Università degli Studi di Salerno
Dipartimento di Matematica

Corso di Dottorato di Ricerca in Matematica (XIII ciclo)

Tesi di dottorato in Algebra

A problem in the Theory of Groups and a question related to Fibonacci-Like sequences

Coordinatore e tutor:
Prof.ssa *Patrizia Longobardi*

Co-tutor:
Prof.ssa *Mercede Maj*

Candidato:
Dott.ssa *Serena Siani*

a.a. 2015-2016

Contents

Introduction

This thesis is composed of four chapters, two of which, Chapter 3 and Chapter 4, contain original results. In Chapter 1 we recall some basic notions and establish some of the notation and terminology which will be used in the sequel. For example, we recall some useful results about the class X of groups which are isomorphic to their non-abelian subgroups. Every group of this class is infinite and 2-generated. This class of groups has been studied by H. Smith and J. Wiegold([34]). They proved that every insoluble X-group is centre-by-simple and they gave a complete characterization of soluble X-groups. Then we recall some results about finitely generated groups which are isomorphic to their non-trivial normal subgroups. In particular, we will use the result proved by J.C. Lennox, H. Smith and J. Wiegold in [17], for which if G is a finitely generated infinite group that is isomorphic to all its non-trivial normal subgroups and which contains a proper normal subgroup of finite index, then $G \simeq \mathbb{Z}$.

Given a group G, a subgroup K of G is said to be a *derived subgroup* or *commutator subgroup* in G if $K = H'$ where H' is the derived subgroup of H, with H subgroup of G. Recently, many authors have been interested in studying the set of derived subgroups in the lattice of all subgroups.

Let $C(G)$ denote the set of all derived subgroups in G:

$$C(G) = \{H' | H \leq G\}.$$

The influence of $C(G)$ on the structure of the group G has been studied by many authors. For example, F. de Giovanni and D.J.S. Robinson in[8] and M. Herzog, P. Longobardi and M. Maj in[14], have investigated groups G for which $C(G)$ is finite. In particular, they proved that if G is locally graded, then $C(G)$ is finite if and only if G' is finite.

Let n be a positive integer and let D_n denote the class of groups with n isomorphism types of derived subgroups. Clearly D_1 is the class of all abelian groups and a group G belongs to D_2 if and only if G is non abelian and $H' \simeq G'$ whenever H is a non abelian subgroup of G. P. Longobardi, M. Maj, D.J.S. Robinson and H. Smith in [18] focused their attention on groups in D_2 and described in a precise way some large classes of D_2-groups.

In Chapter 2 we recall some results about D_2-groups.

In this thesis we analyse a dual problem. Let $B(G)$ denote the set of the central factors of all subgroups of a group G:

$$B(G) = \{\frac{H}{Z(H)} | H \leq G\}$$

and let B_n denote the class of groups for which the elements of $B(G)$ fall into at most n isomorphism classes, where n is a positive integer. Clearly B_1 is the class of abelian groups and G is a B_2-group if and only if every subgroup of G is abelian or $\frac{H}{Z(H)} \simeq \frac{G}{Z(G)}$ for all non abelian subgroups H of G.

In Chapter 3 of this thesis we study B_2-groups. For example, it is possible to see that if G is a group where $\frac{G}{Z(G)}$ is elementary abelian of order p^2, with p prime, then G is in B_2. Moreover, if G is a group with $\frac{G}{Z(G)} \simeq \mathbb{Z} \times \mathbb{Z}$, then $G \in B_2$. Groups in B_2 can be very complicated, in fact a non abelian group whose proper subgroups are all abelian is also a B_2-group and so Tarski Monster Groups, infinite simple groups with all proper subgroups abelian, whose existence was proved by A.Yu. Ol'shankii in 1979, are B_2-groups. First we proved some elementary results for B_2-groups. For example it may be seen that the class of B_2-groups is closed under the formation of subgroups and not closed under the formation of homomorphic images but if G is a nilpotent group in B_2 then $\frac{G}{S} \in B_2$, for any $S \leq Z(G)$. In addition, $\frac{G}{Z(G)}$ is 2-generated for every G in B_2 and if G is also nilpotent, then $\frac{G}{Z(G)}$ is abelian. Then we analyse nilpotent B_2-groups and we prove that if G is non abelian, then G is a nilpotent group in B_2 if and only if either $\frac{G}{Z(G)}$ is elementary abelian of order p^2, where p is a prime, or $\frac{G}{Z(G)}$ is the direct product of two infinite cyclic groups. We also study locally finite groups in B_2 and we show that if G is locally finite, then G is in B_2 if and only if $G = Z(G)H$ where H is finite, minimal non abelian. Then we study soluble groups. We show that if G is a soluble non nilpotent group in B_2, then G is metabelian and in this hypothesis we prove that $Z(\frac{G}{Z(G)}) = 1$, $G = A < x >$, for a suitable x in G and a normal abelian subgroup A of G, and every non abelian subgroup of $\frac{G}{Z(G)}$ is isomorphic to $\frac{G}{Z(G)}$. Finally, we analyse the case of non soluble B_2-groups and we prove that they do not satisfy the *Tits alternative*, i.e. soluble by finite groups or groups that contain a free subgroup of rank 2. Up to this point none of the special types of B_2-groups we have analysed has involved *Tarski groups*. But in this last case we have proved that if G is a non soluble B_2 group and G' satisfies the minimal condition on subgroups, then $\frac{G}{Z(G)}$ is simple, minimal non abelian, every soluble subgroup of G is abelian and if N is a normal subgroup of G, then either $N \leq Z(G)$ or $G' \leq N$. In particular $\frac{G}{Z(G)}$ is a Tarski group.

In Chapter 4 we show a result about Fibonacci-like sequences, obtained in collaboration with Professor Giovanni Vincenzi. This results appear in a published paper, *Fibonacci-like sequences and generalized Pascal's triangle.* We have studied the properties pertaining to diagonals of generalized Pascal's triangles $T(k_1, k_2)$ created using two complex numbers. We have also introduced a particular Fibonacci-like sequence $\{H_n\}_{n\in\mathbb{N}}$ whose seeds are the complex numbers considerated above. As in the case of Pascal's triangle, we have found a relationship between the Fibonacci sequence $\{F_n\}_{n\in\mathbb{N}}$ and the sequence $\{D_n\}_{n\in\mathbb{N}}$ of diagonals we have created.

In particular we have proved that the sequence $\{D_n\}_{n\in\mathbb{N}}$ of the numbers which arise when we consider the diagonals of a generalized $T(k_1, k_2)$ is recursive and that the following relationship holds:

Theorem Let k_1 and k_2 be complex numbers. Let $\{D_n\}_{n\in\mathbb{N}}$ be the associate sequence to the generalized Pascal's triangle $T(k_1, k_2)$ and $\{H_n\}_{n\in\mathbb{N}}$ be the Fibonacci-like sequence of seeds k_1 and k_2. Then the following identity holds:

$$H_n - D_n = F_{n-3}(k_2 - k_1), \forall n \in \mathbb{N}.$$

Chapter 1

Preliminaries

The purpose of this chapter is to recall some basic notions and to establish some of the notation and terminology which will be used in the sequel.

1.1 Basic concepts and definitions

Let n be a positive integer and let $x_1, x_2, \ldots, x_n$ be elements of a group G. We remind that the *commutator* of x_1 and x_2 is defined by

$$[x_1, x_2] = x_1^{-1} x_2^{-1} x_1 x_2 = x_1^{-1} x_1^{x_2},$$

while for $n > 2$ a *simple commutator of weight* n is defined recursively by the rule

$$[x_1, \ldots, x_n] = [[x_1, \ldots, x_{n-1}], x_n].$$

By convention $[x_1, \ldots, x_n] = x_1$ if $n = 1$.

For every $x, y \in G$ we use the symbol

$$[x,_n y]$$

to denote the simple commutator of weight $n+1$ of x and y, where y appears n times on the right. We also assume $[x,_0 y] = x$.

In the following lemma we summarize the standard commutator properties (see [27]).

Lemma 1.1.1. *Let x, y, z be elements of a group. Then:*

1) $[x, y] = [y, x]^{-1}$;

2) $[xy, z] = [x, z]^y[y, z] = [x, z][x, z, y][y, z]$;

3) $[x, yz] = [x, z][x, y]^z = [x, z][x, y][x, y, z]$;

4) $[x, y^{-1}] = ([x, y]^{y^{-1}})^{-1}$;

5) $[x^{-1}, y] = ([x, y]^{x^{-1}})^{-1}$;

6) $[x, y^{-1}, z]^y[y, z^{-1}, x]^z[z, x^{-1}, y]^x = 1$ *(the Hall-Witt identity);*

7) $[x, y, z^x][z, x, y^z][y, z, x^y] = 1$ *(the Jacobi identity).*

Let $X_1, X_2, \ldots, X_n$ be nonempty subsets of a group G, then $[X_1, X_2]$ denote the *commutator subgroup* of X_1 and X_2, namely the subgroup generated by the set of all commutators of elements of X_1 with elements of X_2:

$$[X_1, X_2] =< [x_1, x_2] | x_1 \in X_1, x_2 \in X_2 > .$$

More generally, let

$$[X_1, \ldots, X_n] = [[X_1, \ldots, X_{n-1}], X_n]$$

where $n > 2$. Moreover, we recall that $G' = [G, G]$ is the *derived subgroup* of the group G, being generated by all commutators in G, and the sequence

$$G = G^{(0)} \geq G^{(1)} \geq G^{(2)} \geq \cdots ,$$

where $G^{(n+1)} = (G^{(n)})'$ for every $n \geq 0$, is called the *derived series* of G, although it need not reach 1 or even terminate.

A group G is said to be *solvable* of *derived length* at most n if $G^{(n)} = 1$. In particular, a solvable group with derived length at most 2 is said to be *metabelian.*

We remind that the *lower central series* of a group G is the descending sequence of the commutator subgroups by repeatedly commuting with G:

$$G = \gamma_1(G) \geq \gamma_2(G) \geq \cdots ,$$

in which $\gamma_{n+1}(G) = [\gamma_n(G), G]$ for every $n \geq 1$. Like the derived series the lower central series does not in general reach 1. Instead, the *upper central series* of a group G is the ascending sequence of subgroups that is dual to the lower central series in the same sense that the centre is dual to the commutator subgroup:

$$1 = Z_0(G) \leq Z_1(G) \leq Z_2(G) \leq \cdots ,$$

defined by $Z_{n+1}(G)/Z_n(G) = Z(G/Z_n(G))$ for every $n \geq 0$. Of course $Z_1(G) = Z(G)$ is the centre of G and each $Z_n(G)$ is called the *nth centre* of G. This series need not reach G, but if G is finite, the series terminates at a subgroup.
For infinite groups, one can extend the two series to infinite ordinal numbers via transfinite recursion: if α is a limit ordinal, then the subgroups $\gamma_\alpha(G)$ and $Z_\alpha(G)$ (also called the *α-centre* of G) are defined by the rules

$$\gamma_\alpha(G) = \bigcap_{\lambda<\alpha} \gamma_\lambda(G)$$

and

$$Z_\alpha(G) = \bigcup_{\lambda<\alpha} Z_\lambda(G).$$

Since the cardinality of G cannot be exceeded, there exists a cardinal β at which the upper central series stabilizes. The terminal group $\overline{Z}(G) = Z_\beta$ is called the *hypercentre* of G.

A group G is said to be *nilpotent* if the lower central series reaches the identity subgroup after a finite number of steps or, equivalently, the upper central series reaches the group itself after a finite number of steps. The nilpotent class is the length of the upper central series or, similarly, the length of the lower central series.

Locally graded group

A group G is called *locally graded* if every finitely generated non-trivial subgroup of G has a proper subgroup of finite index.

The class of locally graded groups is not closed with respect to forming quotients but if G is locally graded then $\frac{G}{Z(G)}$ is locally graded. More generally, the following result holds:

Proposition 1.1.1 (H. Smith [33]). *Let G be a locally graded group and H a subgroup of the centre of G. Then $\frac{G}{H}$ is locally graded.*

Proof. Let G and H be as stated and suppose, for a contradiction, that $\frac{G}{H}$ is not locally graded. Then, there exists a finitely generated subgroup F not contained in H such that $\frac{F}{F \cap H}$ has no nontrivial finite image. We may as well assume that $F = G$ and consider $\frac{\frac{G}{H}}{(\frac{G}{H})'}$ which is a finitely generated abelian group and so it has nontrivial finite image. Since $\frac{G}{H}$ has no nontrivial finite image, it follows that $\frac{G}{H} = (\frac{G}{H})'$, thus $G = HG'$. Suppose that $G =< x_1, \dots, x_n >$, then every generator can be written in the form $x_i = h_i a_i$ where $h_i \in H$ and $a_i \in G'$. Now we consider the finitely generated subgroup $K =< a_1, \dots, a_n > \leq G'$. Thus $G = HK$ and $K \leq G' = K'$, hence $G' = K$. Since $\frac{G}{H}$ is not locally graded, it follows that $G' \neq 1$. Thus there is a normal subgroup N of G' such that $\frac{G'}{N}$ is finite and nontrivial, since G is locally graded. Then $\frac{HK}{HN} \simeq \frac{\frac{HK}{H}}{\frac{HN}{H}} \simeq \frac{HNK}{HN} \simeq \frac{K}{N(H \cap K)}$ by Dedekind. This implies that $G = HN$ and thus $G' = N$ which gives a contradiction. □

Hopfian group

A group G is *Hopfian* if it is not isomorphic to any of its proper quotients.

Proposition 1.1.2. *Let G be a group satisfying the maximal condition for normal subgroups. Then G is Hopfian.*

Proof. We will prove that if G is non Hopfian, then G cannot satisfy the maximal condition for normal subgroups. Assume that G is non Hopfian and suppose that G satisfies the maximal condition for normal subgroups. Now consider $N \lhd G$, $N \neq 1$, such that $\frac{G}{N} \simeq G$, since G is non Hopfian. Then there exists

$M \lhd G$ maximal such that $\frac{G}{M} \simeq G$ and so there exists $\frac{S}{M} \lhd \frac{G}{M}$, $\frac{S}{M} \neq 1$, such that $\frac{\frac{G}{M}}{\frac{S}{M}} \simeq \frac{G}{M} \simeq G$. Therefore there exists $S \lhd G$, with $M \subset S$ and $\frac{G}{S} \simeq G$, a contradiction. □

Locally finite groups

A group G is called *locally finite* if every finitely generated subgroup of G is finite.

1.2 Groups which are isomorphic to their non-abelian subgroups

In this section we recall some useful results about groups which are isomorphic to their non-abelian subgroups.

Let Y denote the class of non-abelian groups in which all proper subgroups are abelian. In [22], G.A. Miller and H.C. Moreno classified finite Y-groups. Infinite Y-groups do exist, in fact in [24], A.Yu. Ol'shankii proved the existence of a class of infinite simple groups, called Tarski Monsters with all proper subgroups abelian. However, B. Bruno and R.E. Phillips proved that an infinite locally graded Y-group is abelian (see [5]).

We consider the related class X of groups G, which contain proper nonabelian subgroups, all of which are isomorphic to G. Clearly, every X-group is infinite and 2-generated. In [34], H. Smith and J. Wiegold studied X-groups. They proved that every insoluble X-group is centre-by-simple and they gave a complete characterisation of soluble X-groups. Their main results are as follows:

Theorem 1.2.1 (H. Smith, J. Wiegold [34]). *Let G be an insoluble X-group and let Z denote the centre of G. Then G is 2-generator and $\frac{G}{Z}$ is infinite simple. Moreover, Z is contained in every non-abelian subgroup of G.*

Theorem 1.2.2 (H. Smith, J. Wiegold [34]). *Let G be a soluble group.*

a) If $G \in X$, then G contains an abelian normal subgroup of prime index.

b) If G is nilpotent, then $G \in X$ if and only if G is isomorphic to a group having one of the following presentations (where nil-2 denotes the pair *of relations* $[a, b, b] = 1$, $[a, b, a] = 1$, *p is an arbitrary prime and k is an arbitrary positive integer):*

i) $< a, b \,|nil\text{-}2, [a, b]^p = 1 >$;

ii) $< a, b \,|nil\text{-}2, [a, b]^p = b^{p^k} = 1 >$;

iii) $< a, b \,|nil\text{-}2, [a, b]^2 = 1, b^{2^k} = [a, b] >$;

iv) $< a, b \,|nil\text{-}2, [a, b]^3 = 1, b^{3^k} = [a, b] >$.

c) If G is not nilpotent, then $G \in X$ *if and only if either*

v) $G =< A, x >$, *where A is a finite elementary abelian p-subgroup of order* p^n *which is minimal normal in G, x is of infinite order and has order q mod* $Z(G)$, *where p, q are distint primes, and for each k in the interval* $1 \leq k \leq q-1$, $\overline{x}$ *is conjugate to* $\overline{x}^k$ *or* $\overline{x}^{-k}$ *in* $GL(n, p)$, *where* $\overline{x}$ *denotes the image of x under the natural map from* $< x >$ *to* $GL(n, p)$;

or

vi) G has a normal abelian subgroup $B = A\times < b >$, *where* $A =< a_1 > \times \ldots \times < a_{p-1} >$ *is free abelian of rank* $p-1$ *and normal in G, b is of infinite order or of order* p^k, *for some non negative integer k, and is central in G, and* $G = A \rtimes < x >$ *for some x, where* $x^p = b$, $a_i^x = a_{i+1}$ *for some* $i = 1, \ldots, p-2$ *and* $a_{p-1}^x = (a_1 \ldots a_{p-1})^{-1}$, *where p is a prime at most 19.*

Remark 1.2.1.

As may be seen from Theorem 1.2.1, the class of soluble X-group is precisely that of locally graded X-groups, the factor group of a finitely generated locally graded group by its centre cannot be infinite simple (see [33]). Note that, in the case where $k = 0$, the group G described in part *vi)* of Theorem 1.2.2 is precisely the central factor group of the wreath product $\mathbb{Z}$ wr C_p. The proof of Theorem 1.2.2 requires the following result on wreath products. Its proof, in turn, depends on a substantial result from Number Theory, whose connection was pointed out by L.G. Kovács. In addition, M.W. Liebeck observed that do indeed exist pairs of primes p, q for which the conditions in *v)* hold, provided that either $n+1$ is an odd prime or n is odd and $2n+1$ is prime: if q is the prime $n+1$ (respectively $2n+1$) then there exists a prime p of order n modulo q, and the pair $(p,\, q)$ can be shown to satisfy the extra hypothesis on conjugates.

Theorem 1.2.3 (H. Smith, J. Wiegold [34])**.** *Let p be a prime and let G be the central factor group of the wreath product of an infinite cyclic group by a group of order p. Then every normal subgroup of G contained in the image of the base group is the normal closure of a single element if and only if p is at most 19.*

In order to prove Theorem 1.2.1, it is useful to show the following result:

Lemma 1.2.1. *Let G be an abelian-by-finite group with $G \in X$. Then G is metabelian.*

Proof. If G is centre-by-finite, then G' is finite and therefore abelian since G is infinite. Otherwise, there exists a noncentral normal abelian subgroup A and then, for some $g \in G$, we have $< A, g >$ nonabelian and thus isomorphic to G. It follows that $G' \simeq (< A, g >)' \subseteq A$ which is abelian, as claimed. □

Proof. of Theorem 1.2.1. Let G and Z be as stated, and let A denote the Hirsch-Plotkin radical of G. Since $G \in X$, G is 2-generator and it is non soluble by the hypothesis. Then G is not locally nilpotent, and so A is abelian. For $A = Z$, otherwise $G \simeq < A, g >$, for some $g \in G$, giving the contradiction that G is soluble since $< A, g >$ is metabelian. By Lemma 1.2.1, then $\frac{G}{Z}$ is infinite.

Now suppose, for a contradiction, that there exists a normal subgroup N of G such that $Z < N < G$. For some $g \in G \setminus N$ we have $< N, g >$ non abelian and hence isomorphic to G, and so G has a non trivial finite image. It follows that G is locally graded and hence, by Proposition 1.1.1, that $\frac{G}{Z}$ is locally graded. Now Z is also the Hirsch-Plotkin radical of N and, since $N \simeq G$ otherwise $N \leq Z$, we deduce that $\frac{N}{Z} \simeq \frac{G}{Z}$, that is, $\frac{G}{Z}$ is isomorphic to all of its non trivial normal subgroups.

Since $\frac{G}{Z}$ has a non trivial finite image, we may apply the main result of [17] to obtain the contradiction that $\frac{G}{Z}$ is cyclic. Thus $\frac{G}{Z}$ is simple and $G'Z = G$, and so $G' = G''$. But $G \simeq G'$ and so G is perfect. Thus if H is any non abelian subgroup of G we have $HZ = (HZ)' = H' = H$, and the proof is complete. □

In order to prove Theorem 1.2.2 we start with the following lemma:

Lemma 1.2.2. *Let G be a non nilpotent, centre-by-finite X-group. Then G has the structure described in part v) of Theorem 1.2.2.*

Proof. As in the proof of Lemma 1.2.1, G' is finite and therefore abelian. Since G' is not central it has a non central Sylow p-subgroup, and we may write $G = A \rtimes < x >$, where A is a finite normal abelian p-subgroup of G and x has infinite order.

Now $G' = [A, < x >]$ and so $[a, x, x] \neq 1$ for some element a of A, and we have $< [a, x], x > \simeq G$. But $[a, x]^p = [a^p, x] = 1$, since $< A^p, x >$ is certainly not isomorphic to G. It follows that A has exponent p. Suppose that x has order n mod $Z(G)$. If $n = rs$, where r, $s > 1$, then $< A, x^r >$ is not abelian and is therefore isomorphic to G. But this easily gives a contradiction, and so $n = q$, a prime.

Certainly $q \neq p$, since G is not nilpotent. Further, if A contains a proper non trivial G-invariant subgroup B then, by Maschke's Theorem, we have $A = B \times C$, where C is also non trivial and G-invariant. Now either $< B, x >$ or $< C, x >$ is isomorphic to G, another contradiction. Finally, if q does not divide k then $< A, x^k >$ is isomorphic to G and so x^k acts like $x^{\pm 1}$ on A and the conjugacy condition follows.

Now suppose that G is a group having the structure indicated, and let H be a non abelian subgroup of G. Then H contains a non trivial element b of A and an element of the form $g = ux^{\lambda}$, where $u \in A$ and $\lambda \not\equiv 0$ mod q. Since A is minimal normal we have $< b >^{<g>} = A$, and so H is normal in G and $H =< A, x^{\mu} >$, for some μ which is not a multiple of q. Clearly then $H \simeq G$, and the result follows. $\square$

All that remains here is to show that a non nilpotent group G which has an abelian normal subgroup of prime index, but which is not centre-by-finite, is an X-group if and only if it is of one of the types described in part *vi)* of the Theorem 1.2.2. We shall use the result of Theorem 1.2.3.

Firstly, we use the following observation:

Remark 1.2.2. Let $W =< u >$wr$< v >$, where u has infinite order and v has prime order p. Viewing the base group D of W in the natural way as the additive group of the group ring $\mathbb{Z} < v >$, we may regard the centre C of W as the ideal of D generated by the element $f(v) = 1 + v + \ldots + v^{p-1}$. Since f is irriducible over $\mathbb{Z}$, and hence over $\mathbb{Q}$, we have that every W-invariant subgroup of D which properly contains C is of finite index in D.

Now let $G = A < x >$ be as in *vi)*, and let H be an arbitrary non abelian subgroup of G. Then $H =< H \cap A, ax^r >$, where $a \in A$ and $(p, r) = 1$. We have $(H \cap A)^G = (H \cap A)^{<x>} = (H \cap A)^{<ax^r>}$, so that $H \cap A$ is normal in G. Applying Theorem 1.2.3 to the group $\frac{G}{<b>}$, we deduce that $H \cap A$ is the normal closure in G, and hence in H, of a single element b, say. Now $H \cap A$ has finite index in A and therefore has rank $p - 1$, while $bb^{(ax^r)} \ldots b^{(ax^r)^{p-1}} = 1$, $(ax^r)^p = x^{rp}$, and $H \cap B = (H \cap A)\times < x^{rp} >$. It follows that the assignment $a_1 \to b$, $x \to ax^r$ determines an isomorphism from G onto H.

Now assume that G is an abelian-by-finite X-group which is neither nilpotent nor centre-by-finite. We may write $G = B < x >$ for some x, where B is abelian and normal of prime index p in G. Then $x^p \in Z = Z(G)$. Consider first the case where $x^p = 1$. There is a positive integer k such that B^k is torsion free and normal in G. We have $G \simeq < B^k, x >$ and so we may write $G = A < x >$, where A is torsion free. We claim that $Z = 1$.

Clearly $Z \leq A$ and if $[A, < x >] \leq Z$ then we have the contradiction that G is nilpotent. Thus $< [A, < x >], x >$ is isomorphic to G, and it follows that the rank of $[A, < x >]$ is the same as that of A. If $Z \neq 1$ there must be a non trivial element z in $[A, < x >] \cap Z$. It is easy to see that z must be of the form $[a, x]$, for some a in A; but then $G \simeq < a, x >$, which is nilpotent, and we have a contradiction which establishes the claim.

Now, for arbitrary non trivial a in A, we have that $aa^x \ldots a^{x^{p-1}}$ is central in G and hence trivial, and $< a, x >$ is isomorphic to G. We may assume that $G =< a, x >$, for some a in A. It follows that G is a homomorphic image of the central factor group of $\mathbb{Z}$ wr $\mathbb{C}_p$ and is therefore isomorphic to this central factor group (since G is infinite and non abelian). Let N be an arbitrary non trivial G-invariant subgroup of A, and let $H =< N, x >$.

Then $H \simeq G$ and it follows that $N =$Fitt$H$ is isomorphic to $A =$Fitt$G$

and thus that N is the normal closure in H of a single element b of A. But H contains some element cx, where $c \in A$, so that $N =< b >^G$. Since N was arbitrary, Theorem 1.2.3 tells us that p is at most 19.

Next, consider the more general case where x has finite order $p^r l$, say, where $(p, l) = 1$. As before, we may write $G = A < x >$, where A is torsion free abelian. Since $x^p \in Z$, $< A, x^l >$ is non abelian and therefore isomorphic to G. It follows that $l = 1$ and x has order p^r. Now $< x^p >$ is the torsion subgroup of Z and of the centre of every non abelian subgroup H containing x^p. It follows easily that $\frac{G}{<x^p>} \in X$. If $\frac{G}{<x^p>}$ is nilpotent then so is G, a contradiction. Also, if $\frac{G}{<x^p>}$ is centre-by-finite then G' is finite and G is centre-by-finite, another contradiction. As for the first case, $\frac{G}{<x^p>}$ has trivial centre and so G has the structure indicated in the theorem.

Suppose then that x has infinite order. Again we write $G = A < x >$, where this time A is torsion free abelian and x has order $p^r \mathrm{mod} A$. As above, $[A, < x >] \cap Z = 1$ and $G \simeq < [A, < x >], x >$, and so we may write $G = D < x >$, where D is torsion free abelian and $D \cap Z = 1$.

Thus $< x^p >= Z$, and we deduce that $\frac{G}{<x^p>} \in X$. If d is any non trivial element of D then $[d, x] \notin Z$, else $< d, x >$ is nilpotent and not abelian, and hence isomorphic to G. It follows once more that $\frac{G}{<x^p>}$ has a trivial centre, and the previous argument now shows that G is of the specified form.

1.3 Groups which are isomorphic to their non-trivial normal subgroups

In this section we recall a result about finitely generated groups which are isomorphic to their non-trivial normal subgroups.

Proposition 1.3.1. *Let G be a finitely generated group which is isomorphic to its non-trivial normal subgroups. Then G satisfies the maximal condition for normal subgroups.*

Proof. Let $N_1 < N_2 < \ldots < N_i < \ldots$ be an ascending chain of normal subgroups of G. Now consider $L = \bigcup_i N_i$, which is a non-trivial normal subgroup of G thus $L \simeq G$ and so L is finitely generated. Let $x_1, x_2, \ldots, x_n$ generate L. Therefore there exist $i_1, \ldots, i_n$ such that $x_1 \in N_{i_1}, \ldots, x_n \in N_{i_n}$. Then there exists n_j such that $L = N_{n_j}$ and so the chain stabilizes. $\square$

During a conversation that took place in the early seventies, Philip Hall asked which infinite groups were isomorphic to all their non-trivial normal subgroups.(see [21])

The obvious examples to this question are the infinite cyclic group $\mathbb{Z}$, simple groups and free groups of infinite rank. Notice that the only soluble group of this type is the infinite cyclic group, since if G is soluble and $A = G^{(n)}$ is the last non trivial derived subgroup, from $G \simeq A$ we get G abelian and then $G \simeq < x >$, for every $x \in G \setminus \{1\}$.

J.C. Lennox, H. Smith and J. Wiegold proved the following theorem:

Theorem 1.3.2 (J.C. Lennox, H. Smith, J. Wiegold [17])**.** *Let G be a finitely generated infinite group that is isomorphic to all its non-trivial normal subgroups. If G contains a proper normal subgroup of finite index, then $G \simeq \mathbb{Z}$.*

Any argument like that at the beginning of the proof of this theorem for the case where G is a finitely generated group with no non-trivial finite images is bound to fail since all powers of G could have the same number of generators as G. J.C. Lennox, H. Smith and J. Wiegold used Proposition 1.3.1 to prove the following theorem:

Theorem 1.3.3 (J.C. Lennox, H. Smith, J. Wiegold [17])**.** *Let G be a finitely generated infinite group that is isomorphic to all its non-trivial normal subgroups. Then every pair of non-trivial normal subgroups intersect non-trivially.*

Proof. Let M and N be non-trivial subgroups of G. Suppose that $M \cap N = 1$. Then $G \simeq < M, N > = M \times N \simeq G \times G$, so that $\frac{G}{M} \simeq \frac{M \times N}{M} \simeq \frac{N}{N \cap M} \simeq N \simeq G$. Thus G is non-Hopfian and so it cannot satisfy the maximal condition for normal subgroups. $\square$

In general the statement of Theorem 1.3.2 is not true, for in [23] Obraztsov constructed a non-cyclic group isomorphic to each of its non-trivial normal subgroups.

Other examples have been constructed by R. Göbel, A.T. Paras and S. Shelah in [13].

Chapter 2

Groups with two isomorphism classes of derived subgroups

In this chapter the structure of groups which have at most two isomorphism classes of derived subgroups is investigated. A *derived subgroup* of a group G is the derived (or commutator) subgroup of a subgroup of G. It is a natural question how important the set of derived subgroups is within the lattice of all subgroups. There has been interest in imposing restrictions on the number of derived subgroups in a group and investigating the resulting effect on the structure of the group. Let C_n denote the class of groups in which there are at most n derived subgroups, and let C denote the union of all the classes C_n. The structure of C_n-groups for small n has been investigated in [14], while it is shown in [8] and [14] that a locally graded C-group has finite derived subgroup.

Let n be a positive integer and let D_n denote the class of groups whose derived subgroups fall into at most n isomorphism classes. Clearly $D_1 = C_1$ is the class of all abelian groups. In this section we focus our attention on the class D_2 that is the class of groups which have two isomorphism types of derived subgroup. These groups have been investigated by P. Longobardi, M. Maj, D.J.S. Robinson and H. Smith in [18], where some large classes of D_2-groups have been described in a precise way. Notice that a group G belongs to D_2 if and only if $H' \simeq G'$ whenever H is a non abelian subgroup of G. This class contains groups of many different types: apart from abelian groups, D_2 contains free groups of countable rank, groups whose derived subgroups are cyclic of prime or infinite order, Tarski groups and a whole range of soluble groups.

2.1 Elementary results

We mention some elementary facts about the class D_2.

Lemma 2.1.1 (P. Longobardi, M. Maj, D.J.S. Robinson and H. Smith [18]).

i) The class D_2 is subgroup closed.

ii) Let $G \in D_2$ and assume that G' satisfies min, the minimal condition on subgroups. If $N \lhd G$, then $\frac{G}{N} \in D_2$.

iii) If $G \in D_2$, then G' is countable.

The following result shows that the class D_2 is not closed with respect to forming quotients.

Lemma 2.1.2 (P. Longobardi, M. Maj, D.J.S. Robinson and H. Smith [18]).
A free group F belongs to D_2 if and only if it has countable rank.

Next result plays a fundamental role in the study of infinite D_2-groups.

Proposition 2.1.1 (P. Longobardi, M. Maj, D.J.S. Robinson and H. Smith [18]).

Let G be a perfect group in D_2. Then G has no proper subgroups of finite index.

Corollary 2.1.1 (P. Longobardi, M. Maj, D.J.S. Robinson and H. Smith [18]).
Let G be a D_2-group and assume that G' has a proper subgroup of finite index. Then the derived series of G reaches the identity subgroup transfinitely, i.e., G is a hypoabelian group.

A nilpotent D_2-group has class at most 2, so that locally nilpotent D_2-groups are nilpotent, as next result shows.

Theorem 2.1.2 (P. Longobardi, M. Maj, D.J.S. Robinson and H. Smith [18]).
Let G be a non abelian group. Then G is nilpotent and belongs to D_2 if and only if G' is cyclic of prime or infinite order and $G' \leq Z(G)$.

2.2 Groups with finite derived subgroups

D_2-groups with finite derived subgroup have been classified by P. Longobardi, M. Maj, D.J.S. Robinson and H. Smith. The essential components of such groups are certain finite metabelian groups constructed from pairs of integers:

Let p be a prime and $m > 1$ an integer not divisible by p. Let F be a field of order p^n where n is the order of p modulo m, which will be written $n = |p|_m$. The multiplicative group F^* contains a unique cyclic subgroup $X =< x >$ of order m. Also $|F : \mathbb{Z}_p| = n = |p|_m = |\mathbb{Z}_p(x) : \mathbb{Z}_p|$ and hence $F = \mathbb{Z}_p(x)$. Now regard $A = F^+$, the additive group of F, as an X-module via the field multiplication. Then it is easy to show that A is a simple X-module and that $C_A(y) = 0$ if $1 \neq y \in X$.
Next form the semidirect product

$$G(p, m) = X \ltimes A.$$

Then $(G(p, m))' = A$ and $|G(p, m)| = mp^n$.

Lemma 2.2.1 (P. Longobardi, M. Maj, D.J.S. Robinson and H. Smith [18]).
The group $G(p, m)$ belongs to D_2 if and only if $|p|_m = |p|_d$ for every divisor $d > 1$ of m.

A pair (p, m) which satisfies the condition in 2.2.1 is said to be *allowable*.

The main result on D_2-groups with finite derived subgroup is now recalled:

Theorem 2.2.1 (P. Longobardi, M. Maj, D.J.S. Robinson and H. Smith [18]).
Let G be a non nilpotent group with G' finite. Then $G \in D_2$ if and only if the following conditions hold:

i) $G = X \ltimes A$ where $A = G'$ is an elementary abelian p-group, $Z(G) = C_X(A)$ and $\frac{X}{Z(G)}$ is cyclic of order m.

ii) (p, m) is an allowable pair and $\frac{G}{Z(G)} \simeq G(p, m)$.

Corollary 2.2.1 (P. Longobardi, M. Maj, D.J.S. Robinson and H. Smith [18]).

If G is a non nilpotent, locally finite group in D_2, then G' is finite and the structure of G is given by Theorem 2.2.1.

Digression on allowable pairs of integers

Allowable pairs play a central role in the theory of D_2-groups with finite derived subgroup, so we start a brief discussion of their properties.

Lemma 2.2.2 (P. Longobardi, M. Maj, D.J.S. Robinson and H. Smith [18]). *Let p be a prime and $m > 1$ an integer not divisible by p. Then (p, m) is allowable if and only if $|p|_m = |p|_q$ for every prime q dividing m.*

Corollary 2.2.2 (P. Longobardi, M. Maj, D.J.S. Robinson and H. Smith [18]). *If $m = q_1^{e_1} \dots q_k^{e_k}$ is the primary decomposition of m, then (p, m) is allowable if and only if each $(p, q_i^{e_i})$ is allowable and $|p|_{q_1} = \dots = |p|_{q_k}$.*

Thus the problem of finding allowable pairs (p, m) is reduced to the case where $m = q^e$, with $q \neq p$ a prime. In this case allowability is expressed by a simple congruence.

Lemma 2.2.3 (P. Longobardi, M. Maj, D.J.S. Robinson and H. Smith [18]). *Let p, q be distinct primes and let e be a positive integer. Then (p, q^e) is allowable if and only if $p^{q-1} \equiv 1 \pmod{q^e}$.*

For distinct primes p and q put $n = |p|_q$ and define

$$e(p, q)$$

to be the largest integer such that $p^n \equiv 1 \pmod{q^{e(p,q)}}$. Note that $1 \leq e(p, q) < p^n$, so $e(p, q)$ is finite. Clearly a pair (p, q^e) is allowable if and only if $e \leq e(p, q)$. At this point a question has been formulated: given any prime p, does there exist a prime q such that $e(p, q) \geq 2$, or equivalently such that $p^{q-1} \equiv 1 \pmod{q^2}$?

Such a prime q is called a *base-p Wieferich prime*, after the German number

theorist Arthur Wieferich. Group theorically they asked if there is a prime q such that $G(p, q^2) \in D_2$.
This is a difficult number theoretic problem. A computer search revealed that the answer is positive for all primes $p < 100$ with the possible exception of 47. The case $p = 2$ is of special interest: $e(2, q) \geq 2$ if and only if $2^{q-1} \equiv 1 \pmod{q^2}$. Only two such primes q are known, 1093 and 3511. There is a curious connection between the Wieferich primes and the so-called first case of Fermat's Last Theorem.

2.3 Soluble groups with two isomorphism classes of derived subgroups

In this section the structure of infinite soluble D_2-groups is analysed.

Theorem 2.3.1 (P. Longobardi, M. Maj, D.J.S. Robinson and H. Smith [18]).
Let G be a non nilpotent, soluble D_2-group and set $A = G'$. Then:

i) A is abelian, so that G is metabelian;

ii) A is an elementary p-group for some p, a free abelian group or a torsion free minimax group;

iii) if A is torsion-free minimax and $x \in G \setminus C_G(A)$, the $C_A(x) = 1$;

iv) if $1 < [B, < x >] \leq B \leq A$ and $x \in G$, then $B \simeq A$;

v) nilpotent subgroups of G are abelian.

There is an easy converse to Theorem 2.3.1.

Proposition 2.3.2 (P. Longobardi, M. Maj, D.J.S. Robinson and H. Smith [18]).

Let G be a metabelian group and set $A = G'$. Assume that the following conditions hold:

i) if $1 < [B, < x >] \leq B \leq A$ for some $x \in G$, then $B \simeq A$;

ii) nilpotent subgroups of G are abelian.

Then $G \in D_2$.

Corollary 2.3.1 (P. Longobardi, M. Maj, D.J.S. Robinson and H. Smith [18]).
Let G be a free soluble group. Then $G \in D_2$ if and only if G is free abelian or free metabelian of countable rank.

Groups of finite rank

Soluble D_2-group with finite rank have an additional structure over and above that described in Theorem 2.3.1. We can restrict ourselves to the case where the derived subgroup is torsion-free minimax in view of Theorem 2.2.1 and Theorem 2.3.1.

Theorem 2.3.3 (P. Longobardi, M. Maj, D.J.S. Robinson and H. Smith [18]).
Let G be a non-nilpotent, soluble D_2-group such that $A = G'$ is a torsion-free minimax group. Then the following hold.

i) If $1 < B \leq A$ and $B = B^x$ where $x \in G \setminus C_G(A)$, then $|A : B|$ is finite: hence A is $< x >$-rationally irreducible.

ii) If $C = C_G(A)$, then $\frac{G}{C}$ is finitely generated and A is a noetherian $\frac{G}{C}$-module.

iii) There is an abelian subgroup U such that $U \cap A = 1$ and $|G : UA|$ is finite.

iv) $\frac{G}{Z(G)}$ is a finitely generated, metabelian minimax group in D_2.

Constructing soluble D_2-groups of finite rank

As is evident from the proof of Theorem 2.3.3, the essential part of an infinite, non nilpotent soluble D_2-group G with finite rank is a factor

$$\overline{G} = \overline{U} \ltimes A_0 :$$

where $A_0 = a^G$ for a fixed $a \neq 1$ in $A = G'$, $\overline{U} = \frac{U}{C_U(A_0)}$ is abelian, A is a torsion-free minimax and $\overline{U}$-rationally irreducible, and $\overline{G}$ is finitely generated.

There is a well-established connection between groups with this structure and algebric number fields. Note that $F = A_0 \otimes \mathbb{Q}$ is a simple $\mathbb{Q}\overline{U}$-module and the assignment $r + I \to (a \otimes 1)r$, where $r \in \mathbb{Q}\overline{U}$, yields a ring isomorphism $\frac{\mathbb{Q}\overline{U}}{I} \to F$ where $I = Ann_{\mathbb{Q}\overline{U}}(a)$, a maximal ideal of $\mathbb{Q}\overline{U}$. Thus F is an algebraic number field and we may identify A_0 and $\overline{U}$ with subgroups of F^+ and F^* respectively. Moreover, $A_0 = Rg < \overline{U} >$ and $F = \mathbb{Q}(\overline{U})$. Conversely, suppose we start with an algebraic number field F and a non trivial finitely generated subgroup X of F^* such that $F = \mathbb{Q}(X)$. Let C be the subring of F generated by X and regard C as an X-module in the natural way. Now form the group

$$G = G(F, X) = X \ltimes C.$$

Since $G =< X, 1_F >$, this is a finitely generated metabelian group. Also $F = \mathbb{Q}(X)$, so we have $r_0(C) = (F : \mathbb{Q})$ and G has finite rank; hance it is a minimax group. Notice that if X is a subgroup of the group of units of F, then G will be polycyclic. It is easy to see that any nilpotent subgroup of G is abelian and that $A := G' = [C, X]$. By Proposition 2.3.2 the group G belongs to D_2 if and only if $B \simeq A$ whenever $0 \neq B = Bx \leq A$ and $1 \neq x \in X$. Let us call the pair (F, X) *allowable* if this condition is valid, the analogy with allowable pairs of integers being evident. In conclusion $G(F, X) \in D_2$ if and only if (F, X) is an allowable pair. Note that if X is a group of units of F, then (F, X) is allowable if and only if $C = Rg < X >$ is $< x >$-rationally irreducible for all $x \neq 1$ in X.

Groups with non perfect derived subgroup

Then D_2-groups with G' not perfect have been considered. Under the additional hypothesis that $\frac{G'}{G''}$ has finite abelian ranks, i.e., the p-rank is finite for $p = 0$ or a prime, it emerges that these groups are soluble, so they fall within the scope of the classification of the previous sections.

Theorem 2.3.4 (P. Longobardi, M. Maj, D.J.S. Robinson and H. Smith [18])**.** *Let G be a D_2-group such that $\frac{G'}{G''}$ is non trivial and has finite abelian ranks. Then G is soluble and G' is either finite elementary abelian or torsion-free abelian minimax group.*

Notice that the hypothesis of finite rank cannot be omitted from the theorem since free groups of countable rank belongs to D_2. During the proof of Theorem 2.3.4 two auxiliary results about nilpotent groups which may be known are used. If n is a positive integer, let $e(n)$ denote the sum of the exponents in the primary decomposition of n.

Lemma 2.3.1 (P. Longobardi, M. Maj, D.J.S. Robinson and H. Smith [18]). *Let G be a nilpotent group, n a positive integer and $S = \gamma_n(G)$. If $\frac{S'}{S''}$ is finite and $e(\left|\frac{S'}{S''}\right|) \leq n$, then S is metabelian.*

Lemma 2.3.2 (P. Longobardi, M. Maj, D.J.S. Robinson and H. Smith [18]). *Let G be a nilpotent group, n a positive integer and $S = \gamma_n(G)$. If $r_0(\frac{S'}{S''}) \leq n$, then S'' is periodic.*

Corollary 2.3.2 (P. Longobardi, M. Maj, D.J.S. Robinson and H. Smith [18]). *Let G be a periodic D_2-group. If G' is not perfect, then G is soluble.*

Corollary 2.3.3 (P. Longobardi, M. Maj, D.J.S. Robinson and H. Smith [18]). *Let G be a periodic D_2-group. If G is locally graded, then G is soluble.*

Elements of finite order in D_2-groups

Elements of finite order in a D_2-group are subject to surprisingly strong restrictions, at least if the group is insoluble and its derived subgroup is not perfect.

Theorem 2.3.5 (P. Longobardi, M. Maj, D.J.S. Robinson and H. Smith [18]). *Let G be an insoluble D_2-group such that G' is not perfect. Then the elements of G with finite order form a subgroup F of $Z(G)$ and $\frac{G}{F}$ is in D_2.*

On the other hand, the elements of finite order in a soluble D_2-group need not form a subgroup, as the infinite dihedral group shows.

2.4 Insoluble groups with two isomorphism classes of derived subgroups

Some classes of insoluble D_2-groups are now considered.

Let T denote the class of groups that satisfy the *Tits alternative*, i.e., $G \in T$ if and only if either G is soluble-by-finite or it contains a free subgroup of rank 2.

Theorem 2.4.1 (P. Longobardi, M. Maj, D.J.S. Robinson and H. Smith [18]). *Let G be a T-group. Then $G \in D_2$ if and only if either G is a soluble D_2-group or else G' is free with countably infinite rank and L' is not finitely generated whenever L is a non abelian subgroup of G.*

Corollary 2.4.1 (P. Longobardi, M. Maj, D.J.S. Robinson and H. Smith [18]). *Let G be a locally free group. Then $G \in D_2$ if and only if G' is a free group of countable rank.*

Groups whose derived subgroup satisfies the minimal condition

Up to this point none of the special types of D_2-group they have studied has involved a Tarski group, even if Tarski groups certainly belong to D_2. This final result shows that every insoluble D_2-group whose derived subgroup satisfies the minimal condition has a factor which is of Tarski type.

Theorem 2.4.2 (P. Longobardi, M. Maj, D.J.S. Robinson and H. Smith [18]). *Let G be an insoluble D_2-group such that G' satisfies the minimal condition. Then G has the following properties.*

i) G' is the unique smallest non abelian subgroup of G.

ii) Soluble subgroups of G are abelian.

iii) G' is finitely generated and perfect.

iv) The subgroup $M := G' \cap Z(G)$ is the unique maximum normal subgroup of G', and $\frac{G'}{M}$ is an infinite simple group.

v) $\frac{G}{M}$ *is a* D_2*-group.*

vi) If $N \lhd G$*, then* $N \leq Z(G)$ *or* $G' \leq N$*.*

Observation 1. Note that this result can be inverted:

a) The group $\frac{G'}{M} \simeq \frac{G'Z(G)}{Z(G)}$ is finitely generated infinite simple group with all its proper subgroups abelian, so it is a Tarski group.

b) In the opposite direction notice that properties i) and ii) imply that $G \in D_2$ because, if H is a non abelian subgroup, $G' \subseteq H$ for i) and hence $H' \geq G'' = G'$. Thus $H' = G'$.

Chapter 3

B_2-groups

In this chapter we study groups G for which the set of isomorphism types of elements in $\{\frac{H}{Z(H)} | H \leq G\}$ is very small. If n is a positive integer, let B_n denote the class of groups G such that the factor groups in $\{\frac{H}{Z(H)} | H \leq G\}$ fall into at most n isomorphism classes. Of course, B_1 is the class of all abelian groups, while a group G belongs to B_2 if and only if $\frac{H}{Z(H)} \simeq \frac{G}{Z(G)}$ whenever H is a non-abelian subgroup of G.

3.1 Elementary results

If G is a minimal non-abelian group, then obviously G is in B_2.

The following proposition gives more examples of groups in B_2.

Proposition 3.1.1. *Let G be a group such that $G = TZ(G)$, where $T \leq G$ is minimal non abelian. Then $G \in B_2$.*

Proof. Assume that $G = TZ(G)$. Then $Z(T) = T \cap Z(G)$. Let $H \leq G$, H non abelian. Therefore $HZ(G) = HZ(G) \cap G = Z(G)(T \cap HZ(G))$. Suppose that $T \cap HZ(G) < T$. Since T is minimal non abelian, then $T \cap HZ(G)$ is abelian, so $Z(G)(T \cap HZ(G))$ is also abelian. Hence $HZ(G)$ is abelian, which gives the contradiction H abelian. Thus $T \cap HZ(G) = T$, so that $T \subseteq HZ(G)$ and $TZ(G) \subseteq HZ(G) \subseteq G$. Then $HZ(G) = G$ and so $Z(H) \subseteq H \cap Z(G)$. Therefore $\frac{G}{Z(G)} = \frac{HZ(G)}{Z(G)} \simeq \frac{H}{H \cap Z(G)} = \frac{H}{Z(H)}$, as required. $\square$

Proposition 3.1.2. *Let G be a group and suppose that either $\frac{G}{Z(G)}$ is elementary abelian of order p^2 (p a prime) or $\frac{G}{Z(G)} \simeq \mathbb{Z} \times \mathbb{Z}$. Then $G \in B_2$.*

Proof. First suppose $\frac{G}{Z(G)}$ elementary abelian, with $\left|\frac{G}{Z(G)}\right| = p^2$, p a prime. Let H be a non-abelian subgroup of G. Then $\frac{HZ(G)}{Z(G)} \leq \frac{G}{Z(G)}$, and $\frac{HZ(G)}{Z(G)} \simeq \frac{H}{H \cap Z(G)}$. If $\frac{HZ(G)}{Z(G)} < \frac{G}{Z(G)}$ then from $\frac{H}{H \cap Z(G)}$ cyclic it follows that H is abelian, a contradiction. Then we have $\frac{HZ(G)}{Z(G)} = \frac{G}{Z(G)}$ and $G = HZ(G)$; in particular $Z(H) \leq H \cap Z(G)$, and $\frac{H}{Z(H)} = \frac{H}{H \cap Z(G)} \simeq \frac{HZ(G)}{Z(G)} = \frac{G}{Z(G)}$, as required.
Now suppose that $\frac{G}{Z(G)} \simeq \mathbb{Z} \times \mathbb{Z}$. Then G is nilpotent of class 2 and $G = Z(G) < x, y >$ for suitable $x, y \in G$. Obviously $G' =< x, y >' =< [x, y] >$. If $o([x, y]) = n$, then $[x, y]^n = [x^n, y] = 1$ and $x^n \in Z(G)$, a contradiction. Therefore G' is infinite cyclic. If H is a non-abelian subgroup of G, then $\frac{H}{Z(H)} \simeq \frac{\frac{H}{Z(G) \cap H}}{\frac{Z(H)}{Z(G) \cap H}}$ is a non cyclic, 2-generated group being a quotient of $\frac{H}{Z(G) \cap H} \simeq \frac{Z(G)H}{Z(G)} \leq \frac{G}{Z(G)}$. Moreover it is torsion free, in fact if $h^n \in Z(H)$ then $[h^n, k] = [h, k]^n = 1$ for every $k \in H$, then $h \in Z(H)$ since G' is torsion free. Hence $\frac{H}{Z(H)} \simeq \mathbb{Z} \times \mathbb{Z} \simeq \frac{G}{Z(G)}$. $\square$

We continue by assembling some elementary facts about the class B_2.

Lemma 3.1.1. *i) The class B_2 is subgroup closed.*

ii) If $G \in B_2$, then $\frac{G}{Z(G)}$ is 2-generated.

iii) If G is a nilpotent group and $G \in B_2$, then $\frac{G}{Z(G)}$ is abelian.

iv) If G is non-nilpotent and $G \in B_2$, then every locally nilpotent subgroup of G is abelian.

v) If G is soluble, non-nilpotent and $G \in B_2$, then G is metabelian.

vi) If G is not soluble and $G \in B_2$, then every soluble subgroup of G is abelian.

vii) If G is not soluble and $G \in B_2$, then every normal soluble subgroup of G is contained in $Z(G)$.

Proof. The first statement is obvious. In order to prove *ii)* consider $a, b \in G$, with $[a, b] \neq 1$, then $\frac{G}{Z(G)} \simeq \frac{<a,b>}{Z(<a,b>)}$ as required. Now assume G nilpotent non abelian and consider $x \in Z_2(G) \setminus Z(G)$, then $[x, g] \neq 1$ for some $g \in G$ and we have $< x, g >$ nilpotent of class 2 since $[x, g] \in Z(G)$; then $\frac{G}{Z(G)} \simeq \frac{<x,g>}{Z(<x,g>)}$ is abelian and $iii)$ holds. In order to prove $iv)$, let F be a locally nilpotent subgroup of G and assume there exist $a, b \in F$, with $[a, b] \neq 1$. Then $\frac{G}{Z(G)} \simeq \frac{<a,b>}{Z(<a,b>)}$, thus $\frac{G}{Z(G)}$ is nilpotent and G is nilpotent, a contradiction. Therefore F is abelian. In order to prove $v)$, suppose that G is a soluble non-nilpotent group in B_2. Write $F = FittG$ the Fitting subgroup of G. Then F is abelian by $iv)$. Moreover $C_G(F) \subseteq F$, by 5.4.4(ii) in [27]. Let $x \in G \setminus F$ and write $H = F < x >$. Then H is not abelian and $H' \leq F$ is abelian by $iv)$. Therefore $\frac{G}{Z(G)} \simeq \frac{H}{Z(H)}$ is metabelian. In addition $(\frac{H}{Z(H)})' \simeq (\frac{G}{Z(G)})' = \frac{G'Z(G)}{Z(G)} \simeq \frac{G'}{G' \cap Z(G)}$ is abelian, hence G' is nilpotent and $G' \leq F$. But F is abelian by $iv)$, thus G' is abelian and G is metabelian. Therefore $v)$ holds. If G is non soluble and S is a subgroup of G, then S is abelian, otherwise $\frac{G}{Z(G)} \simeq \frac{S}{Z(S)}$ is soluble and G soluble. Therefore $vi)$ holds. Finally if G is non soluble and $N \trianglelefteq G$ is soluble, then N is abelian by $vi)$ and $N < g >$ is soluble, hence abelian, for every $g \in G$. Then $N \leq Z(G)$ and $vii)$ holds. □

We will see that the class B_2 is not closed under the formation of homomorphic images. But we have the following useful result.

Proposition 3.1.3. *Let G be a non-nilpotent group in B_2. If $S \leq Z(G)$, then $\frac{G}{S} \in B_2$.*

Proof. Let $\frac{H}{S} \leq \frac{G}{S}$. First we show that $Z(\frac{H}{S}) = \frac{Z(H)}{S}$. In fact obviously $\frac{Z(H)}{S} \leq Z(\frac{H}{S})$. Write $\frac{V}{S} = Z(\frac{H}{S})$. Then $V \leq Z_2(H)$. If $V \not\leq Z(H)$, then there exists $h \in H$ such that $V \not\subseteq C_G(h)$. Then the subgroup $V < h >$ is nilpotent and non-abelian, a contradiction by Lemma 3.1.1. Therefore $Z(\frac{H}{S}) = \frac{Z(H)}{S}$ for every non abelian subgroup $\frac{H}{S}$ of $\frac{G}{S}$. In particular we have $Z(\frac{G}{S}) = \frac{Z(G)}{S}$. Hence, for every non abelian subgroup $\frac{H}{S}$ of $\frac{G}{S}$, we have $\frac{\frac{G}{S}}{Z(\frac{G}{S})} = \frac{\frac{G}{S}}{\frac{Z(G)}{S}} \simeq \frac{G}{Z(G)} \simeq \frac{H}{Z(H)} \simeq \frac{\frac{H}{S}}{\frac{Z(H)}{S}} = \frac{\frac{H}{S}}{Z(\frac{H}{S})}$. Therefore $\frac{G}{S} \in B_2$. □

Of course our aim is to study non-abelian B_2-groups, and it is natural to look first at nilpotent B_2-groups: these admit a very simple description.

Theorem 3.1.4. *Let G be a non abelian group. Then G is nilpotent and belongs to B_2, if and only if either $\frac{G}{Z(G)}$ is elementary abelian of order p^2 (p a prime) or $\frac{G}{Z(G)} \simeq \mathbb{Z} \times \mathbb{Z}$.*

Proof. Assume that either $\frac{G}{Z(G)}$ is elementary abelian of order p^2 (p a prime) or $\frac{G}{Z(G)} \simeq \mathbb{Z} \times \mathbb{Z}$. Then G is obviously nilpotent and $G \in B_2$ by Proposition 3.1.2.
Now assume that $G \in B_2$ is nilpotent and put $Z_i = Z_i(G)$. Then $\frac{G}{Z_1}$ is 2-generated and abelian by Lemma 3.1.1. There exists $a \in Z_2 \setminus Z_1$ and $b \in G$ such that $[a, b] \neq 1$.
Put $H =< a, b >$, then $H' =< [a, b] >$ and $\frac{H}{Z(H)} \simeq \frac{G}{Z(G)}$.
Now suppose that $[a, b]$ is aperiodic; therefore $\frac{H}{Z(H)}$ is torsion-free. If, on the other hand, $[a, b]$ is periodic, then H' is finite. Since H is finitely generated, we have $\frac{H}{Z(H)}$ finite. There exists $cZ(H) \in \frac{H}{Z(H)}$ of order p where p is a suitable prime. There exists $x \in H$ such that $[c^p, x] = [c, x]^p = 1$ but $[c, x] \neq 1$. Now it is easy to see that $\frac{G}{Z(G)}$ has order p^2, as claimed. □

Using Theorem 3.1.1., it is now possible to show that the class B_2 is not closed under quotient.
For, let G be a free 2-generated group, nilpotent of class 2. Then $G \in B_2$. Let A be a nilpotent p-group of class 2, 2-generated, and let B be a nilpotent q-group of class 2 and 2-generated, where p, q are distinct primes. Finally, assume that $H = A \times B$. There exists $N \trianglelefteq G$ such that $\frac{G}{N} \simeq H$ but $H \notin B_2$.

3.2 Locally finite B_2-groups

In this section we will classify locally finite B_2-groups.

Theorem 3.2.1. *Let G be a finite group. Then $G \in B_2$ if and only if $G = Z(G)H$, where H is minimal non abelian.*

Proof. Assume that $G = Z(G)H$ where H is minimal non abelian. Then $G \in B_2$ by Proposition 3.1.1.

Now let $G \in B_2$. Consider $H \leq G$, with H non abelian of minimal order. Therefore $\frac{G}{Z(G)} \simeq \frac{H}{Z(H)} \simeq \frac{\frac{H}{H \cap Z(G)}}{\frac{Z(H)}{H \cap Z(G)}}$, thus $\left|\frac{G}{Z(G)}\right| \leq \left|\frac{H}{H \cap Z(G)}\right| = \left|\frac{HZ(G)}{Z(G)}\right| \leq \left|\frac{G}{Z(G)}\right|$.
Then $HZ(G) = G$ as claimed. □

Corollary 3.2.1. *Let G be a locally finite group. Then $G \in B_2$ if and only if $G = Z(G)H$, where H is finite and minimal non abelian.*

Proof. Suppose that G is a locally finite B_2-group. Then there exist a, $b \in G$ such that $\frac{G}{Z(G)} =< aZ(G), bZ(G) >$ and so $G =< a, b > Z(G)$. Since G is locally finite, $< a, b >$ is finite. By Theorem 3.2.1 we have $< a, b >= Z(< a, b >)H$, where H is minimal non abelian and so $G = Z(G) < a, b >= Z(G)H$, since $Z(< a, b >) \leq Z(G)$.

Now suppose that $G = Z(G)H$, where H is finite and minimal non abelian, then $G \in B_2$ by Theorem 3.2.1. □

Corollary 3.2.2. *Let G be a B_2-group. Then G is locally finite if and only if G is a soluble torsion group.*

Proof. Suppose that G is a soluble torsion group. Then G is locally finite by Proposition 5.4.11 in [27].

Now suppose that G is a locally finite B_2-group. By Corollary 3.2.1, there exists $H \leq G$ finite and minimal non abelian such that $G = Z(G)H$. Then H is soluble by a classical theorem of Miller and Moreno ([22]) and so G is soluble and torsion, as required. □

3.3 Soluble B_2-groups

In this section we will analyse the structure of infinite soluble B_2 group. Every soluble non-nilpotent B_2 group is metabelian, by Lemma 3.1.1 v). Moreover $\frac{G}{Z(G)} \in B_2$ by Proposition 3.1.3. More information will be collected in the following theorem.

Theorem 3.3.1. *Let G be a soluble non-nilpotent B_2-group. Then*

i) $Z(\frac{G}{Z(G)}) = 1$.

ii) $G = A < x >$, *where A is a normal abelian sugroup of G.*

iii) Every non-abelian subgroup of $\frac{G}{Z(G)}$ is isomorphic to $\frac{G}{Z(G)}$.

Proof. *i)* Write $\frac{Z_2(G)}{Z(G)} = Z(\frac{G}{Z(G)})$. For every $g \in G$, the group $Z_2(G) < g >$ is nilpotent and so it is abelian by Lemma 3.1.1 *iv*). Then $Z_2(G) \subseteq C_G(g)$ for every $g \in G$ and $Z_2(G) \leq Z(G)$. Thus $Z_2(G) = Z(G)$.

ii) By Lemma 3.1.1 *v*), G is metabelian. Let B be a maximal normal abelian subgroup of G such that $G' \subseteq B$. If $B \leq Z(G)$ then $G' \subseteq B \subseteq Z(G)$ and so G is nilpotent of class 2, a contradiction. Therefore there exists $g \in G$ such that $g \notin C_G(B)$, thus $B \not\subseteq C_G(g)$. Now consider $H = B < g >$. Since it is non abelian, it follows that $\frac{H}{Z(H)} \simeq \frac{G}{Z(G)}$ and so $\frac{G}{Z(G)}$ is abelian-by-cyclic. Then there exists $\frac{A}{Z(G)} \trianglelefteq \frac{G}{Z(G)}$ such that $\frac{A}{Z(G)}$ is abelian and $\frac{G}{A}$ is cyclic. Thus A is nilpotent. By Lemma 3.1.1 *iv*), A is abelian and G is abelian-by-cyclic.

iii) Let $\frac{H}{Z(G)}$ be a non abelian subgroup of $\frac{G}{Z(G)}$. From Proposition 3.1.3, $\frac{G}{Z(G)} \in B_2$ and $Z(\frac{H}{Z(G)}) = \frac{Z(H)}{Z(G)}$. It follows that $\frac{\frac{H}{Z(G)}}{\frac{Z(H)}{Z(G)}} \simeq \frac{G}{Z(G)}$ so it suffices to prove that $Z(H) \subseteq Z(G)$. Now $G = A < x >$, where A is a normal abelian sugroup of G by *ii)*. Obviously we can suppose that A is maximal.

Firstly suppose that $\frac{G}{A}$ is finite cyclic. Consider $yA \in \frac{G}{A}$ of order a prime p. Then $A < y >$ is non abelian and $\frac{A<y>}{Z(A<y>)} \simeq \frac{G}{Z(G)}$ is abelian-by-(prime order). Therefore there exists $\frac{B}{Z(G)} \leq \frac{G}{Z(G)}$ such that $\frac{B}{Z(G)}$ is abelian and $|\frac{G}{B}| = p$ and so B is nilpotent and by Lemma 3.1.1 *iv)* it is abelian. So we can suppose that $|\frac{G}{A}| = p$. Therefore $x^p \in A$. Suppose that there exists an element $h = ax^r \in Z(H)$ with $a \in A$ and r, p coprime. Then $G = A < ax^r >$ and so $H = < ax^r > (A \cap H)$ which is abelian, a contradiction. Thus $h \in A$. Since H is non abelian, there exists an element $cx^s \in H$ where s and p are coprime, $c \in A$. It follows that $G = A < cx^s >$ and then $h \in Z(G)$.

Now suppose that $G = A < x >$, with $\frac{G}{A}$ infinite cyclic. Write $\overline{G} = \frac{G}{Z(G)}$ and $\overline{A} = \frac{A}{Z(G)}$. Suppose that $\overline{D} = C_{\overline{A}}(x^r) \neq 1$, for all $r \neq 0$. Then $\overline{D} < x >$ is non abelian, since $Z(\overline{G}) = 1$ by *i)* , and $x^r \in Z(\overline{D} < x >)$. Therefore $\frac{\overline{D}<x>}{Z(\overline{D}<x>)} \simeq \overline{G}$ is abelian-by-finite, which is a contradiction because $\frac{\overline{G}}{\overline{A}}$ is infinite cyclic.

Let $\overline{H}$ be a non abelian subgroup of $\overline{G}$. Then $\overline{H} \not\subseteq \overline{A}$ and $\overline{H} \not\subseteq < x >$. Consider an element $h = ax^s \in \overline{H}$, where $s \neq 0$ and suppose that $Z(\overline{H}) \neq 1$. Thus there exists an element bx^r which permutes with every element of $\overline{H} \cap \overline{A} \neq 1$, since $\overline{H}$ is not cyclic. Therefore $C_{\overline{A}}(x^r) \neq 1$, a contradiction. It follows that $r = 0$ and so $bx^r = b$ commutes with ax^s. Then x^s commutes with b and so $s = 0$, the final contradiction.

□

3.4 Insoluble B_2-groups

We start with the following

Theorem 3.4.1. *Let G be a group such that $\frac{G}{Z(G)}$ has a proper subgroup of finite index. If $G \in B_2$, then G is soluble.*

Proof. Suppose G non soluble. Then $\frac{G}{Z(G)}$ is infinite and 2-generated by Lemma 3.1.1. We show that $\frac{N}{Z(G)} \simeq \frac{G}{Z(G)}$ for every non-trivial normal subgroup of $\frac{G}{Z(G)}$.

First notice that $M \cap Z(G) = Z(M)$ for every $M \trianglelefteq G$. For obviously $M \cap Z(G) \leq Z(M)$. Let $g \in G$, then $Z(M) < g >$ is soluble, therefore $Z(M) < g >$ is abelian by Lemma 3.1.1 *vi*), thus $Z(M) \subseteq C_G(g)$; that holds for every $g \in G$, hence $Z(M) \leq Z(G)$.

Now suppose $\frac{N}{Z(G)} \trianglelefteq \frac{G}{Z(G)}$, $\frac{N}{Z(G)} \neq 1$. Then $N \trianglelefteq G$ and $Z(G) \leq N$, therefore $Z(G) = Z(G) \cap N = Z(N)$, by the previous remark. If N is abelian, then $N \leq Z(G)$ and $\frac{N}{Z(G)} = 1$, which is not the case. Then N is not abelian, therefore $\frac{N}{Z(G)} = \frac{N}{Z(N)} \simeq \frac{G}{Z(G)}$, as required.

Therefore $\frac{G}{Z(G)}$ is a finitely generated infinite group that is isomorphic to all its non-trivial normal subgroups and that contains a proper normal subgroup of finite index. Then, by Theorem 1.3.2, $\frac{G}{Z(G)}$ is cyclic and G is soluble, a contradiction. □

Corollary 3.4.1. *Let G be a locally graded group in B_2. Then G is soluble.*

Proof. The group $\frac{G}{Z(G)}$ is 2-generated by Lemma 3.1.1 *ii*). It is also locally graded by [33]. Then it has a normal subgroup of finite index. By Theorem 3.4.1, G is soluble. □

Let T denote the class of groups that satisfy the *Tits alternative*, i.e., $G \in T$ if and only if either G is soluble-by-finite or it contains a free subgroup of rank 2.

Theorem 3.4.2. *Let G be an insoluble B_2-group. Then G is not a T-group.*

Proof. First assume that G has a free subgroup F of rank 2. Then $Z(F) = 1$. Moreover H is free for every non-abelian subgroup H of F. Then $F \simeq \frac{F}{Z(F)} \simeq \frac{H}{Z(H)} \simeq H$. This is impossible since a free group of rank 2 contains a free subgroup of infinite rank (see [27]).

Now assume G soluble-by-finite. Then there exists $N \trianglelefteq G$ with N soluble and $\frac{G}{N}$ finite. Lemma 3.1.1 *vii*) yields $N \leq Z(G)$. Therefore $\frac{G}{Z(G)}$ is finite, so G' is finite by Schur's Lemma. Thus, by Corollary 3.4.1, G is soluble, a contradiction. $\square$

Up to this point none of the special types of B_2-group we have studied has involved a Tarski group, even if Tarski groups certainly belong to B_2. Next result shows that every insoluble B_2 group whose derived subgroup satisfies the minimal condition has $\frac{G}{Z(G)}$ which is of Tarski type.

Theorem 3.4.3. *Let G be an insoluble B_2-group such that G' satisfies the minimal condition. Then G satisfies the following properties:*

i) $\frac{G}{Z(G)}$ *is a simple, minimal non abelian group.*

ii) Soluble subgroups of G are abelian.

iii) If $N \lhd G$, then either $N \leq Z(G)$ or $G' \leq N$.

In particular, $\frac{G}{Z(G)}$ *is a Tarski group.*

Proof. i) If G' is soluble, then G is soluble, which is not the case. Then there exists a minimal non soluble subgroup $S \leq G'$. Then $\frac{G}{Z(G)} \simeq \frac{S}{Z(S)}$ since $G \in B_2$, thus $\frac{G}{Z(G)}$ is minimal non soluble. Let $\frac{H}{Z(G)} < \frac{G}{Z(G)}$, then $\frac{H}{Z(G)}$ is soluble. Therefore H is soluble. By Lemma 3.1.1 *vi*), H is abelian and hence $\frac{H}{Z(G)}$ is abelian.

Now we prove that $\frac{G}{Z(G)}$ is simple. Let $\frac{N}{Z(G)} \lhd \frac{G}{Z(G)}$, then N is abelian. Thus $N \leq Z(G)$, otherwise there exists a $x \in G$ such that $\frac{N<x>}{Z(N<x>)} \simeq \frac{G}{Z(G)}$ so that G is soluble, a contradiction. *ii)* By Lemma 3.1.1*vi*) every soluble subgroup of G is abelian. *iii)* If $N \lhd G$, from *i)* it follows that either $\frac{NZ(G)}{Z(G)} = 1$ or $\frac{NZ(G)}{Z(G)} = \frac{G}{Z(G)}$. Then $N \leq Z(G)$ or $G' \leq N$. $\square$

Conversely, we have:

Proposition 3.4.4. *Let G be an insoluble group such that every nilpotent subgroup is abelian and $\frac{G}{Z(G)}$ is simple, minimal non abelian. Then G is a B_2-group.*

Proof. Let H be a non-abelian subgroup of G. Now consider $\frac{HZ(G)}{Z(G)} \leq \frac{G}{Z(G)}$. If $\frac{HZ(G)}{Z(G)} < \frac{G}{Z(G)}$, then $\frac{HZ(G)}{Z(G)}$ is abelian and hence H is nilpotent. Then H is abelian, which is a contradiction. Thus $\frac{HZ(G)}{Z(G)} = \frac{G}{Z(G)}$ and $\frac{G}{Z(G)} \simeq \frac{H}{H \cap Z(G)}$. Since $\frac{G}{Z(G)}$ is simple, $H \cap Z(G) = Z(H)$. Therefore $\frac{G}{Z(G)} \simeq \frac{H}{Z(H)}$, and we have the result. $\square$

Chapter 4

Fibonacci-like sequences and generalized Pascal's triangle

In this chapter we will show some results about Fibonacci-like sequences and generalized Pascal's triangle. We have studied some properties pertaining diagonals of generalized Pascal's triangles and we have determined combinatorial relationships between Fibonacci-like sequences and the Fibonacci sequence itself, using a new sequence whose elements are the numbers that appear in the diagonals of the generalized Pascal's triangle. We start recalling some basic definitions.

A *recursive sequence* $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of number f_n indexed by a positive integer n and generated by solving a recurrence equation in which later terms are deduced from earlier ones.

The most famous case is the *Fibonacci sequence*, which we will denote by $\{F_n\}$, defined by

$$F_1 = F_2 = 1 \qquad\qquad F_n = F_{n-1} + F_{n-2}, n > 2.$$

This sequence has been of wide interest among mathematicians and in applications as well since its first appearance in the book *Liber Abaci* published in 1202. The Fibonacci sequence is often used as a model of recursive phenomena in botany, see e.g. [26, 12], chemistry, see [2], physics and engineering, see e.g. [1] and references therein, medicine, see [6]. In such studies are of a certain relevance both the asymptotic behaviour of processes, which can be described through linear recurrences (see also [11]) and combinatorial aspects, which can be understood by studying the properties of the related recursive sequences.

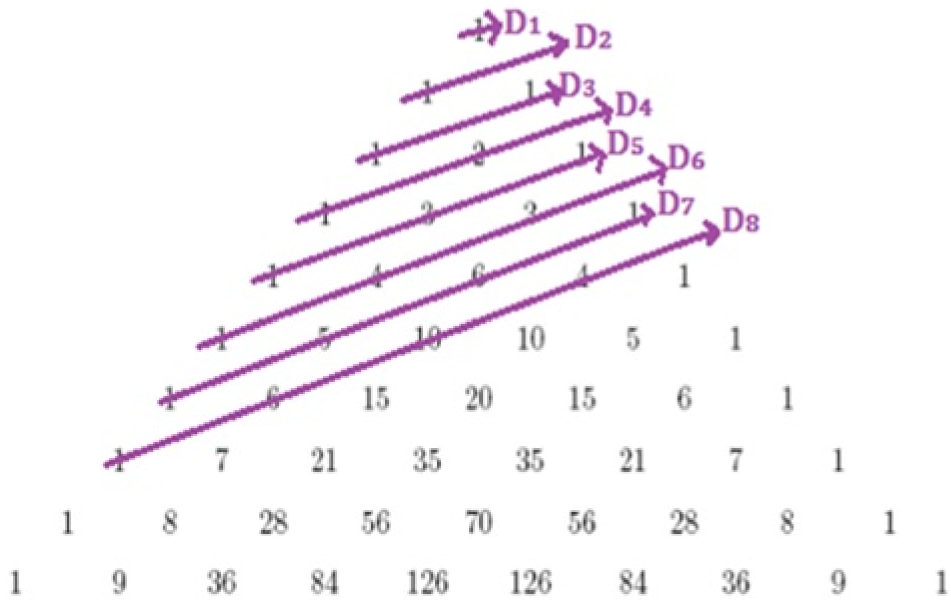
A property due to Lucas (see [7]) shows that

$$F_{n+1} = \binom{n}{0} + \binom{n-1}{1} + \ldots + \binom{n-i}{i} + \ldots$$

for any

$$i : n - i \geq i \geq 0.$$

It is well known that the above identity shows that the Fibonacci numbers can be read from Pascal's triangle (see the figure below). Note that the term on the n-th arrow and i-th place is exactly the choice number $\binom{n-1-i}{i}$. For istance, looking to the figure below, any arrow has 1 on its 0-th place, and the term of the 4-th arrow lying on the 1-st place is 2.



The array in the figure is known as Pascal's triangle because it was intensively studied by Blaise Pascal (1623-1662), who showed his result in this area in his work, published after his death ([25]). In addition, many properties of the Pascal's triangle can be found in *General trattato di numeri et misure* (1556), by Niccoló Tartaglia . In the same period, the German mathematician Eduard Stiefel, to whom the basic combinatorial identity

$$\binom{n}{m} + \binom{n}{m-1} = \binom{n+1}{m},$$

for every $m, n \in \mathbb{N} : m \leq n$

is due, also studied the Pascal's triangle. Much earlier, in 1303, the triangle had already been considered by the Chinese mathematician Zhu Shijie, that called it Yanghui's triangle ([15]) and by Omar Khayyàm. For these reasons, the Pascal's triangle is also called Tartaglia', Stiefel', Yanghui' or Khayyàm's triangle.

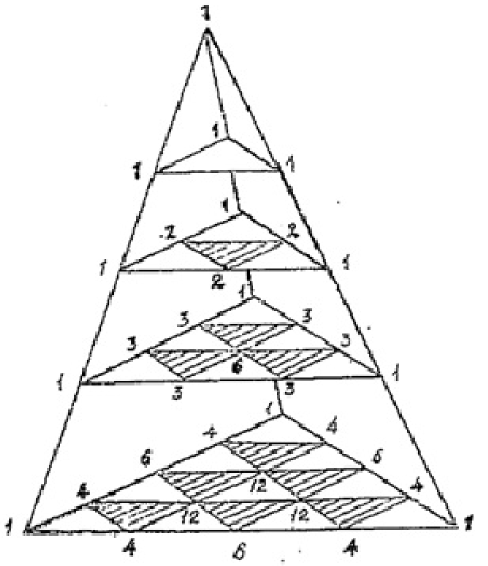
Inside the Pascal's triangle many properties concerning number theory are hidden, so that many mathematicians werw induced to consider its possible generalizations. It is interesting to see how some of these generalizations preserve corresponding results about recursive relations, described by combinatiorial arguments. For instance, the study of Pascal's Triangle of s-th order is strictly connected to binomial coefficients of order s.([3, 4]).

A surprising connection between a special kind of generalized Pascal's triangle and recursive sequences is due to Shannon ([10]). He considered the Pascal Pyramid, constructed in three dimension, and he pointed out that the sequence of the numbers determined by its diagonals is exactly the $Tribonacci$ sequence 1, 1, 1, 3, 5, 9, 17,..., where the n-th term, for $n \geq 3$, is obtained by adding the previous three.

Another generalization of Pascal's Triangle had been studied by Hosoya ([16]), who considered the *Fibonacci triangle* whose edges are the Fibonacci sequence, and the other elements are obtained as the sum of the two elements above (like in Pascal's Triangle). Similarly Sána([29]) and Shapiro([30]), respectively, considered and studied the properties of *Lucas'triangle* and *Catalan's triangle.*

For a rich compendium of other generalizations see the book of Bondarenko ([3, 4]). Also, more recently Falcon and Plaza ([10]) defined the *Pascal two-triangle* and studied it by means of the k-Fibonacci sequences.

Let k_1 and k_2 be two complex numbers, or more generally consider two elements of a commutative ring. Then the *Fibonacci-like sequence* $\{H_n\}_{n\in\mathbb{N}}$ is

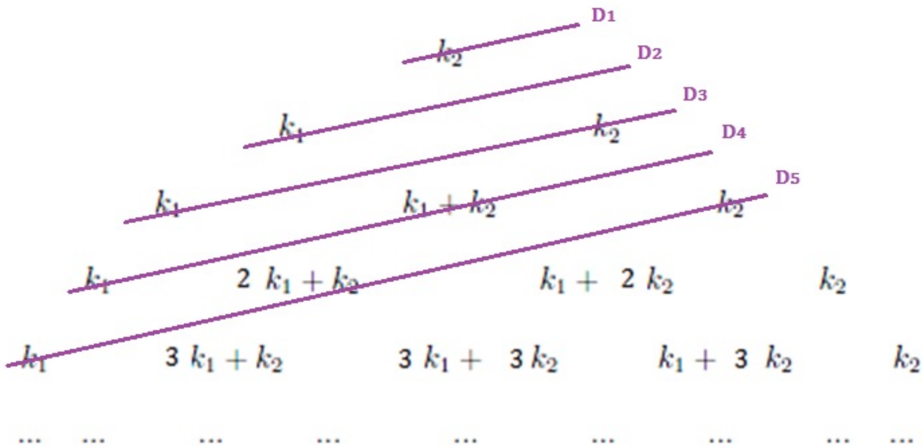


the sequence whose seeds are $H_1 = k_1$ and $H_2 = k_2$ and $H_n = H_{n-1} + H_{n-2}$, for any $n > 2$.

The *generalized Pascal's triangle* $T(k_1, k_2)$ is the triangle which has the value k_2 on the top and the values k_1 and k_2 along the left and right sides respectively (see figure).

$$
\begin{array}{ccccccccc}
 & & & & k_2 & & & & \\
 & & & k_1 & & k_2 & & & \\
 & & k_1 & & k_1 + k_2 & & k_2 & & \\
 & k_1 & & 2\ k_1 + k_2 & & k_1 +\ 2\ k_2 & & k_2 & \\
k_1 & & 3\ k_1 + k_2 & & 3\ k_1 +\ 3\ k_2 & & k_1 +\ 3\ k_2 & & k_2 \\
\cdots\ \cdots & & \cdots & \cdots & \cdots & \cdots & \cdots & & \cdots\ \cdots
\end{array}
$$

It turns out that a sequence $\{D_n\}_{n\in\mathbb{N}}$ associated to the triangle $T(k_1, k_2)$ arises. In a natural way, we define each D_n as the sum of the terms which lie along the n-th arrow of the triangle $T(k_1, k_2)$. In particular $T(1, 1)$ is the usual Pascal's Triangle and its associated sequence is that of Fibonacci.



The following questions arise:

1) Is the sequence $\{D_n\}_{n\in\mathbb{N}}$, of the numbers which arise when we consider the diagonals of a generalized $T(k_1, k_2)$ recursive?

2) Is there any formula that connects the sequences $\{F_n\}_{n\in\mathbb{N}}$, $\{H_n\}_{n\in\mathbb{N}}$ and $\{D_n\}_{n\in\mathbb{N}}$?

In this chapter we give positive answers to both these questions.

4.1 The generalized Pascal's triangle

We can observe that each element of $T(k_1, k_2)$ is the i-th term, from left to the right, of exactly one diagonal D_n as defined before. Therefore we can denote it by $D_{n,i}$: for example $D_{5,1} = 2k_1 + k_2$. Clearly the n-th diagonal has $[\frac{n+1}{2}]$ elements, where $[\frac{n+1}{2}]$ denotes the integer part of $\frac{n+1}{2}$, thus $0 \le i < [\frac{n+1}{2}]$. By definition we have:

$$D_n = \sum_{i=0}^{[\frac{n+1}{2}-1]} D_{n,i} \text{ and } D_{n,i} = D_{n-1,i} + D_{n-2,i-1},\ \forall n \ge 3 \text{ and } \forall i \ge 1.$$

Notice that the first term of every diagonal, $D_{n,0}$, that does not include the vertex of the triangle is k_1 while the last term of "odd" diagonals, $D_{2m-1,m-1}$, is k_2.

The next result will show that each other element of the generalized Pascal's triangle $T(k_1, k_2)$ can be expressed as a linear combination of k_1 and k_2 by means of suitable binomial coefficients.

Proposition 4.1.1 (S. Siani, G. Vincenzi [32]). *For every positive integer $m \geq 3$ the following identities hold:*

$$D_{2m-1,i} = \binom{2m-1-3-(i-1)}{i} k_1 + \binom{2m-1-3-(i-1)}{i-1} k_2, \forall i : 0 < i < m-1$$

$$D_{2m,i} = \binom{2m-3-(i-1)}{i} k_1 + \binom{2m-3-(i-1)}{i-1} k_2, \qquad \forall i : 0 < i < m$$

Proof. If $m = 3$ both the identities are trivially satisfied. Suppose $m > 3$ and proceed by induction on m. Looking to the triangle $T(k_1, k_2)$ it is easy to check, applying Stiefel's identity, that for every $n \geq 5$, $D_{n,1} = \binom{n-3}{1} k_1 + k_2$, therefore if $i = 1$ the identities are satisfied.

If $i = m-1$, the second identity of the statement also holds: applyng Stiefel's identity again, it is not hard to check that the last term lying on the even diagonal D_{2m} is $D_{2m,m-1} = k_1 + \binom{m-1}{1} k_2$, that is equal to:

$$\binom{2m-3-(m-1-1)}{m-1} k_1 + \binom{2m-3-(m-1-1)}{m-2} k_2 =$$

$$\binom{m-1}{m-1} k_1 + \binom{m-1}{m-2} k_2.$$

Thus we may suppose $1 < i < m-1$. Now we can show the first identity of the proposition. By construction we have:

$$D_{2m-1,i} = D_{2m-1-1,i} + D_{2m-1-2,i-1} = D_{2(m-1),i} + D_{2(m-1)-1,i-1}.$$

Clearly $0 < i < m-1$ and $0 < i-1 < m-2$, so that by induction and applying

Stiefel's identity we have:

$$D_{2m-1,i} = \left[\binom{2(m-1)-3-(i-1)}{i} + \binom{2(m-1)-1-3-(i-2)}{i-1}\right]k_1+$$

$$\left[\binom{2(m-1)-3-(i-1)}{i-1} + \binom{2(m-1)-1-3-(i-2)}{i-2}\right]k_2 =$$

$$\binom{2(m-1)-3-(i-1)+1}{i}k_1 + \binom{2(m-1)-3-(i-1)+1}{i-1}k_2 =$$

$$\binom{2m-1-3-(i-1)}{i}k_1 + \binom{2m-1-3-(i-1)}{i-1}k_2.$$

Similarly, it is possible to see that $D_{2m,i} = D_{2m-1,i} + D_{2(m-1),i-1} = D_{2m-1-1,i} + D_{2m-1-2,i-1} + D_{2(m-1),i-1} = D_{2(m-1),i} + D_{2(m-1)-1,i-1} + D_{2(m-1),i-1}$.
Clearly, $0 < i < m-1$ and $0 < i-1 < m-1$, so that by induction we have $D_{2m,i} =$

$$\left[\binom{2(m-1)-3-(i-1)}{i} + \binom{2(m-1)-1-3-(i-2)}{i-1}+\right.$$

$$\left.\binom{2(m-1)-3-(i-2)}{i-1}\right]k_1 + \left[\binom{2(m-1)-3-(i-1)}{i-1}+\right.$$

$$\left.\binom{2(m-1)-1-3-(i-2)}{i-2} + \binom{2(m-1)-3-(i-2)}{i-2}\right]k_2 =$$

$$\binom{2(m-1)-3-(i-1)+2}{i}k_1 + \binom{2(m-1)-3-(i-1)+2}{i-1}k_2 =$$

$$\binom{2m-3-(i-1)}{i}k_1 + \binom{2m-3-(i-1)}{i-1}k_2,$$

as the statement required. □

An immediate consequence of this result is the following combinatorial relation:

Corollary 4.1.1 (S. Siani, G. Vincenzi [32]). *Let k_1 and k_2 be complex numbers, and let $\{D_n\}_{n\in\mathbb{N}}$ be the associate sequence to the generalized Pascal's triangle $T(k_1,k_2)$. Then for every integer $n \geq 5$, we have:*

$$D_n = k_1 + \sum_{i=1}^{m-1}\left[\binom{2m-3-(i-1)}{i}k_1 + \binom{2m-3-(i-1)}{i-1}k_2\right],$$

if $n = 2m$

$$D_n = k_1 + \sum_{i=1}^{m-2}\left[\binom{2m-1-3-(i-1)}{i}k_1 + \binom{2m-1-3-(i-1)}{i-1}k_2\right] + k_2,$$

if $n = 2m-1$.

Remark 4.1.1. It is well known that Lucas' property, recalled in the introduction, can be split in the even and odd case (see Lemma 4.1.1). This can be also detected by Corollary 4.1.1, so we might think the identities stated in Corollary 4.1.1 as a generalized Lucas' property.

Lemma 4.1.1. *Let $\{F_n\}_{n\in\mathbb{N}}$ be the Fibonacci sequence. Then for every positive integer m, we have:*

$$F_{2m} = \sum_{i=0}^{m-1}\binom{2m-1-i}{i}$$

$$F_{2m-1} = \sum_{i=1}^{m} \binom{2m-1-i}{i-1}$$

.

An easy argument by induction also shows the following result.

Lemma 4.1.2. *Let k_1 and k_2 be complex numbers. Let $\{H_n\}_{n\in\mathbb{N}}$ be the Fibonacci-like sequence of seed k_1 and k_2. Then for every integer $n \geq 3$ the following identity holds:*

$$H_n = k_1 F_{n-2} + k_2 F_{n-1}.$$

4.2 The generalized Pascal's triangle and the Fibonacci-like sequence $\{H_n\}_{n\in\mathbb{N}}$

Now we can show the main result.

Theorem 4.2.1 (S. Siani, G. Vincenzi [32])**.** *Let k_1 and k_2 be complex numbers. Let $\{D_n\}_{n\in\mathbb{N}}$ be the associate sequence to the generalized Pascal's triangle $T(k_1, k_2)$ and $\{H_n\}_{n\in\mathbb{N}}$ be the Fibonacci-like sequence of seeds k_1 and k_2. Then the following identity holds:*

$$H_n - D_n = F_{n-3}(k_2 - k_1), \forall n \in \mathbb{N}.$$

Remark 4.2.1. In particular if we consider $\{L_n\}_{n\in\mathbb{N}}$, the *Lucas sequence*, i.e. a Fibonacci-like sequence where $L_0 = 2$ and $L_1 = 1$, we have:

$$L_n - D_n = 2F_{n-3}, \forall n \in \mathbb{N}.$$

Many other relationships of this type can be found in [9]

Proof. It is useful to recall that given a recursive sequence $\{G_n\}_{n\in\mathbb{N}}$, it is possible to define the terms $G_0 = G_2 - G_1, G_{-1} = G_1 - G_0, \dots,$ and so on. For example, if we consider the Fibonacci sequence $\{F_n\}_{n\in\mathbb{N}}$, we may consider $F_0 = 0, F_{-1} = 1, F_{-2} = -1$ and so the statement is true if $n \leq 3$. If $n = 4$, the statement is true since $H_4 - D_4 = (2k_2 + k_1) - (2k_1 + k_2) = k_2 - k_1$.

Suppose now that $n > 4$ and assume that $n = 2m - 1$ where m is a positive integer. By Lemma 4.1.2 and Corollary 4.1.1 we have:

$$H_n - D_n = \left[F_{n-2} - \left(1 + \sum_{i=1}^{m-2} \binom{2m-1-3-(i-1)}{i}\right)\right]k_1 +$$

$$\left[F_{n-1} - \sum_{i=1}^{m-2} \binom{2m-1-3-(i-1)}{i-1} - 1\right]k_2.$$

In addition, Lucas'identities in Lemma 4.1.2 yield:

$$1 + \sum_{i=1}^{m-2} \binom{2(m-1)-i-1}{i} = \sum_{i=0}^{m-2} \binom{2(m-1)-1-i}{i} = F_{n-1},$$

and

$$\sum_{i=1}^{m-2} \binom{2m-1-3-(i-1)}{i-1} + 1 = \sum_{i=1}^{m-2} \binom{2(m-1)-1-i}{i-1} + 1 =$$

$$\sum_{i=1}^{m-1} \binom{2(m-1)-1-i}{i-1} = F_{n-2}.$$

Then it follows that:

$$H_n - D_n = (F_{n-2} - F_{n-1})k_1 + (F_{n-1} - F_{n-2})k_2 = F_{n-3}(k_2 - k_1).$$

Assume now that $n = 2m$ where m is a positive integer. By Lemma 4.1.2 and Corollary 4.1.1 we have:

$$H_n - D_n = \left[F_{n-2} - \left(1 + \sum_{i=1}^{m-1} \binom{2m-3-(i-1)}{i}\right)\right]k_1 +$$

$$\left[F_{n-1} - \sum_{i=1}^{m-1}\binom{2m-3-(i-1)}{i-1}\right]k_2.$$

A new application of Lucas'identities yields:

$$1+\sum_{i=1}^{m-1}\binom{2m-3-(i-1)}{i} = 1+\sum_{j=2}^{m}\binom{2m-1-j}{j-1} = F_{n-1},$$

and

$$\sum_{i=1}^{m-1}\binom{2m-3-(i-1)}{i-1}+1 = \sum_{j=0}^{m-2}\binom{2(m-1)-1-j}{j}+1 = F_{n-2}.$$

Therefore, as in the previous case we have:

$$H_n - D_n = (F_{n-2}-F_{n-1})k_1 + (F_{n-1}-F_{n-2})k_2 = F_{n-3}(k_2-k_1).$$

□

Corollary 4.2.1 (S. Siani, G. Vincenzi [32]). *Let k_1 and k_2 be complex numbers. Let $\{D_n\}_{n\in\mathbb{N}}$ be the associate sequence to the generalized Pascal's triangle $T(k_1,k_2)$. Then*

$$D_n = F_{n-2}k_2 + F_{n-1}k_1, \forall n \in \mathbb{N}.$$

In particular $\{D_n\}_{n\in\mathbb{N}}$ is the Fibonacci-like sequence of seeds k_2 and k_1 and thus it is a recursive sequence.

Proof. The statement is trivial when $n<3$. Let $n>3$ and consider $\{H_n\}_{n\in\mathbb{N}}$, the Fibonacci-like sequence of seeds k_1 and k_2. By Theorem 4.2.1 and Lemma 4.1.2 we have: $D_n = H_n - F_{n-3}(k_2-k_1) = F_{n-2}k_1 + F_{n-1}k_2 - (F_{n-1}-F_{n-2})k_2 + (F_{n-1}-F_{n-2})k_1 = F_{n-2}k_2 + F_{n-1}k_1$.

□

Remark 4.2.2. Another way to prove Theorem4.2.1 and Corollary is the following:

Let $P(0,0) = k_2$ the vertex of $T(k_1, k_2)$, and for any $i \geq 0$ and $0 \leq j \leq i$ let $P(i,j)$ be the (i,j) entry of $T(k_1, k_2)$. Using the Stiefel's identity, an easy argument by induction shows that for any $i \geq 0$ the following relation holds, for any $0 \leq j \leq i$(recall that for $j = i$ and $j = 0$ we have both $\binom{i-1}{i} = 0$ and $\binom{i-1}{-1} = 0$):

$$P(i,j) = k_1\binom{i-1}{j} + k_2\binom{i-1}{j-1}.$$

Now, using the Lucas' property shown in the first paragraph, we can compute the diagonals of $T(k_1, k_2)$:

$$D_{n+1} = P(n,0) + P(n-1,1) + \ldots + P(n-i,i) + \ldots =$$

$$k_1[\binom{n-1}{0} + \binom{n-2}{1} + \ldots + \binom{n-i-1}{i} + \ldots]+$$

$$k_2[\binom{n-2}{0} + \binom{n-3}{1} + \ldots + \binom{n-i-2}{i} + \ldots]$$

and by Lucas'identity we have

$$D_{n+1} = k_1F_n + k_2Fn - 1.$$

Thus,

$$D_n = k_1F_{n-1} + k_2Fn - 2, \forall n \in \mathbb{N}.$$

On the other hand for the generalized Fibonacci-like sequence $\{H_n\}_{n\in\mathbb{N}}$ the well-known closed form gives

$$H_n = F_{n-1}H_2 + F_{n-2}H_1 = k_2F_{n-1} + k_1F_{n-2}, \forall n \in \mathbb{N}.$$

Therefore,

$$H_n - D_n = k_2F_{n-1} + k_1F_{n-2} - k_1F_{n-1} - k_2F_{n-2} = (k_1 - k_2)(F_{n-2} - F_{n-1}) =$$

$$(k_1 - k_2)(-F_{n-3}) = F_{n-3}(k_2 - k_1),$$

and the proof is complete.

Remark 4.2.3. As a consequence of the above results, we may concern any

Fibonacci-like sequence $\{G_n\}_{n\in\mathbb{N}}$ of initial seeds G_1 and G_2, as a sequence appearing as diagonals of the generalized Pascal's triangle $T(G_2, G_1)$.

Example 4.2.1. Let $\mathbb{Z}_5$ be the finite field of order 5, and let k_1 and k_2 be two elements of $\mathbb{Z}_5$. If we have to compute the following

$$s = k_1 + \sum_{i=1}^{10} \left[\binom{22-3-(i-1)}{i} k_1 + \binom{22-3-(i-1)}{i-1} k_2 \right],$$

by Corollary 4.2.1 and Corollary 4.1.1, we have $s = D_22 = F_20k_2 + F_21k_1 = 6765k_2 + 10946k_1 = 0 + k_1 = k_1$.

Bibliography

[1] M. AKBULAK, D. BOZKURT *On the order-m generalized Fibonacci k-numbers.* Chaos Solitons Fract. (3) 42 (2009), 1347-1355.

[2] O. ARAUJO, D.A. MORALES, M. RANDIC *Higher-order Fibonacci numbers.* J. Math. Chem. 20 (1996), 79-94.

[3] B.A. BONDARENKO *Generalized Pascal triangles and pyramids their fractals, graphs and applications.* Tashkent: Izdatel'stvo FAN RUz, 1990. ISBN 5-648-00738-8.

[4] B.A. BONDARENKO *Generalized Pascal triangles and pyramids their fractals, graphs and applications.* http://www.fq.math.ca/Scanned/31-1/bondarenko.pdf.

[5] B. BRUNO, R.E. PHILLIPS *Groups with restricted non-normal subgroups.* Math. Z. 176 (1981), 199-221.

[6] M.S. COHEN, J.J. FERNANDEZ, A.E. PARK, K. SCHMEDDERS *The Fibonacci sequence: relationship to the human hand.* J. Hand Surg. 28A(1) (2002), 157-160.

[7] J.H. CONWAY, R.K. GUY *The book of numbers.* Singapore: World Scientific, 2008.

[8] F. DE GIOVANNI, D.J.S. ROBINSON *Groups with finitely many derived subgroups.* Journal of London Mathematical Society (2) 71 (2005), 658-668.

[9] R.A. DUNLAP *The Golden Ratio and Fibonacci Numbers.* Copernicus, Springer Verlag New York, 1995.

[10] S. FALCON, A. PLAZA *The k-Fibonacci sequence and the Pascal 2-triangle.* Chaos Solitons Fract. 33 (2007), 38-49.

[11] A. FIORENZA, G. VINCENZI *Limit of ratio of consecutive terms for general order-k linear homogeneous recurrences with constant coefficients.* Chaos Solitons Fract. 44 (2011), 147-152.

[12] A.J. FLEMING *Plant mathematics and Fibonaccis flowers.* Nature (2002), 418, 723.

[13] R. GÖBEL, A.T. PARAS, S. SHELAH *Groups isomorphic to all their non-trivial normal subgroups.* Israel J. Math. 129 (2002), 21-27.

[14] M. HERZOG, P. LONGOBARDI, M. MAJ *On the number of commutators in groups.* Ischia Group Theory 2004, 181-192, Contemp. Math. 402, Amer. Math. Soc., Providence, RI, (2006).

[15] L. HODGKIN *A history of mathematics.* Oxford University Press, 2005.

[16] H. HOSOYA *Fibonacci triangle.* Fibonacci Quart. 14 (1976), 173-179.

[17] J.C. LENNOX, H. SMITH, J. WIEGOLD *A problem about normal subgroups.* Journal of Pure and Applied Algebra 88 (1993), 169-171.

[18] P. LONGOBARDI, M. MAJ, D.J.S. ROBINSON, H. SMITH *On groups with two isomorphism classes of derived subgroups.* Glasgow Math. J. (2013), 55 (03).

[19] P. LONGOBARDI, M. MAJ, D.J.S. ROBINSON *Recent results on groups with few isomorphism classes of derived subgroups.* Contemp. Math., 611 (2014), 121-135.

[20] P. LONGOBARDI, M. MAJ, D.J.S. ROBINSON *Locally finite groups with finitely many isomorphism classes of derived subgroups.* Journal of Algebra, 393 (2013), 102-119.

[21] V.D. MAZUROV, E.I. KHUKRHO *Unsolved problems in group theory.* The Kourovka Notebook. arXiv: 1401.0300v5, 2014.

[22] G.A. MILLER, H.C. MORENO *Non-abelian groups in which every subgroup is abelian.* Trans. Amer. Math. Soc., 4 (1903), 398-404.

[23] V.N. OBRAZTSOV *On a problem of P. Hall about groups isomorphic to all their non-trivial normal subgroups.* Proc. London Math. Soc. (3) 75 (1997), 79-98.

[24] A.YU. OL'SHANSKII *Geometry of Defining Relations in Groups.* Nauka, Moscow (1989).

[25] B. PASCAL *Traité du triangle arithmétique avec quelques autres petits traitez sur la mesme matière*. Ouvres completes de Blaise Pascal T3 (1909), 433-593.

[26] J.N. RIDLEY *Packing efficiency in sunflower heads*. Math. Biosci 58 (1982), 129-139.

[27] D.J.S. ROBINSON, *A course in the theory of groups*. Springer-Verlag, 1996.

[28] D.J.S. ROBINSON, *Finiteness conditions and generalized soluble groups, 2 vols.* Springer-Verlag, 1972.

[29] J. SANA *Lucas triangle*. Fibonacci Quart. 21 3 (1983), 192-195.

[30] L.W. SHAPIRO *A Catalan triangle*. Descrete Math. 14 1 (1976), 83-90.

[31] S. SIANI *On Groups with Two Isomorphism Classes of Central Factors*. Submitted.

[32] S. SIANI, G. VINCENZI *Fibonacci-like sequences and generalized Pascal's triangles*. Int. Journal of Math. Education in Science and Technology 45 4 (2014), 609-614.

[33] H. SMITH *On homomorphic images of locally graded groups*. Rend. Sem. Mat. Univ. Padova, Vol. 91 (1994), 53-6.

[34] H. SMITH, J. WIEGOLD *Groups which are isomorphic to their non-abelian subgroups*. Rend. Sem. Mat. Univ. Padova, Vol. 97 (1997).