JCAP **0803**, 024 (2008)

[254] Capozziello S., Cardone V.F., Troisi A., Phys.Rev. D **73**, 104019 (2006)

[255] Schneider P., Ehlers J., Falco E.E., *Gravitational Lenses*, Springer (1999)

[256] Lubini M., Tortora C., Näf J., Jetzer Ph., Capozziello S., arXiv:1104.2851v1 [gr-qc]

[257] Capozziello S., De Laurentis M., *Phyisics Reports* **509**, 167 (2011).

[258] Capozziello S., Francaviglia M., *Gen. Rel. Grav.* **40**,357, (2008).

[259] Capozziello S., De Laurentis M., Faraoni V., *The Open Astr. Jour* **2**, 1874 (2009).

[260] Capozziello S., Faraoni V., *Beyond Einstein gravity: A Survey of gravitational theories for cosmology and astrophysics*, Fundamental Theories of Physics, Vol. 170, Springer, New York (2010).

[261] Quandt I., Schmidt H.J., *Astron. Nachr.* **312**, 97 (1991). Teyssandier P., Class. Quant. Grav. **6**, 219 (1989). S. Capozziello, G. Lambiase, Int. J. Mod. Phys. D **12**, 843 (2003). S. Calchi Novati, S. Capozziello, G. Lambiase, Grav. Cosmol. **6**, 173 (2000).

[262] Capozziello S., De Laurentis M., *Annalen Phys.* **524**, 545 (2012).

[263] Altschul B, Bailey Q.G., et al. to appear in *Advances in Space Research* (2014) arXiv:1404.4307 [gr-qc].

[264] A. Connes, *Noncommutative Geometry*, Academic Press, New York (1994)

[265] A. Connes and M. Marcolli, *Noncommutative Geometry, Quantum Fields and Motives*, Hindustan Book Agency, India (2008)

[266] A. H. Chamseddine and A. Connes, Commun. Math. Phys. **186** (1997) 731 [hep-th/9606001].

[267] M. Sakellariadou, Int. J. Mod. Phys. D **20**, 785 (2011) [arXiv:1008.5348 [hep-th]].

[268] M. Sakellariadou,PoS CORFU **2011**, 053 (2011) [arXiv:1204.5772 [hep-th]].

[269] K. van den Dungen and W. D. van Suijlekom, Rev. Math. Phys. **24** (2012) 1230004 [arXiv:1204.0328 [hep-th]].

[270] A. H. Chamseddine, A. Connes and M. Marcolli, Adv. Theor. Math. Phys. **11**, 991 (2007) [arXiv:hep-th/0610241].

[271] A. H. Chamseddine and A. Connes, JHEP **1209**, 104 (2012) [arXiv:1208.1030 [hep-ph]].

[272] A. H. Chamseddine and A. Connes, Phys. Rev. Lett. **99** (2007) 191601 [arXiv:0706.3690 [hep-th]].

[273] M. Sakellariadou, A. Stabile and G. Vitiello, Phys. Rev. D **84**, 045026 (2011) [arXiv:1106.4164 [math-ph]].

[274] M. V. Gargiulo, M. Sakellariadou and G. Vitiello, [arXiv:1305.0659 [hep-th]].

[275] A. Devastato, F. Lizzi and P. Martinetti, [arXiv:1304.0415 [hep-th]].

[276] A. H. Chamseddine, A. Connes and W. D. van Suijlekom, arXiv:1304.8050 [hep-th].

[277] A. H. Chamseddine and A. Connes, J. Math. Phys. **47** (2006) 063504 [hep-th/0512169].

[278] A. H. Chamseddine and A. Connes, Commun. Math. Phys. **293** (2010) 867 [arXiv:0812.0165 [hep-th]].

[279] W. Nelson and M. Sakellariadou, Phys. Rev. D **81**, 085038 (2010) [arXiv:0812.1657 [hep-th]].

[280] W. Nelson and M. Sakellariadou, Phys. Lett. B **680**, 263 (2009) [arXiv:0903.1520 [hep-th]].

[281] M. Marcolli and E. Pierpaoli, Adv. Theor. Math. Phys. **14** (2010) [arXiv:0908.3683 [hep-th]].

[282] M. Buck, M. Fairbairn and M. Sakellariadou, Phys. Rev. D **82**, 043509 (2010) [arXiv:1005.1188 [hep-th]].

[283] J. F. Donoghue, Phys. Rev. D **50** (1994) 3874 [gr-qc/9405057].

[284] Stabile A., Phys. Rev. D **82**, 064021 (2010).

[285] Stabile A., Stabile An., Phys. Rev. D **85**, 044014 (2012).

[286] Stabile A., Scelza G., Phys. Rev. D **84**, 124023 (2011).

[287] Sanders R. H., Astron. Astrophys. **136**, L21 (1984).

[288] M. Milgrom, Astrophys. J. **270** (1983) 365.

[289] J. D. Bekenstein, Phys. Rev. D **70** (2004) 083509 [Erratum-ibid. D **71** (2005) 069901] [astro-ph/0403694].

[290] I. Ferreras, M. Sakellariadou and M. F. Yusaf, Phys. Rev. Lett. **100** (2008) 031302 [arXiv:0709.3189 [astro-ph]].

[291] N. E. Mavromatos, M. Sakellariadou and M. F. Yusaf, Phys. Rev. D **79** (2009) 081301 [arXiv:0901.3932 [astro-ph.GA]].

[292] I. Ferreras, N. E. Mavromatos, M. Sakellariadou and M. F. Yusaf, Phys. Rev. D **80** (2009) 103506 [arXiv:0907.1463 [astro-ph.GA]].

[293] I. Ferreras, N. E. Mavromatos, M. Sakellariadou and M. F. Yusaf, Phys. Rev. D **86** (2012) 083507 [arXiv:1205.4880 [astro-ph.CO]].

[294] N. Mavromatos and M. Sakellariadou, Phys. Lett. B **652** (2007) 97 [hep-th/0703156 [HEP-TH]].

[295] N. E. Mavromatos, M. Sakellariadou and M. F. Yusaf, JCAP **1303** (2013) 015 [arXiv:1211.1726 [hep-th]].

[296] Napolitano N. R., Capozziello S., Romanowsky A. J., Capaccioli M., Tortora C., ApJ **748**, 87 (2012).

[297] Roy A.E., Orbital Motion, IOP Publishing, Bristol (2005)

[298] Everitt C.W.F., *et al.*, Phys. Rev. Lett. **106**, 221101 (2011).

[299] Lambiase G., Sakellariadou M., Stabile An., JCAP **12**, 020 (2013).

[300] http://www.asi.it/it/attivita/cosmologia/lares

[301] W. Nelson, J. Ochoa and M. Sakellariadou, Phys. Rev. Lett. **105**, 101602 (2010) [arXiv:1005.4279 [hep-th]].

[302] Besides GP-B and LARES experiments, it should be also mentioned GINGER experiment [303], which is an Earth based experiment that aims to evaluate the response to the gravitational field of a ring laser array. GINGER forthcoming data will therefore allow to determine independent constraints on the parameters characterzing theories that generalize GR (see *e.g.* [304]).

[303] See for example http://www.df.unipi.it/ginger.

[304] N. Radicella, G. Lambiase, L. Parisi, and G. Vilasi, arXiv:1408.1247 [gr-qc].

[305] Riess A.G. *et al.*, Observational Evidence from Supernovae for an Accelerating Universe. *The Astronomical Journal* 1998, **116**, 1009-1038.

[306] Perlmutter S. *et al.*, Masurements of Ω and Λ form 42 High-redshift Supernovae. *The Astrophysical Journal* 1999, **517**, 565-586.

[307] Cole S. *et al.*, The 2dF Galaxy Redshift Survey: Power-spectrum analysis of the final dataset and cosmological implications. *Monthly Notices of the Royal Astronomical Society* 2005, **362**, 505-534.

[308] Spergel D.N. *et al.*, Three-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Implications for Cosmology. *Astrophysical Journal Supplement Series* 2007, **170**, 377-408.

[309] Carroll S.M.; Press W.H.; Turner E.L., The cosmological Constant. *Annual Review of Astronomy and Astrophysics* 1992, **30**, 499-542.

[310] Sahni V.; Starobinski A., The Case for a Positive Cosmological Λ-term. *International Journal of Modern Physics D* 2000, **9**, 373-443.

[311] Navarro J.F.; Frenk C.S.; White S.D.M., The Structure of Cold Dark Matter Halos. *Astrophysical Journal* 1996, **462**, 563-575.

[312] Capozziello S.; De Laurentis M., Extended theories of gravity. *Physics Reports* 2011, **509**, 167-321.

[313] Nojiri S.; Odintsov S.D., Properties of singularities in the (phantom) dark energy universe. *Physics Reports* 2011, **505**, 59-104.

[314] Olmo G., Palatini Approach to Modified Gravity: $f(R)$ Theories and Beyond. *International Journal of Modern Physics D* 2011, **20**, 413-462.

[315] Weyl H., Reine Infinitesimalgeometrie. *Mathematische Zeitschrift* 1918, **2**, 384-411.

[316] Weyl H., *Raum zeit Materie: Vorlesungen uuber allgemeine Relativitatstheorie*, Springer, Berlin, 1921.

[317] Pauli W., Zur Theorie der Gravitation und der Elektrizitat von Hermann Weyl. *Phys. Zeit.* 1919, **20**, 457-467.

[318] Bach R., Zur Weylschen Relativitatstheorie und der Weylschen Erweiterung des Krummungstensorbegriffs. *Mathematische Zeitschrift* 1921, **9**, 110-135.

[319] Eddington A.S., *The Mathematical Theory of Relativity*, Cambridge University Press London, 1924.

[320] Lanczos C., Elektromagnetismus als natrliche Eigenschaft der Riemannschen Geometrie. *Zeitschrift fur Physik A Hadrons and Nuclei* 1932, **73**, 147-168.

[321] Buchdahl H.A., On the Gravitational Field Equations Arising from the Square of the Gaussian Curvature. *Il Nuovo Cimento* 1962, **23**, 141-157.

[322] Bicknell G.V., Non-viability of gravitational theory based on a quadratic lagrangian. *Journal of Physics A: Mathematical, Nuclear and General* 1974, **7**, 1061.

[323] Guth A., Inflationary universe: A possible solution to the horizon and flatness problems. *Physical Review D* 1981, **23**, 347-356.

[324] Capozziello S.; Faraoni V., *Beyond Einstein Gravity*, Fundamental Theories of Physics Vol. 170, Springer, Dordrecht, 2011.

[325] Magnano G.; Ferraris M.; Francaviglia M., Nonlinear Gravitational Lagrangians. *General Relativity and Gravitation* 1987, **19**, 465-479.

[326] Amarzguioui M.; Elgaroy O.; Mota D. F.; Multamaki T., Cosmological Constraints on $f(R)$ Gravity Theories within the Palatini Approach. *Astronomy and Astrophysics* 2006, **454**, 707-714.

[327] Allemandi G.; Borowiec A.; Francaviglia M., Accelerated Cosmological Models in Ricci Squared Gravity. *Physical Review D* 2004, **70**, 103503(13).

[328] Sotiriou T.P., Constraining $f(R)$ Gravity in the Palatini Formalism. *Classical and Quantum Gravity* 2006, **23**, 1253.

[329] Sotiriou T.P.; Liberati S., Metric-affine $f(R)$ Theories of Gravity. *Annals of Physics* 2007, **322**, 935-966.

[330] Damour T.; Esposito - Farèse G., Tensor-multi-scalar Theories of Gravitation. *Classical and Quantum Gravity* 1992, **9**, 2093.

[331] Stabile A., Stabile An., Capozziello S., Conformal transformations and weak field limit of scalar-tensor gravity. *Physical Review D* 2013, **88**, 124011 (9).

[332] Olmo G.J., Post-Newtonian constraints on $f(R)$ cosmologies in metric and Palatini formalism. *Physical Review D* 2005, **72**, 08350589(17).

[333] Olmo G.J., The Gravity Lagrangian According to Solar System Experiments. *Physics Review Letters* 2005, **95**, 261102(4).

[334] Olmo G.J., Limit to General Relativity in $f(R)$ Theories of Gravity. *Physical Review D* 2007, **75**, 023511(8).

[335] Clifton T., Parametrized post-Newtonian Limit of Fourth-order Theories of Gravity. *Physical Review D* 2008, **77**, 024041(11).

[336] Nojiri S.; Odintsov S.D., Modified Gauss - Bonnet Theory as Gravitational Alternative for Dark Energy. *Physics Letters B* 2005, **631**, 1.

[337] Capozziello S.; Stabile A.; Troisi A., Newtonian Limit of $f(R)$ Gravity. *Physical Review D* 2007, **76**, 104019(12).

[338] Stabile A., Post-Newtonian Limit of $f(R)$ Gravity in the Harmonic Gauge. *Physical Review D* 2010, **82**, 064021(12).

[339] Stabile A., Most General Fourth-Order Theory of Gravity at Low Energy. *Physical Review D* 2010, **82**, 124026(7).

[340] Capozziello S.; Stabile A., The Newtonian Limit of Metric Gravity Theories with Quadratic Lagrangians. *Classical and Quantum Gravity* 2009, **26**, 085019.

[341] Capozziello S.; Cardone V.F.; Carloni S.; Troisi A.; Can Higher Order Curvature Theories Explain Rotation Curves of Galaxies?. *Physicis Letters A* 2004, **326**, 292-296.

[342] Capozziello S.; Stabile A.; Troisi A.; The Post-Minkowskian Limit of $f(R)$ Gravity. *International Journal of Theoretical Physics* 2010, **49**, 1251-1261.

[343] De Laurentis M.; Capozziello S., Quadrupolar Gravitational Radiation as a Test-Bed for $f(R)$ Gravity. *Astroparticle Physics*, **35**, 257-265.

[344] Capozziello S.; Cardone V.F.; Troisi A., Low Surface Brightness Galaxy Rotation Curves in the Low Energy Limit of R^n Gravity: No Need for Dark Matter? *Monthly Notices of the Royal Astronomical Society* 2007, **375**, 1423-1440.

[345] Boehmer C.G.; Harko T; Lobo F.S.N., Dark Matter as a Geometric Effect in $f(R)$ Gravity. *Astroparticle Physics* 2008, **29**, 386-392.

[346] Boehmer C.G.; Harko T; Lobo F.S.N., The Generalized Virial Theorem in $f(R)$ Gravity. *Journal of Cosmology and Astroparticle Physics* 2008, **0803**, 024.

[347] Stabile A.; Scelza G., Rotation Curves of Galaxies by Fourth Order Gravity. *Physical Review D* 2012, **84**, 124023(11).

[348] Landau Lev D., Lifšits E. M, *Theoretical physics* vol. II

[349] Stabile A., Capozziello S., Galaxy rotation curves in $-f(R,\phi)$ Gravity. *Physical Review D* 2013, **87**, 064002 (13).

[350] StabileA., Capozziello S., Post-Minkowskian Limit and Gravitational Waves Solutions of Fourth Order Gravity: a complete study. arXiv: 1501.02187

[351] Capozziello S., Lambiase G., Sakellariadou M., Stabile A., Stabile An., Constraining Models of Extended Gravity Using Gravity Probe B and LARES Experiments. *Physical Review D* 2015, **91**, 044012 (15).

[352] Prakash S., Solution of Generalized wave equations of Order Higher than 2, *Proc. Math. Sci.* 1950, **31**, 240 (21).

[353] Weinberg S., *Gravitation and Cosmology*, Wiley 1972, New York

[354] Nelson W., Ochoa J.m Sakellariadou M., Gravitational waves in the spectral action of noncommutative geometry William. *Physical Review D* 2010, **82**, 085021 (14).

[355] Nelson W., Ochoa J.m Sakellariadou M., Constraining the Noncommutative Spectral Action via Astrophysical Observations. *Physical Review Letters* 2010, **105**, 101602 (4).

[356] Stabile A., Stabile An., Weak Gravitational Lensing in Fourth Order Gravity. *Phyical Review D* 2012 **85**, 044014 (10).

[357] Antoniadis J. et al, A Massive Pulsar in a Compact Relativistic Binary. *Science* 2013, **340**, 1233232 (10).

[358] Hulse R.A., Taylor J.H., Discovery of a pulsar in a binary system. *Astrophysical Journal* 1975, **195**, L51 (3).

[359] Weisberg J.M., Nice D.J., Taylor J.H., Timing Measurements of the Relativistic Binary Pulsar PSR B1913+16. *Astrophysical Journal* 2010, **722**, 1030 (5).

[360] Starobinskii, A.A., The Perturbation Spectrum Evolving from a Nonsingular Initially De-Sitter Cosmology and the Microwave Background Anisotropy. *Soviet Astronomy Letters* 1983, **9**, 302 (6).

[361] Connes A., *Noncommutative Geometry*, Academic Press, New York (1994).

[362] Connes A., Marcolli M., *Noncommutative Geometry, Quantum Fields and Motives*, Hindustan Book Agency, India (2008)

[363] Chamseddine A.H., Connes A., The Spectral Action Principle. *Communications in Mathematical Physics* 1997, **186**, 731-750.

[364] Lambiase G., Sakellariadou M., Stabile An., Constraints on noncommutative spectral action from Gravity Probe B and torsion balance experiments. *Journal of Cosmology and Astroparticle Physics* 2013, **12**, 020 ().

[365] Antoniadis J. et al, *Science* 2013, **340**, 1233232 (10).

[366] Hulse R.A., Taylor J.H., *Astrophysical Journal* 1975, **195**, L51 (3).

[367] Weisberg J.M., Nice D.J., Taylor J.H., *Astrophysical Journal* 2010, **722**, 1030 (5).